

The Stochastic Characteristics Method

Applied to a Stochastic Schrödinger Equation

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ABSTRACT. We study a stochastic Schrödinger equation using the method of stochastic characteristics, existence and uniqueness results are provided transforming the stochastic equation into a random family of deterministic equations. As a byproduct of the peculiar technique we easily prove that the solution defines a stochastic flow.

1. INTRODUCTION

In this paper we study the following equation:

$$(1) \quad i du(t, x) + \frac{\Delta}{2} u(t, x) dt - V(x) u(t, x) dt = \sum_{k=1}^m B_k(x) u(t, x) \circ dW^k(t)$$

where the stochastic integral has to be interpreted as a Stratonovich integral⁽¹⁾. We will assume that the operators $B_k = i \sum_{j=1}^d b_{kj}(x) \partial_j$ are self adjoint in $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d; \mathbb{C})$, then equation (1) turns out to be an example of the *quantum filtering equation* considered in [2] by Hudson and Parthasarathy in a completely different framework. Truman and Zastawniak consider in [8, 9] the particular case $k = 1$, $d = 1$ and $b \equiv 1$: using a path integral approach they obtain a rigorous representation to the 1-dimensional stochastic Schrödinger equation in the whole space, see also [10] and the bibliography therein. Our aim is to find an existence and uniqueness result for Eq.(1), applying the method of stochastic characteristics, that consists in transforming the stochastic equation into a random family of equations, see for instance [6, 3, 7] where the same method is developed in the *parabolic* case.

The present paper is organized as follows: in section 2 we introduce the equation for the stochastic characteristic and we illustrate some properties of the solution, in section 3 we prove an equivalence result between Eq.(1) and the random family of equations. In section 4 we provide an existence and uniqueness result for Eq.(1) starting from $u_0 \in H^2(\mathbb{R}^d; \mathbb{C})$ at time $t = 0$ and with $V \equiv 0$, finally in the last section we extend this result to the case with a potential $V \in C_b^2(\mathbb{R}^d)$.

¹See the definition in Kunita,[4], for instance.

2. THE TRANSFORMATION

We consider the following Cauchy problem:

$$(2) \quad \begin{cases} i du(t, x) + \frac{1}{2} \Delta u(t, x) dt - V(x) u(t, x) dt = \\ i \sum_{k=1}^m \sum_{j=1}^d b_{k,j}(x) \partial_j u(t, x) \circ dW^k(t) \\ u(0) = u_0, \end{cases}$$

where V is a real function that acts on u as a multiplication operator, and we will assume that $V \in C_b^2(\mathbb{R}^d)$ and that $W = (W^1, \dots, W^m)$ is an m dimensional Brownian motion. Let \mathcal{F}_t be its natural filtration completed.

We denote by $\langle \cdot, \cdot \rangle$, without indexes, the scalar product in \mathbb{R}^d .

Hypothesis 2.1. *We assume that:*

$A_1)$ $b_{k,j} \in C^4(\mathbb{R}^d; \mathbb{R})$;

$A_2)$ each $b_{k,i}$ is a linear function;

$A_3)$ each vector function b_k , with $b_k = (b_{k,1}, \dots, b_{k,n})^*$, is such that $\text{div } b_k = 0$.

We remark that, eventhought these assumptions are rather restrictive, we can deal with the cases $B = i \frac{d}{dx}$ in dimension 1 and $B = i(x_2 \partial_1 - x_1 \partial_2)$ in dimension 3, that corresponds respectively, to the momentum and the third component of the angular momentum and have some relevance in physics.

Definition 2.1. *Let $u_0 \in H^1(\mathbb{R}^d)$. We say that u is a solution to (2) if $u \in L_a^2(0, T; H^1(\mathbb{R}^d))^{(2)}$ and:*

$$\begin{aligned} (u(t) - u_0, v)_{L^2(\mathbb{R}^d)} &= \frac{i}{2} \int_0^t {}_{H^{-1}(\mathbb{R}^d)} \langle \Delta u(s), v \rangle_{{}_{H^1(\mathbb{R}^d)}} ds \\ &- i \int_0^t (u(s), Vv)_{L^2(\mathbb{R}^d)} ds + \int_0^t \sum_{k=1}^m (\langle b_k, \nabla u(s) \rangle, v)_{L^2(\mathbb{R}^d)} \circ dW^k(s) \end{aligned}$$

for all $v \in H^1(\mathbb{R}^d)$,

We associate to (2) the following equation:

$$(3) \quad \xi(t; 0, x) = x - \sum_{k=1}^m \int_0^t b_k(\xi(s; 0, x)) \circ dW^k(s)$$

the solution $\xi(t, x)$ defines a stochastic flow of homeomorphisms⁽³⁾, such that almost surely :

$$(4) \quad \nu \leq \sup_{x \in \mathbb{R}^d \ t \in [0, T]} |D_i \xi(t, x)| \leq C$$

²We denote by $L_a^2(0, T; H^1(\mathbb{R}^d))$, the space of $L^2(\mathbb{R}^d)$ valued adapted processes ϕ such that $\mathbb{P}\{||\phi(\omega)||_{L^2(0, T; H^1(\mathbb{R}^d))} < \infty\} = 1$

³The Stratonovich integral can be reduced to an Itô integral plus a bounded variation term under our regularity assumptions on the integrand.

for any $i = 1, \dots, d$ and the constants ν and C may depend on ω . These estimates are consequence of the hypothesis on $b_{k,j}$, because:

- (1) $\partial_h \xi$ solves an equation that does not depend on the x variable;
- (2) $\partial_{ih}^2 \xi$ vanishes in every direction.

Similar properties has the inverse process $\eta(t, x)$, that is the solution to the following backward Stratonovich equation (see again [4]):

$$(5) \quad d\eta(s; t, x) = \sum_{k=1}^m b_k(\xi(s; t, x)) \circ dW_b^k(s).$$

Note that under the condition $\operatorname{div} b_{k,j} \equiv 0$, the determinant of the Jacobian of $\xi(t; 0, \cdot)$, denoted by $J_{0,t}$ is identically 1, as it is the solution to:

$$(6) \quad J_{0,t} = 1 - \int_0^t \operatorname{div} b_0(\xi(r; 0, x)) J_{0,r}(x) dr \\ - \sum_{k=1}^m \int_0^t \operatorname{div} b_k(\xi(r; 0, x)) J_{0,r}(x) \circ dW_k(r) = 1,$$

where the final condition is given by the relation: $\eta(s; t, \xi(t; s, x)) = x$.

We introduce the random operator $\Phi_t(\omega) : L^2(\mathbb{R}^d) \mapsto L^2(\mathbb{R}^d)$ defined as

$$\Phi_t(\omega)f(\cdot) \doteq f(\xi(t, \cdot, \omega)),$$

from the properties described above it follows that this operator is a bounded linear and unitary operator on $L^2(\mathbb{R}^d)$.

Moreover its restriction to $D(\Delta) = H^2(\mathbb{R}^d)$ is still a bounded operator, actually $\Phi_t : H^n(\mathbb{R}^d) \rightarrow H^n(\mathbb{R}^d)$, for all integers n , is a bounded operator, as the second derivatives of ξ vanish in every direction. We recall and summarize the properties of the family $\{\Phi_t\}_{t \in [0, T]}$ in the following proposition (we also use (4) and (6) to obtain these properties).

Proposition 2.1. *The family $\{\Phi_t\}_{t \geq 0}$ enjoy the following properties:*

- (1) *for each $t \in [0, T]$ and \mathbb{P} -a.s. the operator Φ_t defines an homeomorphism of \mathbb{R}^d ;*
- (2) *$\Phi_t \in \mathcal{L}(L^2(\mathbb{R}^d))$ and in $L^2(\mathbb{R}^d)$ it holds $\Phi_t^* = \Phi_t^{-1}$;*
- (3) *$\Phi_t \in \mathcal{L}(H^n(\mathbb{R}^d))$ for any integer $n \geq 1$.*

3. AN EQUIVALENCE RESULT

In this section we assume the existence of a weak solution of (2) and we compute the differential of $u(t, \xi(t, x))$. In order to do so, following [1] we compute an Itô-Wentzell formula, using a Galerkin approximation. In this case, as the problem is not of parabolic type, we do not have any smoothing property for the solution, therefore we must adapt the Itô-Wentzell formula for a process that does not have a version in the space of $C^2(\mathbb{R}^d)$.

Before stating the result of this section we give the definition of solution for the random family of equations, consider the following problem:

$$(7) \quad \begin{cases} i \frac{\partial}{\partial t} v(t, x, \omega) = -(\Phi_t^{-1})^*(\omega) \frac{\Delta}{2} \Phi_t^{-1}(\omega) v(t, x, \omega) \\ \quad + \Phi_t(\omega) V \Phi_t^{-1}(\omega) v(t, x, \omega) & t \in [0, T], x \in \mathbb{R}^d \\ v(0, x) = u_0 \end{cases}$$

Definition 3.1. $v(t, x, \omega)$ is a solution of (7) if for all $\omega \in \Omega$, $v(\cdot, \cdot, \omega)$ is a weak solution of (7), in the sense that:

$$(v(t, \omega), \psi)_{L^2(\mathbb{R}^d)} - (u_0, \psi)_{L^2(\mathbb{R}^d)} = \frac{i}{2} \int_0^t \langle (\Phi_s^{-1})^* \Delta \Phi_s^{-1}(\omega) v(s, \omega), \psi \rangle_{H^1(\mathbb{R}^d)} ds \\ - i \int_0^t (v(s, \omega), \Phi_s V \Phi_s^{-1}(\omega) \psi)_{L^2(\mathbb{R}^d)} ds$$

for all $\psi \in H^1(\mathbb{R}^d)$. Moreover v is a process in $L_a^2(0, T; H^1(\mathbb{R}^d))$.

Theorem 3.1. Let $u_0 \in H^1(\mathbb{R}^d)$ then the stochastic differential equation:

$$(8) \quad \begin{cases} i du(t) + \frac{\Delta}{2} u(t) dt - V u(t) dt = \\ \quad i \sum_{k=1}^m \langle b_k, \nabla u(t) \rangle \circ dW(t) \quad t \in [0, T], x \in \mathbb{R}^d \\ u(0, x) = u_0 \end{cases}$$

is equivalent to the following evolution equation, depending on the parameter ω ⁽⁴⁾:

$$(9) \quad \begin{cases} i \frac{\partial}{\partial t} v(t, x, \omega) + (\Phi_t^{-1})^*(\omega) \frac{\Delta}{2} \Phi_t^{-1}(\omega) v(t, x, \omega) = \\ \quad \Phi_t(\omega) V \Phi_t^{-1}(\omega) v(t, x, \omega) = 0 \quad t \in [0, T], x \in \mathbb{R}^d \\ v(0, x) = u_0, \end{cases}$$

in the sense that if $u(t, x, u_0)$ is a solution of Eq.(2) then $v(t, x, \omega) = u(t, \xi(t, x), \omega)$ defines a solution for Eq.(9) and vice-versa.

Proof: For simplicity we will restrict to consider an one dimensional Wiener process. Let e_1, \dots, e_n, \dots be a basis in $H^1(\mathbb{R}^d)$ that is orthonormal in $L^2(\mathbb{R}^d)$. We denote by $K_n = \text{span}\{e_1, \dots, e_n, \dots\}$, and by P_n the corresponding canonical projections of $L^2(\mathbb{R}^d)$ into K_n . In this way, for any $v \in L^2(\mathbb{R}^d)$

$$P_n v(x) = v_n(x) = \sum_{j=1}^n (v, e_j)_{L^2(\mathbb{R}^d)} e_j(x).$$

⁴in the random equation we are considering the adjoint operator of the restriction of Φ_t^{-1} on $H^1(\mathbb{R}^d)$

Using the notation $u(t, \xi(t, x), u_0) = \Phi_t(\omega)u(t, x, u_0)$, we have

$$(10) \quad (P_n(\Phi_t u_n(t, x, u_0)), v)_{L^2(\mathbb{R}^d)} = \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)}$$

therefore:

$$\begin{aligned} & d((P_n(\Phi_t(\omega)P_n u(t, x, u_0)), v)_{L^2(\mathbb{R}^d)}) \\ &= d \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\ &= \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (du(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\ &\quad + \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (d\Phi_t(\omega)e_k, e_j)_{L^2(\mathbb{R}^d)} \\ &= \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (b(x)u_x(t, x, u_0) \circ dW(t), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\ &\quad + i \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)H^{-1}(\mathbb{R}^d)} \left\langle \frac{\Delta}{2} u(t, x, u_0) dt, e_k \right\rangle_{H^1(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\ &\quad - i \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (Vu(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\ &\quad + \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, x, u_0), e_k)_{L^2(\mathbb{R}^d)} (d\Phi_t(\omega)e_k, e_j)_{L^2(\mathbb{R}^d)}. \end{aligned}$$

By the Itô formula we obtain

$$d_s \Phi_t(\omega)e_k = de_k(\xi(t, \cdot)) = \nabla e_k(\xi(t, \cdot))d_s \xi(t, \cdot) = -\langle \nabla e_k(\xi(t, \cdot)), b(\xi(t, \cdot)) \rangle \circ dW(t).$$

Therefore, knowing that $\operatorname{div} b = 0$:

$$\begin{aligned} & \sum_{j,k=1}^n e_j(u(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (d\Phi_t(\omega)e_k, e_j)_{L^2(\mathbb{R}^d)} \\ &= - \sum_{j,k=1}^n e_j(u(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (\langle \nabla e_k(\xi(t)), b(\xi(t)) \rangle \circ dW(t), e_j)_{L^2(\mathbb{R}^d)} \\ &= \sum_{j,k=1}^n e_j(u(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \langle b, \nabla(\Phi_t^{-1}(\omega)e_j) \rangle \circ dW(t))_{L^2(\mathbb{R}^d)} \end{aligned}$$

and substituting the latter formula in the former one, we get

$$\begin{aligned}
& d((P_n(\Phi_t(\omega)P_n u(t, u_0))), v)_{L^2(\mathbb{R}^d)} \\
&= d \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&= \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} \langle b, \nabla u(t, u_0) \rangle \circ dW(t), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&\quad + i \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} {}_{H^{-1}(\mathbb{R}^d)} \langle \frac{\Delta}{2} u(t, u_0) dt, e_k \rangle_{H^1(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&\quad + \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (u(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \langle b, \nabla(\Phi_t^{-1}(\omega)e_j) \rangle \circ dW(t))_{L^2(\mathbb{R}^d)} \\
&\quad - i \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (Vu(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t(\omega)^{-1}e_j)_{L^2(\mathbb{R}^d)} \\
&= \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (\langle b, \nabla u(t, u_0) \rangle \circ dW(t), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&\quad + \frac{i}{2} \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} {}_{H^{-1}(\mathbb{R}^d)} \langle \Delta u(t, u_0) dt, e_k \rangle_{H^1(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&\quad - \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (\langle b, \nabla u(t, u_0) \rangle \circ dW(t), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} \\
&\quad - i \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} (Vu(t, u_0), e_k)_{L^2(\mathbb{R}^d)} (e_k, \Phi_t^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)}
\end{aligned}$$

We write it in the integral form:

$$\begin{aligned}
& (P_n(\Phi_t(\omega)P_n u(t, u_0)), v)_{L^2(\mathbb{R}^d)} = (P_n u_0, v) \\
& + \frac{i}{2} \int_0^t \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} {}_{H^{-1}(\mathbb{R}^d)} \langle \Delta u(s, u_0), e_k \rangle_{H^1(\mathbb{R}^d)} (e_k, \Phi_s^{-1}(\omega)e_j)_{L^2(\mathbb{R}^d)} ds \\
& - i \int_0^t (P_n \Phi_s P_n V u(s, u_0), v)_{L^2(\mathbb{R}^d)} ds.
\end{aligned}$$

The left hand side converges to $(\Phi_t(\omega)u(t, u_0), v)_{L^2(\mathbb{R}^d)}$ as n goes to infinity, as $P_n \rightarrow I$ in $L^2(\mathbb{R}^d)$.

For the same reason also the first and the last members at the right hand side converge respectively to $(u_0, v)_{L^2(\mathbb{R}^d)}$ and to $\int_0^t (\Phi_s V u(s, u_0), v)_{L^2(\mathbb{R}^d)} ds$. We rewrite

the second member at the right hand side:

$$\begin{aligned} & \int_0^t \sum_{j,k=1}^n (e_j, v)_{L^2(\mathbb{R}^d)} \langle \frac{i\Delta}{2} u(s, u_0), e_k \rangle_{H^1(\mathbb{R}^d)} (e_k, \Phi_s^{-1}(\omega) e_j)_{L^2(\mathbb{R}^d)} ds \\ &= \int_0^t \langle \frac{i\Delta}{2} u(s, u_0), P_n \Phi_s^{-1} P_n v \rangle_{H^1(\mathbb{R}^d)} ds, \end{aligned}$$

where Φ_t is the restriction of Φ_t to $H^1(\mathbb{R}^d)$. Passing to the limit, we obtain that the latter integral converges to:

$$(11) \quad \frac{i}{2} \int_0^t \langle (\Phi_s^{-1})^* \Delta u(s, u_0) ds, v \rangle_{H^1(\mathbb{R}^d)}$$

all these convergences hold for almost all $\omega \in \Omega$.

Finally, denoting by $v(s, x, u_0) = \Phi_s(\omega) u(s, x, u_0)$ we have that, for all $v \in H^1(\mathbb{R}^d)$:

$$\begin{aligned} & (v(t, u_0), v)_{L^2(\mathbb{R}^d)} = (u_0, v)_{L^2(\mathbb{R}^d)} \\ & + \frac{i}{2} \int_0^t \langle (\Phi_s^{-1})^* \Delta \Phi_s^{-1}(\omega) v(s, u_0), v \rangle_{H^1(\mathbb{R}^d)} ds \\ & - i \int_0^t (v(s, u_0), \Phi_s V \Phi_s^{-1}(\omega) v)_{L^2(\mathbb{R}^d)} ds. \end{aligned}$$

The converse implication follows exactly in the same way. □

Remark 3.1. Note that the operator $\frac{1}{2}(\Phi_t^{-1})^* \Delta \Phi_t^{-1} - \Phi_t V \Phi_t^{-1}$ is still self adjoint in the space $L^2(\mathbb{R}^d)$ and so, by Stone theorem (see for instance [5]), the operator $i[\frac{1}{2}(\Phi_t^{-1})^* \Delta \Phi_t^{-1} - \Phi_t V \Phi_t^{-1}]$ generates a unitary group at each t fixed.

4. EXISTENCE RESULT FOR THE RANDOM EQUATION

4.1. The case with zero potential. Now we look for a solution of the random family of deterministic equations, starting with the case $V \equiv 0$.

In most of the section we will omit the ω dependence as we will consider the equation pathwise. We rewrite problem (9) in abstract notation:

$$(12) \quad \begin{cases} u'(t) = A(t)u(t) & 0 \leq t \leq T \\ u(0) = u_0 \end{cases}$$

where $A(t) = i\frac{1}{2}(\Phi_t^{-1})^* \Delta \Phi_t^{-1}$. Following [5], we recall some results that we will use in the sequel:

Theorem 4.1 ([5], Theorem 2.2). For $t \in [0, T]$ let $A(t)$ be the infinitesimal generator of a C_0 semigroup $T_t(s)$ on the Banach space X . The family of generators

$\{A(t)\}_{t \in [0, T]}$ is stable if and only if there are constants $M \geq 1$ and ω such that $\rho(A(t)) \supset]\omega, \infty]$ for $t \in [0, T]$ and either one of the following conditions is satisfied:

$$(13) \quad \left\| \prod_{j=1}^k R(\lambda_j, A(t_j)) \right\|_{\mathcal{L}(X)} \leq M \prod_{j=1}^k (\lambda_j - \omega)^{-1} \quad \forall \lambda_j > \omega$$

for any finite sequence $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k = 1, 2, \dots$, or

$$(14) \quad \left\| \prod_{j=1}^k T_{t_j}(s_j) \right\|_{\mathcal{L}(X)} \leq M \exp \left\{ \omega \sum_{j=1}^k s_j \right\} \quad \forall \lambda_j > \omega$$

for any finite sequence $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k = 1, 2, \dots$.

If $T_{t_j}(s_j)$ are contraction semigroups for every $t_j \in [0, T]$, then estimate (14) is verified with $M = 1$ and $\omega = 0$.

We recall the notion of *admissible* subspace of X :

Definition 4.1. *There are given a C_0 -semigroup $T(t)$ and its infinitesimal generator A . A subspace Y of X is called A -admissible if it is an invariant subspace of $\{T(t), t \geq 0\}$, and the restriction of $T(t)$ to Y is a C_0 -semigroup in Y .*

Finally we recall the existence result for the *evolution system*; we are given the following assumptions:

- $H_1)$ $\{A(t)\}_{t \in [0, T]}$ is a stable family with stability constants M, ω ,
- $H_2)$ There exists a dense subspace $Y \subset X$ s.t. Y is $A(t)$ -admissible for every $t \in [0, T]$ and the family $\{\tilde{A}(t)\}_{t \in [0, T]}$ of parts $\tilde{A}(t)^{(5)}$ of $A(t)$ in Y , is a stable family in Y with stability constants $\tilde{M}, \tilde{\omega}$,
- $H_3)$ for $t \in [0, T]$, $D(A(t)) \supset Y$, $A(t)$ is a bounded operator from Y into X and $t \rightarrow A(t)$ is continuous in the $\mathcal{L}(Y, X)$ norm $\|\cdot\|_{Y \rightarrow X}$.

Theorem 4.2 ([5], Theorem 3.1). *Let $A(t)$, $0 \leq t \leq T$, be the infinitesimal generator of a C_0 semigroup $\{T(s), s \geq 0\}$, on X . If the family $\{A(t)\}_{t \in [0, T]}$ satisfies the conditions $H_1 - H_3$ then there exists a unique evolution system $U(t, s)$, $0 \leq s \leq t \leq T$, in X satisfying:*

$$(15) \quad \|U(t, s)\|_{\mathcal{L}(X)} \leq M \exp\{\omega(t - s)\} \quad 0 \leq s \leq t \leq T$$

$$(16) \quad \frac{\partial^+}{\partial t} U(t, s)v \Big|_{t=s} = A(s)v \quad v \in Y, \quad 0 \leq s \leq t \leq T$$

$$(17) \quad \frac{\partial}{\partial s} U(t, s)v = -U(t, s)A(s)v \quad v \in Y, \quad 0 \leq s \leq t \leq T.$$

We will use also the following:

⁵The part of A in Y is defined as the restriction of A to Y , $\tilde{A} : D(\tilde{A}) = \{z \in D(A) \cap Y\} \rightarrow Y$, and $\tilde{A}y = Ay$ for all $y \in D(A)$.

Theorem 4.3 ([5], Theorem 4.3). *Let $A(t)$, $0 \leq t \leq T$, satisfy the conditions of the previous theorem and let $U(t, s)$ be the evolution system defined above. If*

$$(18) \quad U(t, s)Y \subset Y \quad 0 \leq s \leq t \leq T$$

and

$$(19) \quad \text{for } v \in Y \quad U(t, s)v \text{ is continuous in } Y \text{ for } 0 \leq s \leq t \leq T,$$

then for every $v \in Y$, $U(t, s)v$ is the unique Y -valued solution of the initial value problem:

$$(20) \quad \begin{cases} u'(t) = A(t)u(t) & 0 \leq s \leq t \leq T \\ u(s) = v \end{cases}$$

By Y -valued solution we mean the following:

Definition 4.2. *A function $u \in C([0, T]; Y)$ is a Y -valued solution of (20) if $u \in C^1([0, T]; X)$ and (20) is satisfied by u in X .*

We apply this abstract theory to problem (9) derived in the previous section. Actually as we follow a semigroup approach we are lead to consider more regular solutions, i.e. with value in the domain of the operator $A(t) = \frac{i}{2}(\Phi(t)^{-1})^* \Delta \Phi(t)^{-1}$. This domain is constant in time and equivalent to $H^2(\mathbb{R}^d)$ hence, recalling that Φ_t is unitary in $L^2(\mathbb{R}^d)$, we rewrite $A(t)$ as $A(t) \doteq \frac{i}{2}\Phi(t)\Delta\Phi(t)^{-1}$. As first result we prove:

Theorem 4.4. *The family of operators $\{A(t), t \in [0, T]\}$ defined above generates a unique evolution system $U(t, s)$ such that:*

$$\|U(t, s)\|_{\mathcal{L}(X)} \leq C.$$

Proof: We will show that the hypotheses $H_1 - H_3$ are verified in our situation. We will denote by $T_{t_0}(t)$ the group generated by $A(t_0)$, from the definition of C_0 -semigroup it follows that $T_{t_0}(t) = \Phi(t_0)e^{\frac{i}{2}\Delta t}\Phi(t_0)^{-1}$. Hypothesis H_1 corresponds to the stability for the family of $\{A(t)\}_{t \in [0, T]}$ and can be replaced by the two equivalent conditions of theorem 4.1, that are easier to verify in this case. First, by remark 3.1, the condition:

$$\Pi_{i+1}^k \|T_{t_i}(s_i)\|_{\mathcal{L}(X)} \leq 1 \quad s_i \geq 0$$

is fulfilled (with constant $M = 1$ and $\omega = 0$) for any finite sequence $0 \leq t_1 \leq t_2 \leq \dots \leq t_k \leq T$, $k \in \mathbb{N}$. The other condition, $\rho(A(t)) \supset]0, +\infty[$ follows from the observation that:

$$(\lambda I - A(t))^{-1} = \Phi(t)^{-1}(\lambda I - i\Delta)^{-1}\Phi(t)$$

and then $\lambda \in \rho(i\Delta) \Leftrightarrow \lambda \in \rho(A(t))$. Then $\rho(A(t)) = \rho(i\Delta) \supset]0, \infty[$.

Hypothesis H_2 requires that there exists a subset Y of $L^2(\mathbb{R}^d)$ such that for $t \in [0, T]$, Y is $A(t)$ -admissible and the family $\{\tilde{A}(t)\}_{t \in [0, T]}$ of the parts of $A(t)$ is a

stable family in Y with some uniform constants \tilde{M} and $\tilde{\omega}$. In our case we will take as Y the constant domain $H^2(\mathbb{R}^d)$ of each $A(t)$. Since the first derivatives of the η and ξ do not depend on x , the operator $iA(t)$ at every t is a second order operator in divergence form whose matrix of coefficients is constant and symmetric. Therefore $iA(t)$ is still selfadjoint on $H^2(\mathbb{R}^d)$ so $A(t)$ generates a unitary group on $H^2(\mathbb{R}^d)$ at every fixed t ; this conclude the proof of the admissibility. Moreover we can prove the following estimate

$$\|T_{t_0}(t)|_{H^2(\mathbb{R}^d)} u\|_{H^2(\mathbb{R}^d)} \leq \|u\|_{H^2(\mathbb{R}^d)},$$

where $\tilde{M} = 1$ and $\tilde{\omega} = 0$. This verifies the stability condition for the family of the parts $\tilde{A}(t)$, thus H_2 is satisfied.

Hypothesis H_3 requires that $t \mapsto A(t)$ is continuous in the norm of the space $\mathcal{L}(H^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$, the space of bounded and linear operators from $H^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$, but this follows from the properties of $\Phi(t)$ and of the Laplacian. Then we can define the evolution system $U(t, s)$.

□

Our intention is to find a representation for the solution of (12) in terms of the evolution system $U(t, s)$ generated by $A(t)$. In order to apply theorem 4.3 we still have to prove that $U(t, 0)(H^2(\mathbb{R}^d)) \subset H^2(\mathbb{R}^d)$ and that $t \rightarrow U(t, 0)v$ is continuous in $H^2(\mathbb{R}^d)$ for $0 \leq t \leq T$. We collect these results in following lemma:

Lemma 4.1. *The domain $H^2(\mathbb{R}^d)$ is invariant under the action of the evolution system $U(t, s)$ and $t \rightarrow U(t, 0)v$ is a continuous mapping in $H^2(\mathbb{R}^d)$ for any $y \in H^2(\mathbb{R}^d)$.*

Proof: Let us take now as $X = H^2(\mathbb{R}^2)$ and as $Y = H^4(\mathbb{R}^2)$, the constant domain of $\{iA(t), t \in [0, T]\}$ in X . Since $iA(t)$ is also self-adjoint in $H^4(\mathbb{R}^2)$ at each $t \in [0, T]$, then theorem 4.4 holds also in this case. Therefore $\|U(t, s)v\|_{H^2(\mathbb{R}^2)} \leq C\|v\|_{H^2(\mathbb{R}^2)}$, for every $v \in H^2(\mathbb{R}^2)$, by the property of the evolution system $U(t, s)$. This concludes the proof.

□

Finally we can state the following existence theorem:

Theorem 4.5. *Problem (9), with $V \equiv 0$, has a unique $H^2(\mathbb{R}^d)$ -valued solution $v(t, \cdot, \omega) = U(t, 0, \omega)u_0$ for all $u_0 \in H^2(\mathbb{R}^d)$ and almost all $\omega \in \Omega$. $U(t, s, \omega)$ is the evolution system defined in theorem 4.4 that is unitary, i.e. $\|U(t, s, \omega)u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}$ for all $u_0 \in H^2(\mathbb{R}^d)$ and for almost all ω . This solution $v \in C([0, T]; H^2(\mathbb{R}^d)) \cap C^1([0, T]; L^2(\mathbb{R}^d))$ is \mathcal{F}_t adapted.*

Proof: The existence, uniqueness and regularity results for the solution are consequence of the previous results. Finally, if one evaluate the energy of the solution

v of (12):

$$\operatorname{Re} \left[(v'(t), v(t))_{L^2(\mathbb{R}^d)} - i(\Phi_t \frac{\Delta}{2} \Phi_t^{-1} v(t), v(t))_{L^2(\mathbb{R}^d)} \right] = \frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2(\mathbb{R}^d)}^2 = 0$$

this implies, passing to the real part of the equation, that

$$(21) \quad \|u(t, \omega)\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}.$$

At this level of regularity this solution is in particular a solution of problem (9) in the sense of definition 3.1, so $\Phi_t^{-1}v$ defines a solution for problem (2). The adaptedness of v is a consequence of the peculiar construction of the solution: $U(t, 0)u_0$ is obtained as a limit of unitary groups depending only on $s \leq t$. In this way, starting from an adapted initial datum the solution remains always adapted. \square

As a byproduct of the equivalence result, we have:

Corollary 4.1. *The unique solution $u(t, x) = \Phi_t^{-1}(v)(t, x)$ is a stochastic flow.*

Proof: Let $u(t; s, u_0)$ be the solution to (2) starting at time $s > 0$. From the property of backward flow η , $\eta(s; t, x) = \eta(s; r, \eta(t, x))$ we get that there exists the random evolution operator $G(t, s)u_0 = u(t; s, u_0) = \Phi_t^{-1}U(t, s)u_0$. Therefore as both Φ_t^{-1} and $U(t, s)$ are unitary we have that

$$\|G(t, s)u_0\|_{L^2(\mathbb{R}^d)} = \|\Phi_t^{-1}U(t, s)u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}.$$

\square

4.2. The case with bounded potentials. In this section we add the potential V assuming that it is a real function in $C_b^2(\mathbb{R}^d)$, then the family of deterministic equations becomes, written in abstract form:

$$(22) \quad \begin{cases} u'(t) = A(t)u(t) - iV(t)u(t) & 0 \leq t \leq T \\ u(0) = u_0 \end{cases}$$

where $V(t) = \Phi_t V \Phi_t^{-1}$. Let us notice first that $\|V(t)\|_{C_b^2(\mathbb{R}^d)} \leq C\|V\|_{C_b^2(\mathbb{R}^d)}$. We treat the potential as a perturbation of the operator A , and we can prove the following:

Theorem 4.6. *If $V \in C_b^2(\mathbb{R}^d)$, then problem (9) has a solution with the same properties as the solution found in theorem 4.5.*

Proof: The function V is a bounded self adjoint operator from $L^2(\mathbb{R}^d)$ into itself and a bounded operator in $H^2(\mathbb{R}^d)$. Hence it is a standard fact that the sum $iA(t) + V(t)$ is still self-adjoint in $L^2(\mathbb{R}^d)$ for every $t \in [0, T]$, therefore $A(t) - iV(t)$ generates a unitary group in $L^2(\mathbb{R}^d)$, and hypothesis H_1 is verified. As the domain is still $H^2(\mathbb{R}^d)$ we choose again it as the $[A(t) - iV(t)]$ -admissible subspace, required

in hypothesis H_2 .

We need to show that the part of $A(t) - iV(t)$ in $H^2(\mathbb{R}^d)$, that is the sum of the parts $\tilde{A}(t)$ and $-i\tilde{V}(t)$, is still an infinitesimal generator of a C_0 semigroup. For this purpose we will apply the Hille-Yoshida Theorem, see [5, Theorem 5.3]. We verify the two conditions required to prove the theorem. The first is immediately verified, as $i\tilde{V}(t)$ is continuous in $H^2(\mathbb{R}^d)$ and $\tilde{A}(t)$ is a generator in $H^2(\mathbb{R}^d)$ hence the sum $\tilde{A}(t) - i\tilde{V}(t)$ is a closed operator in $H^2(\mathbb{R}^d)$ for all $t \in [0, T]$.

Then we need to show that there exists a λ_0 such that the ray $\{\lambda \in \mathbb{C}, \operatorname{Im} \lambda = 0, \lambda > \lambda_0\}$ is contained in the resolvent set $\rho(\tilde{A}(t))$. Moreover we have to prove an estimate for the resolvent $R(\lambda; A(t) - iV(t))$ of the following form:

$$(23) \quad \|R(\lambda; \tilde{A}(t) - i\tilde{V}(t))^n\|_{H^2(\mathbb{R}^d)} \leq \frac{M}{(\lambda - \lambda_0)^n} \quad \forall \lambda > \lambda_0.$$

Since $\tilde{A}(t)$ is the generator of a C_0 semigroup with $\tilde{\omega} = 0$ and $\tilde{M} = 1$, then

$$\|R(\lambda; \tilde{A}(t))\|_{H^2(\mathbb{R}^d)} \leq \frac{1}{\lambda^n}, \quad \forall \lambda > 0,$$

in our case we have that:

$$(24) \quad \begin{aligned} \|R(\lambda; \tilde{A}(t) - i\tilde{V}(t))^n\|_{H^2(\mathbb{R}^d)} &= \|R(\lambda; \tilde{A}(t))^n (I - i\tilde{V}(t)R(\lambda; \tilde{A}(t)))^{-n}\|_{H^2(\mathbb{R}^d)} \\ &\leq \frac{1}{\lambda^n} \leq \frac{1}{(\lambda - \lambda_0)^n} \end{aligned}$$

for all $\lambda > \|\tilde{V}\|_{C_b^2(\mathbb{R}^d)} = \lambda_0$. Therefore the operator $A(t) - iV(t)$ generates a C_0 semigroup of contraction in $H^2(\mathbb{R}^d)$, with constants $\tilde{M} = 1$ and $\tilde{\omega} = 0$, hence the family of parts $\tilde{A}(t) - i\tilde{V}(t)$ is stable, as it is required in H_2 .

The proofs of condition H_3 and of lemma 4.1 are easily deducible from the free case ($V \equiv 0$), as the domain $H^2(\mathbb{R}^d)$ is equivalent the graph of the operator $A(t)$ and the solution can be written as:

$$(25) \quad v(t; 0, u_0) = U(t, 0)u_0 + \int_0^t U(t, s)v(s; 0, u_0) ds$$

where $U(t, s)$ is generated by the $\{A(t), t \geq 0\}$.

□

We conclude with a remark: using the same technique it is possible to treat also an equation of the type⁽⁶⁾:

$$(26) \quad \begin{cases} i du(t, x) + \frac{1}{2} \Delta u(t, x) dt - V(x)u(t, x) dt = \sum_{k=1}^m c_k(t, x)u(t, x) \circ dW_k(t) \\ u(0) = u_0 \end{cases}$$

⁶This equation has been considered also in [8, 9].

where c_k are real regular functions.

In this case we transform it into a deterministic family of equations evaluating the Itô formula for the product $u(t, x)\rho(t, x)$, where ρ is the process defined as $\rho(t, x) = \exp(i \sum_{k=1}^m c_k(s, x) \circ dW_k(s))$. The flow ρ defines a unitary operator $\Phi_t v(\cdot) = v(\cdot)\rho(t, \cdot)$, and using the same approach we get results similar to those found in the case considered here.

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