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A C^2-Finite Element Formulation for Strain Gradient Kirchhoff Nanoplates Using Hierarchical Hermite Basis Functions

C.A. Yan*, R. Vescovini*, N. Fantuzzi+, R. Luciano⋆

*Politecnico di Milano, Via La Masa 34, 20156, Milan, Italy
E-mails: chengangelo.yan@polimi.it, riccardo.vescovini@polimi.it

+Università di Bologna, Viale del Risorgimento 2, 40136, Bologna, Italy
E-mail: nicholas.fantuzzi@unibo.it

⋆Università degli Studi di Napoli Parthenone, Centro Direzionale ISOLA C4, 80133, Naples, Italy
E-mail: raimondo.luciano@uniparthenone.it

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1 Introduction

Classical continuum theories are widely used in mechanics to efficiently model solid behavior via field variables like displacement, strain, and stress. However, they inherently neglect microstructural effects, making them inadequate for capturing size-dependent phenomena that arise at nanoscales.

Strain gradient elasticity addresses this limitation by incorporating size effects into the continuum framework, while retaining the advantages of field-based modeling [1]. Nevertheless, its practical use in Finite Element (FE) analysis remains limited due to the higher inter-element continuity required by the governing equations – specifically, C^2-continuity is required in the case of Kirchhoff strain gradient elasticity. This poses significant challenges, as standard FEs are formulated to accommodate only C^0-continuity.

This work introduces a novel FE formulation based on hierarchical Hermite polynomials that explicitly enforce C^2-continuity. The method is applied to isotropic Kirchhoff nanoplates and benchmarked against the conventional C^2-Hermite formulation.

2 Methodology

To illustrate the proposed FE formulation, we study the bending of a Kirchhoff-type strain gradient nanoplate. The governing equation is derived via the Principle of Virtual Work:

$$\begin{aligned}\int_\Omega \Bigg\{ -D\left[\frac{\partial^2\delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\delta w}{\partial y^2}\frac{\partial^2 w}{\partial y^2}+2(1-\nu)\frac{\partial^2\delta w}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\nu\left(\frac{\partial^2\delta w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2\delta w}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)\right]+\\ -Dl^2\Bigg[\frac{\partial^3\delta w}{\partial x^3}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3\delta w}{\partial y^3}\frac{\partial^3 w}{\partial y^3}+(3-2\nu)\left(\frac{\partial^3\delta w}{\partial x^2\partial y}\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial^3\delta w}{\partial x\partial y^2}\frac{\partial^3 w}{\partial x\partial y^2}\right)+\\ +\nu\left(\frac{\partial^3\delta w}{\partial x\partial y^2}\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3\delta w}{\partial x^3}\frac{\partial^3 w}{\partial x\partial y^2}+\frac{\partial^3\delta w}{\partial x^2\partial y}\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3\delta w}{\partial y^3}\frac{\partial^3 w}{\partial x^2\partial y}\right)\Bigg]\Bigg\}\mathrm{d}\Omega+\int_\Omega \delta w q \mathrm{d}\Omega\end{aligned} \tag{1}$$

where $D=\frac{Eh^3}{12(1-\nu^2)}$ is the bending rigidity with Young's modulus E, thickness h and Poisson ratio ν; w is the deflection; l is the nonlocal parameter; q is an external pressure acting on the plate domain Ω.

The variational statement in Equation 1 involves third-order derivatives. As result, C^2-continuity is required across element boundaries to have a well-posed FE approximation. This requirement is met here by employing a hierarchical Hermite function space, S^p, defined as:

$$\begin{aligned} f_k(\xi) &= a_{k0}+a_{k1}\xi+a_{k2}\xi^2+a_{k3}\xi^3+a_{k4}\xi^4+a_{k5}\xi^5 && \text{for} \quad k=1,\dots,6\\ f_{i+1}(\xi) &= \sqrt{\frac{2i-1}{2}}\iiint_{-1}^{\xi} P_{i-3}(\xi')\mathrm{d}\xi' && \text{for} \quad i=6,5,\dots,p \end{aligned} \tag{2}$$

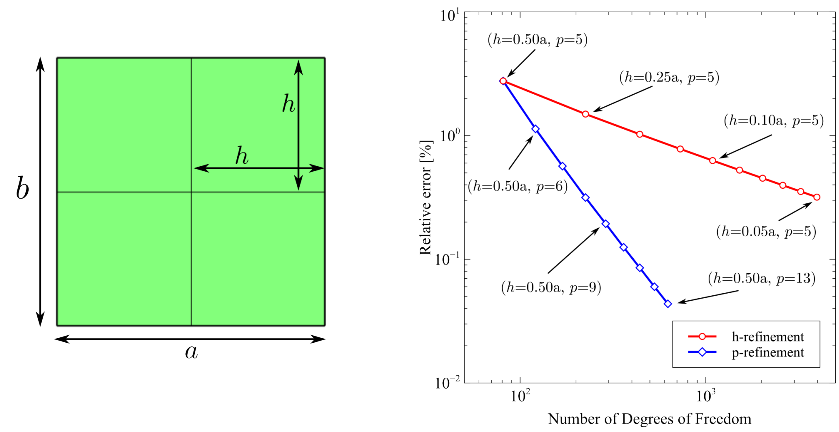
where p denotes the order of the polynomial space, the functions f_1-f_6 are the conventional C^2-Hermite polynomials of order five which ensure continuity of displacement and its first two derivatives [2]. The higher-order functions, f_{i+1}, are constructed from the triple integrals of Legendre polynomials P_{i-3}, yielding shape functions of order i.

The function space defined by Equation 2 represents a C^2-extension of the hierarchical series first proposed in [3], and previously applied in [4] within a C^0-continuous finite element framework for composite shells. It retains the favorable properties of the original series, including hierarchical structure, good numerical stability of the stiffness matrix, and spectral convergence of the FE solution.

3 Results

To evaluate the convergence of the proposed FE formulation, we consider a simply supported square nanoplate ($b/a = 1$) subjected to uniform pressure $q = q_0$. The mesh configuration is shown in Figure 1a. In the convergence analysis, shown in Figure 1b, two approaches are employed to refine the FE model: (I) h-refinement, where the element size h is reduced while keeping the polynomial order fixed at $p = 5$; (II) p-refinement, where the mesh is fixed and the polynomial order p is increased. The former is the refinement approach typically employed in C^2-Hermite elements [2], while the latter is adopted for the present hierarchical C^2-Hermite formulation.

Convergence is assessed using the dimensionless central deflection $\alpha = Dw_{\max}/(q_0 a^4)$ computed from the FE solution and exact solutions from [5]. The results demonstrate that the proposed method achieves significantly faster convergence than the conventional C^2-Hermite approach.



(a) (b)

Figure 1. (a) Finite element mesh; (b) Convergence study using h- and p-refinements.

4 Conclusion

This work introduced a FE formulation based on hierarchical Hermite shape functions for modeling Kirchhoff-type strain gradient nanoplates. The formulation enforces C^2-continuity, enabling to handle the high-order governing equations inherent to gradient elasticity. Numerical results demonstrate superior convergence rates compared to the conventional C^2-Hermite formulation, confirming the effectiveness of the approach for nanoscale structural analysis of thin plates. Future work will focus on extending the proposed formulation to the analysis of strain gradient nanoshells.

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