



Quasi-invariant States with Uniformly Bounded Cocycles

Ameur Dhahri¹ , Éric Ricard²

¹ Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo di Vinci 32, 20133 Milan, Italy.
E-mail: ameur.dhahri@polimi.it

² UNICAEN, CNRS, LMNO, Caen 14000, France.
E-mail: eric.ricard@unicaen.fr

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Abstract: We investigate the notion of quasi-invariant states introduced in [2] from an analytic viewpoint. We give the structures of quasi-invariant states with uniformly bounded cocycles. As a consequence, we can apply a Theorem of Kovács and Szücs to get a conditional expectation on fixed points and another of Størmer to get an invariant semifinite trace under extra assumptions.

1. Introduction

Let (S, \mathcal{B}) be a measurable space and G a group of $*$ -automorphisms acting on S by measurable transformation

$$s \in S \mapsto gs \in S, \quad g \in G.$$

A measure μ on (S, \mathcal{B}) is called G -quasi-invariant if the measures $\mu \circ g$ and μ are absolutely continuous one with respect to the other.

Quasi-invariant measures are a much wider class than invariant, and that is why they constitute the natural environment for the theory of dynamical systems and ergodic theory. Therefore to develop a real quantum analogue of the classical theory of dynamical systems, requires developing a theory of quasi-invariant states on general operator algebras. Recently, in [2] the authors introduced the theory of quasi-invariant states under the action of a group of $*$ -automorphisms on a von Neumann algebra (or a $*$ -algebra). The relationship between the group G and modular automorphism group, invariant subalgebras, ergodicity, modular theory, and abelian subalgebras is studied in [6]. Moreover, a martingale convergence theorem for strongly quasi-invariant states is given in [7]. Finally, in [15], quasi-invariant actions are used in the setting of discrete quantum groups. In particular, the main technical results in [15] is the computation of the Radon-Nikodym cocycle for the action of the dual of $SUq(2)$ on its Martin boundary with respect to the canonical harmonic state defined by a quantum random walk.

In this note, we wish to clarify the notion of G -quasi-invariant state under a natural boundedness assumption. The paper is structured as follows. First, we give the structure of quasi-invariant states with uniformly bounded cocycle. Moreover, for G -quasi-invariant states, we show that one can always spatially implement the action of G . We further deal with a Theorem of Kovács and Szücs. Then, we extend the main result of Størmer in [20] to strongly quasi-invariant states with uniformly bounded cocycles.

2. Quasi-invariant States

We briefly review some definitions and properties of quasi-invariant states, respectively, strongly quasi-invariant states with respect to a group of $*$ -automorphisms on a von Neumann algebra (cf [2]).

Let \mathcal{A} be a von Neumann algebra and G be a group of normal $*$ -automorphisms of \mathcal{A} equipped with its usual strong-topology.

Definition 2.1. A faithful normal state φ on \mathcal{A} is said to be G -quasi-invariant if for all $g \in G$ there exists $x_g \in \mathcal{A}$ such that

$$\varphi(g(a)) = \varphi(x_g a), \quad a \in \mathcal{A}. \tag{2.1}$$

We say that φ is G -strongly quasi-invariant if it is G -quasi-invariant and the Radon-Nikodym derivatives x_g are self-adjoint: $x_g = x_g^*$, $g \in G$.

It is proved in [2] the following properties:

- (i) When φ is G -quasi-invariant, the map $g \mapsto x_g$ is a normalized multiplicative left G -1-cocycle satisfying

$$x_e = \mathbf{1}, \quad x_{g_2 g_1} = x_{g_1} g_1^{-1}(x_{g_2}), \quad g_1, g_2 \in G, \tag{2.2}$$

and x_g is invertible with its inverse

$$x_g^{-1} = g^{-1}(x_{g^{-1}}). \tag{2.3}$$

Furthermore, for all $a \in \mathcal{A}$ and $g \in G$ it holds

$$\varphi(x_g a) = \varphi(ax_g^*). \tag{2.4}$$

- (ii) If φ is G -strongly quasi-invariant, it holds that x_g is positive invertible. It also holds that x_g commutes with x_h for all $g, h \in G$. Hence the von Neumann algebra \mathcal{C} generated by $\{x_g : g \in G\}$ is commutative. It can be also easily shown that \mathcal{C} is a subalgebra of $\text{Centr}(\varphi)$, the centralizer of φ , which is defined by

$$\text{Centr}(\varphi) := \{x \in \mathcal{A} : \varphi(xy) = \varphi(yx) \text{ for all } y \in \mathcal{A}\}. \tag{2.5}$$

Now we introduce the following lemma which plays a crucial rule later.

Lemma 2.2 ([19], 5.20. Lemma). *Let φ be a positive form on a C^* -algebra \mathcal{A} and $a \in \mathcal{A}$. If the linear form $L_a \varphi$ defined on \mathcal{A} by $L_a \varphi(x) = \varphi(ax)$ is positive, then*

$$L_a \varphi(x) \leq \|a\| \varphi(x),$$

for all positive element $x \in \mathcal{A}$.

Remark 2.3. If \mathcal{A} is a von Neumann algebra and φ, ψ are two states on \mathcal{A} , we say that φ is ψ -absolutely continuous if $\psi(a^*a) = 0$ implies to $\varphi(a^*a) = 0$ for $a \in \mathcal{A}$. We say that φ and ψ are equivalent ($\varphi \sim \psi$) if they are absolutely continuous one with respect to the other. A state φ is G -invariant (resp. G -quasi-invariant) if $\varphi \circ g = \varphi$ (resp. $\varphi \circ g \sim \varphi$) for all $g \in G$. Indeed, if φ is as in Definition 2.1, then by Lemma 2.2 one has

$$\varphi \circ g(a^*a) = \varphi(x_g a^*a) \leq \|x_g\| \varphi(a^*a), \quad \varphi(a^*a) \leq \|x_g^{-1}\| \varphi \circ g(a^*a),$$

$$\forall g \in G, \forall a \in \mathcal{A},$$

which proves that $\varphi \circ g \sim \varphi$ for all $g \in G$.

3. Quasi-invariant States with Uniformly Bounded Cocycles

We will make the following assumption.

Assumption. Assume that there exists $\lambda > 0$ such that

$$\|x_g\| \leq \lambda, \quad \forall g \in G. \tag{A}$$

Since $x_e = \mathbf{1}$, it is clear that $\lambda \geq 1$. By (2.3), we also have $\|x_g^{-1}\| \leq \lambda$.

Lemma 3.1. *Under assumption (A), for any $g \in G$ and any positive operator $a \in \mathcal{A}$*

$$\frac{1}{\lambda} \varphi(a) \leq \varphi(x_g a) \leq \lambda \varphi(a) \quad \text{and} \quad \frac{1}{\lambda} \varphi(a) \leq \varphi(a(x_g^{-1})^*) \leq \lambda \varphi(a).$$

Proof. Let $g \in G$ and $a \in \mathcal{A}$ be a positive operator. Then by assumption (A) and Remark 2.3

$$0 \leq \varphi(g(a)) = \varphi(x_g a) \leq \|x_g\| \varphi(a) \leq \lambda \varphi(a).$$

Moreover, one has

$$\begin{aligned} \varphi(a) &= \varphi(g^{-1}(g(a))) \\ &= \varphi(x_{g^{-1}}(g(a))) \\ &\leq \|x_{g^{-1}}\| \varphi(g(a)) \end{aligned}$$

which proves that

$$\varphi(g(a)) \geq \frac{1}{\|x_{g^{-1}}\|} \varphi(a) \geq \frac{1}{\lambda} \varphi(a).$$

Since $\varphi_g = \varphi \circ g : a \mapsto \varphi(x_g a)$ is a state, the linear form $a \mapsto \varphi(a(x_g^{-1})^*) = \varphi_g(x_g^{-1} a(x_g^{-1})^*)$ is positive. Then, for all $a \in \mathcal{A}$, we have

$$\varphi(a(x_g^{-1})^*) = \overline{\varphi(a^*(x_g^{-1})^*)} = \overline{\varphi((x_g^{-1} a)^*)} = \varphi(x_g^{-1} a) = L_{x_g^{-1}} \varphi(a)$$

thanks to the positivity of the above-mentioned linear form and φ , and we can use Lemma 2.2 again as $L_{x_g^{-1}} \varphi \geq 0$. Similarly $\varphi = L_{x_g}(L_{x_g^{-1}} \varphi) \geq 0$ gives the last inequality. \square

Our main result states that this assumption is equivalent to have an invariant normal faithful state for the action. We borrow some ideas of [1].

Theorem 3.2. *If φ is a normal G -quasi-invariant state on \mathcal{A} with cocycle x_g 's satisfying (A), then there exist a normal faithful G -invariant state ψ on \mathcal{A} and a bounded operator $d \in \mathcal{A}$ such that*

$$\psi(a) = \varphi(da), \quad a \in \mathcal{A}.$$

Conversely, if there exist a normal faithful G -invariant state ψ on \mathcal{A} and a bounded invertible operator $d \in \mathcal{A}$ such that for all $g \in G$

$$\psi(a) = \varphi(da), \quad a \in \mathcal{A},$$

then φ is a normal G -quasi-invariant state on \mathcal{A} with cocycle x_g 's satisfying (A).

Proof. On \mathcal{A} , define the linear map

$$\Gamma_g(a) := x_{g^{-1}}g(a), \quad a \in \mathcal{A}.$$

Since g is normal and $x_{g^{-1}}$ is bounded, it follows that Γ_g is σ -weakly continuous. Moreover, the maps Γ_g satisfy the following properties:

(i) For all $g, h \in G$, $\Gamma_g(x_h) = x_{hg^{-1}}$. This is exactly (2.2).

(ii) For all $h, g \in G$, one has $\Gamma_{gh} = \Gamma_g\Gamma_h$.

If $a \in \mathcal{A}$, computing with (2.2)

$$\begin{aligned} \Gamma_{gh}(a) &= x_{(gh)^{-1}}gh(a) \\ &= x_{h^{-1}g^{-1}}gh(a) \\ &= x_{g^{-1}}g(x_{h^{-1}})g(h(a)) \\ &= x_{g^{-1}}g(\Gamma_h(a)) \\ &= \Gamma_g(\Gamma_h(a)). \end{aligned}$$

(iii) For all $g \in G$, $\varphi \circ \Gamma_g = \varphi$.

If $a \in \mathcal{A}$, easy computations give

$$\varphi(\Gamma_g(a)) = \varphi(x_{g^{-1}}g(a)) = \varphi(g^{-1}(g(a))) = \varphi(a).$$

(iv) For all $g \in G$ and $a, b \in \mathcal{A}$, $\Gamma_g(ab) = \Gamma_g(a)(x_{g^{-1}})^{-1}\Gamma_g(b)$.

(v) For all $g \in G$ and $a \in \mathcal{A}$

$$(\Gamma_g(a))^* = (x_{g^{-1}})^{-1}\Gamma_g(a^*)(x_{g^{-1}})^*.$$

Let K be the σ -weak closure of $\text{conv}\{x_g\}_{g \in G}$. Since the set $\{x_g\}_{g \in G}$ is uniformly bounded, K is convex and compact with respect to the σ -weak topology.

Moreover, by (i), $\Gamma_g(K) \subset K$ for all $g \in G$.

Now define a continuous norm ρ on \mathcal{A} by $\rho(x) = \varphi(xx^*)^{1/2}$. Let $a \neq b \in K$, thanks to the second estimate in Lemma 3.1 applied to $\Gamma_g(a - b)(\Gamma_g(a - b))^* \geq 0$ for g^{-1} :

$$\rho(\Gamma_g(a) - \Gamma_g(b))^2 \geq \frac{1}{\lambda} \varphi(\Gamma_g(a - b)(\Gamma_g(a - b))^*((x_{g^{-1}})^{-1})^*).$$

Using the algebraic identities (v), (iv) and (iii)

$$\begin{aligned} \lambda \rho(\Gamma_g(a) - \Gamma_g(b))^2 &\geq \varphi(\Gamma_g(a - b)(x_{g^{-1}})^{-1}\Gamma_g((a - b)^*)) \\ &= \varphi(\Gamma_g((a - b)(a - b)^*)) \\ &= \varphi((a - b)(a - b)^*) > 0. \end{aligned}$$

Therefore Γ_G is non-contracting and by the Ryll-Nardzewski fixed point theorem [17] there exists $d \in K$ such that $\Gamma_g(d) = d$, i.e. $d = x_{g^{-1}g}(d)$ for all $g \in G$. Notice that for all $g \in G$, $\varphi(x_g) = 1$. Then for all $x \in K$

$$\varphi(x) = 1 = \varphi(d).$$

It follows that the operator d is non zero. Define

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}.$$

Then for all $g \in G$

$$\psi(g(a)) = \varphi(dg(a)) = \varphi(g(g^{-1}(d)a)) = \varphi(x_g g^{-1}(d)a) = \varphi(da) = \psi(a)$$

which proves that ψ is G -invariant. By Lemma 3.1 for any $y \in K$, we have $\frac{1}{\lambda}\varphi \leq \varphi \cdot y \leq \lambda\varphi$. In particular ψ is faithful as φ is.

Conversely, let d be a bounded invertible element of \mathcal{A} such that

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}$$

is G -invariant. It follows that for all $a \in \mathcal{A}$, $\varphi(a) = \psi(d^{-1}a)$, and for all $g \in G$

$$\begin{aligned} \varphi(g(a)) &= \psi(d^{-1}g(a)) \\ &= \psi(g(g^{-1}(d^{-1})a)) \\ &= \psi(g^{-1}(d^{-1})a) \\ &= \varphi(dg^{-1}(d^{-1})a). \end{aligned}$$

This proves that φ is G -quasi-invariant state with cocycle

$$x_g = dg^{-1}(d^{-1}). \tag{3.1}$$

Since d is a bounded invertible operator, then for any $g \in G$

$$g^{-1}(d^{-1}(d^{-1})^*) \leq \|d^{-1}\|^2 \quad \text{and} \quad dg^{-1}(d^{-1}(d^{-1})^*)d^* \leq \|d\|^2 \|d^{-1}\|^2.$$

It follows that for any $g \in G$

$$\|x_g\|^2 = \|x_g x_g^*\| \leq \|d\|^2 \|d^{-1}\|^2.$$

Thus the family $\{x_g\}_{g \in G}$ satisfies (A). □

Now, let φ be a normal faithful G -strongly quasi-invariant state on \mathcal{A} with cocycle x_g 's satisfying assumption (A). Since x_g is a positive operator, one has

$$x_g \leq \lambda, \quad \forall g \in G.$$

Thanks to (2.3), this is equivalent to

$$x_{g^{-1}} \geq \frac{1}{\lambda}, \quad \forall g \in G. \tag{3.2}$$

Therefore it is clear that assumption (A) is equivalent to

$$\frac{1}{\lambda} \leq x_g \leq \lambda, \quad \forall g \in G. \tag{3.3}$$

As a consequence of Theorem 3.2, we prove the following.

Theorem 3.3. φ is a normal faithful G -strongly quasi-invariant state with cocycle x_g 's satisfying (A) if and only if there exist a normal faithful G -invariant state ψ and a bounded positive invertible operator $d \in \mathcal{A}$ such that for all $g \in G$

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}.$$

Moreover, in this case for all $g \in G$, $[d, g(d)] = 0$.

Proof. Since φ is a normal faithful G -strongly quasi-invariant state with cocycle x_g 's satisfying assumption (A) then by Theorem 3.2, there exist a normal faithful G -invariant state ψ and a bounded operator $d \in \mathcal{A}$ such that for all $g \in G$

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}.$$

Remember that from the proof of Theorem 3.2, K is the σ -weak closure of $\text{conv}\{x_g\}_{g \in G}$ and the cocycle x_g 's satisfy (3.3), it follows that d satisfies (3.3) and d is a positive invertible operator in \mathcal{A} . Notice that in this case $x_g = dg^{-1}(d^{-1}) = x_g^*$ which is equivalent to $[d, g(d)] = 0$.

Conversely, let ψ be a normal faithful G -invariant state

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}$$

for some normal state φ on \mathcal{A} , where d is a positive invertible operator in \mathcal{A} such that for all $g \in G$, $[d, g(d)] = 0$. Then it is clear that φ is a normal faithful G -strongly quasi-invariant state on \mathcal{A} with cocycle $x_g = dg^{-1}(d^{-1}) = x_g^*$. Since d is a positive invertible operator, it is clear that the cocycle x_g 's satisfies (A). □

4. Unitary Implementation Associated to Quasi-invariant States

When φ is G -strongly quasi-invariant state on a von Neumann algebra \mathcal{A} (for a given action) with cocycle x_g 's, it is proved in [2] that, in the GNS representation of φ , there exists a unique unitary representation of G implementing the action by conjugation. We will recover it in this section and show that for G -quasi-invariant states, one can always spatially implement the action of G (but we don't know if the map $g \mapsto U_g^*$ is a representation). Note that in [10], it is showed that the group G has always a canonical unitary implementation which is different to the one introduced here. In particular in [Theorem 3.2, [10]], with the usual notation from there, the condition $u_g J = J u_g$ is not satisfied in our case.

Let G be a group of normal $*$ -automorphisms of a von Neumann algebra \mathcal{A} and φ be a normal G -quasi-invariant state on \mathcal{A} with cocycle x_g 's.

We will use Haagerup formalism in this section for the standard form of \mathcal{A} as in section II of [23], this will avoid heavy formulas. We denote by \mathcal{N} the crossed product $\mathcal{A} \rtimes_{\sigma^\varphi} \mathbb{R}$ of \mathcal{A} by the modular group σ^φ associated with φ . We denote by θ the dual action of \mathbb{R} on \mathcal{N} . Then θ satisfies $\theta_s(x) = x$ ($s \in \mathbb{R}$) iff $x \in \mathcal{A}$. Let τ be the trace on \mathcal{N} such that $\tau \circ \theta_s = e^{-s} \tau$ and denote by $L_0(\mathcal{N}, \tau)$ the space of τ -measurable operators x affiliated with \mathcal{N} . Define for $p = 1, 2$

$$L_p(\mathcal{A}) := \{x \in L_0(\mathcal{N}, \tau) \mid \theta_s(x) = e^{-s/p} x, \quad s \in \mathbb{R}\}.$$

There is an identification of \mathcal{A}_* with $L_1(\mathcal{A})$ assigning a positive operator $D_\gamma \in L_1(\mathcal{A})$ to a normal linear form $\gamma \in \mathcal{A}_*$ (Proposition 15 [23]). The duality is defined through the use of a trace-like functional $\text{tr} : L_1(\mathcal{A}) \rightarrow \mathbb{C}$ with

$$\langle D_\gamma, a \rangle_{L_1(\mathcal{A}), \mathcal{A}} = \text{tr}(D_\gamma a) = \gamma(a).$$

Since φ is faithful state on \mathcal{A} , its density $D_\varphi \in L_1(\mathcal{A})$ satisfies $\text{supp}(D_\varphi) = 1$ by Proposition 4(2c) in [23]. Note that, by duality, G acts on $L_1(\mathcal{A})$ and we have

$$\langle g^* D_\varphi, a \rangle_{L_1(\mathcal{A}), \mathcal{A}} = \varphi(g(a)).$$

The space $L_2(\mathcal{A})$ is naturally a Hilbert space with scalar product $\langle A, B \rangle = \text{tr}(A^*B)$. Moreover, by Theorem 36 in [23], the GNS representation of (\mathcal{A}, φ) is the triple $(\mathcal{A} \subset \mathcal{N}, L_2(\mathcal{A}), D_\varphi^{1/2})$.

Lemma 4.1. *There exists a positive invertible operator a_g in \mathcal{A} such that*

$$D_{\varphi \circ g} = D_\varphi^{1/2} a_g^2 D_\varphi^{1/2} = x_g^* D_\varphi. \tag{4.1}$$

Proof. Since φ is normal G -quasi-invariant and for all $a \in \mathcal{A}$ and $g \in G$, $\varphi(x_g a) = \varphi(ax_g^*)$, it follows that

$$g^* D_\varphi := D_{\varphi \circ g} = x_g^* D_\varphi = D_\varphi x_g. \tag{4.2}$$

On the other hand, for any positive operator $a \in \mathcal{A}$ and any $g \in G$, one has

$$\varphi(g(a)) = \text{tr}(D_{\varphi \circ g} a) = \varphi(x_g a) \leq \|x_g\| \varphi(a) = \|x_g\| \text{tr}(D_\varphi^{1/2} a D_\varphi^{1/2}).$$

By Theorem 7 [23], we have $0 \leq D_{\varphi \circ g} \leq \|x_g\| D_\varphi$. As D_φ has full support, there exists a positive element a_g of \mathcal{N} such that

$$D_{\varphi \circ g} = D_\varphi^{1/2} a_g^2 D_\varphi^{1/2}.$$

Considering the dual action yields that $\theta_s(a_g) = a_g$ for all $s \in \mathbb{R}$ and a_g is a positive operator in \mathcal{A} . Now, notice that by the proof of Lemma 3.1, $\varphi \leq \|x_{g^{-1}}\| \varphi \circ g$. Hence one gets

$$D_{\varphi \circ g} = D_\varphi^{1/2} a_g^2 D_\varphi^{1/2} \geq \alpha D_\varphi,$$

where $\alpha = \|x_{g^{-1}}\|^{-1}$. This proves that $a_g^2 \geq \alpha$ and a_g is invertible. □

Actually, the equation $x_g^* D_\varphi = D_\varphi x_g$ implies that x_g sits in the domain of σ_{-i}^φ and $\sigma_{-i}^\varphi(x_g) = x_g^*$ (see Theorem 3.25 in [22]). In particular, x_g sits in the domain of $\sigma_{-i/2}^\varphi$ and one can check that

$$a_g^2 = \sigma_{-i/2}^\varphi(x_g). \tag{4.3}$$

Now we prove the following.

Theorem 4.2. *For all $g \in G$, the map*

$$\begin{aligned} U_g : \mathcal{A} D_\varphi^{1/2} \subset L_2(\mathcal{A}) &\longrightarrow L_2(\mathcal{A}) \\ x D_\varphi^{1/2} &\longmapsto g^{-1}(x) D_\varphi^{1/2} a_g \end{aligned}$$

extends to a unitary transformation on $L_2(\mathcal{A})$ (still denoted by U_g). Moreover for any $x \in \mathcal{A}$

$$U_g^* x U_g = g(x).$$

Proof. For all $x, y \in \mathcal{A}$, one has

$$\begin{aligned} \langle U_g(xD_\varphi^{1/2}), U_g(yD_\varphi^{1/2}) \rangle_{L_2(\mathcal{A})} &= \text{tr}(a_g D_\varphi^{1/2} g^{-1}(x)^* g^{-1}(y) D_\varphi^{1/2} a_g) \\ &= \text{tr}(D_\varphi^{1/2} a_g^2 D_\varphi^{1/2} g^{-1}(x)^* g^{-1}(y)) \\ &= \varphi(x^* y) \\ &= \langle xD_\varphi^{1/2}, yD_\varphi^{1/2} \rangle_{L_2(\mathcal{A})}. \end{aligned}$$

Since $\mathcal{A} D_\varphi^{1/2}$ is dense in $L_2(\mathcal{A})$, U_g can be extended to an isometry on $L_2(\mathcal{A})$. Now, let $h \in L_2(\mathcal{A}) \cap \text{Ran}(U_g)^\perp$. Then, for all $a \in \mathcal{A}$

$$0 = \langle h, U_g(aD_\varphi^{1/2}) \rangle_{L_2(\mathcal{A})} = \text{tr}(h^* g^{-1}(a) D_\varphi^{1/2} a_g) = \text{tr}(a_g h^* g^{-1}(a) D_\varphi^{1/2}).$$

Notice that $\mathcal{A} D_\varphi^{1/2}$ is dense in $L_2(\mathcal{A})$ and a_g is invertible. Therefore, one gets $a_g h^* = 0$ and $h = 0$. Hence, U_g is a unitary operator on $L_2(\mathcal{A})$.

Now, let $x, y, z \in \mathcal{A}$. Then, one has

$$\begin{aligned} \langle U_g^* x U_g(yD_\varphi^{1/2}), zD_\varphi^{1/2} \rangle_{L_2(\mathcal{A})} &= \langle xg^{-1}(y)D_\varphi^{1/2} a_g, g^{-1}(z)D_\varphi^{1/2} a_g \rangle_{L_2(\mathcal{A})} \\ &= \text{tr}(a_g D_\varphi^{1/2} (g^{-1}(y))^* x^* g^{-1}(z) D_\varphi^{1/2} a_g) \\ &= \varphi(y^* g(x)^* z) \\ &= \langle g(x)yD_\varphi^{1/2}, zD_\varphi^{1/2} \rangle_{L_2(\mathcal{A})}, \end{aligned}$$

which proves that $U_g^* x U_g = g(x)$. □

In the setting of Theorem 3.2, we have that there exists an invertible $a \in \mathcal{A}$ such that $D_\psi = D_\varphi^{1/2} a^2 D_\varphi^{1/2}$. Thus there is a partial isometry v such that $D_\psi^{1/2} = v a D_\varphi^{1/2}$. Since both operators have full support, v is unitary and we deduce that there exists an invertible $\gamma \in \mathcal{A}$ such that $D_\psi^{1/2} = \gamma D_\varphi^{1/2}$. Thus we get $D_\psi = d^* D_\varphi = \gamma D_\varphi \gamma^*$. As above, we get that γ^* sits in the domain of σ_{-i}^φ and $\sigma_{-i}^\varphi(\gamma^*) = \gamma^{-1} d^*$. We conclude that $d^* = \gamma \sigma_{-i}^\varphi(\gamma^*)$. In the same way, let $\gamma_g = g^{-1}(\gamma^{-1})\gamma$ for $g \in G$, then one can also check that γ_g^* sits in the domain of σ_{-i}^φ and that $x_g^* = \gamma_g \sigma_{-i}^\varphi(\gamma_g^*)$.

Proposition 4.3. *If φ is G -strongly quasi-invariant then $g \mapsto U_g^*$ is a unitary representation of the group G .*

Proof. Notice that if φ is G -strongly quasi-invariant, it is showed in [2] that the cocycle $x_g, g \in G$, are in the centralizer of φ . Therefore by (4.3) it follows that $a_g^2 = x_g = x_g^*$ (i.e. $a_g = x_g^{1/2}$). Moreover from (4.2), one has

$$a_g D_\varphi^{1/2} = D_\varphi^{1/2} a_g$$

and hence for all $x \in \mathcal{A}, g \in G$

$$U_g(xD_\varphi^{1/2}) = g^{-1}(x)x_g^{1/2} D_\varphi^{1/2}.$$

By using the cocycle property, for any $g, h \in G, x \in \mathcal{A}$, one has

$$\begin{aligned} U_g U_h(xD_\varphi^{1/2}) &= U_g(h^{-1}(x)x_h^{1/2} D_\varphi^{1/2}) \\ &= g^{-1}(h^{-1}(x))g^{-1}(x_h^{1/2})x_g^{1/2} D_\varphi^{1/2} = U_{hg}(xD_\varphi^{1/2}). \end{aligned}$$

It follows that $g \mapsto U_g^* = U_{g^{-1}}$ is a unitary representation of the group G . □ □

5. A Theorem of Kovács and Szücs for Quasi-invariant States with Uniformly Bounded Cocycle

By Theorem 3.2, if φ is a normal faithful G -quasi-invariant state on a von Neumann algebra \mathcal{A} with cocycle satisfying assumption (A), then there exist a normal faithful G -invariant state ψ on \mathcal{A} and a bounded operator $d \in \mathcal{A}$ such that for all $g \in G$

$$\psi(a) := \varphi(da), \quad a \in \mathcal{A}.$$

Since ψ is a normal faithful G -invariant state on \mathcal{A} , it follows from [8, Theorem 1] that there exists a normal faithful G -invariant Umegaki conditional expectation $\Phi : \mathcal{A} \rightarrow \mathcal{B}$, where \mathcal{B} is a sub-von Neumann algebra of \mathcal{A} , such that

$$\psi(\cdot) = (\psi|_{\mathcal{B}}) \circ \Phi(\cdot)$$

If moreover d is invertible, one gets

$$\varphi = (\varphi(d\cdot)|_{\mathcal{B}}) \circ \Phi(d^{-1}\cdot) \tag{5.1}$$

Now, if φ is a G -strongly quasi-invariant state and if $g \mapsto U_g^*$ is the unitary representation of G defined by Theorem 4.2, we will show that the conditional expectation Φ in (5.1) can also be characterized by U_g , $g \in G$. Define

$$u_g(a) := U_g^* a U_g = g(a) \quad \forall g \in G \quad \forall a \in \mathcal{A} \tag{5.2}$$

Let E_0 be the orthogonal projection onto the subspace of all $\xi \in L_2(\mathcal{A})$ invariant under all U_g . We denote by \mathcal{B} the set of fixed points of \mathcal{A} under the action of the automorphisms u_g , $g \in G$. Let $F_0 = [\mathcal{B}' E_0]$ be the orthogonal projection onto $\overline{\text{span}(\mathcal{B}' E_0 L_2(\mathcal{A}))}$. Finally, let \mathcal{M} be a Godement mean over G . The following result follows from a theorem of Kovács and Szücs (see [14] or [1] and [8, Theorem 1]).

Theorem 5.1. *Assume that φ is a faithful normal G -strongly quasi-invariant state with uniformly bounded cocycle (i.e. cocycle satisfying (A)). Then*

- (1) *There exists a unique normal faithful G -invariant conditional expectation $\Phi : \mathcal{A} \rightarrow \mathcal{B}$. Moreover, $\Phi(b)$ can be defined (equivalently) as*
 - (i) *the unique element of \mathcal{A} such that $\Phi(b)E_0 = E_0 b E_0$*
 - (ii) *the unique element of \mathcal{A} such that*

$$(\xi_1, \Phi(b)\xi_2) = \mathcal{M}\{(\xi_1, U_{g^{-1}} b U_g \xi_2)\}$$

- (2) *A faithful normal state w on \mathcal{A} is G -strongly quasi-invariant with uniformly bounded cocycle if and only if there exists a positive invertible bounded operator d such that $w(\cdot) = (w(d\cdot)|_{\mathcal{B}}) \circ \Phi(d^{-1}\cdot)$*

Proof. Let φ is a faithful normal G -strongly quasi-invariant state with uniformly bounded cocycle. By applying Theorem 3.3, there exists a bounded positive invertible operator d and a state a G -invariant state ψ such that

$$\psi(a) = \varphi(da) = \langle d^{1/2} D_\varphi^{1/2}, a d^{1/2} D_\varphi^{1/2} \rangle_{L_2(\mathcal{A})} \quad \forall a \in \mathcal{A}.$$

Remember that $x_g = d g^{-1} (d^{-1}) = x_g^*$. Moreover for all $g \in G$

$$U_g d^{1/2} D_\varphi^{1/2} = g^{-1} (d^{1/2}) x_g^{1/2} D_\varphi^{1/2} = d^{1/2} D_\varphi^{1/2}.$$

Let p be the orthogonal projection onto $\overline{\text{span}\{B'd^{1/2}D_\varphi^{1/2}\}}$. Then $p \in \mathcal{B}$ and $p \leq F_0$. Moreover, one has

$$\begin{aligned} \psi(p) &= \langle d^{1/2}D_\varphi^{1/2}, pd^{1/2}D_\varphi^{1/2} \rangle_{L_2(\mathcal{A})} \\ &= \langle d^{1/2}D_\varphi^{1/2}, d^{1/2}D_\varphi^{1/2} \rangle_{L_2(\mathcal{A})} \\ &= \varphi(d) = \psi(I) = 1. \end{aligned}$$

Since ψ is faithful, $p = I$ and $F_0 = I$. Finally the results of the above theorem follows from [8, Theorem 1]. □

Now we provide a commutative easy example of faithful normal G -quasi-invariant state such that the cocycle is not uniformly bounded. Moreover, we show in this example that there is no G -invariant conditional expectation.

Example. On $\mathcal{A} = L^\infty(\mathbb{R})$, define the state

$$\varphi(f) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(s)}{1+s^2} ds.$$

The additive group \mathbb{R} acts on \mathcal{A} by translation:

$$\tau_t(f)(s) := f(s - t)$$

It is clear that φ is \mathbb{R} -quasi-invariant state with bounded cocycle

$$x_t(s) = \frac{1+s^2}{1+(s+t)^2}.$$

The cocycle $(x_t)_t$ does not satisfy assumption (A) since $\|x_t\|, \|x_t^{-1}\| \geq 1+t^2$. The fixed point of the action are constant functions. Thus every conditional expectation Φ on $L^\infty(\mathbb{R})$ has to be of the form

$$\Phi(f) = \psi(f)1 \tag{5.3}$$

for some normal state ψ . But there are no functions in L_1 that are \mathbb{R} -invariant.

This example also shows that it seems unlikely to improve Theorem 3.2.

6. Strongly Quasi-invariant States and Semifinite Trace

In this section we extend the result developed by Størmer [20] to the case of strongly quasi-invariant states with uniformly bounded cocycles. Given a semifinite von Neumann algebra \mathcal{A} , a group G of $*$ -automorphisms is said to act ergodically on the center $\mathcal{Z} = \mathcal{A} \cap \mathcal{A}'$ of \mathcal{A} if $\mathcal{F}(G) \cap \mathcal{Z} = \mathbb{C}1$, where $\mathcal{F}(G)$ is the set of the fixed points of G in \mathcal{A} .

Theorem 6.1. *Let \mathcal{A} be a semifinite von Neumann algebra and let G be an amenable group of $*$ -automorphisms of \mathcal{A} and assume that G acts ergodically on the center \mathcal{Z} of \mathcal{A} . Suppose that φ is a faithful normal G -quasi-invariant state on \mathcal{A} with cocycle x_g 's satisfying assumption (A). Then, there exists up to a scalar multiple a unique faithful normal G -invariant semifinite trace τ of \mathcal{A} . Furthermore the density $c \in L_1(\mathcal{A}, \tau)^+$ of φ with respect to τ satisfies*

$$(g^{-1})^*(c) = cx_{g^{-1}}, \quad x_g^*c = cx_g, \quad \forall g \in G, \tag{6.1}$$

where $(g^{-1})^* : L_1(\mathcal{A}, \tau) \rightarrow L_1(\mathcal{A}, \tau)$ is the predual map of g^{-1} on \mathcal{A} .

Proof. Under assumptions of the above theorem, it follows from Theorem 3.2 that there exists a normal faithful G -invariant state ψ . Applying [[20], Theorem 1] for ψ , there exists up to a scalar multiple a unique faithful normal G -invariant semifinite trace τ on \mathcal{A} . Let $c \in L_1(\mathcal{A}, \tau)^+$ be the density of φ . For all $a \in \mathcal{A}$, we have

$$\tau((g^{-1})^*(c)a) = \tau(cg^{-1}(a)) = \varphi(g^{-1}(a)) = \varphi(x_{g^{-1}a}) = \tau(cx_{g^{-1}a}).$$

This proves the first equality (6.1). For the second, note that for all $a \in \mathcal{A}$

$$\tau(cx_g a) = \varphi \circ g(a) = \overline{\varphi \circ g(a^*)} = \overline{\tau(cx_g a^*)} = \tau(x_g^* ca).$$

This implies that $x_g^* c = cx_g$ in $L_1(\mathcal{A}, \tau)$. □

Example. Taking again $\mathcal{A} = L^\infty(\mathbb{R})$ and G to be the $ax + b$ group of all automorphisms of \mathcal{A} of the form $\pi_{(a,b)} f(t) = f(at + b)$ for $(a, b) \in \mathbb{R}^* \times \mathbb{R}$. Then G is amenable (since it is solvable), it admits a cocycle for the state given by the measure $\frac{1}{\pi} \frac{1}{1+t^2} dt$; $x_{(a,b)}(t) = \frac{a(1+t^2)}{a^2+(t-b)^2}$ that is not uniformly bounded. Moreover, one can check that the fixed point algebra by G consists only in constant functions. Similarly, one also easily sees that there are no measures on \mathbb{R} invariant by G . Thus one cannot remove the assumption (A) even in a commutative situation.

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