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Optimally Controlled Moving Sets with Geographical Constraints

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Abstract. The paper is concerned with a family of geometric evolution problems, modeling the spatial control of an invasive population within a region $V \subset \mathbb{R}^2$ bounded by geographical barriers. If no control is applied, the contaminated set $\Omega(t) \subset V$ expands with unit speed in all directions. By implementing a control, a region of area M can be cleared up per unit time. Given an initial set $\Omega(0) = \Omega_0 \subseteq V$, three main problems are studied: (1) existence of an admissible strategy $t \mapsto \Omega(t)$ which eradicates the contamination in finite time, so that $\Omega(T) = \emptyset$ for some $T > 0$. (2) Optimal strategies that achieve eradication in minimum time. (3) Strategies that minimize the average area of the contaminated set on a given time interval $[0, T]$. For these optimization problems, a sufficient condition for optimality is proved, together with several necessary conditions. Based on these conditions, optimal set-valued motions $t \mapsto \Omega(t)$ are explicitly constructed in a number of cases.

1. Introduction

We consider a family of geometric evolution problems, modeling the spatial control of an invasive population [8]. For each time $t \in [0, T]$, we denote by $\Omega(t) \subset \mathbb{R}^2$ a set moving in the plane. This can be regarded as a "contaminated region", which we would like to shrink as much as possible. To control the evolution of this set, we assign the velocity $\beta = \beta(t, x)$ in the inward normal direction at every boundary point $x \in \partial\Omega(t)$.

A function $E(\beta) \geq 0$ is given, describing the *effort* needed to push the boundary of $\Omega(t)$ inward, with speed β in the normal direction (see Fig. 1). The *total control effort* at time $t \in [0, T]$ is then defined as

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)} E\big(\beta(t, x)\big)\, \mathcal{H}^1(dx), \tag{1.1}$$

where the integral is computed w.r.t. the 1-dimensional Hausdorff measure along the boundary of $\Omega(t)$. In this paper we focus on the case where

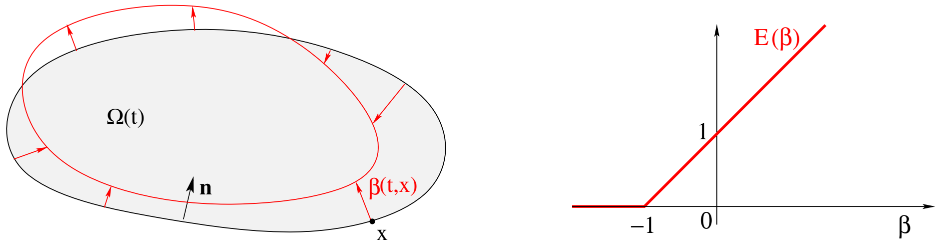

FIGURE 1 Left: a moving set, where the evolution is determined by assigning the inward normal speed β at each boundary point. Right: the effort function $E(\beta)$ in (1.2).

$$E(\beta) = \begin{cases} 1+\beta & \text{if } \beta \geq -1, \\ 0 & \text{if } \beta < -1. \end{cases} \tag{1.2}$$

Given a constant $M > 0$ accounting for the maximum control effort, we consider set motions $t \mapsto \Omega(t)$ which satisfy the constraint

$$\mathcal{E}(t) \leq M \quad \text{for all } t \in [0, T]. \tag{1.3}$$

This models a situation where:

- If the control effort is everywhere zero: $E(\beta) = 0$, then the inward normal speed is $\beta = -1$ at every point. Hence the contaminated set $\Omega(t)$ expands with unit speed in all directions. In particular, its area increases at a rate equal to the perimeter:
$$\frac{d}{dt}\mathcal{L}^2\big(\Omega(t)\big) = -\int_{\partial\Omega(t)} \beta(t,x)\,\mathcal{H}^1(dx) = \mathcal{H}^1\big(\partial\Omega(t)\big).$$
- By implementing a control with total effort $\mathcal{E}(t) = M$, we can clean up a region of area M per unit time, hence
$$\begin{aligned}\frac{d}{dt}\mathcal{L}^2\big(\Omega(t)\big) &= -\int_{\partial\Omega(t)} \beta(t,x)\,\mathcal{H}^1(dx) = \int_{\partial\Omega(t)} \Big[1 - E\big(\beta(t,x)\big)\Big]\,\mathcal{H}^1(dx) \\ &= \mathcal{H}^1\big(\partial\Omega(t)\big) - M. \end{aligned} \tag{1.4}$$

Here and in the sequel, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure while $\mathcal{L}^2$ is the 2-dimensional Lebesgue measure. Given a set $S \subset \mathbb{R}^2$, we write

$$B_\delta(S) \doteq \big\{x \in \mathbb{R}^2;\ d(x,S) \leq \delta\big\}$$

for the δ-neighborhood around the set S. Motivated by (1.4), following [5] we introduce

Definition 1.1. Given a constant $M > 0$, we say that the set motion $t \mapsto \Omega(t)$ is *admissible* if

$$\Omega(t+h) \ \subseteq\ B_h\big(\Omega(t)\big) \quad \text{for all } 0 \leq t < t+h \leq T, \tag{1.5}$$

$$\limsup_{h\to 0+} \frac{\mathcal{L}^2\Big(B_h\big(\Omega(t)\big)\setminus\Omega(t+h)\Big)}{h} \leq M \quad \text{for all } 0 \leq t < T. \tag{1.6}$$

In the above setting, two basic problems can be formulated.

(EP) Eradication problem. Let an initial set $\Omega_0 \subset \mathbb{R}^2$ and a constant $M > 0$ be given. Find an admissible set-valued function $t \mapsto \Omega(t)$ such that, for some $T > 0$,

$$\Omega(0) = \Omega_0, \quad \Omega(T) = \emptyset, \tag{1.7}$$

(MTP) Minimum time problem. Among all admissible strategies that satisfy (1.7), find one which minimizes the time T.

Notice that the constant M puts an upper bound on the total control effort at each time t. For example, in a pest eradication model, there will be an upper bound on the amount of pesticides that can be sprayed per unit time. If M is too small, compared with the size of the contaminated region, it may not be possible to completely eradicate the invasive population.

More generally, even if the Eradication Problem does not have a solution, one can consider an optimization problem, minimizing the size of the contaminated set over time.

(OP) Optimization problem. Given an initial set Ω_0 and two constants $\kappa_1, \kappa_2 > 0$, find an admissible motion $t \mapsto \Omega(t)$ that minimizes the cost functional

$$J(\Omega) \doteq \kappa_1 \int_0^T \mathcal{L}^2\big(\Omega(t)\big)\,dt + \kappa_2\,\mathcal{L}^2\big(\Omega(T)\big). \tag{1.8}$$

with the initial data $\Omega(0) = \Omega_0$.

The main results in [9] provide the existence of optimal solutions, together with necessary conditions for optimality. In the case where the initial set $\Omega_0 \subset \mathbb{R}^2$ is convex, the optimal solution to *(MTP)* and *(OP)* has been explicitly determined in [5].

In the present paper we study similar control problems for moving sets, but with geographical constraints. These models describe an invasive biological population within an island, where the sea provides a natural barrier to its expansion. More precisely, given an open set $V \subset \mathbb{R}^2$ we impose the additional constraint $\Omega(t) \subseteq V$. Since the population cannot propagate outside V, the instantaneous control effort (1.1) is now replaced by

$$\mathcal{E}(t) \doteq \int_{\partial\Omega(t)\cap V} E\big(\beta(t,x)\big)\,\mathcal{H}^1(dx). \tag{1.9}$$

Notice that in (1.9) the effort is integrated only over the relative boundary of $\Omega(t)$, contained inside the open set V. In this setting, Definition 1.1 is replaced by

Definition 1.2. Given a constant $M > 0$ and a bounded open domain $V \subset \mathbb{R}^2$, we say that the set motion $t \mapsto \Omega(t) \subseteq V$ is *admissible* if

$$\Omega(t+h) \;\subseteq\; V \cap B_h\big(\Omega(t)\big) \quad \text{for all } 0 \le t < t+h \le T, \tag{1.10}$$

$$\limsup_{h\to 0+} \frac{\mathcal{L}^2\Big(\Big(V \cap B_h\big(\Omega(t)\big)\Big)\backslash\Omega(t+h)\Big)}{h} \le M \quad \text{for all } 0 \le t < T. \tag{1.11}$$

The three problems *(EP)*, *(MTP)* and *(OP)* can now be formulated in the same way as before, replacing (1.1) with (1.9).

Aim of this paper is to provide an extensive analysis of the above problems, in the presence of geographical constraints. In the first part, several results are proved on the existence of solutions, sufficient conditions for optimality, and various necessary conditions. In the second part, we use the necessary conditions in order to explicitly construct optimal solutions in a variety of cases.

More in detail, in Sect. 2 we give an equivalent notion of admissible strategy and review the definition of weak solutions to the optimization problem, introduced in [9]. Section 3 deals with the eradication problem *(EP)*. Concerning the existence of an admissible eradication strategy, sufficient conditions as well as necessary conditions are proved in Theorem 3.1.

In Sect. 4 we study the minimum time problem *(MTP)*. A sufficient condition, for a strategy to eradicate the contamination in minimum time, is proved in Theorem 4.1. Examples are given, showing cases where such condition can be used. Here the optimal strategy amounts to piecing together a family of solutions to the classical Dido's problem. Namely, at each time $t \in [0, T]$, the relative boundary of the set $\Omega(t) \subseteq V$ has minimum length, compared with all other subsets $W \subseteq V$ of the same area.

In Sect. 5 we prove a general result on the existence of optimal solutions for the problem *(OSM)* and for the minimum time problem *(MTP)*, within a class of multifunctions with bounded variation.

Necessary conditions for optimality in a form similar to the Pontryagin maximum principle, are stated in Sect. 6. An intuitive argument is first given, motivating the result and explaining the role of the adjoint variable, which can be interpreted as a "shadow price for a cleaning service". A detailed proof of the necessary conditions is worked out in the following Sects. 7 and 8. Optimality conditions for the minimum time problem are then derived in Sect. 9.

We should point out from the outset that the regularity of the set-valued map $t \mapsto \Omega(t)$ is a major issue. At the present time, the existence of optimal strategies is proved within a class of maps whose characteristic function

$$\mathbf{1}_\Omega(t, x_1, x_2) = \begin{cases} 1 & \text{if } (x_1, x_2) \in \Omega(t), \\ 0 & \text{if } (x_1, x_2) \notin \Omega(t), \end{cases} \tag{1.12}$$

has bounded variation [4,19,23]. However, the optimality conditions obtained in [9] require a much stronger regularity assumption. Namely, the boundaries $\partial\Omega(t)$ should be $\mathcal{C}^2$ curves. More recently, the explicit solutions constructed in [5] show that these conditions are too restrictive. To handle the majority of relevant cases, is thus important to relax these regularity assumptions. Here we assume that the boundary $\partial\Omega(t)$ is only $\mathcal{C}^1$ with Lipschitz continuous normal vector. Its curvature $\omega(t, x)$ is defined almost everywhere, and the inward normal velocity $\beta(t, x)$ is continuous. Because of these weaker assumptions, a more careful proof is needed.

In spite of this improvement, there still remains a major gap between the regularity provided by the existence theorem and the regularity required by the optimality conditions. To partially bridge this gap, in Sects. 10 and 11 we prove additional necessary conditions, which must be satisfied at boundary points assuming only piecewise $\mathcal{C}^1$ regularity. Roughly speaking, Theorem 10.1 shows that the boundary $\partial\Omega(t)$ cannot have corner points in the interior of V. Moreover, Theorem 11.1 implies that, at points where $\partial\Omega(t)$ meets the boundary of the constraining set V, either the junction is perpendicular, or the control effort must vanish.

In the second part of the paper, Sects. 12 to 16, we study the problem of how to construct admissible motions $t \mapsto \Omega(t) \subset V$ which satisfy all the previous optimality conditions. Here the discussion is partly rigorous, partly heuristic. We make no attempt to cover all possible situations. Rather, we provide guidelines for constructing these set-motions. Explicit formulas are obtained in a number of specific cases. The reader should be aware that, although these motions are the unique ones that satisfy all our necessary conditions for optimality, this does not necessarily imply their optimality. In principle, there may be other strategies, with only BV regularity, which achieve a lower cost.

We outline the main ideas. At each time $t \in [0, T]$ the boundary $\partial\Omega(t)$ is assumed to be the union of finitely many "controlled arcs", where the control is active, and "free arcs", where no control is applied and the set $\Omega(t)$ thus expands with unit speed. If at some intermediate time $\tau \in [0, T]$ a maximally extended free arc is identified, in turn this determines the free arcs at all times t in a neighborhood of τ. The remaining portions of the boundary can now be constructed, since they must be arcs of circumferences, all with the same radius $r(t)$, whose endpoints either (i) meet tangentially a free arc, or (ii) cross perpendicularly the boundary ∂V. Thanks to the identity (1.4) we obtain an ODE for the radius $r(t)$. In suitable cases, the boundaries $\partial\Omega(t)$ can thus be completely determined.

This general approach is described in Sect. 12, together with some examples. Details are discussed in the following sections. The initial stage of an optimal strategy is worked out in Sect. 13, while in Sect. 14 we derive a set of equations describing a maximally extended free interface, assuming that the set V has a smooth boundary.

In Sect. 15 we assume that V is a polygon, and take up the more challenging task of constructing a maximal free arc through one of its vertices. In this setting, maximal arcs are determined by a system of ODEs which is of second order, implicit, and singular. In Sect. 16 a 1-parameter family of local solutions is constructed, in a neighborhood of a vertex.

Finally, Sect. 17 contains some concluding remarks and a brief discussion of open problems.

Geometric optimization problem have been the subject of a rich literature. We refer to [13,18,24] for comprehensive monographs, with further references. More specifically, several different models related to the control of a moving set have recently been considered in [6,7,10,12,16,17]. Eradication problems for invasive biological species are studied in [1–3].

2. An Equivalent Formulation of Admissible Set Motions

The concept of admissible set-motion, introduced is Definitions 1.1 and 1.2, is very general. Indeed, it remains meaningful for any measurable set-valued map $t \mapsto \Omega(t)$. However, toward the analysis of optimal strategies, it is convenient to work within a class of more regular maps. Following [9], in this section we consider a weak formulation of the optimization problem *(OP)*, within the class of BV functions.

Let a bounded open set $V \subset \mathbb{R}^2$ be given, with finite perimeter. Consider the family of subsets

$$\mathcal{F} \doteq \Big\{\Omega \subset]0,T[\,\times V;\ \Omega \text{ has finite perimeter}\Big\}. \tag{2.1}$$

Calling $\mathbf{1}_\Omega$ the characteristic function of Ω, this implies that $\mathbf{1}_\Omega \in BV$. In other words, the distributional gradient $\mu_\Omega \doteq D\,\mathbf{1}_\Omega$ is a finite $\mathbb{R}^3$-valued Radon measure:

$$\int_\Omega \operatorname{div}\varphi\,dx = -\int \varphi\cdot d\mu_\Omega \quad \text{for all } \varphi \in \mathcal{C}^1_c\big(]0,T[\,\times\mathbb{R}^2;\ \mathbb{R}^3\big). \tag{2.2}$$

Given a set $\Omega \in \mathcal{F}$, we consider the multifunction

$$t \ \mapsto\ \Omega(t) \doteq \big\{x \in \mathbb{R}^2;\ (t,x) \in \Omega\big\}. \tag{2.3}$$

By possibly modifying $\mathbf{1}_\Omega$ on a set of 3-dimensional measure zero, the map $t \mapsto \mathbf{1}_{\Omega(t)}$ has bounded variation from $]0,T[$ into $\mathbf{L}^1(\mathbb{R}^2)$. In particular, for every $0 < t < T$, the one-sided limits

$$\mathbf{1}_{\Omega(t+)} \doteq \lim_{t\to t+} \mathbf{1}_{\Omega(t)}, \quad \mathbf{1}_{\Omega(t-)} \doteq \lim_{t\to t-} \mathbf{1}_{\Omega(t)}, \tag{2.4}$$

are well defined in $\mathbf{L}^1(\mathbb{R}^2)$. This uniquely defines the sets $\Omega(t+)$, $\Omega(t-)$, up to a set of 2-dimensional Lebesgue measure zero. Throughout the following, we define the sets $\Omega(0)$ and $\Omega(T)$ in terms of

$$\mathbf{1}_{\Omega(0)} \doteq \lim_{t\to 0+} \mathbf{1}_{\Omega(t)}, \quad \mathbf{1}_{\Omega(T)} \doteq \lim_{t\to T-} \mathbf{1}_{\Omega(t)}. \tag{2.5}$$

In the following, by $B(y,r)$ we denote the open ball centered at y with radius r, while S^2 is the sphere of unit vectors in $\mathbb{R}^3$. If

$$u(t,x) \doteq \mathbf{1}_\Omega = \begin{cases} 1 & \text{if } x \in \Omega(t),\\ 0 & \text{if } x \notin \Omega(t),\end{cases}$$

is the characteristic function of the set Ω, the distributional derivatives of u will be denoted by

$$D_t u, \qquad D_x u(t,\cdot)\ =\ \nu\cdot\mathcal{H}^1 \llcorner \partial\Omega(t). \tag{2.6}$$

Here $\nu = \nu(t,x) \in \mathbb{R}^2$ is the unit inner normal to $\Omega(t)$ at a boundary point x, while the last expression indicates the restriction of the 1-dimensional Hausdorff measure to the boundary $\partial\Omega(t)$.

For every set of finite perimeter $\Omega \in \mathcal{F}$, its reduced boundary $\partial^*\Omega$ is defined to be the set of points $y = (t,x) \in]0,T[\,\times\mathbb{R}^2$ such that

$$\nu_\Omega(y) \doteq \lim_{r\downarrow 0} \frac{\mu_\Omega\big(B(y,r)\big)}{|\mu_\Omega|\big(B(y,r)\big)} \tag{2.7}$$

exists in $\mathbb{R}^3$ and satisfies $|\nu_\Omega(y)| = 1$. The function $\nu_\Omega : \partial^*\Omega \mapsto S^2$ is called the *generalized inner normal* to Ω. A fundamental theorem of De Giorgi [4,23] implies that $\partial^*\Omega$ is countably 2-rectifiable and $|D\mathbf{1}_\Omega| = \mathcal{H}^2 \llcorner \partial^*\Omega$.

To formulate an admissibility condition for a set $\Omega \in \mathcal{F}$, we observe that, in the smooth case, the (inward) normal velocity of the set $\Omega(t)$ at the point $(t, x) \in \partial^*\Omega$ is computed by

$$\beta = \frac{-\nu_0}{\sqrt{\nu_1^2 + \nu_2^2}}. \tag{2.8}$$

Recalling that $E(\beta) \doteq \max\{1 + \beta, 0\}$, the instantaneous effort is thus computed by

$$\begin{aligned}\mathcal{E}(t) &= \int_{\partial\Omega(t)\cap V} E\left(\frac{-\nu_0}{\sqrt{\nu_1^2+\nu_2^2}}\right) \mathcal{H}^1(dx)\\ &= \int_{\partial\Omega(t)\cap V} \max\left\{\frac{-\nu_0 + \sqrt{\nu_1^2+\nu_2^2}}{\sqrt{\nu_1^2+\nu_2^2}},\ 0\right\} \mathcal{H}^1(dx).\end{aligned}$$

Therefore, integrating over a time interval $t \in]t_1, t_2[$ one finds

$$\int_{t_1}^{t_2} \mathcal{E}(t)\, dt = \int_{\partial^*\Omega\cap\{(t,x);\, t_1<t<t_2,\, x\in V\}} \max\left\{-\nu_0 + \sqrt{\nu_1^2+\nu_2^2},\ 0\right\} d\mathcal{H}^2.$$

To impose the requirement that $\mathcal{E}(t) \leq M$ for all t, we consider the convex, positively homogeneous function $L : \mathbb{R}^3 \mapsto \mathbb{R}$, defined as

$$L(v) = L(v_0, v_1, v_2) \doteq \max\left\{-v_0 + \sqrt{v_1^2+v_2^2},\ 0\right\}. \tag{2.9}$$

Definition 2.1. Let $V \subset \mathbb{R}^2$ be a bounded open set with finite perimeter. Given $M > 0$, we say that a set $\Omega \in \mathcal{F}$ in (2.1) represents an *admissible motion*, and write $\Omega \in \mathcal{A}$, if for every $0 \leq t_1 < t_2 \leq T$ one has

$$\int_{\partial^*\Omega\cap\{(t,x);\, t_1<t<t_2,\, x\in V\}} L(\nu)\, d\mathcal{H}^2 \leq M(t_2 - t_1). \tag{2.10}$$

The original problem *(OP)* can now be reformulated as

(OSM) *Optimal set motion problem.* Given $T, \kappa_1, \kappa_2 > 0$ and an initial set $\Omega_0 \subseteq V$, find an admissible set $\Omega \in \mathcal{A}$ which minimizes the functional

$$\mathcal{J}(\Omega) \doteq \kappa_1 \mathcal{L}^3(\Omega) + \kappa_2\, \mathcal{L}^2\big(\Omega(T)\big), \tag{2.11}$$

subject to $\Omega(0) = \Omega_0$.

Notice that the sets $\Omega(0)$ and $\Omega(T)$ are well defined in terms of (2.5).

3. Existence of Eradication Strategies

We consider here a null-controllability problem where all sets $\Omega(t)$ remain within a given domain $V \subset \mathbb{R}^2$. This models the problem of eradicating a pest population over an island.

(CEP) Constrained eradication problem. Consider a bounded open set $V \subset \mathbb{R}^2$ with finite perimeter. Find a set-valued function $t \mapsto \Omega(t) \subseteq V$ such that, for some $T > 0$,

$$\Omega(0) = V, \quad \Omega(T) = \emptyset, \tag{3.1}$$

$$\int_{V \cap \partial\Omega(t)} E\big(\beta(t,x)\big)\, \mathcal{H}^1(dx) \leq M, \quad \text{for all } t \in [0,T]. \tag{3.2}$$

Note that in (3.2) the effort $E(\beta)$ is integrated only along the portion of the boundary $\partial\Omega(t)$ which is contained in the interior of V. Since we are thinking of V as an island, the contamination will never spread outside V.

We seek conditions that imply the existence (or non-existence) of an admissible strategy which eradicates the contamination. Toward this goal, two geometric invariants associated with the set V play a key role.

(i) For any given $\lambda \in [0,1]$, we choose a set $V_\lambda \subset V$ with area $\mathcal{L}^2(V_\lambda) = \lambda\, \mathcal{L}^2(V)$, so that the length of its relative boundary is as small as possible. Then we take the supremum of these lengths over all λ. This leads to the constants

$$\kappa(V,\lambda) \doteq \inf\Big\{\mathcal{H}^1\big(\partial V_\lambda \cap V\big);\ V_\lambda \subseteq V,\ \mathcal{L}^2(V_\lambda) = \lambda\, \mathcal{L}^2(V)\Big\}, \tag{3.3}$$

$$\kappa(V) \doteq \sup_{\lambda \in [0,1]} \kappa(V,\lambda). \tag{3.4}$$

(ii) Next, we slice the set V in terms of a continuous map $\phi : V \mapsto [0,1]$. Here the slices are the pre-images $\phi^{-1}(\lambda)$, $\lambda \in [0,1]$. We choose ϕ such that the maximum length of these slices is as small as possible. This yields the constant

$$K(V) \doteq \inf_{\phi:\, V \mapsto [0,1]} \left(\sup_{\lambda \in [0,1]} \mathcal{H}^1\big(\phi^{-1}(\lambda)\big) \right). \tag{3.5}$$

Some results on the optimal constant $\kappa(V)$ can be found in [15]. The invariant $K(V)$ in (3.5) is known as 1-*width* in dimension 2, see [21,22]. It is easy to see that $\kappa(V) \leq K(V)$. In several cases a strict inequality holds.

Example 3.1. Let V be an equilateral triangle with side of unit length (Fig. 2). Then

$$K(V) = \frac{\sqrt{3}}{2} \ \approx\ 0.866\ldots \tag{3.6}$$

coincides with the height of the triangle. On the other hand, for any $\lambda \in [0,1]$ we have

$$\kappa(V,\lambda) \leq \kappa\Big(V, \frac{1}{2}\Big) = \sqrt{\frac{3\sqrt{3}}{4\pi}} = \kappa(V) \ \approx\ 0.643\ldots \tag{3.7}$$

According to Theorem 3.1 below, the contamination can be eradicated from V if $M > \frac{\sqrt{3}}{2}$, and it cannot be eradicated if $M < \sqrt{\frac{3\sqrt{3}}{4\pi}}$.

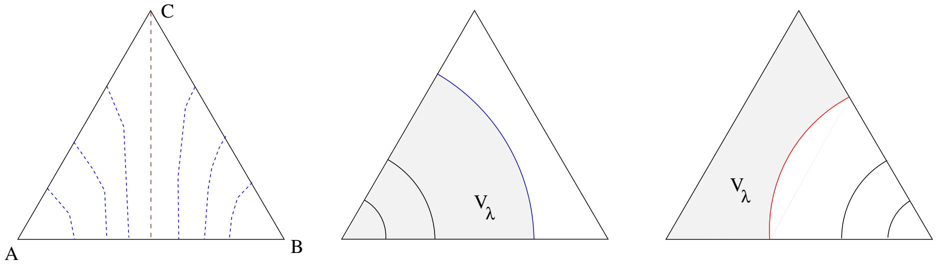

FIGURE 2 The two invariants (3.4) and (3.5) in the case of an equilateral triangle with unit side. Left: one of the level sets of the function ϕ in (3.5) must go through the vertex C. Hence it will have length $\geq \sqrt{3}/2$. Center: for any $\lambda \in [0, 1/2]$ we can cut a sector V_λ with area $\lambda\mathcal{L}^2(V)$, so that its boundary is an arc with length $\leq \sqrt{3\sqrt{3}/4\pi}$. Right: for $\lambda \in [1/2, 1]$ we can take V_λ to be the complement of a sector with the same property.

Theorem 3.1. (Existence of eradication strategies) *Let $V \subset \mathbb{R}^2$ be a compact set with finite perimeter. Consider the following statements:*

(i) *There exists a continuous map $\varphi : V \mapsto [0, 1]$ whose level sets satisfy*

$$\sup_{s\in[0,1]} \mathcal{H}^1\Big(\{x \in V;\ \varphi(x) = s\}\Big) \ <\ M. \tag{3.8}$$

(ii) *The constrained eradication problem (CEP) on V has a solution which satisfies the additional monotonicity property*

$$s < t \quad \Longrightarrow \quad \Omega(s) \supseteq \Omega(t). \tag{3.9}$$

(iii) *The problem (CEP) on V has a solution.*

(iv) *For every $\lambda \in [0, 1]$ there exists a subset $V_\lambda \subset V$ such that*

$$\mathcal{L}^2(V_\lambda) = \lambda\mathcal{L}^2(V), \quad \mathcal{H}^1(V \cap \partial V_\lambda) \leq M. \tag{3.10}$$

Then we have the implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv).

Proof. 1. We begin by proving the implication (i)$\Longrightarrow$(ii). Assume that (i) holds (see Fig. 3, left). As a first step, we observe that the function

$$s \ \mapsto\ f(s) \doteq \mathcal{L}^2\Big(\big\{x \in V;\ \varphi(x) \geq s\big\}\Big) \tag{3.11}$$

is continuous. Indeed, the function f is clearly decreasing. If there exists s_0 such that

$$f(s_0)\ <\ \inf_{s<s_0-} f(s),$$

then

$$\mathcal{L}^2\Big(\big\{x \in V;\ \varphi(x) = s_0\big\}\Big) \ = \inf_{s<s_0}\ \mathcal{L}^2\Big(\big\{x \in V;\ \varphi(x) \geq s\big\}\Big)$$
$$-\mathcal{L}^2\Big(\big\{x \in V;\ \varphi(x) \geq s_0\big\}\Big) > 0.$$

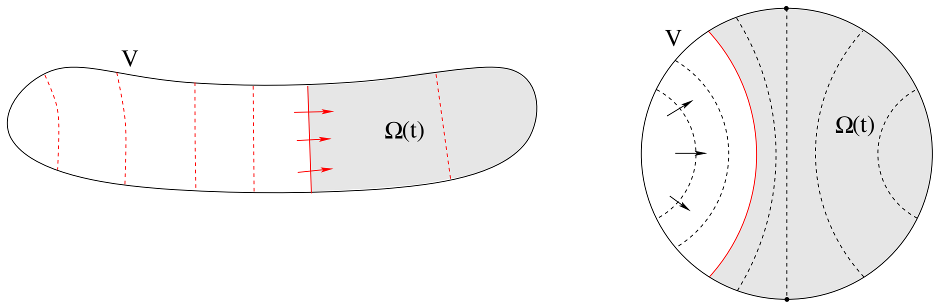

FIGURE 3 Left: in the case where $M > K(V)$, an eradication strategy can be constructed Right: if (3.16) holds and $M < \kappa(V)$, then the area of the set $\Omega(t)$ can never become smaller than $\lambda^* \mathcal{L}^2(V)$.

Hence

$$\mathcal{H}^1\Big(\big\{x \in V;\ \varphi(x) = s_0\big\}\Big) = +\infty,$$

reaching a contradiction. A similar argument rules out the possibility that

$$f(s_0) > \sup_{s > s_0-} f(s).$$

2. To construct a strategy satisfying (3.1)-(3.2) and (3.9), we proceed as follows. By (3.8) we can find $b_1 > 0$ small enough such that

$$\mathcal{H}^1\Big(\big\{x \in V;\ \varphi(x) = s\big\}\Big) \ < \ \frac{M}{1+b_1} \quad \text{for all } s \in [0,1]. \tag{3.12}$$

We now define

$$\Omega(t) = \big\{x \in V;\ \varphi(x) \geq s(t)\big\}, \tag{3.13}$$

for a suitable function $t \mapsto s(t)$, with $s(0) = 0$, $s(T) = 1$ for some time $T > 0$. This function $s(\cdot)$ is chosen so that

$$\mathcal{L}^2\big(\Omega(t)\big) \doteq \mathcal{L}^2\Big(\big\{x \in V;\ \varphi(x) \geq s(t)\big\}\Big) = \mathcal{L}^2(V) - \frac{b_1 M}{1+b_1} t. \tag{3.14}$$

3. It remains to prove that the above multifunction $t \mapsto \Omega(t)$ satisfies all requirements.

By construction, the property (3.9) is trivially satisfied. This implies $\beta(t,x) \geq 0$ for all t and all $x \in V \cap \partial\Omega(t)$. For any $0 \leq t_1 < t_2 \leq 1$, using (3.12) and then (3.14) we compute

$$\begin{aligned}
\int_{t_1}^{t_2} \int_{V\cap\partial\Omega(t)} E\big(\beta(t,x)\big)\, \mathcal{H}^1(dx)\, dt &= \int_{t_1}^{t_2} \int_{V\cap\partial\Omega(t)} \Big(1 + \beta(t,x)\Big) \mathcal{H}^1(dx)\, dt \\
&\leq \ \frac{M}{1+b_1}(t_2 - t_1) + \Big[\mathcal{L}^2\big(\Omega(t_1)\big) - \mathcal{L}^2\big(\Omega(t_2)\big)\Big] \\
&= \frac{M}{1+b_1}(t_2 - t_1) + \frac{b_1 M}{1+b_1}(t_2 - t_1) = M(t_2 - t_1).
\end{aligned}$$

Since this is true for all $t_1 < t_2$, this yields (3.2).

4. The implication (ii) $\Longrightarrow$ (iii) is trivial.

5. To prove the implication (iii) $\Longrightarrow$ (iv), assume that, on the contrary, (iv) fails (see Fig. 3, right). Hence there exists $0 < \lambda^* < 1$ and $\varepsilon > 0$ such that

$$\mathcal{H}^1(V \cap \partial W) > M + 2\varepsilon \quad \text{for all } W \subset V \text{ with } \mathcal{L}^2(W) = \lambda^* \mathcal{L}^2(V). \tag{3.15}$$

We claim that there exists $\delta > 0$ such that,

$$\mathcal{H}^1(V \cap \partial W) \geq M + \varepsilon \quad \text{for all } W \subset V \text{ with } \mathcal{L}^2(W) = \lambda \mathcal{L}^2(V),\ |\lambda - \lambda^*| \leq \delta. \tag{3.16}$$

Indeed, if (3.16) fails, we could find a sequence of subsets $W_n \subset V$ such that

$$\lim_{n\to\infty} \mathcal{L}^2(W_n) = \lambda^* \mathcal{L}^2(V), \qquad \limsup_{n\to\infty} \mathcal{H}^1(V \cap \partial W_n) \leq M + \varepsilon. \tag{3.17}$$

Since all these sets have uniformly bounded perimeters, taking a subsequence we can assume the $\mathbf{L}^1$ convergence of the characteristic functions: $\mathbf{1}_{W_n} \to \mathbf{1}_{W^*}$, for a set $W^* \subset V$ such that

$$\mathcal{H}^1(V \cap \partial W^*) \ \leq \ M + \varepsilon \ \mathcal{L}^2(W^*) = \lambda^* \mathcal{L}^2(V).$$

This yields a contradiction with (3.15).

6. Using (3.16), we claim that, for any admissible strategy $t \mapsto \Omega(t)$ one has the implication

$$\frac{\mathcal{L}^2(\Omega(t))}{\mathcal{L}^2(V)} \ \in \ [\lambda^* - \delta, \lambda^* + \delta] \quad \Longrightarrow \quad \frac{d}{dt}\mathcal{L}^2(\Omega(t)) > 0.$$

Indeed,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}^2(\Omega(t)) = &\ -\int_{\partial\Omega(t)\cap V} \beta(t,x)\, \mathcal{H}^1(dx) = \int_{\partial\Omega(t)\cap V} \Big[1 - E\big(\beta(t,x)\big)\Big]\, \mathcal{H}^1(dx) \\ &\geq \ \mathcal{H}^1\big(\partial\Omega(t) \cap V\big) - M \geq \varepsilon.\end{aligned}$$

As a consequence, the area $\mathcal{L}^2(\Omega(t))$ can never become smaller than $\lambda^* \mathcal{L}^2(V)$. In particular, this area cannot decrease to zero. □

4. A Sufficient Condition for Optimality

In this section we consider the minimum time eradication problem *(MTP)*, constrained to a set $V \subset \mathbb{R}^2$ with finite perimeter. The following result provides a sufficient condition for a set motion $t \mapsto \Omega(t)$ to be optimal for the minimum time problem.

Theorem 4.1. *Let $t \mapsto \Omega(t) \subseteq V$ be an admissible set motion with $\Omega(0) = V$, $\Omega(T) = \emptyset$, and such that, for all $0 < t < t' < T$,*

$$\mathcal{L}^2\big(\Omega(t')\big) \ < \ \mathcal{L}^2\big(\Omega(t)\big), \quad \Omega(t') \ \subseteq \ B_{t'-t}\big(\Omega(t)\big), \tag{4.1}$$

$$\frac{d}{dt}\mathcal{L}^2\big(\Omega(t)\big) \ = \ \mathcal{H}^1\big(\partial\Omega(t) \cap V\big) - M. \tag{4.2}$$

Assume that

$$\mathcal{H}^1\big(V \cap \partial\Omega(t)\big) \ = \ \min\Big\{\mathcal{H}^1(V \cap \partial W); \quad W \subseteq V, \quad \mathcal{L}^2(W) = \mathcal{L}^2\big(\Omega(t)\big)\Big\} \tag{4.3}$$

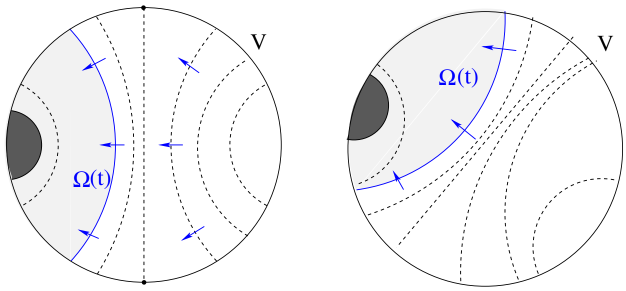

FIGURE 4 Two optimal solutions for the minimum time problem on a disc. For every $t \in]0,T[$, the boundary $\partial\Omega(t)$ should be an arc of circumference, crossing the boundary ∂V perpendicularly.

for all $t \in [0,T]$. Then this motion is optimal for the minimum time problem.

Proof. Let $A = \mathcal{L}^2(V)$ be the area of the set V. For every $a \in [0,A]$ define

$$g(a) \doteq \min\Big\{\mathcal{H}^1(V \cap \partial W);\ W \subseteq V, \quad \mathcal{L}^2(W) = a\Big\}. \tag{4.4}$$

Consider any admissible strategy $t \mapsto \widetilde{\Omega}(t)$, $t \in [0,\widetilde{T}]$. We then have

$$\frac{d}{dt}\mathcal{L}^2(\widetilde{\Omega}(t)) \geq \mathcal{H}^1\big(V \cap \partial\widetilde{\Omega}(t)\big) - M \geq g\Big(\mathcal{L}^2\big(\widetilde{\Omega}(t)\big)\Big) - M. \tag{4.5}$$

Calling $\widetilde{a}(t) \doteq \mathcal{L}^2\big(\widetilde{\Omega}(t)\big)$, for every t one has

$$\frac{d}{dt}\widetilde{a}(t) \geq g\big(\widetilde{a}(t)\big) - M,$$

hence

$$t \geq \int_{\widetilde{a}(t)}^{A} \frac{da}{M - g(a)}. \tag{4.6}$$

In particular, if $\widetilde{\Omega}(0) = V$ and $\widetilde{\Omega}(\widetilde{T}) = \emptyset$, this implies

$$\widetilde{T} \ \geq \ \int_0^A \frac{da}{M - g(a)}. \tag{4.7}$$

On the other hand, if the strategy $t \mapsto \Omega(t)$ satisfies (4.1)–(4.3), then

$$T = \int_0^A \frac{da}{M - g(a)} \leq \widetilde{T}. \tag{4.8}$$

Hence $\Omega(\cdot)$ is optimal for the minimum time problem. □

Remark 4.2. Notice that the minimality condition in (4.3) implies that, for each $t \in]0,T[$, the set $\Omega(t)$ should provide a solution to the classical Dido's problem [25]. As a consequence, the boundary $\partial\Omega(t)$ should be an arc of circumference, crossing the boundary ∂V perpendicularly at both endpoints (see Fig. 4).

More generally, the arguments used in the proof of Theorem 4.1 can be used to obtain necessary and sufficient conditions for a family of *minimum time transfer problems* with prescribed initial and terminal conditions.

(MTTP) Given a bounded open set $V \subset \mathbb{R}^2$ and two subsets

$$\Omega_{final} \subset \Omega_{initial} \subseteq V,$$

consider the family of all admissible motions $t \mapsto \Omega(t) \subseteq V$ such that

$$\Omega(0) = \Omega_{initial}, \quad \Omega(T) = \Omega_{final} \tag{4.9}$$

for some $T > 0$. Among all such motions, minimize the time T.

Corollary 4.1. *In connection with the optimization problem (MTTP), assume that there exists an admissible motion $t \mapsto \Omega^*(t)$, $t \in [0, T^*]$ that satisfies (4.9) together with (4.2)–(4.3), for every t. Then Ω^* is optimal. Moreover, any other optimal motion must satisfy (4.3) as well.*

Example 4.1. In the case where V is disc with diameter $\operatorname{diam}(V) < M$, the two constants $\kappa(V)$ and $K(V)$ in (3.4)–(3.5) both coincide with the diameter of V. In this setting, two strategies satisfying the sufficient conditions stated in Theorem 4.1 are shown in Fig. 4. On the other hand, if $\operatorname{diam}(V) > M$, by Theorem 3.1 the contamination cannot be eradicated.

In the borderline case where $\operatorname{diam}(V) = M$, the contamination still cannot be eradicated. Indeed, recalling the function g at (4.4), for any admissible strategy $t \mapsto \Omega(t)$, if $\Omega(T) = \emptyset$, this implies

$$T \ \geq \ \int_0^{\pi M^2/4} \frac{da}{M - g(a)} = \frac{2}{M} \int_0^{\pi} \frac{d\theta}{1 - \dfrac{\pi - \theta}{2} \tan\left(\dfrac{\theta}{2}\right)} = +\infty. \tag{4.10}$$

5. Existence of Optimal Solutions

In the case without geographical constraints, the existence of solutions for a general class of optimal set motion problems was proved in [9]. The same arguments can be adapted to the present setting.

Theorem 5.1. *Let $V \subset \mathbb{R}^2$ be a bounded open set with finite perimeter. Then for any set $\Omega_0 \subseteq V$ with finite perimeter and any $T, \kappa_1, \kappa_2 > 0$, the set motion problem (OSM) has an optimal solution.*

Proof. 1. We observe that the functional $\mathcal{J}(\Omega)$ at (2.11) is non-negative. Moreover, the strategy $t \mapsto \Omega(t) \doteq B_t(\Omega_0) \cap V$ is admissible and has finite cost. Indeed, setting

$$\Omega \doteq \Big\{(t, x);\ x \in B_t(\Omega_0) \cap V,\ t \in [0, T]\Big\}, \tag{5.1}$$

one has

$$\mathcal{J}(\Omega) \leq \kappa_1 \int_0^T \mathcal{L}^2\big(B_t(\Omega_0)\big)\, dt + \kappa_2 \mathcal{L}^2\big(B_T(\Omega_0)\big) \ < \ +\infty.$$

Moreover, by (5.1) the inner unit normal satisfies

$$-\nu_0 + \sqrt{\nu_1^2 + \nu_2^2} = 0$$

at a.e. boundary point $(t, x) \in \Omega \cap \big(]0, T[\times V\big)$. Hence the admissibility condition in Definition 2.1 is trivially satisfied: $\Omega \in \mathcal{A}$.

We can thus consider a minimizing sequence of sets $\Omega_n \in \mathcal{A}$ such that, as $n \to \infty$,

$$\mathcal{J}(\Omega_n) \;\to\; \mathcal{J}_{min} \doteq \inf_{\Omega\in\mathcal{A}} \mathcal{J}(\Omega).$$

Without loss of generality we can assume that

$$\Omega_n(t) \doteq \big\{x \in \mathbb{R}^2;\ (t,x) \in \Omega_n\big\} \;\subseteq\; B_t(\Omega_0) \cap V \tag{5.2}$$

for all $t \in [0,T]$ and $n \geq 1$. Otherwise, we can simply replace each set $\Omega_n(t)$ with the intersection $\Omega_n(t) \cap B_t(\Omega_0)$, without increasing the total cost.

2. We now prove a uniform bound on the perimeters of the sets $\Omega_n \subset \mathbb{R}^3$. For every $n \geq 1$ we split the reduced boundary in the form

$$\partial^*\Omega_n \cap \big(]0,T[\times V\big) = \Sigma_n^- \cup \Sigma_n^+, \tag{5.3}$$

so that the following holds. Calling $\nu = (\nu_0, \nu_1, \nu_2)$ the unit inner normal vector at the point $(t,x) \in \partial^*\Omega_n \cap V$, and defining the inner normal velocity $\beta_n = \beta_n(t,x)$ as in (2.8), one has

$$\begin{cases} \beta_n(t,x) \;\leq\; -1/2 \text{ if } & (t,x) \in \Sigma_n^-\,, \\ \beta_n(t,x) > -1/2 \text{ if } & (t,x) \in \Sigma_n^+. \end{cases} \tag{5.4}$$

Integrating the effort over Σ_n^+ one obtains

$$MT \geq \iint_{\Sigma_n^+} \max\Big\{ -\nu_0 + \sqrt{\nu_1^2 + \nu_2^2},\ 0\Big\}\, d\mathcal{H}^2 \geq \iint_{\Sigma_n^+} \frac{1}{2}\, d\mathcal{H}^2. \tag{5.5}$$

Hence

$$\mathcal{H}^2\big(\Sigma_n^+\big) \leq 2MT. \tag{5.6}$$

On the other hand, on the domain Σ_n^- one has the lower bound

$$\nu_0 \geq \frac{1}{2}. \tag{5.7}$$

We can now write

$$\begin{aligned} \mathcal{L}^2\Big(B_T(\Omega_0)\Big) &\geq \mathcal{L}^2\big(\Omega_n(T)\big) - \mathcal{L}^2\big(\Omega_n(0)\big) = \int_0^T \int_{\partial\Omega(t)} -\beta(t,x)\mathcal{H}^1(dx)\, dt \\ &= \int_0^T \int_{\partial\Omega(t)} \frac{\nu_0}{\sqrt{\nu_1^2+\nu_2^2}} \mathcal{H}^1(dx)\, dt = \int_{\Sigma_n^+\cup\Sigma_n^-} \nu_0\, d\mathcal{H}^2 \;\geq\; \frac{1}{2}\int_{\Sigma_n^-} d\mathcal{H}^2 - \int_{\Sigma_n^+} d\mathcal{H}^2. \end{aligned} \tag{5.8}$$

Combining (5.8) with (5.6) one obtains

$$\mathcal{H}^2\big(\Sigma_n^-\big) \leq 2\mathcal{L}^2\Big(B_T(\Omega_0)\Big) + 4MT. \tag{5.9}$$

Together, the two inequalities (5.6) and (5.9) yield a uniform bound on the 2-dimensional measure $\mathcal{H}^2\Big(\partial^*\Omega_n \cap \big(]0,T[\times V\big)\Big)$ of the relative boundary of Ω_n inside V.

Since V has finite perimeter, this implies that the characteristic functions $\mathbf{1}_{\Omega_n}$ have uniformly bounded variation.

3. By the uniform BV bound proved in the previous step, by possibly taking a subsequence, a compactness argument (see Theorem 12.26 in [23]) yields the existence of a set with finite perimeter $\Omega \in \mathcal{F}$ such that the following holds. As $n \to \infty$, one has the convergence

$$\Big\| \mathbf{1}_{\Omega_n} - \mathbf{1}_{\Omega} \Big\|_{\mathbf{L}^1\left(]0,T[\,\times \mathbb{R}^2\right)} \;\to\; 0, \tag{5.10}$$

together with the weak convergence of measures

$$\mu_{\Omega_n} \;\overset{*}{\rightharpoonup}\; \mu_{\Omega}. \tag{5.11}$$

4. In this step we check that the limit strategy $\Omega \in \mathcal{F}$ is admissible. Since the function L in (2.9) is convex, for any $0 \le t_1 < t_2 < T$ we can use a lower semicontinuity result for anisotropic functionals (see Theorem 20.1 in [23]) and conclude

$$\begin{aligned} \int_{\partial^*\Omega \cap \{t_1 < t < t_2, x \in V\}} L(\nu)\, d\mathcal{H}^2 &\le \liminf_{n\to\infty} \int_{\partial^*\Omega_n \cap \{t_1 < t < t_2, x \in V\}} L(\nu_n)\, d\mathcal{H}^2 \\ &\le M(\tau' - \tau). \end{aligned} \tag{5.12}$$

Therefore $\Omega \in \mathcal{A}$.

5. It remains to check that the limit set Ω is optimal. As $n \to \infty$, the convergence (5.10) immediately implies

$$\mathcal{L}^3(\Omega_n) \;\to\; \mathcal{L}^3(\Omega). \tag{5.13}$$

Moreover, since the map $t \mapsto \mathbf{1}_{\Omega(t)} \in \mathbf{L}^1(\mathbb{R}^2)$ has bounded variation, given $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\left| \mathcal{L}^2\big(\Omega(T)\big) - \frac{1}{\delta} \int_{T-\delta}^{T} \mathcal{L}^2\big(\Omega(t)\big)\, dt \right| \;<\; \varepsilon. \tag{5.14}$$

On the other hand, for every $n \ge 1$ and $t < T$ we have the one-sided estimate

$$\mathcal{L}^2\big(\Omega_n(t) \backslash \Omega_n(T)\big) \le M(T - t).$$

Therefore, choosing $\delta < \varepsilon / M$ we obtain

$$\mathcal{L}^2\big(\Omega_n(T)\big) \ge \frac{1}{\delta} \int_{T-\delta}^{T} \Big[\mathcal{L}^2\big(\Omega_n(t)\big) - M(T-t)\Big]\, dt \ge \frac{1}{\delta} \int_{T-\delta}^{T} \mathcal{L}^2\big(\Omega_n(t)\big)\, dt - \varepsilon$$

for every $n \ge 1$. Using the convergence

$$\frac{1}{\delta} \int_{T-\delta}^{T} \mathcal{L}^2\big(\Omega_n(t)\big)\, dt \;\to\; \frac{1}{\delta} \int_{T-\delta}^{T} \mathcal{L}^2\big(\Omega(t)\big)\, dt,$$

we obtain

$$\mathcal{L}^2\big(\Omega(T)\big) \le \liminf_{n\to\infty} \mathcal{L}^2\big(\Omega_n(T)\big) + 2\varepsilon. \tag{5.15}$$

Combining (5.13) and (5.15), since $\varepsilon > 0$ was arbitrary we conclude

$$\mathcal{J}(\Omega) \doteq \kappa_1\, \mathcal{L}^3(\Omega) + \kappa_2\, \mathcal{L}^2\big(\Omega(T)\big) \le \liminf_{n\to\infty} \mathcal{J}(\Omega_n). \tag{5.16}$$

6. It remains to prove that the initial condition $\Omega(0) = \Omega_0$ is satisfied. Since Ω_0 has bounded perimeter, by (5.2) we immediately have

$$\lim_{t\to 0+} \mathcal{L}^2\big(\Omega(t)\backslash\Omega_0\big) \le \lim_{t\to 0+} \mathcal{L}^2\Big(B_t(\Omega_0)\backslash\Omega_0\Big) = 0. \tag{5.17}$$

On the other hand, for every $n \ge 1$ and $t > 0$ the admissibility condition $\Omega_n \in \mathcal{A}$ implies

$$\mathcal{L}^2\big(\Omega_0\backslash\Omega_n(t)\big) \le Mt.$$

Taking the limit as $n \to \infty$ one obtains

$$\mathcal{L}^2\big(\Omega_0\backslash\Omega(t)\big) \ \le \ Mt. \tag{5.18}$$

Together, (5.17) and (5.18) yield the convergence $\mathbf{1}_{\Omega(t)} \to \mathbf{1}_{\Omega_0}$ in $\mathbf{L}^1(\mathbb{R}^2)$, completing the proof. □

Entirely similar arguments yield the existence of an optimal solution for the minimum time problem.

Theorem 5.2. *Let $V \subset \mathbb{R}^2$ be a bounded open set with finite perimeter, and let $M > 0$ be given. If the constrained eradication problem (CEP) has a solution, then the minimum time problem (MTP) has an optimal solution.*

Proof. By the assumption, there exists a minimizing sequence $(\Omega_n)_{n\ge 1}$, with $\Omega_n \in \mathcal{A}$, $\Omega_n(0) = V$, $\Omega_n(T_n) = \emptyset$ for all $n \ge 1$. Here $T_n \to T$ converge to the infimum among all eradication times.

Since all sets Ω_n are admissible, by possibly taking a subsequence, the same arguments used in the proof of Theorem 5.1 yield the convergence (5.10)–(5.11). By (5.12) we again conclude that $\Omega \in \mathcal{A}$.

Since $\Omega_n(t) \subseteq V$ for all $t \ge 0$, from the inequality

$$\mathcal{L}^2\big(V\backslash\Omega_n(t)\big) \le Mt,$$

letting $n \to \infty$ one obtains $\Omega(0) = V$.

Finally, using the fact that $T < T_n$ and $\Omega_n(T_n) = \emptyset$, since every Ω_n is admissible we obtain

$$\mathcal{L}^2\big(\Omega_n(T)\backslash\Omega_n(T_n)\big) = \mathcal{L}^2\big(\Omega_n(T)\big) \le M(T_n - T).$$

Letting $n \to \infty$ we conclude that $\mathcal{L}^2\big(\Omega(T)\big) = 0$, hence $\Omega(T) = \emptyset$. This completes the proof. □

6. Necessary Conditions for Optimality

In order to derive a set of necessary conditions for optimality, some regularity conditions on the domain V and on the moving sets $\Omega(t)$ will be needed. A key assumption is that the unit outer normal $\mathbf{n}(x)$ to a boundary point $x \in \partial\Omega(t) \cap V$ depends Lipschitz continuously on x. This allows us to parameterize the relative boundary $\partial\Omega(t)\cap V$ as $\xi \mapsto x(t, \xi)$, in such a way that the maps $t \mapsto x(t, \xi)$ describe perpendicular curves.

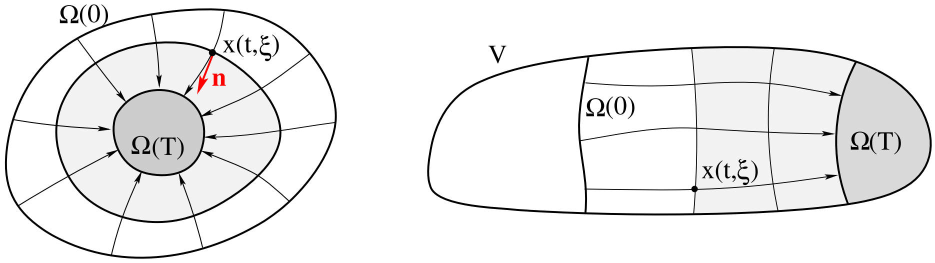

FIGURE 5 Two examples where the boundaries of the sets $\Omega(t)$ can be parameterized as $\xi \mapsto x(t,\xi)$, according to *(A1)–(A3)*. Left: a case where $(t,\xi) \in W = [0,T] \times S^1$. Right: a case with geographical constraints, where $(t,\xi) \in W = [0,T]\times]0,1[$.

In the following, $S^1 \doteq \{\xi \in \mathbb{R}^2;\ |\xi| = 1\}$ denotes the unit circumference. Given a vector $v = (v_1, v_2) \in \mathbb{R}^2$, we write $v^\perp = (-v_2, v_1)$ for the perpendicular vector, obtained by a rotation of $\pi/2$. Moreover, $\mathcal{C}^{1,1}$ denotes a space of continuously differentiable functions with locally Lipschitz continuous partial derivatives.

We shall assume (see Fig. 5):

(A1) Either $V = \mathbb{R}^2$, or $V \subset \mathbb{R}^2$ is a bounded open set with piecewise $\mathcal{C}^1$ boundary.

(A2) The relative boundaries of the sets $\Omega(t)$, $t \in [0,T]$, admit a $\mathcal{C}^{1,1}$ parameterization of the form

$$W \ \ni\ (t,\xi)\ \mapsto\ x(t,\xi)\ \in\ \partial\Omega(t)\cap V \tag{6.1}$$

such that the following holds.

(i) Either $W = [0,T] \times S^1$ if $V = \mathbb{R}^2$, or $W = [0,T]\times]0,1[$ in the case with geographical constraints.

(ii) At each time $0 < t < T$, the map

$$\xi\ \mapsto\ x(t,\xi)\ \in\ \partial\Omega(t)\cap V \tag{6.2}$$

is one-to-one. Its range covers all the relative boundary $\partial\Omega(t) \cap V$.

(iii) For every $(t,\xi) \in W$ with $t > 0$, the partial derivative $x_\xi(t,\xi)$ is a nonzero tangent vector to the boundary $\partial\Omega(t)$ at the point $x(t,\xi)$. For any given ξ and any $\tau > 0$ there holds

$$\min_{t\in[\tau,T]} \big|x_\xi(t,\xi)\big| > 0. \tag{6.3}$$

(iv) The perpendicular vector

$$\mathbf{n}(t,\xi) \doteq \left(\frac{x_\xi(t,\xi)}{\big|x_\xi(t,\xi)\big|}\right)^\perp \tag{6.4}$$

yields the unit inner normal to the set $\Omega(t)$ at the boundary point $x(t,\xi)$.

(v) For each ξ, the trajectory $t \mapsto x(t,\xi)$ is orthogonal to the boundary $\partial\Omega(t)$ at every time $t \in [0,T]$. Namely, there exists a continuous, scalar function $\beta : W \mapsto \mathbb{R}$ such that

$$x_t(t,\xi) = \beta(t,\xi)\,\mathbf{n}(t,\xi) \quad \text{for all } (t,\xi) \in W. \tag{6.5}$$

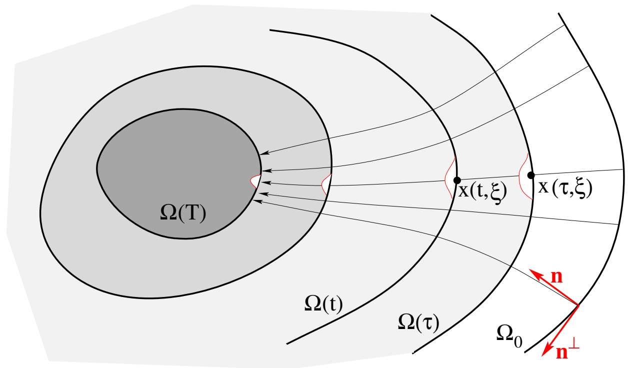

FIGURE 6 At each time $t \in [0, T]$, the boundary of the set $\Omega(t)$ is parameterized by $\xi \mapsto x(t, \xi)$. The curves $t \mapsto x(t, \xi)$ cross the boundaries perpendicularly.

We denote by

$$\omega(t,\xi) \doteq \frac{\big\langle \mathbf{n}(t,\xi),\, x_{\xi\xi}(t,\xi)\big\rangle}{\big|x_\xi(t,\xi)\big|^2} \tag{6.6}$$

the curvature of the boundary $\partial\Omega(t)$ at the point $x(t,\xi)$. Notice that this curvature is well defined for a.e. $(t,\xi) \in W$. Indeed, the functions x_ξ and $\mathbf{n}$ are locally Lipschitz continuous, while $|x_\xi|$ is a continuous, strictly positive function. By Rademacher's theorem [19], $x_{\xi\xi}$ exists almost everywhere. The following regularity property will be assumed:

(A3) For each ξ, the curvature function $t \mapsto \omega(t,\xi)$ is measurable and bounded. Moreover,

$$\lim_{\varepsilon\to 0} \int_0^T \sup_{|\zeta-\xi|<\varepsilon} \big|\omega(t,\zeta) - \omega(t,\xi)\big|\, dt = 0. \tag{6.7}$$

Before stating a precise set of necessary conditions, we outline the main ideas. As shown in Fig. 6, assume that at some time $\tau \in [0, T]$ we change our control $\beta(\cdot)$ in a neighborhood of a point (τ, ξ). This corresponds to a "needle variation" used in the proof of the Pontryagin Maximum Principle [11,14,20]. However, in the present case the perturbation is localized in time and also in space. As a result of this perturbation, at time τ the set $\Omega(\tau)$ will be replaced by a (possibly smaller) set $\Omega^\varepsilon(\tau)$. Its relative boundary will be parameterized by

$$\xi \mapsto x^\varepsilon(\tau,\xi) \;\in\; \partial\Omega^\varepsilon(\tau) \cap V.$$

Afterwards, for $t \in [\tau, T]$, we keep the same control effort as before at all points of the boundary $\partial\Omega^\varepsilon(t)$. More precisely, calling

$$\beta^\varepsilon(t,x) \doteq \big\langle x^\varepsilon_t(t,\xi),\, \mathbf{n}^\varepsilon(t,\xi)\big\rangle$$

the inward speed of the boundary point $x^\varepsilon(t,\xi) \in \partial\Omega^\varepsilon(t)$, we impose

$$\beta^\varepsilon(t,x)\,\big|x^\varepsilon_\xi(t,\xi)\big| = \beta(t,x)\,\big|x_\xi(t,\xi)\big|. \tag{6.8}$$

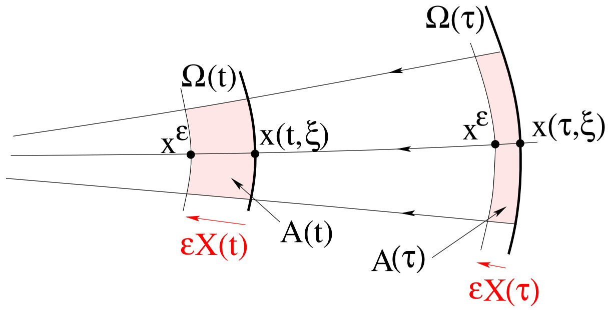

FIGURE 7 In the perturbed motion, at time t the boundary $\partial\Omega(t)$ is pushed inward in the amount $\varepsilon X(t)$. To leading order, the change $A(t)$ in the area is described by the ODE (6.14).

This will guarantee that the total effort in the perturbed strategy is admissible. Namely, for every $t \in]\tau, T[$,

$$\mathcal{E}^\varepsilon(t) = \int E\big(\beta^\varepsilon(t,x)\big)\,\big|x^\varepsilon_\xi(t,\xi)\big|\,d\xi = \int E\big(\beta(t,x)\big)\,\big|x_\xi(t,\xi)\big|\,d\xi = \mathcal{E}(t) = M. \tag{6.9}$$

We wish to estimate the reduction in the total cost, due to the fact that all sets $\Omega^\varepsilon(t)$, $t \in [\tau, T]$ are now smaller than in the original optimal solution. Toward this goal, consider the first order approximation

$$x^\varepsilon(t,\xi) = x(t,\xi) + \varepsilon X(t,\xi)\mathbf{n}(t,\xi) + o(\varepsilon). \tag{6.10}$$

This implies

$$\big|x^\varepsilon_\xi(t,\xi)\big| \;\approx\; \big|x_\xi(t,\xi)\big|\big(1 - \omega(t,\xi)\varepsilon X(t,\xi)\big), \tag{6.11}$$

where ω is the boundary curvature. In view of (6.8) one obtains

$$\beta^\varepsilon(t,\xi) \;\approx\; \frac{\big(1+\beta(t,\xi)\big)\big|x_\xi(t,\xi)\big|}{\big|x^\varepsilon_\xi(t,\xi)\big|} - 1 \;\approx\; \big(1+\beta(t,\xi)\big)\big(1+\omega(t,\xi)\varepsilon X(t,\xi)\big) - 1. \tag{6.12}$$

To leading order we thus find

$$\partial_t X(t,\xi) = \frac{\beta^\varepsilon(t,\xi) - \beta(t,\xi)}{\varepsilon} \;\approx\; \big(1+\beta(t,\xi)\big)\omega(t,\xi)X(t,\xi). \tag{6.13}$$

Calling $A(t,\xi) = X(t,\xi)\,\big|x_\xi(t,\xi)\big|$ the infinitesimal decrease in the area produced by the perturbation (see Fig. 7), differentiating w.r.t. time $t \in [\tau, T]$, we find

$$\partial_t\,\big|x_\xi(t,\xi)\big| \;=\; -\beta(t,\xi)\omega(t,\xi)\big|x_\xi(t,\xi)\big|,$$

$$\begin{aligned}\partial_t\,A(t,\xi) &\approx \big(1+\beta(t,\xi)\big)\omega(t,\xi)X(t,\xi)\big|x_\xi(t,\xi)\big| - X(t,\xi)\beta(t,\xi)\omega(t,\xi)\big|x_\xi(t,\xi)\big|\\ &= \omega(t,\xi)A(t,\xi).\end{aligned} \tag{6.14}$$

The linear ODE (6.14) is the key to understanding the optimality condition. If at time τ we are able to clean up an additional area $\bar{a}$ in a neighborhood of the point $x(\tau,\xi)$, the total cost would decrease in the amount $\bar{a}\cdot Y(\tau,\xi)$, where

$$Y(\tau,\xi) \doteq \int_\tau^T \kappa_1 A(t,\xi)\,dt + \kappa_2 A(T,\xi). \tag{6.15}$$

Here A is the solution to the ODE (6.14) with initial data $A(\tau,\xi)=1$. The adjoint function Y can be computed as the unique solution to the backward, linear ODE

$$\partial_t Y(t,\xi) = -\,\omega(t,\xi)\,Y(t,\xi) - \kappa_1, \quad Y(T,\xi)=\kappa_2. \tag{6.16}$$

Indeed, (6.14) and (6.16) together imply

$$\frac{d}{dt}(AY) = -\kappa_1 AY,$$

$$Y(T,\xi)A(T,\xi) - Y(\tau,\xi)A(\tau,\xi) = -\int_\tau^T \kappa_1 A(t,\xi)\,dt,$$

and this yields (6.15) because $Y(T,\xi)=\kappa_2$, $A(\tau,\xi)=1$.

Remark 6.1. The adjoint variable $Y>0$ introduced at (6.16) can be interpreted as a "shadow price". Namely (see Fig. 6), think of $\Omega(t)$ as the contaminated set, and assume that at time τ an external contractor offered to "clean up" a neighborhood of the point $x(\tau,\xi)$, thus replacing the set $\Omega(\tau)$ with a smaller set $\Omega^\varepsilon(\tau)$, at a price of $Y(\tau,\xi)$ per unit area. In this case, accepting or refusing the offer would make no difference in the total cost.

To obtain an optimality condition, consider two boundary points $P_i = x(\tau,\xi_i) \in \partial\Omega(\tau)$, $i=1,2$. Assume that the control is active in a neighborhood of both points: $\beta(\tau,\xi_i) > -1$. If $Y(\tau,\xi_1) > Y(\tau,\xi_2)$ we can then rule out optimality. Indeed, we can increase the effort (i.e., the value of β) in a neighborhood of P_1 and decrease it by the same amount in a neighborhood of P_2. This will produce an admissible perturbation with a strictly lower total cost.

We conclude that, at each time τ, the control $\beta(\tau,\cdot)$ can be active only at points where $Y(\tau,\cdot)$ attains its global maximum. Otherwise stated, one has the implication

$$\beta(\tau,\xi) > -1 \quad \Longrightarrow \quad Y(\tau,\xi) = \max_\zeta Y(\tau,\zeta) \doteq Y^*(\tau). \tag{6.17}$$

We are now ready to state our main result, providing necessary conditions for optimality.

Theorem 6.1. *Assume that $t\mapsto\Omega(t)$ provides an optimal solution to (OP). Let $(t,\xi)\mapsto x(t,\xi)$ be a parameterization of the relative boundaries of the sets $\Omega(t)$, satisfying the assumptions (A1)–(A2). Let $Y=Y(t,\xi)$ be the adjoint function introduced at* (6.16).

Then, for every $(t,\xi)\in W$ the inward normal velocity $\beta=\beta(t,\xi)$ satisfies (6.17).

A proof will be worked out in the next two sections.

Remark 6.2. The condition (6.17) does not explicitly determine the values $\beta(\tau, \xi)$. However, it implies that at any time t the portion of the boundary $\partial\Omega(t)$ where the control is active must be the union of arcs with constant curvature $\omega(t)$.

Indeed, by the assumptions (A2)–(A3), both $\beta(t,\xi)$ and the solution $Y = Y(t,\xi)$ of the ODE (6.16) are continuous functions of (t,ξ). Consider the open subset where the control is active:

$$W^{active} \doteq \big\{(t,\xi);\ \beta(t,\xi) > -1\big\}. \tag{6.18}$$

Assume $(t,\xi_1),(t,\xi_2) \in W^{active}$ for all $t \in \,]a,b[\,$. Then

$$Y(t,\xi_1) = Y(t,\xi_2) = \max_{\zeta} Y(t,\zeta). \quad \text{for all } t \in \,]a,b[.$$

In view of (6.16), this implies

$$\omega(t,\xi_1) = \omega(t,\xi_2) \quad \text{for a.e. } t \in \,]a,b[. \tag{6.19}$$

Since ξ_1,ξ_2 are arbitrary, we conclude that $\omega(t,\xi) = \omega(t)$ is constant for all ξ such that $(t,\xi) \in W^{active}$. In particular: if $\omega(t) \neq 0$, this means that at time t the control effort is concentrated on the union of arcs of circumferences with radius $r(t) = \big|\omega(t)\big|^{-1}$. If $\omega(t) = 0$, at time t the effort is concentrated on a family of straight lines.

In view of (6.16), one expects that the effort should be concentrated precisely on the portion of the boundary $\partial\Omega(t)$ where the curvature is maximum. The recent paper [5] proves that this is indeed the case when $V = \mathbb{R}^2$ (i.e., without geographical constraints), assuming that the initial set Ω_0 is convex.

7. A Family of Admissible Perturbations

Toward a proof of Theorem 6.1 we will construct a family of admissible perturbations of a set motion $t \mapsto \Omega(t)$, satisfying the regularity conditions *(A1)-(A2)*.

1. Define the functions (see Fig. 8)

$$\varphi(s) \doteq \begin{cases} \frac{1}{2} - s^2 & \text{for } |x| \leq \frac{1}{2}, \\ \big(1-|s|\big)^2 & \text{for } \frac{1}{2} \leq |s| \leq 1, \\ 0 & \text{for } |s| \geq 1, \end{cases} \qquad \varphi_\varepsilon(s) \doteq \varphi\left(\frac{s}{\varepsilon}\right). \tag{7.1}$$

Notice that these are $\mathcal{C}^{1,1}$ functions, continuously differentiable with Lipschitz derivative and with compact support. Moreover, they satisfy

$$\int \varphi(s)\,ds = \frac{1}{2}, \qquad \int \varphi_\varepsilon(s)\,ds = \frac{\varepsilon}{2}, \tag{7.2}$$

$$\big[\varphi'(s)\big]^2 \leq 4\varphi(s), \qquad \big[\varphi_\varepsilon'(s)\big]^2 \leq 4\varepsilon^{-2}\varphi_\varepsilon(s) \quad \text{for all } s \in \mathbb{R}. \tag{7.3}$$

Assume that, at a time $\tau \in \,]0,T[\,$, we can perturb the boundary of the set $\Omega(\tau)$ in a neighborhood of a point $x(\tau,\xi_0)$, so that the new boundary is described by

$$x^\varepsilon(\tau,\xi) = x(\tau,\xi) \pm \varepsilon^\gamma \varphi_\varepsilon(\xi - \xi_0)\mathbf{n}(\tau,\xi), \tag{7.4}$$

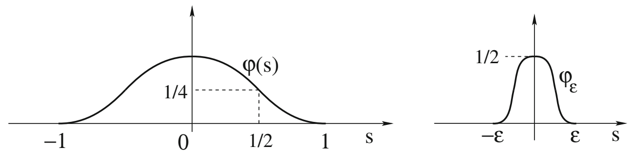

FIGURE 8 The functions φ and φ_ε introduced at (7.1).

for some exponent $\gamma > 0$ whose value will be determined later. As a consequence, the boundary $\partial\Omega(t)$ will be perturbed for all $t \in [\tau, T]$. We seek a function $\sigma : [\tau, T] \mapsto \mathbb{R}_+$, with

$$\sigma(\tau) = \pm 1, \tag{7.5}$$

and such that the perturbed motion described by

$$x^\varepsilon(t,\xi) = x(t,\xi) + \varepsilon^\gamma \sigma(t)\varphi_\varepsilon(\xi-\xi_0)\mathbf{n}(t,\xi) \tag{7.6}$$

is admissible.

Notice that, since the normal vector $\mathbf{n}$ is inward pointing, the perturbed sets $\Omega^\varepsilon(t)$ will be smaller in the case $\sigma(t) > 0$, and larger when $\sigma(t) < 0$.

To estimate the total effort $\mathcal{E}^\varepsilon(t)$ for the perturbed motion, we write

$$\mathbf{n}^\varepsilon(t,\xi) \doteq \frac{\big[x^\varepsilon_\xi(t,\xi)\big]^\perp}{\big|x^\varepsilon_\xi(t,\xi)\big|}, \qquad \beta^\varepsilon(t,\xi) \doteq \big\langle \mathbf{n}^\varepsilon(t,\xi),\, x^\varepsilon_t(t,\xi)\big\rangle.$$

Since $x^\varepsilon(t,\xi) = x(t,\xi)$ for $|\xi-\xi_0| \geq \varepsilon$, it suffices to find a function σ such that

$$E\big(\beta^\varepsilon(t,\xi)\big)\big|x^\varepsilon_\xi(t,\xi)\big| \;\leq\; E\big(\beta(t,\xi)\big)\big|x_\xi(t,\xi)\big| \tag{7.7}$$

for all $(t,\xi) \in [\tau, T] \times [\xi_0-\varepsilon,\, \xi_0+\varepsilon]$.

2. Observing that $\langle x_\xi, \mathbf{n}\rangle = \langle \mathbf{n}_\xi, \mathbf{n}\rangle = 0$, recalling the formula (6.6) for the curvature, one obtains

$$\big\langle \mathbf{n}_\xi(t,\xi),\, x_\xi(t,\xi)\big\rangle = -\big\langle \mathbf{n}(t,\xi),\, x_{\xi\xi}(t,\xi)\big\rangle = -\omega(t,\xi)\big|x_\xi(t,\xi)\big|^2. \tag{7.8}$$

Differentiating (7.6) w.r.t. ξ we compute

$$x^\varepsilon_\xi = x_\xi + \varepsilon^\gamma\sigma\varphi'_\varepsilon\mathbf{n} + \varepsilon^\gamma\sigma\varphi_\varepsilon\mathbf{n}_\xi, \tag{7.9}$$

$$\begin{aligned} |x^\varepsilon_\xi|^2 &= |x_\xi|^2 + 2\big\langle x_\xi,\, \varepsilon^\gamma\sigma\varphi_\varepsilon\mathbf{n}_\xi\big\rangle + \varepsilon^{2\gamma}\sigma^2(\varphi'_\varepsilon)^2 + \varepsilon^{2\gamma}\sigma^2\varphi_\varepsilon^2|\mathbf{n}_\xi|^2 \\ &= \big(1-2\omega\varepsilon^\gamma\sigma\varphi_\varepsilon\big)\,|x_\xi|^2 + \varepsilon^{2\gamma}\sigma^2(\varphi'_\varepsilon)^2 + \varepsilon^{2\gamma}\sigma^2\varphi_\varepsilon^2|\mathbf{n}_\xi|^2. \end{aligned} \tag{7.10}$$

Taking square roots, since we are assuming that $|x_\xi|$ remains uniformly positive, we obtain

$$|x^\varepsilon_\xi| = \big(1-\omega\varepsilon^\gamma\sigma\varphi_\varepsilon\big)\,|x_\xi| + \mathcal{O}(1)\cdot\varepsilon^{2\gamma}\sigma^2\big[(\varphi'_\varepsilon)^2+\varphi_\varepsilon^2\big]. \tag{7.11}$$

3. Next, we need to estimate the new inward speed $\beta^\varepsilon = \langle x^\varepsilon_t, \mathbf{n}^\varepsilon\rangle$. For this purpose we compute

$$x^\varepsilon_t = x_t + \varepsilon^\gamma\dot\sigma\varphi_\varepsilon\mathbf{n} + \varepsilon^\gamma\sigma\varphi_\varepsilon\mathbf{n}_t. \tag{7.12}$$

$$\mathbf{n}^\varepsilon = \frac{(x_\xi^\varepsilon)^\perp}{|x_\xi^\varepsilon|}. \tag{7.13}$$

It is convenient to write

$$\begin{aligned} \beta^\varepsilon = & \ \langle x_t^\varepsilon, \mathbf{n}^\varepsilon\rangle = \langle x_t^\varepsilon, \mathbf{n}\rangle + \langle x_t^\varepsilon, \mathbf{n}^\varepsilon - \mathbf{n}\rangle \\ = & \ \beta + \varepsilon^\gamma \dot\sigma \varphi_\varepsilon + \langle x_t^\varepsilon, \mathbf{n}^\varepsilon - \mathbf{n}\rangle. \end{aligned} \tag{7.14}$$

Since $|\mathbf{n}^\varepsilon| = |\mathbf{n}| = 1$, we have

$$\begin{aligned} |\mathbf{n}^\varepsilon - \mathbf{n}| &= \mathcal{O}(1)\cdot \big\langle \mathbf{n},\, (x_\xi^\varepsilon - x_\xi)^\perp\big\rangle = \mathcal{O}(1)\cdot\big\langle x_\xi,\, (x_\xi^\varepsilon - x_\xi)\big\rangle \\ &= \mathcal{O}(1)\cdot\big\langle x_\xi,\, \varepsilon^\gamma \sigma\varphi_\varepsilon \mathbf{n}_\xi\big\rangle. \end{aligned} \tag{7.15}$$

Moreover, since x_t is parallel to $\mathbf{n}$, assuming that $|x_\xi|$ remains uniformly positive, the last term on the right hand side of (7.14) can be bounded as

$$\begin{aligned} \langle x_t^\varepsilon, \mathbf{n}^\varepsilon - \mathbf{n}\rangle = & \ \mathcal{O}(1)\cdot\big(|x_t| + \varepsilon^\gamma|\dot\sigma|\varphi_\varepsilon\big)\cdot|\mathbf{n}^\varepsilon - \mathbf{n}|^2 + \mathcal{O}(1)\cdot \varepsilon^\gamma\sigma\varphi_\varepsilon\, |\mathbf{n}_t|\, |\mathbf{n}^\varepsilon - \mathbf{n}| \\ = & \ \mathcal{O}(1)\cdot\varepsilon^{2\gamma}\varphi_\varepsilon^2. \end{aligned} \tag{7.16}$$

4. Combining (7.11), (7.13) and (7.15), and then using (7.3), we obtain

$$\begin{aligned} &(1+\beta^\varepsilon)|x_\xi^\varepsilon| - (1+\beta)|x_\xi| \\ &= \ \Big(1+\beta+\varepsilon^\gamma\dot\sigma\varphi_\varepsilon + \mathcal{O}(1)\cdot\varepsilon^{2\gamma}\sigma^2\varphi_\varepsilon^2\Big)\cdot\Big(\big(1-\omega\varepsilon^\gamma\sigma\varphi_\varepsilon\big)\,|x_\xi| \\ &\quad + \mathcal{O}(1)\cdot\varepsilon^{2\gamma}\sigma^2\big[\varphi_\varepsilon^2 + (\varphi_\varepsilon')^2\big]\Big) - (1+\beta)|x_\xi| \\ &= \ \varepsilon^\gamma\big(\dot\sigma - (1+\beta)\omega\sigma\big)\varphi_\varepsilon|x_\xi|\cdot\varepsilon^{2\gamma}\sigma^2\varphi_\varepsilon^2 + \mathcal{O}(1) \\ &\quad + \mathcal{O}(1)\cdot(1+\beta+\varepsilon^\gamma\dot\sigma\varphi_\varepsilon)\,\varepsilon^{2\gamma-2}\varphi_\varepsilon \\ &= \ \varepsilon^\gamma\big(\dot\sigma - (1+\beta)\omega\sigma\big)\varphi_\varepsilon|x_\xi| + \mathcal{O}(1)\cdot\varepsilon^{2\gamma-2}\varphi_\varepsilon. \end{aligned} \tag{7.17}$$

5. We now choose

$$\alpha \,=\, 1, \quad \gamma \,=\, 4, \tag{7.18}$$

$$\dot\sigma(t) \ \doteq \ \min_{|\zeta-\xi_0|\le\varepsilon} \big(1+\beta(t,\zeta)\big)\omega(t,\zeta)\sigma(t) - \varepsilon^\alpha. \tag{7.19}$$

We claim that, with the above choices, the left hand side of (7.17) is non-positive, for all $t,\xi \in [\tau,T]\times[\xi_0-\varepsilon, \xi_0+\varepsilon]$ and all $\varepsilon > 0$ sufficiently small. Indeed, from (7.17), it now follows

$$\begin{aligned} &\big(1+\beta^\varepsilon(t,\xi)\big)\big|x_\xi^\varepsilon(t,\xi)\big| - \big(1+\beta(t,\xi)\big)\big|x_\xi(t,\xi)\big| \\ &\le \ -\varepsilon^{\gamma+\alpha}\varphi_\varepsilon(\xi-\xi_0)\big|x_\xi(t,\xi)\big| + \mathcal{O}(1)\cdot\varepsilon^{2\gamma-2}\varphi_\varepsilon(\xi-\xi_0) \\ &= \ \Big(-\varepsilon^5\big|x_\xi(t,\xi)\big| + \mathcal{O}(1)\cdot\varepsilon^6\Big)\varphi_\varepsilon(\xi-\xi_0) \ \le \ 0, \end{aligned} \tag{7.20}$$

since we are assuming that $|x_\xi|$ remains uniformly positive.

8. Proof of Theorem 6.1

In this section we give a proof of Theorem 6.1, in several steps.

1. Let $t \mapsto \Omega(t)$ be an optimal strategy for the problem *(OP)*, and let $Y = Y(t,\xi)$ be the adjoint function, obtained by solving the backward Cauchy problem (6.16). Assume that at some time $\tau \in [0,T]$ the condition (6.17) is violated. Namely, there exist $(\tau,\xi_1),(\tau,\xi_2) \in W$ such that

$$\beta(\tau,\xi_1) > -1, \quad Y(\tau,\xi_1) < Y(\tau,\xi_2). \tag{8.1}$$

By continuity, without loss of generality we can assume that $0 < \tau < T$. In this case, by reducing the control effort in a neighborhood of ξ_1 (where it is less effective) and increasing the effort near ξ_2 (where it is more effective), we will construct another admissible strategy $t \mapsto \Omega^\varepsilon(t)$ with strictly smaller total cost, thus achieving a contradiction.

2. For $\varepsilon > 0$ small we let $\Omega^\varepsilon(t) = \Omega(t)$ for $t \in [0, \tau-\varepsilon]$. Calling $x^\varepsilon(t,\xi)$ the parameterization of the boundary $\partial\Omega^\varepsilon(t)$. On the interval $t \in [\tau-\varepsilon,\tau]$, we define

$$x^\varepsilon(t,\xi) = x(t,\xi) - \varepsilon^\gamma \sigma_1(t)\mathbf{n}(t,\xi) + \varepsilon^\gamma\sigma_2(t)\mathbf{n}(t,\xi), \tag{8.2}$$

for suitable functions σ_1,σ_2. These must be chosen so that the perturbed motion is also admissible. Recalling (7.17), for every $t \in [\tau-\varepsilon,\tau]$ this requires

$$\begin{aligned} 0 \;&\geq\; \sum_{i=1,2} \int_{|\xi-\xi_i|<\varepsilon} \Big\{ (1+\beta^\varepsilon)|x^\varepsilon_\xi| - (1+\beta)|x_\xi| \Big\}\, d\xi \\ &= \sum_{i=1,2} \int_{|\xi-\xi_i|<\varepsilon} \Big\{ \varepsilon^\gamma \big(\dot\sigma_i - (1+\beta)\omega\sigma\big)\varphi_\varepsilon |x_\xi| + \mathcal{O}(1)\cdot\varepsilon^{2\gamma-2}\,\varphi_\varepsilon \Big\}\, d\xi \\ &= \sum_{i=1,2} \int_{|\xi-\xi_i|<\varepsilon} \Big\{ \varepsilon^\gamma \dot\sigma_i \varphi_\varepsilon |x_\xi| + \mathcal{O}(1)\cdot\varepsilon^{\gamma}\,\varphi_\varepsilon \Big\}\, d\xi, \end{aligned} \tag{8.3}$$

To achieve (8.3), for $i = 1,2$ we first define

$$\eta_i(\varepsilon) \doteq \sup \left\{ \frac{\big|x_\xi(t,\xi) - x_\xi(\tau,\xi_i)\big|}{\big|x_\xi(\tau,\xi_i)\big|}; \quad t \in [\tau-\varepsilon,\tau],\; |\xi-\xi_i| < \varepsilon \right\}. \tag{8.4}$$

Then we let $\sigma_1,\sigma_2 : [\tau-\varepsilon,\tau] \mapsto \mathbb{R}$ be the affine functions such that

$$\begin{cases} \dot\sigma_i(t) = \dfrac{-1}{\varepsilon\,\big|x_\xi(\tau,\xi_1)\big|} - \dfrac{\eta_1(\varepsilon)}{\varepsilon} - \dfrac{1}{\sqrt{\varepsilon}}, \\[2mm] \dot\sigma_2(t) = \dfrac{1}{\varepsilon\,\big|x_\xi(\tau,\xi_2)\big|} - \dfrac{\eta_2(\varepsilon)}{\varepsilon} - \dfrac{1}{\sqrt{\varepsilon}}, \end{cases} \qquad \sigma_1(\tau-\varepsilon) = \sigma_2(\tau-\varepsilon) = 0. \tag{8.5}$$

Inserting (8.5) in the right hand side of (8.3) and observing that the leading order terms cancel, for every $\varepsilon > 0$ sufficiently small we obtain

$$\begin{aligned} &\sum_{i=1,2} \int_{|\xi-\xi_i|<\varepsilon} \Big\{ \varepsilon^\gamma \dot\sigma_i(t) \Big[|x_\xi(\tau,\xi_i)| + \Big(|x_\xi(t,\xi)| - |x_\xi(\tau,\xi_i)| \Big) \Big] + \mathcal{O}(1)\cdot\varepsilon^\gamma \Big\} \varphi_\varepsilon(\xi-\xi_i)\, d\xi \\ &\qquad \leq \sum_{i=1,2} \int_{|\xi-\xi_i|<\varepsilon} \Big\{ -\varepsilon^{\gamma-\frac12} + \mathcal{O}(1)\cdot\varepsilon^\gamma \Big\} \varphi_\varepsilon(\xi-\xi_i)\, d\xi \;<\; 0, \end{aligned} \tag{8.6}$$

showing that the perturbed motion is admissible.

3. On the remaining interval $[\tau, T]$, recalling (7.19) we let the functions σ_i be the solutions to the linear ODEs

$$\dot{\sigma}_i(t) = \min_{|\zeta-\xi_i|\leq\varepsilon} \big(1+\beta(t,\zeta)\big)\omega(t,\zeta)\sigma_i(t) - \varepsilon^\alpha, \tag{8.7}$$

with initial data given at $t = \tau$ corresponding to the function constructed in step 2, namely

$$\begin{aligned} \sigma_1(\tau) = \bar{\sigma}_1 \ &\doteq \ \frac{-1}{\big|x_\xi(\tau,\xi_1)\big|} - \eta_1(\varepsilon) - \sqrt{\varepsilon}, \\ \sigma_2(\tau) = \bar{\sigma}_2 \ &\doteq \ \frac{1}{\big|x_\xi(\tau,\xi_2)\big|} - \eta_2(\varepsilon) - \sqrt{\varepsilon}. \end{aligned} \tag{8.8}$$

As shown by the analysis in Sect. 7, by (7.20) this motion is admissible.

4. It remains to prove that, for $\varepsilon > 0$, the new strategy achieves a strictly lower cost. Let the functions $\sigma_i^\sharp : [\tau, T] \mapsto \mathbb{R}$, $i = 1, 2$, be the solutions to

$$\dot{\sigma}_i^\sharp(t) = \big(1+\beta(t,\xi_1)\big)\sigma_1^\sharp(t), \quad \sigma_i^\sharp(\tau) = 1. \tag{8.9}$$

Thanks to the assumption (6.7), our construction implies the uniform convergence

$$\sigma_{1,\varepsilon}(t) \ \to \ -\frac{\sigma^\sharp(t)}{\big|x_\xi(\tau,\xi_1)\big|}, \quad \sigma_{2,\varepsilon}(t) \ \to \ \frac{\sigma_2^\sharp(t)}{\big|x_\xi(\tau,\xi_2)\big|}, \tag{8.10}$$

uniformly for $t \in [\tau, T]$.

Recalling (7.2) and then using (6.15)–(6.16), the difference in the total cost over the interval $[\tau, T]$ can be estimated as

$$\begin{aligned} &\kappa_1 \int_\tau^T \Big[\mathcal{L}^2\big(\Omega^\varepsilon(t)\big) - \mathcal{L}^2\big(\Omega(t)\big)\Big]\,dt + \kappa_2\Big[\mathcal{L}^2\big(\Omega^\varepsilon(T)\big) - \mathcal{L}^2\big(\Omega(T)\big)\Big] \\ &\quad = \ \kappa_1 \int_\tau^T \varepsilon^\gamma \cdot \frac{\varepsilon}{2}\big[\sigma_1^\sharp(t) - \sigma_2^\sharp(t)\big]\,dt + \kappa_2\varepsilon^\gamma \cdot \frac{\varepsilon}{2}\big[\sigma_1^\sharp(T) - \sigma_2^\sharp(T)\big] + o(\varepsilon^{\gamma+1}) \\ &\quad = \ \frac{\varepsilon^{\gamma+1}}{2}\Big(Y(\tau,\sigma_1) - Y(\tau,\sigma_2)\Big) + o(\varepsilon^{\gamma+1}) \ < \ 0, \end{aligned} \tag{8.11}$$

As usual, here $o(\varepsilon^{\gamma+1})$ denotes an infinitesimal of higher order, as $\varepsilon \to 0$.

On the other hand, during the time interval $[\tau - \varepsilon, \tau]$ the difference in cost can be bounded as

$$\int_{\tau-\varepsilon}^\tau \Big[\mathcal{L}^2\big(\Omega^\varepsilon(t)\big) - \mathcal{L}^2\big(\Omega(t)\big)\Big]\,dt = \mathcal{O}(1)\cdot\varepsilon^{\gamma+2}. \tag{8.12}$$

Combining (8.11)-(8.12) we conclude

$$J(\Omega^\varepsilon) - J(\Omega) \ < \ 0$$

for all $\varepsilon > 0$ sufficiently small. This contradicts optimality, proving the theorem.

□

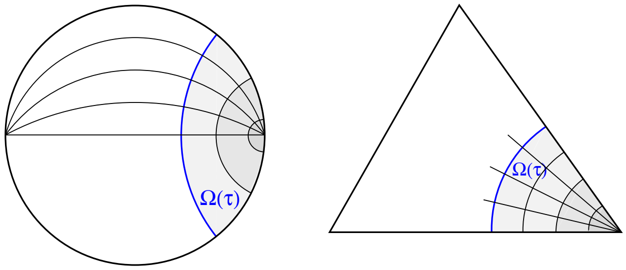

FIGURE 9 As $t \to T-$ and the eradication is nearly completed, the length of boundary $\partial\Omega(t)$ approaches zero. Hence the assumption that $|x_\xi(t,\xi)|$ remains uniformly positive as $t \to T-$ is never satisfied. On the other hand, in most cases the optimal set motion satisfies the necessary and sufficient conditions in Corollary 4.1, on some terminal interval $[\tau, T]$.

9. Optimality Conditions for the Minimum Time Problem

If one tries to apply the optimality conditions derived in the previous sections to the minimum time eradication problem, the construction invariably breaks down. Indeed, a key assumption used in the construction of admissible perturbations is that the arc-length $|x_\xi(t,\xi)|$ remains uniformly positive for $t \in [\tau, T]$. However, in all examples (see Fig. 9) one finds $|x_\xi(t,\xi)| \to 0$ for all ξ, as $t \to T-$. To obtain necessary conditions, a different approach is needed.

We start with the simple observation that, if $t \mapsto \Omega(t)$, $t \in [0,T]$, is an optimal solution to the minimum time eradication problem, then for any intermediate time $0 < T_1 < T$ one has:

(1) Restricted to the subinterval $[0,T_1]$, the strategy $t \mapsto \Omega(t)$ is optimal for the minimum time transfer problem *(MTTP)*, with

$$\Omega_{initial} = V, \quad \Omega_{final} = \Omega(T_1), \tag{9.1}$$

(2) Restricted to the subinterval $[T_1, T]$, the strategy $t \mapsto \Omega(t)$ is optimal for the minimum time transfer problem *(MTTP)*, with

$$\Omega_{initial} = \Omega(T_1), \quad \Omega_{final} = \emptyset. \tag{9.2}$$

In many situations, one can choose $T_1 < T$ so that the optimal transfer problem with data (9.2) has a solution satisfying the assumptions of Corollary 4.1. This provides necessary and sufficient conditions for optimality, restricted to the subinterval $[T_1, T]$.

It thus remains to derive necessary conditions for the problem (9.1). This will require an additional assumption on the parameterization $(t,\xi) \mapsto x(t,\xi)$ of the boundaries $\partial\Omega(t)$, for $t \in [0, T_1]$.

(A4) At time T_1 the effort is uniformly positive along the boundary $\partial\Omega(T_1) \cap V$. Namely, there exists $\delta_0 > 0$ such that

$$\beta(T_1, \xi) \geq -1 + \delta_0 \quad \text{for all } \xi. \tag{9.3}$$

The next result shows that, if $t \mapsto \Omega(t)$ is optimal for the minimum time transfer problem (9.2) and the additional assumptions (A4) hold, then it satisfies the necessary conditions for the problem of minimizing the terminal area at time $t = T_1$.

$$\text{Minimize: } \mathcal{L}^2\big(\Omega(T_1)\big) \quad \text{subject to: } \Omega(0) = V, \tag{9.4}$$

among all admissible motions. Notice that this is precisely the problem *(OSM)*, in the special case where $\kappa_1 = 0$, $\kappa_2 = 1$ in (2.11). The corresponding adjoint equations (6.16) reduce to

$$\partial_t Y(t,\xi) = -\omega(t,\xi)Y(t,\xi), \quad Y(T_1) = 1. \tag{9.5}$$

Theorem 9.1. *Assume that $t \mapsto \Omega(t)$ provides an optimal solution to the minimum time transfer problem (MTTP) with data (9.1). Let $(t,\xi) \mapsto x(t,\xi)$ be a parameterization of the boundaries $\partial\Omega(t)$, satisfying the regularity assumptions in (A1)–(A3) with $T = T_1$. Moreover, assume that (A4) holds. Let $Y = Y(t,\xi)$ be the adjoint function at (9.5).*

Then, for every $(t,\xi) \in]0,T_1[\times]0,1[$, the inward normal velocity $\beta = \beta(t,\xi)$ satisfies the implication

$$\beta(t,\xi) > -1 \quad \Longrightarrow \quad Y(t,\xi) = \max_{\zeta\in]0,1[} Y(t,\zeta). \tag{9.6}$$

Proof. 1. Assume that at some time $\tau \in [0,T]$ the condition (9.6) is violated. Namely, there exist $(\tau,\xi_1),(\tau,\xi_2)$ with $0 < \tau < T_1$, such that

$$\beta(\tau,\xi_1) > -1, \quad Y(\tau,\xi_1) < Y(\tau,\xi_2).$$

For $\varepsilon > 0$ small we then let $\Omega^\varepsilon(t) = \Omega(t)$ for $t \in [0,\tau-\varepsilon]$. Calling $x^\varepsilon(t,\xi)$ the parameterization of the boundary $\partial\Omega^\varepsilon(t)$. On the interval $t \in [\tau-\varepsilon,\tau]$, we define

$$x^\varepsilon(t,\xi) = x(t,\xi) - \varepsilon^\gamma\sigma_1(t)\mathbf{n}(t,\xi) + \varepsilon^\gamma\sigma_2(t)\mathbf{n}(t,\xi), \tag{9.7}$$

for suitable functions σ_1,σ_2, constructed as in steps 2 and 3 of the proof of Theorem 6.1. This guarantees that the perturbed strategy (9.7) is admissible.

The same analysis as in (8.11) now yields

$$\mathcal{L}^2\big(\Omega^\varepsilon(T_1)\big) - \mathcal{L}^2\big(\Omega(T_1)\big) = \frac{\varepsilon^{\gamma+1}}{2}\Big(Y(\tau,\sigma_1) - Y(\tau,\sigma_2)\Big) + o(\varepsilon^{\gamma+1}) \;<\; 0. \tag{9.8}$$

This shows that $\Omega(\cdot)$ is not optimal for problem of minimizing the terminal area at time T_1. In particular, the previous construction yields

$$\sigma_2(T_1)\left|x_\xi(T_1,\xi_2)\right| - \sigma_1(T_1)\left|x_\xi(T_1,\xi_1)\right| \;\geq c_1 > 0, \tag{9.9}$$

for some constant $c_1 > 0$ independent of $\varepsilon > 0$. As shown in Fig. 10, this means that the perturbed set $\Omega^\varepsilon(T_1)$ contains an additional small region in a neighborhood of the point $x(T_1,\xi_1)$, while a larger region is removed in a neighborhood of $x(T_1,\xi_2)$.

2. By a further modification, we now construct a set motion $t \mapsto \widehat{\Omega}^\varepsilon(t)$ such that

$$\widehat{\Omega}^\varepsilon(T_1-\varepsilon) = \Omega^\varepsilon(T_1-\varepsilon), \quad \widehat{\Omega}^\varepsilon(T_1) = \Omega(T_1), \tag{9.10}$$

while, for all $t \in [T_1-\varepsilon,T_1]$, a strictly smaller effort is required:

$$\widehat{\mathcal{E}}^\varepsilon(t) = \int_0^1 \big(1+\widehat{\beta}^\varepsilon(t,\xi)\big)\left|\widehat{x}^\varepsilon_\xi(t,\xi)\right| d\xi \leq \mathcal{E}^\varepsilon(t) - \delta_\varepsilon \leq M - \delta_\varepsilon, \tag{9.11}$$

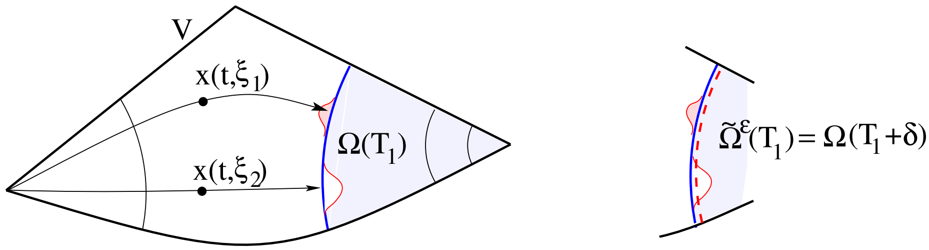

FIGURE 10 Left: the perturbed set $\Omega^\varepsilon(T_1)$, whose boundary is parameterized as in (9.7), is obtained from $\Omega(T_1)$ by adding a small region near $x(T_1,\xi_1)$ and removing a larger region near $x(T_1,\xi_2)$. Right: by a further modification, we obtain a family of sets such that $\widetilde\Omega^\varepsilon(t) = \Omega(t')$, with $T_1 \le t < t'$. This rules out optimality for the minimum time problem.

for some $\delta_\varepsilon > 0$.

This new motion is obtained replacing (9.7) with

$$\widehat{x}^\varepsilon(t,\xi) = x(t,\xi) - \varepsilon^\gamma\widehat\sigma_1(t)\mathbf{n}(t,\xi) + \varepsilon^\gamma\widehat\sigma_2(t)\mathbf{n}(t,\xi), \tag{9.12}$$

where

$$\widehat\sigma_i(t) = \begin{cases} \sigma_i(t) & \text{if} \quad t \le T_1-\varepsilon, \\ \frac{T_1-t}{\varepsilon}\,\sigma_i(T_1-\varepsilon) & \text{if} \quad t\in[T_1-\varepsilon, T_1]. \end{cases} \tag{9.13}$$

This clearly implies (9.10).

Concerning the total effort, by the assumption *(A4)* for $\varepsilon > 0$ small by continuity we can assume $\widehat\beta^\varepsilon > -1$, hence $E(\widehat\beta^\varepsilon) = 1 + \widehat\beta^\varepsilon$. Moreover, for $t\in[T_1-\varepsilon, T_1]$, similar computations as in (8.11) yield

$$\begin{aligned}
\widehat{\mathcal{E}}^\varepsilon(t) - \mathcal{E}^\varepsilon(t) &= \varepsilon^\gamma \int \Big(\dot\sigma_2(t)\big|x_\xi(t,\xi_2)\big|\varphi_\varepsilon(\xi-\xi_2) - \dot\sigma_1(t)\big|x_\xi(t,\xi_1)\big|\varphi_\varepsilon(\xi-\xi_1)\Big)d\xi + o(\varepsilon^\gamma)\\
&= \frac{\varepsilon^\gamma}{2}\Big(- \sigma_2(T_1-\varepsilon)\big|x_\xi(t,\xi_2)\big| + \sigma_1(T_1-\varepsilon)\big|x_\xi(t,\xi_1)\big|\Big) + o(\varepsilon^\gamma)\\
&= \frac{\varepsilon^\gamma}{2}\Big(- \sigma_2(T_1)\big|x_\xi(T_1,\xi_2)\big| + \sigma_1(T_1)\big|x_\xi(T_1,\xi_1)\big|\Big) + o(\varepsilon^\gamma)\\
&\le -\frac{c_1\varepsilon^\gamma}{2} + o(\varepsilon^\gamma) \ < \ 0,
\end{aligned} \tag{9.14}$$

where we used continuity and finally (9.9).

3. In the previous step we constructed a set-motion that reaches the same terminal configuration $\Omega(T_1)$ but with a strictly smaller effort (9.11). In this step we use the additional available effort $M - \widehat{\mathcal{E}}^\varepsilon(t) > 0$ to construct a further motion $t\mapsto \widetilde\Omega^\varepsilon(t)$ that reaches $\Omega(T_1)$ in a strictly shorter time. This motion will have the form

$$t \ \mapsto \ \widetilde\Omega^\varepsilon(t) = \widehat\Omega^\varepsilon\big(\tau(t)\big) \quad t\in[T_1-\varepsilon,\, T_1], \tag{9.15}$$

with $\tau(t) > t$ for $t > T_1-\varepsilon$.

For this purpose, notice that the corresponding inner normal velocity for the motion (9.15) is

$$\widetilde\beta^\varepsilon(t,\xi) = \dot\tau(t)\cdot\widehat\beta^\varepsilon\big(\tau(t),\xi\big).$$

By the assumption (9.3) we can assume that $E(\beta^\varepsilon), E(\widehat{\beta}^\varepsilon)$ remain uniformly positive for $T_1 - \delta_1 < t < T$. The instantaneous total efforts are thus computed by

$$\widetilde{\mathcal{E}}^\varepsilon(t) = \int_0^1 \Big(1 + \widetilde{\beta}^\varepsilon(t,\xi)\Big)\big|\widetilde{x}^\varepsilon_\xi(t,\xi)\big|\, d\xi = \int_0^1 \Big(1 + \dot{\tau}(t)\cdot \widehat{\beta}^\varepsilon\big(\tau(t),\xi\big)\Big)\big|\widehat{x}^\varepsilon_\xi(t,\xi)\big|\, d\xi$$
$$= \widehat{\mathcal{E}}^\varepsilon(t) + \int_0^1 \big(\dot{\tau}(t) - 1\big)\cdot \widehat{\beta}^\varepsilon\big(\tau(t),\xi\big)\,\big|\widehat{x}^\varepsilon_\xi(t,\xi)\big|\, d\xi. \tag{9.16}$$

In view of (9.11), the right hand side is $\leq M$ provided we choose

$$1 \;<\; \dot{\tau}(t) \;<\; 1 + \delta_\varepsilon \left[\int_0^1 \widehat{\beta}^\varepsilon\big(\tau(t),\xi\big)\,\big|\widehat{x}^\varepsilon_\xi(t,\xi)\big|\, d\xi\right]^{-1}. \tag{9.17}$$

Let $T_1' < T_1$ be such that $\tau(T_1') = T_1$. Then the motion $t \mapsto \widetilde{\Omega}^\varepsilon(t)$ is admissible and satisfies $\widetilde{\Omega}^\varepsilon(T_1') = \Omega(T_1)$, showing that Ω is not time optimal. This contradiction proves the theorem. □

Remark 9.2. In the above proof, the assumption (9.3) was only used to conclude that the effort $E\big(\beta(T_1,\xi)\big)$ is strictly positive in a neighborhood of the point $x(T_1,\xi_2)$. A sharper version of the above result is as follows. Let $t \mapsto \Omega(t)$ be optimal for the minimum time problem. Assume

$$\beta(T_1,\xi_2) \;>\; -1 \tag{9.18}$$

and let $Y = Y(t,\xi)$ be as in (9.5). Then, at any point (τ,ξ_1) where $\beta(\tau,\xi_1) > -1$ one has

$$Y(\tau,\xi_1) \geq Y(\tau,\xi_2). \tag{9.19}$$

10. Optimality Conditions at Junctions

In Theorem 4.1, the existence of optimal solutions was proved within a class of functions with BV regularity. On the other hand, the necessary conditions for optimality derived in Theorem 6.1 require that the sets $\Omega(t)$ have $\mathcal{C}^{1,1}$ boundary. Indeed, we used this assumption to uniquely define the perpendicular curves $t \mapsto x(t,\xi)$.

Aim of this section is to partially fill this regularity gap, ruling out certain configurations where the sets $\Omega(t)$ have corners. The following situation will be considered:

(A5) There exists $\tau, \delta_0 > 0$ such that, for $|t-\tau| < \delta_0$, the boundary $\partial\Omega(t)$ contains two adjacent arcs $\gamma_1(t,\cdot)$, $\gamma_2(t,\cdot)$ joining at a point $P(t)$ at an angle $\theta(t)$. Each of these arcs admits a $\mathcal{C}^1$ parameterization of the form

$$\begin{cases} s \mapsto \gamma_1(t,s), & s \leq 0,\\ s \mapsto \gamma_2(t,s), & s \geq 0,\end{cases} \qquad \gamma_1(t,0) = \gamma_2(t,0) = P(t). \tag{10.1}$$

The tangent vectors to the curves $\gamma_1(\tau,\cdot)$ and $\gamma_2(\tau,\cdot)$ at the intersection point $P(\tau)$ will be denoted by

$$\mathbf{w}_1 = \gamma_{1,s}(\tau,0-), \quad \mathbf{w}_2 = \gamma_{2,s}(\tau,0+). \tag{10.2}$$

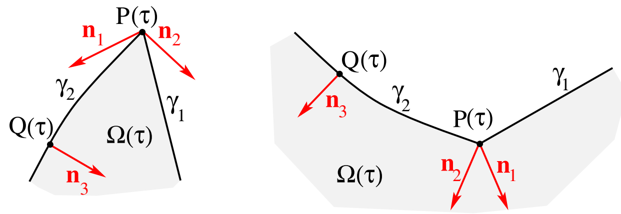

FIGURE 11 Left: the case of outward corner. Right: the case of an inward corner.

Moreover, we call $\mathbf{w}_1^\perp, \mathbf{w}_2^\perp$ the orthogonal vectors (rotated by 90^o counterclockwise). Notice that the two curves γ_1, γ_2 form an outward corner at $P(\tau)$ if the vector product satisfies (see Fig. 11, left)

$$\mathbf{w}_1 \times \mathbf{w}_2 \doteq \langle \mathbf{w}_1^\perp, \mathbf{w}_2 \rangle > 0.$$

On the other hand, if $\mathbf{w}_1 \times \mathbf{w}_2 < 0$ one has an inward corner, as shown in Fig. 11, right.

As before, we say that the control is *active* on a portion of the boundary $\partial\Omega(t)$ if the inward normal speed is $\beta > -1$, hence the effort is strictly positive: $E(\beta) = 1+\beta > 0$. The main result of this section shows that, for an optimal motion $t \mapsto \Omega(t)$, non-parallel junctions cannot be optimal if the control is active on at least one of the adjacent arcs.

Remark 10.1. If the control is not active along any of the two arcs γ_1, γ_2, then in a neighborhood of $P(t)$ the set $\Omega(t)$ expands with unit speed all along the boundary. This implies that for $t > \tau$ the set $\Omega(t)$ satisfies an interior ball condition, hence it can only have inward corners.

The next result shows that non-tangential junctions between two arcs cannot be optimal. An intuitive explanation comes from the basic formula (1.4). By "cutting the corners", as shown in Figs. 13 and 14, we can reduce the perimeter $\partial\Omega(t)$ in a neighborhood of the junction point $P(t)$. After this perturbation, the control effort spent in this neighborhood of $P(t)$ is reduced, hence $\mathcal{E}(t) < M$. We can then use the additional available effort $M - \mathcal{E}(t)$ to shrink the set $\Omega(t)$ in the neighborhood of some other point $Q(t)$ where the boundary is smooth. This will produce a bigger decrease in the area, thus decreasing the total cost $J(\Omega)$ at (1.8).

Theorem 10.1. *Assume that $t \mapsto \Omega(t)$ provides an optimal solution to (OP), with $\kappa_1 > 0$.*

In the setting described at (A5), if the control is active along at least one of the two arcs γ_1, γ_2, then these two arcs must be tangent at $P(t)$.

Proof. 1. Assume that the two arcs γ_1, γ_2 are not tangent, forming an angle $\theta(t) \neq \pi$ for $|\tau - t| < \delta$. We will then construct an admissible strategy $t \mapsto \Omega^\varepsilon(t)$ with smaller total cost.

To fix ideas, assume that the control is active along γ_2 and choose a point $Q(\tau) = \gamma_2(\tau, \bar{s})$, with $\bar{s} > 0$. Consider the tangent vectors

$$\mathbf{w}_1 \doteq \gamma_{1,s}(\tau, 0), \quad \mathbf{w}_2 \doteq \gamma_{1,s}(\tau, 0),$$

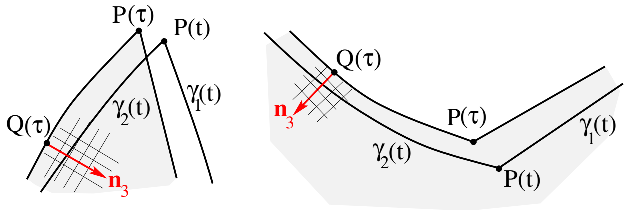

FIGURE 12 In a neighborhood of the point $Q(\tau)$, the curves $\gamma_2(t,\cdot)$ are parameterized according to a set of coordinates with axes parallel to $\mathbf{n}_3, \mathbf{n}_3^\perp$.

and the three inward-pointing unit normal vectors (see Fig. 11)

$$\mathbf{n}_1 \doteq \frac{\gamma_{1,s}(\tau,0)^\perp}{\big|\gamma_{1,s}(\tau,0)\big|}, \quad \mathbf{n}_2 \doteq \frac{\gamma_{2,s}(\tau,0)^\perp}{\big|\gamma_{2,s}(\tau,0)\big|}, \quad \mathbf{n}_3 \doteq \frac{\gamma_{2,s}(\tau,\bar{s})^\perp}{\big|\gamma_{2,s}(\tau,\bar{s})\big|}. \tag{10.3}$$

To simplify the computations, without loss of generality we can assume that, for some $\rho > 0$ small enough:

- For $|t-\tau| \le \rho$ and $s \in [-\rho, 0]$, the arc $s \mapsto \gamma_1(t,s)$ is parameterized by arc-length.
- For $|t-\tau| \le \rho$ and $s \in [0,\rho]$, the arc γ_2 is parameterized by arc-length.
- For $|t-\tau| \le \rho$ and $s \in [\bar{s}-\rho, \bar{s}+\rho]$, the arc γ_2 is parameterized according to a system of coordinates with coordinate axes parallel to $\mathbf{n}_3, \mathbf{n}_3^\perp$. More precisely (see Fig. 12), this means:

$$\langle \gamma_{2,t}, \mathbf{n}_3^\perp \rangle = 0 \text{ and } \langle \gamma_{2,s}, \mathbf{n}_3^\perp \rangle = -1 \quad \text{for } |t-\tau| \le \rho, \ |s-\bar{s}| \le \rho. \tag{10.4}$$

Notice that the above arc-length parameterization implies

$$|\mathbf{w}_1| = |\mathbf{w}_2| = 1, \quad \mathbf{n}_1 = \mathbf{w}_1^\perp, \quad \mathbf{n}_2 = \mathbf{w}^\perp.$$

Given $0 < \varepsilon << \delta$, for $t \in [\tau-\varepsilon, \tau+\delta+\varepsilon]$ consider the points

$$P_1(t) = \begin{cases} \gamma_1\big(t, \tau - t - \varepsilon\big) & \text{if } t \in [\tau-\varepsilon, \tau], \\ \gamma_1(t, -\varepsilon) & \text{if } t \in [\tau, \tau+\delta], \\ \gamma_1\big(t, t-\tau-\delta-\varepsilon\big) & \text{if } t \in [\tau+\delta, \tau+\delta+\varepsilon], \end{cases} \tag{10.5}$$

$$P_2(t) = \begin{cases} \gamma_2\big(t, t-\tau+\varepsilon\big) & \text{if } t \in [\tau-\varepsilon, \tau], \\ \gamma_2(t, \varepsilon) & \text{if } t \in [\tau, \tau+\delta], \\ \gamma_2\big(t, \tau+\delta+\varepsilon-t\big) & \text{if } t \in [\tau+\delta, \tau+\delta+\varepsilon]. \end{cases} \tag{10.6}$$

Observe that $P_1(t) = P_2(t) = P(t)$ for $t = \tau - \varepsilon$ and for $t = \tau+\delta+\varepsilon$. We now construct a modified set motion $t \mapsto \widehat{\Omega}(t)$ by the following rules.

(i) For $t \notin [\tau-\varepsilon, \tau+\delta+\varepsilon]$, one has $\widehat{\Omega}(t) = \Omega(t)$.

(ii) If $\mathbf{w}_1 \times \mathbf{w}_2 > 0$ (an outward corner), then for $t \in [\tau-\varepsilon, \tau+\delta+\varepsilon]$ the set $\widehat{\Omega}(t)$ is obtained from $\Omega(t)$ by removing the triangular region with vertices $P_1(t), P(t), P_2(t)$, as shown in Fig. 13.

(iii) If $\mathbf{w}_1 \times \mathbf{w}_2 < 0$ (an inward corner), then for $t \in [\tau-\varepsilon, \tau+\delta+\varepsilon]$ the set $\widehat{\Omega}(t)$ is obtained from $\Omega(t)$ by adding the triangular region with vertices $P_1(t), P(t), P_2(t)$, as shown in Fig. 14.

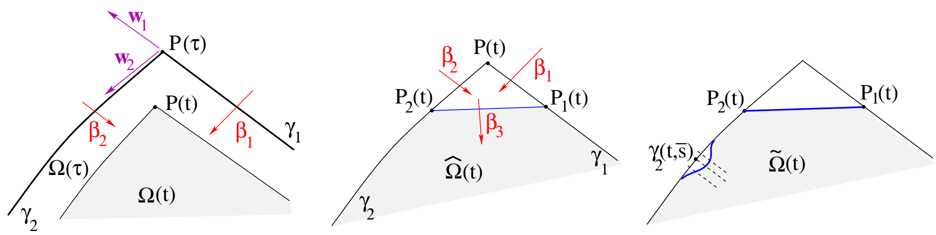

FIGURE 13 The case of an outward corner, with $\mathbf{w}_1 \times \mathbf{w}_2 > 0$. Left: the original set $\Omega(t)$, for $t \in [\tau, \tau + \delta]$. Center: the set $\widehat{\Omega}(t)$ obtained from $\Omega(t)$ by removing the triangular region $\widehat{P_1 P P_2}$. Right: the set $\widetilde{\Omega}(t)$ obtained by removing an additional region in a neighborhood of $\gamma_2(t, \bar{s})$.

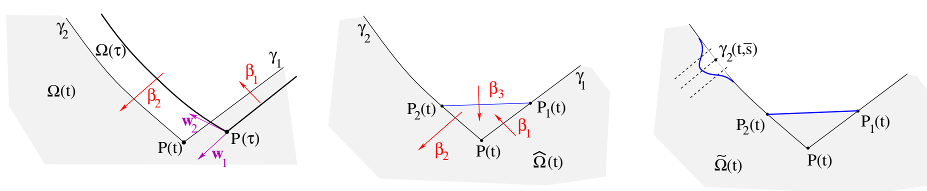

FIGURE 14 The case of an inward corner, with $\mathbf{w}_1 \times \mathbf{w}_2 < 0$. Left: the original set $\Omega(t)$, for $t \in [\tau, \tau + \delta]$. Center: the set $\widehat{\Omega}(t)$ obtained from $\Omega(t)$ by adding the triangular region $\widehat{P P_1 P_2}$. Right: the set $\widetilde{\Omega}(t)$ obtained by removing an additional region in a neighborhood of $\gamma_2(t, \bar{s})$.

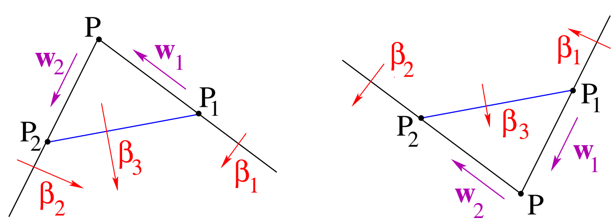

FIGURE 15 Computing the normal speed β_3 of the side AB, as in (10.7).

2. To estimate the change in the total effort $\widehat{\mathcal{E}}(t)$ required by the strategy $\widehat{\Omega}$, we need to compute the inward normal speed β_3 along the segment $\overline{P_1 P_2}$.

Referring to Fig. 15, left, consider a triangle with vertices

$$P(t), \quad P_1(t) = P(t) - \mathbf{w}_1, \quad P_2(t) = P(t) + \mathbf{w}_2.$$

Call $\beta_1, \beta_2, \beta_3$ respectively the normal speeds of the three sides $\overline{P_1 P}$, $\overline{P_2 P}$ and $\overline{P_1 P_2}$. Knowing the velocity $\dot{P} = dP/dt$, these are computed by

$$\begin{aligned} \beta_1 &= \langle \dot{P}, \mathbf{w}_1^\perp \rangle, \quad \beta_2 = \langle \dot{P}, \mathbf{w}_2^\perp \rangle, \\ \beta_3 &= \left\langle \dot{P}, \frac{(\mathbf{w}_1 + \mathbf{w}_2)^\perp}{|\mathbf{w}_1 + \mathbf{w}_2|} \right\rangle = \frac{\beta_1 + \beta_2}{|\mathbf{w}_1 + \mathbf{w}_2|}. \end{aligned} \tag{10.7}$$

During the time interval $[\tau, \tau+\delta]$, the change in the total effort is thus computed as

$$\begin{aligned}\widehat{\mathcal{E}}(t)-\mathcal{E}(t) = & \ \varepsilon\Big(|\mathbf{w}_1+\mathbf{w}_2|\,E(\beta_3)-E(\beta_1)-E(\beta_2)+\mathcal{O}(1)\cdot(\delta+\varepsilon)\Big)+o(\varepsilon)\\ = & \ \varepsilon\left(|\mathbf{w}_1+\mathbf{w}_2|\,E\left(\frac{\beta_1+\beta_2}{|\mathbf{w}_1+\mathbf{w}_2|}\right)-E(\beta_1)-E(\beta_2)\right)+\mathcal{O}(1)\cdot\delta\varepsilon+o(\varepsilon).\end{aligned} \tag{10.8}$$

We claim that, for ε, δ sufficiently small, the right hand side of (10.8) is strictly negative. Indeed, set

$$\overline{\beta} \doteq \frac{\beta_1+\beta_2}{2} > -1 \qquad \lambda \ \doteq \ \frac{2}{|\mathbf{w}_1+\mathbf{w}_2|} > 1. \tag{10.9}$$

Notice that the first inequality follows from the assumption that at least one of the normal speeds β_1, β_2 is strictly larger than -1. The second inequality is trivially true because $\mathbf{w}_1, \mathbf{w}_2$ are non-parallel unit vectors. By (10.9) and the convexity of effort function $E(\beta) \doteq \max\{0,\, 1+\beta\}$ it now follows

$$\begin{aligned}\frac{1}{2}\left[|\mathbf{w}_1+\mathbf{w}_2|\,E\left(\frac{\beta_1+\beta_2}{|\mathbf{w}_1+\mathbf{w}_2|}\right)-E(\beta_1)-E(\beta_2)\right] &= \frac{1}{\lambda}E\left(\lambda\cdot\frac{\beta_1+\beta_2}{2}\right)-\frac{E(\beta_1)+E(\beta_2)}{2}\\ &< E\left(\frac{\beta_1+\beta_2}{2}\right)-\frac{E(\beta_1)+E(\beta_2)}{2}\leq 0.\end{aligned} \tag{10.10}$$

By choosing $0<\varepsilon<<\delta$ sufficiently small in (10.8), our claim is proved.

By the above analysis, for $\tau-\varepsilon \leq t \leq \tau+\delta+\varepsilon$ we have the bounds

$$\begin{cases}\widehat{\mathcal{E}}(t)-\mathcal{E}(t) \leq \ -c_1\varepsilon \text{ for } t\in[\tau,\tau+\delta],\\ \widehat{\mathcal{E}}(t)-\mathcal{E}(t) \leq \ \ c_2\varepsilon \quad \text{for } t\in[\tau-\varepsilon,\tau]\cup[\tau+\delta,\tau+\varepsilon+\delta],\end{cases} \tag{10.11}$$

for suitable constants $c_1, c_2 > 0$.

3. In this step we construct a further set $\widetilde{\Omega}(t)$, adding a perturbation in a neighborhood of the point $Q(\tau) = \gamma_2(\tau, \bar{s})$ where the boundary is smooth (see Figs. 13 and 14, right). At time $t \in [\tau-\varepsilon,\, \tau+\delta+\varepsilon]$, this perturbed boundary is described by

$$\widetilde{\gamma}_2(t,s) = \gamma_2(t,s) + \sigma(t)\varphi_\rho(s-\bar{s})\mathbf{n}_3, \tag{10.12}$$

where $\mathbf{n}_3$ is the unit normal vector at (10.3), while $\varphi_\rho(s) = \varphi(s/\rho)$ is the function considered at (7.1). Assuming

$$0 < \varepsilon << \rho << \delta << 1, \tag{10.13}$$

by a suitable choice of $\sigma(\cdot)$ and $\rho > 0$, we claim that this motion $\widetilde{\Omega}$ is admissible and yields a strictly lower cost than Ω.

4. To compare the effort of the two strategies $\widetilde{\Omega}(\cdot)$ and $\widehat{\Omega}(\cdot)$, we observe that along the boundary arc γ_2 by continuity one has

$$\begin{aligned}\big|\mathbf{n}(t,s)-\mathbf{n}_3\big| &= o(t-\tau)+o(s-\bar{s}),\\ \big|\gamma_{2,s}(t,s)-1\big| &= o(t-\tau)+o(s-\bar{s}).\end{aligned} \tag{10.14}$$

Recalling (7.2), by (10.12) the difference in the effort of the two strategies in a neighborhood of $Q(\tau)=\gamma_2(\tau,\bar s)$ can be estimated by

$$\int_{\bar s-\rho}^{\bar s+\rho}\bigl[\widetilde E(t,s)-E(t,s)\bigr]\,ds=\frac{\rho}{2}\dot\sigma(t)+\rho\Bigl(\bigl|\sigma(t)\bigr|+\bigl|\dot\sigma(t)\bigr|\Bigr)\cdot o(\rho+\varepsilon+\delta). \tag{10.15}$$

We now define $\sigma:[\tau-\varepsilon,\tau+\delta+\varepsilon]\mapsto\mathbb{R}$ to be the unique piecewise affine function such that

$$\begin{cases}\sigma(\tau-\varepsilon)\ =\sigma(\tau+\delta+\varepsilon)=0,\\ \dot\sigma\ =\ -3c_2\varepsilon/\rho \text{ if } t\in[\tau-\varepsilon,\tau],\\ \dot\sigma\ =\ c_1\varepsilon/2\rho \text{ if } t\in[\tau,\tau+\delta],\\ \dot\sigma\ =\ -c_3\varepsilon/\rho \text{ if } t\in[\tau+\delta,\tau+\delta+\varepsilon],\end{cases} \tag{10.16}$$

for some constant $c_3>2c_2$, uniquely determined by the terminal condition $\sigma(\tau+\delta+\varepsilon)=0$.

Using (10.15), one now checks that, with the choice of the constants ε,δ,ρ as in (10.13), one has $\widetilde{\mathcal{E}}(t)\le\mathcal{E}(t)=M$ for all t. Hence this new strategy is admissible.

5. It remains to check that the perturbed strategy $\widetilde\Omega$ yields a strictly lower total cost. Since the perturbation is effective only over the time interval $[\tau-\varepsilon,\tau+\delta+\varepsilon]$, we have to show that

$$\int_{\tau-\varepsilon}^{\tau+\delta+\varepsilon}\Bigl[\mathcal{L}^2\bigl(\Omega(t)\bigr)-\mathcal{L}^2\bigl(\widetilde\Omega(t)\bigr)\Bigr]dt>0. \tag{10.17}$$

At any time t, the triangular region with vertices P,P_1,P_2 has area $\le\varepsilon^2$.

Moreover, comparing the portions of the sets $\Omega(t)$ and $\widetilde\Omega(t)$ contained in a suitable ball $B(Q,r)$ centered at the point $Q=\gamma_2(\tau,\bar s)$, we find

$$\mathcal{L}^2\bigl(\Omega(t)\cap B(Q,r)\bigr)-\mathcal{L}^2\bigl(\widetilde\Omega(t)\cap B(Q,r)\bigr)=\frac{\rho}{2}\Bigl(\sigma(t)+o\bigl(\sigma(t)\bigr)\Bigr). \tag{10.18}$$

We now observe that

- On the interval $[\tau-\varepsilon,\tau]$, one has $\sigma(t)=\mathcal{O}(1)\cdot\varepsilon^2/\rho$.
- On the middle interval, choosing $\varepsilon>0$ much smaller than the other parameters, we have

$$\begin{aligned}\int_\tau^{\tau+\delta}\frac{\rho}{2}\Bigl(\sigma(t)+o\bigl(\sigma(t)\bigr)\Bigr)&\ge\frac{\rho}{2}\int_\tau^{\tau+\delta}\Bigl[\mathcal{O}(1)\cdot\frac{\varepsilon^2}{\rho}+\frac{c_1\varepsilon}{3\rho}(t-\tau)\Bigr]dt\\&=\frac{\rho}{2}\Bigl[\mathcal{O}(1)\cdot\frac{\delta\varepsilon^2}{\rho}+c_1\frac{\varepsilon\delta^2}{6\rho}\Bigr]>0.\end{aligned}$$

- On the final interval $[\tau-\varepsilon,\tau+\delta,\tau+\delta+\varepsilon]$, one has $\sigma(t)\ge0$.

Combining the above estimates we conclude that $J(\widetilde\Omega)<J(\Omega)$, showing that the original strategy was not optimal. □

11. Optimality Conditions at Boundary Points

Theorem 11.1. *Let $t\mapsto\Omega(t)$ be an optimal strategy for the minimization problem (OP), with $\kappa_1>0$. Assume that, for t in a neighborhood of τ, the boundary $\partial\Omega(t)$*

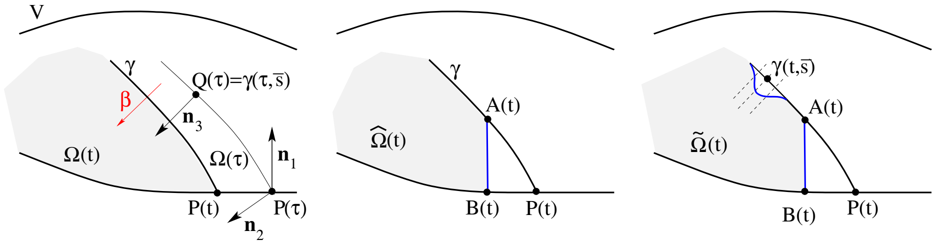

FIGURE 16 Left: the original sets $t \mapsto \Omega(t)$. Center: the sets $\widehat{\Omega}(t)$ obtained by removing the triangular region $\widehat{PAB}$. Right: the sets $\widetilde{\Omega}(t)$ obtained by further perturbing the boundary in a neighborhood of the point $Q(\tau)$.

contains a $\mathcal{C}^1$ arc $\gamma = \gamma(t,s)$ which meets the boundary ∂V at one of its endpoints, say at $\gamma(t,0)$. Then

(i) *either the junction is perpendicular,*

(ii) *or else at the junction point the control effort vanishes, i.e., the inward normal speed is $\beta(t,s) = -1$.*

Proof. The main argument is similar to the one used in the proof of Theorem 10.1. Assuming that the intersection is not perpendicular and the control effort does not vanish, we first perform a perturbation $\widehat{\Omega}$ which decreases the length of the boundary, so that the total effort is strictly reduced: $\widehat{\mathcal{E}}(t) < M$, for $t \in [\tau, \tau+\delta]$. We then use the additional available effort to remove from $\widehat{\Omega}(t)$ a region in a neighborhood of some other point $Q = \gamma(\tau, \bar{s})$. This will yield an admissible motion $t \mapsto \widetilde{\Omega}(t)$ with strictly smaller total cost.

1. To fix ideas, assume that, near the boundary of V, the curve $s \mapsto \gamma(t,s)$ is parameterized by arc-length, so that $\gamma(t,0) \in \partial V$, $\gamma(t,s) \in \partial\Omega(t)$ for $s \geq 0$ small.

As shown in Fig. 16, left, consider the points $P(\tau) = \gamma(\tau,0)$ and choose some other point $Q(\tau) = \gamma(\tau,\bar{s})$ with $\bar{s} > 0$. The following notation will be used:

- $\mathbf{n}_1$ is the inward pointing, unit vector, perpendicular to the boundary ∂V at $P(\tau)$.
- $\mathbf{n}_2 = \frac{\gamma_s(\tau,0)^\perp}{\left|\gamma_s(\tau,0)\right|}$ is the unit vector perpendicular to γ at $P(\tau)$.
- $\mathbf{n}_3$ is the inward pointing, unit vector, perpendicular to the boundary $\partial\Omega(\tau)$ at $Q(\tau)$.

2. Let Ω be an optimal strategy, and assume, by contradiction, that the intersection at $P(\tau)$ is not perpendicular. To derive a contradiction, we first consider the case where $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle < 0$, so that the tangent cone to $\Omega(\tau)$ at $P(\tau)$ has an opening $\theta < \pi/2$.

As shown in Fig. 16, center, given $0 < \varepsilon << \delta$, for $t \in [\tau-\varepsilon, \tau+\delta+\varepsilon]$ we consider the points

$$A(t) = \begin{cases} \gamma\big(t,\, t-\tau+\varepsilon\big) & \text{if } t \in [\tau-\varepsilon,\, \tau], \\ \gamma(t,\varepsilon) & \text{if } t \in [\tau,\, \tau+\delta], \\ \gamma\big(t,\, \tau+\delta+\varepsilon-t\big) & \text{if } t \in [\tau+\delta,\, \tau+\delta+\varepsilon], \end{cases} \tag{11.1}$$

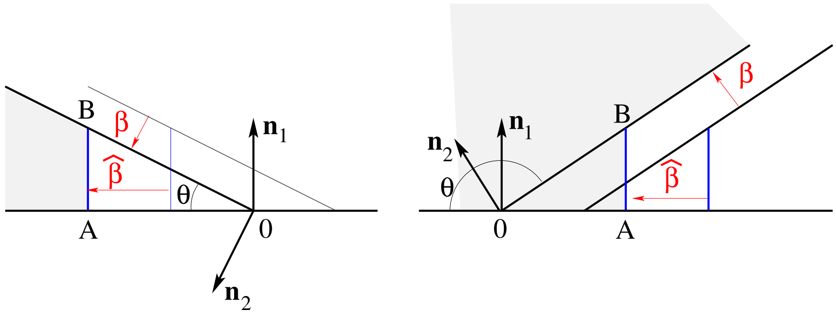

FIGURE 17 Computing the reduction in the total effort as in (11.4). Left: the case where $0 < \theta < \pi/2$. Right: the case where $\pi/2 < \theta < \pi$.

We then define $B(t)$ to be the projection of $A(t)$ on the boundary of V. More precisely, $B(t)$ is the unique point which satisfies

$$B(t) = A(t) + h(t)\mathbf{n}_1, \quad B(t) \in \partial V. \tag{11.2}$$

By the implicit function theorem, such a point is uniquely determined.

Finally, we let $\widehat{\Omega}(t)$ be the set obtained by removing from $\Omega(t)$ the triangular set with vertices $P(t), A(t), B(t)$ (see Fig. 16, center).

Performing a similar analysis as in step 2 of the proof of Theorem 10.1, for $\tau - \varepsilon \le t \le \tau + \delta + \varepsilon$ we obtain the bounds on the total effort:

$$\begin{cases} \widehat{\mathcal{E}}(t) - \mathcal{E}(t) \le -c_1\varepsilon \text{ for } t \in [\tau, \tau+\delta], \\ \widehat{\mathcal{E}}(t) - \mathcal{E}(t) \le \;\; c_2\varepsilon \quad \text{for } t \in [\tau-\varepsilon, \tau] \cup [\tau+\delta, \tau+\varepsilon+\delta], \end{cases} \tag{11.3}$$

for suitable constants $c_1, c_2 > 0$.

Indeed, referring to Fig. 17 left, for $t \in [\tau, \tau+\delta]$ to leading order the difference in the instantaneous effort is measured by $E(\widehat{\beta}) \cdot |AB| - E(\beta) \cdot |0B|$. We have

$$|AB| \;=\; |0B| \sin\theta, \quad \widehat{\beta} \;=\; \frac{\beta}{\sin\theta},$$

$$\begin{aligned} E(\widehat{\beta}) \cdot |AB| - E(\beta) \cdot |0B| &= \max\left\{1 + \frac{\beta}{\sin\theta}, 0\right\} \sin\theta \cdot |0B| - (1+\beta) \cdot |0B| \\ &= \;\max\{\sin\theta + \beta, 0\} \cdot |0B| - (1+\beta) \cdot |0B| \;<\; 0, \end{aligned} \tag{11.4}$$

because we are assuming $\beta > -1$. This yields the first inequality in (11.3). The second one is clear, choosing c_2 sufficiently large.

3. The remainder of the proof is the same as for the previous theorem. Without loss of generality, we can assume that, for $|t-\tau| \le \rho$ and $s \in [\bar{s}-\rho, \bar{s}+\rho]$, the arc γ is parameterized according to a system of coordinates with coordinate axes parallel to $\mathbf{n}_3, \mathbf{n}_3^\perp$. This means:

$$\langle \gamma_t, \mathbf{n}_3^\perp \rangle \;=\; 0 \text{ and } \langle \gamma_s, \mathbf{n}_3^\perp \rangle \;=\; -1 \quad \text{for } |t-\tau| \le \rho, \;\; |s-\bar{s}| \le \rho. \tag{11.5}$$

We now construct a further set $\widetilde{\Omega}(t)$, adding a perturbation in a neighborhood of the point $Q(\tau) = \gamma_2(\tau, \bar{s})$ where the boundary is smooth (see Fig. 16, right). At

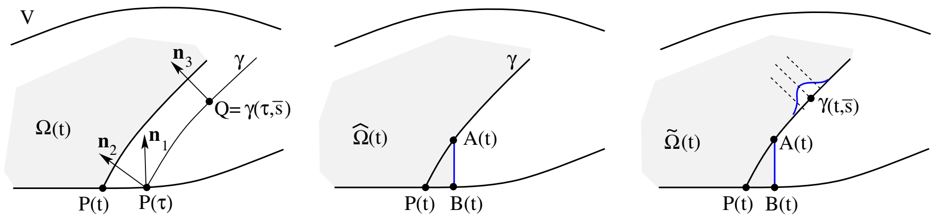

FIGURE 18 The sets $\Omega(t)$, $\widehat{\Omega}(t)$ and $\widetilde{\Omega}(t)$ in the case where the tangent cone to the set $\Omega(\tau)$ at $P(\tau)$ covers an angle $\theta > \pi/2$.

time $t \in [\tau - \varepsilon,\, \tau + \delta + \varepsilon]$, this perturbed boundary is described by

$$\widetilde{\gamma}(t,s) = \gamma(t,s) + \sigma(t)\varphi_\rho(s - \bar{s})\mathbf{n}_3, \tag{11.6}$$

where $\varphi_\rho(s) = \varphi(s/\rho)$ is the function at (7.1), and $\sigma : [\tau - \varepsilon, \tau + \delta + \varepsilon] \mapsto \mathbb{R}$ is the piecewise affine function defined at (10.16). Repeating the analysis in the last two steps of the proof of Theorem 10.1, we conclude that the set motion $t \mapsto \widetilde{\Omega}(t)$ is admissible and yields a strictly smaller total cost.

4. It remains to consider the case where $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle > 0$, so that the tangent cone to $\Omega(\tau)$ at $P(\tau)$ has an opening $\theta > \pi/2$, as shown in Fig. 18. The points $A(t)$, $B(t)$ are defined as before. The only difference is that the set $\widehat{\Omega}(t)$ is obtained by adding to $\Omega(t)$ the triangular region $\widehat{PAB}$. During the time interval $[\tau, \tau + \delta]$ this reduces the length of the boundary $\partial\widehat{\Omega}(t) \cap V$, hence reducing the effort: $\widehat{\mathcal{E}}(t) < \mathcal{E}(t) = M$.

As before, the additional available effort is used to remove from $\widehat{\Omega}(t)$ a region in a neighborhood of some other point $Q = \gamma(\tau, \bar{s})$. This will yield an admissible motion $t \mapsto \widetilde{\Omega}(t)$ with strictly smaller total cost. □

Remark 11.1. In the above construction, the sets $\widetilde{\Omega}(t)$ coincide with $\Omega(t)$ for $t \notin [\tau - \varepsilon, \tau + \delta + \varepsilon]$. Hence the modified strategy would achieve eradication within the same minimum time.

However, the same orthogonality condition can be shown to hold for the minimum time problem. Indeed, for $t \in [\tau, \tau + \delta]$ we can use the additional available effort $M - \widetilde{\mathcal{E}}(t) > 0$ to construct a perturbed strategy $\Omega^\sharp(t)$ so that $\Omega(\tau+\delta) = \Omega^\sharp(\tau')$ for some $t' < \tau + \delta$.

More in detail, consider a new strategy obtained by shifting time: $\Omega^\sharp(t) = \widetilde{\Omega}\big(t^\sharp(t)\big)$. The inward normal speed, at any boundary point $x \in \partial\Omega^\sharp(t^\sharp(t))$ is

$$\beta^\sharp(t^\sharp, x) \;=\; \widetilde{\beta}(t,x) \cdot \frac{dt^\sharp}{dt}.$$

Hence the corresponding total effort is

$$\begin{aligned}\mathcal{E}^\sharp(t^\sharp) &= \int_{\partial\Omega^\sharp(t^\sharp)} \max\left\{1 + \beta^\sharp(t,x),\ 0\right\}\, d\mathcal{H}^1(x) \\ &= \int_{\partial\widetilde{\Omega}(t)} \max\left\{1 + \widetilde{\beta}(t,x)\cdot \frac{dt^\sharp}{dt},\ 0\right\}\, d\mathcal{H}^1(x).\end{aligned} \tag{11.7}$$

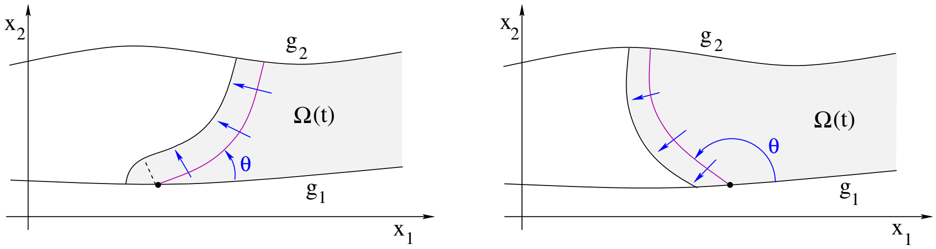

FIGURE 19 Left: if no control is active, an angle $\theta < \pi/2$ is immediately replaced by a perpendicular intersection. Right: An angle $\theta > \pi/2$ can persist, even if no control is active.

We recall that, for $t \in [\tau, \tau + \varepsilon]$, the previous construction yields

$$\widetilde{\mathcal{E}}(t) = \int_{\partial\widetilde{\Omega}(t)} \max\Big\{1 + \widetilde{\beta}(t,x),\ 0\Big\}\, d\mathcal{H}^1(x) \ < \ M.$$

By continuity, we can thus choose a function $t \mapsto t^\sharp(t)$ with

$$\begin{cases} dt^\sharp/dt \ > \ 1 \text{ for } \tau < t < \tau + \varepsilon, \\ dt^\sharp/dt \ = \ 1 \text{ for } t > \tau + \varepsilon, \end{cases} \qquad t^\sharp(\tau) = \tau,$$

and such that the right hand side of (11.7) remains $\leq M$ for $t \in [\tau, \tau + \varepsilon]$. This contradicts the optimality of the original strategy, for the minimum time problem.

Remark 11.2. If the control is not active on a portion of the boundary of $\Omega(t)$ which touches the boundary of V, the angle θ between the two boundaries $\partial\Omega(t)$ and ∂V immediately becomes $\geq \pi/2$ (see Fig. 19). However, orthogonality may fail.

12. Optimal Strategies Determined by the Necessary Conditions

In the remaining sections of this paper we study motions $t \mapsto \Omega(t) \subseteq V$ that satisfy all the necessary conditions derived in the previous sections. Of course, these motions will be natural candidates for optimality. However, we should point out that our necessary conditions are all based on regularity assumptions that may not be always verified. It is only in the setting of Theorem 4.1 that our constructions are guaranteed to yield the globally optimal solutions.

Throughout the following we consider an admissible strategy $t \mapsto \Omega(t) \subseteq V$ such that the following holds.

(A6) For every $t \in [0, T]$, the relative boundary $\partial\Omega(t) \cap V$ is the concatenation of finitely many $\mathcal{C}^2$ arcs

$$\gamma_i(t) = \Big\{\gamma_i(t,\xi);\ \xi \in \big[a_i(t), b_i(t)\big]\Big\}, \quad i = 1, \ldots, N.$$

Each γ_i can be
– either a *free arc*, where the inner normal velocity is $\beta = -1$ at every point,
– or a *controlled arc*, where $\beta > -1$ at every point.

Notice that, along a free arc, no control effort is present. Along this portion of its boundary, the set $\Omega(t)$ thus freely expands with unit speed. On the other hand, along a controlled arc, the evolution of the boundary depends on the pointwise control effort $E = 1 + \beta(t, \xi) > 0$.

Given an initial set $\Omega_0 \subseteq V$, we shall seek admissible set motions $t \mapsto \Omega(t)$ which satisfy the regularity assumptions *(A6)* together with the optimality conditions derived in previous sections.

(A7) At a.e. time $t \in [0, T]$ the following optimality conditions hold.

(i) The area of the set $\Omega(t)$ varies according to

$$\frac{d}{dt}\mathcal{L}^2\big(\Omega(t)\big) = \mathcal{H}^1\big(\partial\Omega(t) \cap V\big) - M. \tag{12.1}$$

(ii) All controlled arcs γ_i have same constant curvature: $\omega_i(t, \xi) = \omega(t)$ for all ξ.

(iii) Controlled arcs join tangentially with free arcs, at their endpoints in the interior of V.

(iv) At a point $Q = \gamma_i(t, \xi)$ where a controlled arc touches the boundary ∂V, either the junction is perpendicular, or else the effort vanishes: $E\big(\beta(t, \xi)\big) = 0$.

Assume that, at some time τ, a configuration

$$\partial\Omega(\tau) = \gamma_1(\tau) \cup \gamma_2(\tau) \cup \cdots \cup \gamma_N(\tau)$$

is given. Based on the above conditions *(i)–(iv)*, for t in a neighborhood of τ the motion $t \mapsto \Omega(t)$ can then be determined as follows (see Fig. 20).

- As a first step, we determine the portion of the boundary $\partial\Omega(t)$ covered by free arcs. Since the inward normal speed is $\beta = -1$, these free arcs $\gamma_i(t)$ can be constructed by shifting the points $\gamma_i(\tau, \xi)$ in the normal direction $\mathbf{n}_i(t, \xi)$, in the amount $\tau - t$.
- As a second step, we construct the controlled arcs $\gamma_j(t)$. These are arcs of circumferences, all with the same radius $r(t)$. At their endpoints they are tangent to the free arcs, and perpendicular to the boundary ∂V. The common radius $r(t)$ is uniquely determined by the area identity (12.1), which yields an ODE for the time derivative $\dot{r}(t)$.

In our approach, a key role is played by maximal extended free arcs.

Definition 12.1. We say that a free arc $\gamma_i(\tau)$ is a *maximal free arc* if, for all times t in a neighborhood of τ one has

$$\gamma_i(t) \;\subseteq\; \Big\{\gamma_i(\tau, \xi) - (t - \tau)\mathbf{n}_i(\tau, \xi);\; \xi \in [a_i(\tau), b_i(\tau)]\Big\}. \tag{12.2}$$

Here $\mathbf{n}_i(t, \xi)$ denotes the unit inner normal vector at the point $\gamma_i(\tau, \xi) \in \partial\Omega(t) \cap V$.

If the maximal free arcs $\gamma_i(\tau)$ can be found, in turn these determine the free arcs $\gamma_i(t)$ at all other times $t \in [0, T]$. In a further step, we can then construct the family of circumferences $\gamma_j(t)$ tangent to the free arcs and perpendicular to the boundary ∂V, thus completely solving the optimal set motion problem (see Fig. 20). In the remaining sections we shall focus in more detail on some aspects of this construction. The next simple example illustrates the main ideas.

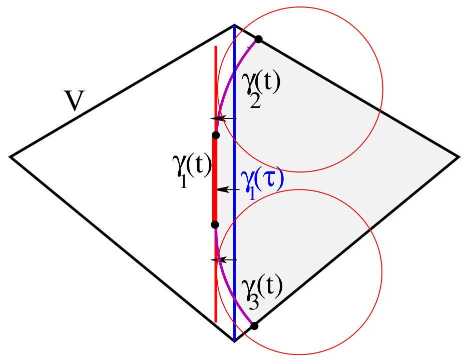

FIGURE 20 Here the segment $\gamma_1(\tau)$ is a maximal free arc. At the later time $t > \tau$, the boundary $\partial\Omega(t) \cap V$ is the union of a free arc γ_1 and the controlled arcs, γ_2, γ_3. We first construct $\gamma_1(t)$ by shifting the free arc $\gamma_1(\tau)$ in the amount $t-\tau$ in the orthogonal direction. The controlled arcs $\gamma_2(t), \gamma_3(t)$ are then two arcs of circumferences, with the same radius $r(t)$, tangent to $\gamma_1(t)$ and perpendicular to ∂V at the endpoints. There is a 1-parameter family of such arcs, depending on the radius r. A unique radius $r(t)$ can be determined by imposing the identity (12.1) on the rate of decrease of the area.

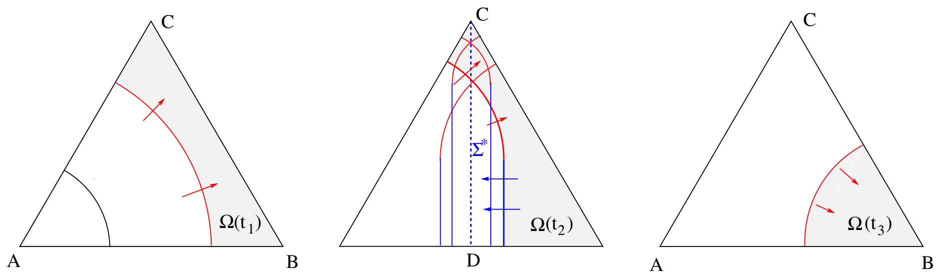

FIGURE 21 The (conjectured) optimal strategy, for the minimum time eradication problem, where the set $\Omega(t)$ is confined within the triangle ABC. Left: the set $\Omega(t_1)$ for $0 < t_1 < t^*$. Center: the set $\Omega(t_2)$ for $t^* < t_2 < T/2$. Right: the set $\Omega(t_3)$ for $T - t^* < t_3 < T$.

Example 12.2. In the x-y plane, let V be an equilateral triangle with sides of unit length, and vertices $A = (-1/2, 0)$, $B = (1/2, 0)$, $C = (0, \sqrt{3}/2)$. In this case, there is no way to satisfy the sufficient conditions stated in (4.3). A possible eradication strategy $t \mapsto \Omega(t)$, $t \in [0, T]$ is the one shown in Fig. 21. For a suitable $t^* \in]0, T/2[$ this strategy can be described as follows.

(i) For $t \in [0, t^*]$ one has $\Omega(t) = V \setminus B\big(A, r(t)\big)$, where by (12.1) the radius $r(t)$ evolves according to

$$\dot{r}(t) = \frac{3}{\pi}\frac{M}{r(t)} - 1. \tag{12.3}$$

(ii) For $t^* < t < T/2$, the boundary $\partial\Omega(t)$ is the union of a vertical segment along the line where $x = x(t) = r(t^*) - (t - t^*)$, and an arc of circumference.

(iii) For $t = T/2$, the set $\Omega(T/2) = \{(x,y) \in V;\ x > 0\}$ is the right half of the triangle.

(iv) For $t \in [T/2, T]$, the set $\Omega(t)$ can be defined by the symmetry property

$$\Omega(t) = \{(x,y) \in V;\ (-x,y) \notin \Omega(T-t)\}.$$

Calling $r^* \doteq r(t^*)$, we observe that (12.3) implies

$$r^* \ < \ \frac{3M}{\pi}. \tag{12.4}$$

Otherwise $\dot r \le 0$ and the set $\Omega(t)$ does not shrink. Moreover, by (iii) it follows

$$\frac{T}{2} - t^* = r^* - \frac{1}{2}. \tag{12.5}$$

For $t^* < t < T/2$, using the rate of decrease of the area (12.1), one finds that the radius of the boundary arc shrinks at the rate

$$\dot r(t) = -1 - \frac{M}{\left(\sqrt{3} - \frac{\pi}{3}\right) r(t)}. \tag{12.6}$$

Indeed, as shown in Fig. 22, we have

$$\begin{aligned}
\mathcal{L}^2(\Omega(t)) \ &= \ \mathcal{L}^2(V) - \left[\frac{\pi}{6} r^2(t) + \frac{\sqrt{3}}{2}\big(x(t) - r(t)\big) \cdot \big[x(t) + r(t)\big]\right] \\
&= \ \mathcal{L}^2(V) - \frac{\sqrt{3}}{2} x^2(t) + \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) r^2(t),
\end{aligned}$$

$$\mathcal{H}^1\Big(V \cap \partial\Omega(t)\Big) = \frac{\pi}{3} r(t) + \sqrt{3}\big(x(t) - r(t)\big).$$

Since $\dot x(t) = -1$, from (12.1) it follows

$$\left(\sqrt{3} - \frac{\pi}{3}\right) r(t) \dot r(t) = \left(\frac{\pi}{3} - \sqrt{3}\right) r(t) - M,$$

which yields (12.6).

We seek a solution to (12.6) with $r(t^*) = r^*$ and $r(T/2) = 0$, where r^* satisfies (12.4) and t^*, T are related by (12.5). Toward this goal, we observe that the solution of (12.6) with initial condition $r(0) = r^\sharp = 3M/\pi$ satisfies the implicit equation

$$t = (r^\sharp - r(t)) + \frac{1}{\lambda} \ln\left(\frac{\lambda r(t) + 1}{\lambda r^\sharp + 1}\right), \qquad \lambda \doteq \frac{1}{M}\left(\sqrt{3} - \frac{\pi}{3}\right). \tag{12.7}$$

Hence the time $t^\sharp$ at which $r(t^\sharp) = 0$ is given by

$$t^\sharp = r^\sharp - \frac{1}{\lambda} \ln\left(\lambda r^\sharp + 1\right).$$

The value of M for which we have the identity $t^\sharp = r^\sharp - \frac{1}{2}$, is

$$M^\sharp = \frac{(3\sqrt{3} - \pi)}{6} \left[\ln\left(\frac{3\sqrt{3}}{\pi}\right)\right]^{-1} \ \approx 0.6805. \tag{12.8}$$

The previous analysis shows that the eradication problem for the equilateral triangle is solvable if $M > M^\sharp$. Notice that $\kappa(V) < M^\sharp < K(V)$, in accordance with Theorem 4.1.

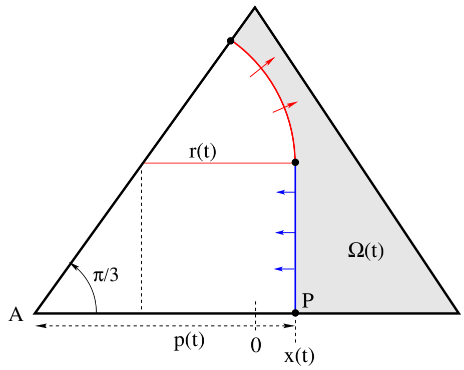

FIGURE 22 Computing the time derivative of the radius $r(t)$, at (12.6).

We conjecture that the above eradication strategy is the optimal one, for the minimum time problem. Indeed, it satisfies all the necessary conditions for optimality derived in the later sections. However, at present we cannot rule out the possibility that some other strategy (possibly with lower regularity) may achieve eradication in a shorter time.

Remark 12.3. If the boundary of the constraining set has a corner C, as in Example 12.2, the best strategy has a curious behavior. Namely, as the boundary of the set $\Omega(t)$ crosses the corner, all the effort gets concentrated near the singular point. The portion of the boundary where the control is active shrinks to a point, then grows again. This remains true even if in (1.3) the constant M is very large. In other words, even if there exist admissible eradication strategies where the set $\Omega(t)$ is monotonically shrinking, these are never optimal if the domain V has corners.

Example 12.2. Let V be the triangle

$$V = \big\{(x_1, x_2);\ 0 < x_1 < K, \quad 0 < x_2 < x_1\big\}.$$

As shown in Fig. 23, left, let the initial contaminated set be

$$\Omega(0) = \left\{(x_1, x_2);\ \frac{K}{2} < x_1 < K, \quad 0 < x_2 < x_1\right\}.$$

As shown in Fig. 23, right. an optimal strategy for *(OP)* proceeds in two stages. For every $t \in [0, T]$, the boundary $\partial\Omega(t) \cap V$ is an arc of circumference $\gamma(t)$ with endpoints $A(t), B(t)$.

- On an initial time interval $t \in [0, t^*]$, this arc is perpendicular to ∂V at $A(t)$, while at $B(t)$ the control effort is zero. This first stage is continued up to the time t^* where the intersection at $B(t)$ becomes perpendicular as well.
- For $t \in [t^*, T]$, $\gamma(t)$ is an arc of circumference centered at the origin, perpendicular to ∂V at both endpoints.

Notice that this motion satisfies all the necessary conditions for optimality proved in the previous sections. The arcs $\gamma(t)$ are uniquely determined by the above conditions, together with the identity $\mathcal{E}(t) = M$ on the total effort (1.9).

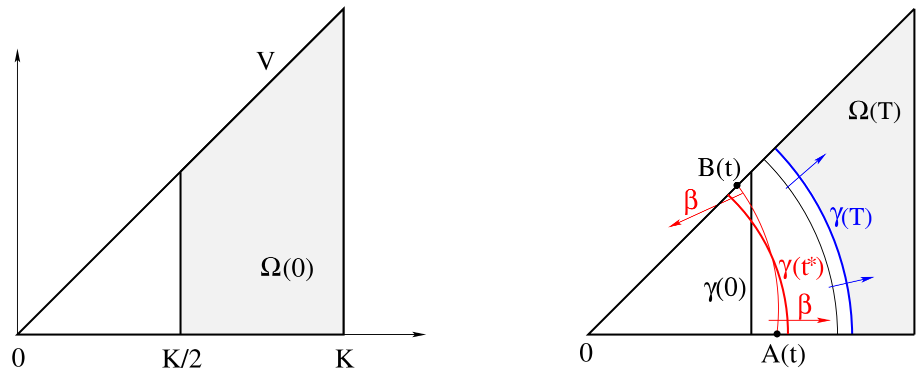

FIGURE 23 The optimal strategy in Example 12.2.

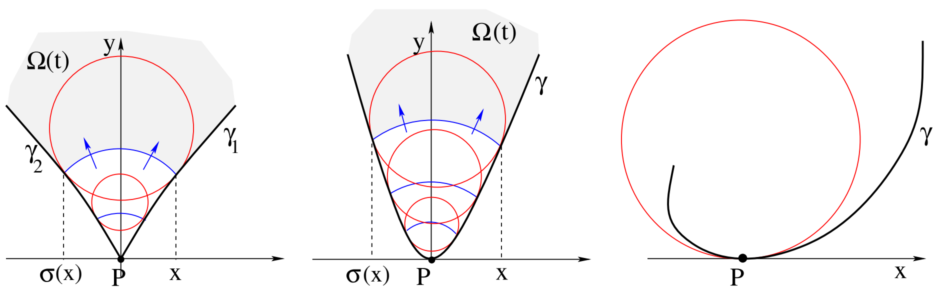

FIGURE 24 In order to construct a family of circumferences that cross perpendicularly a curve γ, it suffices to construct a family of circumferences tangent to γ at two distinct points. Left and center: this is possible near a point P where γ has a corner, or where the curvature is maximum (or minimum). Right: this construction is not possible in case of a spiral-like curve γ, where the curvature is monotone increasing.

13. The Initial Stages of an Optimal Strategy

In this section we focus on the initial stages of an optimal eradication strategy. Assuming that V is an open set with piecewise smooth boundary, we seek an admissible set motion $t \mapsto \Omega(t) \subseteq V$ with $\Omega(0) = V$, which satisfies the assumptions of Corollary 4.1 on some initial interval $t \in [0, T_1]$.

As shown in Fig. 24, for $t > 0$ small the complementary set $V \backslash \Omega(t)$ will be strictly increasing, bounded by an arc of circumference which crosses the boundary ∂V perpendicularly at both endpoints.

The next two propositions show that such a family of circumferences can be constructed in the neighborhood of a boundary point $P \in \partial V$ in two main cases, shown in Fig. 24, right and center:

(i) The boundary ∂V has an outward corner at P.

(ii) P is a point where the curvature of the boundary ∂V attains a local maximum.

Proposition 13.1. *Assume that the boundary ∂V contains two $\mathcal{C}^2$ curves $s \mapsto \gamma_i(s)$, $i = 1, 2$, joining at an angle $\theta \in]0, \pi[$ at a point $P = \gamma_1(0) = \gamma_2(0)$. Then, there* *exists* $s_0 > 0$ *and a family of circumferences which cross perpendicularly the curves* γ_i *at points* $\gamma_1(s)$, $\gamma_2(\sigma(s))$, *for all* $s \in]0, s_0]$

Proof. 1. As shown in Fig. 24, left, using cartesian coordinates x, y we can assume that $P = (0,0)$ and the curves γ_1 and γ_2 are parametrized by

$$\begin{aligned} x \mapsto \gamma_1(x) &= \big(x, \phi_1(x)\big), \quad \text{for } x \in [0, x_0],\\ x \mapsto \gamma_2(x) &= \big(x, \phi_2(x)\big), \quad \text{for } x \in [-x_0, 0], \end{aligned}$$

for some $\mathcal{C}^2$ functions ϕ_1, ϕ_2 which satisfy

$$\phi_1(0) = \phi_2(0) = 0, \quad \phi_1'(0) = -\phi_2'(0) > 0. \tag{13.1}$$

To find a family of circumferences which cross perpendicularly the curves γ_1, γ_2, it clearly suffices to construct a family of circumferences which are tangent to both curves.

Toward this goal, for any $x > 0$ small, consider a circle with radius $r > 0$ tangent to γ_1 at the point $\gamma_1(x)$. We claim that, for a suitable choice of $r = r(x)$, this circle will be also tangent to γ_2 at some point $\gamma_2\big(\sigma(x)\big)$.

2. In the following, tangent and normal vectors to the curves γ_i at the points $\gamma_i(x)$, $i = 1, 2$, are denoted by

$$\mathbf{t}_1(x) = \big(1, \phi_1'(x)\big), \quad \mathbf{n}_1(x) = (-\phi_1'(x), 1), \quad \mathbf{t}_2(x) = \big(1, \phi_2'(x)\big), \quad \mathbf{n}_2(x) = \big(-\phi_2'(x), 1\big).$$

To uniquely determine the value of $\sigma(x)$, we impose that the center of the circumference tangent to the two graphs lies at the intersection of two lines: one parallel to $\mathbf{n}_1(x)$ through $(x, \phi_1(x))$ and the other parallel to to $\mathbf{n}_2\big(\sigma(x)\big)$ through $(\sigma, \phi_1(\sigma))$. These can be parameterized as

$$t \mapsto \big(x, \phi_1(x)\big) + t\big(-\phi_1'(x), 1\big), \quad s \mapsto \big(\sigma, \phi_2(\sigma)\big) + s\big(-\phi_2'(\sigma), 1\big).$$

For any $x > 0$ small, we thus seek a solution $t, s > 0$, $\sigma < 0$, of the system

$$\begin{cases} x - \phi_1'(x)t = \sigma - \phi_2'(\sigma)s,\\ \phi_1(x) + t = \phi_2(\sigma) + s,\\ \big((\phi_1'(x))^2 + 1\big)t^2 = \big(\big(\phi_2'(\sigma)\big)^2 + 1\big)s^2. \end{cases} \tag{13.2}$$

The above identities hold if and only if

$$\begin{cases} s = \dfrac{x - \sigma + \phi_1'(x)\big(\phi_1(x) - \phi_2(\sigma)\big)}{\phi_1'(x) - \phi_2'(\sigma)},\\ t = \dfrac{x - \sigma + \phi_2'(\sigma)\big(\phi_1(x) - \phi_2(\sigma)\big)}{\phi_1'(x) - \phi_2'(\sigma)},\\ \sqrt{(\phi_1'(x))^2 + 1} \cdot t = \sqrt{(\phi_2'(\sigma))^2 + 1} \cdot s. \end{cases} \tag{13.3}$$

Inserting the two expressions for t, s in the third equation, one obtains

$$\begin{aligned} &\sqrt{(\phi_1'(x))^2 + 1} \cdot \big(x - \sigma + \phi_2'(\sigma)\big(\phi_1(x) - \phi_2(\sigma)\big)\\ &\quad = \sqrt{(\phi_2'(\sigma))^2 + 1} \cdot \big(x - \sigma + \phi_1'(x)\big(\phi_1(x) - \phi_2(\sigma)\big). \end{aligned}$$

This holds if and only if

$$\Big(\sqrt{(\phi_1'(x))^2+1}-\sqrt{(\phi_2'(\sigma))^2+1}\Big)(x-\sigma) \\ +\Big(\phi_2'(\sigma)\sqrt{(\phi_1'(x))^2+1}-\phi_1'(x)\sqrt{(\phi_2'(\sigma))^2+1}\Big)\big(\phi_1(x)-\phi_2(\sigma)\big)=0.$$

Multiplying by $\Big(\sqrt{(\phi_1'(x))^2+1}+\sqrt{(\phi_2'(\sigma))^2+1}\Big)$ and dividing by $\big(\phi_1'(x)-\phi_2'(x)\big)$, we eventually obtain the equation

$$F(x,\sigma)=\big(\phi_1'(x)+\phi_2'(\sigma)\big)\big(x-\sigma\big) \\ +\big(\phi_1(x)-\phi_2(\sigma)\big)\Big(\phi_1'(x)\phi_2'(\sigma)-1-\sqrt{(\phi_2'(\sigma))^2+1}\sqrt{(\phi_1'(x))^2+1}\Big)=0. \tag{13.4}$$

By (13.1), at the origin the partial derivatives are computed by

$$\begin{aligned}\frac{\partial F(x,\sigma)}{\partial x}\Big|_{x=\sigma=0} &= (\phi_1'(0)+\phi_2'(0))+\phi_1'(0)\Big(\phi_1'(0)\phi_2'(0)-1-\sqrt{(\phi_2'(0))^2+1}\sqrt{(\phi_1'(0))^2+1}\Big)\\ &= \phi_1'(0)\Big(-\big(\phi_1'(0)\big)^2-1-[(\phi_1'(0))^2+1]\Big) < 0.\end{aligned}$$

$$\begin{aligned}\frac{\partial F(x,\sigma)}{\partial \sigma}\Big|_{x=\sigma=0} &= -\big(\phi_1'(0)+\phi_2'(0)\big)-\phi_2'(0)\Big(\phi_1'(0)\phi_2'(0)-1-\sqrt{(\phi_2'(0))^2+1}\sqrt{(\phi_1'(0))^2+1}\Big)\\ &= \phi_1'(0)\Big(-\big(\phi_1'(0)\big)^2-1-\big[(\phi_1'(0))^2+1\big]\Big) < 0.\end{aligned}$$

By the implicit function theorem, the equation $F(x,\sigma)=0$ thus uniquely determines the function $\sigma(x)$, for $x>0$ small. In fact, $\sigma(0)=0$ and $\frac{d}{dx}\sigma(x)=-1$. □

Next, we consider the case where the boundary ∂V is smooth, and show that a family of circumferences can be constructed in a neighborhood of a point P where the curvature of ∂V attains a local maximum.

Proposition 13.2. *Assume that a portion of the boundary ∂V is a $\mathcal{C}^5$ curve $s\mapsto\gamma(s)$, whose curvature $\omega(s)$ attains a positive, strict local maximum at a point $P=\gamma(\bar s)$. Namely, $\omega'(\bar s)=0$, $\omega''(\bar s)<0$.*

Then there exists $\varepsilon_0>0$ and a family of circumferences γ^ε with radii $r(\varepsilon)$ depending continuously on $\varepsilon\in[0,\varepsilon_0]$, with the following properties. Each arc γ^ε crosses the boundary ∂V perpendicularly at two points $Q_1(\varepsilon),Q_2(\varepsilon)$. Moreover, as $\varepsilon\to 0$ one has the convergence $r(\varepsilon)\to 0$ and $Q_1(\varepsilon),Q_2(\varepsilon)\to P$.

Proof. Working in a cartesian coordinates, we parametrize the boundary γ by $x\mapsto\big(x,\phi(x)\big)$. Without loss of generality we can assume that $\bar s=0$, $\gamma(\bar s)=(0,0)$ and $\phi(0)=\phi'(0)=0$. Higher order derivatives will be denoted by $\phi^{(j)}(x)$, for $j=3,4,5$. Notice that, since $\omega(x)=\frac{\phi''(x)}{\big(1+(\phi'(x))^2\big)^{3/2}}$, the assumption $\phi'(0)=0$ implies

$$\omega(0)=\phi''(0)>0,\quad \omega'(0)=\phi^{(3)}(0)=0, \\ \omega''(0)=\phi^{(4)}(0)-3\big(\phi''(0)\big)^3<0. \tag{13.5}$$

Performing the same calculations as in the proof of Proposition 13.1, to construct the family of circumferences we need to solve (13.2), where now we simply define

$$\begin{cases}\phi_1(x)=\phi(x) & \text{if } x\ge 0,\\ \phi_2(x)=\phi(x) & \text{if } x<0.\end{cases}$$

In view of (13.5) we have the Taylor approximations

$$\phi(x)=\frac{\phi''(0)}{2}x^2+\frac{\phi^{(4)}(0)}{4!}x^4+\mathcal{O}(1)\cdot x^5,\quad \phi'(x)=\phi''(0)x+\frac{\phi^{(4)}(0)}{3!}x^3+\mathcal{O}(1)\cdot x^4,$$

The same expression for $F(x,\sigma)$ used at (13.4) now yields

$$\begin{aligned}F(x,\sigma)&=\left(\phi''(0)\big(x+\sigma\big)+\frac{\phi^{(4)}(0)}{3!}\big(x^3+\sigma^3\big)+\mathcal{O}(1)\cdot(x^4+\sigma^4)\right)(x-\sigma)\\&\quad+\left(\frac{\phi''(0)}{2}\big(x^2-\sigma^2\big)+\frac{\phi^{(4)}(0)}{4!}\big(x^4-\sigma^4\big)+\mathcal{O}(1)\cdot\big(|x|^5+|\sigma|^5\big)\right)\\&\quad\times\left(-2+\frac{\phi''(0)^2}{2}\big(x-\sigma\big)^2+\mathcal{O}(1)\cdot(x^4+\sigma^4)\right)\\&=\frac{\phi^{(4)}(0)}{12}\Big(2\big(x^3+\sigma^3\big)\big(x-\sigma\big)-\big(x^4-\sigma^4\big)\Big)\\&\quad-\frac{(\phi''(0))^3}{4}\big(x^2-\sigma^2\big)\big(x-\sigma\big)^2+R(x,\sigma)\\&=\frac{1}{12}\Big(\phi^{(4)}(0)-3(\phi''(0))^3\Big)\big(x-\sigma\big)^3\big(x+\sigma\big)+R(x,\sigma)\\&=\big(x-\sigma\big)^3\left(\frac{1}{12}\omega''(0)\big(x+\sigma\big)+\frac{R(x,\sigma)}{\big(x-\sigma\big)^3}\right).\end{aligned}$$

Here the remainder R is a $\mathcal{C}^5$ function that satisfies

$$R(x,\sigma)=\mathcal{O}(1)\cdot\big(|x|^5+|\sigma|^5\big).$$

The equation $F(x,\sigma)=0$ is equivalent to

$$\sigma=-x+\frac{12R(x,\sigma)}{\omega''(0)\cdot(x-\sigma)^3}\doteq G(x,\sigma). \tag{13.6}$$

In the region where $0<x<1$, $\sigma\le 0$, one has

$$G(x,-x)\;=\;-x+\mathcal{O}(1)\cdot x^2,\quad \frac{\partial}{\partial\sigma}G(x,\sigma)=\mathcal{O}(1)\cdot\big(|x|+|\sigma|\big).$$

In this setting, $\sigma(x)$ is the fixed point of the map $\sigma\mapsto G(x,\sigma)$. By the contraction mapping theorem, a unique fixed point $\sigma(x)$ exists for all $x>0$ small enough. Moreover, the map $x\mapsto\sigma(x)$ is $\mathcal{C}^5$ and satisfies

$$\big|\sigma(x)+x\big|=\mathcal{O}(1)\cdot x^2.$$

This establishes the existence of a family of circumferences tangent to the graph of ϕ at the points $x,\sigma(x)$. In turn, this yields the existence of circumferences which cross the graph of ϕ perpendicularly at the same points. □

Remark 13.1. In Proposition 13.1, one can still carry out the same construction in the case where the angle satisfies $\theta\in\,]\pi,2\pi[$. Such a strategy will satisfy the necessary conditions, but we do not expect that it will be globally optimal. Similarly, in Proposition 13.2, one can consider the case where the curvature has a local minimum. The above construction will yield a strategy which satisfies the necessary conditions, but it will likely not be optimal.

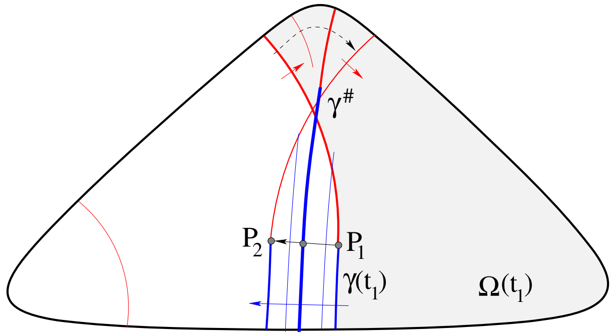

FIGURE 25 The setting considered in Remark 14.1.

14. Maximally Extended Free Arcs

This section is focused on the construction of maximally extended free boundary arcs. Our goal is to derive a system of ODEs that describes these special curves. The starting point is provided by the following remark.

Remark 14.1. Fix ξ and consider a trajectory $t \mapsto x(t,\xi)$, perpendicular to the boundary $\partial\Omega(t)$. As shown in Fig. 25, assume that

(i) For $t \in [t_1, t_2]$, the point $x(t,\xi)$ lies on the free portion of the boundary, where no control effort is present.

(ii) For $i = 1, 2$, the point $P_i = x(t_i, \xi)$ lies at the edge of the arcs where the control is active.

The necessary conditions (6.17) imply

$$Y(t_1,\xi) = Y^*(t_1), \quad Y(t_2,\xi) = Y^*(t_2). \tag{14.1}$$

Calling $\omega(t,\xi)$ the curvature of the boundary $\partial\Omega(t)$ at the point $x(t,\xi)$, and calling $\omega^*(t)$ the curvature of the portion of the boundary $\partial\Omega(t)$ where the control is active, by (14.1) and (9.5) it follows

$$\int_{t_1}^{t_2} \omega(t,\xi)\, dt = \int_{t_1}^{t_2} \omega^*(t)\, dt. \tag{14.2}$$

14.1. The Symmetric Case

As a first step, we consider the case where the domain V is symmetric w.r.t. the y-axis. More precisely (see Fig. 26, left)

$$V = \big\{(x,y);\ 0 < y < g(x)\big\}, \tag{14.3}$$

for some $\mathcal{C}^2$ function $g : \mathbb{R} \mapsto \mathbb{R}$ such that

$$g(x) = g(-x), \quad g(0) > 0, \quad g''(x) < 0 \quad \text{for all } x \in \mathbb{R}.$$

In this setting, a natural guess is that the maximal free boundary $\overline{\gamma}$ should be a vertical segment contained in the axis of symmetry. For notational convenience we shift time, so that at time $t = 0$ the relative boundary is contained in the y-axis: $\Omega(0) = \big\{(x,y);\ 0 < y < g(x),\ x > 0\big\}$.

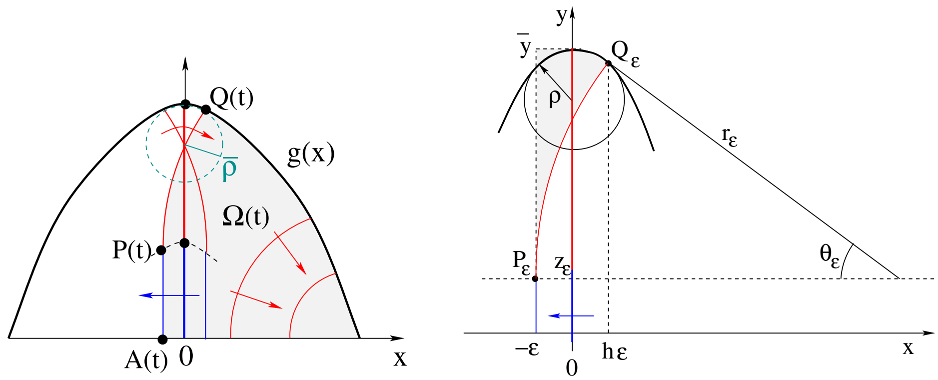

FIGURE 26 Left: the relative boundary $\partial\Omega(t)\cap V$ contains a free arc: the vertical segment with endpoints $A(t), P(t)$, and a controlled arc: the arc of circumference with endpoints $P(t), Q(t)$. Right: computing the active portion of the boundary, at the time $t = 0$ when this boundary is contained in the y-axis.

As a consequence, as shown in Fig. 26, left, at each time t the relative boundary $\partial\Omega(t) \cap V$ will be the union of a free portion: the vertical segment with endpoints $A(t), P(t)$, and a controlled arc: the arc of circumference with endpoints $P(t), Q(t)$. Notice that, for a fixed t, there is a 1-parameter family of circumferences that are tangent to the vertical line $\{x = -t\}$ and cross perpendicularly the boundary ∂V. A unique choice of this circumference is determined by the area identity (12.1).

We observe that, in this symmetric setting, the identity (14.2) is trivially satisfied. Indeed, the curvature of the free arc is $\omega(t,\xi) \equiv 0$. On the other hand, by symmetry we have $t_1(\xi) = -t_2(\xi)$, while $\omega^*(-t) = -\omega^*(t)$ for all t. Hence the right hand side of (14.2) vanishes as well.

Next, we wish to determine the lengths of the arcs AP and PQ at time $t = 0$, when they are both vertical segments on the y-axis. These should depend on:

- The radius of curvature $\overline{\rho}$ of the boundary ∂V at the point Q.
- The bound M on the total effort.

With reference to Fig. 26, right, for any $\varepsilon > 0$, consider the point $Q_\varepsilon = (x_\varepsilon, y_\varepsilon)$ where a circumference tangent to the vertical line $\{x = -\varepsilon\}$ crosses the boundary ∂V perpendicularly. We denote by $P_\varepsilon = (-\varepsilon, z_\varepsilon)$ the point where this circumference has a vertical tangent.

As $\varepsilon \to 0$, the area identity (12.1) yields

$$[\text{area of shaded region}] = \frac{1}{2}(y_\varepsilon - z_\varepsilon)\cdot(\varepsilon + x_\varepsilon) + o(\varepsilon) = M\varepsilon + o(\varepsilon)$$

We now set $x_\varepsilon = h\varepsilon + o(\varepsilon)$ and denote by $\ell \doteq \lim_{\varepsilon\to 0}(y_\varepsilon - z_\varepsilon)$ the length of the free arc at time $t = 0$. Letting $\varepsilon \to 0$, we obtain the identity

$$\frac{\ell}{2}(1 + h) = M. \tag{14.4}$$

Next, we impose the orthogonality condition at the boundary. Setting $\bar{y} = g(0)$ and calling $\rho = -1/g''(0) > 0$ the radius of curvature of the boundary ∂V at the point

$Q = \big(0, g(0)\big)$, recalling that g is an even function, a Taylor expansion yields

$$y = g(x) = \bar{y} - \frac{1}{2\rho}x^2 + o(x^3).$$

Referring to Fig. 26, right, call r_ε the radius of the circumference through P_ε and Q_ε, and $\theta_\varepsilon > 0$ the angle of the corresponding arc between Q_ε and P_ε. By standard trigonometric identities we find

$$r_\varepsilon \sin\theta_\varepsilon = \ell - \frac{h^2\varepsilon^2}{2\rho} + o(\varepsilon^2), \quad r_\varepsilon - r_\varepsilon\cos\theta_\varepsilon = (1+h)\varepsilon + o(\varepsilon), \tag{14.5}$$

$$\frac{1-\cos\theta_\varepsilon}{\sin\theta_\varepsilon}\,\ell = (1+h)\varepsilon + o(\varepsilon). \tag{14.6}$$

On the other hand, the orthogonality condition at Q_ε implies

$$\theta_\varepsilon = \frac{h\varepsilon}{\rho} + o(\varepsilon). \tag{14.7}$$

Combining the two identities (14.6)-(14.7) one obtains

$$\frac{\theta_\varepsilon^2/2}{\theta_\varepsilon}\,\ell = \frac{\theta_\varepsilon}{2}\ell = \frac{h\varepsilon}{2\rho}\,\ell + o(\varepsilon) = (1+h)\varepsilon + o(\varepsilon).$$

Letting $\varepsilon \to 0$ we conclude

$$h\ell = 2\rho(1+h). \tag{14.8}$$

Together with the first identity (14.4), this determines the two constants ℓ, h:

$$h \;=\; \frac{2M}{\ell} - 1, \quad \ell \;=\; M - \sqrt{M^2 - 4\rho M}. \tag{14.9}$$

Notice that the sign in front of the square root is consistent with the fact that, as the radius $\rho \to 0$, the length of the controlled arc also approaches zero:

$$\ell = M\left(1 - \sqrt{1 - \frac{4\rho}{M}}\right) \;\approx\; 2\rho \;\to\; 0.$$

14.2. Maximal Free Arcs: The Non-symmetric Case

If the set V is not symmetric, we do not expect that a maximal free arc $\gamma^\sharp$ should be a straight line. We seek a system of ODEs describing this arc. This will be achieved in several steps.

1. Consider the configuration in Fig. 27. At time $t = 0$ the relative boundary $\partial\Omega(0) \cap V = \gamma^* \cup \gamma^\sharp$, is the union of a controlled arc γ^* and a free arc $\gamma^\sharp$ where the set $\Omega(t)$ expands with unit speed. By a suitable choice of coordinates, we assume that the arc of circumference γ^* is tangent to $\gamma^\sharp$ at the point $P = (0, 0)$, and perpendicular to ∂V at the other endpoint Q. We choose the x-axis so that it is perpendicular to $\gamma^\sharp$ and γ^* at P.

Let $\gamma^\sharp$ be parameterized in terms of arc-length, as $\xi \mapsto \gamma^\sharp(\xi)$, with $\gamma^\sharp(0) = P$, and call $\mathbf{n}(\xi)$ the unit inward normal vector. Then at time $t = \varepsilon > 0$ the corresponding free arc will be parameterized by

$$\gamma^\sharp_\varepsilon(\xi) = \gamma^\sharp(\xi) - \varepsilon\mathbf{n}(\xi). \tag{14.10}$$

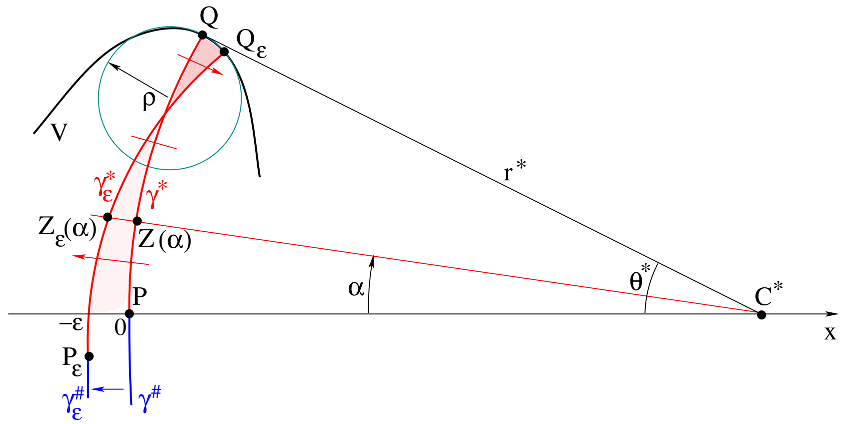

Figure 27 The shaded region yields the signed area swept by the moving arc of circumference for $t \in [0, \varepsilon]$, up to higher order infinitesimals.

In addition, at time ε the controlled arc γ^*_ε will be a portion of circumference with endpoints $Q_\varepsilon \in \partial V$ and $P_\varepsilon = \gamma^*_\varepsilon\big(\xi(\varepsilon)\big)$, for some $\xi(\varepsilon)$. We seek a formula for the time derivatives

$$\dot P = \lim_{\varepsilon\to 0+} \frac{P_\varepsilon - P}{\varepsilon}, \qquad \dot Q = \lim_{\varepsilon\to 0+} \frac{Q_\varepsilon - Q}{\varepsilon}. \tag{14.11}$$

This will rely on three properties:

- The junction at P_ε is tangential.
- The intersection at Q_ε is perpendicular.
- By the identity (12.1), the signed area swept by the moving arc of circumference equals $M - \mathcal{H}^1(\gamma^*)$.

2. To fix ideas, assume that at time $t = 0$ the controlled arc γ^* has center at $C^* = (r^*, 0)$, radius r^* and spans an angle θ^*. As shown in Fig. 27, for every angle $\alpha \in [0, \theta^*]$ let $Z(\alpha)$ and $Z_\varepsilon(\alpha)$ be the intersections of the circumferences $\gamma^*, \gamma^*_\varepsilon$ with the line through C^*, forming an angle α with the x-axis, and call $r^*_\varepsilon(\alpha) = \big|Z_\varepsilon(\alpha) - C^*\big|$. Notice that $\big|Z(\alpha) - C^*\big| = r^*$ for all α.

The signed area swept (in the inward direction) by the moving arc of circumference during the time interval $[0, \varepsilon]$ is computed by

$$A_\varepsilon = \int_0^{\theta^*} \big(r^*_\varepsilon(\alpha) - r^*\big)\, r^* d\alpha + o(\varepsilon) = \varepsilon\big(|\dot Q| - 1\big)\frac{\theta^* r^*}{2} + o(\varepsilon). \tag{14.12}$$

Indeed, the endpoint P_ε moves outward with unit speed, while the other endpoint Q_ε moves inward with speed $\dot Q$.

The area identity (12.1) now yields

$$A_\varepsilon = \varepsilon\big[M - \mathcal{H}^1(\gamma^*)\big] + o(\varepsilon) = \varepsilon\big[M - \theta^* r^*\big] + o(\varepsilon). \tag{14.13}$$

Combining the two above formulas and letting $\varepsilon \to 0$ we obtain

$$M - \theta^* r^* = \big(|\dot Q| - 1\big)\frac{\theta^* r^*}{2}, \qquad |\dot Q| = \frac{2M}{\theta^* r^*} - 1. \tag{14.14}$$

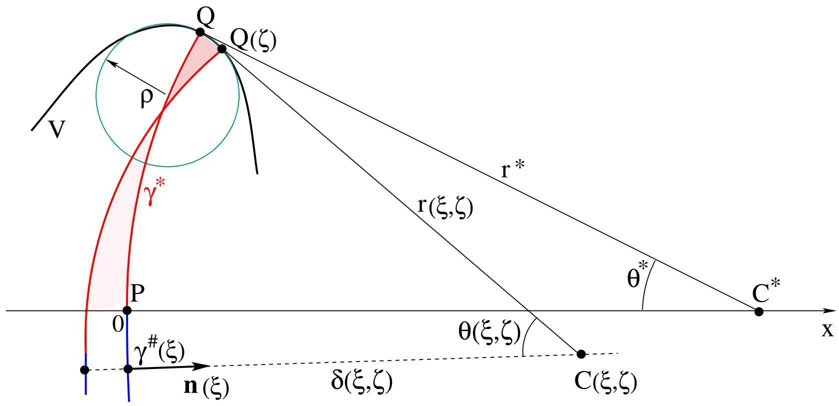

FIGURE 28 The shaded region is the area swept by the moving arc of circumference for $t \in [0, \varepsilon]$.

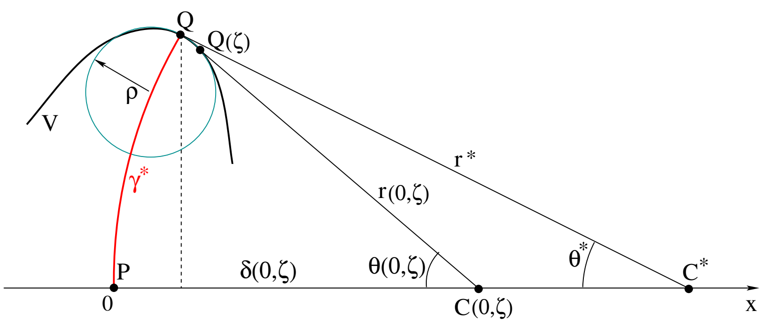

FIGURE 29 Computing the partial derivatives of the distances $r(\xi, \zeta)$ and $\delta(\xi, \zeta)$ w.r.t. ζ.

3. Next, referring to Fig. 28, let the boundary ∂V be parameterized by arc-length as $\zeta \mapsto Q(\zeta)$, with $Q(0) = Q$. For any pair $(\xi, \zeta) \approx (0, 0)$, denote by $C(\xi, \zeta)$ the intersection of

- the perpendicular line to $\gamma^\sharp$ at the point $\gamma^\sharp(\xi)$,
- the tangent line to ∂V at the point $Q(\zeta)$.

The distances between $C(\xi, \zeta)$ and the points $\gamma^\sharp(\xi)$, $Q(\zeta)$ will be denoted respectively by

$$\delta(\xi, \zeta) = \big|C(\xi, \zeta) - \gamma^\sharp(\xi)\big|, \quad r(\xi, \zeta) = \big|C(\xi, \zeta) - Q(\zeta)\big|. \tag{14.15}$$

When $\xi = \zeta = 0$ we clearly have

$$\delta(0, 0) = |C^* - P| = r^* = \big|C^* - Q\big| = r(0, 0). \tag{14.16}$$

To compute the derivatives of these distances w.r.t. ξ and ζ, we let $\omega = 1/\rho$ be the curvature of the boundary ∂V at the point Q, and let $\omega^\sharp$ be the curvature of $\gamma^\sharp$ at the point P. At the point $C(\xi, \zeta)$, the angle formed by the lines through $\gamma^\sharp(\xi)$ and $Q(\zeta)$ is thus

$$\theta(\xi, \zeta) = \theta^* + \omega^\sharp \xi + \omega \zeta + o(\xi) + o(\zeta). \tag{14.17}$$

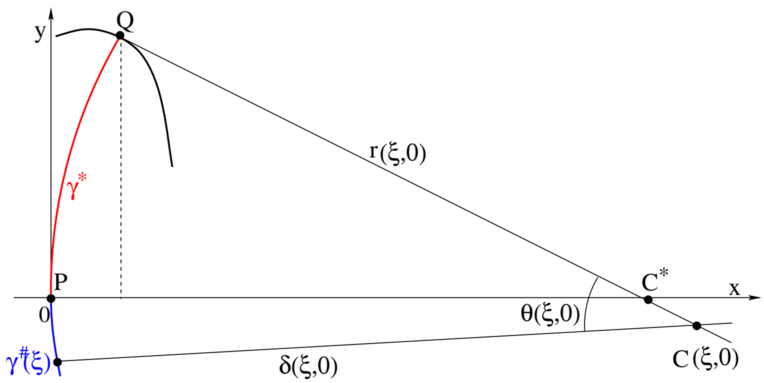

FIGURE 30 Computing the partial derivatives of the distances $r(\xi,\zeta)$ and $\delta(\xi,\zeta)$ w.r.t. ξ. Here $P=(0,0)$, $C^*=(r^*,0)$.

At $\xi=\zeta=0$, recalling that the point $Q(\zeta)$ is parameterized by arc-length, by elementary trigonometric identities (see Fig. 29) one finds

$$\frac{\partial}{\partial\zeta}r(0,\zeta) = -r^*\omega\,\cot\theta^* - 1. \tag{14.18}$$

Moreover, observing that

$$\big|Q-C(0,\zeta)\big|\,\sin\theta(0,\zeta) = r^*\sin\theta^* + o(\zeta),\qquad \frac{\partial}{\partial\zeta}\theta(0,\zeta)=\omega,$$

at $\xi=\zeta=0$ we obtain

$$\frac{\partial}{\partial\zeta}\delta(0,\zeta) \;=\; \frac{\partial}{\partial\zeta}\big[r^*\sin\theta^*\cot\theta(0,\zeta)\big] = r^*\sin\theta^*\cdot\frac{-\omega}{\sin^2\theta^*} = -\frac{r^*\omega}{\sin\theta^*}. \tag{14.19}$$

Next, referring to Fig. 30, we observe that the point $C(\xi,0)$ lies at the intersection of two straight lines with equations

$$y \;=\; -\tan\theta^*\,(x-r^*),\qquad y\;=\;-\xi+\omega^\sharp\xi\,x+o(\xi)\cdot x. \tag{14.20}$$

At the intersection point, this yields

$$\tan\theta^*(x-r^*)+(\omega^\sharp r^*-1)\xi = \omega^\sharp\xi(x-r^*)+o(\xi)\cdot x.$$

Observing that $x-r^*=\mathcal{O}(1)\cdot\xi$, we obtain $C(\xi,0)=\big(x(\xi),y(\xi)\big)$, where

$$x(\xi)\;=\;r^*+\frac{1-\omega^\sharp r^*}{\tan\theta^*}\,\xi+o(\xi),\qquad y(\xi)\;=\;(\omega^\sharp r^*-1)\,\xi+o(\xi). \tag{14.21}$$

Using (14.21), the partial derivatives of δ and r w.r.t. ξ at $\xi=\zeta=0$ are computed by

$$\frac{\partial}{\partial\xi}\delta(\xi,0)=\frac{\partial}{\partial\xi}x(\xi)=\frac{1-\omega^\sharp r^*}{\tan\theta^*},\qquad \frac{\partial}{\partial\xi}r(\xi,0)=\frac{1}{\cos\theta^*}\,\frac{\partial}{\partial\xi}x(\xi)=\frac{1-\omega^\sharp r^*}{\sin\theta^*}. \tag{14.22}$$

4. For $t\approx 0$, denote by

$$Q(t)=Q\big(\zeta(t)\big),\qquad P(t)=\gamma^\sharp\big(\xi(t)\big)-t\,\mathbf{n}\big(\xi(t)\big),$$

the endpoints of the controlled arc of circumference $\gamma^*(t)$ at time t. By the previous analysis we have

$$\frac{d}{dt}\zeta(t)\Big|_{t=0} = |\dot{Q}| = \frac{2M}{\theta^* r^*} - 1.$$

We now use the identity

$$r\big(\xi(t),\zeta(t)\big) \;=\; t + \delta\big(\xi(t),\zeta(t)\big) \tag{14.23}$$

which is valid for all t, to compute the derivative $d\xi(t)/dt$. Inserting (14.18), (14.19) and (14.22) in (14.23), we obtain

$$\frac{\partial r}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial r}{\partial \zeta}\frac{d\zeta}{dt} = 1 + \frac{\partial \delta}{\partial \xi}\frac{d\xi}{dt} + \frac{\partial \delta}{\partial \zeta}\frac{d\zeta}{dt},$$

$$\begin{aligned}\frac{d}{dt}\xi(t)\Big|_{t=0} &= \left[\frac{\partial r}{\partial \xi} - \frac{\partial \delta}{\partial \xi}\right]^{-1}\left(1 + \frac{\partial \delta}{\partial \zeta}\frac{d\zeta}{dt} - \frac{\partial r}{\partial \zeta}\frac{d\zeta}{dt}\right)\\ &= \left[(1-\omega^\sharp r^*)\frac{1-\cos\theta^*}{\sin\theta^*}\right]^{-1}\cdot\left[1 + \left(-\frac{r^*\omega}{\sin\theta^*} + r^*\omega\cot\theta^* + 1\right)\left(\frac{2M}{\theta^* r^*} - 1\right)\right].\end{aligned} \tag{14.24}$$

5. We are now ready to derive a second order system of ODEs satisfied by the maximal free arc $\gamma^\sharp$. As before, let $\xi \mapsto \gamma^\sharp(\xi)$ be an arc-length parameterization, with $\gamma^\sharp(0) = P$. For any $\xi > 0$, let $t_1(\xi) < 0 < t_2(\xi)$ be the times where the points

$$P_i(\xi) = \gamma^\sharp(\xi) - \big(t_i(\xi) - \tau\big)\mathbf{n}(\xi)$$

lie at the junction between the active and the free arcs (see Fig. 25).

To derive an ODE for $\gamma^\sharp$, let $\omega^\sharp(\xi)$ be the curvature of $\gamma^\sharp$ at the point $\gamma^\sharp(\xi)$, and denote by $\omega(t)$ the curvature of the controlled arc at time t. Differentiating (14.2) w.r.t. ξ we obtain

$$\frac{\omega^\sharp(\xi)}{1+t_2\omega^\sharp(\xi)}\frac{dt_2(\xi)}{d\xi} - \frac{\omega^\sharp(\xi)}{1+t_1\omega^\sharp(\xi)}\frac{dt_1(\xi)}{d\xi} = \omega\big(t_2(\xi)\big)\frac{dt_2(\xi)}{d\xi} - \omega\big(t_1(\xi)\big)\frac{dt_1(\xi)}{d\xi}. \tag{14.25}$$

Here the time derivatives $dt_i(\xi)/d\xi$ can be computed by inverting (14.24). Notice that this is a second order, highly nonlinear ODE. Indeed, this is an implicit equation that can be solved for the curvature $\omega(\xi)$ in terms of the other variables.

Remark 14.2. At a time τ when the free arc is maximal, assume that the boundary $\partial\Omega(\tau) \cap V$ is the union of the free arc $\gamma^\sharp$ together with a controlled arc of circumference γ^* with endpoints P, Q. From the necessary conditions for optimality we deduce:

(i) The arc of circumference γ^* is *perpendicular* to the boundary ∂V at the point Q.

(ii) The two arcs $\gamma^\sharp$ and γ^* have a *second order tangency* at the point P.

Indeed, (i) follows from Theorem 11.1. The first order tangency condition in (ii) follows from Theorem 10.1. Both of this facts remain valid at all times. On the other hand, the second order tangency is a unique property of maximal arcs. Indeed, let

$$P(t,\xi) = \gamma^\sharp(\xi) - (t-\tau)\mathbf{n}(\xi)$$

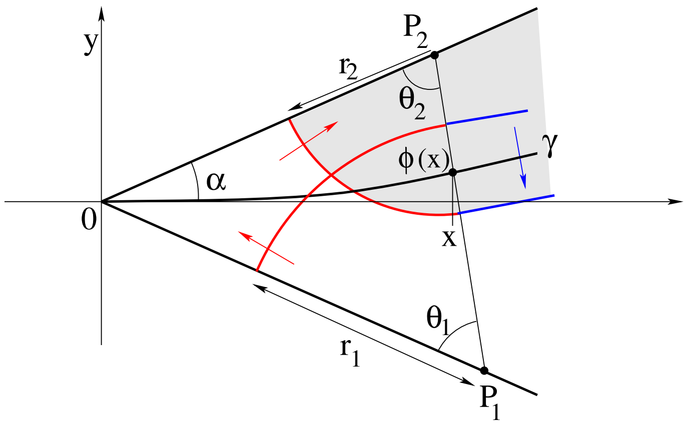

Figure 31 Evolution of the free and the controlled portions of the boundary $\partial\Omega(t)$. Here $y = \phi(x)$ is the equation of the maximal free interface.

denote a point on the free arc for $t \in \big[t_1(\xi),\, t_2(\xi)\big]$, which coincides with the endpoint of the controlled arc for $t = t_1(\xi)$ and $t = t_2(\xi)$, with

$$t_1(\xi) < \tau < t_2(\xi), \qquad \lim_{\xi\to\bar\xi-} t_1(\xi) = \lim_{\xi\to\bar\xi-} t_1(\xi) = \tau.$$

By (14.2) the curvatures are related by

$$\frac{1}{t_2(\xi)-t_1(\xi)}\cdot\int_{t_1(\xi)}^{t_2(\xi)} \omega(t,\xi)\,dt = \frac{1}{t_2(\xi)-t_1(\xi)}\cdot\int_{t_1(\xi)}^{t_2(\xi)} \omega^*(t)\,dt.$$

Taking the limit as $\xi \to \bar\xi-$ one obtains (ii).

15. The Free Interface Near a Corner Point

In the x-y plane let the motion take place within a set V having a corner point. After a change of coordinates, as shown in Fig. 31 we assume

$$V = \big\{(x,y);\ |y| \le x\tan\alpha\big\}, \tag{15.1}$$

for some $0 < \alpha < \pi/2$. Our goal is to describe the maximal free interfaces.

To fix ideas, assume that at time $t = 0$ a smooth free interface

$$y = \phi(x) \tag{15.2}$$

is given. Then for small times $t \in [-\delta,\delta]$ we can uniquely determine a family a circumferences, tangent to the free curves at points $\big(x(t),\, \phi(x(t))\big)$. Indeed, the radii of these circumferences will be determined by the tangency requirement, together with the swept area equation (12.1). In addition, the optimality conditions require that the identities (14.2) be satisfied.

Our goal is to show that there exists a 1-parameter family of admissible maximal interfaces, all satisfying $\phi(0) = \phi'(0) = 0$.

As shown in Fig. 31, let $x \mapsto \gamma(x) = (x, \phi(x))$ be a parameterization of the maximal free interface γ, with $\phi(0) = 0$. For each $x > 0$, call $P_1(x)$, $P_2(x)$ the points

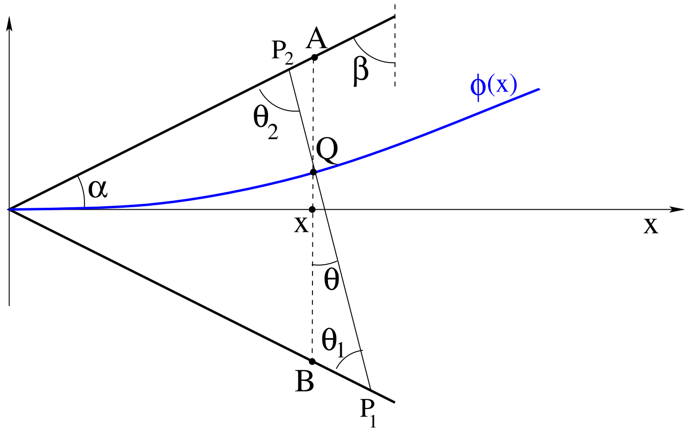

FIGURE 32 Deriving the identities in (15.6).

where the straight line perpendicular to γ at the point $\gamma(x) = (x, \phi(x))$ crosses the lower and the upper boundary of V, respectively. Moreover, we call α the angle between the x-axis and the upper boundary of V, and set $\beta = \frac{\pi}{2} - \alpha$

Call

$$\theta(x) = \arctan \phi'(x), \tag{15.3}$$

so that

$$\sqrt{1 + [\phi'(x)]^2} = \frac{1}{\cos\theta(x)}. \tag{15.4}$$

The angles at $P_1(x)$ and $P_2(x)$ are then computed by

$$\begin{cases} \theta_1(x) = & \beta - \theta(x), \\ \theta_2(x) = & \beta + \theta(x). \end{cases} \tag{15.5}$$

With reference to Fig. 32, according to the law of sines one has

$$\frac{|P_1 - Q|}{\sin(\pi - \beta)} = \frac{|B - Q|}{\sin(\beta - \theta)}, \quad \frac{|P_2 - Q|}{\sin\beta} = \frac{|A - Q|}{\sin(\pi - \beta - \theta)}. \tag{15.6}$$

Let $t_1 = t_1(x) < 0$ be the time when the center of the active circumference is at $P_1(x)$, and call $r_1 = r_1(x)$ the radius of this circumference. Similarly, let $t_2 = t_2(x) > 0$ be the time when the center of the active circumference is at $P_2(x)$, and let $r_2 = r_2(x)$ be the radius. Using (15.6) we obtain

$$\begin{cases} r_1(x, t_1) = & |P_1 - Q| - t_1(x) = \dfrac{\sin\beta\big(x\tan\alpha + \phi(x)\big)}{\sin\big(\beta - \theta(x)\big)} - t_1, \\ r_2(x, t_2) = & |P_2 - Q| + t_2(x) = \dfrac{\sin\beta\big(x\tan\alpha - \phi(x)\big)}{\sin\big(\beta + \theta(x)\big)} + t_2. \end{cases} \tag{15.7}$$

Imposing the area identity (12.1), we obtain (see Fig. 33)

$$\frac{1}{2}\theta_2(r_2 + \varepsilon)^2 + (x_\varepsilon - x)\sqrt{1 + [\phi'(x)]^2}\, r_2 - (\theta_{2,\varepsilon} - \theta_2)\frac{r_2^2}{2} \;\approx\; \theta_{2,\varepsilon}\frac{1}{2}r_{2,\varepsilon}^2 + M\varepsilon.$$

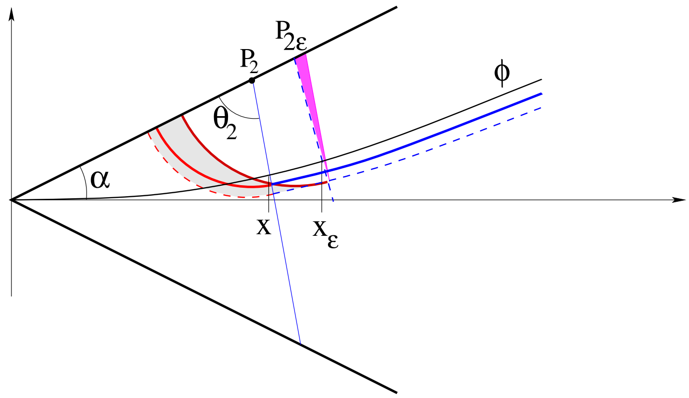

FIGURE 33 During the time interval $[t,\ t+\varepsilon]$ the center of the active circumference moves from P to P_ε. The shaded region has area $M\varepsilon$.

Denoting the derivatives w.r.t. time by an upper dot, letting $\varepsilon \to 0$ we obtain

$$\theta_2 r_2 + \dot{x} r_2 \sqrt{1+[\phi'(x)]^2} - \dot{\theta}_2 \frac{r_2^2}{2} = \frac{1}{2}\dot{\theta}_2 r_2^2 + \theta_2 r_2 \dot{r}_2 + M. \tag{15.8}$$

Observing that

$$\dot{\theta}_2 = \frac{d}{dx}\Big[\arctan \phi'(x)\Big]\dot{x} = \frac{\phi''(x)}{1+[\phi'(x)]^2}\,\dot{x},$$

and dividing both sides of (15.8) by r_2, one finds

$$\theta_2(\dot{r}_2 - 1) + \frac{M}{r_2} = \left(\sqrt{1+[\phi'(x)]^2} - \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_2\right)\dot{x}. \tag{15.9}$$

Recalling (15.5) and (15.7), since $\tan\alpha = \cot\beta$ we obtain

$$\dot{r}_2 - 1 = \left[\frac{\sin\beta\big(\cot\beta - \phi'(x)\big)}{\sin\theta_2(x)} - \frac{\sin\beta\big(x\cot\beta - \phi(x)\big)}{\sin^2\theta_2(x)}\cos\big(\theta_2(x)\big)\frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\dot{x}. \tag{15.10}$$

Combined with (15.9) and (15.5), this yields

$$\begin{aligned}\frac{M}{r_2} &= \left(\sqrt{1+[\phi'(x)]^2} - \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_2\right)\dot{x}\\ &\quad -(\beta+\theta(x))\left[\frac{\sin\beta\big(\cot\beta-\phi'(x)\big)}{\sin\big(\beta+\theta(x)\big)} - \frac{\sin\beta\big(x\cot\beta-\phi(x)\big)}{\sin^2\big(\beta+\theta(x)\big)}\cos\big(\beta+\theta(x)\big)\frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\dot{x}.\end{aligned} \tag{15.11}$$

Solving for $\dot{x} = dx/dt_2$ we obtain

$$\begin{aligned}\frac{dt_2}{dx} &= \mathcal{T}_2(x, t_2, \phi, \phi', \phi'')\\ &\doteq \frac{r_2}{M}\Bigg\{\left(\sqrt{1+[\phi'(x)]^2} - \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_2\right)\\ &\quad -\big(\beta+\theta(x)\big)\left[\frac{\cos\beta - \phi'(x)\sin\beta}{\sin\big(\beta+\theta(x)\big)} - \frac{x\cos\beta - \phi(x)\sin\beta}{\sin^2\big(\beta+\theta(x)\big)}\cos\big(\beta+\theta(x)\big)\frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\Bigg\}.\end{aligned}$$

(15.12)

An entirely similar computation can be done for $t < 0$. To treat this case, it is simpler to reverse time and replace ϕ by $-\phi$.

$$\theta_1 r_1 - \dot{x} r_1 \sqrt{1+[\phi'(x)]^2} + \frac{1}{2}\dot\theta_1 r_1^2 = -\frac{1}{2}\dot\theta_1 r_1^2 - \theta_1 r_1 \dot r_1 + M.$$

$$\theta_1(\dot r_1 + 1) - \frac{M}{r_1} = \left(\sqrt{1+[\phi'(x)]^2} + \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_1\right)\dot{x}.$$

$$\dot r_1 + 1 = \left[\frac{\sin\beta\big(\cot\beta + \phi'(x)\big)}{\sin\theta_1(x)} - \frac{\sin\beta\big(x\cot\beta + \phi(x)\big)}{\sin^2\theta_1(x)}\cos\big(\theta_1(x)\big)\frac{-\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\dot{x},$$

$$\begin{aligned}-\frac{M}{r_1} &= \left(\sqrt{1+[\phi'(x)]^2} + \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_1\right)\dot{x}\\ &\quad -\big(\beta-\theta(x)\big)\left[\frac{\sin\beta\big(\cot\beta+\phi'(x)\big)}{\sin\big(\beta-\theta(x)\big)} + \frac{\sin\beta\big(x\cot\beta+\phi(x)\big)}{\sin^2\big(\beta-\theta(x)\big)}\cos\big(\beta-\theta(x)\big)\frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\dot{x}.\end{aligned}$$

Eventually we obtain

$$\begin{aligned}\frac{dt_1}{dx} &= \mathcal{T}_1(x, t_1, \phi, \phi', \phi'')\\ &\doteq -\frac{r_1}{M}\left\{\left(\sqrt{1+[\phi'(x)]^2} + \frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\, r_1\right)\right.\\ &\quad \left. -\big(\beta-\theta(x)\big)\left[\frac{\cos\beta+\phi'(x)\sin\beta}{\sin\big(\beta-\theta(x)\big)} + \frac{x\cos\beta+\phi(x)\sin\beta}{\sin^2\big(\beta-\theta(x)\big)}\cos\big(\beta-\theta(x)\big)\frac{\phi''(x)}{1+\big[\phi'(x)\big]^2}\right]\right\}.\end{aligned} \tag{15.13}$$

Remark 15.1. In the case where $\phi(x) \equiv 0$, the identity (15.11) reduces to

$$\frac{M}{r_2} = \dot{x} - (\beta\cot\beta)\dot{x} = [1-\beta\cot\beta](\dot r_2 - 1)\tan\beta = [\tan\beta - \beta](\dot r_2 - 1).$$

15.1. Necessary Conditions

Next, we impose the necessary condition (14.2). The curvature of the maximal free interface γ at the point $\gamma(x) = \big(x, \phi(x)\big)$ is computed by

$$\omega(x) = \frac{1}{\rho(x)} = \frac{\phi''(x)}{\big(1+(\phi'(x))^2\big)^{3/2}}. \tag{15.14}$$

By (14.2) it now follows that, for every $x > 0$,

$$\int_0^{t_2(x)} \frac{dt}{r_2(t)} - \int_{t_1(x)}^0 \frac{dt}{r_1(t)} = \int_{t_1(x)}^{t_2(x)} \frac{dt}{\rho(x)+t}. \tag{15.15}$$

We wish to use these identities to obtain a second order ODE describing the curve $\gamma(\cdot)$. Differentiating (15.15) w.r.t. x, one obtains

$$\frac{1}{r_2}\frac{dt_2}{dx}+\frac{1}{r_1}\frac{dt_1}{dx}=\frac{\omega}{1+t_2\omega}\cdot\frac{dt_2}{dx}-\frac{\omega}{1+t_1\omega}\cdot\frac{dt_1}{dx}. \tag{15.16}$$

Remark 15.2. We choose the time $t=0$ as the time when the free interface is maximal, touching the vertex C. In this case, it is precisely the graph of $\phi(\cdot)$.

We have

$$\frac{dt_1(x)}{dx}\ <\ 0,\qquad \frac{dt_2(x)}{dx}>0.$$

The radii r_1, r_2 are always taken to be positive. However, for $t>0$ the curvature is $\omega=\frac{1}{r_2}$, while for $t<0$ it is $\omega=-\frac{1}{r_1}$.

There is a huge cancellation of the terms on the left hand sides of (15.15) or equivalently (15.16), while there is no cancellation on the right hand sides.

In view of (15.12)–(15.13), multiplying both sides of (15.16) by M we obtain an equation of the form

$$F(x,t_1,t_2,\phi,\phi',\phi'')=G(x,t_1,t_2,\phi,\phi',\phi''). \tag{15.17}$$

Setting

$$\begin{cases} r_1=\ \dfrac{x\cos\beta+\phi\sin\beta}{\sin\big(\beta-\theta\big)}-t_1,\\ r_2=\ \dfrac{x\cos\beta-\phi\sin\beta}{\sin\big(\beta+\theta\big)}+t_2,\end{cases}\qquad \theta\ =\ \arctan\phi',\qquad \omega=\frac{\phi''}{\big(1+|\phi'|^2\big)^{3/2}}, \tag{15.18}$$

the left hand side of (15.17) is computed by

$$\begin{aligned}F(x,t_1,t_2,\phi,\phi',\phi'')=&\Bigg\{\left(\sqrt{1+|\phi'|^2}-\frac{\phi''}{1+|\phi'|^2}\,r_2\right)\\ &-(\beta+\theta)\left[\frac{\cos\beta-\phi'\sin\beta}{\sin(\beta+\theta)}-\frac{x\cos\beta-\phi\sin\beta}{\sin^2(\beta+\theta)}\cos(\beta+\theta)\frac{\phi''}{1+|\phi'|^2}\right]\Bigg\}\\ &-\Bigg\{\left(\sqrt{1+|\phi'|^2}+\frac{\phi''}{1+|\phi'|^2}\,r_1\right)\\ &-(\beta-\theta)\left[\frac{\cos\beta+\phi'\sin\beta}{\sin(\beta-\theta)}+\frac{x\cos\beta+\phi\sin\beta}{\sin^2(\beta-\theta)}\cos(\beta-\theta)\frac{\phi''}{1+|\phi'(x)|^2}\right]\Bigg\}.\end{aligned} \tag{15.19}$$

After some cancellations, one obtains

$$\begin{aligned}F(x,t_1,t_2,\phi,\phi',\phi'')=&-\frac{\phi''}{1+|\phi'|^2}\cdot(r_1+r_2)\\ &-\frac{\beta+\theta}{\sin(\beta+\theta)}\left[(\cos\beta-\phi'\sin\beta)-\frac{x\cos\beta-\phi\sin\beta}{\tan(\beta+\theta)}\cdot\frac{\phi''}{1+|\phi'|^2}\right]\\ &+\frac{\beta-\theta}{\sin(\beta-\theta)}\left[(\cos\beta+\phi'\sin\beta)+\frac{x\cos\beta+\phi\sin\beta}{\tan(\beta-\theta)}\cdot\frac{\phi''}{1+|\phi'|^2}\right].\end{aligned} \tag{15.20}$$

On the other hand, recalling (15.12)–(15.13), the right hand side is computed by

$$\begin{aligned} G(x,t_1,t_2,\phi,\phi',\phi'') &= M\omega\left(\frac{dt_2}{dx}-\frac{dt_1}{dx}\right) - M\omega\left[\frac{t_2\omega}{1+t_2\omega}\frac{dt_2}{dx}-\frac{t_1\omega}{1+t_1\omega}\frac{dt_1}{dx}\right] \\ &= M\omega\left[\frac{t_1\omega}{1+t_1\omega}\mathcal{T}_1-\frac{t_2\omega}{1+t_2\omega}\mathcal{T}_2\right]+\frac{\phi''}{1+|\phi'|^2}(r_1+r_2)+\frac{\phi''}{1+|\phi'|^2}\omega(r_1^2-r_2^2) \\ &\quad -\frac{\beta+\theta}{\sin(\beta+\theta)}\left[\cos\beta-\phi'\sin\beta-\frac{x\cos\beta-\phi\sin\beta}{\tan(\beta+\theta)}\cdot\frac{\phi''}{1+|\phi'|^2}\right]\omega\, r_2 \\ &\quad -\frac{\beta-\theta}{\sin(\beta-\theta)}\left[\cos\beta+\phi'\sin\beta+\frac{x\cos\beta+\phi\sin\beta}{\tan(\beta-\theta)}\cdot\frac{\phi''}{1+|\phi'|^2}\right]\omega\, r_1. \end{aligned} \tag{15.21}$$

Summarizing, we need to solve a system of three ODEs for the three functions $\phi(x), t_1(x), t_2(x)$. Namely: the two first order ODEs (15.12)-(15.13) for the functions t_2, t_1, respectively, together with the second order ODE (15.17) for ϕ. These are supplemented by the identities in (15.18). They are to be solved with boundary conditions

$$t_1(0)=t_2(0)=\phi(0)=0. \tag{15.22}$$

Since the equations also involve the second order derivative $\phi''(x)$, we expect to find a 1-parameter family of solutions.

The main difficulty in constructing solutions stems from the fact that the above equations (15.12)–(15.13) and (15.17) are implicit, and singular at the initial point $x=0$. To gain some insight, we make the guess

$$\big|\phi(x)\big| \;<<\; x \quad \text{as} \quad x\to 0. \tag{15.23}$$

Starting with (15.12)–(15.13) and neglecting higher order infinitesimals, we are led to

$$\frac{dt_2}{dx} \;\approx\; \frac{r_2}{M}\left\{1-\beta\cot\beta-\phi'(x)\big(\cot\beta-\beta-\beta\cot^2\beta\big)-x\phi''(x)\left(\cot\beta-\beta\cot^2\beta\right)\right\}, \tag{15.24}$$

$$\frac{dt_1}{dx} \;\approx\; -\frac{r_1}{M}\left\{1-\beta\cot\beta+\phi'(x)\big(\cot\beta-\beta-\beta\cot^2\beta\big)+x\phi''(x)\left(\cot\beta-\beta\cot^2\beta\right)\right\}. \tag{15.25}$$

In view of (15.7), this yields

$$t_i(x)=\mathcal{O}(1)\cdot x^2, \quad r_i(x)=x\cot\beta+o(x), \quad i=1,2. \tag{15.26}$$

As usual, $o(x)$ denotes a higher order infinitesimal. Inserting these expressions in (15.16) we obtain

$$F(x,t_1,t_2,\phi,\phi',\phi'') \;\approx\; \left\{-2(\cot\beta-\beta-\beta\cot^2\beta)\phi'-2(\cot\beta-\beta\cot^2\beta)x\phi''\right\}. \tag{15.27}$$

On the other hand, the right hand side of (15.16) can be approximated by

$$G(x,t_1,t_2,\phi,\phi',\phi'') \;\approx\; 2\frac{\tan\beta-\beta}{\tan^2\beta}x\phi''=2(\cot\beta-\beta\cot^2\beta)\,x\phi''. \tag{15.28}$$

To leading order, (15.27) and (15.28) yield the approximate ODE

$$x\phi''(x) = \sigma\phi'(x), \quad \phi(0) \,=\, 0, \tag{15.29}$$

where

$$\begin{aligned}\sigma = \sigma(\beta) &= 2\frac{\cot\beta - \beta - \beta\cot^2\beta}{4\beta\cot^2\beta - 4\cot\beta}\\ &= \frac{\beta + \beta\tan^2\beta - \tan\beta}{2\tan\beta - 2\beta} = \frac{\beta - \sin\beta\cos\beta}{2\cos\beta(\sin\beta - \beta\cos\beta)}.\end{aligned} \tag{15.30}$$

We claim that $\sigma(\beta) > 1$ for every $\beta \in]0, \pi/2]$. Indeed, a Taylor approximation yields

$$\lim_{\beta\to 0+} \sigma(\beta) = \lim_{\beta\to 0+} \frac{\beta - \big(\beta - \beta^3/6\big)\big(1 - \beta^2/2\big)}{2\big(\beta - \beta^3/6\big) - \beta\big(1 - \beta^2/2\big)} = 1. \tag{15.31}$$

Moreover, by elementary differentiations one obtains

$$\frac{d}{d\beta}\Big(\beta\tan^2\beta + \beta - \tan\beta\Big) = \frac{d}{d\beta}\left(\frac{\beta}{\cos^2\beta} - \frac{\sin\beta}{\cos\beta}\right) = \beta\,\frac{2\sin\beta}{\cos^3\beta},$$

hence

$$\begin{aligned}\frac{d\sigma(\beta)}{d\beta} &= \frac{\beta\frac{2\sin\beta}{\cos^3\beta}(\tan\beta - \beta) - (\beta + \beta\tan^2\beta - \tan\beta)\tan^2\beta}{2(\tan\beta - \beta)^2}\\ &= \frac{\sin\beta}{2(\tan\beta - \beta)^2\cos^4\beta}\big(2\beta(\sin\beta - \beta\cos\beta) - \sin\beta(\beta - \sin\beta\cos\beta)\big)\\ &> \frac{\sin^2\beta}{2(\tan\beta - \beta)^2\cos^4\beta}\big(2(\sin\beta - \beta\cos\beta) - (\beta - \sin\beta\cos\beta)\big) > 0,\end{aligned} \tag{15.32}$$

in view of the fact that

$$\frac{d}{d\beta}\big(2(\sin\beta - \beta\cos\beta) - (\beta - \sin\beta\cos\beta)\big) = 2\beta\sin\beta - 2\sin^2\beta > 0.$$

Together, (15.31) and (15.32) imply that $\sigma(\beta) > 1$.

The general solution to the linear equation (15.29) is

$$\phi(x) = cx^{\sigma+1}, \tag{15.33}$$

where c is an arbitrary constant. Since $\sigma > 1$, this yields a 1-parameter family of solutions, for which the asymptotic condition (15.23) holds.

16. Local Solutions to the Equations of Maximal Free Interfaces

Aim of this section is to prove a local existence theorem for solutions to the singular, implicit Cauchy problem

$$\begin{cases} H(x,t_1,t_2,\phi,\phi',\phi'')=0,\\ t_1' = \mathcal{T}_1(x,t_1,\phi,\phi',\phi''),\\ t_2' = \mathcal{T}_2(x,t_2,\phi,\phi',\phi''),\end{cases} \qquad \begin{cases} \phi(0) = 0,\\ \lim_{x\to 0+} \dfrac{\phi(x)}{x^{\sigma+1}} = c,\\ t_1(0) = 0,\\ t_2(0) = 0.\end{cases} \tag{16.1}$$

Here $\mathcal{T}_2, \mathcal{T}_1$ are the functions introduced at (15.12)–(15.13), while $H = G - F$, with F, G as in (15.20)–(15.21).

Theorem 16.1. *Let $0 < \beta < \pi/2$ and $c \in \mathbb{R}$ be given, and let $\sigma > 0$ be the constant in (15.30). Then there exists $x^\dagger > 0$ and a local solution $x \mapsto \big(\phi(x), t_1(x), t_2(x)\big)$ to the initial value problem (16.1), defined for $x \in [0, x^\dagger]$.*

Proof. 1. In view of the previous analysis, to leading order the equation (15.17) reduces to (15.29). We thus expect that solutions will satisfy

$$\phi(x) = \mathcal{O}(1)\cdot x^{\sigma+1}, \quad \phi'(x) = \mathcal{O}(1)\cdot x^{\sigma}, \quad \phi''(x) = \mathcal{O}(1)\cdot x^{\sigma-1}, \quad t_1(x), t_2(x) = \mathcal{O}(1)\cdot x^2.$$

We thus consider a domain of the form

$$\mathcal{D} = \Big\{(t_1,t_2,\phi,\phi');\ \big|t_i(x)\big| \le C_0\, x^2, \quad \big|\phi(x)\big| \le C_0\, x^{\sigma+1},\ \big|\phi'(x)\big| \le C_0\, x^{\sigma}\Big\}, \tag{16.2}$$

for a suitable constant C_0.

Within this domain, the local solution will be obtained as the fixed point of a Picard-type operator

$$\Big(\mathcal{P}(t_1,t_2,\phi,\phi')\Big)(x) = (\widetilde{t}_1, \widetilde{t}_2, \widetilde{\phi}, \widetilde{\phi}')(x), \tag{16.3}$$

where the functions on the right hand side are defined as follows. We begin by constructing the function $\Upsilon = \Upsilon(x,t_1,t_2,\phi,\phi')$, implicitly defined by the identity

$$H(x,t_1,t_2,\phi,\phi',\Upsilon) = 0. \tag{16.4}$$

Details of the construction of Υ will be worked out in step 2. We then define

$$T_i(x,t_1,t_2,\phi,\phi') = \mathcal{T}_i\big(x,t_i,\phi,\phi',\Upsilon(x,t_1,t_2,\phi,\phi')\big), \tag{16.5}$$

and set

$$\widetilde{t}_i(x) \doteq \int_0^x T_i(x,t_1,t_2,\phi,\phi')\,dy, \quad i = 1,2. \tag{16.6}$$

Next, we write the equation $H = 0$ in the equivalent form

$$x\phi''(x) - \sigma\phi'(x) = K(x,t_1,t_2,\phi,\phi',\phi''). \tag{16.7}$$

for a suitable function K. Solving (16.7) with $\phi'' = \Upsilon$, in view of the second boundary condition at (16.1) this leads to

$$\widetilde{\phi}'(x) \doteq x^{\sigma}\left((\sigma+1)c + \int_0^x y^{-\sigma-1}\,K(y,t_1,t_2,\phi,\phi',\Upsilon)\,dy\right) \tag{16.8}$$

$$\widetilde{\phi}(x) \doteq \int_0^x \widetilde{\phi}'(y)dy. \tag{16.9}$$

In the remainder of the proof, we will show that the above formulas (16.6), (16.8), (16.9) yield a strictly contractive map, on an interval $[0, x^\dagger]$ small enough.

2. In this step, using the implicit function theorem we construct the function Υ and provide estimates on the function K in (16.7). Computing the partial derivatives of $H = G - F$, in view of (15.20)–(15.21) and (15.18) we obtain

$$\frac{\partial}{\partial \phi''} H = 4x(\cot\beta - \beta\cot^2\beta) + \mathcal{O}(1)\cdot x^{1+\varepsilon}, \tag{16.10}$$

$$\frac{\partial}{\partial \phi'} H = -2(\beta + \beta\cot^2\beta - \cot\beta) + \mathcal{O}(1)\cdot x^{\varepsilon}, \tag{16.11}$$

$$\frac{\partial}{\partial \phi} H = \mathcal{O}(1)\cdot|\phi''| + \mathcal{O}(1)\cdot\big(x + |\phi|\big)|\phi''|^2 = \mathcal{O}(1)\cdot x^{\sigma-1}. \tag{16.12}$$

$$\frac{\partial}{\partial t_i} H = \mathcal{O}(1)\cdot|\phi''| + \mathcal{O}(1)\cdot\big(x + |\phi|\big)|\phi''|^2 = \mathcal{O}(1)\cdot x^{\sigma-1}. \tag{16.13}$$

Since $\partial H/\partial\phi'' > 0$, this yields the local existence of the function Υ implicitly defined by (16.4). Here, $\varepsilon = \min\{1, \sigma\}$.

Moreover, we have

$$\frac{\partial \Upsilon}{\partial \phi'} = -\frac{\partial H/\partial\phi'}{\partial H/\partial\phi''} = \frac{2(\beta + \beta\cot^2\beta - \cot\beta) + \mathcal{O}(1)\cdot x^{\varepsilon}}{4x(\cot\beta - \beta\cot^2\beta) + \mathcal{O}(1)\cdot x^{1+\varepsilon}} = \sigma + \mathcal{O}(1)\cdot x^{\varepsilon-1}. \tag{16.14}$$

Similarly, we obtain

$$\frac{\partial \Upsilon}{\partial \phi} = -\frac{\partial H/\partial\phi}{\partial H/\partial\phi''} = \mathcal{O}(1)\cdot x^{\sigma-2}, \quad \frac{\partial \Upsilon}{\partial t_i} = -\frac{\partial H/\partial t_i}{\partial H/\partial\phi''} = \mathcal{O}(1)\cdot x^{\sigma-2}. \tag{16.15}$$

Next, we observe that the equation $H = 0$ is equivalent to

$$\phi''(x) = \Upsilon\big(x, t_1, t_2, \phi(x), \phi'(x)\big).$$

In turn, this yields

$$x\phi'' = x\Upsilon, \quad x\phi'' - \sigma\phi' = K,$$

where

$$K(x, t_1, t_2, \phi, \phi') = x\,\Upsilon(x, t_1, t_2, \phi, \phi') - \sigma\phi'. \tag{16.16}$$

In view of (16.14)–(16.15), the partial derivatives of K thus satisfy the bounds

$$\frac{\partial K}{\partial \phi'} = O(1)\cdot x^{\varepsilon}, \quad \frac{\partial K}{\partial \phi} = \mathcal{O}(1)\cdot x^{\sigma-1}, \quad \frac{\partial K}{\partial t_i} = \mathcal{O}(1)\cdot x^{\sigma-1}. \tag{16.17}$$

3. We now provide estimates on the functions T_1, T_2 introduced at (16.5). Recalling (15.12)–(15.13) and (15.18), we begin by estimating the partial derivatives

$$\frac{\partial \mathcal{T}_i}{\partial \phi} = \mathcal{O}(1), \quad \frac{\partial \mathcal{T}_i}{\partial t_i} = \mathcal{O}(1), \tag{16.18}$$

$$\frac{\partial \mathcal{T}_i}{\partial \phi'} = \mathcal{O}(1) \cdot x, \quad \frac{\partial \mathcal{T}_i}{\partial \phi''} = \mathcal{O}(1) \cdot x^2. \tag{16.19}$$

By (15.12)–(15.13) and (16.14)–(16.15) it follows

$$\frac{\partial T_i}{\partial \phi} = \frac{\partial \mathcal{T}_i}{\partial \phi} + \frac{\partial \mathcal{T}_i}{\partial \phi''}\frac{\partial \Upsilon}{\partial \phi} = \mathcal{O}(1) \tag{16.20}$$

$$\frac{\partial T_i}{\partial \phi'} = \frac{\partial \mathcal{T}_i}{\partial \phi'} + \frac{\partial \mathcal{T}_i}{\partial \phi''}\frac{\partial \Upsilon}{\partial \phi'} \;=\; \mathcal{O}(1) \cdot (x + x^2 x^{\varepsilon-1}) = \mathcal{O}(1) \cdot x, \tag{16.21}$$

$$\frac{\partial T_i}{\partial t_j} = \frac{\partial \mathcal{T}_i}{\partial t_j} + \frac{\partial \mathcal{T}_i}{\partial \phi''}\frac{\partial \Upsilon}{\partial t_j} = \mathcal{O}(1) \cdot (1 + x^2\, x^{\sigma-2}) = \mathcal{O}(1). \tag{16.22}$$

4. We need to prove that the transformation $\Big(\mathcal{P}(t_1, t_2, \phi, \phi')\Big)(x) = (\widetilde{t}_1, \widetilde{t}_2, \widetilde{\phi}, \widetilde{\phi}')(x)$, as in (16.3), maps the domain (16.2) into itself. Let $(t_1, t_2, \phi, \phi') \in \mathcal{D}$. Then, recalling (16.17), we obtain

$$\begin{aligned}
\widetilde{\phi}'(x) &= x^\sigma(\sigma+1)c + x^\sigma \int_0^x y^{-\sigma-1}\bigg\{\int_0^1 \frac{\partial K\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial \phi}\,d\lambda \cdot \phi(y) \\
&\quad + \int_0^1 \frac{\partial K\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial \phi'}\,d\lambda \cdot \phi'(y) + \sum_{i=1,2}\int_0^1 \frac{\partial K\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial t_i}\,d\lambda \cdot t_i(y)\bigg\}dy \\
&= x^\sigma(\sigma+1)c + x^\sigma O(1)\big\{x^\varepsilon + x^\sigma + x\big\} = (\sigma+1)cx^\sigma + O(1)x^{\sigma+\varepsilon}.
\end{aligned} \tag{16.23}$$

Hence,

$$\widetilde{\phi}(x) = \int_0^x \widetilde{\phi}'(y)dy = x^{\sigma+1}c + O(1)x^{\sigma+1+\varepsilon}.$$

For $i = 1, 2$, working as in (16.23) and using (16.20)–(16.22), one gets

$$\begin{aligned}
\widetilde{t}_i(x) &= \int_0^x \bigg\{\int_0^1 \frac{\partial T_i\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial \phi}\,d\lambda \cdot \phi(y) \\
&\quad + \int_0^1 \frac{\partial T_i\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial \phi'}\,d\lambda \cdot \phi'(y) + \sum_{j=1,2}\int_0^1 \frac{\partial T_i\big(\lambda(t_1,t_2,\phi,\phi')(y)\big)}{\partial t_j}\,d\lambda \cdot t_j(y)\bigg\}dy \\
&= O(1)\big\{x^{\sigma+2} + x^3\big\} = O(1)x^{2+\varepsilon}.
\end{aligned}$$

Taking $C_0 \geq 2(\sigma+1)c$ and $x^\dagger$ small enough, we obtain that, for any $x \in [0, x^\dagger]$, $\Big(\mathcal{P}(t_1, t_2, \phi, \phi')\Big)(x) \in \mathcal{D}$.

5. To achieve the contraction property, consider the norm

$$\big\|(t_1, t_2, \phi, \phi')\big\| = \sup_{0<x<x^\dagger} \max\left\{\frac{|\phi(x)|}{x^{\sigma+1}},\ \frac{|\phi'(x)|}{x^\sigma},\ \frac{|t_1(x)|}{x^2},\ \frac{|t_2(x)|}{x^2}\right\}. \tag{16.24}$$

Given two sets of functions $(t_{j1}, t_{j2}, \phi_j, \phi'_j)$, $j = 1, 2$, such that

$$\Big\|(t_{11}, t_{12}, \phi_1, \phi'_1) - (t_{21}, t_{22}, \phi_2, \phi'_2)\Big\| = \delta > 0, \tag{16.25}$$

we will prove that the corresponding Picard iterates satisfy

$$\Big\|(\widetilde t_{11}, \widetilde t_{12}, \widetilde\phi_1, \widetilde\phi'_1) - (\widetilde t_{21}, \widetilde t_{22}, \widetilde\phi_2, \widetilde\phi'_2)\Big\| \le \frac{\delta}{2}. \tag{16.26}$$

6. We are now ready to prove the contraction property (16.26), with distance induced by the norm (16.24). If (16.25) holds, then

$$\big|\phi_1(x) - \phi_2(x)\big| \le \delta x^{\sigma+1}, \quad \big|\phi'_1(x) - \phi'_2(x)\big| \le \delta x^{\sigma}, \quad \big|t_{1i}(x) - t_{2i}(x)\big| \le \delta x^2, \quad i = 1, 2. \tag{16.27}$$

Consider the intermediate points

$$Z(\lambda, x) \doteq \lambda(t_{11}, t_{12}, \phi_1, \phi'_1)(x) + (1-\lambda)(t_{21}, t_{22}, \phi_2, \phi'_2)(x).$$

By (16.8), the difference between the two values of $\widetilde\phi'$ is computed by

$$\begin{aligned}\widetilde\phi'_1(x) - \widetilde\phi'_2(x) = x^\sigma \int_0^x y^{-\sigma-1}\Bigg\{ &\int_0^1 \frac{\partial K\big(Z(\lambda,y)\big)}{\partial\phi}\, d\lambda \cdot (\phi_1(y) - \phi_2(y)) \\ &+ \int_0^1 \frac{\partial K\big(Z(\lambda,y)\big)}{\partial\phi'}\, d\lambda \cdot (\phi'_1(y) - \phi'_2(y)) \\ &+ \sum_{i=1,2}\int_0^1 \frac{\partial K\big(Z(\lambda,y)\big)}{\partial t_i}\, d\lambda \cdot (t_{i1}(y) - t_{i2}(y))\Bigg\} dy. \end{aligned} \tag{16.28}$$

Recalling (16.17) and (16.27), one obtains

$$\widetilde\phi'_1(x) - \widetilde\phi'_2(x) = \mathcal{O}(1)\cdot\Big\{x^{\sigma-1}\,\delta x^{\sigma+1} + x^{\varepsilon}\,\delta x^{\sigma} + x^{\sigma-1}\delta x^2\Big\} = \mathcal{O}(1)\cdot\delta x^{\sigma+\varepsilon}. \tag{16.29}$$

In turn, this yields

$$\widetilde\phi_1(x) - \widetilde\phi_2(x) = \int_0^x \big[\widetilde\phi'_1(y) - \widetilde\phi'_2(y)\big]\, dy = \mathcal{O}(1)\cdot\int_0^x \delta y^{\sigma+\varepsilon}\, dy = \mathcal{O}(1)\cdot\delta x^{\sigma+1+\varepsilon}. \tag{16.30}$$

Finally we estimate the change in the t_i variables. For $i = 1, 2$, the same approach as in (16.28) now yields

$$\begin{aligned}\widetilde t_{1i}(x) - \widetilde t_{2i}(x) = \int_0^x \Bigg\{ &\int_0^1 \frac{\partial T_i\big(Z(\lambda,y)\big)}{\partial\phi}\, d\lambda \cdot (\phi_1(y) - \phi_2(y)) \\ &+ \int_0^1 \frac{\partial T_i\big(Z(\lambda,y)\big)}{\partial\phi'}\, d\lambda \cdot (\phi'_1(y) - \phi'_2(y)) \\ &+ \sum_{j=1,2}\int_0^1 \frac{\partial T_i\big(Z(\lambda,y)\big)}{\partial t_j}\, d\lambda \cdot (t_{j1}(y) - t_{j2}(y))\Bigg\} dy. \end{aligned}$$

Using (16.20)–(16.22) and recalling (16.27), we now obtain

$$\widetilde{t}_{1i}(x) - \widetilde{t}_{2i}(x) = \int_0^x \mathcal{O}(1) \cdot \left[1 \cdot \delta y^{\sigma+1} + y \cdot \delta y^{\sigma} + 1 \cdot y^2\right] dy = \mathcal{O}(1) \cdot x^{2+\varepsilon}. \tag{16.31}$$

As x ranges in an interval $[0, x^\dagger]$ small enough, the three inequalities (16.29)–(16.31) yield the contractive property (16.26).

6. The unique fixed point of the contractive transformation defined at (16.6), (16.8) and (16.9) now provides the desired solution to the Cauchy problem (16.1). □

17. Concluding Remarks

In this paper we analyzed a family of set-motion problems, discussing existence of optimal solutions and necessary conditions toward their explicit computation. We like to think of these as "time dependent isoperimetric problems". Indeed, by the area formula (12.1), to minimize the area one should reduce the perimeter as fast as possible. In the optimal motions considered in Theorem 4.1, at each time $t \in [0, T]$ the set $\Omega(t)$ has the shortest relative boundary $\partial\Omega(t) \cap V$, compared with all other subsets with the same area.

Among several problems which are left open, a key issue is the regularity of optimal solutions. While the existence results proved in Sect. 5 yield optimal solutions in the class of BV functions, it seems reasonable to conjecture that for a.e. $t \in [0, T]$ the set $\Omega(t)$ should satisfy an interior ball condition. That is: $\Omega(t)$ should be the union of discs of radius $r(t) > 0$. Such regularity would likely suffice for deriving necessary conditions for optimality.

A second goal would be to use the analysis in Sects. 13–16 in order to construct set motions that satisfy all optimality conditions *globally in time*. Two cases appear to be particularly interesting, where either (i) V is a polygon, or (ii) V is a convex set with smooth boundary, whose curvature has finitely many local maxima and minima.

Both of these problems are left for future investigation.

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