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On the Design of Terminal Ingredients for Linear Time Varying Model Predictive Control: Theory and Experimental Application

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Abstract: The use of Linear Time Varying (LTV) Model Predictive Control (MPC) to stabilize a set of trajectories of a nonlinear system is considered. This technique has been successfully applied in simulations and experiments, but only few contributions investigate stability aspects and the essential involved quantities: the terminal penalty and terminal constraint. Deriving the former is not always thoroughly addressed or it is based on the -rather restrictive- assumption that the whole set of linearized dynamics is quadratically stabilizable. In this article, we propose Linear Matrix Inequality (LMI) conditions to co-design a gain-scheduled auxiliary feedback and Lyapunov function, used to derive offline terminal set conditions and a terminal penalty constraint for an LTV MPC scheme guaranteeing stability and recursive constraint satisfaction. Recent results by the authors are extended to the case of a varying stage cost, such that the controller can be tuned to meet time-varying trade-offs between tracking accuracy and input activity. The approach is demonstrated in embedded hardware running on a CrazyFlie drone.

Keywords: Trajectory Stabilization, Experimental Application, Nonlinear Control, Model Predictive Control.

1. INTRODUCTION

A common control approach for industrial systems is a two Degrees-of-Freedom (2-DoF) design, in which a feed-forward control action achieves a desired output trajectory of the disturbance-free, nominal model and an accompanying feedback law achieves trajectory tracking, i.e., disturbance rejection and stabilization of the reference trajectory for the real system. Designing the feedback law for nonlinear systems in presence of constraints, which is the focus of this article, is still under active research to this date, in particular when a control decision needs to be made at relatively fast sampling rates. Linear Time Varying (LTV) Model Predictive Control (MPC) is one method that promises to deliver robust and safe tracking under external disturbances and bounded inputs while also optimizing tracking performance. It has been applied successfully in practice, for example for quadruped control in Chignoli and Wensing (2020) and helicopter control in Kunz et al. (2013). Further, recursive feasibility and stability have been investigated in Falcone et al. (2008), considering assumptions on two main ingredients: a terminal set constraint and a terminal penalty. However, to the best of the authors' knowledge, a general, non-restrictive, and tractable approach to obtain these ingredients for LTV MPC is not available. This is supported by the finding of the survey Morato et al. (2020) on Linear Parameter Varying MPC, which states that LTV MPC through solving of an LTV MPC scheme seems to be a strong candidate for wide adoption in the industry, mainly due to the efficiency and quality of QP solvers, but tools for offline computation of the essential ingredients are involved or inefficient. There is a gap in procedural and efficient means to compute terminal penalties and terminal sets. One contribution of this paper aims to provide a more general approach, exploiting Linear Matrix Inequalities (LMIs) to co-design an auxiliary feedback and terminal penalty for trajectory tracking LTV MPC. The approach entails deriving an uncertain LTV model along the reference trajectory, used to formulate the LMIs. To robustify against external disturbances, we further show, that the approach can be extended to a tube-MPC technique. Therefore, the bounds assumed during the design of the MPC scheme are satisfied and safety as well as maximum tracking errors can be guaranteed. Moreover, existing results pertaining to gain scheduling for trajectory stabilization are extended to the case, that the stage costs varies along the reference trajectories. This is a common scenario in several industrial control systems, such as car suspension control or aircraft control, where different phases pose different performance goals on the controller. While one could perform a hard switching, we derive conditions guaranteeing stability during a smooth switching. Furthermore, we point out that the proposed design method is not limited to a fixed reference trajectory, but can be implemented for a set of reference trajectories. Finally, another contribution is the experimental test of the approach, where the MPC scheme is implemented onboard a CrazyFlie drone for longitudinal flight control

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with good results, thus demonstrating the applicability of the method on an embedded hardware platform with rather low computational capacity.

2. PROBLEM FORMULATION

We consider a nonlinear, time-invariant system

$$x(k+1) = f(x(k), u(k), d(k)), \tag{1}$$

where $k \in \mathbb{Z}$ is the discrete time, $x \in \mathbb{X} \subseteq \mathbb{R}^n$ is the state, $u \in \mathbb{U} \subseteq \mathbb{R}^m$ the input vector and $d \in \mathbb{D} \subset \mathbb{R}^n$ a bounded disturbance.We assume that $d = 0$ corresponds to a disturbance-free ("nominal") situation. Next we consider a dynamically feasible reference trajectory $(x_{\mathrm{ref}}(k), u_{\mathrm{ref}}(k)), k = 0, \dots, N$, which is to be tracked.

For a given state-input pair $(x(k), u(k))$, $\Delta x(k) := x(k) - x_{\mathrm{ref}}(k)$ denotes the tracking error and $\Delta u(k) := u(k) - u_{\mathrm{ref}}(k)$ denotes the input deviation from the reference. We consider the presence of desired sets where these deviations shall lie during the whole considered time interval, given by $\Delta x(k) \in \Delta\mathbb{X}$, $\Delta u(k) \in \Delta\mathbb{U}, \forall k = 0, \dots, N$, containing the origin in their interior.

We consider the following assumptions, where $\mathbb{A} \oplus \mathbb{B}$ is the Minkowski sum of the two generic sets $\mathbb{A}$ and $\mathbb{B}$:

$$\mathbb{A} \oplus \mathbb{B} \doteq \{a + b : a \in \mathbb{A}, b \in \mathbb{B}\}.$$

Assumption 1. The sets $\Delta\mathbb{X}$ and $\Delta\mathbb{U}$ are compact, nonempty and contain the origin in their interior.

Assumption 2. $\forall k = 0, \dots, N$, function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ in (1) is differentiable in the set

$$(\{x_{\mathrm{ref}}(k)\} \oplus \Delta\mathbb{X}) \times (\{u(k)\} \oplus \Delta\mathbb{U}) \times \mathbb{D}.$$

Further, the input deviations shall lie within a hypercuboid, as is common in MPC, $\Delta u_{\min} \le \Delta u(k) \le \Delta u_{\max}$.

In addition to stability, we aim to achieve optimal performance with respect to a quadratic, time-varying performance objective. For all time steps $k = 0, \dots, N$, consider positive (semi-)definite weight matrices $L(k) \in \mathbb{S}_{+}^{n\times n}$ and $R(k) \in \mathbb{S}_{++}^{m\times m}$ and formulate the stage cost

$$l(\Delta x(k), \Delta u(k), k) = \Delta x^T(k) L(k) \Delta x(k) + \Delta u^T(k) R(k) \Delta u(k).$$

The precise task is to design a stabilizing LTV MPC, for which we shall propose an auxiliary feedback law in the form of a time-varying, feedback controller $\mathcal{C}(\Delta x, k)$, as well as a terminal penalty, such that the origin of the closed-loop system describing the error dynamics

$$\Delta x(k+1) = f\left(\Delta x(k) + x_{\mathrm{ref}}(k), u_{\mathrm{ref}}(k) + \mathcal{C}(\Delta x(k)), 0\right) - f\left(x_{\mathrm{ref}}(k), u_{\mathrm{ref}}(k), 0\right) \tag{2}$$

is asymptotically stable along the reference trajectory and robust bound satisfaction under the disturbance $d(k) \in \mathbb{D}$ is guaranteed.

3. PROPOSED APPROACH AND MAIN RESULTS

For the purpose of deriving an uncertain LTV model, we utilize the approach in Kessler and Fagiano (2023b). Let the total linearization in each time step be

$$x(k+1) \in \{Ax(k) + Bu(k) | A \in \mathbb{A}_k, B \in \mathbb{B}_k\} \tag{3}$$

with

$$\forall k = (0, \dots, N):$$

$$\mathbb{A}_k = \left\{ A = \frac{\partial f}{\partial x}\bigg|_{(x,u,0)} : (x, u) \in \{x_{\mathrm{ref}}(k)\} \oplus \Delta\mathbb{X} \times \{u_{\mathrm{ref}}(k)\} \oplus \Delta\mathbb{U} \right\}$$

$$\mathbb{B}_k = \left\{ B = \frac{\partial f}{\partial u}\bigg|_{(x,u,0)} : (x, u) \in \{x_{\mathrm{ref}}(k)\} \oplus \Delta\mathbb{X} \times \{u_{\mathrm{ref}}(k)\} \oplus \Delta\mathbb{U} \right\}$$

Define for each k an uncertain linear system $\mathbb{S}_k = (\mathbb{A}_k \times \mathbb{B}_k)$ as well as the nominal linear system $\mathbb{S}_{k,\mathrm{ref}} = (A_k, B_k)$ with

$$\forall k = 0, \dots, N:$$

$$A_k = \frac{\partial f}{\partial x}\bigg|_{(x_{\mathrm{ref}}(k), u_{\mathrm{ref}}(k), 0)}, \; B_k = \frac{\partial f}{\partial u}\bigg|_{(x_{\mathrm{ref}}(k), u_{\mathrm{ref}}(k), 0)}.$$

For the sake of simplicity, we specialize on trajectories terminating at a steady-state, such that the MPC can utilize a fixed horizon length. General, dynamically feasible trajectories can be handled as well, but require a shrinking-horizon formulation. Finally, for a given time step k, let us introduce its successor $\bar{S}(k)$:

$$\bar{S}(k) = \min(k+1, N)$$

Then, we consider the following sequence of sets

$$\begin{aligned} \mathbb{S} &= \{\mathbb{S}_1, \dots, \mathbb{S}_N\} \\ \mathbb{S}_k &= (\mathbb{A}_k) \times (\mathbb{B}_k), \; k = 1, \dots, N. \end{aligned} \tag{4}$$

We refer to $\mathbb{S}$ (4) as the uncertain LTV system.

Consider now a linear, time-varying auxiliary feedback:

$$\mathcal{C}(\Delta x, k) = -K(k)\Delta x(k). \tag{5}$$

To co-design the auxiliary feedback and a terminal penalty for the MPC, we propose an extension of Daafouz and Bernussou (2001); Cuzzola et al. (2002) for the LTV case. For the purpose of formulating a tractable optimization problem, we employ a finite horizon cost structure with

$$J(\Delta x(k), \Delta u(k)) = \sum_{k=0}^{N_H - 1} l(\Delta x(k), \Delta u(k), k) + \Delta x^T(N_H) P(k) \Delta x(N_H),$$

where $P \in \mathbb{S}_{++}^{n\times n}$ is to be designed such that the terminal penalty is proven to be monotonously decreasing (De Nicolao et al. (1998)), i.e.,

$$\forall k = 0, \dots : \; \forall \Delta x(k) \in \Delta\mathbb{X}_f :$$

$$l(\Delta x(k), \Delta u(k), k) < V(\Delta x(k), k) - V(\Delta x(k+1), k+1), \tag{6}$$

with $V(\Delta x, k) = \Delta x^T P(k) \Delta x$. For an LTV system with terminal penalty matrix $P(k)$ the conditions on $K(k)$, anticipating $K(k) = Y(k)G^{-1}(k)$, and $P(k)$ can be expressed as the matrix inequality

$$\begin{aligned} \forall k:\; & L(k) + (Y(k)G^{-1}(k))^T R(k) Y(k) G^{-1}(k) \\ & \prec P(k) - (A(k) - B(k)Y(k)G^{-1}(k))^T P(k+1) \\ & \qquad (A(k) - B(k)Y(k)G^{-1}(k)). \end{aligned} \tag{7}$$

We obtain a suitable structure for $P(k)$ by associating with each k a terminal penalty P_k and evaluating

$$P(k) = P_{\bar{S}(k+N_H-1)}, \tag{8}$$

which in all $k = 0, \dots, N - N_H$ evaluates to $P(k) = P_{k+N_H}$. Similarly, $K(k) = K_{\bar{S}(k)}$. The feedback and terminal penalty are co-designed by applying the following Theorem:

Theorem 1. (Gain-scheduled Stabilizability of the uncertain LTV System). The uncertain LTV system (4) can be asymptotically stabilized at the origin by a controller of the form (5), if there exist matrices $G_k \in \mathbb{R}^{n\times n}$, $Y_k \in \mathbb{R}^{m\times n}$, $Q_k \in \mathbb{S}_{++}^{n\times n}$ and scalars $\gamma_k \in (0,1], k = 1,\dots,N$ such that

$$0 \prec \begin{bmatrix} G_k + G_k^T - Q_k & (\cdot) & (\cdot) & (\cdot) \\ AG_k - BY_k & Q_j & (\cdot) & (\cdot) \\ L_k^{\frac{1}{2}} G_k & 0 & \gamma_k \mathbb{I}_n & (\cdot) \\ R_k^{\frac{1}{2}} Y_k & 0 & 0 & \gamma_k \mathbb{I}_n \end{bmatrix} \quad \begin{matrix} \forall k = 1,\dots,N \\ \forall j \in \{k, \bar{S}(k)\} \\ \forall (A,B) \in \mathbb{S}_k \end{matrix} \tag{9}$$

In this case, a stabilizing controller is the feedback law (5) with the gains $K(k) = Y_k G_k^{-1}$. Further, the Lyapunov inequality (6) is fulfilled in each time step with $P_k = Q_k^{-1}$ and the nonlinear system is stabilized along the reference trajectory.

Proof. The below proof is a sketch. Employing the same reformulations as used in Daafouz and Bernussou (2001); Cuzzola et al. (2002) on LMI (9), we can recover the nonlinear matrix Inequality (7), from which we recover Condition (6): LMI (9) resembles the result in Cuzzola et al. (2002) applied to each time-step k, but we find, that the term Q_j in the second row and second column includes k, but also $\bar{S}(k)$. If it was including only Q_k, the result in Cuzzola et al. (2002) would yield, that in each node, the cost for the uncertain system is minimized by the feedback. Transitions between time-steps are considered by letting $j \in \bar{S}(k)$ and using the result in Daafouz and Bernussou (2001). Reformulating the LMI Condition (9) into Condition 7 through a change in coordinates $K_k = Y_k G_k^{-1}$ and application of the Schur-complement, one obtains that Q_j^{-1} is required to fulfill for all $\bar{S}(k)$ the nonlinear matrix Inequality (7). With $P(k)$ given in Equation (8), $V(\Delta x, k) = \Delta x^T(k) P(k) \Delta x(k)$ serves as a terminal penalty function. Further, LMI (9) implies the stability condition in Daafouz and Bernussou (2001), thus, stability of the closed-loop uncertain LTV system is established.

Remark 2. We will first remark on finding suitable stage costs L_i and R_i. In practice, different control schemes tuned for different performance trade-offs are available to the operator at equilibria, which a designer may wish to integrate into a trajectory tracking scheme. In the case, that the controller is based on a full-state feedback, an LMI condition to recover the stage cost producing the feedback gains ensuring numerical stability is presented in Priess et al. (2014). An example, where an MPC scheme was designed in this manner for $\mathcal{H}_\infty$-optimal aircraft control is found in Latif et al. (2022).

Remark 3. The second remark considers the extension to an unknown set of trajectories. As in Kessler and Fagiano (2023a), one can consider a graph of nodes pertaining to reference states and edges (i,j), $j = \bar{S}(i)$ pertaining to transitions from one node to another in one time step given a specific input. The reference trajectories then emerge as walks in the drafted graph. The modification of the LMI condition (9) is rather immediate: instead of considering $j \in \{k, \bar{S}(k)\}$, consider j to be in the set of out-neighbors in each node.

4. TUBE-MPC FORMULATION

To achieve robustness to external disturbances and thus robust bound satisfaction, we extend the gain-scheduled feedback to a tube-MPC scheme. An essential ingredient in tube-MPC design is a Robust Positive Invariant (RPI) set. Since it is intractable to compute the minimal RPI, outer, invariant approximations are employed in practice Rakovic et al. (2005). A result for an outer, polytopic approximation of the mRPI on Linear Difference Inclusions (LDI), such as equation (3) under feedback (5), is presented in Kouramas et al. (2005). A RPI set for an LDI is defined as

Definition 4. (RPI Set of an LDI A Kolmanovsky and Gilbert (1998)).A set $\Omega \subset \mathbb{R}^n$ is called a RPI set of a Lyapunov stable LDI $\mathbb{A} \subset \mathcal{P}\{\mathbb{R}^{n\times n}\}$ under a disturbance $d \in \mathbb{D} \subset \mathbb{R}^n$ if and only if

$$\forall x \in \Omega,\ \forall d \in \mathbb{D},\ \forall A \in \mathbb{A},$$
$$Ax + d \in \Omega.$$

In order to apply the result of Kouramas et al. (2005), we need to show that the error-dynamics under feedback is simultaneously Lyapunov stable, which is equivalent to implying $\forall i:\ Q_i = Q$ constant. This is less general then the condition in (9), but still allows for a node-dependent feedback and has been verified to hold in the experimental application presented in this paper.

Denote an RPI set of the uncertain LTV System (4) under Feedback (5) by $\mathbb{Z}_x$ and the union of the images of $\mathbb{Z}_x$ under matrices K_i by $\mathbb{Z}_u$. Employing the Pontryagin-difference

$$\mathbb{A} \ominus \mathbb{B} = \{x \in \mathbb{R}^n \mid x \oplus \mathbb{B} \subseteq \mathbb{A}\}.$$

we define the tightened constraints

$$\Delta\mathbb{X}_- = \Delta\mathbb{X} \ominus \mathbb{Z}_x$$
$$\Delta\mathbb{U}_- = \Delta\mathbb{U} \ominus \mathbb{Z}_u.$$

The terminal set constraint for MPC is required to be Maximal Output Admissible Set (MOAS) under the auxiliary feedback law in order to establish recursive feasibility Gilbert and Tan (1991). A result to derive a polytopic MOAS for LDIs is presented in Pluymers et al. (2005) and applies under the standing assumption, that simultaneous Lyapunov stability is established. We denote the MOAS under constraint tightening by $\Delta\mathbb{X}_f$.

We shall now propose a fixed-horizon MPC scheme with horizon length N_H to stabilize the system along the trajectory. Let us denote with $\Delta\bar{x}(i|k)$ the predictions at time $k+i$ and with k the current time step. The open-loop optimal control problem to correct the tracking error $\Delta x(k)$ in a given time step k is given by

$$\begin{aligned} & V_N^0(\Delta x(k)) = \underset{\Delta\bar{x}(k),\Delta\bar{u}(k)}{\text{minimize}} \quad J(\Delta\bar{x}(k), \Delta\bar{u}(k), k) = \\ & \sum_{i=0}^{N_H} \left(\Delta\bar{x}(i|k)^T L(k) \Delta\bar{x}(i|k) + \Delta\bar{u}(i|k)^T R(k) \Delta\bar{u}(i|k)\right) + \\ & \Delta\bar{x}(N_H|k)^T P(k) \Delta\bar{x}(N_H|k) \\ & \text{subject to} \\ & \Delta\bar{x}(0|k) \in \{\Delta x(k)\} \oplus \mathbb{Z}_x \\ & \Delta\bar{x}(i+1|k) = A_i \Delta\bar{x}(i|k) + B_i \Delta\bar{u}(i|k)\ \forall i = 0,\dots,N_H \\ & \Delta\bar{x}(i+N_H|k) \in \Delta\mathbb{X}_f \\ & \Delta\bar{x}(i|k) \in \Delta\mathbb{X}_-\ \forall i = 0,\dots,N_H \\ & \Delta\bar{u}(i|k) \in \Delta\mathbb{U}_-\ \forall i = 0,\dots,N_H - 1. \end{aligned} \tag{10}$$

Solving the optimal control Problem (10) in a given time step k yields the optimal, nominal solution $\Delta\bar{\mathbf{X}}^* = \{\Delta\bar{x}^*(k), \Delta\bar{x}^*(k+1), \ldots, \Delta\bar{x}^*(k+N_H)\}$ and $\Delta\bar{\mathbf{U}}^* = \{\Delta\bar{u}^*(k), \Delta\bar{u}^*(k+1), \ldots, \Delta\bar{u}^*(k+N_H-1)\}$. The auxiliary control law is applied to the error $\Delta x(k) - \Delta\bar{x}^*(k)$ of the nominal solution $\Delta\bar{\mathbf{X}}^*$ which in each step k evaluates to

$$\kappa_{\text{aux}}(\Delta x, k) = K(k)(\Delta x(k) - \Delta\bar{x}^*(k)).$$

Only the first input superposed with the auxiliary control law is applied, resulting in

$$\begin{aligned} &\kappa_{\text{MPC}}(x(k), k) = \\ &u_{\text{ref}}(k) - K(k)(\Delta\bar{x}^*(k) - (x(k) - x_{\text{ref}}(k))) + \Delta\bar{u}^*(k). \end{aligned} \tag{11}$$

The closed-loop MPC scheme is implemented by solving the optimal control Problem (10) in each time step k iteratively for the measured $\Delta x(k)$ and applying control Law (11).

Recursive feasibility and stability can be established with the proof-framework in Rawlings et al. (2017). Recursive feasibility is achieved in virtue of the MOAS, which is such that the auxiliary control law is feasible for the tightened constraints and stability in virtue of the terminal penalty providing an upper bound on the infinite-horizon cost. A formal proof will be given in a future journal article.

The control procedure is summarized in Algorithm 1. It is split into two phases: design and control loop. In the design phase, LMI (9) is solved and the RPI and MOAS sets are computed. Both steps are computationally intensive and the MPC scheme has been designed such that they can be done offline. The reference trajectory is then tracked by the inner loop, which can be run at fast sampling rates as a consequence of a proper terminal penalty allowing for low prediction horizons and a (convex) QP formulation.

Algorithm 1. Overview on Control Scheme

{Design Phase}
Require: $\forall k = 0, \ldots, N : \ x_{\text{ref}}(k), u_{\text{ref}}(k)$
Require: $\mathbb{S}, \Delta\mathbb{X}, \Delta\mathbb{U}, \mathbb{D}, \forall i : \ L_i, R_i$
$G = \{G_1, \ldots, G_N\}, Y = \{Y_1, \ldots, Y_N\}, Q = \{Q_1, \ldots, Q_N\} \leftarrow$ *Solve LMI* $(9)(A, B, L_i, R_i, \mathbb{S})$
Compute $\mathbb{Z}_x, \mathbb{Z}_u, \Delta\mathbb{X}_f$
{Control Loop}
Require: $x_{\text{mes}}, k, \mathbb{Z}_x, \mathbb{Z}_u, L_i, R_i, \Delta\mathbb{X}, \Delta\mathbb{U}, \Delta\mathbb{X}_f, N_H, \forall i : A_i, B_i, K_i, P_i$
$\Delta x \leftarrow x_{\text{mes}} - x_{\text{ref}}(k)$
$P \leftarrow P(k)$
$\Delta\bar{x}^*(k), \Delta\bar{u}^*(k) \leftarrow$ *Solve*$(\text{QP}, k, A_i, B_i, L_i, R_i, P(k), \Delta x$
$\kappa = u(k) + \Delta\bar{u}^*(N) - K(k)(\Delta x(k) - \Delta\bar{x}^*(k))$
Control(κ)

5. APPLICATION ON A CRAZYFLIE DRONE

For an experimental application, we implemented the LTV MPC scheme to control the longitudinal dynamics of a CrazyFlie drone powered by an STM32F405 microcontroller. The details of the drone hardware are given in Giernacki et al. (2017). Due to constraints on the available computing power, we opted for a hierarchical setup in which the MPC controls a kinematic model of the drone for pitch dynamics and the desired pitch rate is attained by the drone's builtin PID controller. For motion capture,

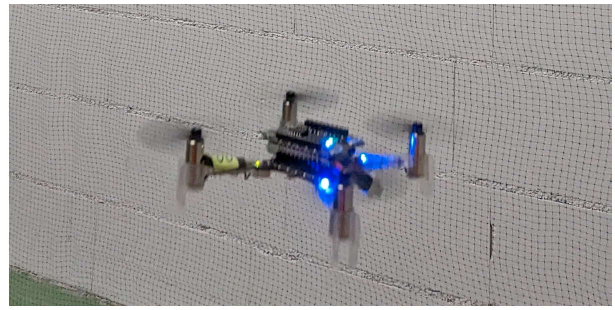

Fig. 1. CrazyFlie drone.

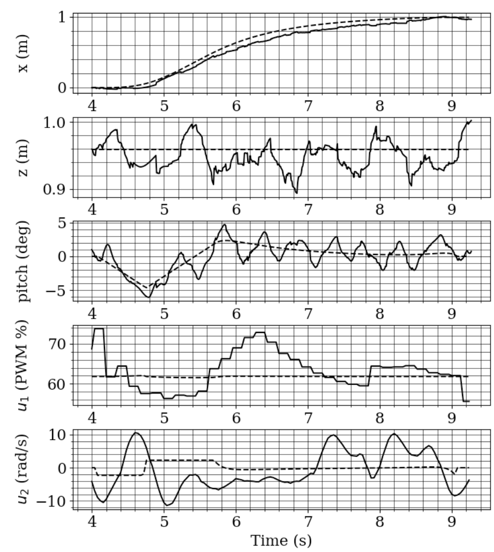

Fig. 2. Data gathered from trajectory tracking on a CrazyFlie drone in the laboratory with 3D positioning system. Dashed line: reference, continuous line: experiment data.

we employ a "lighthouse" deck in combination with HTC-Vive infrared beacons. We present the drone dynamics and a short extension to the case, that the set of reference trajectories is not known a-priori and finally the implementation and experimental results.

The CrazyFlie alongside one infrared beacon in the background is shown in Figure 1. Data is collected via a 2.4GHz radio at 100Hz. The continuous time dynamics with integrator extension of the altitude are given by

$$\dot{x} = \begin{bmatrix} x_4\cos(x_3) + x_5\sin(x_3) \\ -x_4\sin(x_3) + x_5\cos(x_3) \\ u_2 \\ \dfrac{u_1}{m} + g\sin(x_3) \\ \dfrac{u_1}{m} - g\cos(x_3) \\ x_2 \end{bmatrix},$$

where the mass of the drone is given by $m = 0.034$kg and the states are in order position in x, altitude, pitch, body forward speed, body upward speed and altitude absement. In this application, we do not have a-priori knowledge on the set of reference trajectories. However,

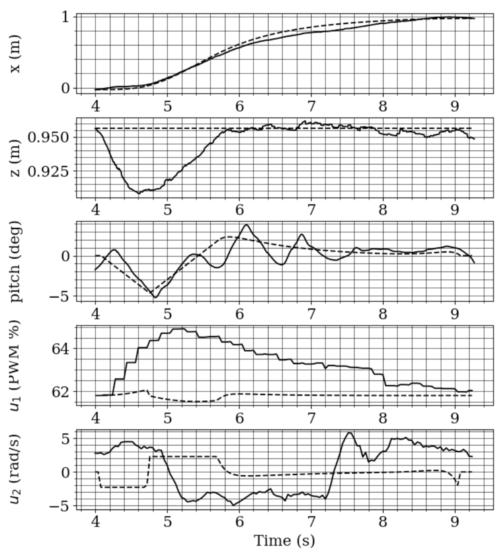

Fig. 3. Data gathered from trajectory tracking on a CrazyFlie drone with optical flow. Dashed line: reference, continuous line: experiment data.

we still succeeded in applying the scheme by partitioning the state space inspired by a symbolic control approach Pola and Di Benedetto (2019). The manifold $\mathbb{M}$ on which the state space evolves on and our proposed partitioning are

$$x_{\mathrm{ref}} \in \mathbb{R}^2 \times \underbrace{\mathbb{S}^1}_{\mathbb{G}} \times \mathbb{R}^2 \doteq \mathbb{M},$$

where $\mathbb{S}^1$ is the unit circle. Consider the projection η

$$\eta : \mathbb{M} \to \mathbb{G}$$
$$\eta(x) = [x_{(3)}],$$

which projects $\mathbb{M}$ to the base space $\mathbb{G}$. Observe that for all elements $\eta(x)$ in the base space, the linearization matrices remain unchanged for any choice of a specific x in the preimage of $\eta(x)$, i.e.

$$\forall z \in \mathbb{G} : \forall x \in \mathrm{preim}_\eta(z), \forall u \in \mathbb{U} :$$
$$\left.\frac{\partial f}{\partial x}\right|_{(x,u)} = const, \ \left.\frac{\partial f}{\partial u}\right|_{(x,u)} = const.$$

We partition $\mathbb{G}$ into eight equally sized intervals and associate with each interval C_i a reference pitch Θ_i. Further, we choose $\Delta\mathbb{X}$ such that

$$\forall x \in \mathbb{M} : \ \exists \Theta_i, \, \Delta x \in \Delta\mathbb{X} : \ \eta(x) = \Theta_i + \eta(\Delta x).$$

We can obtain a reference walk w by evaluating for all k $\eta(x_{\mathrm{ref}}(k))$ and choosing the partition with the closest Θ_i, which alleviates the need to decide on a set of reference trajectories a-priori completely.

The scheme was implemented for the STM32F405 on the drone and run onboard the quadcopter alongside the tasks for state estimation and communication. In virtue of a proper choice for a terminal penalty, we attain stability using a horizon length of $N_H = 6$ steps at a sampling rate of 25Hz. Within this sampling rate, we use OSQP Stellato et al. (2020) to approximately solve the tube LTV MPC problem with a zero-terminal constraint and slack relaxation within 26 iterations. To test the MPC scheme, we generate a trajectory steering the drone one meter forward by solving a nonlinear optimal control problem using multiple shooting for the kinematic model beforehand. The optimal control problem is formulated in CasADi Andersson et al. (2019) and solved with ipopt Wächter and Biegler (2006). The LMI Condition (9) is formulated in CVXPY Agrawal et al. (2018) and solved with CVXOPT Andersen et al. (2020).

A test flight in the laboratory is shown in Figure 2. The tracking error in the altitude lies within a few centimeters and the drone reliably and accurately tracks the trajectory. It is robust to external turbulence, for example those caused by its own down-wash, when flown close to the walls and the air conditioning. Another flight performed with a flow deck outside the laboratory, see Figure 3, shows better performance due to the absence of turbulence and lower frequency measurement noise. The experiment proofs that LTV MPC for low-dimensional systems can be implemented on inexpensive and light microcontrollers and runs at sampling rates suitable for drones.

6. CONCLUSION

We present a novel approach to modeling and control of nonlinear systems along reference trajectories with time-varying stage costs. Modeling is achieved based on an uncertain LTV system with each sample in the reference trajectory pertaining to a time step for which we associate a quadratic stage cost. To control the system along the set of reference trajectories we propose an LMI condition to co-design a time-varying feedback law and terminal penalty matrix. LTV MPC has been applied with wide success in the industry and employing the presented approach, the essential quantities for stability can be rigorously verified. We propose a tractable procedure based on LMI conditions to derive the terminal penalty and outline, how the terminal set can be established with existing methods, filling the gap between theoretical guarantees and practical findings. Under employment of our proposed terminal penalty, we achieve a stabilizing MPC with a horizon close to the theoretical minimum, enabling implementation on the onboard microcontroller.

The proposed approach heavily relies on solvers for optimization problems, the latter growing to large scales for systems with an increasing number of states. First, finding a global linearization requires the solution of a nonlinear program. Optimizers such as Sabug Jr et al. (2021) can be employed, but data driven methods are still being researched. We found that solving LMI (9) is unproblematic, due to the efficiency of semi-definite programming solvers and the graph structure of the problem, which can be exploited Hallac et al. (2017). However, minimal, forward-invariant, outer approximation of the RPI set and MOAS are still challenging to compute. We are looking forward to publish complete proofs of the stated theorems and MPC stability in detail. Further, the way the LTV MPC problem is formulated has a special structure in time, which can be exploited to accelerate online computation of the control law, which we are looking forward to address. We also are looking forward into tackling the problem of deriving an uncertain LTV system from data on a theoretical front. Further, means to deal with model uncertainty need to be addressed to attain safety under model-plant mismatch and optimize performance by learning the real model.

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