Ergodic Optimal Quadratic Control for an Affine Equation with Stochastic and Stationary Coefficients

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1 Introduction

In this paper we study an ergodic quadratic control problem for a linear affine equation with control dependent noise, namely we characterize the ergodic limit where the coefficients of the state equation, which we take random, are assumed to be stationary. We continue our previous work [4], where the infinite horizon case and the ergodic case are studied, but no characterization of the egodic limit was given: in order to do this we assume the coefficients to be stationary in a suitable sense, see [11] and section 2 below.

Backward Stochastic Riccati Equations (BSREs) are naturally linked with stochastic optimal control problems with stochastic coefficients. The first existence and uniqueness result for such a kind of equations has been given by Bismut in [2], but then several works, see e. g. [3], [6], [7], [8], [9] followed. Only very recently Tang in [10] solved the general non singular case corresponding to the linear quadratic problem with random coefficients and control dependent noise, and then in [4], the infinite horizon case and the ergodic case are studied. Namely, we consider a cost functional depending only on the asymptotic behaviour of the state (ergodic control). To do it we first consider the stationary problem, and we are able to prove that there exists an optimal pair $(u^\natural, X^\natural)$, such that

$$(u^\natural, X^\natural) \in \mathcal{U}^\natural = \left\{(u, X) \in L^2_{\mathcal{P}}(\Omega \times [0,1]) \times C([0,1], L^2_{\mathcal{P}}(\Omega)) : X_s = X_0 \circ \theta_s, \forall s \in \mathbb{R}\right\}$$

where θ is the shift operator and $X^\natural$ is the solution of equation

$$dX^\natural_t = A_t X^\natural_t dt + B_t u^\natural_t dt + \sum_{i=1}^d C^i_t X^\natural_t dW^i_t + \sum_{i=1}^d D^i_t u^\natural_t dW^i_t + f_t dt, \quad (1.1)$$

It turns out that the optimal cost is given by

$$\overline{J}^\natural = J^\natural(u^\natural, X^\natural) = 2\mathbb{E}\int_0^1 \langle r^\natural_s, f_s\rangle ds - \mathbb{E}\int_0^1 |(I+\sum_{i=1}^d \left(D^i_t\right)^* \overline{P}_t D^i_t)^{-1}(B^*_t r^\natural_t + \sum_{i=1}^d \left(D^i_t\right)^* g^{\natural,i}_t)|^2 ds. \quad (1.2)$$

and the following feedback law holds true:

$$u^\natural_t = -\left(I + \sum_{i=1}^d \left(D^i_t\right)^* P_t D^i_t\right)^{-1}\left(P_t B_t + \sum_{i=1}^d \left(Q^i_t D^i_t + \left(C^i_t\right)^* Q_t D^i_t\right)\right)^* X^\natural_t + B^*_t r^\natural_t + \sum_{i=1}^d (D^i_t)^* g^{\natural,i}_t.$$

The main technical point of this paper is to prove that the closed loop equation for the stationary control problem,

$$dX_s = H_s X_s ds + \sum_{i=1}^d K^i_s X_s dW^i_s + B_s(B^*_s r^\natural_s + \sum_{i=1}^d D^i_s g^{\natural,i}_s) ds + f_s ds + \sum_{i=1}^d D^i_s (B^*_s r^\natural_s + \sum_{i=1}^d D^i_s g^{\natural,i}_s) dW^i_s, \quad (1.3)$$

admits a unique stationary solution, see proposition 2.10.

In order to study the ergodic control problem, we first consider the discounted cost functional

$$J^\alpha(0, x, u) = \mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\left\langle S_s X^{0,x,u}_s, X^{0,x,u}_s\right\rangle + |u_s|^2] ds, \quad (1.4)$$

where X is solution to equation

$$\begin{cases} dX_s = (A_s X_s + B_s u_s)ds + \sum_{i=1}^{d} \left(C_s^i X_s + D_s^i u_s\right) dW_s^i + f_s ds \; s \geq 0 \\ X_0 = x. \end{cases} \tag{1.5}$$

A, B, C and D are bounded random and stationary processes and $f \in L^\infty_{\mathcal{P}}(\Omega \times [0,+\infty), \mathbb{R}^n)$. It is proved in [4] that

$$\begin{aligned} &\underline{\lim}_{\alpha\to 0}\alpha \overline{J}^\alpha(x) = \underline{\lim}_{\alpha\to 0}\alpha\mathbb{E}\int_0^{+\infty} 2\langle r_s^\alpha, f_s^\alpha\rangle ds \\ &- \overline{\lim}_{\alpha\to 0}\alpha\mathbb{E}\int_0^{+\infty} |(I + \sum_{i=1}^{d} \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1}(B_s^* r_s^\alpha + \sum_{i=1}^{d} \left(D_s^i\right)^* g_s^{\alpha,i})|^2 ds. \end{aligned}$$

Starting from this point, we can prove that in the stationary case this optimal cost is given by (1.2), namely

$$\underline{\lim}_{\alpha\to 0} 2\alpha \overline{J}^\alpha(x) = \overline{J}^\natural$$

The final step is is to minimize the following functional

$$\widehat{J}(x,u) = \underline{\lim}_{\alpha\to 0} 2\alpha J(x,u)$$

over all $u \in \widehat{\mathcal{U}}$, where

$$\widehat{\mathcal{U}} = \left\{ u \in L^2_{loc} : \mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\left\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\right\rangle + |u_s|^2] ds < +\infty, \; \forall \alpha > 0. \right\}$$

We prove that

$$\inf_{u\in\widehat{\mathcal{U}}} \widehat{J}(x,u) = J^\natural(u).$$

and this concludes the characterization of the ergodic optimal cost.

2 Linear Quadratic optimal control in the stationary case

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{F}_t)_{t\geq 0}$. Assume that $W : (-\infty, +\infty) \to \mathbb{R}$ is a d-dimensional brownian motion defined on the whole real axis. For all $s, t \in \mathbb{R}$ with $t \geq s$ we denote by $\mathcal{G}_t^s$ the σ-field generated by $\{W_\tau - W_s, s \leq \tau \leq t\}$. Notice that for all

$s \in \mathbb{R}$, $\{\mathcal{G}_t^s\}_{t \geq s}$ is a filtration in $(\Omega, \mathcal{F})$. Finally we assume that for all $s < 0$, $\mathcal{G}_0^s \subseteq \mathcal{F}_0$.

Next we set a stationary framework: we introduce the semigroup $(\theta_t)_{t \in \mathbb{R}}$ of measurable mappings $\theta_t : (\Omega, \mathcal{E}) \to (\Omega, \mathcal{E})$ verifying

(1) $\theta_0 = \mathrm{Id}$, $\theta_t \circ \theta_s = \theta_{t+s}$, for all $t, s \in \mathbb{R}$
(2) θ_t is measurable: $(\Omega, \mathcal{F}_t) \to (\Omega, \mathcal{F}_0)$ and $\{\{\theta_t \in A\} : A \in \mathcal{F}_0\} = \mathcal{F}_t$
(3) $\mathbb{P}\{\theta_t \in A\} = \mathbb{P}(A)$ for all $A \in \mathcal{F}_0$
(4) $W_t \circ \theta_s = W_{t+s} - W_s$

According to this framework we introduce the definition of stationary stochastic process.

Definition 2.1 *We say that a stochastic process $X : [0, \infty[\times \Omega \to \mathbb{R}^m$, is stationary if for all $s \in \mathbb{R}$*

$$X_t \circ \theta_s = X_{t+s} \qquad \mathbb{P}\text{-}a.s. \text{ for a.e. } t \geq 0$$

We assume all the coefficients A, B, C, D and S to be stationary stochastic processes. Namely on the coefficients we make the following assumptions:

Hypothesis 2.2

A1) $A : [0, +\infty) \times \Omega \to \mathbb{R}^{n \times n}$, $B : [0, +\infty) \times \Omega \to \mathbb{R}^{n \times k}$, $C^i : [0, +\infty) \times \Omega \to \mathbb{R}^{n \times n}$, $i = 1, ..., d$ *and* $D^i : [0, +\infty) \times \Omega \to \mathbb{R}^{n \times k}$, $i = 1, ..., d$, *are uniformly bounded process adapted to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$.

A2) $S : [0, +\infty) \times \Omega \to \mathbb{R}^{n \times n}$ *is uniformly bounded and adapted to the filtration* $\{\mathcal{F}_t\}_{t \geq 0}$ *and it is almost surely and almost everywhere symmetric and nonnegative. Moreover we assume that there exists* $\beta > 0$ *such that* $S \geq \beta I$.

A3) A, B, C, D *and* S *are stationary processes.*

In this case we immediately get:

Lemma 2.3 *Fix* $T > 0$ *and let hypothesis 2.2 holds true. Let* (P, Q) *be* *the solution of the finite horizon BSRE*

$$\begin{cases} -dP_t = G\left(A_t, B_t, C_t, D_t; S_t; P_t, Q_t\right) dt + \sum_{i=1}^{d} Q_t^i dW_t^i, & t \in [0,T] \\ P_T = P_T. \end{cases} \tag{2.1}$$

For fixed $s > 0$ *we define* $\widehat{P}(t+s) = P(t)\theta_s$, $\widehat{Q}(t+s) = Q(t)\theta_s$ *then* $(\widehat{P}, \widehat{Q})$ *is the unique solution in* $[s, T+s]$ *of the equation*

$$\begin{cases} -d\widehat{P}_t = G\left(A_t, B_t, C_t, D_t; S_t; \widehat{P}_t, \widehat{Q}_t\right) dt + \sum_{i=1}^{d} \widehat{Q}_t^i dW_t^i, & t \in [s, T+s] \\ \widehat{P}_T = P_T \circ \theta_s. \end{cases} \tag{2.2}$$

In the stationary assumptions the backward stochastic Riccati equation

$$dP_t = -\left[A_t^* P_t + P_t A_t + S_t + \sum_{i=1}^{d} \left(\left(C_t^i\right)^* P_t C_t^i + \left(C_t^i\right)^* Q_t + Q_t C_t^i\right)\right] dt + \sum_{i=1}^{d} Q_t^i dW_t^i + \tag{2.3}$$

$$\left[P_t B_t + \sum_{i=1}^{d} \left(\left(C_t^i\right)^* P_t D_t^i + Q^i D_t^i\right)\right] \left[I + \sum_{i=1}^{d} \left(D_t^i\right)^* P_t D_t^i\right]^{-1} \left[P_t B_t + \sum_{i=1}^{d} \left(\left(C_t^i\right)^* P_t D_t^i + Q_t^i D_t^i\right)\right] \tag{2.4}$$

admits a minimal solution $(\overline{P}, \overline{Q})$, in the sense that whenever another couple (P, Q) is a solution to the Riccati equation then $P - \overline{P}$ is a non-negative matrix, see also Corollary 3.3 in [5] and definiton 3.2 in [4]. This minimal solution $(\overline{P}, \overline{Q})$ turns out to be stationary.

Proposition 2.4 *Assume Hypothesis 2.2, then the minimal solution* $(\overline{P}, \overline{Q})$ *of the infinite horizon stochastic Riccati equation* (2.3) *is stationary.*

Proof. For all $\rho > 0$ we denote by P^ρ the solution of equation (2.1) in $[0, \rho]$ with final condition $P^\rho(\rho) = 0$. Denoting by $\lfloor \rho \rfloor$ the integer part of ρ, we have, following Proposition 3.2 in [5] that for all N for all $t \in [0, \lfloor N+s \rfloor]$, $P_t^{\lfloor N+s \rfloor} \leq P_t^{N+s} \leq P_t^{\lfloor N+s \rfloor + 1}$, $\mathbb{P}$-a.s.. Thus we can conclude noticing that by lemma 2.2

$$P_{t+s}^{N+s} = P_t^N \circ \theta_s.$$

Thus letting $N \to +\infty$ we obtain that for all $t \geq 0$, and $s > 0$:

$$\mathbb{P}\left\{\overline{P}_{t+s} = \overline{P}_t \circ \theta_s\right\} = 1.$$

Now $\overline{P}_{T+s} = \overline{P}_T \circ \theta_s = \overline{P}_T$ so if one consider (2.1) in the intervall $[s, T+s]$ with final data $\overline{P}_{T+s}$ and (2.2) with final data $\overline{P}_T \circ \theta_s$, by the uniqueness of the solution it follows that $Q_r = \hat{Q}_r$, $\mathbb{P}-$ a.s. and for all $r \in [s, T+s]$. □

We notice that in the BSRDE (2.3) the final condition has been replaced by the stationarity condition on the solution process (P, Q).

Next we give some definitions.

Definition 2.5 *We say that (A, B, C, D) is stabilizable relatively to the observations $\sqrt{S}$ (or $\sqrt{S}$-stabilizable) if there exists a control $u \in L^2_{\mathcal{P}}([0, +\infty) \times \Omega; U)$ such that for all $t \geq 0$ and all $x \in \mathbb{R}^n$*

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} [\langle S_s X_s^{t,x,u}, X_s^{t,x,u} \rangle + |u_s|^2] ds < M_{t,x}. \tag{2.5}$$

for some positive constant $M_{t,x}$ where $X^{t,x,u}$ is the solution of the linear equation

$$\begin{cases} dX_s = (A_s X_s + B_s u_s) ds + \sum_{i=1}^d \left(C_s^i X_s + D_s^i u_s \right) dW_s^i \; s \geq 0 \\ X_0 = x. \end{cases} \tag{2.6}$$

This kind of stabilizability condition has been introduced in [5]. This condition has been proved to be equivalent to the existence of a minimal solution $(\bar{P}, \bar{Q})$ of the Riccati equation (2.3). Moreover whenever the first component $\bar{P}$ is uniformly bounded in time it follows that the constant $M_{t,x}$ appearing in (2.7) can be chosen independent of time.

Definition 2.6 *Let P be a solution to equation* (2.3). *We say that P stabilizes (A, B, C, D) relatively to the identity I if for every $t > 0$ and*

$x \in \mathbb{R}^n$ there exists a positive constant M, independent of t, such that

$$\mathbb{E}^{\mathcal{F}_t} \int_t^{+\infty} |X^{t,x}(r)|^2 dr \leq M \qquad \mathbb{P} - a.s., \tag{2.7}$$

where $X^{t,x}$ is a mild solution to:

$$\begin{cases} d\overline{X}_t = \left[A\overline{X}_t - B_t \left(I + \sum_{i=1}^{d} \left(D_t^i\right)^* P_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^{d} \left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i \right) \right)^* \overline{X}_t \right] dt + \\ \sum_{i=1}^{d} \left[C_s^i \overline{X}_t - D_s^i \left(I + \sum_{i=1}^{d} \left(D_t^i\right)^* P_t D_t^i \right)^{-1} \left(P_t B_t + \sum_{i=1}^{d} \left(Q_t^i D_t^i + \left(C_t^i\right)^* P_t D_t^i \right) \right)^* \overline{X}_t \right] dW_t, \\ \overline{X}_0 = x \end{cases} \tag{2.8}$$

From now on we assume that

Hypothesis 2.7

(i) (A, B, C, D) is $\sqrt{S}$- stabilizable;

(ii) the process $\overline{P}$ is uniformly bounded in time;

(iii) the minimal solution $\bar{P}$ stabilizes (A, B, C, D) with respect to the identity I.

We refer to [4] for cases when P stabilizes (A, B, C, D) relatively to the identity I. Notice that, thanks to the stationarity assumptions the stabilizability condition can be simplified, see Remark 5.7 of [5].

Next we study the dual (costate) equation in the stationary case. We denote by

$$\begin{aligned} &\Lambda\left(t, \overline{P}_t, \overline{Q}_t\right) = -\left(I + \sum_{i=1}^{d} \left(D_t^i\right)^* \overline{P}_t D_t^i \right)^{-1} \left(\overline{P}_t B_t + \sum_{i=1}^{d} \left(\overline{Q}_t^i D_t^i + \left(C_t^i\right)^* \overline{P}_t D_t^i \right) \right)^*, \\ &H_t = A_t + B_t \Lambda\left(t, \overline{P}_t, \overline{Q}_t\right), \\ &K_t^i = C_t^i + D_t^i \Lambda\left(t, \overline{P}_t, \overline{Q}_t\right). \end{aligned} \tag{2.9}$$

Thanks to proposition 2.4, all the coefficients that appear in equation

$$\begin{cases} dr_t = -H_t^* r_t dt - \bar{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \; t \in [0,T] \\ r_T = 0. \end{cases} \tag{2.10}$$

are stationary so exactly as before we deduce that for the solution (r_T, g_T) the following holds:

Lemma 2.8 *Let A, B, C, D and S satisfy hypothesis 2.2 and let $f \in L^\infty_{\mathcal{P}}(\Omega \times [0,+\infty))$ be a stationary process. Fix $T > 0$ and $r_T \in L^\infty_{\mathcal{P}}(\Omega, \mathcal{F}_T; \mathbb{R}^n)$. Let (r, g) a solution to equation*

$$\begin{cases} dr_t = -H_t^* r_t dt - \bar{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \; t \in [0,T] \\ r_T = r_T. \end{cases} \tag{2.11}$$

For fixed $s > 0$ we define $\hat{r}_{t+s} = r_t \circ \theta_s$, $\hat{g}_{t+s} = g_t \circ \theta_s$ then $(\hat{r}, \hat{g})$ is the unique solution in $[s, T+s]$ of the equation

$$\begin{cases} d\hat{r}_t = -H_t^* \hat{r}_t dt - \bar{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* \hat{g}_t^i dt + \sum_{i=1}^{d} \hat{g}_t^i dW_t^i, \; t \in [s, T+s] \\ \hat{r}_T = r_T \circ \theta_s. \end{cases} \tag{2.12}$$

Hence arguing as for the first component $\overline{P}$, we get that the solution of the infinite horizon dual equation is stationary, as stated in the following proposition:

Proposition 2.9 *Assume hypothesis 2.2 and hypothesis 2.7, then the solution $(r^\natural, g^\natural)$ of*

$$dr_t = -H_t^* r_t dt - \overline{P}_t f_t dt - \sum_{i=1}^{d} \left(K_t^i\right)^* g_t^i dt + \sum_{i=1}^{d} g_t^i dW_t^i, \tag{2.13}$$

obtained as the pointwise limit of the solution to equation 2.10 is stationary. Moreover $\left(r^\natural, g^\natural\right) \in L^\infty_{\mathcal{P}}(\Omega \times [0,1], \mathbb{R}^n) \times L^2_{\mathcal{P}}\left(\Omega \times [0,1], \mathbb{R}^{n\times d}\right)$.

Proof. The proof follows from an argument similar to the one in proposition 4.5 in [4]. Stationarity of the solution $(r^\natural, g^\natural)$ follows from the previous lemma. □

Again we notice that in the dual BSDE (2.13) the final condition has been replaced by the stationarity condition on the solution process $(r^\natural, g^\natural)$.

We need to show that in the stationary assumptions, the solution of the closed loop equation is stationary. By using notation (2.9), we consider the following stochastic differential equation, which will turn out to be the closed loop equation:

$$dX_s = H_s X_s ds + \sum_{i=1}^d K_s^i X_s dW_s^i + B_s(B_s^* r_s^\natural + \sum_{i=1}^d D_s^i g_s^{\natural,i}) ds + f_s ds + \sum_{i=1}^d D_s^i (B_s^* r_s^\natural + \sum_{i=1}^d D_s^i g_s^{\natural,i}) dW_s^i, \tag{2.14}$$

where $(r^\natural, g^\natural)$ is the solution of the dual (costate) equation (2.13).

Proposition 2.10 *Assume hypothesis 2.2 and hypothesis 2.7 holds true then there exists a unique stationary solution of equation* (2.14).

Proof. We set $f_s^1 = f_s + B_s(B_s^* r_s^\natural + \sum_{i=1}^d D_s^i g_s^{\natural,i})$ and $f_s^{2,j} = D_s^j (B_s^* r_s^\natural + \sum_{i=1}^d D_s^i g_s^{\natural,i})$, $j = 1, ..., d$. We can extend f^1, f^2 for negative times letting for all $t \in [0,1]$, $f_{-N+t}^i = f_t^i \circ \theta_{-N}$, $i = 1, 2$, $N \in \mathbb{N}$. We notice that $f^i \mid_{[-N,+\infty)}$ is predictable with respect to the filtration $(\mathcal{G}_t^{-N})_{t \geq -N}$. Therefore for all $N \in \mathbb{N}$ equation

$$\begin{cases} dX_s^{-N} = H_s X_s^{-N} ds + \ + \sum_{i=1}^d K_s^i X_s^{-N} dW_s^i + f_s^1 ds + \sum_{i=1}^d f_s^{2,i} dW_s^i, \\ X_{-N}^{-N} = 0, \end{cases}$$

admits a solution $(X_t^{-N,0})_t$, defined for $t \geq -N$ and predictable with respect to the filtration $(\mathcal{G}_t^{-N})_{t \geq -N}$. We extend $X^{-N,0}$ to the whole real axis by setting $X_t^{-N,0} = 0$ for $t < -N$. We want to prove that, fixed $t \in \mathbb{R}$, $(X_t^{-N,0})_N$ is a Cauchy sequence in $L^2(\Omega)$. In order to do this we notice that for $t \geq -N+1$, $X_t^{-N,0} - X_t^{-N+1,0}$ solves the following (linear) stochastic differential equation

$$X_t^{-N,0}-X_t^{-N+1,0}=X_{-N+1}^{-N,0}+\int_{-N+1}^t H_s(X_s^{-N,0}-X_s^{-N+1,0})+\sum_{i=1}^d\int_{-N+1}^t K_s^i(X_s^{-N,0}-X_s^{-N+1,0})dW$$

By the Datko theorem, see e.g. [4] and [5], there exist constants $a,c>0$ such that

$$(\mathbb{E}|X_t^{-N,0}-X_t^{-N+1,0}|^2)^{1/2}\leq Ce^{-\frac{a(t+N-1)}{2}}(\mathbb{E}|X_{-N+1}^{-N,0}|^2)^{1/2}.$$

So, fixed $t\in\mathbb{R}$ and $M,N\in\mathbb{N}$, $M>N$ sufficiently large such that $-N\leq t$,

$$(\mathbb{E}|X_t^{-N,0}-X_t^{-M,0}|^2)^{1/2}\leq\sum_{k=N}^{M-1}(\mathbb{E}|X_t^{-k,0}-X_t^{-k+1,0}|^2)^{1/2}\leq C\sum_{k=N}^{M-1}e^{-\frac{a(t+k-1)}{2}}(\mathbb{E}|X_{-k+1}^{-k,0}|^2)^{1/2}. \tag{2.15}$$

Next we look for a uniform estimate with respect to k of $\mathbb{E}|X_{-k+1}^{-k,0}|^2$. For $s\in[-k,-k+1]$,

$$X_s^{-k,0}=\int_{-k}^s A_rX_r^{-k,0}dr+\int_{-k}^s B_r\bar{u}_rdr+\sum_{i=1}^d\int_{-k}^s C_r^iX_r^{-k,0}dW_r^i+\sum_{i=1}^d\int_{-k}^s D_r^i\bar{u}_rdW_r^i+\int_{-k}^s f_rdr, \tag{2.16}$$

where $\bar{u}$ is the optimal control that minimizes the cost

$$J(-k,0,u)=\int_{-k}^{-k+1}[|\sqrt{S_s}X_s|^2+|u_s|^2]ds.$$

By computing $d\langle\overline{P}_sX_s^{-k,0},X_s^{-k,0}\rangle+2\langle\bar{r}_s^\natural,X_s^{-k,0}\rangle$ we get, for every $T>0$,

$$\mathbb{E}\int_{-k}^{-k+1}[|\sqrt{S_s}X_s|^2+|\bar{u}_s|^2]ds=-\mathbb{E}\langle\overline{P}_{-k+1}X_{-k+1}^{-k,0},X_{-k+1}^{-k,0}\rangle-2\mathbb{E}\int_{-k}^{-k+1}\langle r_s^\natural,f_s\rangle ds$$
$$-\mathbb{E}\int_{-k}^{-k+1}|\left(I+\sum_{i=1}^d\left(D_s^i\right)^*\overline{P}_sD_s^i\right)^{-1}(B_s^*r_s^\natural+\sum_{i=1}^d\left(D_s^i\right)^*g_s^{\natural,i})|^2ds\leq 2|\mathbb{E}\int_{-k}^{-k+1}\langle r_s^\natural,f_s\rangle ds|\leq A$$

where A is a constant independent on k. By (2.16) we get

$$\sup_{-k\leq s\leq -k+1}\mathbb{E}|X_s^{-k,0}|^2\leq C\int_{-k}^s\sup_{-k\leq r\leq s}\mathbb{E}|X_r^{-k,0}|^2dr+C\mathbb{E}\int_{-k}^{-k+1}|\bar{u}_r|^2dr+\mathbb{E}\int_{-k}^{-k+1}|f_r|^2dr,$$

and so by applying the Gronwall lemma, we get

$$\sup_{-k\leq s\leq -k+1}\mathbb{E}|X_s^{-k,0}|^2\leq Ce^C(A+\mathbb{E}\int_{-k}^{-k+1}|f_r|^2dr).$$

Since f is stationary, we can conclude that

$$\sup_{-k\leq s\leq -k+1} \mathbb{E}|X_s^{-k,0}|^2 \leq C,$$

where C is a constant independent on k. By (2.15), we get

$$(\mathbb{E}|X_t^{-N,0} - X_t^{-M,0}|^2)^{1/2} \leq C \sum_{k=N}^{M-1} e^{-\frac{a(t+k-1)}{2}}.$$

So we can conclude that, fixed $t \in \mathbb{R}$, $(X_t^{-N,0})_N$ is a Cauchy sequence in $L^2(\Omega)$, and so it converges in $L^2(\Omega)$ to a random variable denoted by $\zeta_t^\natural$. Notice that for every $t \in \mathbb{R}$ we can define $\zeta_t^\natural$, and we prove that $\zeta^\natural$ is a stationary process. Let $t \in \mathbb{R}$, $-N < t$ and $s > 0$: since the shift θ is measure preserving,

$$\lim_{N\to\infty} \mathbb{E}|X_t^{-N,0} \circ \theta_s - \zeta_t^\natural \circ \theta_s|^2 = 0,$$

moreover $X_t^{-N,0} \circ \theta_s = X_{t+s}^{-N+s,0}$ and

$$\lim_{N\to\infty} \mathbb{E}|X_{t+s}^{-N+s,0} - \zeta_{t+s}^\natural|^2 = 0.$$

By uniqueness of the limit we conclude that $\zeta_t^\natural \circ \theta_s = \zeta_{t+s}^\natural$. Notice that since $N \in \mathbb{N}$ and $\mathcal{F}_0 \supset \mathcal{G}_0^{-N}$, $\zeta_0^\natural$ is $\mathcal{F}_0$-measurable. Let us consider the value of the solution of equation (2.14) starting from $X_0 = \zeta_0^\natural$. By stationarity of the coefficients and of $\zeta^\natural$, we get that X is a periodic solution of equation (2.14), that we denote by $X^\natural$. In order to show the uniqueness of the periodic solution it is enough to notice that if $f^j = 0$, $j = 1, 2$, and $X^\natural$ is a periodic solution of (2.14), then

$$\mathbb{E}|X_0^\natural|^2 = \mathbb{E}|X_N^\natural|^2 \leq Ce^{-aN}\mathbb{E}|\zeta_0^\natural|^2.$$

Therefore $X_0^\natural = 0$ and this concludes the proof. □

We can now treat the following optimal control problem for a periodic cost functional: minimize over all admissible controls $u \in \mathcal{U}^\natural$ the cost functional

$$J^\natural(u, X) = \mathbb{E}\int_0^1 [|\sqrt{S_s}X_s|^2 + |u_s|^2]ds, \quad (u, X) \in \mathcal{U}^\natural, \tag{2.17}$$

where

$$\mathcal{U}^\natural = \left\{(u,X) \in L^2_{\mathcal{P}}(\Omega \times [0,1]) \times C([0,1], L^2_{\mathcal{P}}(\Omega)) : X_s = X_0 \circ \theta_s, \forall s \in \mathbb{R}\right\} \tag{2.18}$$

and X is the solution of equation

$$dX_t = A_t X_t dt + B_t u_t dt + \sum_{i=1}^d C_t^i X_t dW_t^i + \sum_{i=1}^d D_t^i u_t dW_t^i + f_t dt, \tag{2.19}$$

relative to u.

Theorem 2.11 *Let $X^\natural \in C([0,1], L^2(\Omega))$ be the unique stationary solution of equation* (2.14) *and let*

$$u_t^\natural = -\left(I + \sum_{i=1}^d \left(D_t^i\right)^* \overline{P}_t D_t^i\right)^{-1} \left(\overline{P}_t B_t + \sum_{i=1}^d \left(\overline{Q}_t^i D_t^i + \left(C_t^i\right)^* \overline{Q}_t D_t^i\right)\right)^* X_t^\natural + B_t^* r_t^\natural + \sum_{i=1}^d (D_t^i)^* g_t^{\natural,i}. \tag{2.20}$$

Then $(u^\natural, X^\natural) \in \mathcal{U}^\natural$ and it is the unique optimal couple for the cost 2.17, that is

$$J^\natural(u^\natural, X^\natural) = \inf_{(u,X)\in\mathcal{U}^\natural} J^\natural(u,X).$$

The optimal cost is given by

$$\overline{J}^\natural = J^\natural(u^\natural, X^\natural) = 2\mathbb{E}\int_0^1 \langle r_s^\natural, f_s\rangle ds - \mathbb{E}\int_0^1 |(I + \sum_{i=1}^d \left(D_t^i\right)^* \overline{P}_t D_t^i)^{-1} (B_t^* r_t^\natural + \sum_{i=1}^d \left(D_t^i\right)^* g_t^{\natural,i})|^2 ds. \tag{2.21}$$

Proof. By computing $d\langle \overline{P}_s X_s, X_s\rangle + 2\langle r_s^\natural, X_s\rangle$ we get

$$\mathbb{E}\int_0^1 [\langle S_s X_s, X_s\rangle + |u_s|^2] ds = \mathbb{E}\langle \overline{P}_0 X_0, X_0\rangle - \mathbb{E}\langle \overline{P}_1 X_1, X_1\rangle + 2\mathbb{E}\langle r_0^\natural, X_0\rangle - 2\mathbb{E}\langle r_1^\natural, X_1\rangle - 2\mathbb{E}$$

$$+\mathbb{E}\int_0^1 | \left(I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i\right)^{1/2} \left(u_s + (I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i)^{-1} * \right.$$

$$\left. * \left(\overline{P}_s B_s + \sum_{i=1}^d \left(\overline{Q}_s^i D_s^i + \left(C_s^i\right)^* \overline{P}_s D_s^i\right)\right)^* X_s + B_s^* r_s^\natural + \sum_{i=1}^d D_s^i (\bar{g}_s^{\natural,i})^*\right) |^2 ds$$

$$-\mathbb{E}\int_0^1 | \left(I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i\right)^{-1} (B_s^* r_s^\natural + \sum_{i=1}^d \left(D_s^i\right)^* \bar{g}_s^{\natural,i})|^2 ds.$$

Since by proposition 2.4, 2.9, and 2.10 $(u, X) \in \mathcal{U}^\natural$, we get

$$\mathbb{E}\int_0^1 [\langle S_s X_s, X_s\rangle + |u_s|^2] ds = -2\mathbb{E}\int_0^1 \langle r_s^\natural, f_s\rangle ds + \mathbb{E}\int_0^1 | \left(I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i\right)^{1/2} \times$$
$$\times \left(u_s + (I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i)^{-1} \left(\overline{P}_s B_s + \sum_{i=1}^d \left(\overline{Q}_s^i D_s^i + \left(C_s^i\right)^* \overline{P}_s D_s^i\right)\right)^* X_s + B_s^* r_s^\natural + \sum_{i=1}^d D_s^i \right.$$
$$- \mathbb{E}\int_0^1 | \left(I + \sum_{i=1}^d \left(D_s^i\right)^* \overline{P}_s D_s^i\right)^{-1} (B_s^* r_s^\natural + \sum_{i=1}^d \left(D_s^i\right)^* \bar{g}_s^{\natural,i})|^2 ds.$$

So

$$u_t^\natural = -\left(I + \sum_{i=1}^d \left(D_t^i\right)^* \overline{P}_t D_t^i\right)^{-1} \left(\overline{P}_t B_t + \sum_{i=1}^d \left(\overline{Q}_t^i D_t^i + \left(C_t^i\right)^* \overline{Q}_t D_t^i\right)\right)^* X_t^\natural + B_t^* r_t^\natural + \sum_{i=1}^d (D_t^i)^* g_t^{\natural,i} \tag{2.22}$$

is the optimal cost: $u^\natural$ minimizes the cost (2.21), and the corresponding state $X^\natural$ is stationary by proposition 2.10, so that $(u^\natural, X^\natural) \in \mathcal{U}^\natural$. □

3 Ergodic control

In this section we consider cost functionals depending only on the asymptotic behaviour of the state (ergodic control). Throughout this section we assume the following:

Hypothesis 3.1 *The coefficient satisfy hypothesis 2.2, and moreover*

- $S \geq \epsilon I$, *for some* $\epsilon > 0$.
- (A, B, C, D) *is stabilizable relatively to* S.
- *The first component of the minimal solution* P *is bounded in time.*

Notice that these conditions implies that (P, Q) stabilize (A, B, C, D) relatively to the identity.

We first consider discounted cost functional and then we compute a suitable limit of the discounted cost. Namely, we consider the discounted cost functional

$$J_\alpha(0,x,u)=\mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds, \qquad (3.1)$$

where X is solution to equation

$$\begin{cases} dX_s = (A_s X_s + B_s u_s)ds + \sum_{i=1}^d \left(C_s^i X_s + D_s^i u_s\right) dW_s^i + f_s ds \; s \geq t \\ X_t = x. \end{cases}$$

A, B, C and D satisfy hypothesis 2.2 and $f \in L^\infty_\mathcal{P}(\Omega\times[0,+\infty))$ and is a stationary process. When the coefficients are deterministic the problem has been extensively studied, see e.g. [1] and [11].

Our purpose is to minimize the discounted cost functional with respect to every admissible control u. We define the set of admissible controls as

$$\mathcal{U}^\alpha = \left\{ u \in L^2(\Omega\times[0,+\infty)) : \mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\langle S_s X_s^{0,x,u}, X_s^{0,x,u}\rangle + |u_s|^2]ds < +\infty \right\}.$$

Fixed $\alpha > 0$, we define $X_s^\alpha = e^{-\alpha s}X_s$ and $u_s^\alpha = e^{-\alpha s}u_s$. Moreover we set $A_s^\alpha = A_s - \alpha I$ and $f_s^\alpha = e^{-\alpha s}f_s$, and $f^\alpha \in L^2_\mathcal{P}(\Omega\times[0,+\infty)) \cap L^\infty_\mathcal{P}(\Omega\times[0,+\infty))$. X_s^α is solution to equation

$$\begin{cases} dX_s^\alpha = (A_s^\alpha X_s^\alpha + B_s u_s^\alpha)ds + \sum_{i=1}^d \left(C_s^i X_s^\alpha + D_s^i u_s^\alpha\right) dW_s^i + f_s^\alpha ds \; s \geq 0 \\ X_0^\alpha = x, \end{cases} \qquad (3.2)$$

By the definition of X^α, we note that if (A,B,C,D) is stabilizable with respect to the identity, then (A^α,B,C,D) also is. We also denote by (P^α,Q^α) the minimal solution of a stationary backward Riccati equation (2.3) with A^α in the place of A. Since, for $0<\alpha<1$, A^α is uniformly bounded in α, also P^α is uniformly bounded in α. Arguing as in proposition 2.4, (P^α,Q^α) is a stationary process.

Let us denote by (r^α,g^α) the solution of the infinite horizon BSDE

$$dr_t^\alpha = -(H_t^\alpha)^* r_t^\alpha dt - P_t^\alpha f_t^\alpha dt - \sum_{i=1}^d \left(K_t^{\alpha,i}\right)^* g_t^{\alpha,i} dt + \sum_{i=1}^d g_t^{\alpha,i} dW_t^i, \qquad t \geq 0, \qquad (3.3)$$

where H^α and K^α are defined as in (2.9), with A^α, P^α and Q^α respectively in the place of A, P and Q. By [4], section 4, we get that equation (3.3) admits a solution $(r^\alpha, g^\alpha) \in L^2_{\mathcal{P}}(\Omega \times [0,+\infty)) \cap L^2_{\mathcal{P}}(\Omega \times [0,+\infty)) \times L^2_{\mathcal{P}}(\Omega \times [0,T])$, for every fixed $T > 0$.

Moreover by [4], section 6, we know that

$$\underline{\lim}_{\alpha\to 0}\alpha \inf_{u^\alpha\in\mathcal{U}^\alpha} J_\alpha(0,x,u^\alpha) =$$
$$\underline{\lim}_{\alpha\to 0}[\alpha \int_0^{+\infty} 2\langle r^\alpha_s, f^\alpha_s\rangle ds - \alpha \int_0^{+\infty} |(I + \sum_{i=1}^d \left(D^i_s\right)^* P^\alpha_s D^i_s)^{-1}(B^*_s r^\alpha_s + \sum_{i=1}^d \left(D^i_s\right)^* g^{\alpha,i}_s)|^2 ds].$$

We can also prove the following convergence result for (r^α, g^α).

Lemma 3.2 *For all fixed $T > 0$, $r^\alpha\mid_{[0,T]}\to r^\natural\mid_{[0,T]}$ in $L^2_{\mathcal{P}}(\Omega \times [0,T])$. Moreover, for every fixed $T > 0$, as $\alpha \to 0$.*

$$\mathbb{E}\int_0^T |(I + \sum_{i=1}^d \left(D^i_s\right)^* P^\alpha_s D^i_s)^{-1}(B^*_s r^\alpha_s + \sum_{i=1}^d \left(D^i_s\right)^* g^{\alpha,i}_s)|^2 ds \to$$
$$\mathbb{E}\int_0^T |(I + \sum_{i=1}^d \left(D^i_s\right)^* \overline{P}_s D^i_s)^{-1}(B^*_s r^\natural_s + \sum_{i=1}^d \left(D^i_s\right)^* g^{\natural,i}_s)|^2 ds$$

Proof. The first assertion follows from lemma 6.6 in [4]. Notice that stationarity of the coefficients in the limit equation gives stationarity of the solution, and so it allows to identify the limit with the stationary solution of the dual BSDE. For the second assertion for the optimal couple (X^α, u^α) for the optimal control problem on the time interval $[0,T]$:

$$\int_0^T [|\sqrt{S_s}X^\alpha_s|^2 + |u^\alpha_s|^2 ds = \langle P^\alpha_0 x, x\rangle + 2\langle r^\alpha_0, x\rangle + 2\mathbb{E}\int_0^T \langle r^\alpha_s, f^\alpha_s\rangle ds$$
$$\mathbb{E}\langle P^\alpha_T X^\alpha_T, X^\alpha_T\rangle + 2\mathbb{E}\langle r^\alpha_T, X^\alpha_T\rangle - \mathbb{E}\int_0^T |(I + \sum_{i=1}^d \left(D^i_t\right)^* P^\alpha_t D^i_t)^{-1}(B^*_t r^\alpha_t + \sum_{i=1}^d \left(D^i_t\right)^* g^{\alpha,i}_t)|^2 ds. \tag{3.4}$$

Since, as $\alpha \to 0$, in (3.4) all the terms but the last one converge to the corresponding stationary term, and since by [4] (r^α, g^α) is uniformly, with respect to α, bounded in $L^2_{\mathcal{P}}(\Omega \times [0,T]) \times L^2_{\mathcal{P}}(\Omega \times [0,T])$, then

$(r^\alpha \mid_{[0,T]}, g^\alpha \mid_{[0,T]}) \rightharpoonup (r^\natural \mid_{[0,T]}, g^\natural \mid_{[0,T]})$ in $L^2_{\mathcal{P}}(\Omega \times [0,T]) \times L^2_{\mathcal{P}}(\Omega \times [0,T])$, we get the desired convergence.

□

This is enough to characterize the ergodic limit. Indeed we have that:

Theorem 3.3 *We get the following characterization of the optimal cost:*

$$\lim_{\alpha\to 0} 2\alpha \inf_{u\in\mathcal{U}^\alpha} J_\alpha(x,u) = 2\mathbb{E}\left[\langle f(0), r^\natural(0)\rangle - |(I + \sum_{i=1}^d \left(D_0^i\right)^* \overline{P}_0 D_0^i)^{-1}(B_0^* r_0^\natural + \sum_{i=1}^d \left(D_0^i\right)^* g_0^{\natural,i})|^2\right]$$

Proof. Let us define $\tilde{r}_t^\alpha = e^{\alpha t} r_t^\alpha$, $\tilde{g}_t^\alpha = e^{\alpha t} g_t^\alpha$. $(\tilde{r}_t^\alpha, \tilde{g}_t^\alpha)$ is the solution to

$$d\tilde{r}_t^\alpha = -(H_t^\alpha)^* \tilde{r}_t^\alpha dt + \alpha I \tilde{r}_t^\alpha dt - P_t^\alpha f_t dt - \sum_{i=1}^d \left(K_t^{\alpha,i}\right)^* \tilde{g}_t^{\alpha,i} dt + \sum_{i=1}^d \tilde{g}_t^{\alpha,i} dW_t^i, \qquad t \geq 0,$$

and so, arguing as in lemma 2.9, $(\tilde{r}_t^\alpha, \tilde{g}_t^\alpha)$ are stationary processes. Now we compute

$$\begin{aligned}
\underline{\lim}_{\alpha\to 0} 2\alpha \inf_{u^\alpha\in\mathcal{U}^\alpha} J_\alpha(0,x,u^\alpha) =& \underline{\lim}_{\alpha\to 0} \Big[2\alpha \int_0^{+\infty} e^{-2\alpha s} 2\mathbb{E}\langle \tilde{r}_s^\alpha, f_s\rangle ds \\
& -2\alpha \int_0^{+\infty} e^{-2\alpha s} \mathbb{E}|(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1}(B_s^* \tilde{r}_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^* \tilde{g}
\\
&= \underline{\lim}_{\alpha\to 0} \Big[2\alpha \sum_{k=1}^\infty e^{-2\alpha k} \int_0^1 e^{-2\alpha s} 2\mathbb{E}\langle \tilde{r}_s^\alpha, f_s\rangle ds \\
& -2\alpha \sum_{k=1}^\infty e^{-2\alpha k} \int_0^1 e^{-2\alpha s} \mathbb{E}|(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1}(B_s^* \tilde{r}_s^\alpha + \sum_{i=1}^d
\\
&= \underline{\lim}_{\alpha\to 0} \Big[2\alpha \sum_{k=1}^\infty e^{-2\alpha k} \int_0^1 2\mathbb{E}\langle r_s^\alpha, f_s^\alpha\rangle ds \\
& -2\alpha \sum_{k=1}^\infty e^{-2\alpha k} \int_0^1 \mathbb{E}|(I + \sum_{i=1}^d \left(D_s^i\right)^* P_s^\alpha D_s^i)^{-1}(B_s^* r_s^\alpha + \sum_{i=1}^d \left(D_s^i\right)^*
\end{aligned}$$

Since $(r^\alpha_s, g^\alpha_s) \to (r^\natural_s, g^\natural_s)$ in $L^2_{\mathcal{P}}(\Omega\times[0,1])\times L^2_{\mathcal{P}}(\Omega\times[0,1])$ we get that

$$\underline{\lim}_{\alpha\to 0} 2\alpha \inf_{u^\alpha\in\mathcal{U}^\alpha} J_\alpha(0,x,u^\alpha) = 2\mathbb{E}\int_0^1 \langle r^\natural_s, f_s\rangle ds - \mathbb{E}\int_0^1 |(I+\sum_{i=1}^d \left(D^i_s\right)^* P^\alpha_s D^i_s)^{-1}(B^*_s r^\natural_s + \sum_{i=1}^d \left(\right.$$

$$= 2\mathbb{E}\langle r^\natural_0, f_0\rangle - \mathbb{E}|(I+\sum_{i=1}^d \left(D^i_0\right)^* \overline{P}_0 D^i_0)^{-1}(B^*_0 r^\natural_0 + \sum_{i=1}^d \left(D^i_0\right)^* g^{\alpha,i}_0)$$

where the first equality holds also in the periodic case and the second equality holds only in the stationary case. □

In the following we denote $\inf_{u\in\mathcal{U}^\alpha} J_\alpha(x,u) := J^*_\alpha(x,u)$.

The next step is to minimize

$$\hat{J}(x,u) = \underline{\lim}_{\alpha\to 0} 2\alpha J(x,u)$$

over all $u\in\widehat{\mathcal{U}}$, where

$$\widehat{\mathcal{U}} = \left\{u\in L^2_{loc} : \mathbb{E}\int_0^{+\infty} e^{-2\alpha s}[\left\langle S_s X^{0,x,u}_s, X^{0,x,u}_s\right\rangle + |u_s|^2]ds < +\infty,\ \forall\alpha>0.\right\}$$

We will prove that

$$\inf_{u\in\widehat{\mathcal{U}}} \hat{J}(x,u) = J^\natural(u).$$

Let $\widehat{X}$ be solution of

$$\begin{cases} d\widehat{X}^x_s = H_s\widehat{X}^x_s ds + \sum_{i=1}^d K^i_s \widehat{X}^x_s dW^i_s + B_s(B^*_s r^\natural_s + \sum_{i=1}^d D^i_s g^{\natural,i}_s)ds + f_s ds + \sum_{i=1}^d D^i_s(B^*_s r^\natural_s + \sum_{i=1}^d D^i_s g \\ \widehat{X}^x_0 = x, \end{cases}$$

and let

$$\hat{u}^x_s = -\Lambda(s,\overline{P}_s,\overline{Q}_s)\widehat{X}^x_s + (B^*_s r^\natural_s + \sum_{i=1}^d D^i_s g^{\natural,i}_s).$$

Notice that by proposition 2.10 if $x=\zeta^\natural_0$, then $\widehat{X}^{\zeta^\natural_0}$ is stationary and $(\hat{u}^{\zeta^\natural_0}, \widehat{X}^{\zeta^\natural_0})$ is the optimal couple $(u^\natural_0, X^\natural_0)$.

Lemma 3.4 *For all $x\in L^2(\Omega)$, $\hat{u}^x\in\widehat{\mathcal{U}}$ and $\hat{J}(\hat{u}^x,x)$ does not depend on x.*

Proof. Let us consider $X_t^{s,x}$ the solution of equation

$$\begin{cases} dX_t^{s,x} = H_t X_t^{s,x} dt + \sum_{i=1}^{d} K_t^i X_t^{s,x} dW_t^i \\ X_s^{s,x} = x, \end{cases}$$

starting from x at time s. We denote, for every $0 \le s \le t$, $U(t,s)x := X_t^{s,x}$. We notice that

$$\widehat{X}_t^{0,x} - \widehat{X}_t^{0,\zeta_0^\natural} = x - \zeta_0^\natural + \int_0^s H_s(\widehat{X}_s^{0,x} - \widehat{X}_s^{0,\zeta_0^\natural}) ds + \sum_{i=1}^{d} \int_0^s K_s^i (\widehat{X}_s^{0,x} - \widehat{X}_s^{0,\zeta_0^\natural}) dW_s^i = U(t,0)(x - \zeta_0^\natural).$$

So by the Datko theorem there exist constants $a, C > 0$ such that

$$\mathbb{E}|\widehat{X}_t^x - \widehat{X}_t^{\zeta_0^\natural}|^2 \le C e^{-at} \mathbb{E}|x - \zeta_0^\natural|^2.$$

So

$$\mathbb{E}|\widehat{X}^x|^2 \le C e^{-at} \mathbb{E}|x - \zeta_0^\natural|^2 + \mathbb{E}|\widehat{X}^{\zeta_0^\natural}|^2 \le C,$$

where in the last passage we use that $\widehat{X}^{\zeta^\natural} = X^\natural$ and it is stationary.

Again by applying the Datko theorem we obtain

$$\lim_{\alpha \to 0} \alpha \mathbb{E} \int_0^\infty e^{-2\alpha s} (2\langle S X_s^\natural, U(s,0)(x - \zeta_0^\natural)\rangle + |\sqrt{S} U(s,0)(x - \zeta_0^\natural))|^2) ds = 0.$$

Moreover

$$\hat{u}_t = u_t^\natural - \Lambda(t, \overline{P}_t, \overline{Q}_t) U(0,t)(x - \zeta_0^\natural)$$

It is clear that $u^\natural$ belongs to the space of admissible control space $\widehat{\mathcal{U}}$.

The term $\tilde{u}_t = \Lambda(t, \overline{P}_t, \overline{Q}_t) U(0,t)(x - \zeta_0^\natural)$, $t \in (0 + \infty)$ can be proved to be the optimal control for the infinite horizon problem with $f = 0$ and random initial data $x - \zeta_0^\natural$:

$$\inf_{u \in L^2_{\mathcal{P}}((0,+\infty);\mathbb{R}^k)} \mathbb{E} \int_0^{+\infty} (|\sqrt{S}_s X_s^u|^2 + |u(s)|^2) \, ds.$$

Hence Theorem 5.2 of [4] can be extended without any difficulty to get that:

$$\begin{aligned} J(0, x - \zeta_0^\natural, \tilde{u}) &= \mathbb{E}\langle \overline{P}_0(x - \zeta_0^\natural), x - \zeta_0^\natural\rangle + 2\mathbb{E}\langle r_0, x - \zeta_0^\natural\rangle \\ &\quad - \mathbb{E} \int_0^\infty |(I + \sum_{i=1}^{d} \left(D_t^i\right)^* \overline{P}_t D_t^i)^{-1} (B_t^* r_t + \sum_{i=1}^{d} \left(D_t^i\right)^* g_t^i)|^2 ds. \end{aligned}$$

Therefore

$$\mathbb{E}\int_0^\infty e^{-2\alpha s}|\tilde{u}(s)|^2\,ds \le \mathbb{E}\int_0^\infty |\tilde{u}(s)|^2\,ds \le C.$$

This proves that $\hat{u}$ is an admissible control since it follows that

$$\lim_{\alpha\to 0} 2\alpha\mathbb{E}\int_0^\infty e^{-2\alpha s}(|u_s^\natural|^2 - |\hat{u}_s^x|^2)ds = 0.$$

□

We can now conclude as follows:

Theorem 3.5 *For all $x \in L^2(\Omega)$ the couple $(\widehat{X}^x, \hat{u}^x)$ is optimal that is*

$$\hat{J}(\hat{u}^x, x) = \min\{\hat{J}(u,x) : u \in \widehat{\mathcal{U}}\}$$

Moreover the optimal cost, that does not depend on the initial state x, is equal to the optimal cost for the periodic (respectively stationary) problem, i.e.

$$\hat{J}(\hat{u}^x, x) = \overline{J}^\natural.$$

Proof. If $u \in \widehat{\mathcal{U}}$, then for every $\alpha > 0$, $u \in \mathcal{U}^\alpha$. Consequently for every $\alpha > 0$

$$2\alpha J_\alpha(u,x) \ge 2\alpha J_\alpha^*.$$

By taking the limit on both sides we get

$$\hat{J}(x,u) = \underline{\lim}_{\alpha\to 0} 2\alpha J_\alpha \ge \underline{\lim}_{\alpha\to 0} 2\alpha J_\alpha^* = \overline{J}^\natural.$$

By the previous lemma $\hat{J}(x, \hat{u}^x)$ is independent on x so we let $x = \zeta_0^\natural$, which implies that $\hat{u}^x = u^\natural$ and $\widehat{X}^x = X^\natural$. Then

$$\hat{J}(\zeta_0^\natural, u^\natural) = \lim_{\alpha\to 0} 2\alpha \int_0^\infty e^{-2\alpha t}[|\sqrt{S}_t X_t^\natural|^2 + |u_t^\natural|^2]dt$$
$$= \lim_{\alpha\to 0} 2\alpha(\sum_{k=1}^d e^{-2k\alpha}) \int_0^1 e^{-2\alpha t}[|\sqrt{S}_t X_t^\natural|^2 + |u_t^\natural|^2]dt = \overline{J}^\natural,$$

and this concludes the proof. □

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