

Phase space Feynman path integrals

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A rigorous mathematical formulation of the “phase space Feynman path integral” is given in a general setting. This is then applied to yield a representation of solutions of the Schrödinger equation with potential depending both on the position and momentum variables. © 2002 American Institute of Physics.

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I. INTRODUCTION

Let us consider the Schrödinger equation for the time evolution of a d -dimensional quantum particle:

$$\begin{cases} \dot{\psi} = -\frac{i}{\hbar} H \psi, \\ \psi(0, x) = \psi_0(x), \end{cases} \quad (1)$$

where $H = p^2/2m + V$ is the Hamiltonian of the system, $m > 0$ is the mass of the particle, \hbar is the (reduced) Planck' constant, and p is the quantum mechanical momentum operator. Under suitable conditions on the potential V , in the position representation H is realized as the self-adjoint operator in the Hilbert space $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^d)$, obtained by closure of the operator

$$H\psi(x) = -\frac{\hbar^2}{2m} \Delta \psi(x) + V(x)\psi(x), \quad \psi \in D(-\Delta) \cap D(V),$$

$$\psi \in \mathcal{H}, \quad \int_{\mathbb{R}^d} |\psi(x)|^2 dx < +\infty,$$

where $D(A)$ denotes the domain of the operator A in \mathcal{H} (see, e.g. Ref. 1). Let $G(t, x, y)$ be the Green function or propagator, namely the kernel of the unitary group $e^{-itH/\hbar}$,

$$\psi(t, x) = \int_{\mathbb{R}^d} G(t, x, y) \psi_0(y) dy$$

(see, e.g., Ref. 1 for a discussion on sufficient conditions for the existence of G and for the properties of it). In 1942 Feynman gave a suggestive representation of the propagator and showed the connection between the classical Lagrangian description of the physical world and the quantum one. In fact the kernel of the unitary group can be heuristically computed by means of an infinite dimensional path integral of the following form:

$$G(t, x, y) = \text{const} \int e^{(i/\hbar) S(\gamma)} d\gamma, \quad (2)$$

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where the integration is on the set of paths starting in y at time 0 and ending in x at time t and $S(\gamma)$ is the classical action of the system, evaluated on the path γ :

$$S(\gamma) = \int_0^t (\dot{\gamma}(s)^2/2m - V(\gamma(s))) ds.$$

Expression (2) lacks rigor, e.g., neither the constant in front of the integral nor the “infinite dimensional Lebesgue measure” $d\gamma$ are well defined. Nevertheless a rigorous mathematical formulation of the *Feynman functional* can be given (see Refs. 2 and 3, and references therein; for other approaches see also Refs. 4 and 5). Feynman himself (see Ref. 6) gave other heuristic representations of the integral, e.g., the heuristic integration over paths in phase space. For physical discussions of this concept see also Ref. 7.

The aim of the present paper is to give a mathematical definition of a “phase space path integral:” a Hamiltonian instead of a Lagrangian representation of the propagator. We want to give mathematical meaning to the Hamiltonian version of formula (2), namely to

$$G(t, x, y) = \text{const} \int e^{(i/\hbar) S(q, p)} dq dp, \quad (3)$$

where $q(s), p(s)$, $s \in [0, t]$ are paths in the phase space and S is the action in the Hamiltonian fomulation: $S(q, p) = \int_0^t (\dot{q}(s)p(s) - H(q(s), p(s))) ds$.

The Hamiltonian formulation is more convenient for two reasons:

- (1) for many classical systems it is better than the Lagrangian one;
- (2) the discussion of the approach from quantum mechanics to classical mechanics, i.e., the study of the behavior of physical quantities taking into account that \hbar is small, is more natural in a Hamiltonian setting (see, e.g., Refs. 8 and 9 for a discussion of this behavior).

We note that an approach of phase space Feynman path integrals via analytic continuation of “phase space Wiener integrals” has been presented by Daubechies and Klauder.¹⁰ Analytic continuation was also used in other “path space” approaches, see Refs. 11 and 4, and references therein. Our approach is more direct in the spirit of Ref. 2.

II. LIE-TROTTER PRODUCT FORMULA

We first recall an abstract version of the Lie–Trotter product formula.

Lemma 1: Let A and B be self-adjoint operators in a Hilbert space \mathcal{H} and let $A+B$ be essentially self-adjoint on $D(A) \cap D(B)$. Then

$$s - \lim_{n \rightarrow \infty} (e^{itA/n} e^{itB/n})^n = e^{i(A+B)t}, \quad t \in \mathbb{R}. \quad (4)$$

Here $s - \lim$ is the strong operator limit. For a proof and a discussion of this lemma see e.g., Refs. 12 and 1.

Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and let us consider a potential V depending both on the position and on the momentum in the following way: $V = V_1(x) + V_2(p)$. V_1 is defined as a self-adjoint operator in \mathcal{H} , with its natural domain as a multiplication operator. V_2 is the operator in \mathcal{H} with domain

$$D(V_2(p)) = \{\psi \in \mathcal{H} \mid \alpha \rightarrow V_2(\alpha) \hat{\psi}(\alpha) \in \mathcal{H}\},$$

where $\hat{\psi}$ is the Fourier transform of ψ . It coincides with the operator defined by functional calculus as $V_2(p)$, with p the self-adjoint operator $-i\hbar \nabla$ in \mathcal{H} . V is then the sum, as a self-adjoint operator in \mathcal{H} , of the self-adjoint operators V_1 and V_2 . We assume that the functions V_1 and V_2 are such that the corresponding operators have a common dense domain of essential self-adjointness D . This is the case, e.g., when $V_1 \in L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, V_2 is bounded measurable, and $D = C_0^\infty(\mathbb{R}^d)$ or $D = \mathcal{S}(\mathbb{R}^d)$. We assume, in order to apply lemma 1, that V_1 , V_2 are such that $-(\hbar^2/2m) \Delta + V_2$ and $-(\hbar^2/2m) \Delta + V_1 + V_2$ are essentially self-adjoint on D . We denote by H the closure of the

latter operator. H (which we also write simply as $-(\hbar^2/2m)\Delta + V_1 + V_2$) is then the quantum Hamiltonian.

By lemma 1 we have then

$$\exp\left(-\frac{it(p^2/2m + V)}{\hbar}\right) = s\text{-}\lim_{n \rightarrow \infty} \left(\exp\left(-\frac{i\epsilon(p^2/2m + V_2)}{\hbar}\right) \exp\left(-\frac{i\epsilon(V_1)}{\hbar}\right) \right)^n, \quad \epsilon \equiv \frac{t}{n},$$

$$\psi(t) = \exp\left(-\frac{it(p^2/2m + V)}{\hbar}\right) \psi_0 = \lim_{n \rightarrow \infty} \left(\exp\left(-\frac{i\epsilon(p^2/2m + V_2)}{\hbar}\right) \exp\left(-\frac{i\epsilon V_1}{\hbar}\right) \right)^n \psi_0,$$

$$\psi_0 \in C_0^\infty(\mathbb{R}^d),$$

(see, e.g., Refs. 12 and 13 for related uses of the Lie–Trotter formula).

By shifting from the position representation to the momentum representation and vice versa and assuming that V_1 and V_2 are continuous, we can write in the strong $L^2(\mathbb{R}^d)$ -sense, for all $t > 0$:

$$\begin{aligned} \psi(t, x) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \exp\left(-\frac{i\epsilon(p_{n-1}^2/2m + V_2(p_{n-1}))}{\hbar}\right) \cdot \left(\exp\left(-\frac{i\epsilon V_1}{\hbar}\right) \right. \\ &\quad \times \left. \left(\exp\left(-\frac{i\epsilon(p^2/2m + V_2)}{\hbar}\right) \exp\left(-\frac{i\epsilon(V_1)}{\hbar}\right) \right)^{n-1} \psi_0 \right)(p_1) \frac{\exp\left(i \frac{x p_{n-1}}{\hbar}\right)}{(2\pi\hbar)^{d/2}} dp_{n-1} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{2d}} \exp\left(-\frac{i\epsilon(p_{n-1}^2/2m + V_2(p_{n-1}))}{\hbar}\right) \exp\left(-\frac{i\epsilon V_1(x_{n-1})}{\hbar}\right) \\ &\quad \cdot \left(\left(\exp\left(-\frac{i\epsilon(p^2/2m + V_2)}{\hbar}\right) \exp\left(-\frac{i\epsilon(V_1)}{\hbar}\right) \right)^{n-1} \psi_0 \right)(x_{n-1}) \frac{\exp\left(i \frac{x p_{n-1}}{\hbar}\right)}{(2\pi\hbar)^{d/2}} \\ &\quad \times \frac{\exp\left(-i \frac{x_{n-1} p_{n-1}}{\hbar}\right)}{(2\pi\hbar)^{d/2}} dp_{n-1} dx_{n-1} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^{2nd} \cdot \int_{\mathbb{R}^{2nd}} \exp\left(-\frac{i\epsilon}{\hbar} \sum_{j=0}^{n-1} \left(\frac{p_j^2}{2m} + V_1(x_j) + V_2(p_j) \right. \right. \\ &\quad \left. \left. - p_j \frac{(x_{j+1} - x_j)}{\epsilon} \right) \right) \psi_0(x_0) \prod_{j=0}^{n-1} dp_j dx_j, \end{aligned} \quad (5)$$

where $x_n \equiv x$.

Remark: The integrals above are to be understood as limits as $\Lambda \uparrow \mathbb{R}^d$, $n \rightarrow \infty$ in the $L^2(\mathbb{R}^{2nd})$ sense of the corresponding integrals over Λ^{2nd} , with Λ bounded (see Ref. 11). Formula (5) holds first as a strong L^2 -limit, but then (possibly by subsequences) also for Lebesgue a.e.x. It also follows from this that (5) gives the solution to the Cauchy problem (1).

The latter expression suggests the following formula for the limit:

$$\psi(t, x) = \text{const} \int_{q(t)=x} e^{(i/\hbar) S(q, p)} \psi(0, q(0)) dq dp,$$

$$S(q, p) = \int_0^t p(s) \dot{q}(s) - H(q(s), p(s)) ds, \quad (6)$$

which does not yet have a mathematical meaning. It will be rigorously defined in Secs. III and IV.

III. OSCILLATORY INTEGRALS AND THE CAMERON MARTIN FORMULA

In this section we recall for later use some known results, for more details we refer to Refs. 2, 3, and 8.

A. Finite dimensional oscillatory integrals

Let us consider the finite dimensional real Hilbert space \mathbb{R}^n , whose elements are denoted by $x, y \in \mathbb{R}^n$ and the scalar product with $\langle x, y \rangle$. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a nondegenerate symmetric operator.

Definition 1: A function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ is Fresnel integrable with respect to T if and only if for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\phi(0) = 1$ the limit

$$\lim_{\epsilon \rightarrow 0} \int e^{i\langle x, Tx \rangle} f(x) \phi(\epsilon x) dx \quad (7)$$

exists and is independent of ϕ . In this case the limit is called the Fresnel integral of f with respect to T and denoted by

$$\int e^{i\langle x, Tx \rangle} f(x) dx.$$

There is an important class $\mathcal{F}(\mathbb{R}^n)$ of Fresnel integrable functions: those which are Fourier transforms of complex bounded variation measures on \mathbb{R}^n , $\mathcal{M}(\mathbb{R}^n)$:

$$f \in \mathcal{F}(\mathbb{R}^n) \Leftrightarrow f(x) = \int e^{i\langle x, \alpha \rangle} \mu_f(d\alpha), \quad \mu_f \in \mathcal{M}(\mathbb{R}^n), \quad x \in \mathbb{R}^n.$$

$\mathcal{F}(\mathbb{R}^n)$ contains in particular $\mathcal{S}(\mathbb{R}^n)$, hence it is also dense in $L^2(\mathbb{R}^n)$. In this case the Parseval equality gives us the following expression for the limit (7):

$$(2\pi i)^{-n/2} \int e^{(i/2)\langle x, Tx \rangle} f(x) dx = (\det T)^{-1/2} \int e^{(-i/2)\langle x, T^{-1}x \rangle} \mu_f(dx).$$

Analogously one can define the normalized Fresnel integral by means of the following expression:

$$\widetilde{\int} \int e^{(i/2)\langle x, Tx \rangle} f(x) dx := (\det T)^{1/2} (2\pi)^{-n/2} \int e^{(i/2)\langle x, Tx \rangle} f(x) dx = \int e^{(-i/2)\langle x, T^{-1}x \rangle} \mu_f(dx).$$

Note that if we substitute into the latter definition the function $f=1$ we have $\widetilde{\int} e^{(i/2)\langle x, Tx \rangle} f(x) dx = 1$.

B. Infinite dimensional oscillatory integrals

Let us consider an infinite dimensional real Hilbert space of paths \mathcal{H} whose elements are denoted by $\gamma, \eta \in \mathcal{H}$ and the scalar product by $\langle \gamma, \eta \rangle$. Let P_n be a sequence of projectors on n -dimensional subspaces of \mathcal{H} such that $P_n \leq P_{n+1}$ [i.e., $P_{n+1} = I$ on $P_n(\mathcal{H})$] and $P_n \rightarrow I$ strongly as $n \rightarrow \infty$; let $f: \mathcal{H} \rightarrow \mathbb{C}$ be a function on \mathcal{H} and let $T: D(T) \subseteq \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint invertible operator.

Definition 2: A function $f: \mathcal{H} \rightarrow \mathbb{C}$ is Fresnel integrable with respect to T if and only if for each n the following finite dimensional integral

$$\widetilde{\int}_{P_n \mathcal{H}} e^{i\langle P_n \gamma, TP_n \gamma \rangle} f(P_n \gamma) dP_n \gamma \quad (8)$$

is well defined and the limit

$$\lim_{n \rightarrow \infty} \int_{P_n \mathcal{H}} e^{i\langle P_n \gamma, T P_n \gamma \rangle} f(P_n \gamma) dP_n \gamma \quad (9)$$

exists and is independent of the sequence $\{P_n\}$.

In this case the limit is called the Fresnel integral of f with respect to T and is denoted by

$$\int e^{i\langle \gamma, T \gamma \rangle} f(\gamma) d\gamma.$$

Equation (8) is called the normalized finite dimensional approximation of this Fresnel integral.

One can prove that if $f \in \mathcal{F}(\mathcal{H})$ then $f \circ P_n \in \mathcal{F}(P_n(\mathcal{H}))$. Moreover f is Fresnel integrable and the Cameron–Martin-type formula holds:

$$\int e^{(i/2)\langle \gamma, T \gamma \rangle} f(\gamma) d\gamma = \int_{\mathcal{H}} e^{-(i/2)\langle \gamma, T^{-1} \gamma \rangle} \mu_f(d\gamma), \quad (10)$$

see Refs. 8 and 3.

IV. PHASE SPACE FEYNMAN FUNCTIONAL

Let us consider again the expression (6) in the particular case of the free particle, namely when the Hamiltonian is just the kinetic energy: $H = p^2/2m$. In this case we have heuristically

$$\psi(t, x) = \text{const} \int_{q(t)=x} \exp\left(\frac{i}{\hbar} \int_0^t (p(s)\dot{q}(s) - p(s)^2/2m) ds\right) \psi(0, q(0)) dq dp. \quad (11)$$

We can give a precise meaning to this expression: under suitable hypothesis on the initial wave function ψ_0 , it is an infinite dimensional oscillatory integral.¹⁴ In fact, following Ref. 15, let us introduce the Hilbert space $\mathcal{H}_t \times \mathcal{L}_t$, namely the space of paths in the d -dimensional phase space $(q(s), p(s))_{s \in [0, t]}$, such that the path $(q(s))_{s \in [0, t]}$ belongs to the Cameron–Martin space \mathcal{H}_t , namely to the space of the absolutely continuous functions q from $[0, t]$ to \mathbb{R}^d such that $q(t) = 0$ and $\dot{q} \in \mathcal{L}_2([0, t], \mathbb{R}^d)$, with inner product $\langle q_1, q_2 \rangle = \int_0^t \dot{q}_1(s) \dot{q}_2(s) ds$, while the path in the momentum space $(p(s))_{s \in [0, t]}$ belongs to $\mathcal{L}_t = \mathcal{L}_2([0, t], \mathbb{R}^d)$. $\mathcal{H}_t \times \mathcal{L}_t$ is an Hilbert space with the natural inner product

$$\langle q, p; Q, P \rangle = \int_0^t \dot{q}(s) \dot{Q}(s) ds + \int_0^t p(s) P(s) ds.$$

Let us introduce the following bilinear form:

$$[q, p; Q, P] = \int_0^t \dot{q}(s) P(s) ds + \int_0^t p(s) \dot{Q}(s) ds - \int_0^t p(s) P(s) ds = \langle q, p; A(Q, P) \rangle,$$

where A is the following operator in $\mathcal{H}_t \times \mathcal{L}_t$:

$$A(Q, P)(s) = \left(\int_t^s P(u) du, \dot{Q}(s) - P(s) \right). \quad (12)$$

$A(Q, P)$ is densely defined, e.g., on $C^1([0, t]; \mathbb{R}^d) \times C^1([0, t]; \mathbb{R}^d)$. Moreover $A(Q, P)$ is invertible with inverse given by

$$A^{-1}(Q, P)(s) = \left(\int_t^s P(u) du + Q(s), \dot{Q}(s) \right) \quad (13)$$

(on the range of A).

Now expression (6) can be realized rigorously as

$$\int_{\mathcal{H}_t \times \mathcal{L}_t} e^{(i/2\hbar)\langle q, p; A(q, p) \rangle} \psi(0, q(0) + x) dq dp,$$

where $q+x$ denotes the translated path $q(s) \rightarrow q(s) + x$, and the normalized integral is defined by (10).

In this case the heuristic expression (6) is well defined through Lie–Trotter product formula, namely as the limit of a sequence of finite dimensional integrals, as we saw in Sec. II. We are now going to show that it is also the limit of a sequence of finite dimensional oscillatory integrals in the sense of definition 2.

Let us consider a sequence of partitions π_n of the interval $[0, t]$ into n subintervals of amplitude $\epsilon \equiv t/n$:

$$t_0 = 0, t_1 = \epsilon, \dots, t_i = i\epsilon, \dots, t_n = n\epsilon = t.$$

To each π_n we associate a projector $P_n: \mathcal{H}_t \times \mathcal{L}_t \rightarrow \mathcal{H}_t \times \mathcal{L}_t$ onto a finite dimensional subspace of $\mathcal{H}_t \times \mathcal{L}_t$, namely the subspace of polygonal paths. In other words each projector P_n acts on a phase space path $(q, p) \in \mathcal{H}_t \times \mathcal{L}_t$ in the following way:

$$P_n(q, p)(s) = \left(\sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) \left(q(t_{i-1}) + \frac{(q(t_i) - q(t_{i-1}))(s - t_{i-1})}{t_i - t_{i-1}} \right), \sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) p_i \right),$$

where

$$p_i = \frac{\int_{t_{i-1}}^{t_i} p(s) ds}{t_i - t_{i-1}} = \frac{1}{\epsilon} \int_{t_{i-1}}^{t_i} p(s) ds.$$

Theorem 1: For each $n \in \mathbb{N}$, P_n is a projector in $\mathcal{H}_t \times \mathcal{L}_t$. Moreover for $n \rightarrow \infty$ $P_n \rightarrow I$ as a bounded operator.

Proof: P_n is symmetric, indeed for all $(Q, P) \in \mathcal{H}_t \times \mathcal{L}_t$ and all $(q, p) \in \mathcal{H}_t \times \mathcal{L}_t$,

$$\begin{aligned} \langle Q, P; P_n(q, p) \rangle &= \int_0^t \dot{Q}(s) \sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) \frac{(q(t_i) - q(t_{i-1}))(s - t_{i-1})}{t_i - t_{i-1}} ds + \int_0^t P(s) \sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) p_i ds \\ &= \sum_{i=1}^n \frac{(q(t_i) - q(t_{i-1}))(Q(t_i) - Q(t_{i-1}))}{t_i - t_{i-1}} + \sum_{i=1}^n \frac{\int_{t_{i-1}}^{t_i} p(s) ds \int_{t_{i-1}}^{t_i} P(s) ds}{t_i - t_{i-1}} \\ &= \langle P_n(Q, P); q, p \rangle. \end{aligned}$$

$P_n^2 = P_n$, indeed

$$\begin{aligned} P_n^2(q, p)(s) &= \left(\sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) \left(q(t_{i-1}) + \frac{(q(t_i) - q(t_{i-1}))(s - t_{i-1})}{t_i - t_{i-1}} \right), \sum_{i=1}^n \chi_{[t_{i-1}, t_i]}(s) p_i \right) \\ &= P_n(q, p)(s) \end{aligned}$$

$\forall (q, p) \in \mathcal{H}_t \times \mathcal{L}_t$, $\|P_n(q, p) - (q, p)\| \rightarrow 0$ as $n \rightarrow \infty$:

Let us consider the subset $\mathcal{K} \subseteq \mathcal{H}_t \times \mathcal{L}_t$, $\mathcal{K} = \{(q, p) \in \mathcal{H}_t \times \mathcal{L}_t; \|P_n(q, p) - (q, p)\| \rightarrow 0, n \rightarrow \infty\}$. It is enough to prove that the closure of \mathcal{K} is $\mathcal{H}_t \times \mathcal{L}_t$. To prove this it is sufficient to show that \mathcal{K}

is a closed subspace of $\mathcal{H}_t \times \mathcal{L}_t$ and contains a dense subset of $\mathcal{H}_t \times \mathcal{L}_t$. This follows from the density of the piecewise linear paths in \mathcal{H}_t and the density of the piecewise constant paths in \mathcal{L}_t (see, e.g., Ref. 14).

Theorem 2: Let the function $(q, p) \rightarrow \psi_0(x + q(0))$, $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$, be Fresnel integrable with respect to A [with A defined by (12)]. Then the phase space Feynman path integral, namely the limit

$$\lim_{n \rightarrow \infty} \int_{P_n(\mathcal{H}_t \times \mathcal{L}_t)} e^{(i/2\hbar) \langle P_n(q, p), A P_n(q, p) \rangle} \psi_0(x + q(0)) dP_n(q, p) \quad (14)$$

coincides with the limit (5), namely with the solution of the Schrödinger equation with a free Hamiltonian.

Proof: The result follows by direct computation, indeed:

$$\begin{aligned} & \int_{P_n(\mathcal{H}_t \times \mathcal{L}_t)} e^{(i/2\hbar) \langle P_n(q, p), A P_n(q, p) \rangle} \psi_0(x + q(0)) dP_n(q, p) \\ &= \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^{2nd} \int_{\mathbb{R}^{2nd}} \exp \left(-\frac{i\epsilon}{\hbar} \sum_{j=0}^{n-1} \left(\frac{p_j^2}{2m} - p_j \frac{(x_{j+1} - x_j)}{\epsilon} \right) \right) \psi_0(x_0) \prod_{j=0}^{n-1} dp_j dx_j, \end{aligned}$$

and the two limits (5) and (14) coincide. Indeed (14) is a pointwise limit by hypothesis. On the other hand (5) is a limit in L_2 sense, hence, passing if necessary to a subsequence, it is also a pointwise limit.

Remark 1: The latter result is equivalent to the “traditional” formulation of the Feynman path integral in the configuration space. Indeed it can be obtained by means of Fubini theorem² and an integration with respect to the momentum variables:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^{2nd} \int_{\mathbb{R}^{2nd}} \exp \left(-\frac{i\epsilon}{\hbar} \sum_{j=0}^{n-1} \left(\frac{p_j^2}{2m} - p_j \frac{(x_{j+1} - x_j)}{\epsilon} \right) \right) \psi_0(x_0) \prod_{j=0}^{n-1} dp_j dx_j \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{2\pi i \hbar}} \right)^{nd} \int_{\mathbb{R}^{nd}} \exp \left(-\frac{i\epsilon}{\hbar} \sum_{j=0}^{n-1} m \frac{(x_{j+1} - x_j)^2}{2\epsilon^2} \right) \psi_0(x_0) \prod_{j=0}^{n-1} dx_j. \end{aligned}$$

The latter expression yields the Feynman functional on the configuration space, i.e., heuristically $\text{const} \int \exp(\int_0^t \mathcal{L}(q(s), \dot{q}(s)) ds) dq$ (\mathcal{L} being the classical Lagrangian density).

Remark 2: The integration with respect to the momentum variables might seem to be superfluous, but it is very useful when we introduce a potential depending on the momentum.

Theorem 3: Let us consider a semibounded potential V depending explicitly on the momentum: $V = V(p)$ and the corresponding quantum mechanical Hamiltonian $H = -(\hbar^2/2) \Delta + V(p)$. Let us suppose H is an essentially self-adjoint operator on $\mathcal{L}_2(\mathbb{R}^d)$. Let the function $(q, p) \rightarrow \exp(-(i/\hbar) \int_0^t V(P_n(p(s))) ds) \psi_0(x + q(0))$ be Fresnel integrable with respect to the operator A , with A defined by (12). Then the solution to the Schrödinger equation

$$\begin{cases} \dot{\psi} = -\frac{i}{\hbar} H \psi, \\ \psi(0, x) = \psi_0(x), \quad \psi_0 \in \mathcal{S}(\mathbb{R}^d) \end{cases} \quad (15)$$

is given by the phase space path integral

$$\lim_{n \rightarrow \infty} \int_{P_n(\mathcal{H}_t \times \mathcal{L}_t)} \exp\left(\frac{i}{2\hbar} \langle P_n(q, p), A P_n(q, p) \rangle\right) \exp\left(-\frac{i}{\hbar} \int_0^t V(P_n(p(s))) ds\right) \\ \times \psi_0(x + q(0)) dP_n(q, p).$$

Proof: We can proceed in a completely analogous way as in the proof of Theorem 2, therefore we shall omit the details. \square

V. THE PHASE SPACE FEYNMAN-KAC FORMULA

Let us consider a classical potential V depending both on the position $Q \in \mathbb{R}^d$ and on the momentum $P \in \mathbb{R}^d$, but of the special form: $V = V(Q, P) = V_1(Q) + V_2(P)$ (The general case presents problems due to the noncommutativity of the quantized expression of Q and P), for a different approach with more general Hamiltonians see Ref. 16. Moreover let us suppose the function $f: \mathcal{H}_t \times \mathcal{L}_t \rightarrow \mathbb{C}$,

$$f(q, p) = \psi_0(x + q(0)) \exp\left(-\frac{i}{\hbar} \int_0^t V(q(s) + x, p(s)) ds\right), \quad \psi_0 \in \mathcal{S}(\mathbb{R}^d)$$

is the Fourier transform of a complex bounded variation measure μ_f on $\mathcal{H}_t \times \mathcal{L}_t$:¹⁷

$$f(q, p) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{i\langle q, p; Q, P \rangle} d\mu_f(Q, P).$$

Under additional assumptions on V_1 and V_2 we shall see that the phase space Feynman path integral of the function f can be computed and is given by

$$\int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \exp\left(-\frac{i}{\hbar} \int_0^t V(q(s) + x, p(s)) ds\right) \psi(0, q(0) + x) dq dp \\ = \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{-i\hbar}{2} \langle q, p; A^{-1}(q, p) \rangle\right) d\mu_f(q, p). \quad (16)$$

This follows from Sec. III together with the following.

Lemma 2: Let us consider a potential $V(Q, P) = V_1(Q) + V_2(P)$ and an initial wave function ψ_0 such that $V_1, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$ and the function $p(s)_{s \in [0, t]} \rightarrow \int_0^t V_2(p(s)) ds \in \mathcal{F}(\mathcal{L}_t)$. Then the functional

$$f(q, p) = \psi_0(x + q(0)) \exp\left(-\frac{i}{\hbar} \int_0^t V(q(s) + x, p(s)) ds\right)$$

belongs to $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$.

Proof: $f(q, p)$ is the product of two functions: the first, say f_1 , depends only on the first variable q , while the second f_2 depends only on the variable p , more precisely

$$f_1(q) = \psi_0(x + q(0)) \exp\left(-\frac{i}{\hbar} \int_0^t V_1(q(s) + x) ds\right), \quad f_2(p) = \exp\left(-\frac{i}{\hbar} \int_0^t V_2(p(s)) ds\right).$$

Under the given hypothesis on V_1 and ψ_0 , f_1 belongs to $\mathcal{F}(\mathcal{H}_t)$. The proof is given for instance in Ref. 2. For f_2 one must pay more attention: indeed the same proof given for f_1 does not work, as f_2 is defined on a different Hilbert space and we have to require explicitly that $\int_0^t V_2(p(s)) ds \in \mathcal{F}(\mathcal{L}_t)$. Under this hypothesis one can easily prove that (see Ref. 2) $f_2 \in \mathcal{F}(\mathcal{L}_t)$.

Now if $f_1 = \hat{\mu}_{f_1} \in \mathcal{F}(\mathcal{H}_t)$, f_1 can be extended to a function, denoted again by f_1 , in $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$: it is the Fourier transform of the product measure on $\mathcal{H}_t \times \mathcal{L}_t$ of $\mu_{f_1}(dq)$ and $\delta_0(dp)$. The same holds for $f_2 = \hat{\mu}_{f_2}$: $f_2 = (\delta_0(dq)\mu_{f_2}(dp))$.

Finally, as $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$ is a Banach algebra,² the product of two elements $f_1 f_2$ is again an element of $\mathcal{F}(\mathcal{H}_t \times \mathcal{L}_t)$: more precisely it is the Fourier transform of the convolution of the two measures in $\mathcal{M}(\mathcal{H}_t \times \mathcal{L}_t)$ corresponding to f_1 and f_2 , respectively, and the conclusion follows. The next theorem shows that the above oscillatory integral (16) gives the solution to the Schrödinger equation.

Theorem 4: *Let us consider the following Hamiltonian*

$$H(Q;P) = \frac{P^2}{2} + V_1(Q) + V_2(P)$$

in $L^2(\mathbb{R}^d)$ and the corresponding Schrödinger equation

$$\begin{cases} \dot{\psi} = -\frac{i}{\hbar} H \psi, \\ \psi(0, x) = \psi_0(x), \quad x \in \mathbb{R}^d. \end{cases} \quad (17)$$

Let us suppose that $V_1, \psi_0 \in \mathcal{F}(\mathbb{R}^d)$ and $\int_0^t V_2(p(s)) ds \in \mathcal{F}(\mathcal{L}_t)$. Then the solution to the Cauchy problem (17) is given by the phase space Feynman path integral:

$$\int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \exp\left(-\frac{i}{\hbar} \int_0^t (V_1(q(s) + x) + V_2(p(s))) ds\right) \psi(0, q(0) + x) dq dp.$$

Proof: We follow the proof given by Elworthy and Truman in Ref. 3.

For $0 \leq u \leq t$ let $\mu_u(V_1, x) \equiv \mu_u, \nu_u^t(V_1, x) \equiv \nu_u^t, \eta_u^t(V_2) \equiv \eta_u^t$, and $\mu_0(\psi)$ be the measures on $\mathcal{H}_t \times \mathcal{L}_t$ whose Fourier transforms when evaluated at $(q, p) \in \mathcal{H}_t \times \mathcal{L}_t$ are

$$V_1(x + q(u)), \quad \exp\left(-i \int_u^t V_1(x + q(s)) ds\right), \quad \exp\left(-i \int_u^t V_2(p(s)) ds\right), \quad \psi_0(q(0) + x).$$

We set

$$\begin{aligned} U(t) \psi_0(x) &= \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar} \int_0^t (V_1(q(s) + x) + V_2(p(s))) ds\right) \psi(0, q(0) + x) dq dp \end{aligned}$$

and

$$\begin{aligned} U_0(t) \psi_0(x) &= \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar} \int_0^t V_2(p(s)) ds\right) \psi(0, q(0) + x) dq dp. \end{aligned}$$

By Sec. III we have

$$U(t) \psi_0(x) = \int_{\mathcal{H}_t \times \mathcal{L}_t} e^{(-i\hbar/2) \langle q, p; A^{-1}(q, p) \rangle} (\eta_0^t * \nu_0^t * \mu_0(\psi))(dq dp). \quad (18)$$

Now, if $\{\mu_u : a \leq u \leq t\}$ is a family in $\mathcal{M}(\mathcal{H}_t \times \mathcal{L}_t)$, we shall let $\int_a^b \mu_u du$ denote the measure on $\mathcal{H}_t \times \mathcal{L}_t$ given by

$$f \rightarrow \int_a^b \int_{\mathcal{H}_t \times \mathcal{L}_t} f(q, p) d\mu_u(q, p) du$$

whenever it exists.

Since for any continuous path q ,

$$\exp\left(-i \int_0^t V_1(q(s)) ds\right) = 1 - i \int_0^t V_1(q(u)) \exp\left(-i \int_u^t V_1(q(s)) ds\right) du,$$

the following relation holds:

$$\nu_0^t = \delta_0 - i \int_0^t (\mu_u^* \nu_u^t) du, \quad (19)$$

where δ_0 is the Dirac measure at $0 \in \mathcal{H}_t$.

Applying this relation to (18) we obtain

$$\begin{aligned} U(t) \psi_0(x) &= \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{-i\hbar}{2} \langle q, p; A^{-1}(q, p) \rangle\right) (\eta_0^* \mu_0(\psi)) (dq dp) \\ &\quad - i \int_0^t \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{-i\hbar}{2} \langle q, p; A^{-1}(q, p) \rangle\right) (\eta_0^* \mu_u(V_1, x) * \nu_u^* \mu_0(\psi)) (dq dp) du \\ &= U_0(t) \psi_0(x) - i \int_0^t \int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \exp\left(-\frac{i}{\hbar} \int_u^t V_1(q(s) + x) ds\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar} \int_0^t V_2(p(s)) ds\right) V_1(q(u) + x) \psi_0(q(0) + x) dq dp du. \end{aligned}$$

Now we have, by Fubini theorem for Fresnel integrals,²

$$\begin{aligned} &\int_{\mathcal{H}_t \times \mathcal{L}_t} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle\right) \exp\left(-\frac{i}{\hbar} \int_u^t V_1(q(s) + x) ds\right) \exp\left(-\frac{i}{\hbar} \int_0^t V_2(p(s)) ds\right) ds \\ &\quad V_1(q(u) + x) \psi_0(q(0) + x) dq dp \\ &= \int_{\mathcal{H}_{t-u} \times \mathcal{L}_{t-u}} \exp\left(\frac{i}{2\hbar} \langle q, p; A(q, p) \rangle_{\mathcal{H}_{t-u} \times \mathcal{L}_{t-u}}\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar} \int_0^{t-u} V_1(q(s) + x) ds\right) \left(\exp - \frac{i}{\hbar} \int_0^{t-u} V_2(p(s)) ds\right) \\ &\quad \times ds V_1(q(0) + x) \int_{\mathcal{H}_u \times \mathcal{L}_u} \exp\left(\frac{i}{2\hbar} \langle q_1, p_1; A(q_1, p_1) \rangle_{\mathcal{H}_u \times \mathcal{L}_u}\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar} \int_0^u V_2(p_1(s)) ds\right) \psi_0(q_1(0)) dq_1 dp_1 dq dp. \end{aligned}$$

Here $q \in \mathcal{H}_{t-u}$ and $q_1 \in \mathcal{H}_u$ are the integration variables, and \mathcal{H}_s denotes the Cameron–Martin space of paths $\gamma: [0, s] \rightarrow \mathbb{R}^d$.

We have

$$\begin{aligned}
U(t)\psi_0(x) &= U_0(t)\psi_0(x) - i \int_0^t U(t-u)(V_1 U_0(u)\psi_0)(x) du \\
&= U_0(t)\psi_0(x) - i \int_0^t U(u)(V_1 U_0(-u)U_0(t)\psi_0)(x) du.
\end{aligned}$$

The iterative solution of the latter integral equation is the convergent Dyson perturbation series for $U(t)$ with respect to $U_0(t)$, which proves the theorem. \square

Remark: We have stated our results for $\psi_0 \in \mathcal{F}(\mathbb{R}^d)$. They can be extended by density to $L^2(\mathbb{R}^d)$.

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