Multibody System Dynamics
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RESEARCH

Eigensolution continuation by projection sensitivity of minimal coordinate set multibody systems

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Abstract
Constrained dynamics problems naturally result in systems of differential-algebraic equations, which can be reduced to the underlying systems of ordinary differential equations with the minimum required number of coordinates using projection algorithms. Their linearization about a steady reference solution can be interpreted as a generalized eigenproblem whose eigensolution represents the spectrum of the problem in that configuration. The eigensolution sensitivity to the original problem's specific parameters of interest may provide useful insight into the system's dynamic characteristics. Building on an original continuation approach for updating the coordinate projection matrix, the sensitivity of the eigensolution is here formulated and analyzed in analytical and numerical form for simple, yet illustrative, problems.

Keywords Minimal coordinate set · Coordinate projection · Eigenanalysis · Eigensolution sensitivity · QR factorization

1 Introduction

Constrained system dynamics can be described by a minimal set of coordinates using coordinate projection onto the coordinate subspace tangent to the constraint manifold. Among the possible choices for coordinate selection (see, for example, [1, 2]), an optimal one is based on the QR factorization [3] of the constraint Jacobian matrix, as originally proposed by Kim and Vanderploeg [4].

The QR-based coordinate projection subspace must be updated frequently to maintain its optimality. If carelessly implemented, the update algorithm can result in unnecessary, although harmless, discontinuities in the generalized velocities. The algorithm originally proposed in [5, 6] and generalized in [7] to the case of redundant constraints and singular configurations uses the notion of continuation to minimize the modification of the projection subspace during these updates.

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This work, originally presented in [8], addresses the linearization of constrained dynamics problems about reference configurations, their eigenanalysis, and the sensitivity of the resulting eigensolutions when problems are formulated using coordinate projection.

The general problem of linearization and eigenanalysis of constrained dynamics problems was recently discussed in [9], including the case of coordinate projection formulations; however, the sensitivity of its eigensolution was not considered. The latter was addressed, for example, in [10], resorting to a formulation of the sensitivity of the coordinate projection matrix that either requires its analytical expression or needs to be performed numerically, for example, using finite differences.

In this work, the sensitivity of the projection matrix is proficiently derived from the projection subspace continuation formulation proposed in [5–7], with an algorithm that is purely numerical but locally exact, as it relies on the analytical sensitivity of the constraint Jacobian matrix. The sensitivity of the solution of coordinate projection problems using the QR factorization to determine an optimal projection matrix was also recently proposed in [11].

Thanks to its capability to obtain the projection matrix's sensitivity, and that of derived matrices and other quantities, the proposed continuation of the coordinate projection matrix provides a consistent and effective framework to formulate the sensitivity of eigensolutions. It is instrumental in tracking an eigensolution's evolution when the parameters on which the problem depends are modified.

As discussed in [7]—although not dealt with in this paper for conciseness—the proposed formulation can be extended and generalized to the cases of redundant constraints and singular configurations.

The general multibody dynamics problem is formulated in a minimal coordinate set in Sect. 2. Its linearization is described in Sect. 3, and the corresponding eigenproblem and its sensitivity are discussed in Sect. 4. The use of the QR factorization for coordinate projection and its implications on linearization, eigensolution, and sensitivity of the latter are discussed in Sect. 5. The analytical and numerical solutions to problems of increasing complexity are presented in Sect. 6, and conclusions are drawn in Sect. 7.

2 Formulation

2.1 Constrained dynamics

Consider the unconstrained dynamics problem

$$\mathbf{M}(\mathbf{x}, p)\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t, p), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ are the unconstrained coordinates, $\mathbf{M} \in \mathbb{R}^{n\times n}$ is the (symmetric, positive-definite) mass matrix, possibly dependent on the coordinates (such dependence will be dropped in the following, for ease of notation), $\mathbf{f} \in \mathbb{R}^n$ is the vector of the (time-, $t \in \mathbb{R}$, and configuration-dependent) applied loads, and $p \in P$ is a generic real-valued parameter in a parameter set P.

Consider now a set of algebraic (holonomic, possibly rheonomic, i.e., time-dependent) equations,

$$\mathbf{c}(\mathbf{x}, t, p) = \mathbf{0}, \tag{2}$$

with $\mathbf{c} \in \mathbb{R}^m$, the array of algebraic equations that describe the kinematic constraints the original coordinates $\mathbf{x}$ are subjected to. For ease of notation, the dependence on the parameter p is dropped in the following, until the sensitivity problem is addressed.

The resulting constrained problem is a system of differential-algebraic equations (DAE)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{3a}$$

$$\mathbf{c}(\mathbf{x}, t) = \mathbf{0}, \tag{3b}$$

where $\mathbf{A} = \mathbf{c}_{/\mathbf{x}} \in \mathbb{R}^{m \times n}$ is the constraint Jacobian matrix ($\spadesuit_{/\clubsuit}$ indicates partial differentiation of $\spadesuit$ with respect to $\clubsuit$) and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is the array of Lagrange multipliers. The term $\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}$ represents the reaction forces caused by the ideal constraints $\mathbf{c}$.

The problem of Eqs. (3a)–(3b) is expressed according to the so-called *Redundant Coordinate Set* (RCS) approach, which results, for example, from Lagrange's formalism of the first kind (see, for example, [12]).

2.2 Coordinate projection

Consider the time derivative of the constraint equation,

$$\dot{\mathbf{c}} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{c}_{/t} = \mathbf{A}\dot{\mathbf{x}} + \mathbf{b}' = \mathbf{0}, \tag{4}$$

and determine a coordinate projection matrix $\mathbf{T} \in \mathbb{R}^{n \times (n-m)}$, with $\mathbf{AT} \equiv \mathbf{0}$ by construction, which yields

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{t}', \tag{5}$$

where $\mathbf{q} \in \mathbb{R}^{n-m}$ are truly independent Lagrangian coordinates, and

$$\mathbf{A}\left(\mathbf{T}\dot{\mathbf{q}} + \mathbf{t}'\right) + \mathbf{c}_{/t} = \cancel{\mathbf{AT}}\dot{\mathbf{q}} + \mathbf{A}\mathbf{t}' + \mathbf{c}_{/t} = \mathbf{0}. \tag{6}$$

How $\mathbf{t}' \in \mathbb{R}^n$ is computed is inessential at the moment and will be discussed later in Sect. 5.2, Eq. (52), where its formulation using the QR factorization of the constraint Jacobian matrix becomes straightforward.

2.3 Minimal coordinate set

Consider now the second time derivative of the constraint equation,

$$\ddot{\mathbf{c}} = \mathbf{A}\ddot{\mathbf{x}} + \left(\mathbf{A}\dot{\mathbf{x}} + \mathbf{c}_{/t}\right)_{/\mathbf{x}}\dot{\mathbf{x}} + \left(\mathbf{A}\dot{\mathbf{x}} + \mathbf{c}_{/t}\right)_{/t} = \mathbf{A}\ddot{\mathbf{x}} + \mathbf{b}'' = \mathbf{0} \tag{7}$$

and

$$\ddot{\mathbf{x}} = \mathbf{T}\ddot{\mathbf{q}} + \mathbf{t}'', \tag{8}$$

which yields

$$\mathbf{A}\left(\mathbf{T}\ddot{\mathbf{q}} + \mathbf{t}''\right) + \mathbf{b}'' = \cancel{\mathbf{AT}}\ddot{\mathbf{q}} + \mathbf{A}\mathbf{t}'' + \mathbf{b}'' = \mathbf{0}. \tag{9}$$

Again, how $\mathbf{t}'' \in \mathbb{R}^n$ is determined as inessential at the moment and will be discussed later in Sect. 5.2, Eq. (54).

Premultiplying Eq. (3a) by $\mathbf{T}^{\mathrm{T}}$ and replacing in it $\ddot{\mathbf{x}}$ from Eq. (8) yields

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\left(\mathbf{T}\ddot{\mathbf{q}}+\mathbf{t}''\right)+\cancel{\mathbf{T}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}}=\mathbf{T}^{\mathrm{T}}\mathbf{f}, \tag{10}$$

or

$$\hat{\mathbf{M}}\ddot{\mathbf{q}}=\hat{\mathbf{f}}, \tag{11}$$

with

$$\hat{\mathbf{M}}=\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}, \tag{12a}$$

$$\hat{\mathbf{f}}=\mathbf{T}^{\mathrm{T}}\left(\mathbf{f}-\mathbf{M}\mathbf{t}''\right). \tag{12b}$$

The problem projected in the subspace tangent to the constraint manifold is

$$\hat{\mathbf{M}}\ddot{\mathbf{q}}=\hat{\mathbf{f}}, \tag{13a}$$

$$\dot{\mathbf{x}}=\mathbf{T}\dot{\mathbf{q}}+\mathbf{t}'. \tag{13b}$$

It is a system of purely ordinary differential equations (ODE), formulated according to the so-called *Minimal Coordinate Set* (MCS) approach [13], in what are known as Maggi–Kane equations [14–16].

The form of Eqs. (13a)–(13b) implies the simultaneous integration of the minimal set equations of motion, Eq. (13a), and of the definition of the time derivatives of the original unconstrained coordinates, $\dot{\mathbf{x}}$, Eq. (13b), because the original unconstrained coordinates, $\mathbf{x}$, might be needed to determine configuration-dependent forces (e.g., elastic) in $\hat{\mathbf{f}}$.

However, the value of $\mathbf{x}$ resulting from the numerical integration of Eq. (13b) may not exactly satisfy the original constraint equation, Eq. (2), because of numerical drift. Consequently, $\mathbf{x}$ may need to be corrected by bringing it back onto the constraint manifold along some preferable direction, which will be discussed later in Sect. 5.2, Eq. (60).

3 Linearization

3.1 Linearization of the problem in DAE form

Consider a steady equilibrium condition $\mathbf{x}_0$, $\boldsymbol{\lambda}_0$ (namely, with $\ddot{\mathbf{x}}\equiv\mathbf{0}$, $\dot{\mathbf{x}}\equiv\mathbf{0}$, and no explicit time dependence either in the constraint equation, Eq. (2), or in the loads, $\mathbf{f}$), resulting from the solution of

$$\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0=\mathbf{f}\left(\mathbf{x}_0,\mathbf{0}\right), \tag{14a}$$

$$\mathbf{c}\left(\mathbf{x}_0\right)=\mathbf{0}, \tag{14b}$$

which, if exists, yields a reference value $\mathbf{x}_0$ that satisfies Eq. (14b) and the corresponding Lagrange multipliers vector

$$\boldsymbol{\lambda}_0=(\mathbf{A}^{\mathrm{T}})^{+}\mathbf{f}\left(\mathbf{x}_0,\mathbf{0}\right), \tag{15}$$

where $(\spadesuit)^{+}$ denotes the (Moore–Penrose) generalized inverse (MPGI, or pseudo-inverse [17, 18]) of matrix $(\spadesuit)$, namely $(\mathbf{A}^{\mathrm{T}})^{+}=(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{A}$, such that $(\mathbf{A}^{\mathrm{T}})^{+}\mathbf{A}^{\mathrm{T}}\equiv\mathbf{I}$; recall that both $\mathbf{A}$ and $\mathbf{f}$ may depend on $\mathbf{x}$.

Consider now a perturbation of Eqs. (3a)–(3b) about the steady equilibrium condition that solves Eqs. (14a)–(14b), namely

$$\mathbf{M}\Delta\ddot{\mathbf{x}} + \mathbf{A}^{\mathrm{T}}\Delta\boldsymbol{\lambda} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\Delta\mathbf{x} = \mathbf{f}_{/\mathbf{x}}\Delta\mathbf{x} + \mathbf{f}_{/\dot{\mathbf{x}}}\Delta\dot{\mathbf{x}}, \tag{16a}$$

$$\mathbf{A}\Delta\mathbf{x} = \mathbf{0}, \tag{16b}$$

where $\mathbf{f}_{/\mathbf{x}} = -\mathbf{K}$ is the opposite of the stiffness matrix and $\mathbf{f}_{/\dot{\mathbf{x}}} = -\mathbf{C}$ is the opposite of the damping matrix. The expression $\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}$ indicates the gradient of the reaction forces with respect to the unconstrained coordinates, $\mathbf{x}$, under the assumption that the constraint Jacobian matrix, $\mathbf{A}$, may present a further dependence on them; $\boldsymbol{\lambda}_0$ is the value of the Lagrange multipliers in the reference configuration.

In general, the derivative with respect to a vector $\mathbf{x}$ of a vector $\mathbf{v}$ represented by a matrix $\mathbf{U}$ that depends on $\mathbf{x}$, $\mathbf{U}(\mathbf{x})$, multiplied by a vector $\mathbf{u}$, namely $\mathbf{v}_{/\mathbf{x}} = (\mathbf{U}\mathbf{u})_{/\mathbf{x}}$, yields a matrix whose elements are

$$\left[\mathbf{U}_{/\mathbf{x}}\mathbf{u}\right]_{ij} = U_{ik/x_j} u_k, \tag{17}$$

where U_{ik} is the element of $\mathbf{U}$ of row index i and column index k and u_k is the element of $\mathbf{u}$ of row index k, assuming summation of terms with repeated indices.

Considering that $A_{ki} = c_{k/x_i}$, the generic element of matrix $\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}$ is thus

$$\left[\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\right]_{ij} = \left(c_{k/x_i}\lambda_{0_k}\right)_{/x_j} = c_{k/x_i x_j}\lambda_{0_k}, \tag{18}$$

which confirms the structural symmetry of the resulting matrix.

The linearized problem can be rearranged as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\mathbf{x}} \\ \Delta\ddot{\boldsymbol{\lambda}} \end{Bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\dot{\mathbf{x}} \\ \Delta\dot{\boldsymbol{\lambda}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{x} \\ \Delta\boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{19}$$

The DAE nature of the problem of Eq. (19) is testified by the structural singularity of the generalized mass and damping matrices.

3.2 Linearization of the projected problem

Define

$$\Delta\mathbf{x} = \mathbf{T}\Delta\mathbf{q}, \tag{20a}$$

$$\Delta\dot{\mathbf{x}} = \mathbf{T}\Delta\dot{\mathbf{q}}, \tag{20b}$$

$$\Delta\ddot{\mathbf{x}} = \mathbf{T}\Delta\ddot{\mathbf{q}}. \tag{20c}$$

Equations (20b) and (20c) directly stem from the linearization of Eqs. (5) and (8). Equation (20a), instead, stems from the consideration that since by definition

$$\frac{\partial\mathbf{x}}{\partial\mathbf{q}} \equiv \frac{\partial\dot{\mathbf{x}}}{\partial\dot{\mathbf{q}}} = \mathbf{T}, \tag{21}$$

it implies

$$\Delta\mathbf{x} = \frac{\partial\mathbf{x}}{\partial\mathbf{q}}\Delta\mathbf{q} = \frac{\partial\dot{\mathbf{x}}}{\partial\dot{\mathbf{q}}}\Delta\mathbf{q} = \mathbf{T}\Delta\mathbf{q}. \tag{22}$$

Premultiply Eq. (16a) by $\mathbf{T}^{\mathrm{T}}$ as in Eq. (13a), and replace the occurrences of the perturbations of the unconstrained coordinates and their time derivatives with the expressions from Eqs. (20a)–(20c):

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}\Delta\ddot{\mathbf{q}} + \cancel{\mathbf{T}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}}\Delta\boldsymbol{\lambda} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T}\Delta\mathbf{q} = \mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\mathbf{x}}\mathbf{T}\Delta\mathbf{q} + \mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\dot{\mathbf{x}}}\mathbf{T}\Delta\dot{\mathbf{q}}, \tag{23a}$$

$$\cancel{\mathbf{A}\mathbf{T}}\Delta\mathbf{q} = \mathbf{0}. \tag{23b}$$

Equation (23b) is implicitly satisfied by the definition of matrix $\mathbf{T}$. Equation (23a) can be rearranged as

$$\mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}\Delta\ddot{\mathbf{q}} + \mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T}\Delta\dot{\mathbf{q}} + \left(\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T}\right)\Delta\mathbf{q} = \mathbf{0}, \tag{24}$$

or

$$\hat{\mathbf{M}}\Delta\ddot{\mathbf{q}} + \hat{\mathbf{C}}\Delta\dot{\mathbf{q}} + \hat{\mathbf{K}}\Delta\mathbf{q} = \mathbf{0}, \tag{25}$$

with

$$\hat{\mathbf{C}} = \mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T} = -\mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\dot{\mathbf{x}}}\mathbf{T}, \tag{26a}$$

$$\hat{\mathbf{K}} = \mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T} = -\mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\mathbf{x}}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T}, \tag{26b}$$

in addition to matrix $\hat{\mathbf{M}}$ as defined in Eq. (12a).

4 Eigenanalysis

4.1 Generalized eigenanalysis in DAE form

The problem of Eq. (19) can be further rearranged as

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\mathbf{x}} \\ \Delta\dot{\mathbf{x}} \\ \Delta\dot{\boldsymbol{\lambda}} \end{Bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{K} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} & \mathbf{A}^{\mathrm{T}} \\ -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\dot{\mathbf{x}} \\ \Delta\mathbf{x} \\ \Delta\boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{27}$$

Consider a solution of the form

$$\begin{Bmatrix} \Delta\dot{\mathbf{x}} \\ \Delta\mathbf{x} \\ \Delta\boldsymbol{\lambda} \end{Bmatrix} = \mathbf{y}e^{st}. \tag{28}$$

The generalized eigenproblem associated with Eq. (27) is

$$(s\boldsymbol{\mathcal{E}} - \boldsymbol{\mathcal{A}})\,\mathbf{y} = \mathbf{0}, \tag{29}$$

with matrices $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{A}}$ defined accordingly.

This equation may be solved as a generalized eigenproblem using specialized algorithms, such as the QZ (or generalized Schur) decomposition [19], which determines the spectrum of the underlying differential part in the form of pairs of eigenvalues s_k and the corresponding eigenvectors $\mathbf{y}_k$, when defined. The problem has $2(n-m)$ finite-valued eigenvalues, either real or complex conjugated, since the problem's matrices are real-valued. The remaining $3m$ eigenvalues have infinite value (they are not part of the problem spectrum).

Alternatively, the generalized eigenproblem can be directly formulated as second-order differential-algebraic, namely

$$\left(s^2\begin{bmatrix}\mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+s\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}+\begin{bmatrix}\mathbf{K}+\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} & \mathbf{A}^{\mathrm{T}}\\ \mathbf{A} & \mathbf{0}\end{bmatrix}\right)\begin{Bmatrix}\Delta\mathbf{x}\\ \Delta\boldsymbol{\lambda}\end{Bmatrix}=\begin{Bmatrix}\mathbf{0}\\ \mathbf{0}\end{Bmatrix}. \tag{30}$$

4.2 Generalized eigenanalysis of the projected problem

Consider a solution of the form

$$\Delta\mathbf{q}=\mathbf{z}e^{st} \tag{31}$$

and replace it in Eq. (25); this yields

$$\left(s^2\hat{\mathbf{M}}+s\hat{\mathbf{C}}+\hat{\mathbf{K}}\right)\mathbf{z}=\mathbf{0}, \tag{32}$$

namely the second-order generalized eigenvalue problem characteristic of structural dynamics.

Its solution is the set of k pairs, with $k\in 1..2(n-m)$, formed by the eigenvalues s_k and their corresponding eigenvectors $\mathbf{z}_k$. Since the matrices are real-valued, eigenvalues appear either as real values or pairs of complex-conjugated values; in the latter case, also the eigenvectors are complex-conjugated, whereas in the former they are real.

The eigensolutions of the projected problem correspond to those of the problem formulated as a DAE with finite-valued eigenvalues.

4.3 Eigensolution sensitivity

The sensitivity of an eigensolution is studied by forming a nonlinear problem with Eq. (32) and the real scalar eigenvector normalization equation

$$\frac{1}{2}\mathrm{conj}(\mathbf{z})^{\mathrm{T}}\mathbf{z}=\frac{1}{2}, \tag{33}$$

which imposes the eigenvector to have unit norm.

The sensitivity of the nonlinear problem with respect to the scalar parameter p yields[1]

$$\left(s^2\hat{\mathbf{M}}+s\hat{\mathbf{C}}+\hat{\mathbf{K}}\right)\mathbf{z}_{/p}+\left(2s\hat{\mathbf{M}}+\hat{\mathbf{C}}\right)\mathbf{z}s_{/p}+\left(s^2\hat{\mathbf{M}}_{/p}+s\hat{\mathbf{C}}_{/p}+\hat{\mathbf{K}}_{/p}\right)\mathbf{z}=\mathbf{0}, \tag{34a}$$

$$\mathrm{conj}(\mathbf{z})^{\mathrm{T}}\mathbf{z}_{/p}=0, \tag{34b}$$

[1] It is worth noticing that Eq. (34b) is not directly the sensitivity of Eq. (33); in fact,

$$\frac{\partial}{\partial p}\left(\frac{1}{2}\mathrm{conj}(\mathbf{z})^{\mathrm{T}}\mathbf{z}-\frac{1}{2}\right)=\mathrm{conj}(\mathbf{z})^{\mathrm{T}}\mathbf{z}_{/p}+\mathbf{z}^{\mathrm{T}}\mathrm{conj}(\mathbf{z}_{/p})=\mathrm{Re}(\mathbf{z})^{\mathrm{T}}\mathrm{Re}(\mathbf{z})_{/p}+\mathrm{Im}(\mathbf{z})^{\mathrm{T}}\mathrm{Im}(\mathbf{z})_{/p},$$

with $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ respectively indicating the real and imaginary parts of a complex argument. Such an expression cannot be written as a row vector that premultiplies $\mathbf{z}_{/p}$. However, one can show that the real part of Eq. (34b) corresponds to the above reported exact sensitivity of Eq. (33), whereas setting its imaginary part to zero arbitrarily eliminates the indetermination on the phase of the eigenvector.

which can be rearranged as

$$\begin{bmatrix} \left(s^2\hat{\mathbf{M}} + s\hat{\mathbf{C}} + \hat{\mathbf{K}}\right) & \left(2s\hat{\mathbf{M}} + \hat{\mathbf{C}}\right)\mathbf{z} \\ \operatorname{conj}(\mathbf{z})^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{z}_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} -\left(s^2\hat{\mathbf{M}}_{/p} + s\hat{\mathbf{C}}_{/p} + \hat{\mathbf{K}}_{/p}\right)\mathbf{z} \\ 0 \end{Bmatrix}, \tag{35}$$

i.e., a linear problem in the sensitivity of the eigensolution, whose matrix only depends on the original problem and solution, and whose right-hand side depends on the sensitivity of the problem's matrices, namely

$$\hat{\mathbf{M}}_{/p} = \mathbf{T}^{\mathrm{T}}_{/p}\mathbf{M}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\mathbf{M}_{/p}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\mathbf{M}\mathbf{T}_{/p}, \tag{36a}$$

$$\hat{\mathbf{C}}_{/p} = \mathbf{T}^{\mathrm{T}}_{/p}\mathbf{C}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\mathbf{C}_{/p}\mathbf{T} + \mathbf{T}^{\mathrm{T}}\mathbf{C}\mathbf{T}_{/p}, \tag{36b}$$

$$\begin{aligned} \hat{\mathbf{K}}_{/p} = {} & \mathbf{T}^{\mathrm{T}}_{/p}\left(\mathbf{K} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\right)\mathbf{T} + \mathbf{T}^{\mathrm{T}}\left(\mathbf{K}_{/p} + \left(\mathbf{A}^{\mathrm{T}}_{/p}\boldsymbol{\lambda}_0 + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_{0/p}\right)_{/\mathbf{x}}\right)\mathbf{T} \\ & + \mathbf{T}^{\mathrm{T}}\left(\mathbf{K} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\right)\mathbf{T}_{/p}. \end{aligned} \tag{36c}$$

The sensitivity of the constraint reactions must comply with the sensitivity of the equation of motion at the static equilibrium solution,

$$\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/p} = \mathbf{A}^{\mathrm{T}}_{/p}\boldsymbol{\lambda}_0 + \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_{0/p} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{x}_{0/p} = \mathbf{f}_{/\mathbf{x}}\mathbf{x}_{/p} + \mathbf{f}_{/p} = -\mathbf{K}\mathbf{x}_{0/p} + \mathbf{f}_{/p}, \tag{37a}$$

$$\mathbf{A}\mathbf{x}_{0/p} + \mathbf{c}_{/p} = \mathbf{0}, \tag{37b}$$

which can be rearranged as

$$\begin{bmatrix} \mathbf{K} + \left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_{0/p} \\ \boldsymbol{\lambda}_{0/p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{/p} - \mathbf{A}^{\mathrm{T}}_{/p}\boldsymbol{\lambda}_0 \\ -\mathbf{c}_{/p} \end{Bmatrix}. \tag{38}$$

It is worth noticing that to evaluate the sensitivity to several parameters the matrix of Eq. (35) only needs to be factored once, as the explicit sensitivity of the problem's matrices with respect to the parameters only affects the right-hand side of Eq. (35). Similar considerations apply to Eq. (38).

4.4 Alternative formulation of projection linearization

The formulation of the sensitivity of the eigensolution, which implies computing the sensitivity of the projection matrix $\mathbf{T}$ with respect to an arbitrary parameter, suggests an alternative way to formulate the linearization of the projected problem. Indeed, in the projected problem the Lagrange multipliers play no role since the constraint Jacobian matrix is orthogonal to the projection matrix: $\mathbf{A}\mathbf{T} \equiv \mathbf{0}$ by construction.

Starting from the projected problem of Eq. (10), one could consistently linearize it by considering its perturbation as

$$\Delta\left(\hat{\mathbf{M}}\ddot{\mathbf{q}} - \hat{\mathbf{f}}\right) = \mathbf{0}, \tag{39}$$

yielding

$$\hat{\mathbf{M}}\Delta\ddot{\mathbf{q}} - \frac{\partial\hat{\mathbf{f}}}{\partial\dot{\mathbf{q}}}\Delta\dot{\mathbf{q}} + \frac{\partial}{\partial\mathbf{q}}\left(\hat{\mathbf{M}}\ddot{\mathbf{q}} - \hat{\mathbf{f}}\right)\Delta\mathbf{q} = \mathbf{0}. \tag{40}$$

Consider

$$\frac{\partial(\cdot)}{\partial\dot{\mathbf{q}}}=\frac{\partial(\cdot)}{\partial\dot{\mathbf{x}}}\frac{\partial\dot{\mathbf{x}}}{\partial\dot{\mathbf{q}}}=\frac{\partial(\cdot)}{\partial\dot{\mathbf{x}}}\mathbf{T}, \tag{41a}$$

$$\frac{\partial(\cdot)}{\partial\mathbf{q}}=\frac{\partial(\cdot)}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\mathbf{q}}\equiv\frac{\partial(\cdot)}{\partial\mathbf{x}}\mathbf{T}, \tag{41b}$$

as per Eq. (21). Equation (40) becomes

$$\hat{\mathbf{M}}\Delta\ddot{\mathbf{q}}-\frac{\partial\hat{\mathbf{f}}}{\partial\dot{\mathbf{x}}}\mathbf{T}\Delta\dot{\mathbf{q}}+\frac{\partial}{\partial\mathbf{x}}\left(\hat{\mathbf{M}}\ddot{\mathbf{q}}-\hat{\mathbf{f}}\right)\mathbf{T}\Delta\mathbf{q}=\mathbf{0}, \tag{42}$$

with

$$-\frac{\partial\hat{\mathbf{f}}}{\partial\dot{\mathbf{x}}}=-\frac{\partial}{\partial\dot{\mathbf{x}}}\left[\mathbf{T}^{\mathrm{T}}\left(\mathbf{f}-\mathbf{M}\mathbf{t}''\right)\right]=\mathbf{T}^{\mathrm{T}}\left(\mathbf{M}\mathbf{t}''-\mathbf{f}\right)_{/\dot{\mathbf{x}}}, \tag{43}$$

since matrix $\mathbf{T}$ cannot depend on $\dot{\mathbf{x}}$, and

$$\begin{aligned}\frac{\partial}{\partial\mathbf{x}}\left(\hat{\mathbf{M}}\ddot{\mathbf{q}}-\hat{\mathbf{f}}\right)&=\frac{\partial}{\partial\mathbf{x}}\left[\mathbf{T}^{\mathrm{T}}\left(\mathbf{M}\mathbf{T}\ddot{\mathbf{q}}+\mathbf{M}\mathbf{t}''-\mathbf{f}\right)\right]\\&=\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\left(\mathbf{M}\mathbf{T}\ddot{\mathbf{q}}+\mathbf{M}\mathbf{t}''-\mathbf{f}\right)\\&\quad+\mathbf{T}^{\mathrm{T}}\left[\mathbf{M}_{/\mathbf{x}}\left(\mathbf{T}\ddot{\mathbf{q}}+\mathbf{t}''\right)+\mathbf{M}\left(\mathbf{T}_{/\mathbf{x}}\ddot{\mathbf{q}}+\mathbf{t}''_{/\mathbf{x}}\right)-\mathbf{f}_{/\mathbf{x}}\right].\end{aligned} \tag{44}$$

Assuming that the reference solution describes a static equilibrium condition, with $\ddot{\mathbf{q}}=\mathbf{0}$, $\dot{\mathbf{q}}=\mathbf{0}$, and thus $\mathbf{t}''=\mathbf{0}$ as well, the two contributions boil down to

$$-\frac{\partial\hat{\mathbf{f}}}{\partial\dot{\mathbf{x}}}=-\mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\dot{\mathbf{x}}}=\mathbf{T}^{\mathrm{T}}\mathbf{C}, \tag{45a}$$

$$\frac{\partial}{\partial\mathbf{x}}\left(\hat{\mathbf{M}}\ddot{\mathbf{q}}-\hat{\mathbf{f}}\right)=-\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\mathbf{f}-\mathbf{T}^{\mathrm{T}}\mathbf{f}_{/\mathbf{x}}=-\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\mathbf{f}+\mathbf{T}^{\mathrm{T}}\mathbf{K}. \tag{45b}$$

The elements of $\mathbf{T}_{/\mathbf{x}}$ can be found by independently evaluating the sensitivity of matrix $\mathbf{T}$ with respect to each coordinate in $\mathbf{x}$. The procedure will be illustrated later in Sect. 5.3, when dealing with the proposed method to determine the coordinate projection matrix.

Comparing Eqs. (45b) and (26b), one notices the equivalence

$$\mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T}\equiv-\left(\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\mathbf{f}\right)\mathbf{T}, \tag{46}$$

or

$$\left[\mathbf{T}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{T}\right]_{ij}=T_{hi}A_{\ell h/x_k}\lambda_{0_\ell}T_{kj}=-T_{\ell i/x_k}f_\ell T_{kj}=-\left[\left(\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\mathbf{f}\right)\mathbf{T}\right]_{ij}, \tag{47}$$

which will also be further developed later, in Sect. 5.4.

5 Coordinate projection via QR factorization

5.1 QR factorization

As anticipated, a convenient coordinate projection matrix can be determined using the QR factorization [3]. Consider

$$\mathbf{A}^{\mathrm{T}} = \mathbf{Q}\mathbf{R} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_1\mathbf{R}_1, \tag{48}$$

where matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is orthonormal, with its submatrices $\mathbf{Q}_1 \in \mathbb{R}^{n\times m}$ and $\mathbf{Q}_2 \in \mathbb{R}^{n\times(n-m)}$, respectively, and submatrix $\mathbf{R}_1 \in \mathbb{R}^{m\times m}$ is upper triangular.

5.2 Coordinate projection

A valid and optimal choice for the projection matrix $\mathbf{T}$ is submatrix $\mathbf{Q}_2$, since by construction $\mathbf{A}\mathbf{T} = \mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_2 \equiv \mathbf{0}$, thanks to the orthonormality of matrix $\mathbf{Q}$, which implies $\mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_2 \equiv \mathbf{0}$. This approach, originally proposed by Kim and Vanderploeg [4], was further discussed by Shabana [20]. Apart from coordinate partitioning (see, for example, Wehage and Haug [21]), alternative approaches can be developed, for example, using Singular Value Decomposition (SVD) [22], as proposed by Mani et al. [23], the Zero Eigenvalue theorem proposed by Steeves and Walton [24], or a Gram–Schmidt process to generate the nullspace of admissible motions, as suggested by Liang and Lance [25].

Considering the definition of the projected coordinate derivative, Eq. (5), one obtains

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{t}' = \mathbf{Q}_2\dot{\mathbf{q}} + \mathbf{Q}_1\mathbf{p}', \tag{49}$$

which, after replacing the first derivative of the constraint equation, Eq. (6), recalling that, according to Eq. (48), $\mathbf{A} = \mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}$, yields

$$\mathbf{A}\dot{\mathbf{x}} + \mathbf{c}_{/t} = \mathbf{A}\mathbf{t}' + \mathbf{c}_{/t} = \mathbf{A}\mathbf{Q}_1\mathbf{p}' + \mathbf{c}_{/t} = \mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_1\mathbf{p}' + \mathbf{c}_{/t} = \mathbf{0}. \tag{50}$$

The resulting $\mathbf{p}'$ is

$$\mathbf{p}' = -\mathbf{R}_1^{-\mathrm{T}}\mathbf{c}_{/t}, \tag{51}$$

and thus

$$\mathbf{t}' = -\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\mathbf{c}_{/t} = -\mathbf{A}^{+}\mathbf{c}_{/t}, \tag{52}$$

yielding

$$\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{q}} + \mathbf{t}' = \mathbf{Q}_2\dot{\mathbf{q}} - \mathbf{A}^{+}\mathbf{c}_{/t}. \tag{53}$$

Similarly, considering the projected acceleration, Eq. (8), and the second time derivative of the constraint equation, Eq. (9), one obtains

$$\mathbf{t}'' = -\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\mathbf{b}'' = -\mathbf{A}^{+}\mathbf{b}'', \tag{54}$$

yielding

$$\ddot{\mathbf{x}} = \mathbf{T}\ddot{\mathbf{q}} + \mathbf{t}'' = \mathbf{Q}_2\ddot{\mathbf{q}} - \mathbf{A}^{+}\mathbf{b}''. \tag{55}$$

Although not strictly essential in the context of the present work, as anticipated the solution may need to be corrected by bringing it back onto the constraint manifold in case the integration of $\dot{\mathbf{x}}$ drifted away. This can be conveniently done by considering a correction in the subspace $\mathbf{Q}_1$. Consider a perturbation of the constraint equation about $\mathbf{x}^{(0)}$, the result of the integration, which represents a prediction of the actual solution (the iteration number is the superscript in brackets),

$$\mathbf{c}(\mathbf{x}^{(0)}, t) + \mathbf{A}\Delta\mathbf{x} = \mathbf{0}, \tag{56}$$

and express the solution increment, $\Delta\mathbf{x}$, as

$$\Delta\mathbf{x} = \mathbf{Q}_2\Delta\mathbf{q} + \mathbf{Q}_1\Delta\mathbf{p}. \tag{57}$$

Recalling again that, according to Eq. (48), $\mathbf{A} = \mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}$, one obtains

$$\mathbf{c}(\mathbf{x}^{(0)}, t) + \mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_1^{\mathrm{T}}(\mathbf{Q}_2\Delta\mathbf{q} + \mathbf{Q}_1\Delta\mathbf{p}) = \mathbf{c}(\mathbf{x}^{(0)}, t) + \mathbf{R}_1^{\mathrm{T}}\Delta\mathbf{p} = \mathbf{0}, \tag{58}$$

i.e., $\Delta\mathbf{q}$, the contribution in the subspace tangent to the constraint manifold, is inessential and should be set to zero, to avoid corrections compatible with the constraint and thus impacting the results of integrating the equations of motion; thus,

$$\Delta\mathbf{p} = -\mathbf{R}_1^{-\mathrm{T}}\mathbf{c}(\mathbf{x}^{(0)}, t) \tag{59}$$

and, generalizing to the kth iteration of the correction procedure,

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\mathbf{c}(\mathbf{x}^{(k)}, t) = \mathbf{x}^{(k)} - \mathbf{A}^{+}\mathbf{c}(\mathbf{x}^{(k)}, t), \tag{60}$$

where $\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}} = \mathbf{A}^{+}$ is the MPGI of the constraint Jacobian matrix.[2]

5.3 Sensitivity of QR factorization

Since $\mathbf{T} \equiv \mathbf{Q}_2$, we need $\mathbf{Q}_{2/p}$. Consider

$$\mathbf{A}_{/p}^{\mathrm{T}} = \mathbf{Q}_{1/p}\mathbf{R}_1 + \mathbf{Q}_1\mathbf{R}_{1/p}; \tag{61}$$

thanks to the orthonormality of matrix $\mathbf{Q}$, the product $\mathbf{Q}^{\mathrm{T}}\mathbf{Q}_{/p}$ is skew-symmetric, so

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q}_{/p} = \begin{bmatrix} \mathbf{Q}_1^{\mathrm{T}} \\ \mathbf{Q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_{1/p} & \mathbf{Q}_{2/p} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_{1/p} & \mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_{2/p} \\ \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_{1/p} & \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_{2/p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega}_1 & -\boldsymbol{\Omega}_{21}^{\mathrm{T}} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_2 \end{bmatrix}, \tag{62}$$

with $\boldsymbol{\Omega}_1 = -\boldsymbol{\Omega}_1^{\mathrm{T}}$ and $\boldsymbol{\Omega}_2 = -\boldsymbol{\Omega}_2^{\mathrm{T}}$.

Consider now

$$\mathbf{Q}_1^{\mathrm{T}}\mathbf{A}_{/p}^{\mathrm{T}}\mathbf{R}_1^{-1} = \mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_{1/p} + \mathbf{R}_{1/p}\mathbf{R}_1^{-1} = \boldsymbol{\Omega}_1 + \mathbf{R}_{1/p}\mathbf{R}_1^{-1}. \tag{63}$$

The product $\mathbf{R}_{1/p}\mathbf{R}_1^{-1}$ is an upper-triangular matrix, whereas matrix $\boldsymbol{\Omega}_1$ can be decomposed into its strictly lower triangular part, $\boldsymbol{\Omega}_L = \mathrm{stril}(\boldsymbol{\Omega}_1)$, as $\boldsymbol{\Omega}_1 = \boldsymbol{\Omega}_L - \boldsymbol{\Omega}_L^{\mathrm{T}}$, and since $\mathrm{stril}(\mathbf{R}_{1/p}\mathbf{R}_1^{-1}) \equiv \mathbf{0}$, one obtains

$$\boldsymbol{\Omega}_L = \mathrm{stril}(\mathbf{Q}_1^{\mathrm{T}}\mathbf{A}_{/p}^{\mathrm{T}}\mathbf{R}_1^{-1}), \tag{64}$$

[2] Namely, $\mathbf{A}^{+} = \mathbf{A}^{\mathrm{T}}(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{-1} = \mathbf{Q}_1\mathbf{R}_1(\mathbf{R}_1^{\mathrm{T}}\cancel{\mathbf{Q}_1^{\mathrm{T}}\mathbf{Q}_1}\mathbf{R}_1)^{-1} = \mathbf{Q}_1\cancel{\mathbf{R}_1\mathbf{R}_1^{-1}}\mathbf{R}_1^{-\mathrm{T}}$, such that $\mathbf{A}\mathbf{A}^{+} \equiv \mathbf{I}$.

which yields $\boldsymbol{\Omega}_1$.

Consider then

$$\mathbf{Q}_2^{\mathrm{T}}\mathbf{A}_{/p}^{\mathrm{T}}\mathbf{R}_1^{-1} = \mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_{1/p} = \boldsymbol{\Omega}_{21}, \tag{65}$$

thanks to the orthonormality of matrix $\mathbf{Q}$ and thus $\mathbf{Q}_2^{\mathrm{T}}\mathbf{Q}_1 \equiv \mathbf{0}$. Assuming $\boldsymbol{\Omega}_2 \equiv \mathbf{0}$, from $\mathbf{Q}_{/p} = \mathbf{Q}\boldsymbol{\Omega}$ one obtains

$$\left[\mathbf{Q}_{1/p}\ \mathbf{Q}_{2/p}\right] = \left[\mathbf{Q}_1\ \mathbf{Q}_2\right]\begin{bmatrix} \boldsymbol{\Omega}_1 & -\boldsymbol{\Omega}_{21}^{\mathrm{T}} \\ \boldsymbol{\Omega}_{21} & \cancel{\boldsymbol{\Omega}_2} \end{bmatrix} = \left[\mathbf{Q}_1\boldsymbol{\Omega}_1 + \mathbf{Q}_2\boldsymbol{\Omega}_{21}\ -\mathbf{Q}_1\boldsymbol{\Omega}_{21}^{\mathrm{T}} + \cancel{\mathbf{Q}_2\boldsymbol{\Omega}_2}\right], \tag{66}$$

i.e.,

$$\mathbf{Q}_{2/p} = -\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\mathbf{A}_{/p}\mathbf{Q}_2 = -\mathbf{A}^{+}\mathbf{A}_{/p}\mathbf{Q}_2. \tag{67}$$

Matrix $\boldsymbol{\Omega}_2$ can be arbitrarily set to zero since it does not depend on the value of $\mathbf{A}_{/p}$; its only role is to recombine the sensitivity of $\mathbf{Q}_2$, namely $\mathbf{Q}_{2/p}$, within the subspace spanned by $\mathbf{Q}_2$ itself. Setting it to zero minimizes the modification of the subspace by removing whatever is not necessary to preserve its orthonormality and its orthogonality with $\mathbf{Q}_1$.

A further interpretation of Eq. (67) involves the sensitivity of matrix $\mathbf{A}$, namely

$$\mathbf{Q}_{2/p} = -\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\left(\mathbf{Q}_{1/p}\mathbf{R}_1 + \mathbf{Q}_1\mathbf{R}_{1/p}\right)^{\mathrm{T}}\mathbf{Q}_2 = -\mathbf{Q}_1\mathbf{R}_1^{-\mathrm{T}}\mathbf{R}_1^{\mathrm{T}}\mathbf{Q}_{1/p}^{\mathrm{T}}\mathbf{Q}_2 = -\mathbf{Q}_1\mathbf{Q}_{1/p}^{\mathrm{T}}\mathbf{Q}_2, \tag{68}$$

and thus of $\mathbf{Q}_1$. Further development, namely replacing $\mathbf{Q}_{1/p}$ with its expression as a function of $\mathbf{A}_{/p}$, would lead again to Eq. (67); in the form of Eq. (68), it clearly shows that any perturbation of $\mathbf{Q}_2$ must lie in the subspace spanned by $\mathbf{Q}_1$, to preserve orthogonality, as originally enforced by selecting $\boldsymbol{\Omega}_2 \equiv \mathbf{0}$, and is related to the sensitivity of $\mathbf{Q}_1$, which it must compensate.

5.4 Alternative formulation of projection linearization

Following the alternative form proposed in Sect. 4.4 for the linearization of the projected problem, independently considering one by one the sensitivity of the projection matrix to each of the unconstrained coordinates $\mathbf{x}$, one can prove the equivalence of the two formulations in three steps:

1. $$\mathbf{T}_{/\mathbf{x}}^{\mathrm{T}} = \mathbf{Q}_{2/\mathbf{x}}^{\mathrm{T}} = -\left(\mathbf{A}^{+}\mathbf{A}_{/\mathbf{x}}\mathbf{Q}_2\right)^{\mathrm{T}} = -\mathbf{Q}_2^{\mathrm{T}}\mathbf{A}_{/\mathbf{x}}^{\mathrm{T}}\left(\mathbf{A}^{+}\right)^{\mathrm{T}}, \tag{69a}$$
2. $$\mathbf{T}_{/\mathbf{x}}^{\mathrm{T}}\mathbf{f} = \mathbf{Q}_{2/\mathbf{x}}^{\mathrm{T}}\mathbf{f} = -\mathbf{Q}_2^{\mathrm{T}}\mathbf{A}_{/\mathbf{x}}^{\mathrm{T}}\left(\mathbf{A}^{+}\right)^{\mathrm{T}}\mathbf{f} = -\mathbf{Q}_2^{\mathrm{T}}\mathbf{A}_{/\mathbf{x}}^{\mathrm{T}}\boldsymbol{\lambda}_0, \tag{69b}$$
3. $$\left(\mathbf{T}_{/\mathbf{x}}^{\mathrm{T}}\mathbf{f}\right)\mathbf{T} = \left(\mathbf{Q}_{2/\mathbf{x}}^{\mathrm{T}}\mathbf{f}\right)\mathbf{Q}_2 = -\mathbf{Q}_2^{\mathrm{T}}\left(\mathbf{A}_{/\mathbf{x}}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)\mathbf{Q}_2 = -\mathbf{Q}_2^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}}\mathbf{Q}_2, \tag{69c}$$

where $\boldsymbol{\lambda}_0 = (\mathbf{A}^{\mathrm{T}})^{+}\mathbf{f}(\mathbf{x}_0, \mathbf{0})$ is used in Eq. (69b) as the solution of the static equilibrium resulting from Eqs. (14a)–(14b), namely Eq. (15), and as such does not undergo the gradient operation with respect to $\mathbf{x}$.

The main advantage of the alternative form is that it does not require the computation of the Lagrange multipliers in the reference configuration; it only requires algebraic operations on the tangent subspace of the constraint manifold. Notice, however, that the Lagrange multipliers must be computed anyway, as they are part of the solution of the nonlinear algebraic problem of Eqs. (14a)–(14b) that needs to be solved to determine the steady equilibrium configuration.

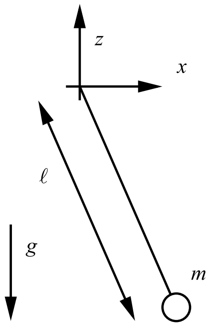


Fig. 1 Sketch of point mass pendulum

6 Results

In this section, simple problems are presented and analyzed to show the soundness of the proposed formulation. The problems are sufficiently simple that analytical solutions can be obtained and compared with those resulting from the proposed procedure. Each example is intended to show specific features of the formulation or challenge it in specific critical conditions, as highlighted in the description of the problem.

6.1 Point mass pendulum

Consider the simplest possible problem, i.e., a point mass pendulum of mass m and length ℓ, sketched in Fig. 1. This example is proposed here for continuity with that described in [6, 7], as the corresponding problem can be solved analytically while following the procedures illustrated in this paper consistently.

The equations of motion of the constrained system are:

$$m\ddot{x} + 2x\lambda = 0, \tag{70a}$$

$$m\ddot{z} + 2z\lambda = -mg, \tag{70b}$$

$$x^2 + z^2 = \ell^2. \tag{70c}$$

Their linearization can be formulated in the matrix form of Eq. (16a)–(16b) with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \tag{71a}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{71b}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{71c}$$

$$\mathbf{f} = \begin{Bmatrix} 0 \\ -mg \end{Bmatrix}, \tag{71d}$$

$$\mathbf{c}(\mathbf{x}, t) = x^2 + z^2 - \ell^2, \tag{71e}$$

$$\mathbf{A} = \begin{bmatrix} 2x & 2z \end{bmatrix}, \tag{71f}$$

$$\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} = \begin{bmatrix} 2\lambda_0 & 0 \\ 0 & 2\lambda_0 \end{bmatrix}. \tag{71g}$$

Consider the stable static equilibrium solution $\mathbf{x}_0 = \{x_0; z_0\} = \{0; -\ell\}$, which implies

$$-2\ell\lambda_0 = -mg \qquad \rightarrow \qquad \lambda_0 = \frac{mg}{2\ell}. \tag{72}$$

6.1.1 Generalized eigenanalysis in DAE form

In second-order differential-algebraic form, the linearized problem is

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{x} \\ \Delta\ddot{z} \\ \Delta\ddot{\lambda} \end{Bmatrix} + \cancel{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \begin{Bmatrix} \Delta\dot{x} \\ \Delta\dot{z} \\ \Delta\dot{\lambda} \end{Bmatrix} + \begin{bmatrix} 2\lambda_0 & 0 & 2x_0 \\ 0 & 2\lambda_0 & 2z_0 \\ 2x_0 & 2z_0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta x \\ \Delta z \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{73}$$

which, evaluated at the stable static equilibrium point, becomes

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{x} \\ \Delta\ddot{z} \\ \Delta\ddot{\lambda} \end{Bmatrix} + \begin{bmatrix} mg/\ell & 0 & 0 \\ 0 & mg/\ell & -2\ell \\ 0 & -2\ell & 0 \end{bmatrix} \begin{Bmatrix} \Delta x \\ \Delta z \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{74}$$

Its characteristic polynomial is

$$\mathcal{P}(s) = -4m\ell\left(g + s^2\ell\right), \tag{75}$$

i.e., second-order in s, although formally resulting from a second-order differential equation of a problem of size three and thus potentially sixth-order; however, this is expected because the underlying problem is actually differential-algebraic. The roots of the polynomial of Eq. (75) are

$$s_{1,2} = \pm\mathrm{j}\sqrt{\frac{g}{\ell}}, \tag{76}$$

where $\mathrm{j} = \sqrt{-1}$ is the imaginary unit, yielding the eigenvectors

$$\begin{Bmatrix} \Delta x \\ \Delta z \\ \Delta\lambda \end{Bmatrix}_{1,2} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \tag{77}$$

i.e., the expected analytical expressions.

In first-order differential-algebraic form, the linearized problem is recast as

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{x} \\ \Delta\ddot{z} \\ \Delta\dot{x} \\ \Delta\dot{z} \\ \Delta\dot{\lambda} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 2\lambda_0 & 0 & 2x_0 \\ 0 & 0 & 0 & 2\lambda_0 & 2z_0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2x_0 & 2z_0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\dot{x} \\ \Delta\dot{z} \\ \Delta x \\ \Delta z \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{78}$$

which, evaluated at the stable static equilibrium point, becomes

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{x} \\ \Delta\ddot{z} \\ \Delta\dot{x} \\ \Delta\dot{z} \\ \Delta\dot{\lambda} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & mg/\ell & 0 & 0 \\ 0 & 0 & 0 & mg/\ell & -2\ell \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\ell & 0 \end{bmatrix} \begin{Bmatrix} \Delta\dot{x} \\ \Delta\dot{z} \\ \Delta x \\ \Delta z \\ \Delta\lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{79}$$

Its characteristic polynomial is again that of Eq. (75), with the same roots, yielding the eigenvectors

$$\begin{Bmatrix} \Delta\dot{x} \\ \Delta x \\ \Delta\dot{z} \\ \Delta z \\ \Delta\lambda \end{Bmatrix}_{1,2} = \begin{Bmatrix} \pm\mathrm{j}\sqrt{g/\ell} \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{80}$$

6.1.2 Generalized eigenanalysis of the projected problem

Consider

$$\mathbf{A}^{\mathrm{T}} = \mathbf{QR} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 2z \end{bmatrix}. \tag{81}$$

Since matrix $\mathbf{Q}$ is subjected to the orthonormality constraint, it can be conveniently expressed as a function of a single parameter θ as

$$\mathbf{Q} = \frac{1}{\ell} \begin{bmatrix} x & -z \\ z & x \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \tag{82}$$

which is intrinsically orthonormal (it corresponds to a planar rotation about axis y by an angle θ in a right-hand triad x–y–z). A valid factorization is

$$R_1 = 2\ell, \qquad \theta = -\tan^{-1}\left(\frac{z}{x}\right), \tag{83}$$

i.e, at the stable static equilibrium point,

$$R_1 = 2\ell, \qquad \theta = \pi/2, \tag{84}$$

and thus

$$\mathbf{Q} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 2\ell \\ 0 \end{bmatrix}, \tag{85}$$

or

$$\mathbf{Q}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{R}_1 = 2\ell. \tag{86}$$

The matrices of the projected problem of Eq. (25) are

$$\hat{\mathbf{M}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = m, \tag{87a}$$

$$\hat{\mathbf{C}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0, \tag{87b}$$

$$\hat{\mathbf{K}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2\lambda_0 & 0 \\ 0 & 2\lambda_0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\lambda_0 = \frac{mg}{\ell}, \tag{87c}$$

$$\hat{\mathbf{f}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -mg \end{Bmatrix} = 0, \tag{87d}$$

and the corresponding projected equation is thus

$$m\Delta\ddot{q} + \frac{mg}{\ell}\Delta q = 0, \tag{88}$$

whose eigensolutions are

$$s = \pm \mathrm{j}\sqrt{\frac{g}{\ell}}, \qquad \mathbf{z} = 1, \tag{89}$$

i.e., the expected eigenvalues of Eq. (76), and a trivial eigenvector. It is worth recalling that the eigenvector in the original coordinates intrinsically corresponds to matrix $\mathbf{Q}_2$ of Eq. (86), which is consistent with the result of the generalized eigenanalysis of the problem in DAE form, Eqs. (77) and (80).

Considering the alternative form of the problem linearization, for the sensitivity of the projection matrix one obtains

$$\mathbf{T}_{/x_k} = \mathbf{Q}_{2/x_k} = -\mathbf{A}^{+}\mathbf{A}_{/x_k}\mathbf{Q}_2 = -\frac{1}{4\ell^2}\begin{bmatrix} 2x \\ 2z \end{bmatrix} \begin{bmatrix} 2\delta_{1k} & 2\delta_{2k} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{1}{\ell^2}\begin{bmatrix} x \\ z \end{bmatrix}\delta_{1k}, \tag{90}$$

where $k = 1, 2$, $x_1 \equiv x$, and $x_2 \equiv z$; δ_{ik} is Kronecker's operator, equal to 1 when $k \equiv i$ and 0 otherwise. At the stable static equilibrium solution, it yields

$$\mathbf{T}_{/x_k} = \frac{1}{\ell}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta_{1k}. \tag{91}$$

Then, for matrix $\hat{\mathbf{K}}$,

$$\begin{aligned} \hat{\mathbf{K}} &= -\left(\mathbf{T}^{\mathrm{T}}_{/\mathbf{x}}\mathbf{f}\right)\mathbf{T} = -\left[\mathbf{T}^{\mathrm{T}}_{/x_1}\mathbf{f}\ \mathbf{T}^{\mathrm{T}}_{/x_2}\mathbf{f}\right]\mathbf{T} \\ &= -\left[\frac{1}{\ell}\begin{bmatrix} 0 \\ 1 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -mg \end{Bmatrix} \frac{1}{\ell}\begin{bmatrix} 0 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ -mg \end{Bmatrix}\right] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= -\begin{bmatrix} -\frac{mg}{\ell} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{mg}{\ell}, \end{aligned} \tag{92}$$

i.e., the same value of Eq. (87c), thus yielding the same projected problem eigensolution.

6.1.3 Sensitivity with respect to gravity

Consider the sensitivity of the analytical eigensolution of Eq. (76) with respect to gravity, $p := g$; the "eigenvector" does not change, whereas the eigenvalue's sensitivity yields

$$s_{/p} = \pm \mathrm{j} \frac{1}{2\sqrt{g\ell}}. \tag{93}$$

The same result can be obtained following the whole proposed sensitivity analysis process.

First of all, to determine the sensitivity of the Lagrange multiplier, Eq. (38) must be solved, namely

$$\begin{bmatrix} 2\lambda_0 & 0 & 2x_0 \\ 0 & 2\lambda_0 & 2z_0 \\ 2x_0 & 2z_0 & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -m \\ 0 \end{Bmatrix}, \tag{94}$$

which, in the previously selected static equilibrium position, becomes

$$\begin{bmatrix} mg/\ell & 0 & 0 \\ 0 & mg/\ell & -2\ell \\ 0 & -2\ell & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -m \\ 0 \end{Bmatrix}. \tag{95}$$

Its solution is $x_{0/p} = 0$, $z_{0/p} = 0$, $\lambda_{0/p} = m/(2\ell)$ (the latter corresponds to the analytical sensitivity of λ_0 with respect to g, namely the partial derivative of Eq. (72) with respect to g).

The sensitivity of the projection matrix is zero, since the constraint Jacobian matrix, $\mathbf{A}$, does not depend on g; thus,

$$\hat{\mathbf{M}}_{/p} = 0, \tag{96a}$$

$$\hat{\mathbf{C}}_{/p} = 0, \tag{96b}$$

$$\hat{\mathbf{K}}_{/p} = \mathbf{Q}_2^{\mathrm{T}} \left(\cancel{\mathbf{A}^{\mathrm{T}}_{/p}} \boldsymbol{\lambda}_0 + \mathbf{A}^{\mathrm{T}} \boldsymbol{\lambda}_{0/p} \right)_{/\mathbf{x}} \mathbf{Q}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m/\ell & 0 \\ 0 & m/\ell \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{m}{\ell}. \tag{96c}$$

The eigenvalue sensitivity problem of Eq. (35) thus becomes

$$\begin{bmatrix} s^2 m + mg/\ell & 2sm \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} -m/\ell \\ 0 \end{Bmatrix}, \tag{97}$$

or, substituting the value of s,

$$\begin{bmatrix} \cancel{-mg/\ell + mg/\ell} & \pm 2\mathrm{j}\sqrt{g/\ell}\,m \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} -m/\ell \\ 0 \end{Bmatrix}, \tag{98}$$

i.e., $q_{/p} = 0$ and

$$s_{/p} = \mp \frac{m/\ell}{2\mathrm{j}\sqrt{g/\ell}\,m} = \pm \mathrm{j} \frac{1}{2\sqrt{g\ell}}, \tag{99}$$

which is the exact result directly obtained through the sensitivity analysis of the analytical expression of the problem's eigenvalues, Eq. (93).

6.1.4 Sensitivity with respect to length

Consider the sensitivity of the eigensolution with respect to the length, $p := \ell$; the "eigenvector" does not change, whereas the eigenvalue's sensitivity, namely the partial derivative of Eq. (76) with respect to ℓ, yields

$$s_{/p} = \mp \mathrm{j} \frac{1}{2\ell} \sqrt{\frac{g}{\ell}}. \tag{100}$$

Consider now the whole sensitivity analysis process.

First of all, to determine the sensitivity of the Lagrange multipliers, Eq. (38) must be solved, namely

$$\begin{bmatrix} 2\lambda_0 & 0 & 2x_0 \\ 0 & 2\lambda_0 & 2z_0 \\ 2x_0 & 2z_0 & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 2\ell \end{Bmatrix}, \tag{101}$$

which, in the steady reference position, becomes

$$\begin{bmatrix} mg/\ell & 0 & 0 \\ 0 & mg/\ell & -2\ell \\ 0 & -2\ell & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 2\ell \end{Bmatrix}. \tag{102}$$

Its solution is $x_{0/p} = 0$, $z_{0/p} = -1$, $\lambda_{0/p} = -mg/(2\ell^2)$; the same solution could have been obtained analytically by computing the partial derivative of Eq. (72) with respect to ℓ.

The sensitivity of the projection matrix is zero; thus,

$$\hat{\mathbf{M}}_{/p} = 0, \tag{103a}$$

$$\hat{\mathbf{C}}_{/p} = 0, \tag{103b}$$

$$\hat{\mathbf{K}}_{/p} = \mathbf{Q}_2^{\mathrm{T}} \left(\cancel{\mathbf{A}^{\mathrm{T}}_{/p}} \boldsymbol{\lambda}_0 + \mathbf{A}^{\mathrm{T}} \boldsymbol{\lambda}_{0/p} \right)_{/\mathbf{x}} \mathbf{Q}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -mg/\ell^2 & 0 \\ 0 & -mg/\ell^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -\frac{mg}{\ell^2}. \tag{103c}$$

The eigenvalue sensitivity problem of Eq. (35) thus becomes

$$\begin{bmatrix} s^2 m + mg/\ell & 2sm \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} mg/\ell^2 \\ 0 \end{Bmatrix}, \tag{104}$$

or, substituting the value of s,

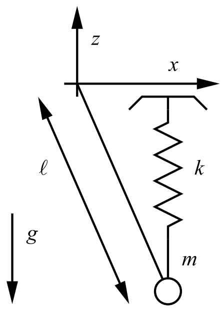
$$\begin{bmatrix} \cancel{-mg/\ell + mg/\ell} & \pm 2\mathrm{j}\sqrt{g/\ell}m \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} mg/\ell^2 \\ 0 \end{Bmatrix}, \tag{105}$$

i.e., $q_{/p} = 0$ and

$$s_{/p} = \pm \frac{mg/\ell^2}{2\mathrm{j}\sqrt{g/\ell}m} = \mp \mathrm{j} \frac{1}{2\ell} \sqrt{\frac{g}{\ell}}, \tag{106}$$

which again is exactly the result directly obtained through the analytical expression of the problem's eigenvalues, Eq. (100).

Fig. 2 Sketch of point mass pendulum with vertical spring



6.2 Point mass pendulum with vertical spring

Consider now the same pendulum proposed in the previous example, with the point mass suspended to a spring of stiffness k that remains vertical, as sketched in Fig. 2. The spring is unloaded when $z = z_{\text{ref}}$. This example is discussed to test the proposed formulation when the constraint Jacobian matrix depends on the system parameters.

The equations of motion are as follows:

$$m\ddot{x} + 2x\lambda = 0, \tag{107a}$$

$$m\ddot{z} + k(z - z_{\text{ref}}) + 2z\lambda = -mg, \tag{107b}$$

$$x^2 + z^2 = \ell^2. \tag{107c}$$

Their linearization can be formulated in the matrix form of Eq. (16a)–(16b) with

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \tag{108a}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{108b}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}, \tag{108c}$$

$$\mathbf{f} = \begin{Bmatrix} 0 \\ -mg - k(z - z_{\text{ref}}) \end{Bmatrix}, \tag{108d}$$

$$\mathbf{c}(\mathbf{x}, t) = x^2 + z^2 - \ell^2, \tag{108e}$$

$$\mathbf{A} = \begin{bmatrix} 2x & 2z \end{bmatrix}, \tag{108f}$$

$$\left(\mathbf{A}^{\text{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} = \begin{bmatrix} 2\lambda_0 & 0 \\ 0 & 2\lambda_0 \end{bmatrix}. \tag{108g}$$

Consider $z_{\text{ref}} = 0$ (spring at rest when the pendulum is horizontal) and the static equilibrium solution $\mathbf{x}_0 = \{\sqrt{\ell^2 - m^2g^2/k^2}; -mg/k\}$, which implies

$$\lambda_0 = 0, \tag{109}$$

i.e., the point mass is completely suspended by the spring, while the kinematic constraint expressed by Eq. (107c) simply prescribes the trajectory of the point mass when the static equilibrium solution is perturbed. This solution is feasible only if $k > mg/\ell$, in which case it is stable. The solution $\mathbf{x}_0 = \{0; -\ell\}$ is also feasible but stable only if $k < mg/\ell$, and thus not considered in the following.

Consider

$$\mathbf{A}^{\mathrm{T}} = \mathbf{QR} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x \\ 2z \end{bmatrix}. \tag{110}$$

Since matrix $\mathbf{Q}$ is subjected to the orthonormality constraints, it can be conveniently expressed as

$$\mathbf{Q} = \frac{1}{\ell} \begin{bmatrix} x & -z \\ z & x \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{111}$$

A valid factorization, at the selected equilibrium point, is

$$R_1 = 2\ell, \qquad \theta_0 = -\tan^{-1}\left(\frac{z_0}{x_0}\right) = \sin^{-1}\left(\frac{mg}{k\ell}\right), \tag{112}$$

and thus

$$\mathbf{Q} = \begin{bmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} 2\ell \\ 0 \end{bmatrix}, \tag{113}$$

with

$$\cos\theta_0 = x_0/\ell = (1 - (mg/(k\ell))^2)^{1/2}, \tag{114a}$$

$$\sin\theta_0 = -z_0/\ell = mg/(k\ell), \tag{114b}$$

or

$$\mathbf{Q}_1 = \frac{1}{\ell} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \cos\theta_0 \\ -\sin\theta_0 \end{bmatrix} = \begin{bmatrix} (1 - (mg/(k\ell))^2)^{1/2} \\ -mg/(k\ell) \end{bmatrix}, \tag{115a}$$

$$\mathbf{Q}_2 = \frac{1}{\ell} \begin{bmatrix} -z_0 \\ x_0 \end{bmatrix} = \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = \begin{bmatrix} mg/(k\ell) \\ (1 - (mg/(k\ell))^2)^{1/2} \end{bmatrix}, \tag{115b}$$

$$\mathbf{R}_1 = 2\ell. \tag{115c}$$

The gradient of $\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0$ is now

$$\left(\mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda}_0\right)_{/\mathbf{x}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{116}$$

since $\lambda_0 = 0$. The matrices of the projected problem are

$$\hat{\mathbf{M}} = \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = m, \tag{117a}$$

$$\hat{\mathbf{C}} = 0, \tag{117b}$$

$$\hat{\mathbf{K}} = \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix}^{\mathrm{T}} \left(\begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix} = k\cos^2\theta_0 = k(1 - (mg/(k\ell))^2), \tag{117c}$$

$$\hat{\mathbf{f}} = \begin{bmatrix} \sin\theta_0 \\ \cos\theta_0 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ \cancel{-mg - kz_0} \end{Bmatrix} = 0, \tag{117d}$$

and the corresponding perturbed equation is thus

$$m\Delta\ddot{q} + k\cos^2\theta_0\Delta q = 0. \tag{118}$$

The resulting eigensolutions are

$$s = \pm \mathrm{j}\cos\theta_0\sqrt{\frac{k}{m}} = \pm\mathrm{j}\left(1 - \left(\frac{mg}{k\ell}\right)^2\right)^{1/2}\sqrt{\frac{k}{m}}. \tag{119}$$

The alternative form of the problem linearization discussed in Sect. 5.4 is pointless in this context since, according to Eq. (117d), the projected force vector $\hat{\mathbf{f}}$ is zero, much like the Lagrange multiplier of Eq. (109). As such, the corresponding stiffness contribution vanishes accordingly.

6.2.1 Sensitivity with respect to spring stiffness

Consider for example the sensitivity of the eigensolution of Eq. (119) to the stiffness of the spring, namely $p := k$,

$$\begin{aligned} s_{/p} &= \frac{\partial s}{\partial k} = \frac{\partial}{\partial k}\left(\pm\mathrm{j}\left(1 - \left(\frac{mg}{k\ell}\right)^2\right)^{1/2}\sqrt{\frac{k}{m}}\right) \\ &= \pm\mathrm{j}\left(1 - \left(\frac{mg}{k\ell}\right)^2\right)^{-1/2}\left(1 + \left(\frac{mg}{k\ell}\right)^2\right)\frac{1}{2\sqrt{mk}}. \end{aligned} \tag{120}$$

The analytical sensitivity of the solution is

$$x_{0/p} = \left(1 - \left(\frac{mg}{k\ell}\right)^2\right)^{-1/2}\left(\frac{mg}{k\ell}\right)^2\frac{\ell}{k} = \frac{z_0^2}{kx_0}, \tag{121a}$$

$$z_{0/p} = \frac{mg}{k\ell}\frac{\ell}{k} = -\frac{z_0}{k}, \tag{121b}$$

$$\lambda_{0/p} = 0. \tag{121c}$$

The constraint Jacobian matrix at the static solution is

$$\mathbf{A} = 2\ell\left[\left(1 - (mg/(k\ell))^2\right)^{1/2} \; -mg/(k\ell)\right]. \tag{122}$$

Its MPGI is

$$\mathbf{A}^+ = \frac{1}{2\ell}\begin{bmatrix}\left(1 - (mg/(k\ell))^2\right)^{1/2} \\ -mg/(k\ell)\end{bmatrix}. \tag{123}$$

Its gradient with respect to $p := k$ yields

$$\mathbf{A}_{/p} = \left[\left(1 - (mg/(k\ell))^2\right)^{-1/2} mg/(k\ell) \;\; 1 \right] \frac{2mg}{k^2}. \tag{124}$$

Considering $\mathbf{Q}_2$ from Eq. (115b), one obtains

$$\begin{aligned} \mathbf{Q}_{2/p} &= -\mathbf{A}^+ \mathbf{A}_{/p} \mathbf{Q}_2 \\ &= \begin{bmatrix} -mg/(k\ell) \\ (mg/(k\ell))^2 \left(1 - (mg/(k\ell))^2\right)^{-1/2} \end{bmatrix} \frac{1}{k}. \end{aligned} \tag{125}$$

The sensitivity of the projection matrix is not zero; thus

$$\hat{\mathbf{M}}_{/p} = \cancel{\mathbf{Q}_{2/k}^T \mathbf{M} \mathbf{Q}_2} + \cancel{\mathbf{Q}_2^T \mathbf{M}_{/k} \mathbf{Q}_2} + \cancel{\mathbf{Q}_2^T \mathbf{M} \mathbf{Q}_{2/k}} = 0, \tag{126a}$$

$$\hat{\mathbf{C}}_{/p} = 0, \tag{126b}$$

$$\begin{aligned} \hat{\mathbf{K}}_{/p} &= \mathbf{Q}_{2/k}^T \mathbf{K} \mathbf{Q}_2 + \mathbf{Q}_2^T \mathbf{K}_{/k} \mathbf{Q}_2 + \mathbf{Q}_2^T \mathbf{K} \mathbf{Q}_{2/k} \\ &= \left(\frac{mg}{k\ell}\right)^2 + \left(1 - \left(\frac{mg}{k\ell}\right)^2\right) + \left(\frac{mg}{k\ell}\right)^2 = 1 + \left(\frac{mg}{k\ell}\right)^2. \end{aligned} \tag{126c}$$

The eigenvalue sensitivity problem of Eq. (35) thus becomes

$$\begin{bmatrix} s^2 m + k(1 - (mg/(k\ell))^2) & 2sm \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} q_{/p} \\ s_{/p} \end{Bmatrix} = \begin{Bmatrix} -\left(1 + (mg/(k\ell))^2\right) \\ 0 \end{Bmatrix}, \tag{127}$$

whose result is $q_{/p} = 0$ and

$$s_{/p} = -\frac{1 + \left(\dfrac{mg}{k\ell}\right)^2}{2sm}, \tag{128}$$

which is equivalent to the result of Eq. (120) when the eigenvalues of Eq. (119) are used for s.

The sensitivity of matrix $\mathbf{Q}_2$ through Eq. (125) yields the components of the reference solution's sensitivity, Eqs. (121a) and (121b), namely

$$\mathbf{Q}_{2/p} = \frac{1}{\ell} \begin{bmatrix} -z_{0/k} \\ x_{0/k} \end{bmatrix} = \frac{1}{k} \frac{z_0}{x_0} \frac{1}{\ell} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix} = -\frac{\tan\theta_0}{k} \mathbf{Q}_1. \tag{129}$$

6.3 Rotating point mass

Consider a point mass (sketched in Fig. 3, left) connected to a hub whose position, in a rotating x–z frame that rotates with the hub about global axis $\hat{z}$ (sketched in Fig. 3, right) at constant angular velocity ω, namely $\boldsymbol{\omega} = \{0; 0; \omega\}$, is

$$\mathbf{p}(\mathbf{x}) = \begin{Bmatrix} e + x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} e \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \mathbf{e} + \mathbf{x}, \tag{130}$$

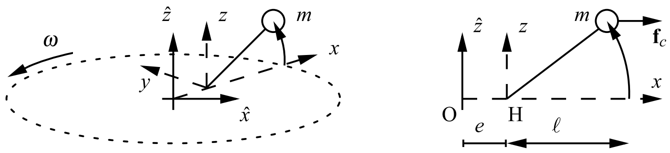


Fig. 3 Rotating point mass

where $\mathbf{x} = \{x; y; z\}$ and e is a constant offset of the point mass connection point, H, from the rotation axis in the direction of the rotating x axis. The point mass is subjected to the (scalar) constraint

$$c(\mathbf{x}) = x^2 + y^2 + z^2 - \ell^2 = \mathbf{x}^{\mathrm{T}}\mathbf{x} - \ell^2 = 0, \tag{131}$$

with $\ell = R - e$.

The constraint Jacobian matrix is

$$\mathbf{A} = \frac{\partial c}{\partial \mathbf{x}} = \left[2x \; 2y \; 2z \right] = 2\mathbf{x}^{\mathrm{T}}, \tag{132}$$

and

$$\mathbf{b}' = 0, \qquad \mathbf{b}'' = 2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = 2\dot{\mathbf{x}}^{\mathrm{T}}\dot{\mathbf{x}}. \tag{133}$$

The relative acceleration of the point mass with respect to the rotating frame is

$$\ddot{\mathbf{p}} = \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} = \ddot{\mathbf{x}}. \tag{134}$$

Its virtual displacement is

$$\delta\mathbf{p} = \begin{Bmatrix} \delta x \\ \delta y \\ \delta z \end{Bmatrix} = \delta\mathbf{x}. \tag{135}$$

The inertia force in the rotating frame is

$$\begin{aligned} \mathbf{f}_{\mathrm{ine}} &= -m\left(\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{p} + 2\boldsymbol{\omega} \times \dot{\mathbf{p}} + \ddot{\mathbf{p}}\right) \\ &= -m\left(-\omega^2 \begin{Bmatrix} e + x \\ y \\ 0 \end{Bmatrix} + 2\omega \begin{Bmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{Bmatrix} + \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix}\right), \end{aligned} \tag{136}$$

with

$$\boldsymbol{\omega}\times = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{137}$$

and $\dot{\mathbf{p}} = \dot{\mathbf{x}}$, consisting of a centrifugal force contribution,

$$\mathbf{f}_{\mathrm{g}} = -m\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{p} = m\omega^2 \begin{Bmatrix} e + x \\ y \\ 0 \end{Bmatrix}, \tag{138}$$

part of what some authors refer to as a consequence of a "guidance" acceleration contribution [26] ("trascinamento" in Italian), and a Coriolis acceleration contribution,

$$\mathbf{f}_{\mathrm{c}} = -2m\boldsymbol{\omega} \times \dot{\mathbf{p}} = 2m\omega \begin{Bmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{Bmatrix}, \tag{139}$$

in addition to the relative acceleration contribution,

$$\mathbf{f}_{\mathrm{r}} = -m\ddot{\mathbf{p}} = -m \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix}. \tag{140}$$

For completeness, consider also a force that acts against the motion of the point mass in the rotating frame, namely

$$\mathbf{f}_{\mathrm{d}} = -r\dot{\mathbf{x}} = -r \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix}, \tag{141}$$

loosely representative of velocity-dependent aerodynamic drag, which will play the role of damping, and one that remains perpendicular to the direction of connection of the point with the hub,

$$\mathbf{f}_{\mathrm{a}} = \begin{Bmatrix} -a_t y - a_n z \\ a_t x \\ a_n x \end{Bmatrix}, \tag{142}$$

loosely representative of lift (a_n) and drag (a_t) for a prescribed pitch angle.

The virtual work principle,

$$\delta\mathcal{W} = \delta\mathbf{p}^{\mathrm{T}} \left(\mathbf{f}_{\mathrm{ine}} + \mathbf{f}_{\mathrm{d}} + \mathbf{f}_{\mathrm{a}} - \mathbf{A}^{\mathrm{T}}\lambda\right) = 0, \tag{143}$$

yields

$$m \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{Bmatrix} + 2m\omega \begin{Bmatrix} -\dot{y} \\ \dot{x} \\ 0 \end{Bmatrix} + r \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix} - m\omega^2 \begin{Bmatrix} e + x \\ y \\ 0 \end{Bmatrix} - \begin{Bmatrix} -a_t y - a_n z \\ a_t x \\ a_n x \end{Bmatrix} + \begin{Bmatrix} 2x \\ 2y \\ 2z \end{Bmatrix} \lambda = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{144a}$$

$$x^2 + y^2 + z^2 - \ell^2 = 0. \tag{144b}$$

This example resembles a helicopter rotor blade hinged in H with coincident flap and lag hinges, without the complexity of finite rotation handling. Furthermore, since the unconstrained problem has three degrees of freedom and only one kinematic constraint equation is introduced, the constrained problem subspace is of dimension 2, with matrix $\mathbf{T} \equiv \mathbf{Q}_2 \in \mathbb{R}^{3\times 2}$ (subjected to the orthonormality constraint), instead of just a column vector like in the previous examples.

To cast the problem in the matrix form of Eq. (16a)–(16b), define

$$\mathbf{M} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{145a}$$

$$\mathbf{C} = 2m\omega \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + r \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{145b}$$

$$\mathbf{K} = -m\omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_t & a_n \\ -a_t & 0 & 0 \\ -a_n & 0 & 0 \end{bmatrix}, \tag{145c}$$

$$\left(\mathbf{A}^{\mathrm{T}}\lambda_0\right)_{/\mathbf{x}} = 2\lambda_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{145d}$$

$$\mathbf{f} = \begin{Bmatrix} 2m\omega\dot{y}_0 - r\dot{x}_0 + m\omega^2(e + x_0) - a_t y_0 - a_n z_0 \\ -2m\omega\dot{x}_0 - r\dot{y}_0 + m\omega^2 y_0 + a_t x_0 \\ -r\dot{z}_0 + a_n x_0 \end{Bmatrix}. \tag{145e}$$

Temporarily setting $a_t = a_n = 0$, consider the steady (i.e., with $\ddot{\mathbf{x}} = \mathbf{0}$, $\dot{\mathbf{x}} = \mathbf{0}$) solution $\mathbf{x}_0 = \{\ell; 0; 0\}$, with the corresponding multiplier

$$\lambda_0 = \frac{m\omega^2}{2}\left(1 + \frac{e}{\ell}\right). \tag{146}$$

In that reference condition, consider

$$\mathbf{A}^{\mathrm{T}} = \mathbf{QR} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\ell \\ 0 \\ 0 \end{bmatrix}, \tag{147}$$

i.e.,

$$R_1 = 2\ell, \tag{148a}$$

$$Q_{11} = 1, \tag{148b}$$

$$Q_{21} = 0, \tag{148c}$$

$$Q_{31} = 0, \tag{148d}$$

which implies

$$Q_{12} = 0, \tag{149a}$$

$$Q_{13} = 0, \tag{149b}$$

and thus

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \tag{150}$$

where the obvious choice of $\alpha = 0$ yields q_1 along the y axis and q_2 along the z axis, but any other choice is legit. Thus,

$$\mathbf{Q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 0 & 0 \\ \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}. \tag{151}$$

The projected problem involves:

$$\hat{\mathbf{M}} = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{152a}$$

$$\hat{\mathbf{C}} = r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{152b}$$

$$\begin{aligned} \hat{\mathbf{K}} &= -m\omega^2 \begin{bmatrix} \cos^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix} + 2\lambda_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= m\omega^2 \begin{bmatrix} \sin^2\alpha + e/\ell & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha + e/\ell \end{bmatrix}, \end{aligned} \tag{152c}$$

$$\hat{\mathbf{f}} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{152d}$$

The corresponding linearized projected equation is thus

$$m \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + r \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + m\omega^2 \begin{bmatrix} \sin^2\alpha + e/\ell & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha + e/\ell \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{153}$$

whose eigensolution is

$$s_{1,2} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \omega^2\frac{e}{\ell}}, \qquad \mathbf{z}_{1,2} = \begin{Bmatrix} \cos\alpha \\ -\sin\alpha \end{Bmatrix}, \tag{154a}$$

$$s_{3,4} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \omega^2\left(1 + \frac{e}{\ell}\right)}, \qquad \mathbf{z}_{3,4} = \begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix}. \tag{154b}$$

As expected, the eigenvalues do not depend on α, whereas the eigenvectors only depend on it in the measure needed to identify the tangential and axial directions.

6.3.1 Sensitivity with respect to angular velocity

Consider the sensitivity of the eigensolution of Eqs. (154a)–(154b) with respect to the value of the angular velocity, $p := \omega$; the "eigenvectors" do not change, thus

$$s_{1,2/p} = \mp \left(\left(\frac{r}{2m} \right)^2 - \omega^2 \frac{e}{\ell} \right)^{-1/2} \omega \frac{e}{\ell}, \qquad \mathbf{z}_{1,2/p} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{155a}$$

$$s_{3,4/p} = \mp \left(\left(\frac{r}{2m} \right)^2 - \omega^2 \left(1 + \frac{e}{\ell} \right) \right)^{-1/2} \omega \left(1 + \frac{e}{\ell} \right), \qquad \mathbf{z}_{3,4/p} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{155b}$$

Consider now the whole sensitivity analysis process.

First of all, and considering

$$\mathbf{M}_{/p} = \mathbf{0}, \tag{156a}$$

$$\mathbf{C}_{/p} = 2m \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{156b}$$

$$\mathbf{K}_{/p} = -2m\omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{156c}$$

$$\mathbf{f}_{/p} = \begin{Bmatrix} 2m\omega\,(e + x_0) \\ 2m\omega y_0 \\ 0 \end{Bmatrix}, \tag{156d}$$

$$\mathbf{A}_{/p} = \mathbf{0}, \tag{156e}$$

$$c_{/p} = 0, \tag{156f}$$

to determine the sensitivity of the Lagrange multipliers, Eq. (38) must be solved, namely

$$\begin{bmatrix} 2\lambda_0 - m\omega^2 & 0 & 0 & 2x_0 \\ 0 & 2\lambda_0 - m\omega^2 & 0 & 2y_0 \\ 0 & 0 & 2\lambda_0 & 2z_0 \\ 2x_0 & 2y_0 & 2z_0 & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ y_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 2m\omega\,(e + x_0) \\ 2m\omega y_0 \\ 0 \\ 0 \end{Bmatrix}, \tag{157}$$

which, in the static equilibrium position, becomes

$$\begin{bmatrix} m\omega^2 e/\ell & 0 & 0 & 2\ell \\ 0 & m\omega^2 e/\ell & 0 & 0 \\ 0 & 0 & m\omega^2\,(1 + e/\ell) & 0 \\ 2\ell & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_{0/p} \\ y_{0/p} \\ z_{0/p} \\ \lambda_{0/p} \end{Bmatrix} = \begin{Bmatrix} 2m\omega\,(e + \ell) \\ 0 \\ 0 \\ 0 \end{Bmatrix}, \tag{158}$$

and whose solution is $x_{0/p} = 0$, $y_{0/p} = 0$, $z_{0/p} = 0$, and $\lambda_{0/p} = m\omega(1 + e/\ell)$.

The sensitivity of the projection matrix is zero; thus,

$$\hat{\mathbf{M}}_{/p} = \mathbf{0}, \tag{159a}$$

Table 1 Rotating point mass data

Property		Value	
Angular velocity	ω	40	radian s^{-1}
Mass	m	50	kg
Damping characteristic	r	2	N s m^{-1}
Length	ℓ	5	m
Offset	e	1	m
Tangential force coefficient	a_t	-100	N m^{-1}
Normal force coefficient	a_n	1000	N m^{-1}

$$\hat{\mathbf{C}}_{/p} = \mathbf{0}, \tag{159b}$$

$$\begin{aligned}\hat{\mathbf{K}}_{/p} &= \mathbf{Q}_2^{\mathrm{T}} \left[\mathbf{K}_{/p} + \left(\cancel{\mathbf{A}^{\mathrm{T}}_{/p}} \boldsymbol{\lambda}_0 + \mathbf{A}^{\mathrm{T}} \boldsymbol{\lambda}_{0/p} \right)_{/\mathbf{x}} \right] \mathbf{Q}_2 \\ &= \begin{bmatrix} 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 2m\omega e/\ell & 0 & 0 \\ 0 & 2m\omega e/\ell & 0 \\ 0 & 0 & 2m\omega\left(1 + e/\ell\right) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ &= 2m\omega \begin{bmatrix} \sin^2\alpha + e/\ell & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \cos^2\alpha + e/\ell \end{bmatrix}. \end{aligned} \tag{159c}$$

The eigenvalue sensitivity problem of Eq. (35) thus becomes

$$\begin{aligned} &\begin{bmatrix} s_k^2 m + s_k r + m\omega^2 \left(\sin^2\alpha + e/\ell\right) & m\omega^2 \sin\alpha\cos\alpha & \left(2s_k m + r\right) q_{1_k} \\ m\omega^2 \sin\alpha\cos\alpha & s_k^2 m + s_k r + m\omega^2 \left(\cos^2\alpha + e/\ell\right) & \left(2s_k m + r\right) q_{2_k} \\ \mathrm{conj}(q_{1_k}) & \mathrm{conj}(q_{2_k}) & 0 \end{bmatrix} \\ &\times \begin{Bmatrix} q_{1_{k/p}} \\ q_{2_{k/p}} \\ s_{k/p} \end{Bmatrix} = \begin{Bmatrix} -2m\omega\left[\left(\sin^2\alpha + e/\ell\right) q_{1_k} + \sin\alpha\cos\alpha q_{2_k}\right] \\ -2m\omega\left[\sin\alpha\cos\alpha q_{1_k} + \left(\cos^2\alpha + e/\ell\right) q_{2_k}\right] \\ 0 \end{Bmatrix} \end{aligned} \tag{160}$$

for each of the four eigensolutions s_k, $\mathbf{z}_k = \{q_{1_k}; q_{2_k}\}$, $k = 1, 2, 3, 4$, of the original problem, which yields exactly the result of Eqs. (155a)–(155b).

6.3.2 Numerical solution in generic configuration

Considering the data reported in Table 1, a steady solution is found with

$$\mathbf{x}_0 = \{4.9901; -0.0313; 0.3125\}$$

and $\lambda_0 = 4.7984\mathrm{e}{+}4$. The corresponding constraint Jacobian matrix is

$$\mathbf{A} = \left[9.9803 \; -0.0625 \; 0.6250 \right]. \tag{161}$$

The eigensolution of the problem in DAE form is

$$s_{1,2} = -0.1992 \pm \mathrm{j}17.8349 \qquad \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\lambda \end{Bmatrix}_{1,2} = \begin{Bmatrix} 4.6758\mathrm{e}{-7} \mp \mathrm{j}1.7458\mathrm{e}{-6} \\ -3.1218\mathrm{e}{-6} \mp \mathrm{j}2.7877\mathrm{e}{-4} \\ -7.7786\mathrm{e}{-6} \pm \mathrm{j}0.0000\mathrm{e}{+0} \\ 9.9719\mathrm{e}{-1} \mp \mathrm{j}2.8105\mathrm{e}{-3} \end{Bmatrix}, \tag{162a}$$

$$s_{3,4} = -0.2008 \pm \mathrm{j}43.8244 \qquad \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta\lambda \end{Bmatrix}_{3,4} = \begin{Bmatrix} 3.3108\mathrm{e}{-6} \mp \mathrm{j}1.4288\mathrm{e}{-3} \\ 1.5643\mathrm{e}{-3} \pm \mathrm{j}8.9478\mathrm{e}{-6} \\ 1.0356\mathrm{e}{-4} \pm \mathrm{j}2.2817\mathrm{e}{-2} \\ -1.0918\mathrm{e}{-1} \mp \mathrm{j}1.0703\mathrm{e}{-2} \end{Bmatrix}. \tag{162b}$$

The eigenvectors associated with the complex conjugated eigenvalues $s_{1,2}$ show substantial participation of motion in the y direction, whereas those associated with eigenvalues $s_{3,4}$ are dominated by motion in the z direction. The combinations of eigenvalues and eigenvectors are consistent with typical lead-lag and flap dynamics of helicopter rotor blades. The remaining three eigenvalues of the generalized eigenproblem in the form of Eq. (27) have infinite value.

The QR factorization of the constraint Jacobian matrix's transpose, performed using Matlab's `qr()` function, yields

$$\mathbf{R}_1 = -10, \qquad \mathbf{Q}_1 = \begin{bmatrix} -9.9803\mathrm{e}{-1} \\ 6.2500\mathrm{e}{-3} \\ -6.2500\mathrm{e}{-2} \end{bmatrix}, \qquad \mathbf{Q}_2 = \begin{bmatrix} 6.2500\mathrm{e}{-3} & -6.2500\mathrm{e}{-2} \\ 9.9998\mathrm{e}{-1} & 1.9551\mathrm{e}{-4} \\ 1.9551\mathrm{e}{-4} & 9.9804\mathrm{e}{-1} \end{bmatrix}. \tag{163}$$

The problem's projected matrices are

$$\hat{\mathbf{M}} = \begin{bmatrix} 50 & 0 \\ 0 & 50 \end{bmatrix}, \qquad \hat{\mathbf{C}} = \begin{bmatrix} 20 & -125 \\ 125 & 20 \end{bmatrix}, \qquad \hat{\mathbf{K}} = \begin{bmatrix} 1.5968\mathrm{e}{+4} & 0.0016\mathrm{e}{+4} \\ 0.0016\mathrm{e}{+4} & 9.5656\mathrm{e}{+4} \end{bmatrix}. \tag{164}$$

The resulting eigensolution is

$$s_{1,2} = -0.1992 \pm \mathrm{j}17.8349, \qquad \mathbf{z}_{1,2} = \begin{Bmatrix} -6.2501\mathrm{e}{-4} \mp \mathrm{j}5.5953\mathrm{e}{-2} \\ -1.5642\mathrm{e}{-3} \pm \mathrm{j}1.0979\mathrm{e}{-5} \end{Bmatrix}, \tag{165a}$$

$$s_{3,4} = -0.2008 \pm \mathrm{j}43.8244, \qquad \mathbf{z}_{1,2} = \begin{Bmatrix} 1.5573\mathrm{e}{-5} \pm \mathrm{j}4.3708\mathrm{e}{-6} \\ 1.0427\mathrm{e}{-4} \mp \mathrm{j}2.2759\mathrm{e}{-2} \end{Bmatrix}. \tag{165b}$$

The eigenvalues of Eqs. (165a)–(165b) are identical to those of Eqs. (162a)–(162a), obtained from the generalized eigenanalysis in DAE form. By projecting the eigenvectors of Eqs. (165a)–(165b) back into the physical coordinates' subspace through matrix $\mathbf{Q}_2$ and normalizing the largest eigenvector component to unit value, one obtains

$$\begin{gathered} \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix}_{1,2_{\mathrm{DAE}}} = \begin{Bmatrix} 6.2429\mathrm{e}{-3} \pm \mathrm{j}1.7472\mathrm{e}{-3} \\ 1.0000\mathrm{e}{+0} \pm \mathrm{j}0.0000\mathrm{e}{+0} \\ 3.1147\mathrm{e}{-4} \mp \mathrm{j}2.7900\mathrm{e}{-2} \end{Bmatrix}, \\ \mathbf{Q}_2\mathbf{z}_{1,2} = \begin{Bmatrix} 6.2429\mathrm{e}{-3} \pm \mathrm{j}1.7472\mathrm{e}{-3} \\ 1.0000\mathrm{e}{+0} \pm \mathrm{j}0.0000\mathrm{e}{+0} \\ 3.1147\mathrm{e}{-4} \mp \mathrm{j}2.7900\mathrm{e}{-2} \end{Bmatrix}, \end{gathered} \tag{166a}$$

$$\begin{gathered} \begin{Bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{Bmatrix}_{3,4_{\mathrm{DAE}}} = \begin{Bmatrix} -6.2619\mathrm{e}{-2} \mp \mathrm{j}4.2932\mathrm{e}{-4} \\ 7.0331\mathrm{e}{-4} \mp \mathrm{j}6.8555\mathrm{e}{-2} \\ 1.0000\mathrm{e}{+0} \pm \mathrm{j}0.0000\mathrm{e}{+0} \end{Bmatrix}, \\ \mathbf{Q}_2\mathbf{z}_{3,4} = \begin{Bmatrix} -6.2619\mathrm{e}{-2} \mp \mathrm{j}4.2932\mathrm{e}{-4} \\ 7.0331\mathrm{e}{-4} \mp \mathrm{j}6.8555\mathrm{e}{-2} \\ 1.0000\mathrm{e}{+0} \pm \mathrm{j}0.0000\mathrm{e}{+0} \end{Bmatrix}, \end{gathered} \tag{166b}$$

i.e., exactly the same result down to machine precision (the largest error is of the order of 10^{-15}).

The corresponding sensitivity analysis with respect to $p := \omega$, using Eqs. (35), yields

$$s_{1,2/p} = 3.0239\mathrm{e}{-3} \pm \mathrm{j}2.4903\mathrm{e}{+0}, \qquad \mathbf{z}_{1,2/p} = \left\{ \begin{array}{c} -1.6249\mathrm{e}{-3} \mp \mathrm{j}9.9915\mathrm{e}{-6} \\ 5.9868\mathrm{e}{-4} \mp \mathrm{j}5.8123\mathrm{e}{-2} \end{array} \right\}, \tag{167a}$$

$$s_{1,2/p} = -3.0239\mathrm{e}{-3} \pm \mathrm{j}1.4401\mathrm{e}{+0}, \quad \mathbf{z}_{1,2/p} = \left\{ \begin{array}{c} -5.4566\mathrm{e}{-4} \pm \mathrm{j}1.3331\mathrm{e}{-1} \\ 9.1217\mathrm{e}{-3} \mp \mathrm{j}3.0056\mathrm{e}{-5} \end{array} \right\}. \tag{167b}$$

This result is consistent with the eigensolution sensitivity that can be directly obtained from the problem in DAE form, namely from the partial derivative of Eq. (30) and the corresponding eigenvector normalization equation with respect to the angular velocity, ω.

7 Conclusions

The general constrained dynamics problem formulated using redundant and minimal coordinate set approaches results in DAE and ODE problems. Both have been consistently linearized about steady reference solutions, producing generalized eigenproblems. The corresponding eigensolutions have been computed and discussed. The minimal coordinate set problem uses coordinate projection; in previous papers, an original continuation approach was formulated to track the evolution of the coordinate projection matrix, minimizing its modifications during the updates required to maintain it tangent to the constraint manifold. An analogous concept was applied in this work to consistently study the sensitivity of the said eigensolutions to the parameters that characterize the problem. The proposed methodology is applied to simple, yet representative, analytical and numerical problems. Its application to more complex numerical problems of engineering interest is foreseen in future developments.

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Declarations

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