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New Covariance Models for Local Applications of Collocation

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Abstract. The least-squares collocation method, used to predict or filter a signal, is based on the estimation of the empirical covariance function and the fitting of the empirical values with a proper model function. Generally, with the standard methods on the sphere, we reach a good fitting only up to the first zero of the empirical function. In this work we have investigated how much the collocation filtering is affected by a poor fitting of the empirical covariance.

Numerical tests have been done both on 1D observed and simulated data to quantify the combined impact on filtering of non stationarity and covariance fitting.

Furthermore, a new model function on the sphere has been developed which is able to fit in an optimal way the empirical values.

Simulations have been also carried out on the sphere to test the effectiveness of the collocation filtering using the new covariance model.

1 Introduction

Collocation is widely used in Geodesy for estimating functionals of the anomalous potential of the Earth. Very often, local solution are computed based on the well known remove-restore procedure. Thus, the collocation estimate refers to the so called residual geoid computed using the residual gravity anomaly Δg_r, which reflects the local features of the gravity field.

Local covariance models have been proposed in the past (Knudsen (1987); Tscherning and Rapp (1974)) and are nowadays commonly used when computing geoid in local areas.

The basic models, implemented in the COVFIT program, contained in the GRAVSOFT package, are of the type:

$$\mathrm{COV}_{TT}(P,Q) = \\ = \sum_{i=2}^{\infty} \sigma_i^2 \left(\frac{R^2}{rr'}\right)^{i+1} P_i(\cos\Psi) \quad (1)$$

where:

$$\sigma_i^2 = \begin{cases} \varepsilon_i \text{ error degree variances} \\ \\ \text{degree variances, e.g.} \\ \qquad \dfrac{A}{(i-1)(i-2)(i-B)} \left(\dfrac{R_B}{R}\right) \end{cases}$$

R is the Earth radius

r, r' are respectively the radial distances of points in space P, Q

P_i the Legendre Polynomial of degree i

Ψ the spherical distance between P and Q

Generally, with this model we reach a good fitting only up to the first zero of the empirical function, being the two functions quite different for larger values of the spherical distance. In Figure 1 is represented an example of a fitting of an empirical covariance function computed on local data.

Starting from an idea of Albertella et al. (1994), we have developed a new model function for local application on the sphere, that is a finite combinations of Legendre Polynomial taken at fixed step Δ with positive c_n coefficients to guarantee the positive definiteness of the covariance function. So our new model is represented by the relationship (2):

$$\mathrm{COV}(P,Q) = \sum_{n\in\Lambda} c_n P_n(\cos\Psi) \qquad (2)$$

where

$$P_n(\cos\Psi) \in \aleph = \{P_n(\cos\Psi) \in \Im | c_n \geq 0\},$$
$$\Im = \{P_{N\,min+\Delta}, P_{N\,min+2\Delta}, \dots, P_{N\,max} = \\ = \{P_n(\cos\Psi), n \in \Lambda\}$$

The fitting[1] of the empirical values in Fig. 1 have been repeated with the new model, and, as showed in Fig. 2, we have reached a very good fit, being the model function completely superimposed to the empirical values, even at large values of the spherical distance.

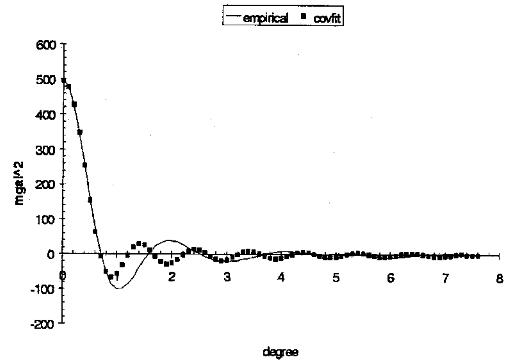


Figure 1: Empirical covariance function fitting with the COVFIT program of the GRAVSOFT package

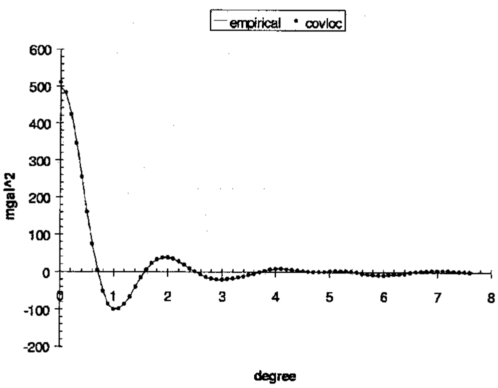
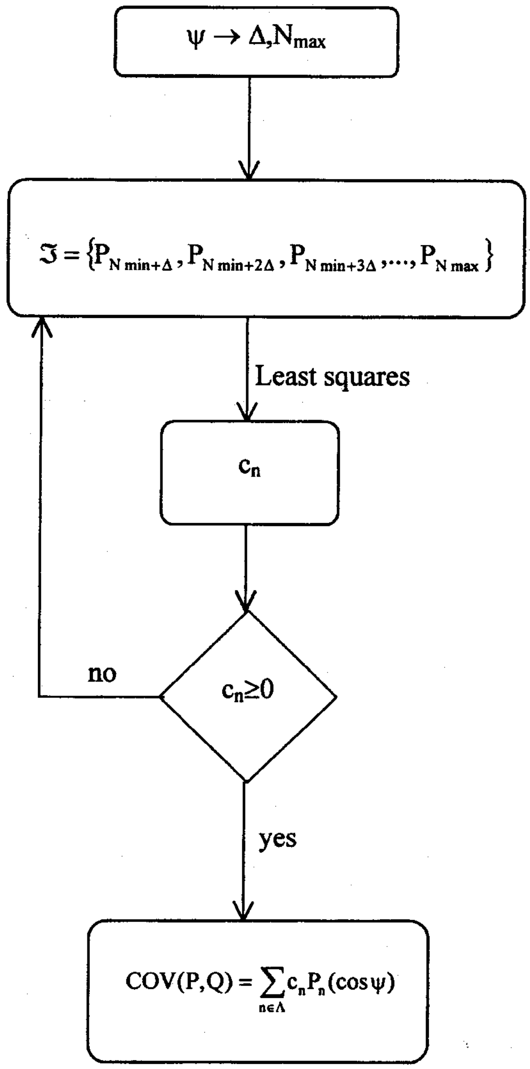


Figure 2: Empirical covariance function fitting with the COVLOC program

The model function has been estimated using the COVLOC program that we developed for fitting the empirical values with the model (2).

In the following flow-chart is presented the COVLOC program structure.



The Legendre Polynomial of the set $\Im$ in Formula (2) are taken at a fixed step Δ, chosen in relationship to the cap width Ψ by the following relation (Albertella et al. (1994))

$$\Delta = \frac{180}{\Psi} \tag{3}$$

Based on Δ and on the number of empirical covariance values, the maximum degree N_{max} of the Legendre Polynomial is fixed and we can compute the complete set $\Im$.

By fitting the empirical covariance function values using least squares, the coefficients c_n are computed: then their positiveness is checked to guarantee the positive definiteness of the covariance function.

If one or more coefficients are negative, for instance $c_{\tilde{n}}$, the relative polynomial, $P_{\tilde{n}}$, is elim-

[1]This is done by a least square fitting considering weighted $C(P_i, Q_j)$ observations, being the weight the number products used in computing $C(P_i, Q_j)$.

inated from the set $\Im$ and the coefficients computation is repeated.

When all the estimated c_n are positive, the new model covariance function is computed as a finite combination of Legendre Polynomial.

With the two covariance models (1) and (2), tests on filtering accuracy as related to the fit of the empirical values have been carried out. Particularly, the hypothesis of stationarity of the data has been checked to test a possible interaction between stationarity and covariance fitting. We wanted to quantify the effect on the collocation filtering, due to model covariance fitting, both in stationary and non stationary cases. The stationarity hypothesis has been checked through the following condition:

$$2C(0) = E\{[y(t)]^2\} + E\{y(t+\tau)]^2\} \qquad (4)$$

that is we tested if the $C(0)$ is constant along the whole data domain.

2 Applications on 1D domain

Before testing the filtering accuracy on the sphere, some simpler cases on the 1D domain were analysed. In particular, the covariance fitting effect on filtering is tested both with stationary and non stationary data. Although it is not theoretically correct to estimate a covariance function in non stationary conditions, we have done that because usually, in practical applications, no testing on stationarity is performed and the collocation filtering is used straightforwardly.

So, in the following, the empirical covariance function will be estimate also in non stationarity data even though, in this non stationary case, this must be understood in a wider an improper sense[2].

Hence, we wanted to quantify the relevance of stationarity on filtering through covariance fitting.

2.1 Non stationary signal

The stationarity hypothesis for a set of real altimetric data (Fig. 3) has been checked. As one can see in Figure 4, this condition is not satisfied; in fact the variance values are constant only up to 500 km and after they rapidly increase.

As we pointed out previously, in spite of the non stationarity of the data, the empirical *covariance* function has been computed, as generally, in current applications, the stationarity check is left out. The empirical values have been fitted by the function called *best fit* (Fig. 5). In addition to this function which is close to the empirical values, a Normal one has been drawn to analyse how much the filtering accuracy in this case depends on the empirical function fitting. To define the parameters of the Normal covariance function the following procedure has been adopted. The value in the origin has been fixed, as usual, equal to the variance of the data. The correlation length has been tuned on the covariance function of the residuals of the data obtained through a moving average (i.e. a moving average has been applied to the data, the empirical covariance of the residuals computed and its correlation length assumed as the correlation for the Normal function). In this way a kind of *deterministic-stochastic* filter is designed, its deterministic feature being the correlation length of the Normal covariance which is fixed through an *a priori* assumption on the features of the moving average.

In Picture 8 are plotted the filtered data by least-square collocation. The signal obtained with the *best fit* function is smoother than the Normal filtering. This one reproduces better

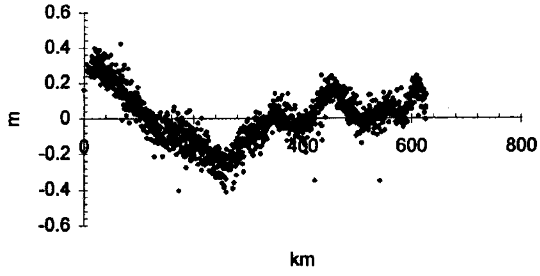


Figure 3: Real altimetric data

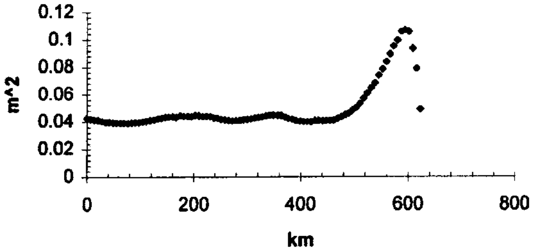


Figure 4: Stationarity test

[2]We shall use the notation *covariance* function to identify the empirical function estimated with non stationary data.

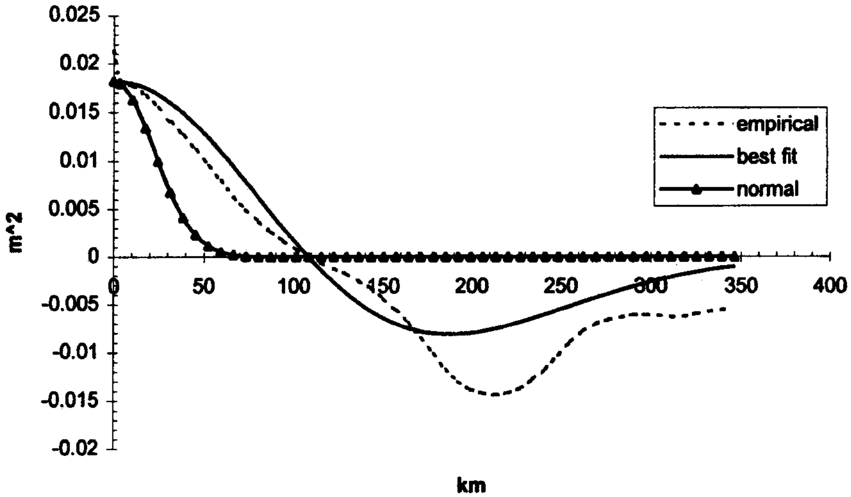


Figure 5: Covariance functions

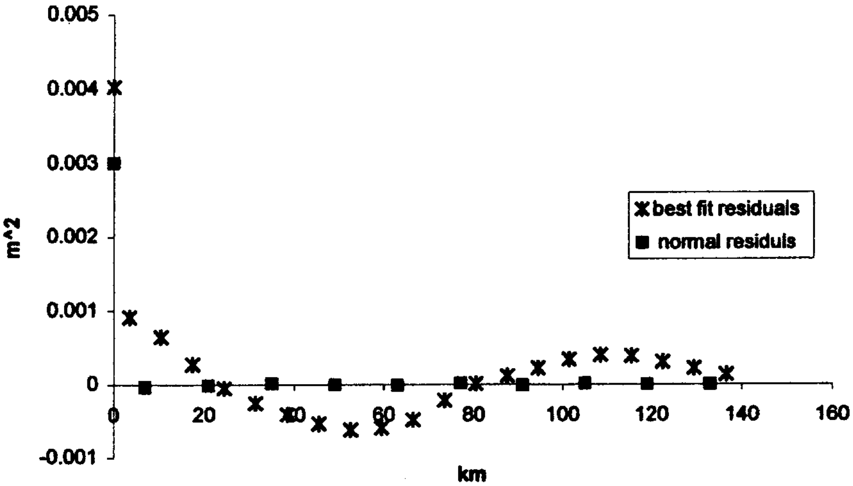


Figure 6: Empirical covariance functions of the filtered residuals

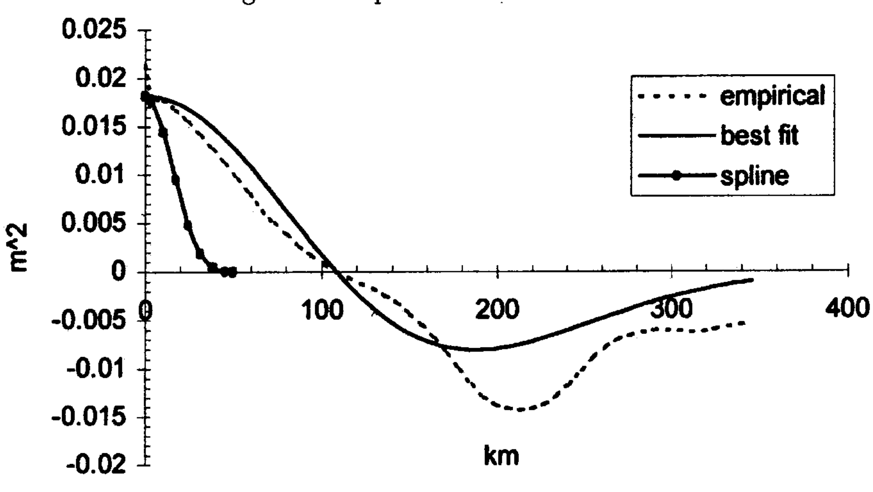


Figure 7: Covariance functions

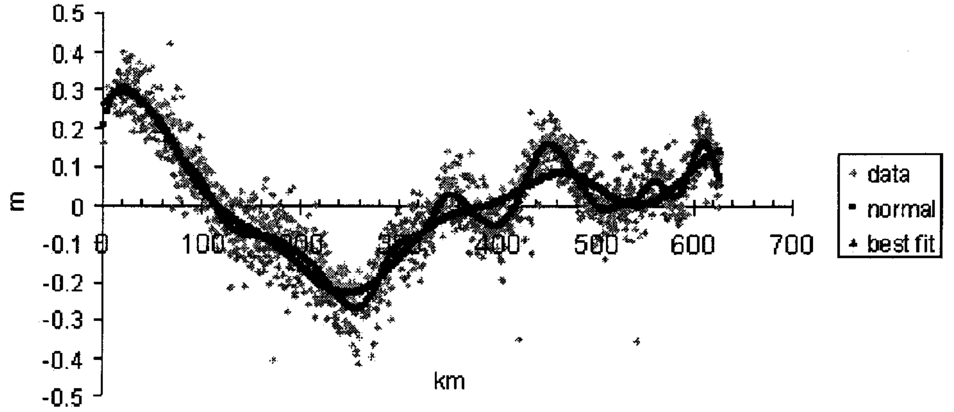


Figure 8: 1^{st} step filtered data

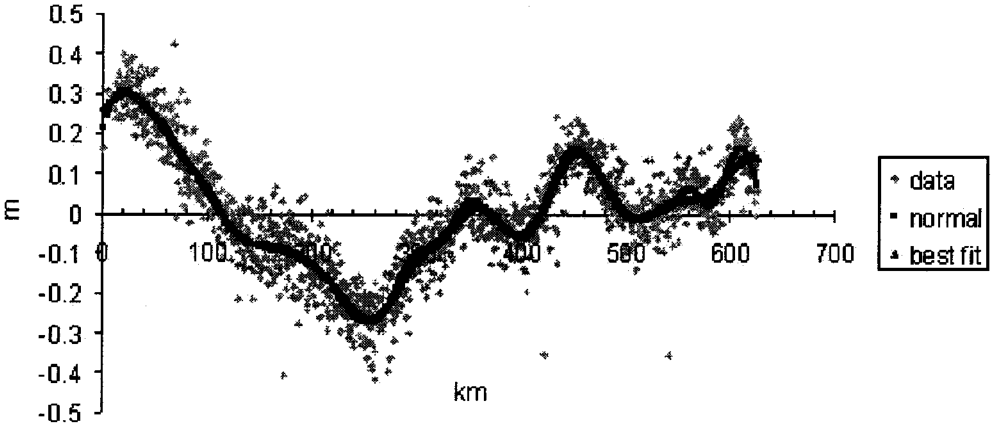


Figure 9: Filtered data with two steps of collocation

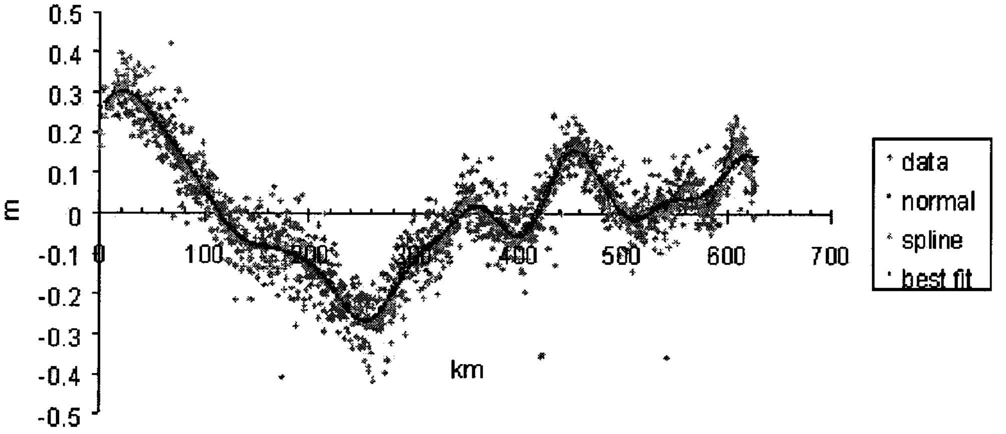


Figure 10: Filtered data with the spline function

	best fit (m) 1 step	best fit (m) 2 step	normal (m) 1 step	Spline (m) finite colloc.
♯	1380	1380	1380	1380
E	0.000	0.000	0.000	0.000
σ	0.064	0.056	0.055	0.054
Min	-0.395	-0.382	-0.370	-0.370
Max	0.247	0.242	0.241	0.234

Table 1: Statistic of the filtered data

the high frequencies in the data, and, as consequence, the standard deviation of the Normal residuals is lower than the *best fit* standard deviation value (Tab. 1). Besides, the normal filtering residuals don't show any correlation, while the *best fit* residuals do (Fig. 6); so a second collocation step has been computed on these residuals. After two step of computation with the *best fit* covariance functions we have obtained the same standard deviation value as with one collocation step using the Normal function (Tab. 3 and Fig. 9).

Furthermore, with the same data set, a filtering based on a finite covariance model has been tested. Following the same scheme used in the Normal function case, the parameters of a cubic spline have been fixed. In this way, the *deterministic-stochastic* filter has been further investigated to understand its stability with respect to the covariance model function selected. Naturally, in this case, a sharp improvement in computation time is also reached (in the end the CPU time was 40 time less than the one we had in the Normal case). The outcome of such a filtering is represented in Fig. 10 and the statistics are summarised in Tab. 1; as one can see, the two *deterministic-stochastic* filters (based on Normal and spline covariance models) perform better than two steps of *best-fit* covariance filter.

2.2 Stationary signal

To understand if in a stationary case the filtering accuracy has the same behaviour as showed in the non stationary case, a signal has been simulated satisfying the stationary condition (Fig. 11).

The empirical covariance function has been estimated and fitted by a model, called again *best fit* function (Fig. 12).

By lest-square collocation the synthetic data have been filtered using the *best fit* covariance function and a Normal function. The standard deviation value of the residuals obtained using the filter based on the Normal covariance is a bit lower than the *best fit* one, but the Normal filtering of the data with some noise added (about 15% of the signal) is more perturbed (Fig. 13 and 14). In fact, looking at the statistic of the differences between the two filtering, with and without noise, the standard deviation of the residuals coming from the Normal filter has a greater value than the *best fit* standard deviation (Tab. 2).

The filtering has been repeated after adding some outliers to the data, and even in this case the Normal filtering is more disturbed than the *best fit* filtering (Fig. 15 and 16).

	data-filtered signal (without noise)		filtering differences (data with noise - without noise)	
	best fit (m)	Normal (m)	best fit (m)	Normal (m)
♯	586	586	586	586
E	-0.004	0.000	0.002	0.002
σ	0.080	0.077	0.008	0.013
Min	-0.162	-0.175	-0.021	-0.030
Max	0.153	0.152	-0.019	0.033

Table 2: Statistic of the filtering

So, in conclusions, these numerical experiments proved that the stationarity condition is crucial and that only with stationary data it is important to fit at the best the empirical covariance values.

3 Applications on a 2D spherical domain

The same test has been performed in a 2D spherical domain to see if the same effects that we had in 1D are present in a 2D spherical case. In particular, having proved that the stationary condition is so critical, we simulated a stationary grav-

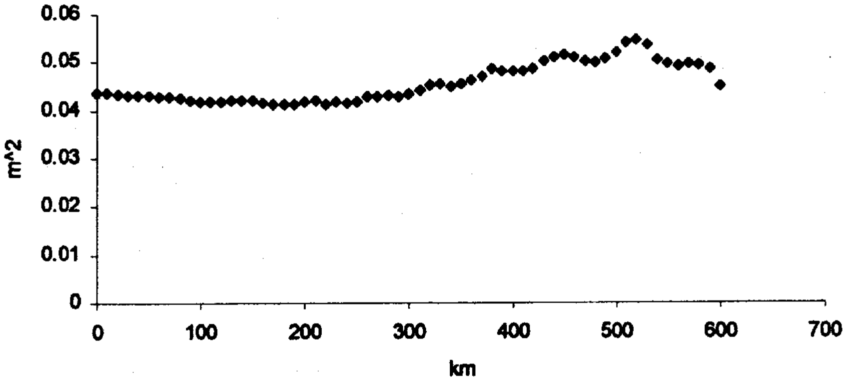


Figure 11: Stationary condition

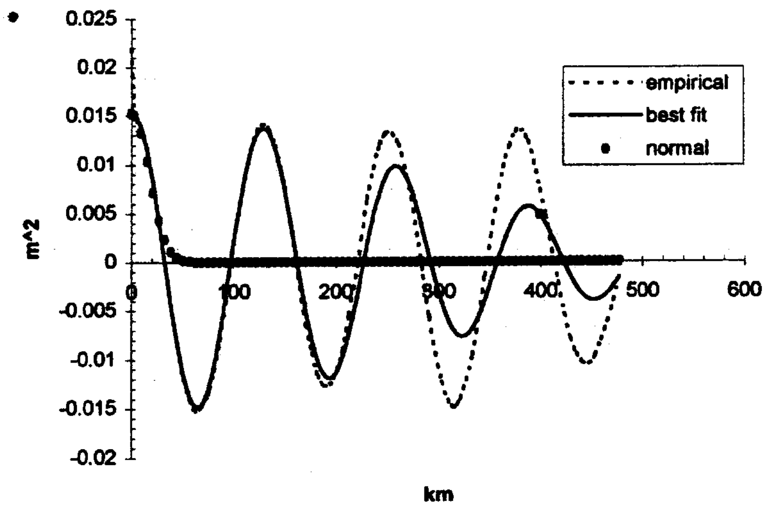


Figure 12: Covariance functions

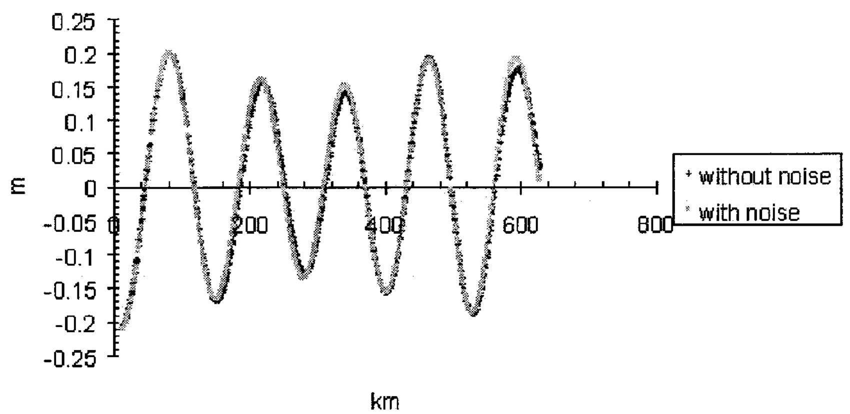


Figure 13: Filtered data with the *best fit* covariance function

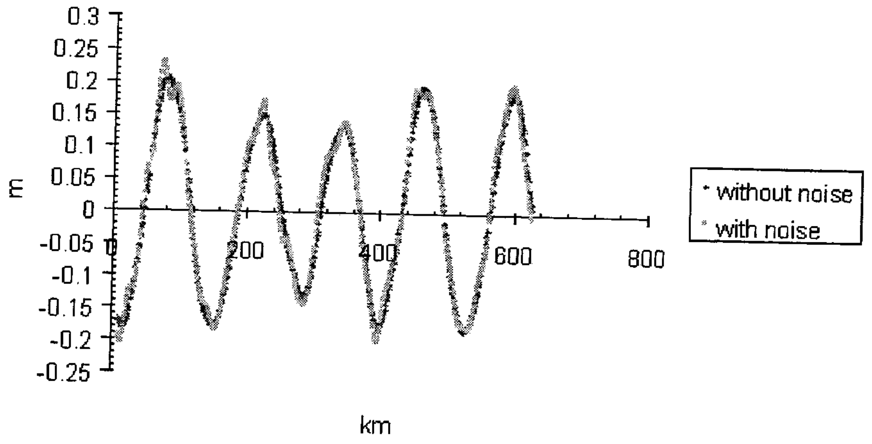


Figure 14: Filtered data with the normal function

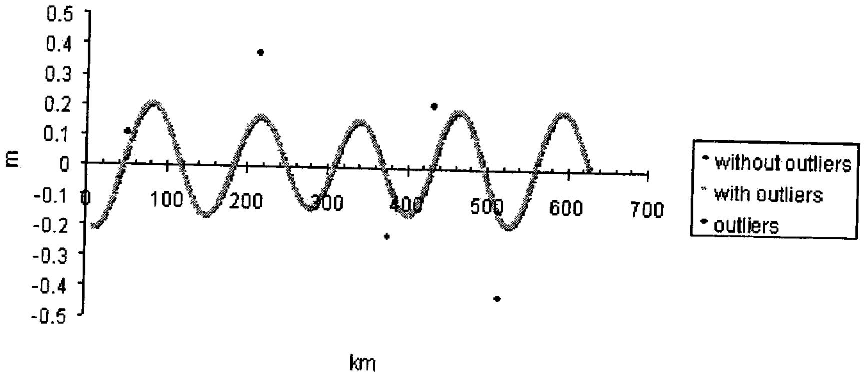


Figure 15: Filtered data with the *best fit* covariance function

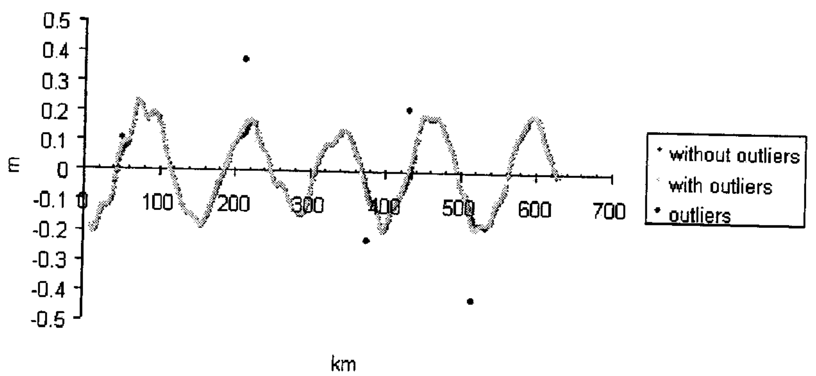


Figure 16: Filtered data with the normal function

	data - filtered signal (without noise)		filtering differences (data with noise - data without noise)		filtering differences (data with - without outliers)	
	gravsoft (m)	covloc (m)	gravsoft (m)	covloc (m)	gravsoft (m)	covloc (m)
♯	423	423	423	423	423	423
E	0.006	1.215	-0.209	-0.328	-0.020	-1.112
σ	3.324	3.773	1.044	0.889	0.272	0.145
Min	-8.488	-8.640	-2.482	-2.985	-1.084	-1.500
Max	6.854	10.452	3.961	1.945	1.484	-0.665

Table 3: Statistic of the filtered data

ity signal to quantify the effect of an improved empirical covariance fitting on filtering accuracy. We also remark that in this paragraph the sentence *empirical covariance function* is used properly because our data satisfies the stationarity condition. Synthetic gravity data have been simulated on the sphere in a square of 1 degree of side in latitude. Looking at Figure 17, we can say that the synthetic signal satisfies the stationary condition.

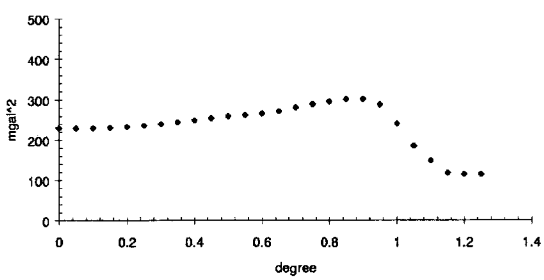


Figure 17: Stationarity condition with 2D spherical data

In fact the variance values are nearly constant up to 1 degree and after decrease; however only few data are distant more than 1 degree, so that values computed with (4) larger than 1 degree are unreliable. The empirical covariance function has been fitted by the COVFIT program of the GRAVSOFT package (Tscherning (1994)) and by the COVLOC program, as described in Paragraph 1. As you can see from Picture 18, COVLOC fitting is closer to the empirical values than the GRAVSOFT one. The data have been filtered by least-square collocation; the value of standard deviation obtained using the COVLOC

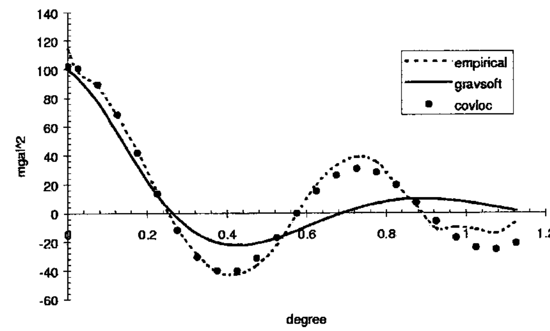


Figure 18: Covariance functions

fitting function, is greater than the GRAVSOFT value (Tab. 3), but, as in the 1D case, after adding some noise to the data, the best fit function based filtering is less perturbed by the noise (Fig. 19). In Fig. 21 the differences between the two filtering, with noise and without noise, have been plotted: the COVLOC function differences are smoother than the GRAVSOFT, as their standard deviation values are 0.89 mgal and 1.04 mgal respectively (Tab. 3).

The outliers test has been repeated (Fig. 21). The GRAVSOFT filtering is again more perturbed by the outliers; in fact the standard deviation value of the differences is twice than the COVLOC value (Tab. 3).

4 Conclusions

The 1D and 2D tests proved that the stationarity condition is critical in designing an optimal filter even from the numerical point of view. When this condition is satisfied, the best fit covariance

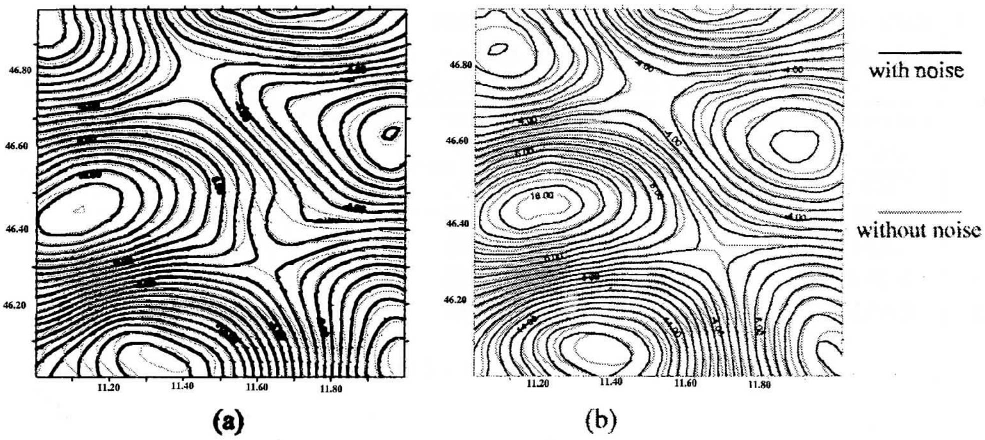


Figure 19: Filtered data: a) with GRAVSOFT covariance function; b) with COVLOC covariance function

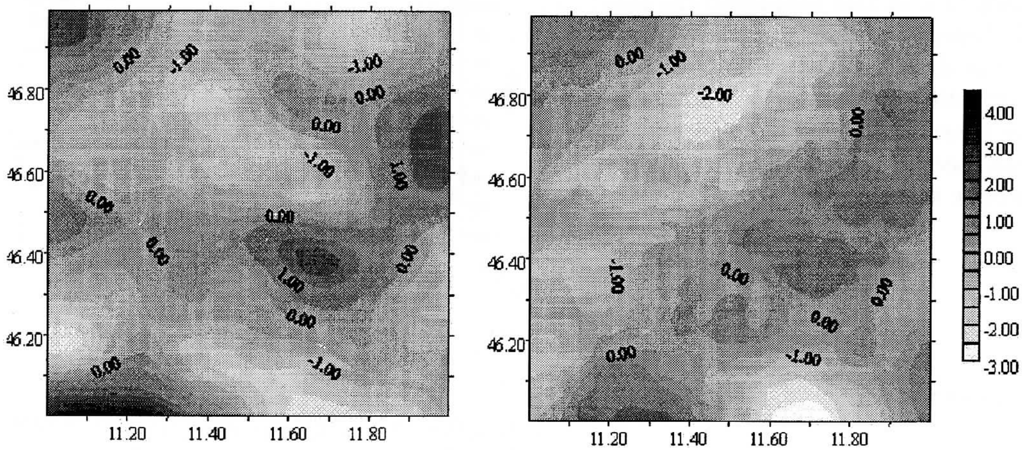


Figure 20: Differences between the filtering with noise and without: a) with GRAVSOFT covariance function; b) with COVLOC covariance function

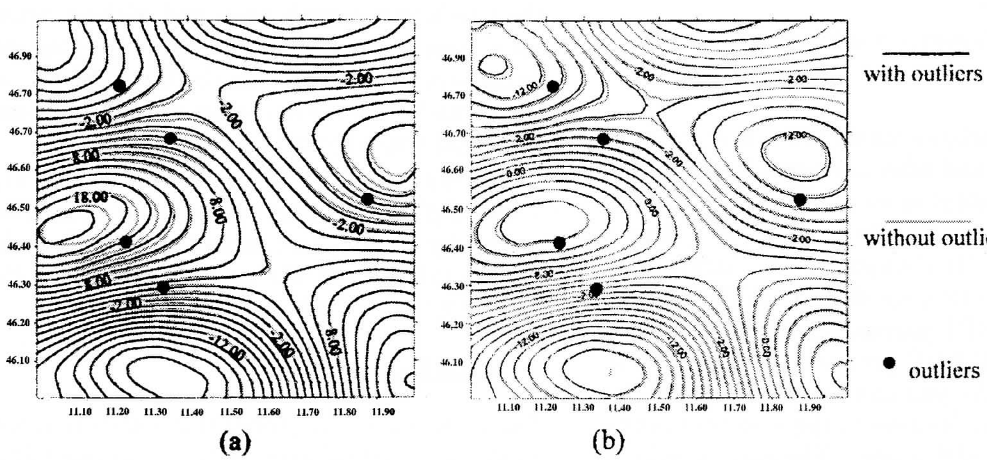


Figure 21: Filtered data: a) with GRAVSOFT covariance function; b) with COVLOC covariance function

model leads to a filtering which is less perturbed by noise and outliers. So, we came to the conclusion that modelling all the features of the empirical values helps in designing a more stable filter. On the contrary, with non stationary data, this requirement, as it is quite obvious, is not relevant at all; *optimal* filter can be defined which are based on model covariances that very loosely resemble the empirical *covariance*.

Further steps must be done to completely define the impact of such arguments and the new model covariances in the geodetic case. In particular, in the COVLOC program, the functional transformations (e.g. from the gravity covariance function to the geoid undulation covariance function) must be studied and implemented. Moreover, we have to perform tests on the accuracy of the prediction of different functionals as related to the fit of the empirical values, as we have done for the filtering. Some software improvements are also necessary in the spherical case to reduce the Legendre Polynomial computation time.

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