Stability Analysis and Stabilization of Semi-Markov Jump Linear Systems With Improved Efficiency of Probabilistic Information Utilization

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***Abstract*—This article establishes a systematic methodology to improve the utilization efficiency of probabilistic information for the stability analysis and stabilizing control of discrete-time semi-Markov jump linear systems (SMJLSs). The transition and sojourn information is incompletely known, and the coupling between the known (or unknown) transition and unknown (or known) sojourn information renders the known probabilistic information difficult to be fully leveraged, which can lead to conservative results in system analysis and synthesis. To approximate the unknown transition and sojourn information, a polyhedral approach is developed, which facilitates the incorporation of the known probabilistic information coupled with unknown information. Accordingly, novel vertex-based Lyapunov functions are proposed to establish stability conditions. New criteria are established for the stability analysis and control of SMJLSs by incorporating all the jointly known transition and sojourn information, all the known probabilistic information, and both the known and the approximation of unknown probabilistic information, respectively. The effectiveness and superiority of the theoretical results are illustrated by a numerical example and a simulated continuous stirred tank reactor process. The results demonstrate that both the improvement in information utilization efficiency and the implementation of the proposed vertex-based Lyapunov functions can reduce the conservativeness in the stability analyses.**

***Index Terms*—Incomplete transition and sojourn information, polyhedral approximation, probabilistic information utilization, semi-Markov jump linear systems, time-dependent control.**

I. Introduction

In various real-world systems, abrupt changes in structures and/or parameters are frequently encountered. These changes may lead to switchings among different operating modes, which can have a significant impact on the system dynamics [1]. The systems that exhibit stochastic switching behaviors are referred to as jump systems [2]–[4]. As two important classes of jump systems, Markov jump systems (MJSs) and semi-Markov jump systems (SMJSs) have gained extensive attention due to their modeling efficacy. In an MJS, the sojourn time between any two consecutive jumps follows exponential distributions for a continuous-time MJS and geometric distributions for a discrete-time MJS, while the sojourn time for SMJSs may follow any possible probability distributions. Therefore, SMJSs are more general, allowing for greater flexibility in modeling and analysis of industrial systems/processes. For the past twenty years, there have been a significant amount of research on MJSs, e.g., [5]–[10], and SMJSs, e.g., [11]–[21]. As compared to MJSs, the switching signals for SMJSs are more complex, which leads to a greater challenge in research on SMJSs.

In the context of MJSs, transition probabilities (TPs) are crucial for understanding the system dynamics. However, according to [2], [22], [23], it is challenging to determine TPs accurately. If inaccurate TPs are used for control system development, then the closed-loop performance of the underlying system may be degraded or even unstable [22]. To overcome this problem, the authors in [24], [25] developed stochastic stability and stabilization conditions for MJSs with incomplete knowledge of TPs by separating the sub-conditions corresponding to the known TPs from the unknown TPs. Regarding continuous-time MJSs for which all the off-diagonal entries of the TP rate matrix are unknown, a polyhedral approach was proposed in [26]. This approach uses a set of linearly independent matrix vertices that are determined by known diagonal TP rates to approximate these off-diagonal TP rates. Accordingly, a necessary and sufficient mean-square stability condition for the MJSs was established via analyzing the stability of a higher-dimensional deterministic switched system derived from the original MJS. Approximation algorithms based on polyhedral TP matrices can also be found in [2], [27]; however, the linear independence of the matrix vertices cannot be guaranteed. Moreover, the previously discussed approaches can hardly be extended to more generalized jump system models, including SMJSs [28], [29].

SMJSs have been employed to model various real-world systems, such as computer systems [30], [31] and process control systems [32]–[34]. In an SMJS, there are two sources of probabilistic information: transition information and sojourn information. Transition information refers to the TPs for the embedded Markov chain associated with the switching signal, which amount to the TPs specific to jump instants; sojourn information means the probabilistic distribution information of sojourn time. The two types of information can be integrated to form a semi-Markov kernel (SMK) that was leveraged for the

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analysis and synthesis of SMJSs in [35]–[39]. So far, the majority of the existing results on SMJSs have been established based on an ideal assumption of completely and precisely known probabilistic information. However, the samples are often inadequate and the available statistical information is generally insufficient in practice, and several studies, such as [29], [40], [41], highlighted the difficulty in obtaining accurate probabilistic information. Therefore, it is common that only partial information on the transition and/or sojourn is accessible. The above observations highlight the importance of considering incomplete probabilistic information when conducting analysis and control designs for SMJSs. Accordingly, initial efforts were made to develop methods that aim to handle the scenario of incomplete probabilistic information, see, e.g., [29], [40]–[43]. However, the results in [41]–[43] were restricted to specific constraints on the cumulative distribution of each sojourn-time probability mass function (STPMF). If the STPMF is fragmentarily known, it becomes challenging to ensure the satisfaction of this assumption. Although this limitation was addressed in [29], [40], the known probabilistic information was not fully utilized in these studies on SMJSs with incomplete probabilistic information; this can lead to additional conservativeness.

In particular, previous studies on SMJS analysis and synthesis neglected certain known probabilistic information, including fragmentarily known STPMFs, TPs[1] coupled with the unknown or abandoned part of their corresponding STPMFs, and STPMFs coupled with their unknown corresponding TPs. For example, in [29], [40]–[43], only entirely known STPMFs were utilized in analysis and synthesis; if the sojourn time takes a value such that its STPMF is unknown, then the entire STPMF needs to be disregarded. Also, based on the approaches in [29], [40]–[43], each SMK entry is expressed as the product of the TP and STPMF with the same pair of the current and the target modes. If either of the TP or STPMF is unknown, then the corresponding SMK entry will be inaccessible, and the known TP or STPMF will not be usable. The exclusion of these types of known probabilistic information from system analysis and control design can lead to additional conservativeness and limit the applicability of the developed approaches, since the known information is replaced with uncertainty to cover all the possible cases. Therefore, it is crucial to develop new approaches that can effectively utilize all the known information, which we aim to address. Moreover, the ranges of the unknown TPs and the unknown portion of STPMFs can be confirmed based on the known probabilistic information. The use of appropriate approximations can enhance the utilization efficiency of probabilistic information, offering potential to reduce the conservativeness of the results for SMJSs and improve the applicability of the theoretical findings. However, how to incorporate unknown information into SMJS analysis and synthesis remains an open problem. This gap will also be bridged in this research.

In this article, we establish new criteria on stochastic stability and time-dependent control for discrete-time semi-Markov jump linear systems (SMJLSs). We offer three levels of improvement in the utilization of probabilistic information: i) the utilization of jointly known transition and sojourn information; ii) the complete utilization of known probabilistic information, and iii) the joint utilization of both known and unknown probabilistic information. A polyhedral modeling method is developed to approximate the unknown probabilistic information. To derive the stability conditions for the cases in ii) and iii), two forms of vertex-based Lyapunov functions are proposed. The main contributions are fourfold. *First*, the established stability analysis and control criteria for SMJLSs are made less conservative by improving the utilization efficiency of probabilistic information, leading to improved feasibility and enhanced applicability of the established criteria. *Second*, the proposed polyhedral modeling approach provides good approximations of the unknown transition and sojourn information, and enables the coupled known information accessible for system analysis and synthesis. *Third*, with the dependence on the polyhedrons of the approximation of the unknown information, the proposed vertex-based Lyapunov functions further reduce the conservativeness. *Fourth*, we reveal the relationship between the utilization efficiency of probabilistic information and the level of conservativeness in stability conditions, which provides valuable research directions for the less conservative analysis criteria for various other systems.

Notation: The superscript $\top$ denote the vector/matrix transpose and $(?)$ indicates unknown parameters. $\mathbb{R}$ and $\mathbb{I}$ symbolize the sets of real numbers and integers, respectively. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. $\mathbb{I}_{\geq N_1}$ and $\mathbb{I}_{[N_1,N_2]}$ signify the sets $\{n \in \mathbb{I} | n \geq N_1\}$ and $\{n \in \mathbb{I} | N_1 \leq n \leq N_2\}$, respectively. When $N_1 > N_2$, $\mathbb{I}_{[N_1,N_2]}$ is an empty set $\varnothing$. $\mathbb{X} \backslash \mathbb{Y}$ expresses the set subtraction of set $\mathbb{Y}$ from set $\mathbb{X}$. $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}_{[N_1,N_2]}$ represents the space defined on $m \times n$ real matrices with each entry taking a value from the set $\mathbb{R}_{[N_1,N_2]} \triangleq \{n \in \mathbb{R} | N_1 \leq n \leq N_2\}$. $\|\cdot\|$ denotes the Euclidean vector norm. In a matrix, each $\star$ simplifies an off-diagonal entry/block of the symmetric matrix. $\mathrm{diag}\{\cdots\}$ means a block-diagonal matrix. The symbol $\times$ between two sets denotes the Cartesian product, and $\otimes$ is the operator of Kronecker product. $\mathsf{E}\{\cdot\}|_x$ denotes the expectation conditioned on x. I_n signifies the n-dimensional identity matrix. $\mathbf{0}$ is a zero matrix. The expressions $P \succ Q$, $P \prec Q$, and $P \preceq Q$ indicate that $P - Q$ is positive definite, negative definite, and non-positive definite, respectively. Given $n \times n$ matrices $A(i)$, $i \in \mathbb{I}_{[N_1,N_2]}$, $\prod_{i=N_1}^{N_2} A(i) \triangleq A(N_2)A(N_2-1)\cdots A(N_1)$, and we define $\prod_{i=N_1}^{N_2} A(i) \triangleq I_n$ and $\sum_{i=N_1}^{N_2} A(i) \triangleq \mathbf{0}$ if $N_1 > N_2$.

II. Preliminary Knowledge

We denote the index of a discrete-time semi-Markov chain $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$ at the nth jump by R_n, the time at the nth jump with $k_0 = 0$ by k_n, and the sojourn time in mode R_n by S_n. τ is used to represent the value of $S_n \triangleq k_{n+1} - k_n$.

Consider a discrete-time homogeneous Markov renewal chain (HMRC) $\{(R_n, k_n)\}_{n\in\mathbb{I}_{\geq 0}}$ satisfying the following property: $\forall a, b \in \mathbb{M}$ $(a \neq b)$, $\forall \tau \in \mathbb{I}_{\geq 1}$, $\forall n \in \mathbb{I}_{\geq 0}$, $\mathrm{Prob}(R_{n+1} = b, S_n = \tau | R_n = a, k_n, R_{n-1}, k_{n-1}, \ldots, R_0, k_0)$

[1]In the remainder of this article, TPs refer to the TPs for the associated embedded Markov chain in SMJSs unless otherwise specified.

$=\mathrm{Prob}\left(R_{n+1}=b,S_n=\tau|R_n=a\right)=\mathrm{Prob}(R_1=b,S_0=\tau|R_0=a)$. Associated with the HMRC $\{(R_n,k_n)\}_{n\in\mathbb{I}_{\geq 0}}$, the definition of semi-Markov chain is given below.

Definition 1: [35] The stochastic process $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$ is a discrete-time semi-Markov chain (SMC) associated with the discrete-time HMRC $\{(R_n,k_n)\}_{n\in\mathbb{I}_{\geq 0}}$, if $\forall k\in\mathbb{I}_{\geq 0}$, $r(k)=R_{\mathcal{N}_{\max}(k)}$, where $\mathcal{N}_{\max}(k)\triangleq\max\{n\in\mathbb{I}_{\geq 0}|k\geq k_n\}$.

Additional definitions and associated properties are introduced before the problem is formulated.

Definition 2: [35] Given an HMRC $\{(R_n,k_n)\}_{n\in\mathbb{I}_{\geq 0}}$, the stochastic process $\{R_n\}_{n\in\mathbb{I}_{\geq 0}}$ is referred to as the embedded Markov chain (EMC) of the HMRC.

Definition 3: [35] Consider an SMC $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$ taking values in a finite set $\mathbb{M}\triangleq\{1,2,\ldots,M\}$ associated with the HMRC $\{(R_n,k_n)\}_{n\in\mathbb{I}_{\geq 0}}$.

i) The transition probabilities (TPs) for the EMC $\{R_n\}_{n\in\mathbb{I}_{\geq 0}}$ are defined as $\pi_{ab}\triangleq\mathrm{Prob}(R_{n+1}=b|R_n=a)$ and $\pi_{aa}\triangleq 0$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$.

ii) The sojourn-time probability mass functions (STPMFs) from the current mode a to the target mode b are defined as $f_{ab}(\tau)\triangleq\mathrm{Prob}(S_n=\tau|R_{n+1}=b,R_n=a)$ and $f_{aa}(\tau)\triangleq 0$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$, $\forall\tau\in\mathbb{I}_{\geq 1}$.

iii) The entries of the semi-Markov kernel (SMK) for the SMC are defined as $\theta_{ab}(\tau)\triangleq\mathrm{Prob}(R_{n+1}=b,S_n=\tau|R_n=a)$ and $\theta_{aa}(\tau)\triangleq 0$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$, $\forall\tau\in\mathbb{I}_{\geq 1}$. The SMK is described as $\Theta\left(\tau\right)\in\mathbb{R}_{[0,1]}^{M\times M}$ with the ath row, bth column entry $\theta_{ab}(\tau)$.

The TPs π_{ab} satisfy $\sum_{b\in\mathbb{M},b\neq a}\pi_{ab}=1$ with $0\leq\pi_{ab}\leq 1$, $\forall a,b\in\mathbb{M}$. Also, as $\sum_{\tau=1}^{\infty}f_{ab}(\tau)=1$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$, it holds that $\sum_{\tau=1}^{\infty}\sum_{b\in\mathbb{M},b\neq a}\pi_{ab}f_{ab}(\tau)=1$ with $0\leq\pi_{ab}f_{ab}(\tau)\leq 1$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$, $\forall\tau\in\mathbb{I}_{\geq 1}$. Due to $\pi_{aa}=0$ and $f_{aa}(\tau)=0$, $\forall a\in\mathbb{M}$, $\forall\tau\in\mathbb{I}_{\geq 1}$, we have $\sum_{b\in\mathbb{M}}\pi_{ab}=1$ and $\sum_{\tau=1}^{\infty}\sum_{b\in\mathbb{M}}\pi_{ab}f_{ab}(\tau)=1$, $\forall a\in\mathbb{M}$. The TPs π_{ab} are different from the TPs of the SMC $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$, where the latter TPs within the same mode can be nonzero.

III. Problem Formulation

A. System Description

Consider a semi-Markov jump linear system (SMJLS) on a complete probability space:

$$x\left(k+1\right)=A_{r(k)}x\left(k\right)+B_{r(k)}u\left(k\right) \tag{1}$$

where $x(k)\in\mathbb{R}^{n_x}$ and $u(k)\in\mathbb{R}^{n_u}$ represent the system state and control input, respectively; $r(k)$ denotes the switching signal subject to an SMC that takes values from a finite set $\mathbb{M}\triangleq\{1,2,\ldots,M\}$; both $A_{r(k)}$ and $B_{r(k)}$ are real-valued parameter matrix functions of the switching signal $r(k)$. Since the sojourn time in practical systems or processes is generally bounded, the sojourn time is truncated by an upper bound $\bar{\tau}_a\in\mathbb{I}_{\geq 1}$ for each mode $a\in\mathbb{M}$. We consider that not all the STPMFs or the TPs are fully accessible. To be specific, $\exists a,b\in\mathbb{M}$ $(a\neq b)$, such that the TP π_{ab} is unknown and/or the STPMF $f_{ab}(\tau),\exists\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$ is unknown.

In terms of the TPs, the index set of system modes can be categorized as $\mathbb{M}=\bar{\mathbb{M}}_a\cup\tilde{\mathbb{M}}_a,\forall a\in\mathbb{M}$, where

$$\left\{\begin{array}{l}\bar{\mathbb{M}}_a\triangleq\{b\in\mathbb{M}|\pi_{ab}\text{ is known}\},\\ \tilde{\mathbb{M}}_a\triangleq\{b\in\mathbb{M}|\pi_{ab}\text{ is unknown}\}.\end{array}\right. \tag{2}$$

The elements in the set $\bar{\mathbb{M}}_a$ are listed as $\{\bar{m}_a(\xi)|\xi\in\mathbb{I}_{[1,\bar{M}_a]}\}$ subject to $1\leq\bar{m}_a(1)<\bar{m}_a(2)<\ldots<\bar{m}_a(\bar{M}_a)\leq M$, where $\bar{m}_a(\xi)$ is the ξth element in $\bar{\mathbb{M}}_a$, and is also the $\bar{m}_a(\xi)$th element in $\mathbb{M}$; $\bar{M}_a\in\mathbb{I}_{[1,M]}$ is the number of the elements contained in $\bar{\mathbb{M}}_a$. Since $\pi_{aa}=0$, $\forall a\in\mathbb{M}$, we can readily know $a\in\bar{\mathbb{M}}_a$.

For an STPMF $f_{ab}(\tau)$, $\forall a,b\in\mathbb{M}$, the values of sojourn time are classified as $\mathbb{I}_{[1,\bar{\tau}_a]}=\bar{\mathbb{S}}_{ab}\cup\tilde{\mathbb{S}}_{ab}$, where

$$\left\{\begin{array}{l}\bar{\mathbb{S}}_{ab}\triangleq\left\{\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}|f_{ab}(\tau)\text{ is known}\right\},\\ \tilde{\mathbb{S}}_{ab}\triangleq\left\{\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}|f_{ab}(\tau)\text{ is unknown}\right\}.\end{array}\right. \tag{3}$$

In (3), we describe $\bar{\mathbb{S}}_{ab}=\{\bar{s}_{ab}(\delta)|\delta\in\mathbb{I}_{[1,\bar{\tau}_{ab}]}\}$, where $\bar{s}_{ab}(\delta)$ denotes the sojourn time for the known $f_{ab}(\bar{s}_{ab}(\delta))$, $\forall a,b\in\mathbb{M}$, $\forall\delta\in\mathbb{I}_{[1,\bar{\tau}_{ab}]}$; $\bar{\tau}_{ab}$ represents the number of elements in $\bar{\mathbb{S}}_{ab}$ satisfying $\bar{\tau}_{ab}\leq\bar{\tau}_a$. Since $f_{aa}(\tau)=0$, $\forall a\in\mathbb{M}$, $\forall\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$, it holds that $\bar{\tau}_{aa}=\bar{\tau}_a$, $\bar{\mathbb{S}}_{aa}=\mathbb{I}_{[1,\bar{\tau}_a]}$, $\tilde{\mathbb{S}}_{aa}=\varnothing$, $\forall a\in\mathbb{M}$. Specifically, we let $\bar{\tau}_{ab}=0$ and $\bar{s}_{ab}(0)=0$ if $\exists b\in\mathbb{M}$ $(b\neq a)$ such that $\bar{\mathbb{S}}_{ab}=\varnothing$, $\forall a\in\mathbb{M}$. For an STPMF $f_{ab}(\tau)$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$, we introduce the following concepts:

Definition 4: Consider an STPMF $f_{ab}(\tau)$, $\forall a,b\in\mathbb{M}$.

i) The STPMF is said to be entirely known if the values of $f_{ab}(\tau)$ are known for all $\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$.

ii) The STPMF is said to be fragmentarily known if $\exists\tau\in\tilde{\mathbb{S}}_{ab}\subsetneqq\mathbb{I}_{[1,\bar{\tau}_a]}$, such that $f_{ab}(\tau)$ $(a\neq b)$ is unknown.

iii) The STPMF is said to be entirely unknown if the values of $f_{ab}(\tau)$ $(a\neq b)$ are unknown for all $\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$.

Remark 1: From (2), we know $\tilde{\mathbb{M}}_a=\mathbb{M}\backslash\bar{\mathbb{M}}_a$ and $\bar{\mathbb{M}}_a\cap\tilde{\mathbb{M}}_a=\varnothing$, $\forall a\in\mathbb{M}$. If $\bar{\mathbb{M}}_a=\mathbb{M}$ and $\tilde{\mathbb{M}}_a=\varnothing$, then $\bar{m}_a(\xi)=\xi,\forall\xi\in\mathbb{I}_{[1,\bar{M}_a]}$ with $\bar{M}_a=M$. Also, from (3), there hold that $\tilde{\mathbb{S}}_{ab}=\mathbb{I}_{[1,\bar{\tau}_a]}\backslash\bar{\mathbb{S}}_{ab}$ and $\bar{\mathbb{S}}_{ab}\cap\tilde{\mathbb{S}}_{ab}=\varnothing$, $\forall a,b\in\mathbb{M}$. If $\exists b\in\mathbb{M}$ such that $\bar{\mathbb{S}}_{ab}=\mathbb{I}_{[1,\bar{\tau}_a]}$, then $\tilde{\mathbb{S}}_{ab}=\varnothing$, $\bar{\tau}_{ab}=\bar{\tau}_a$, $\forall a\in\mathbb{M}$.

B. Problem Statement

Since $\theta_{ab}(\tau)=\pi_{ab}f_{ab}(\tau)$, $\forall a,b\in\mathbb{M}$, $\forall\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$, from (2) and (3), we know that $\theta_{ab}(\tau)$ is inaccessible when $b\in\tilde{\mathbb{M}}_a$ (corresponding to unknown π_{ab}) or when $b\in\mathbb{M}$ $(b\neq a)$, $\tau\in\tilde{\mathbb{S}}_{ab}$ (corresponding to the unknown $f_{ab}(\tau)$). In view of this, $\theta_{ab}(\tau)$ is accessible only when $a\in\mathbb{M}$, $(b,\tau)\in\bar{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab}$. However, in most of the previous studies involving inaccessible probabilistic information, such as [40]–[42], $\bar{\mathbb{S}}_{ab}=\mathbb{I}_{[1,\bar{\tau}_a]}$ is required for the accessibility of $\theta_{ab}(\tau)$, $\forall a\in\mathbb{M}$, $\forall(b,\tau)\in\bar{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab}$. In these studies, if $\exists\tau\in\tilde{\mathbb{S}}_{ab}$, then the entire STPMF $f_{ab}(\tau)$, $\forall a,b\in\mathbb{M}$ $(a\neq b)$ will not be incorporated in the subsequent system analysis and synthesis. The neglect of known probabilistic information can increase the level of resulting conservativeness. This motivates us to incorporate the disregarded fragmentarily known STPMFs $f_{ab}(\tau)$, $a\in\mathbb{M}$, $(b,\tau)\in\bar{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab}$ when the corresponding $\tilde{\mathbb{S}}_{ab}\neq\varnothing$, such that the utilization ratio of known probabilistic information, which is mathematically represented as the proportion of known probabilistic information effectively incorporated into analysis and synthesis, can be improved.

In addition, when $a\in\mathbb{M}$, $(b,\tau)\in\tilde{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab}$, the STPMF $f_{ab}(\tau)$ is known while the TP π_{ab} is unknown, in which case, $\theta_{ab}(\tau)$ is inaccessible. Similarly, when $a\in\mathbb{M}$, $(b,\tau)\in\bar{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, the TP π_{ab} is known yet the STPMF $f_{ab}(\tau)$

is unknown, thus we still do not have access to the resulting $\theta_{ab}(\tau)$. Therefore, resorting to explicit SMK entries $\theta_{ab}(\tau)$ in the SMJLS analysis and synthesis can result in the neglect of a portion of the known probabilistic information, which also introduces additional conservativeness to the analysis and synthesis results, unless we approximate the TPs π_{ab} corresponding to $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b, \tau) \in \tilde{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab}$ or the STPMFs $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b, \tau) \in \bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}$. Therefore, it is crucial to propose an effective approach to tackle the problem that π_{ab} and $f_{ab}(\tau)$ are not simultaneously known for the same $a, b \in \mathbb{M}$ $(a \neq b)$, such that all the known probabilistic information therein is utilizable in system analysis and synthesis.

To present our objectives formally, we invoke the notion of stochastic stability in the framework of bounded sojourn time.

Definition 5: A jump system with bounded sojourn time is stochastically stable if

$$\mathsf{E}\left\{\sum_{k=0}^{\infty} \|x(k)\|^2\right\}\Bigg|_{x(0)\in\mathbb{R}^{n_x}, r(0)\in\mathbb{M}} < \infty \tag{4}$$

holds, where the bounded sojourn time satisfying $k_{\mathcal{N}_{\max}(k)+1} - k_{\mathcal{N}_{\max}(k)} \in \mathbb{I}_{[1,\bar{\tau}_{r(k)}]}$, $r(k) \in \mathbb{M}$.

In the sequel, we use k_n to represent the time when the latest jump occurs. To stabilize SMJLS (1), we employ the following elapsed-time-dependent controller:

$$u(k) = K_{r(k)}(k - k_n)x(k) \tag{5}$$

where $K_{r(k)}(k - k_n)$ is the controller gain that is computed offline and has dependence on both the current system mode $r(k)$ and the elapsed time $k - k_n$ since the latest mode jump. By substituting (5) into (1), the following closed-loop system can be formed:

$$x(k+1) = \mathcal{A}_{r(k)}(k - k_n)x(k) \tag{6}$$

where $\mathcal{A}_{r(k)}(k - k_n) \triangleq A_{r(k)} + B_{r(k)}K_{r(k)}(k - k_n)$.

The main problems to be addressed include:

i) To establish relaxed conditions for the stochastic stability of closed-loop SMJLS (6) by incorporating all the simultaneously known TPs and their corresponding STPMFs that are entirely or fragmentarily known; and to design an elapsed-time-dependent controller as the form in (5) to ensure the stochastic stability of the resulting closed-loop SMJLS in (6).

ii) To develop an approach to approximate the unknown TPs whose corresponding STPMFs are entirely or fragmentarily known and approximate the unknown portion of the STPMFs corresponding to known TPs, such that the utilization efficiency of the known probabilistic information can reach 100%; and to derive stability criteria for closed-loop SMJLS (6) and implement the elapsed-time-dependent control scheme based on the approximated unknown TPs and STPMFs.

iii) To propose an approximation approach for all the unknown probabilistic information, such that all the known and unknown probabilistic information can be jointly incorporated, based on which the stochastic stability analysis and elapsed-time-dependent control synthesis for SMJLS (1) are addressed.

Remark 2: If we set $K_{r(k)}(k - k_n) = K_{r(k)}$, $\forall k \in \mathbb{I}_{[k_n,k_{n+1}-1]}$, $\forall n \in \mathbb{I}_{\geq 0}$, then the controller in (5) will reduce to $u(k) = K_{r(k)}x(k)$. Therefore, our elapsed-time-dependent controller can be viewed as a more general control paradigm compared to the conventional mode-dependent controller independent of the elapsed time.

C. Preliminaries for Stochastic Stability Analysis

Before addressing the stability analysis for SMJLSs, we invoke the notion of class $\mathcal{K}_\infty$ functions.

Definition 6: [44] A continuous function $\mathcal{K}(\cdot) \in \mathbb{R}_{\geq 0}$, which is defined on $\mathbb{R}_{\geq 0}$, is a class $\mathcal{K}_\infty$ function if it is strictly increasing and satisfies $\mathcal{K}(0) = 0$ and $\lim_{x\to\infty} \mathcal{K}(x) \to \infty$.

Additionally, a general criterion is provided below for the stochastic stability analysis of SMJLSs.

Lemma 1: [41] Given $x(k_n) \in \mathbb{R}^{n_x}$ and $r(k_n) = R_n \in \mathbb{M}$, $\forall n \in \mathbb{I}_{\geq 0}$, a discrete-time SMJLS of form (1) is stochastically stable if there exists a Lyapunov function $\mathcal{L}_{r(k),k-k_n}(\cdot) : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$, $\forall k \in \mathbb{I}_{[k_n,k_{n+1}-1]}$, $\forall n \in \mathbb{I}_{\geq 0}$, satisfying

$$\mathcal{K}_1(\|x(k)\|) \leq \mathcal{L}_{r(k),k-k_n}(x(k)) \leq \mathcal{K}_2(\|x(k)\|) \tag{7}$$

such that $\forall n \in \mathbb{I}_{\geq 0}$, the following inequality holds:

$$\begin{aligned} &\mathsf{E}\left\{\mathcal{L}_{R_{n+1},0}(x(k_{n+1}))\right\}|_{x(k_n)\in\mathbb{R}^{n_x}, R_n\in\mathbb{M}} \\ &\quad - \mathcal{L}_{R_n,0}(x(k_n)) \leq -\mathcal{K}_3\left(\|x(k_n)\|\right) \end{aligned} \tag{8}$$

under the initial condition $x(0) \in \mathbb{R}^{n_x}$, $r(0) \in \mathbb{M}$, where $\mathcal{K}_1(\cdot)$, $\mathcal{K}_2(\cdot)$ and $\mathcal{K}_3(\cdot)$ are class $\mathcal{K}_\infty$ functions of $\|x(k)\|$.

Remark 3: According to the linear characteristic of an SMJLS, the associated Lyapunov function cannot reach infinity during any bounded sojourn time. Hence, the stochastic stability can be ensured if the expectation of the Lyapunov function decreases with each jump, and we need not to account for the Lyapunov function during the sojourn time.

IV. Full Utilization of Jointly Known Transition and Sojourn Information

A. Stochastic Stability Analysis

We construct a Lyapunov function as presented below (ref. [29], [35], [41]–[43]), which has dependence on both the current mode and the time elapsed in the current mode:

$$\mathcal{L}_{r(k),k-k_n}(x(k)) \triangleq x^\top(k)P_{r(k)}(k - k_n)x(k). \tag{9}$$

For the derivation of the subsequent results, we assign the values of the involved random variables: $R_n = a \in \mathbb{M}$, $R_{n+1} = b \in \mathbb{M}$ $(b \neq a)$, $S_n = \tau \in \mathbb{I}_{[1,\bar{\tau}_a]}$, $k - k_n = \varepsilon \in \mathbb{I}_{[0,\tau-1]}$. Then, for $r(k) = a \in \mathbb{M}$, $\forall k \in \mathbb{I}_{[k_n,k_{n+1}-1]}$, $\forall n \in \mathbb{I}_{\geq 0}$, the relation $\lambda_1 \|x(k)\|^2 \leq \mathcal{L}_{a,\varepsilon}(x(k)) \leq \lambda_2 \|x(k)\|^2$ can be satisfied, where $\lambda_1 \triangleq \min_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]}} \{\lambda_{\min}(P_a(\varepsilon))\}$ and $\lambda_2 \triangleq \max_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]}} \{\lambda_{\max}(P_a(\varepsilon))\}$. Based on Lemma 1, the criterion on the stochastic stability of SMJLS (6) is presented below, which incorporates all the simultaneously known TPs and their corresponding STPMFs.

Theorem 1: SMJLS (6) is stochastically stable if there exist matrices $P_a(0) \succ \mathbf{0}$, $a \in \mathbb{M}$, such that $\forall a \in \mathbb{M}$,

$$\sum_{b\in\bar{\mathbb{M}}_a} \sum_{\tau\in\bar{\mathbb{S}}_{ab}} \frac{\pi_{ab} f_{ab}(\tau)}{\bar{\eta}_a} \left(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\right)^\top P_b(0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon) - P_a(0)$$

$$\prec \mathbf{0}\ (\bar{\eta}_a > 0) \tag{10}$$

and $\forall a \in \mathbb{M}$, $\forall (b,\tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}) \cup (\tilde{\mathbb{M}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]})$,

$$\left(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\right)^{\top} P_b(0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon) - P_a(0) \prec \mathbf{0} \tag{11}$$

where $\bar{\eta}_a \triangleq \sum_{b\in\bar{\mathbb{M}}_a} \sum_{\tau\in\bar{\mathbb{S}}_{ab}} \pi_{ab} f_{ab}(\tau)$.

Proof: See Appendix A. ■

If all the fragmentarily known STPMFs, i.e., $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab}$ ($\bar{\mathbb{S}}_{ab} \subsetneqq \mathbb{I}_{[1,\bar{\tau}_a]}$), are not incorporated, as in [29], [41]–[43], then Theorem 1 will reduce to Corollary 1, disregarding a portion of the known probabilistic information.

Corollary 1: SMJLS (6) is stochastically stable if there exist matrices $P_a(0) \succ \mathbf{0}$, $a \in \mathbb{M}$, such that $\forall a \in \mathbb{M}$,

$$\sum_{b\in\bar{\bar{\mathbb{M}}}_a} \sum_{\tau=1}^{\bar{\tau}_a} \frac{\pi_{ab} f_{ab}(\tau)}{\bar{\bar{\eta}}_a} \left(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\right)^{\top} P_b(0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon) - P_a(0)$$
$$\prec \mathbf{0}\ (\bar{\bar{\eta}}_a > 0) \tag{12}$$

and $\forall a \in \mathbb{M}$, $\forall b \in \mathbb{M}\backslash\bar{\bar{\mathbb{M}}}_a$, $\forall \tau \in \mathbb{I}_{[1,\bar{\tau}_a]}$, (11) holds, where $\bar{\bar{\mathbb{M}}}_a \triangleq \{b \in \mathbb{M} | \text{both } \pi_{ab} \text{ and } f_{ab}(\tau) \text{ are known}, \forall \tau \in \mathbb{I}_{[1,\bar{\tau}_a]}\}$, $\bar{\bar{\eta}}_a \triangleq \sum_{b\in\bar{\bar{\mathbb{M}}}_a} \sum_{\tau=1}^{\bar{\tau}_a} \pi_{ab} f_{ab}(\tau)$.

If $\bar{\mathbb{M}}_a = \{a\}$, $\tilde{\mathbb{M}}_a = \mathbb{M}\backslash\{a\}$ or if $\bar{\mathbb{S}}_{ab} = \varnothing$, $\tilde{\mathbb{S}}_{ab} = \mathbb{I}_{[1,\bar{\tau}_a]}$, $\forall a, b \in \mathbb{M}$ ($a \neq b$), then no probabilistic information is involved. In this case, the SMJLS jumps arbitrarily within $\tau \in \mathbb{I}_{[1,\bar{\tau}_a]}$, and accordingly, (10) and (12) are excluded.

Remark 4: Corollary 1 is derived based on the technique in [40], [42], which only incorporates the TPs and entire STPMFs that are jointly known. Since $\bar{\bar{\mathbb{M}}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]} \subseteq \bar{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab}$, some specific probabilistic information is disregarded in Corollary 1 in comparison to Theorem 1 that takes into account all the jointly known transition and sojourn information. Consequently, Corollary 1 represents more conservative findings than Theorem 1. In addition, Corollary 1 can be inferred from Theorem 1 by setting $\bar{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab} = \bar{\bar{\mathbb{M}}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]}$ and $\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab} = (\bar{\mathbb{M}}_a\backslash\bar{\bar{\mathbb{M}}}_a) \times \mathbb{I}_{[1,\bar{\tau}_a]}$.

B. Stabilizing Control Synthesis

Based on Theorem 1, the criterion for the closed-loop stabilization of SMJLS (6) is presented below, in which the zero-filling method is proposed to tackle the unknown sojourn information $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}$.

Theorem 2: Closed-loop SMJLS (6) is stochastically stable, if there exist matrices $G_a(0,\varsigma) \succ \mathbf{0}$, $\mathcal{G}_a(\imath,\jmath) \succ \mathbf{0}$ and $\mathfrak{G}_a(\bar{\tau}_a) \succ \mathbf{0}$, $a \in \mathbb{M}$, $\imath \in \mathbb{I}_{[\jmath+1,\bar{\tau}_a]}$, $\varsigma, \jmath \in \mathbb{I}_{[0,\bar{\tau}_a]}$, matrices $Y_a(\varepsilon)$ and X, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, and admissible controller gains for (5) calculated as $K_a(\varepsilon) = Y_a(\varepsilon)X^{-1}$, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, such that $\forall a \in \mathbb{M}$, $\forall \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, there hold

$$\begin{bmatrix} -\mathfrak{G}_a(\jmath) & \star \\ \hat{\Theta}_a(\jmath+1) \otimes (A_a X + B_a Y_a(\jmath)) & \mathbf{F}_a(\jmath+1) \end{bmatrix} \prec \mathbf{0} \tag{13}$$

$$\mathfrak{G}_a(0) - G_a(0,0) \prec \mathbf{0} \tag{14}$$

and $\forall a \in \mathbb{M}$, $\forall \varsigma \in \mathbb{I}_{[0,\tau-1]}$, $\forall (b,\tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}) \cup (\tilde{\mathbb{M}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]})$, there hold

$$\begin{bmatrix} -G_a(0,\varsigma) & * \\ A_a X + B_a Y_a(\varsigma) & G_a(0,\varsigma+1) - X - X^{\top} \end{bmatrix} \prec \mathbf{0} \tag{15}$$

$$G_b(0,0) - G_a(0,\tau) \prec \mathbf{0} \tag{16}$$

where $\mathfrak{G}_a(\jmath) \triangleq \sum_{\imath=\jmath+1}^{\bar{\tau}_a} \mathcal{G}_a(\imath,\jmath)$, $\mathbf{F}_a(\jmath+1) \triangleq \mathbf{G}_a(\jmath+1) - I_{\bar{M}_a+1} \otimes X - I_{\bar{M}_a+1} \otimes X^{\top}$, $\hat{\Theta}_a(\jmath+1) \triangleq \Big[\sqrt{\hat{\theta}_{a\bar{m}_a(1)}(\jmath+1)/\bar{\eta}_a}\ \ \sqrt{\hat{\theta}_{a\bar{m}_a(2)}(\jmath+1)/\bar{\eta}_a}\ \cdots\ \sqrt{\hat{\theta}_{a\bar{m}_a(\bar{M}_a)}(\jmath+1)/\bar{\eta}_a}\ \ \mathbf{1}_a(\jmath+1)\Big]^{\top}$ ($\bar{\eta}_a > 0$) with

$$\mathbf{G}_a(\jmath+1) \triangleq \operatorname{diag}\{G_{\bar{m}_a(1)}(0,0), G_{\bar{m}_a(2)}(0,0), \ldots, G_{\bar{m}_a(\bar{M}_a)}(0,0), \mathfrak{G}_a(\jmath+1)\}$$

$$\hat{\theta}_{ab}(\jmath+1) \triangleq \begin{cases} \pi_{ab} f_{ab}(\jmath+1), & \jmath+1 \in \bar{\mathbb{S}}_{ab},\ b \in \bar{\mathbb{M}}_a \\ 0, & \jmath+1 \in \tilde{\mathbb{S}}_{ab},\ b \in \bar{\mathbb{M}}_a \end{cases}$$

$$\mathbf{1}_a(\jmath+1) \triangleq \begin{cases} 1, & \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-2]} \\ 0, & \jmath = \bar{\tau}_a - 1. \end{cases}$$

Proof: See Appendix B. ■

If we disregard all the fragmentarily known STPMFs $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab}$ ($\bar{\mathbb{S}}_{ab} \subsetneqq \mathbb{I}_{[1,\bar{\tau}_a]}$), then only π_{ab} and $f_{ab}(\tau)$, $a \in \mathbb{M}$, $b \in \bar{\bar{\mathbb{M}}}_a$, $\tau \in \mathbb{I}_{[1,\bar{\tau}_a]}$ remain available. If this is the case, Theorem 2 will reduce to the following stabilization criterion.

Corollary 2: Closed-loop SMJLS (6) is stochastically stable, if there exist matrices $G_a(0,\varsigma) \succ \mathbf{0}$, $\mathcal{G}_a(\imath,\jmath) \succ \mathbf{0}$ and $\mathfrak{G}_a(\bar{\tau}_a) \succ \mathbf{0}$, $a \in \mathbb{M}$, $\imath \in \mathbb{I}_{[\jmath+1,\bar{\tau}_a]}$, $\varsigma, \jmath \in \mathbb{I}_{[0,\bar{\tau}_a]}$, matrices $Y_a(\varepsilon)$ and X, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, and admissible controller gains for (5) calculated as $K_a(\varepsilon) = Y_a(\varepsilon)X^{-1}$, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, such that $\forall a \in \mathbb{M}$, $\forall \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$,

$$\begin{bmatrix} -\mathfrak{G}_a(\jmath) & \star \\ \Theta_a(\jmath+1) \otimes (A_a X + B_a Y_a(\jmath)) & \boldsymbol{F}_a(\jmath+1) \end{bmatrix} \prec \mathbf{0}$$

and (14) hold, and $\forall a \in \mathbb{M}$, $\forall b \in \mathbb{M}\backslash\bar{\bar{\mathbb{M}}}_a$, $\forall \varsigma \in \mathbb{I}_{[0,\tau-1]}$, $\forall \tau \in \mathbb{I}_{[1,\bar{\tau}_a]}$, (15) and (16) hold, where $\boldsymbol{F}_a(\jmath+1) \triangleq \boldsymbol{G}_a(\jmath+1) - I_{\bar{\bar{M}}_a+1} \otimes X - I_{\bar{\bar{M}}_a+1} \otimes X^{\top}$, $\Theta_a(\jmath+1) \triangleq \Big[\sqrt{\pi_{a\bar{\bar{m}}_a(1)} f_{a\bar{\bar{m}}_a(1)}(\jmath+1)/\bar{\bar{\eta}}_a}\ \ \sqrt{\pi_{a\bar{\bar{m}}_a(2)} f_{a\bar{\bar{m}}_a(2)}(\jmath+1)/\bar{\bar{\eta}}_a}\ \cdots\ \sqrt{\pi_{a\bar{\bar{m}}_a(\bar{\bar{M}}_a)} f_{a\bar{\bar{m}}_a(\bar{\bar{M}}_a)}(\jmath+1)/\bar{\bar{\eta}}_a}\ \ \mathbf{1}_a(\jmath+1)\Big]^{\top}$ ($\bar{\bar{\eta}}_a > 0$) with $\boldsymbol{G}_a(\jmath+1) \triangleq \operatorname{diag}\{G_{\bar{\bar{m}}_a(1)}(0,0), G_{\bar{\bar{m}}_a(2)}(0,0), \ldots, G_{\bar{\bar{m}}_a(\bar{\bar{M}}_a)}(0,0), \mathfrak{G}_a(\jmath+1)\}$ and $\{\bar{\bar{m}}_a(1), \bar{\bar{m}}_a(2), \ldots, \bar{\bar{m}}_a(\bar{\bar{M}}_a)\} = \bar{\bar{\mathbb{M}}}_a$.

Remark 5: To establish Theorem 2, we leverage the zero-filling method to tackle a part of unknown probabilistic information, i.e., $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}$. Owing to the zero-filling method, the relation between $\sum_{b\in\bar{\mathbb{M}}_a} \sum_{\tau\in\bar{\mathbb{S}}_{ab}} (\pi_{ab} f_{ab}(\tau)/\bar{\eta}_a) \big(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\big)^{\top} O_b(0,0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)$ and $\sum_{\tau=1}^{\bar{\tau}_a} \big(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\big)^{\top} \sum_{b\in\bar{\mathbb{M}}_a} \big(\hat{\theta}_{ab}(\tau)/\bar{\eta}_a\big) O_b(0,0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)$ can be revealed, which facilitates a smoother development of Theorem 2.

V. Complete Utilization of Known Probabilistic Information

In Section IV, π_{ab}, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}$ and $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in (\tilde{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab})$ are known yet not leveraged. Aiming for 100% utilization ratio of the known probabilistic information, this section is dedicated to developing a new method that can also utilize the previously disregarded information π_{ab}, $a \in \mathbb{M}$, $(b,\tau) \in \bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}$ and $f_{ab}(\tau)$, $a \in \mathbb{M}$, $(b,\tau) \in (\tilde{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab})$.

A. Key Idea

Let us denote the unknown TPs by $\pi_{ab}^{(?)}$ and the unknown part of STPMFs by $f_{ab}^{(?)}(\tau)$. $\pi_{ab}^{(?)}$ and $f_{ab}^{(?)}(\tau)$ are bounded as

$$0 \le \pi_{ab}^{(?)} \le 1 - \sum_{b \in \bar{\mathbb{M}}_a} \pi_{ab}, \ \forall a \in \mathbb{M}, \ \forall b \in \tilde{\mathbb{M}}_a$$

$$0 \le f_{ab}^{(?)}(\tau) \le 1 - \sum_{\tau \in \bar{\mathbb{S}}_{ab}} f_{ab}(\tau), \ \forall a, b \in \mathbb{M} \ (a \ne b), \ \forall \tau \in \tilde{\mathbb{S}}_{ab}$$

due to $\sum_{b\in\tilde{\mathbb{M}}_a} \pi_{ab}^{(?)} = 1 - \sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab}$ and $\sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(?)}(\tau) = \mathfrak{f}_{ab} - \sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) \to 1 - \sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau)$, where $\mathfrak{f}_{ab} \triangleq \sum_{\tau=1}^{\bar{\tau}_a} f_{ab}(\tau)$ is unknown but satisfies $\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) \le \mathfrak{f}_{ab} \to 1$, $\forall a, b \in \mathbb{M} \ (a \ne b)$.

Integrating all the unknown TPs, it forms the vectors below.

$$\boldsymbol{\pi}_a^{(?)} \triangleq \begin{bmatrix} \pi_{a\tilde{m}_a(1)}^{(?)} & \pi_{a\tilde{m}_a(2)}^{(?)} & \cdots & \pi_{a\tilde{m}_a(M-\bar{M}_a)}^{(?)} \end{bmatrix}, \ a \in \mathbb{M}.$$

Since $\boldsymbol{\pi}_a^{(?)}$ is unknown, and its entries $\pi_{a\tilde{m}_a(1)}^{(?)}, \pi_{a\tilde{m}_a(2)}^{(?)}, \dots, \pi_{a\tilde{m}_a(M-\bar{M}_a)}^{(?)}$ are all bi-boundary, $\boldsymbol{\pi}_a^{(?)}$ can be characterized by a convex combination of vertices as

$$\boldsymbol{\pi}_a^{(?)} = \sum_{p_a=1}^{M-\bar{M}_a} \alpha_a^{(p_a)} \boldsymbol{\pi}_a^{(p_a)} \tag{17}$$

where $\alpha_a^{(p_a)} \ge 0$, $\sum_{p_a=1}^{M-\bar{M}_a} \alpha_a^{(p_a)} = 1$, $\forall a \in \mathbb{M}$. The vertices $\boldsymbol{\pi}_a^{(p_a)}$, $p_a \in \mathbb{P}_a \triangleq \{1, 2, \dots, M - \bar{M}_a\}$ are formulated as

$$\boldsymbol{\pi}_a^{(p_a)} \triangleq \begin{bmatrix} \pi_{a\tilde{m}_a(1)}^{(p_a)} & \pi_{a\tilde{m}_a(2)}^{(p_a)} & \cdots & \pi_{a\tilde{m}_a(M-\bar{M}_a)}^{(p_a)} \end{bmatrix} \tag{18}$$

where

$$\pi_{a\tilde{m}_a(\tilde{\xi})}^{(p_a)} \triangleq \begin{cases} 1 - \sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab}, & p_a = \tilde{\xi} \\ 0, & \text{else} \end{cases}$$

for $1 \le \tilde{\xi} \le M - \bar{M}_a$. It can be seen that $\forall p_a = \tilde{\xi} \in \mathbb{P}_a$, the $\tilde{\xi}$th entry of $\boldsymbol{\pi}_a^{(p_a)}$ takes the value of $1 - \sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab}$, while the other entries are 0, that is,

$$\boldsymbol{\pi}_a^{(\tilde{\xi})} = \begin{bmatrix} 0 & \cdots & 0 & \underset{\uparrow \text{ the } \tilde{\xi}\text{th entry}}{1 - \sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab}} & 0 & \cdots & 0 \end{bmatrix}.$$

In what follows, we present an example to illustrate the above approximation approach.

Example 1: Consider an SMC with four modes, for which the transition probability matrix of the associated EMC is

$$\begin{bmatrix} 0 & \pi_{12}^{(?)} & \pi_{13}^{(?)} & 0.2 \\ 0.6 & 0 & 0.1 & 0.3 \\ \pi_{31}^{(?)} & \pi_{32}^{(?)} & 0 & \pi_{34}^{(?)} \\ 0.7 & \pi_{42}^{(?)} & \pi_{43}^{(?)} & 0 \end{bmatrix}$$

from which we can know $\boldsymbol{\pi}_1^{(?)} = \begin{bmatrix} \pi_{12}^{(?)} & \pi_{13}^{(?)} \end{bmatrix}$, $\boldsymbol{\pi}_3^{(?)} = \begin{bmatrix} \pi_{31}^{(?)} & \pi_{32}^{(?)} & \pi_{34}^{(?)} \end{bmatrix}$ and $\boldsymbol{\pi}_4^{(?)} = \begin{bmatrix} \pi_{42}^{(?)} & \pi_{43}^{(?)} \end{bmatrix}$. The vertices of $\boldsymbol{\pi}_1^{(?)}, \boldsymbol{\pi}_3^{(?)}$ and $\boldsymbol{\pi}_4^{(?)}$ are $\boldsymbol{\pi}_1^{(1)} = \begin{bmatrix} 0.8 & 0 \end{bmatrix}$, $\boldsymbol{\pi}_1^{(2)} = \begin{bmatrix} 0 & 0.8 \end{bmatrix}$, $\boldsymbol{\pi}_3^{(1)} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $\boldsymbol{\pi}_3^{(2)} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\boldsymbol{\pi}_3^{(3)} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $\boldsymbol{\pi}_4^{(1)} = \begin{bmatrix} 0.3 & 0 \end{bmatrix}$ and $\boldsymbol{\pi}_4^{(2)} = \begin{bmatrix} 0 & 0.3 \end{bmatrix}$.

For the fragmentarily known STPMFs, $\forall a, b \in \mathbb{M} \ (a \ne b)$, we define the following vector:

$$\boldsymbol{f}_{ab}^{(?)} \triangleq \begin{bmatrix} f_{ab}^{(?)}(\tilde{s}_{ab}(1)) & f_{ab}^{(?)}(\tilde{s}_{ab}(2)) & \cdots & f_{ab}^{(?)}(\tilde{s}_{ab}(\bar{\tau}_a - \bar{\tau}_{ab})) \end{bmatrix}$$

which can be expressed in a convex polyhedral form:

$$\boldsymbol{f}_{ab}^{(?)} = \sum_{q_{ab}=1}^{\bar{\tau}_a - \bar{\tau}_{ab}} \beta_{ab}^{(q_{ab})} \boldsymbol{f}_{ab}^{(q_{ab})} \tag{19}$$

where $\beta_{ab}^{(q_{ab})} \ge 0$, $\sum_{q_{ab}=1}^{\bar{\tau}_a - \bar{\tau}_{ab}} \beta_{ab}^{(q_{ab})} = 1$, $\forall a, b \in \mathbb{M} \ (a \ne b)$. Denote the index set of the vertices of $\boldsymbol{f}_{ab}^{(?)}$ by $\mathbb{Q}_{ab} \triangleq \{1, 2, \dots, \bar{\tau}_a - \bar{\tau}_{ab}\}$. The vertices of $\boldsymbol{f}_{ab}^{(?)}$ can be described by

$$\begin{aligned} \boldsymbol{f}_{ab}^{(q_{ab})} \triangleq \Big[& f_{ab}^{(q_{ab})}(\tilde{s}_{ab}(1)) \quad f_{ab}^{(q_{ab})}(\tilde{s}_{ab}(2)) \quad \cdots \\ & f_{ab}^{(q_{ab})}(\tilde{s}_{ab}(\bar{\tau}_a - \bar{\tau}_{ab})) \Big], \ q_{ab} \in \mathbb{Q}_{ab} \end{aligned} \tag{20}$$

where

$$f_{ab}^{(q_{ab})}(\tilde{s}_{ab}(\tilde{\delta})) \triangleq \begin{cases} 1 - \sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau), & q_{ab} = \tilde{\delta} \\ 0, & \text{else} \end{cases} \tag{21}$$

for $1 \le \tilde{\delta} \le \bar{\tau}_a - \bar{\tau}_{ab}$. According to (20) and (21), we know

$$\boldsymbol{f}_{ab}^{(\tilde{\delta})} = \begin{bmatrix} 0 & \cdots & 0 & \underset{\uparrow \text{ the } \tilde{\delta}\text{th entry}}{1 - \sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau)} & 0 & \cdots & 0 \end{bmatrix}$$

in which the $\tilde{\delta}$th entry is $1 - \sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau)$, and the other entries are 0. The following example gives a more concrete illustration of the approach used to approximate the unknown parts of the fragmentarily known STPMFs.

Example 2: Let us consider an STPMF $f_{ab}(\tau)$, $\forall a, b \in \mathbb{M}$ $(a \ne b)$, where $f_{ab}(1) = 0.2$, $f_{ab}(2) = 0.35$, $f_{ab}(4) = 0.05$, $f_{ab}(6) = 0.1$ and $\bar{\tau}_a = 7$. The values of $f_{ab}(3)$, $f_{ab}(5)$ and $f_{ab}(7)$ are unknown. Then, we have

$$\begin{aligned} \boldsymbol{f}_{ab}^{(?)} &= \begin{bmatrix} f_{ab}^{(?)}(\tilde{s}_{ab}(1)) & f_{ab}^{(?)}(\tilde{s}_{ab}(2)) & f_{ab}^{(?)}(\tilde{s}_{ab}(3)) \end{bmatrix} \\ &= \begin{bmatrix} f_{ab}^{(?)}(3) & f_{ab}^{(?)}(5) & f_{ab}^{(?)}(7) \end{bmatrix} \end{aligned}$$

and the vertices of $\boldsymbol{f}_{ab}^{(?)}$ are presented below.

$$\boldsymbol{f}_{ab}^{(1)} = \begin{bmatrix} 0.3 & 0 & 0 \end{bmatrix}, \ \boldsymbol{f}_{ab}^{(2)} = \begin{bmatrix} 0 & 0.3 & 0 \end{bmatrix}, \ \boldsymbol{f}_{ab}^{(3)} = \begin{bmatrix} 0 & 0 & 0.3 \end{bmatrix}.$$

Remark 6: The values of $\bar{\tau}_a$, $a \in \mathbb{M}$ should be chosen to be sufficiently large to ensure $\mathfrak{f}_{ab} \to 1$, $a, b \in \mathbb{M}$ $(a \ne b)$. To achieve this goal, an adequate number of samples is necessary to guarantee sufficiently accurate statistical distributions of sojourn time. By this means, appropriate upper bounds for $f_{ab}^{(?)}(\tau)$, $\forall \tau \in \tilde{\mathbb{S}}_{ab}$ can be achieved. With the approximation approach developed above, all the possible values of $\pi_{ab}^{(?)}$, $a \in \mathbb{M}$, $b \in \tilde{\mathbb{M}}_a$ and $f_{ab}^{(?)}(\tau)$, $a, b \in \mathbb{M}$ $(a \ne b)$, $\tau \in \tilde{\mathbb{S}}_{ab}$ can be confined within the corresponding bounds. In the case of continuous-time S-MJSs, the transition rates are time-varying [13], [19], which can be tackled by referring to [26].

B. Stochastic Stability Analysis

We shall analyze the closed-loop stability of SMJLS (6) based on the proposed modeling approach for the unknown TPs $\pi_{ab}^{(?)}$, $a \in \mathbb{M}$, $b \in \tilde{\mathbb{M}}_a$ and the unknown part of STPMFs $f_{ab}^{(?)}(\tau)$, $a \in \mathbb{M}$, $(b, \tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab})$ as convex polyhedral uncertainties. To achieve this, we develop a vertex-based Lyapunov function in a polyhedral form that has dependence on both the polyhedrons of the approximated transition information and sojourn information:

$$\mathcal{L}'_{r(k),k-k_n}\left(x(k)\right) \triangleq x^\top(k)\bar{\mathfrak{P}}_{r(k)}(k-k_n)x(k) \tag{22}$$

where $\bar{\mathfrak{P}}_{r(k)}(k-k_n) \triangleq \sum_{p_{r(k)}\in\mathbb{P}_{r(k)}}\sum_{\bar{\boldsymbol{q}}_{r(k)}\in\bar{\mathbb{Q}}'_{r(k)}} \alpha_{r(k)}^{(p_{r(k)})} \boldsymbol{\beta}_{r(k)}^{(\bar{\boldsymbol{q}}_{r(k)})} P_{r(k)}^{(p_{r(k)},\bar{\boldsymbol{q}}_{r(k)})}(k-k_n)$ with $\bar{\boldsymbol{q}}_{r(k)} \triangleq (q_{r(k)\bar{m}_{r(k)}(1)}, q_{r(k)\bar{m}_{r(k)}(2)},\ldots,q_{r(k)\bar{m}_{r(k)}(\bar{M}_{r(k)})})$, $\bar{\mathbb{Q}}'_{r(k)} \triangleq \mathbb{Q}_{r(k)\bar{m}_{r(k)}(1)}\times\mathbb{Q}_{r(k)\bar{m}_{r(k)}(2)}\times\cdots\times\mathbb{Q}_{r(k)\bar{m}_{r(k)}(\bar{M}_{r(k)})}$ and $\boldsymbol{\beta}_{r(k)}^{(\bar{\boldsymbol{q}}_{r(k)})} \triangleq \prod_{\xi=1}^{\bar{M}_{r(k)}} \beta_{r(k)\bar{m}_{r(k)}(\xi)}^{(q_{r(k)\bar{m}_{r(k)}(\xi)})}$. Assigning $r(k)=a\in\mathbb{M}$, the following relations are satisfied: $\alpha_a^{(p_a)}, \boldsymbol{\beta}_a^{(\bar{\boldsymbol{q}}_a)} \geq 0$, $\forall p_a\in\mathbb{P}_a$, $\forall \bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a$ and $\sum_{p_a\in\mathbb{P}_a}\sum_{\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\alpha_a^{(p_a)}\boldsymbol{\beta}_a^{(\bar{\boldsymbol{q}}_a)}=1$.

According to the Lyapunov function in (22), the relation $\lambda_5\left\|x(k)\right\|^2 \leq \mathcal{L}'_{a,\varepsilon}\left(x(k)\right) \leq \lambda_6\left\|x(k)\right\|^2$ holds, where

$$\lambda_5 \triangleq \min_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]},p_a\in\mathbb{P}_a,\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\left\{\lambda_{\min}\left(P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(\varepsilon)\right)\right\}$$
$$\lambda_6 \triangleq \max_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]},p_a\in\mathbb{P}_a,\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\left\{\lambda_{\max}\left(P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(\varepsilon)\right)\right\}.$$

By applying the proposed convex polyhedral approximation for the unknown transition and sojourn information, and by leveraging the Lyapunov function in (22), we present the stochastic stability criterion of SMJLS (6), which contains all the known information of TPs and STPMFs.

Theorem 3: SMJLS (6) is stochastically stable if there exist matrices $P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\succ\mathbf{0}$, $a\in\mathbb{M}$, $(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, such that $\forall a\in\mathbb{M}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(p_1,\bar{\boldsymbol{q}}_1)\in\mathbb{P}_1\times\bar{\mathbb{Q}}'_1$, $\forall(p_2,\bar{\boldsymbol{q}}_2)\in\mathbb{P}_2\times\bar{\mathbb{Q}}'_2,\ldots,\forall(p_M,\bar{\boldsymbol{q}}_M)\in\mathbb{P}_M\times\bar{\mathbb{Q}}'_M$,

$$\begin{aligned}&\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\pi_{ab}f_{ab}(\tau)}{\bar{\eta}'_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\&+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\pi_{ab}^{(p_a)}f_{ab}(\tau)}{\bar{\eta}'_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\&+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\frac{\pi_{ab}f_{ab}^{(q_{ab})}(\tau)}{\bar{\eta}'_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\&-P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\prec\mathbf{0}\ (\bar{\eta}'_a>0)\end{aligned} \tag{23}$$

and $\forall a\in\mathbb{M}$, $\forall(b,\tau)\in\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, $\forall(p_{\tilde{m}_a(1)},\bar{\boldsymbol{q}}_{\tilde{m}_a(1)})\in\mathbb{P}_{\tilde{m}_a(1)}\times\bar{\mathbb{Q}}'_{\tilde{m}_a(1)}$, $\forall(p_{\tilde{m}_a(2)},\bar{\boldsymbol{q}}_{\tilde{m}_a(2)})\in\mathbb{P}_{\tilde{m}_a(2)}\times\bar{\mathbb{Q}}'_{\tilde{m}_a(2)},\ldots$, $\forall(p_{\tilde{m}_a(M-\bar{M}_a)},\bar{\boldsymbol{q}}_{\tilde{m}_a(M-\bar{M}_a)})\in\mathbb{P}_{\tilde{m}_a(M-\bar{M}_a)}\times\bar{\mathbb{Q}}'_{\tilde{m}_a(M-\bar{M}_a)}$,

$$\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)-P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\prec\mathbf{0} \tag{24}$$

where $\bar{\eta}'_a \triangleq \min_{p_a\in\mathbb{P}_a,\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\{\bar{\eta}_a^{(p_a,\bar{\boldsymbol{q}}_a)}\}$ with $\bar{\eta}_a^{(p_a,\bar{\boldsymbol{q}}_a)} \triangleq \sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)$.

Proof: See Appendix C. ■

An alternative approach is to choose Lyapunov function (9) for the development of stochastic stability conditions for SMJLS (6). Accordingly, $P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)$ and $P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)$ in Theorem 3 are replaced by $P_a(0)$ and $P_b(0)$, respectively.

Remark 7: Theorem 3 involves larger numbers of matrix variables and inequalities, and is more computationally expensive in comparison to the results based on Lyapunov function (9). Meanwhile, Theorem 3 provides a greater degree of freedom in choosing matrix variables, thus it can be less conservative than the results based on Lyapunov function (9). In addition, owing to the polyhedral approximation to the unknown probabilistic information, the known transition information π_{ab}, $a\in\mathbb{M}$, $b\in\bar{\mathbb{M}}_a$ corresponding to $\tau\in\tilde{\mathbb{S}}_{ab}$ and sojourn information $f_{ab}(\tau)$, $a\in\mathbb{M}$, $(b,\tau)\in(\tilde{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab})$, which is not exploited in Theorem 1, is incorporated into the stability analysis in Theorem 3. In view of this, as compared to Theorem 1, the utilization ratio of the known probabilistic information for Theorem 3 has been improved.

C. Stabilizing Control Synthesis

To establish the closed-loop stabilization condition that utilizes all the known probabilistic information for SMJLS (6), we define, $\forall a\in\mathbb{M}$, $\forall(p_a,q_{ab})\in\mathbb{P}_a\times\mathbb{Q}_{ab}$,

$$\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau) \triangleq \begin{cases}\pi_{ab}f_{ab}(\tau), & (b,\tau)\in(\bar{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab})\\ \pi_{ab}^{(p_a)}f_{ab}(\tau), & (b,\tau)\in(\tilde{\mathbb{M}}_a\times\bar{\mathbb{S}}_{ab})\\ \pi_{ab}f_{ab}^{(q_{ab})}(\tau), & (b,\tau)\in(\bar{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab})\\ 0, & (b,\tau)\in(\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab})\end{cases} \tag{25}$$

for which the zero-filling technique is employed to tackle the simultaneously unknown π_{ab} and $f_{ab}(\tau)$, $a\in\mathbb{M}$, $b\in\tilde{\mathbb{M}}_a$, $\tau\in\tilde{\mathbb{S}}_{ab}$. Denoting the augmented vertex indices by $\boldsymbol{p}\triangleq(p_1,p_2,\ldots,p_M)\in\mathbb{P}'$, $\bar{\mathbf{q}}\triangleq(\bar{\boldsymbol{q}}_1,\bar{\boldsymbol{q}}_2,\ldots,\bar{\boldsymbol{q}}_M)\in\bar{\mathbb{Q}}''$ and $\boldsymbol{q}_a\triangleq(q_{a1},q_{a2},\ldots,q_{aM})\in\mathbb{Q}'_a$, where $\mathbb{P}'\triangleq\mathbb{P}_1\times\mathbb{P}_2\times\ldots\times\mathbb{P}_M$, $\bar{\mathbb{Q}}''\triangleq\bar{\mathbb{Q}}'_1\times\bar{\mathbb{Q}}'_2\times\ldots\times\bar{\mathbb{Q}}'_M$ and $\mathbb{Q}'_a\triangleq\mathbb{Q}_{a1}\times\mathbb{Q}_{a2}\times\ldots\times\mathbb{Q}_{aM}$, the following criterion for the stabilization of SMJLS (6) can be derived by extending Theorem 3.

Theorem 4: Closed-loop SMJLS (6) is stochastically stable, if there exist matrices $Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)\succ\mathbf{0}$, $\mathcal{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)\succ\mathbf{0}$ and $\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\bar{\tau}_a)\succ\mathbf{0}$, $a\in\mathbb{M}$, $\imath\in\mathbb{I}_{[\jmath+1,\bar{\tau}_a]}$, $\varsigma,\jmath\in\mathbb{I}_{[0,\bar{\tau}_a]}$, $(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$, matrices $Y_a(\varepsilon)$ and X, $a\in\mathbb{M}$, $\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, and admissible controller gains for (5) calculated to be $K_a(\varepsilon)=Y_a(\varepsilon)X^{-1}$, $a\in\mathbb{M}$, $\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, such that $\forall a\in\mathbb{M}$, $\forall\jmath\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$, there hold

$$\begin{bmatrix}-\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath) & \star\\ \hat{\Theta}_a^{(p_a,\boldsymbol{q}_a)}(\jmath+1)\otimes(A_aX+B_aY_a(\jmath)) & \mathbf{L}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\end{bmatrix}\prec\mathbf{0} \tag{26}$$

$$\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(0)-Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)\prec\mathbf{0} \tag{27}$$

and $\forall a\in\mathbb{M}$, $\forall\varsigma\in\mathbb{I}_{[0,\tau-1]}$, $\forall(b,\tau)\in\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(p_b,\bar{\boldsymbol{q}}_b)\in\mathbb{P}_b\times\bar{\mathbb{Q}}'_b$, there hold

$$\begin{bmatrix}-Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma) & *\\ A_aX+B_aY_a(\varsigma) & Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)-X-X^\top\end{bmatrix}\prec\mathbf{0} \tag{28}$$

$$Q_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)-Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\tau)\prec\mathbf{0} \tag{29}$$

where $\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)\triangleq\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\mathcal{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)$ and

$$\mathbf{L}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\triangleq\mathbf{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)-I_{M+1}\otimes\left(X+X^\top\right)$$

$$\mathbf{Q}_a^{(\boldsymbol{p},\bar{\boldsymbol{q}},\boldsymbol{q}_a)}(\jmath+1) \triangleq \mathrm{diag}\{Q_1^{(p_1,\bar{\boldsymbol{q}}_1)}(0,0), Q_2^{(p_2,\bar{\boldsymbol{q}}_2)}(0,0),\ldots, Q_M^{(p_M,\bar{\boldsymbol{q}}_M)}(0,0), \mathfrak{Q}_a^{(\boldsymbol{p},\bar{\boldsymbol{q}},\boldsymbol{q}_a)}(\jmath+1)\}$$

$$\hat{\Theta}_a^{(p_a,\boldsymbol{q}_a)}(\jmath+1) \triangleq \Bigg[\sqrt{\frac{\hat{\theta}_{a1}^{(p_a,q_{a1})}(\jmath+1)}{\bar{\eta}'_a}} \quad \sqrt{\frac{\hat{\theta}_{a2}^{(p_a,q_{a2})}(\jmath+1)}{\bar{\eta}'_a}} \quad \ldots \quad \sqrt{\frac{\hat{\theta}_{aM}^{(p_a,q_{aM})}(\jmath+1)}{\bar{\eta}'_a}} \quad \mathbf{1}_a(\jmath+1) \Bigg]^\top \ (\bar{\eta}'_a > 0).$$

Proof: See Appendix D. ■

VI. Joint Utilization of Known and Unknown Probabilistic Information

With the proposed polyhedral approximation method for the unknown transition and sojourn information, we address the stability analysis and control problems for SMJLSs by taking advantage of all the approximated probabilistic information.

A. Stochastic Stability Analysis

To further lower the conservativeness in the stability analysis of SMJLS (6) by leveraging both the known probabilistic information and the approximation of the unknown transition and sojourn information, we propose a vertex-based Lyapunov function that depends on both the uncertainties of the approximated transition and sojourn information :

$$\mathcal{L}''_{r(k),k-k_n}(x(k)) \triangleq x^\top(k)\mathfrak{P}_{r(k)}(k-k_n)x(k) \tag{30}$$

Here $\mathfrak{P}_{r(k)}(k-k_n) \triangleq \sum_{p_{r(k)}\in\mathbb{P}_{r(k)}}\sum_{\boldsymbol{q}_{r(k)}\in\mathbb{Q}'_{r(k)}} \alpha_{r(k)}^{(p_{r(k)})} \boldsymbol{\beta}_{r(k)}^{(\boldsymbol{q}_{r(k)})} P_{r(k)}^{(p_{r(k)},\boldsymbol{q}_{r(k)})}(k-k_n)$ is in a polyhedral form, where $\boldsymbol{\beta}_{r(k)}^{(\boldsymbol{q}_{r(k)})} \triangleq \prod_{b=1}^{M} \beta_{r(k)b}^{(q_{r(k)b})}$ with $\alpha_a^{(p_a)}, \boldsymbol{\beta}_a^{(\boldsymbol{q}_a)} \geq 0$, $\forall p_a \in \mathbb{P}_a$, $\forall \boldsymbol{q}_a \in \mathbb{Q}'_a$ and $\sum_{p_a\in\mathbb{P}_a}\sum_{\boldsymbol{q}_a\in\mathbb{Q}'_a} \alpha_a^{(p_a)}\boldsymbol{\beta}_a^{(\boldsymbol{q}_a)} = 1$, $\forall a \in \mathbb{M}$.

The Lyapunov function in (30) satisfies $\lambda_9 \|x(k)\|^2 \leq \mathcal{L}''_{a,\varepsilon}(x(k)) \leq \lambda_{10}\|x(k)\|^2$, where

$$\lambda_9 \triangleq \min_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]},p_a\in\mathbb{P}_a,\boldsymbol{q}_a\in\mathbb{Q}'_a} \left\{\lambda_{\min}\left(P_a^{(p_a,\boldsymbol{q}_a)}(\varepsilon)\right)\right\}$$

$$\lambda_{10} \triangleq \max_{a\in\mathbb{M},\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]},p_a\in\mathbb{P}_a,\boldsymbol{q}_a\in\mathbb{Q}'_a} \left\{\lambda_{\max}\left(P_a^{(p_a,\boldsymbol{q}_a)}(\varepsilon)\right)\right\}.$$

Then, we present stochastic stability condition for SMJLS (6).

Theorem 5: SMJLS (6) is stochastically stable if there exist matrices $P_a^{(p_a,\boldsymbol{q}_a)}(0) \succ \mathbf{0}$, $a \in \mathbb{M}$, $(p_a,\boldsymbol{q}_a) \in \mathbb{P}_a \times \mathbb{Q}'_a$, such that $\forall a \in \mathbb{M}$, $\forall (p_a,\boldsymbol{q}_a) \in \mathbb{P}_a \times \mathbb{Q}'_a$, $\forall (p_1,\boldsymbol{q}_1) \in \mathbb{P}_1 \times \mathbb{Q}'_1$, $\forall (p_2,\boldsymbol{q}_2) \in \mathbb{P}_2 \times \mathbb{Q}'_2, \ldots, \forall (p_M,\boldsymbol{q}_M) \in \mathbb{P}_M \times \mathbb{Q}'_M$,

$$\begin{aligned}&\sum_{b\in\bar{\mathbb{M}}_a}\frac{\pi_{ab}}{\eta'_a}\Bigg[\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau)\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\boldsymbol{q}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\ &+\sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau)\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\boldsymbol{q}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\Bigg]\\ &+\sum_{b\in\tilde{\mathbb{M}}_a}\frac{\pi_{ab}^{(p_a)}}{\eta'_a}\Bigg[\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau)\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\boldsymbol{q}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\ &+\sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau)\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\boldsymbol{q}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\Bigg]\\ &-P_a^{(p_a,\boldsymbol{q}_a)}(0) \prec \mathbf{0}\end{aligned} \tag{31}$$

where $\eta'_a \triangleq \min_{p_a\in\mathbb{P}_a,\boldsymbol{q}_a\in\mathbb{Q}'_a}\{\eta_a^{(p_a,\boldsymbol{q}_a)}\}$ with $\eta_a^{(p_a,\boldsymbol{q}_a)} \triangleq \sum_{b\in\bar{\mathbb{M}}_a}\pi_{ab}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau)\big) + \sum_{b\in\tilde{\mathbb{M}}_a}\pi_{ab}^{(p_a)}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau)\big)$.

Proof: See Appendix E. ■

If we choose Lyapunov function (9) to develop the stochastic stability conditions for SMJLS (6) instead of (30), $P_a^{(p_a,\boldsymbol{q}_a)}(0)$ and $P_b^{(p_b,\boldsymbol{q}_b)}(0)$ in Theorem 5 will become $P_a(0)$ and $P_b(0)$, respectively.

Remark 8: Theorem 5 and Theorem 3 are equivalent only if $\tilde{\mathbb{S}}_{ab} = \varnothing$, $\forall b \in \tilde{\mathbb{M}}_a$, $\forall a \in \mathbb{M}$, in which π_{ab} and $f_{ab}(\tau)$ are not simultaneously unknown. In addition, since $x(k_n)$ is arbitrarily given, according to the proof of Theorem 5, one has $\Gamma_a \triangleq \sum_{b\in\mathbb{M},b\neq a}\sum_{\tau=1}^{\bar{\tau}_a}\theta_{ab}(\tau)\mathcal{D}_{ab}(\tau)/\eta_a - \mathfrak{P}_a(0) \prec \mathbf{0}$. If the sojourn time of each mode obeys geometric distribution satisfying $f_{ab}(\tau) = \phi_a(1-\phi_a)^{\tau-1}$, $0 < \phi_a < 1$, $\forall a,b \in \mathbb{M}$, then the SMJLS under study will become an uncertain Markov jump linear system (MJLS). Accordingly, the TPs of $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$, denoted by $\varkappa_{ab}$, will be memoryless. Letting $\bar{\tau}_a \to \infty$ and $K_a(\varepsilon) = \mathbf{0}$, $\forall a \in \mathbb{M}$, $\forall \varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, and leveraging the same method as Remark 5 in [35], we can know that $\Gamma_a \prec \mathbf{0}$ is equivalent to $\Upsilon_a \prec \mathbf{0}$, where $\Upsilon_a \triangleq A_a^\top\sum_{b\in\mathbb{M},b\neq a}\varkappa_{ab}\mathfrak{P}_b(0)A_a + \varkappa_{aa}A_a^\top\mathfrak{P}_a(0)A_a - \mathfrak{P}_a(0)$. In general, the STPMFs $f_{ab}(\tau)$ should be entirely known in MJLSs, otherwise, we cannot confirm that the sojourn time obeys geometric distribution for all the modes. In view of this, $\mathbb{Q}_{ab} = \varnothing$. For the known TPs π_{ab} and $\varkappa_{aa}$, we define $\pi_{ab} \triangleq \pi_{ab}^{(p_a)}$ and $\varkappa_{aa} \triangleq \varkappa_{aa}^{(p_a)}$, $\forall a \in \mathbb{M}$, $\forall b \in \bar{\mathbb{M}}_a$, $\forall p_a \in \mathbb{P}_a$. According to $\varkappa_{ab} = \phi_a\pi_{ab}$ and $\varkappa_{aa} = 1-\phi_a$, and defining $\varkappa_{ab}^{(p_a)} \triangleq \phi_a\pi_{ab}^{(p_a)}$, $\forall a,b \in \mathbb{M}$ $(a \neq b)$, $\forall p_a \in \mathbb{P}_a$, one has $\varkappa_{ab} = \sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}\varkappa_{ab}^{(p_a)}$ and $\varkappa_{aa} = \sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}\varkappa_{aa}^{(p_a)}$, $\forall a,b \in \mathbb{M}$ $(a \neq b)$, $\forall p_a \in \mathbb{P}_a$. In addition, we have $\mathfrak{P}_a(0) \triangleq \sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}P_a^{(p_a)}(0)$, $\forall a \in \mathbb{M}$. Therefore, $\Upsilon_a \prec \mathbf{0}$ is equivalent to $A_a^\top\sum_{b\in\mathbb{M}}\big(\sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}\varkappa_{ab}^{(p_a)}\big)\sum_{p_b\in\mathbb{P}_b}\alpha_b^{(p_b)}P_b^{(p_b)}(0)A_a - \sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}P_a^{(p_a)}(0) \prec \mathbf{0}$, which is Proposition 1 in [27].

B. Stabilizing Control Synthesis

To establish the closed-loop stabilization condition that incorporates all the known probabilistic information and the approximation of the unknown information for SMJLS (6), we denote that $\forall a \in \mathbb{M}$, $\forall (p_a, q_{ab}) \in \mathbb{P}_a \times \mathbb{Q}_{ab}$,

$$\theta_{ab}^{(p_a,q_{ab})}(\tau) \triangleq \begin{cases}\pi_{ab}f_{ab}(\tau), & (b,\tau) \in (\bar{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab})\\ \pi_{ab}^{(p_a)}f_{ab}(\tau), & (b,\tau) \in (\tilde{\mathbb{M}}_a \times \bar{\mathbb{S}}_{ab})\\ \pi_{ab}f_{ab}^{(q_{ab})}(\tau), & (b,\tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab})\\ \pi_{ab}^{(p_a)}f_{ab}^{(q_{ab})}(\tau), & (b,\tau) \in (\tilde{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}).\end{cases} \tag{32}$$

Denoting the augmented vertex indices by $\mathbf{q}$, i.e., $\mathbf{q} \triangleq (\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_M) \in \mathbb{Q}''$ where $\mathbb{Q}'' \triangleq \mathbb{Q}'_1 \times \mathbb{Q}'_2 \times \ldots \times \mathbb{Q}'_M$, the following criterion for the stabilization of SMJLS (6) can be derived based on Theorem 5.

Theorem 6: Closed-loop SMJLS (6) is stochastically stable, if there exist matrices $U_a^{(p_a,\boldsymbol{q}_a)}(0,\varsigma) \succ \mathbf{0}$, $\mathcal{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\imath,\jmath) \succ \mathbf{0}$ and $\mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\bar{\tau}_a) \succ \mathbf{0}$, $a \in \mathbb{M}$, $\imath \in \mathbb{I}_{[\jmath+1,\bar{\tau}_a]}$, $\varsigma, \jmath \in \mathbb{I}_{[0,\bar{\tau}_a]}$,

$(p_a, \boldsymbol{q}_a) \in \mathbb{P}_a \times \mathbb{Q}'_a$, $(\boldsymbol{p}, \mathbf{q}, \boldsymbol{q}_a) \in \mathbb{P}' \times \mathbb{Q}'' \times \mathbb{Q}'_a$, matrices $Y_a(\varepsilon)$ and X, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, and admissible controller gains for (5) calculated to be $K_a(\varepsilon) = Y_a(\varepsilon)X^{-1}$, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, such that $\forall a \in \mathbb{M}$, $\forall \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, $\forall (p_a, \boldsymbol{q}_a) \in \mathbb{P}_a \times \mathbb{Q}'_a$, $\forall (\boldsymbol{p}, \mathbf{q}, \boldsymbol{q}_a) \in \mathbb{P}' \times \mathbb{Q}'' \times \mathbb{Q}'_a$, there hold

$$\begin{bmatrix} -\mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath) & \star \\ \Theta_a^{(p_a,\boldsymbol{q}_a)}(\jmath+1) \otimes (A_a X + B_a Y_a(\jmath)) & \mathbf{V}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath+1) \end{bmatrix} \prec \mathbf{0} \quad (33)$$

$$\mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(0) - U_a^{(p_a,\boldsymbol{q}_a)}(0,0) \prec \mathbf{0} \quad (34)$$

where $\mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath) \triangleq \sum_{\imath=\jmath+1}^{\bar{\tau}_a} \mathcal{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\imath,\jmath)$ and

$$\mathbf{V}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath+1) \triangleq \mathbf{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath+1) - I_{M+1} \otimes \left(X + X^\top\right)$$

$$\mathbf{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath+1) \triangleq \operatorname{diag}\{U_1^{(p_1,\boldsymbol{q}_1)}(0,0), U_2^{(p_2,\boldsymbol{q}_2)}(0,0), \ldots, U_M^{(p_M,\boldsymbol{q}_M)}(0,0), \mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath+1)\}$$

$$\Theta_a^{(p_a,\boldsymbol{q}_a)}(\jmath+1) \triangleq \left[\sqrt{\frac{\theta_{a1}^{(p_a,q_{a1})}(\jmath+1)}{\eta'_a}} \quad \sqrt{\frac{\theta_{a2}^{(p_a,q_{a2})}(\jmath+1)}{\eta'_a}} \quad \cdots \quad \sqrt{\frac{\theta_{aM}^{(p_a,q_{aM})}(\jmath+1)}{\eta'_a}} \quad \mathbf{1}_a(\jmath+1) \right]^\top.$$

Proof: See Appendix F. ■

Remark 9: Intuitively, we have more freedom in determining the controller gains depending on both the system mode and elapsed time, compared with the control scheme that only considers the system mode. This way, the elapsed-time-dependent controller in (5) can provide more flexibility in solving control synthesis problems and contribute to reducing the conservativeness of control design for SMJLSs. All our proposed criteria will remain applicable if we employ a controller independent of the elapsed time, i.e., $u(k) = K_{r(k)}x(k)$. Then, $K_a(\varepsilon) = K_a$, $Y_a(\varepsilon) = Y_a$, $\mathcal{A}_a(\varepsilon) = A_a + B_a K_a$, $\forall \varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, $a \in \mathbb{M}$. From this perspective, these criteria are more general than their counterparts based on the controller independent of the elapsed time.

Remark 10: The theorems in Sections IV through VI exhibit a gradual improvement in the utilization efficiency of probabilistic information. Since directly quantifying the utilized probabilistic information can be challenging, we resort to comparing the utilization ratios between two approaches, where one approach fully encompasses the probabilistic information utilized in the other. While all the utilization ratios of known probabilistic information in Theorems 3–6 reach 100%, Theorems 5 and 6 additionally include all the approximation of unknown probabilistic information; this way, their utilization ratios of unknown probabilistic information are improved in comparison to Theorems 3 and 4. Therefore, Theorems 5 and 6 are less conservative than Theorems 3 and 4, respectively. Meanwhile, Theorems 5 and 6 introduce additional approximated probabilistic information in the convex polyhedral form, which leads to potentially higher computational costs compared to Theorems 3 and 4, respectively.

VII. Simulation Studies

We provide a numerical example and a simulated continuous stirred tank reactor (CSTR) to testify our theoretical results.

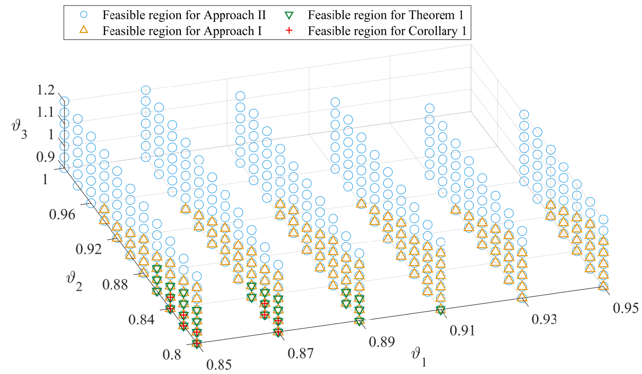


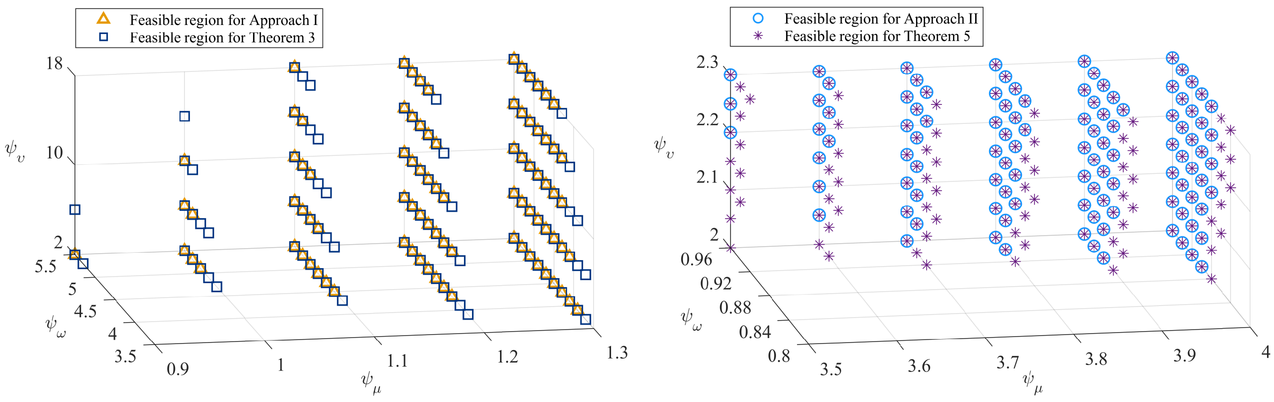
Fig. 1. Feasible regions of $(\vartheta_1, \vartheta_2, \vartheta_3)$ computed by the stability criteria with different levels of utilization efficiency of probabilistic information.

Example 3: Consider an SMJLS in the form of (6), where

$$\mathcal{A}_{r(k)}(k-k_n) = \vartheta_{r(k)} \begin{bmatrix} 1 & \frac{\hbar(k-k_n)}{\omega_{r(k)}} \\ -\frac{\hbar(k-k_n)}{\nu_{r(k)}} & 1 - \frac{\mu\hbar(k-k_n)}{\nu_{r(k)}} \end{bmatrix} \quad (35)$$

with $\hbar(k-k_n) = 0.1 + 0.01(k-k_n)$; $\vartheta_{r(k)}$, μ, $\omega_{r(k)}$ and $\nu_{r(k)}$ in (35) are positive system parameters, among which, $\vartheta_{r(k)}$ $\omega_{r(k)}$ and $\nu_{r(k)}$ are governed by a three-mode SMC $\{r(k)\}_{k\in\mathbb{I}_{\geq 0}}$. The known TPs are assigned to be $\pi_{12} = 0.7$, $\pi_{13} = 0.3$, $\pi_{21} = 0.4$, $\pi_{23} = 0.6$ and $\pi_{aa} = 0$, $a \in \mathbb{I}_{[1,3]}$, while π_{31} and π_{32} are unknown TPs; the upper bounds for the sojourn time are $(\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3) = (8, 6, 5)$; the entirely known STPMFs are $f_{13}(\tau) = 0.2^{\tau-1}0.8$, $f_{23}(\tau) = 0.7^{\tau-1}0.3^{6-\tau}5!/[(6-\tau)!(\tau-1)!]$, $f_{31}(\tau) = 0.6^{(\tau-1)^5} - 0.6^{\tau^5}$ and $f_{aa}(\tau) = 0$, $a \in \mathbb{I}_{[1,3]}$. Both $f_{12}(\tau)$ and $f_{21}(\tau)$ are fragmentarily known STPMFs. Specifically, we know $f_{12}(\tau) = 4^\tau \exp(-4)/\tau!$, $\tau \in \mathbb{I}_{[1,4]}$, $f_{12}(6) = 0.1$ and $f_{21}(\tau) = 0.5^5 \cdot 5!/[(6-\tau)!(\tau-1)!]$, $\tau \in \{1, 2\}$, $f_{21}(\tau) = 0.5^\tau$, $\tau \in \{3, 4\}$, however, we cannot know $f_{12}(\tau)$, $\tau \in \{5, 7, 8\}$ and $f_{21}(\tau)$, $\tau \in \{5, 6\}$. $f_{32}(\tau)$ is an entirely unknown STPMF. The values of the system parameters are assigned to be $\mu = 0.1\psi_\mu$, $(\omega_1, \omega_2, \omega_3) = (0.4\psi_\omega, 0.8\psi_\omega, \psi_\omega)$ and $(\nu_1, \nu_2, \nu_3) = (\psi_\nu, 0.1\psi_\nu, 0.2\psi_\nu)$.

We evaluate the feasibility and conservativeness for the stability criteria with different levels of utilization efficiencies of probabilistic information. To exclude the factor of the adopted Lyapunov functions, we name the stability criterion reduced from Theorem 3 by Lyapunov function (9) as Approach I, and the criterion reduced from Theorem 5 by Lyapunov function (9) as Approach II. By assign $\psi_\mu = \psi_\omega = \psi_\nu = 1$, Fig. 1 presents the feasible regions of $(\vartheta_1, \vartheta_2, \vartheta_3)$ computed by Corollary 1, Theorem 1, Approach I and Approach II, where $\vartheta_1 \in \{0.85, 0.87, \ldots, 0.95\}$, $\vartheta_2 \in \{0.8, 0.82, \ldots, 1\}$ and $\vartheta_3 \in \{0.9, 0.95, \ldots, 1.2\}$. We use the symbols $\mathbf{R}_{\text{Col1}}$, $\mathbf{R}_{\text{Thm1}}$, $\mathbf{R}_{\text{AprchI}}$ and $\mathbf{R}_{\text{AprchII}}$ to represent the feasible regions for Corollary 1, Theorem 1, Approach I and Approach II, respectively, and we can see from Fig. 1 that $\mathbf{R}_{\text{Col1}} \subsetneqq \mathbf{R}_{\text{Thm1}} \subsetneqq \mathbf{R}_{\text{AprchI}} \subsetneqq \mathbf{R}_{\text{AprchII}}$. This result illustrates that increasing the utilization efficiency of probabilistic information can help improve the feasibility and reduce the conservativeness for the stability conditions of SMJLS, which verifies the advantage of our proposed stability criteria.



(a) (b)

Fig. 2. Comparisons between the feasible regions of $(\psi_\mu, \psi_\omega, \psi_\nu)$ based on different Lyapunov functions. (a) Lyapunov functions (9) and (22). (b) Lyapunov functions (9) and (30).

Now, we set $\vartheta_1 = \vartheta_2 = \vartheta_3 = 1$. To demonstrate the superiority of our proposed vertex-based Lyapunov functions (22) and (30), we present the feasible regions of $(\psi_\mu, \psi_\omega, \psi_\nu)$ computed by Theorem 3 and Approach I in the region $\{0.9, 1, \ldots, 1.3\} \times \{3.5, 3.7, \ldots, 5.5\} \times \{2, 6, \ldots, 18\}$ in Fig. 2(a), and the feasible regions of $(\psi_\mu, \psi_\omega, \psi_\nu)$ by Theorem 5 and Approach II in the region $\{3.5, 3.6, \ldots, 4\} \times \{0.84, 0.86, \ldots, 0.96\} \times \{2, 2.05, \ldots, 2.3\}$ in Fig. 2(b). Here, we symbolize the feasible regions for Approach I and Approach II by $\mathbf{R}'_{\mathrm{AprchI}}$ and $\mathbf{R}'_{\mathrm{AprchII}}$, respectively, and the feasible regions for Theorem 3 and Theorem 5 by $\mathbf{R}_{\mathrm{Thm3}}$ and $\mathbf{R}_{\mathrm{Thm5}}$, respectively. The results $\mathbf{R}'_{\mathrm{AprchI}} \subsetneqq \mathbf{R}_{\mathrm{Thm3}}$ and $\mathbf{R}'_{\mathrm{AprchII}} \subsetneqq \mathbf{R}_{\mathrm{Thm5}}$ manifest the improved feasibility and reduced conservativeness owing to Lyapunov function (22) and Lyapunov function (30), respectively.

Example 4 (Control of a CSTR): Consider a continuous C-STR that operates under non-isothermal and well-mixed conditions with a constant liquid volume. The reactor contains an exothermic and irreversible reaction. The reactant is premixed with a solvent and converted into product at a rate that is proportional to the concentration of the reactant, with negligible shaft work. Under the regulation of a selector valve, the reactor is fed by a single inlet stream among three different source streams at one time. The reactant concentrations, the flow rates and the temperature are different for the three source streams. Accordingly, the reactor exhibits three operating modes, and a supervisory mechanism regulate the switching among these operating modes in a stochastic manner. We present the dynamic model of the CSTR for the ath mode below (ref. [45]), which is established based on mass and energy balances:

$$\begin{cases} \dot{c} = \frac{F_a}{V}(c_{\mathrm{f},a} - c) - \epsilon_{\mathrm{rr}} \exp\left(-\frac{E}{\epsilon_{\mathrm{g}} T}\right) c \\ \dot{T} = \frac{F_a}{V}(T_{\mathrm{f},a} - T) - \frac{\Delta H}{\rho C_{\mathrm{h}}} \epsilon_{\mathrm{rr}} \exp\left(-\frac{E}{\epsilon_{\mathrm{g}} T}\right) c \\ \quad + \frac{\epsilon_{\mathrm{ht}}}{V \rho C_{\mathrm{h}}}(T_{\mathrm{ctr}} - T), \ a \in \{1, 2, 3\} \end{cases} \tag{36}$$

where c and T denote the reactant concentration and the reactor temperature, respectively, both of which are the variables to be controlled; $c_{\mathrm{f},a}$, F_a and $T_{\mathrm{f},a}$ denote the concentrations of the reactants, the feed flow rates and the temperature in the feed stream provided by the ath stream source, respectively, and their values are $(c_{\mathrm{f},1}, c_{\mathrm{f},2}, c_{\mathrm{f},3}) = (1.5, 1, 0.75)$ mol/L, $(F_1, F_2, F_3) = (50, 100, 200)$ L/min, $(T_{\mathrm{f},1}, T_{\mathrm{f},2}, T_{\mathrm{f},3}) = (300, 350, 400)$ K; the other parameters and their values involved in (36) are given in Table I.

TABLE I
NOMENCLATURE OF THE CONSTANT PARAMETERS OF THE CSTR SYSTEM AND THEIR VALUES.

Notation	Physical meaning	Value
C_{h}	Heat capacity of the fluid	0.239 J/(g · K)
E	Activation energy	72.7475 kJ/mol
V	Volume of the reactor	100 L
ΔH	Enthalpy of the reaction	-5×10^4 J/mol
ϵ_{g}	Gas constant	8.314 J/(mol · K)
ϵ_{ht}	Heat transfer constant	50 kJ/(min · K)
ϵ_{rr}	Reaction rate constant	7.2×10^{10} min^{-1}
ρ	Density of the fluid	1000 g/L

By treating the state variables as $x_1 \triangleq c - c^{(\mathrm{e})}$, $x_2 \triangleq T - T^{(\mathrm{e})}$ and the control input variable as $u \triangleq T_{\mathrm{ctr}} - T_{\mathrm{ctr}}^{(\mathrm{e})}$, the CSTR in (36) can be transformed into

$$\begin{cases} \dot{x}_1 = \frac{F_a}{V}\left(c_{\mathrm{f},a} - c^{(\mathrm{e})} - x_1\right) \\ \qquad - \epsilon_{\mathrm{rr}}\left(x_1 + c^{(\mathrm{e})}\right) \exp\left(-\frac{E}{\epsilon_{\mathrm{g}}\left(x_2 + T^{(\mathrm{e})}\right)}\right) \\ \dot{x}_2 = \frac{F_a}{V}\left(T_{\mathrm{f},a} - T^{(\mathrm{e})} - x_2\right) \\ \qquad - \frac{\Delta H}{\rho C_{\mathrm{h}}} \epsilon_{\mathrm{rr}}\left(x_1 + c^{(\mathrm{e})}\right) \exp\left(-\frac{E}{\epsilon_{\mathrm{g}}\left(x_2 + T^{(\mathrm{e})}\right)}\right) \\ \qquad + \frac{\epsilon_{\mathrm{ht}}}{V\rho C_{\mathrm{h}}}\left(T_{\mathrm{ctr}}^{(\mathrm{e})} - T^{(\mathrm{e})} - x_2\right) + \frac{\epsilon_{\mathrm{ht}}}{V\rho C_{\mathrm{h}}} u \end{cases} \tag{37}$$

where $c^{(\mathrm{e})} = 0.5$ mol/L, $T^{(\mathrm{e})} = 350$ K and $T_{\mathrm{ctr}}^{(\mathrm{e})} = 300$ K constitute an unstable equilibrium. After performing Taylor series expansion on nonlinear CSTR model (37) around the point $(c^{(\mathrm{e})}, T^{(\mathrm{e})}, T_{\mathrm{ctr}}^{(\mathrm{e})})$, and discretizing the obtained linearized model via a sampling period of 0.04 min, a discrete-time CSTR as the form in (1) can be obtained. We employ an SMC $\{r(k)\}_{k \in \mathbb{I}_{\geq 0}}$, which has been successfully applied to the operation of CSTRs in [32], [33], to characterize the stochastic switching behavior in the discrete-time CSTR model. The upper bounds for the sojourn time and the knowledge of probabilistic information are the same as Example 3.

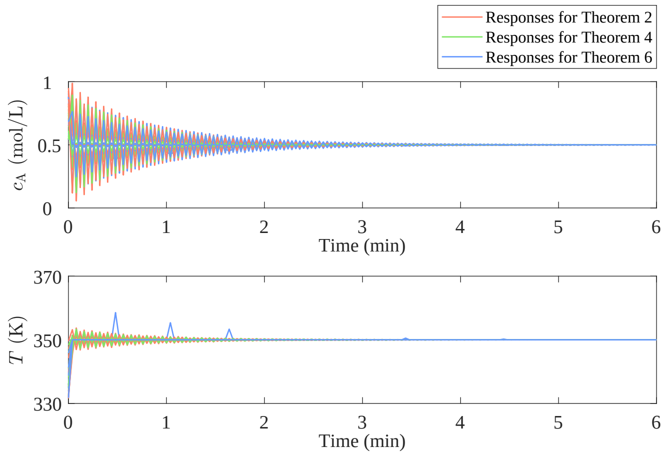


Fig. 3. State responses of closed-loop CSTR process based on Theorems 2, 4 and 6 subject to 10 randomly generated jumping sequences.

The linearized discrete-time CSTR model is unstable without closed-loop control. We aim to design controllers in the form of (5) based on the proposed control methods. To realize the simulation, the values of unknown TPs π_{31} and π_{32} are randomly assigned in $\mathbb{R}_{[0,1]}$, and the values of unknown sojourn information are randomly taken satisfying $f_{12}(5), f_{12}(7), f_{12}(8) \in \mathbb{R}_{[0,0.2895]}$, $f_{21}(5), f_{21}(6) \in \mathbb{R}_{[0,0.6250]}$ and $f_{32}(\tau) \in \mathbb{R}_{[0,1]}$, $\forall \tau \in \mathbb{I}_{[1,3]}$. We choose the initial reactant concentration $c(0)$ randomly between 0.5 $\mathrm{mol/L}$ and 1 $\mathrm{mol/L}$, the initial reactor temperature $T(0)$ randomly between 330 K and 350 K, and the initial operating mode $r(0)$ randomly in $\mathbb{I}_{[1,3]}$ for each realization. Fig. 3 depicts 10 realizations of the state responses of the CSTR regulated by the controllers designed via Theorems 2, 4 and 6, respectively. Since all the state responses in Fig. 3 converge to the equilibrium asymptotically, the effectiveness of our proposed control method is validated in tackling the scenario of incomplete knowledge of probabilistic information.

VIII. Conclusion

In this article, we presented the conditions on the stochastic stability and the existence of time-dependent stabilizing controllers for the SMJLSs for three different levels of improved utilization efficiency of incomplete probabilistic information. By proposing a polyhedral modeling method to approximate the unknown probabilistic information, the known probabilistic information coupled with the unknown information became available for utilization. The improvement in the utilization efficiency of probabilistic information can help reduce the conservativeness for the conditions of the stochastic stability of SMJLSs. We constructed two vertex-based Lyapunov functions depending on the approximated unknown information respectively for the full usage of all the known probabilistic information and the joint usage of both known and unknown probabilistic information, which can further lower the conservativeness. A numerical example and a simulated CSTR process are adopted as case studies to showcase the benefits of our established methodology of improving the information utilization efficiency and the effectiveness of our proposed control scheme.

Appendix

A. Proof of Theorem 1

Considering the Lyapunov function (9) and defining the normalization factor $\eta_a \triangleq \sum_{b\in\mathbb{M}}\sum_{\tau=1}^{\bar{\tau}_a}\pi_{ab}f_{ab}(\tau)$ that is unknown but tends to 1, together with $\theta_{aa}(\tau)=0$, $\forall a\in\mathbb{M}$, $\forall\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$, it can be deduced that

$$\begin{aligned}
&\mathsf{E}\{\mathcal{L}_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}_{a,0}(x(k_n))\\
&=x^\top(k_n)\Bigg[\sum_{b\in\mathbb{M},b\neq a}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\theta_{ab}(\tau)}{\eta_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b(0)\\
&\qquad\times\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\Bigg]x(k_n)-x^\top(k_n)P_a(0)x(k_n)\\
&=\frac{\bar{\eta}_a}{\eta_a}x^\top(k_n)\left(\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\pi_{ab}f_{ab}(\tau)}{\bar{\eta}_a}C_{ab}(\tau)\right)x(k_n)\\
&\quad+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\frac{\theta_{ab}(\tau)}{\eta_a}x^\top(k_n)C_{ab}(\tau)x(k_n)\\
&\quad+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\theta_{ab}(\tau)}{\eta_a}x^\top(k_n)C_{ab}(\tau)x(k_n)
\end{aligned}$$

where $C_{ab}(\tau)\triangleq\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)-P_a(0)$. Further, one can derive from (10) and (11) that

$$\begin{aligned}
&\mathsf{E}\{\mathcal{L}_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}_{a,0}(x(k_n))\\
&\leq\left[-\frac{1}{\eta_a}\lambda_3-\left(\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\theta_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau=1}^{\bar{\tau}_a}\theta_{ab}(\tau)\right)\frac{\lambda_4}{\eta_a}\right]\\
&\quad\times\|x(k_n)\|^2\\
&\leq-\lambda_3\|x(k_n)\|^2
\end{aligned}$$

where $\lambda_3\triangleq\min_{a\in\mathbb{M}}\{\lambda_{\min}(\bar{\eta}_aP_a(0)-\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon))^\top P_b(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon))\}$ and $\lambda_4\triangleq\min\{\lambda_{4,1},\lambda_{4,2}\}$ in which $\lambda_{4,1}\triangleq\min_{a\in\mathbb{M},(b,\tau)\in\bar{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}}\{\lambda_{\min}(-C_{ab}(\tau))\}$ and $\lambda_{4,2}\triangleq\min_{a\in\mathbb{M},b\in\tilde{\mathbb{M}}_a,\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}}\{\lambda_{\min}(-C_{ab}(\tau))\}$. Thus, (8) can be ensured by assigning the class $\mathcal{K}_\infty$ function $\mathcal{K}_3(\cdot)=\lambda_3(\cdot)^2$, which implies the stochastic stability of SMJLS (6) according to Lemma 1.

B. Proof of Theorem 2

Since $G_a(0,\varsigma)\succ\mathbf{0}$ and $\mathfrak{G}_a(\jmath)\succ\mathbf{0}$ hold for all $a\in\mathbb{M}$, $\varsigma,\jmath\in\mathbb{I}_{[0,\bar{\tau}_a]}$, the following inequalities hold: $(G_b(0,0)-X)G_b^{-1}(0,0)(G_b(0,0)-X)^\top\succ\mathbf{0}$, $(\mathfrak{G}_a(\jmath+1)-X)\mathfrak{G}_a^{-1}(\jmath+1)(\mathfrak{G}_a(\jmath+1)-X)^\top\succ\mathbf{0}$ and $(G_a(0,\varsigma+1)-X)G_a^{-1}(0,\varsigma+1)(G_a(0,\varsigma+1)-X)^\top\succ\mathbf{0}$, $\forall a\in\mathbb{M}$, $\forall b\in\bar{\mathbb{M}}_a$, $\forall\jmath,\varsigma\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$. Then, we can obtain the inequalities: $G_b(0,0)-X-X^\top\succ-XG_b^{-1}(0,0)X^\top$, $\mathfrak{G}_a(\jmath+1)-X-X^\top\succ-X\mathfrak{G}_a^{-1}(\jmath+1)X^\top$ and $G_a(0,\varsigma+1)-X-X^\top\succ-XG_a^{-1}(0,\varsigma+1)X^\top$. Meanwhile, by defining $O_a(0,\varsigma)\triangleq X^{-\top}G_a(0,\varsigma)X^{-1}$ and

$\mathfrak{O}_a(\jmath) \triangleq X^{-\top}\mathfrak{G}_a(\jmath)X^{-1}$, $a \in \mathbb{M}$, $\varsigma, \jmath \in \mathbb{I}_{[0,\bar{\tau}_a]}$, it yields the following inequalities:

$$\begin{aligned} -O_b^{-1}(0,0) &\prec G_b(0,0) - X - X^\top \\ -\mathfrak{O}_a^{-1}(\jmath+1) &\prec \mathfrak{G}_a(\jmath+1) - X - X^\top \\ -O_a^{-1}(0,\varsigma+1) &\prec G_a(0,\varsigma+1) - X - X^\top. \end{aligned} \tag{38}$$

From (13)–(16), and (38), by replacing $Y_a(\varepsilon)$ with $K_a(\varepsilon)X$, $a \in \mathbb{M}$, $\varepsilon \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$, one has $\forall a \in \mathbb{M}$, $\forall \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$,

$$\begin{bmatrix} -X^\top\mathfrak{O}_a(\jmath)X & \star \\ \hat{\Theta}_a(\jmath+1)\otimes(\mathcal{A}_a(\jmath)X) & -\mathbf{O}_a^{-1}(\jmath+1) \end{bmatrix} \prec \mathbf{0} \tag{39}$$

$$X^\top\mathfrak{O}_a(0)X - X^\top O_a(0,0)X \prec \mathbf{0} \tag{40}$$

and $\forall a \in \mathbb{M}$, $\forall \varsigma \in \mathbb{I}_{[0,\tau-1]}$, $\forall (b,\tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}) \cup (\tilde{\mathbb{M}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]})$,

$$\begin{bmatrix} -X^\top O_a(0,\varsigma)X & * \\ \mathcal{A}_a(\varsigma)X & -O_a^{-1}(0,\varsigma+1) \end{bmatrix} \prec \mathbf{0} \tag{41}$$

$$X^\top O_b(0,0)X - X^\top O_a(0,\tau)X \prec \mathbf{0} \tag{42}$$

where $\mathbf{O}_a(\jmath+1) \triangleq \mathrm{diag}\{O_{\bar{m}_a(1)}(0,0), O_{\bar{m}_a(2)}(0,0), \ldots, O_{\bar{m}_a(\bar{M}_a)}(0,0), \mathfrak{O}_a(\jmath+1)\}$.

Now, we define $\mathcal{O}_a(\jmath+1,\jmath+1) \triangleq \sum_{b\in\bar{\mathbb{M}}_a}\hat{\theta}_{ab}(\jmath+1)O_b(0,0)/\bar{\eta}_a$, and then perform congruence transformations to (39) by $\mathrm{diag}\left\{X^{-1}, I_{(\bar{M}_a+1)n_x}\right\}$ and to (40) by X^{-1}. After that, by applying Schur complement, it yields that $\forall a \in \mathbb{M}$, $\forall \jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$,

$$\begin{aligned} &\mathcal{A}_a^\top(\jmath)\mathcal{O}_a(\jmath+1,\jmath+1)\mathcal{A}_a(\jmath) \\ &\quad + \mathbf{1}_a(\jmath+1)\mathcal{A}_a^\top(\jmath)\mathfrak{O}_a(\jmath+1)\mathcal{A}_a(\jmath) - \mathfrak{O}_a(\jmath) \prec \mathbf{0} \end{aligned} \tag{43}$$

$$\mathfrak{O}_a(0) - O_a(0,0) \prec \mathbf{0} \tag{44}$$

where $\mathfrak{O}_a(\jmath) \triangleq \sum_{\imath=\jmath+1}^{\bar{\tau}_a}\mathcal{O}_a(\imath,\jmath)$, $\jmath \in \mathbb{I}_{[0,\bar{\tau}_a-1]}$. Similarly, it can be derived from (41) and (42) that $\forall a \in \mathbb{M}$, $\forall \varsigma \in \mathbb{I}_{[0,\tau-1]}$, $\forall (b,\tau) \in (\bar{\mathbb{M}}_a \times \tilde{\mathbb{S}}_{ab}) \cup (\tilde{\mathbb{M}}_a \times \mathbb{I}_{[1,\bar{\tau}_a]})$,

$$\mathcal{A}_a^\top(\varsigma)O_a(0,\varsigma+1)\mathcal{A}_a(\varsigma) - O_a(0,\varsigma) \prec \mathbf{0} \tag{45}$$

$$O_b(0,0) - O_a(0,\tau) \prec \mathbf{0}. \tag{46}$$

Due to $\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\left(\mathcal{A}_a^\top(\jmath)\mathcal{O}_a(\imath,\jmath+1)\mathcal{A}_a(\jmath) - \mathcal{O}_a(\imath,\jmath)\right) = \mathcal{A}_a^\top(\jmath)\mathcal{O}_a(\jmath+1,\jmath+1)\mathcal{A}_a(\jmath) + \mathbf{1}_a(\jmath+1)\mathcal{A}_a^\top(\jmath)\mathfrak{O}_a(\jmath+1)\mathcal{A}_a(\jmath) - \mathfrak{O}_a(\jmath)$, one can derive from (43) that

$$\begin{aligned} &\sum_{\tau=1}^{\bar{\tau}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{O}_a(\tau,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) - \sum_{\tau=1}^{\bar{\tau}_a}\mathcal{O}_a(\tau,0) \\ &= \sum_{\jmath=0}^{\bar{\tau}_a-1}\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\Bigg[\left(\prod_{\varepsilon=0}^{\jmath}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{O}_a(\imath,\jmath+1)\prod_{\varepsilon=0}^{\jmath}\mathcal{A}_a(\varepsilon) \\ &\qquad - \left(\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{O}_a(\imath,\jmath)\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\Bigg] \\ &= \sum_{\jmath=0}^{\bar{\tau}_a-1}\left(\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\right)^\top \\ &\quad \times\left[\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\left(\mathcal{A}_a^\top(\jmath)\mathcal{O}_a(\imath,\jmath+1)\mathcal{A}_a(\jmath) - \mathcal{O}_a(\imath,\jmath)\right)\right]\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon) \\ &\prec \mathbf{0}. \end{aligned}$$

which implies

$$\begin{aligned} &\sum_{\tau=1}^{\bar{\tau}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\sum_{b\in\bar{\mathbb{M}}_a}\frac{\hat{\theta}_{ab}(\tau)}{\bar{\eta}_a}O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) - \mathfrak{O}_a(0) \\ &\prec \mathbf{0}. \end{aligned} \tag{47}$$

Since the following equalities

$$\begin{aligned} &\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\pi_{ab}f_{ab}(\tau)}{\bar{\eta}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \\ &= \sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\hat{\theta}_{ab}(\tau)}{\bar{\eta}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \\ &\quad + \sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\frac{\hat{\theta}_{ab}(\tau)}{\bar{\eta}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \\ &= \sum_{\tau=1}^{\bar{\tau}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\sum_{b\in\bar{\mathbb{M}}_a}\frac{\hat{\theta}_{ab}(\tau)}{\bar{\eta}_a}O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \end{aligned}$$

hold, it can be obtained from (44) and (47) that

$$\begin{aligned} &\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\pi_{ab}f_{ab}(\tau)}{\bar{\eta}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \\ &- O_a(0,0) \prec \mathbf{0}. \end{aligned} \tag{48}$$

In addition, according to (45), it can be derived that

$$\begin{aligned} &\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_a(0,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) - O_a(0,0) \\ &= \sum_{\varsigma=0}^{\tau-1}\left(\prod_{\varepsilon=0}^{\varsigma-1}\mathcal{A}_a(\varepsilon)\right)^\top \\ &\quad \times\left(\mathcal{A}_a^\top(\varsigma)O_a(0,\varsigma+1)\mathcal{A}_a(\varsigma) - O_a(0,\varsigma)\right)\prod_{\varepsilon=0}^{\varsigma-1}\mathcal{A}_a(\varepsilon) \\ &\prec \mathbf{0}. \end{aligned} \tag{49}$$

Moreover, by pre- and post-multiplying (46) with $\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top$ and $\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)$, respectively, the following inequality holds:

$$\begin{aligned} &\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \\ &- \left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_a(0,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) \prec \mathbf{0}. \end{aligned} \tag{50}$$

Taking the sum of (49) and (50) yields

$$\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top O_b(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon) - O_a(0,0) \prec \mathbf{0}. \tag{51}$$

Assigning $O_a(0,0) \triangleq P_a(0)$, $O_b(0,0) \triangleq P_b(0)$, $a, b \in \mathbb{M}$, (48) and (51) are transformed into (10) and (11), respectively. The remainder of the proof follows the proof of Theorem 1.

C. Proof of Theorem 3

According to (22), we have

$$\begin{aligned}&\mathsf{E}\{\mathcal{L}'_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}'_{a,0}(x(k_n))\\&=x^\top(k_n)\Bigg[\sum_{b\in\mathbb{M},b\neq a}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\theta_{ab}(\tau)}{\eta_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\bar{\mathfrak{P}}_b(0)\\&\qquad\times\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\Bigg]x(k_n)-x^\top(k_n)\bar{\mathfrak{P}}_a(0)x(k_n).\end{aligned}$$

For brevity, we define $\bar{\mathcal{D}}_{ab}(\tau)\triangleq\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\bar{\mathfrak{P}}_b(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)$, $a,b\in\mathbb{M}$, $\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$. Since $\sum_{b\in\mathbb{M}}\sum_{\tau=1}^{\bar{\tau}_a}\theta_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)=\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(?)}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\bar{\mathcal{D}}_{ab}(\tau)$ with $\pi_{ab}^{(?)}=\sum_{p_a\in\mathbb{P}_a}\alpha_a^{(p_a)}\pi_{ab}^{(p_a)}$, $f_{ab}^{(?)}(\tau)=\sum_{q_{ab}\in\mathbb{Q}_{ab}}\beta_{ab}^{(q_{ab})}f_{ab}^{(q_{ab})}(\tau)$, together with $\bar{\mathfrak{P}}_a(0)=\sum_{p_a\in\mathbb{P}_a}\sum_{\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\alpha_a^{(p_a)}\boldsymbol{\beta}_a^{(\bar{\boldsymbol{q}}_a)}P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)$, we can deduce

$$\begin{aligned}&\mathsf{E}\{\mathcal{L}'_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}'_{a,0}(x(k_n))\\&=x^\top(k_n)\left(\sum_{b\in\mathbb{M},b\neq a}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\theta_{ab}(\tau)}{\eta_a}\bar{\mathcal{D}}_{ab}(\tau)-\bar{\mathfrak{P}}_a(0)\right)x(k_n)\\&=\frac{x^\top(k_n)}{\eta_a}\sum_{p_a\in\mathbb{P}_a}\sum_{\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\alpha_a^{(p_a)}\boldsymbol{\beta}_a^{(\bar{\boldsymbol{q}}_a)}\Bigg[\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)\\&\quad+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)\\&\quad\times\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\bar{\mathcal{D}}_{ab}(\tau)-\eta_aP_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\Bigg]\\&\quad\times x(k_n).\end{aligned}\tag{52}$$

In terms of the normalization factor, it yields that $\eta_a=\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(?)}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)=\sum_{p_a\in\mathbb{P}_a}\sum_{\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a}\alpha_a^{(p_a)}\boldsymbol{\beta}_a^{(\bar{\boldsymbol{q}}_a)}[\bar{\eta}_a^{(p_a,\bar{\boldsymbol{q}}_a)}+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)]\geq\bar{\eta}'_a+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)$. Then, one has the following inequality:

$$\begin{aligned}&\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)\bar{\mathcal{D}}_{ab}(\tau)\\&+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\bar{\mathcal{D}}_{ab}(\tau)\\&-\eta_aP_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\\&\preceq\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)\bar{\mathcal{D}}_{ab}(\tau)\\&\quad+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)\bar{\mathcal{D}}_{ab}(\tau)-\bar{\eta}'_aP_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\\&\quad+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\left(\bar{\mathcal{D}}_{ab}(\tau)-P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\right).\end{aligned}\tag{53}$$

Moreover, as $\bar{\mathcal{D}}_{ab}(\tau)$ can also be expressed as $\sum_{p_b\in\mathbb{P}_b}\sum_{\bar{\boldsymbol{q}}_b\in\bar{\mathbb{Q}}'_b}\alpha_b^{(p_b)}\boldsymbol{\beta}_b^{(\bar{\boldsymbol{q}}_b)}D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)$, where $D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)\triangleq\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)$, combining (52) and (53), we obtain

$$\begin{aligned}&\mathsf{E}\{\mathcal{L}'_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}'_{a,0}(x(k_n))\\&=\frac{x^\top(k_n)}{\eta_a}\sum_{p_1\in\mathbb{P}_1}\sum_{\bar{\boldsymbol{q}}_1\in\bar{\mathbb{Q}}'_1}\cdots\sum_{p_M\in\mathbb{P}_M}\sum_{\bar{\boldsymbol{q}}_M\in\bar{\mathbb{Q}}'_M}\left(\prod_{b=1}^M\alpha_b^{(p_b)}\boldsymbol{\beta}_b^{(\bar{\boldsymbol{q}}_b)}\right)\\&\quad\times\Bigg[\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)\\&\qquad+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)\\&\qquad+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)-\bar{\eta}'_aP_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\\&\qquad+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\left(D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)-P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\right)\Bigg]x(k_n).\end{aligned}$$

From (23) and (24), it can be ensured that

$$\begin{aligned}&\mathsf{E}\{\mathcal{L}'_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}'_{a,0}(x(k_n))\\&\leq\left[-\frac{1}{\eta_a}\lambda_7-\left(\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}^{(?)}f_{ab}^{(?)}(\tau)\right)\frac{\lambda_8}{\eta_a}\right]\|x(k_n)\|^2\\&\leq-\lambda_7\|x(k_n)\|^2\end{aligned}$$

where $\lambda_7\triangleq\min_{a\in\mathbb{M},p_1\in\mathbb{P}_1,\ldots,p_M\in\mathbb{P}_M,\bar{\boldsymbol{q}}_1\in\bar{\mathbb{Q}}'_1,\ldots,\bar{\boldsymbol{q}}_M\in\bar{\mathbb{Q}}'_M}\{\lambda_{\min}(\bar{\eta}'_aP_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)-\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)-\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\pi_{ab}f_{ab}^{(q_{ab})}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)-\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\pi_{ab}^{(p_a)}f_{ab}(\tau)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau))\}$ and

$$\lambda_8\triangleq\min_{a\in\mathbb{M},b\in\tilde{\mathbb{M}}_a,\tau\in\tilde{\mathbb{S}}_{ab},p_a\in\mathbb{P}_a,p_b\in\mathbb{P}_b,\bar{\boldsymbol{q}}_a\in\bar{\mathbb{Q}}'_a,\bar{\boldsymbol{q}}_b\in\bar{\mathbb{Q}}'_b}\left\{\lambda_{\min}\left(P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)-D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)\right)\right\}.$$

The inequality in (8) can be guaranteed and the stochastic stability has thus been proved.

D. Proof of Theorem 4

Due to $Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)\succ\mathbf{0}$ and $\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)\succ\mathbf{0}$, $a\in\mathbb{M}$, $\varsigma,\jmath\in\mathbb{I}_{[0,\bar{\tau}_a]}$, $(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$, by defining $J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)\triangleq X^{-\top}Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)X^{-1}$ and $\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)\triangleq X^{-\top}\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)X^{-1}$, it can be deduced that $\forall a,b\in\mathbb{M}$, $\forall\jmath,\varsigma\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$,

$$\begin{aligned}-\left(J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)\right)^{-1}&\prec Q_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)-X-X^\top\\-\left(J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)\right)^{-1}&\prec Q_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)-X-X^\top\\-\left(\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\right)^{-1}&\prec\mathfrak{Q}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)-X-X^\top.\end{aligned}\tag{54}$$

From (26)–(29), (54) and replacing $Y_a(\varepsilon)$ with $K_a(\varepsilon)X$, $a\in\mathbb{M}$, $\varepsilon\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, the following holds: $\forall a\in\mathbb{M}$, $\forall\jmath\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$,

$$\begin{bmatrix}-X^\top\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)X&\star\\\hat{\Theta}_a^{(p_a,\boldsymbol{q}_a)}(\jmath+1)\otimes(\mathcal{A}_a(\jmath)X)&-\left(\mathbf{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\right)^{-1}\end{bmatrix}\prec\mathbf{0}\tag{55}$$

$$X^\top\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(0)X-X^\top J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)X\prec\mathbf{0}\tag{56}$$

and $\forall a\in\mathbb{M}$, $\forall\varsigma\in\mathbb{I}_{[0,\tau-1]}$, $\forall(b,\tau)\in\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(p_b,\bar{\boldsymbol{q}}_b)\in\mathbb{P}_b\times\bar{\mathbb{Q}}'_b$,

$$\begin{bmatrix}-X^\top J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)X & *\\ \mathcal{A}_a(\varsigma)X & -\big(J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)\big)^{-1}\end{bmatrix}\prec\mathbf{0} \quad (57)$$

$$X^\top J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)X-X^\top J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\tau)X\prec\mathbf{0} \quad (58)$$

where $\mathbf{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\triangleq\mathrm{diag}\{J_1^{(p_1,\bar{\boldsymbol{q}}_1)}(0,0), J_2^{(p_2,\bar{\boldsymbol{q}}_2)}(0,0),\ldots, J_M^{(p_M,\bar{\boldsymbol{q}}_M)}(0,0), \mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\}$.

Perform congruence transformations to (55) by $\mathrm{diag}\{X^{-1}, I_{(M+1)n_x}\}$ and to (56) by X^{-1}. After that, we denote $\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1,\jmath+1)\triangleq\sum_{b\in\mathbb{M}}\hat{\theta}_{ab}^{(p_a,q_{ab})}(\jmath+1)$ $J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)/\bar{\eta}'_a$ and apply Schur complement, and we have that $\forall a\in\mathbb{M}$, $\forall\jmath\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)\in\mathbb{P}'\times\bar{\mathbb{Q}}''\times\mathbb{Q}'_a$,

$$\begin{aligned}&\mathcal{A}_a^\top(\jmath)\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1,\jmath+1)\mathcal{A}_a(\jmath)+\mathbf{1}_a(\jmath+1)\\&\times\mathcal{A}_a^\top(\jmath)\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\mathcal{A}_a(\jmath)-\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)\prec\mathbf{0} \quad (59)\end{aligned}$$

$$\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(0)-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)\prec\mathbf{0} \quad (60)$$

where $\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)\triangleq\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)$, $\jmath\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$. Similarly, from (57) and (58), we can infer that $\forall a\in\mathbb{M}$, $\forall\varsigma\in\mathbb{I}_{[0,\tau-1]}$, $\forall(b,\tau)\in\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, $\forall(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $\forall(p_b,\bar{\boldsymbol{q}}_b)\in\mathbb{P}_b\times\bar{\mathbb{Q}}'_b$,

$$\mathcal{A}_a^\top(\varsigma)J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)\mathcal{A}_a(\varsigma)-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)\prec\mathbf{0} \quad (61)$$

$$J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\tau)\prec\mathbf{0}. \quad (62)$$

Since the equality $\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\big(\mathcal{A}_a^\top(\jmath)\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath+1)\mathcal{A}_a(\jmath)-\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)\big)=\mathcal{A}_a^\top(\jmath)\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1,\jmath+1)\mathcal{A}_a(\jmath)+\mathbf{1}_a(\jmath+1)\mathcal{A}_a^\top(\jmath)\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath+1)\mathcal{A}_a(\jmath)-\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\jmath)$ holds, by considering $\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(0)\triangleq\sum_{\tau=1}^{\bar{\tau}_a}\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\tau,0)$, one can derive from (59) that

$$\begin{aligned}&\sum_{\tau=1}^{\bar{\tau}_a}\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\tau,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)-\mathfrak{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(0)\\&=\sum_{\jmath=0}^{\bar{\tau}_a-1}\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\Bigg[\left(\prod_{\varepsilon=0}^{\jmath}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath+1)\prod_{\varepsilon=0}^{\jmath}\mathcal{A}_a(\varepsilon)\\&\qquad-\left(\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\right)^\top\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\Bigg]\\&=\sum_{\jmath=0}^{\bar{\tau}_a-1}\left(\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\right)^\top\Bigg[\sum_{\imath=\jmath+1}^{\bar{\tau}_a}\Big(\mathcal{A}_a^\top(\jmath)\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath+1)\mathcal{A}_a(\jmath)\\&\qquad-\mathcal{J}_a^{(\boldsymbol{p},\bar{\mathbf{q}},\boldsymbol{q}_a)}(\imath,\jmath)\Big)\Bigg]\prod_{\varepsilon=0}^{\jmath-1}\mathcal{A}_a(\varepsilon)\\&\prec\mathbf{0}. \quad (63)\end{aligned}$$

Combining (60) and (63), and setting

$$J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)\triangleq P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0), J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)\triangleq P_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0) \quad (64)$$

for all $a,b\in\mathbb{M}$ $(a\neq b)$, $(p_a,\bar{\boldsymbol{q}}_a)\in\mathbb{P}_a\times\bar{\mathbb{Q}}'_a$, $(p_b,\bar{\boldsymbol{q}}_b)\in\mathbb{P}_b\times\bar{\mathbb{Q}}'_b$, it is further derived that

$$\sum_{b\in\mathbb{M}}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)}{\bar{\eta}'_a}D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)-P_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0)\prec\mathbf{0}. \quad (65)$$

Since $\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)=0$ holds for all $a\in\mathbb{M}$, $(b,\tau)\in\tilde{\mathbb{M}}_a\times\tilde{\mathbb{S}}_{ab}$, $(p_a,q_{ab})\in\mathbb{P}_a\times\mathbb{Q}_{ab}$, we have $\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}(\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)/\bar{\eta}'_a)D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)=\mathbf{0}$. Accordingly, the following equality can be satisfied:

$$\begin{aligned}&\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\bar{\mathbb{S}}_{ab}}\frac{\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)}{\bar{\eta}'_a}D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)+\sum_{b\in\bar{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\frac{\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)}{\bar{\eta}'_a}\\&\times D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)+\sum_{b\in\tilde{\mathbb{M}}_a}\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}\frac{\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)}{\bar{\eta}'_a}D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau)\\&=\sum_{b\in\mathbb{M}}\sum_{\tau=1}^{\bar{\tau}_a}\frac{\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)}{\bar{\eta}'_a}D_{ab}^{(p_b,\bar{\boldsymbol{q}}_b)}(\tau).\end{aligned}$$

According to the definition of $\hat{\theta}_{ab}^{(p_a,q_{ab})}(\tau)$ in (25), the inequality in (65) can be converted into (23).

Moreover, it can be deduced from (61) that

$$\begin{aligned}&\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)\\&=\sum_{\varsigma=0}^{\tau-1}\left(\prod_{\varepsilon=0}^{\varsigma-1}\mathcal{A}_a(\varepsilon)\right)^\top\Big(\mathcal{A}_a^\top(\varsigma)J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma+1)\mathcal{A}_a(\varsigma)\\&\qquad-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\varsigma)\Big)\prod_{\varepsilon=0}^{\varsigma-1}\mathcal{A}_a(\varepsilon)\\&\prec\mathbf{0}. \quad (66)\end{aligned}$$

By pre- and post-multiplying (62) by $\big(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\big)^\top$ and $\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)$, respectively, the following inequality holds:

$$\begin{aligned}&\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\\&-\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,\tau)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\prec\mathbf{0}. \quad (67)\end{aligned}$$

Adding (66) and (67) yields

$$\left(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\right)^\top J_b^{(p_b,\bar{\boldsymbol{q}}_b)}(0,0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)-J_a^{(p_a,\bar{\boldsymbol{q}}_a)}(0,0)\prec\mathbf{0}. \quad (68)$$

Consequently, one can obtain (24) from (68) by considering (64). The remainder of this proof can be completed following the proof of Theorem 3.

E. Proof of Theorem 5

Define $\mathcal{D}_{ab}(\tau)\triangleq\big(\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)\big)^\top\mathfrak{P}_b(0)\prod_{\varepsilon=0}^{\tau-1}\mathcal{A}_a(\varepsilon)$, $a,b\in\mathbb{M}$, $\tau\in\mathbb{I}_{[1,\bar{\tau}_a]}$. Since $\sum_{b\in\mathbb{M}}\sum_{\tau=1}^{\bar{\tau}_a}\theta_{ab}(\tau)\ \mathcal{D}_{ab}(\tau)=\sum_{b\in\bar{\mathbb{M}}_a}\pi_{ab}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}}f_{ab}(\tau)\mathcal{D}_{ab}(\tau)+\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}f_{ab}^{(?)}(\tau)\mathcal{D}_{ab}(\tau)\big)+\sum_{b\in\tilde{\mathbb{M}}_a}\pi_{ab}^{(?)}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}}f_{ab}(\tau)\mathcal{D}_{ab}(\tau)+\sum_{\tau\in\tilde{\mathbb{S}}_{ab}}f_{ab}^{(?)}(\tau)\mathcal{D}_{ab}(\tau)\big)$ holds, based on Lyapunov function (30), together with (17), (19) and the definition of $\mathfrak{P}_a(0)$, it can be derived that

$$\mathsf{E}\{\mathcal{L}''_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a}-\mathcal{L}''_{a,0}(x(k_n))$$

$$= \sum_{b\in\mathbb{M}, b\neq a} \sum_{\tau=1}^{\bar{\tau}_a} x^\top(k_n) \frac{\theta_{ab}(\tau)}{\eta_a} \mathcal{D}_{ab}(\tau) x(k_n) - x^\top(k_n) \mathfrak{P}_a(0) x(k_n)$$
$$= \frac{x^\top(k_n)}{\eta_a} \sum_{p_a\in\mathbb{P}_a} \sum_{\boldsymbol{q}_a\in\mathbb{Q}'_a} \alpha_a^{(p_a)} \boldsymbol{\beta}_a^{(\boldsymbol{q}_a)} \Bigg[\sum_{b\in\bar{\mathbb{M}}_a} \sum_{\tau\in\bar{\mathbb{S}}_{ab}} \pi_{ab} f_{ab}(\tau) \mathcal{D}_{ab}(\tau) + \sum_{b\in\bar{\mathbb{M}}_a} \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} \pi_{ab} f_{ab}^{(q_{ab})}(\tau) \mathcal{D}_{ab}(\tau) + \sum_{b\in\tilde{\mathbb{M}}_a} \sum_{\tau\in\bar{\mathbb{S}}_{ab}} \pi_{ab}^{(p_a)} f_{ab}(\tau) \times \mathcal{D}_{ab}(\tau) + \sum_{b\in\tilde{\mathbb{M}}_a} \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} \pi_{ab}^{(p_a)} f_{ab}^{(q_{ab})}(\tau) \mathcal{D}_{ab}(\tau) - \eta_a P_a^{(p_a,\boldsymbol{q}_a)}(0) \Bigg] x(k_n). \tag{69}$$

Due to $\eta_a = \sum_{p_a\in\mathbb{P}_a} \sum_{\boldsymbol{q}_a\in\mathbb{Q}'_a} \alpha_a^{(p_a)} \boldsymbol{\beta}_a^{(\boldsymbol{q}_a)} \eta_a^{(p_a,\boldsymbol{q}_a)} \geq \eta'_a$ and $\mathcal{D}_{ab}(\tau) = \sum_{p_b\in\mathbb{P}_b} \sum_{\boldsymbol{q}_b\in\mathbb{Q}'_b} \alpha_b^{(p_b)} \boldsymbol{\beta}_b^{(\boldsymbol{q}_b)} D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau)$, where $D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) \triangleq \left(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)\right)^\top P_b^{(p_b,\boldsymbol{q}_b)}(0) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon)$, we can obtain from (69) that

$$\mathsf{E}\{\mathcal{L}''_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a} - \mathcal{L}''_{a,0}(x(k_n)) = \frac{x^\top(k_n)}{\eta_a} \sum_{p_1\in\mathbb{P}_1} \sum_{\boldsymbol{q}_1\in\mathbb{Q}'_1} \cdots \sum_{p_M\in\mathbb{P}_M} \sum_{\boldsymbol{q}_M\in\mathbb{Q}'_M} \left(\prod_{b=1}^{M} \alpha_b^{(p_b)} \boldsymbol{\beta}_b^{(\boldsymbol{q}_b)} \right) \times \Bigg[\sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab} \Bigg(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau) \times D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) \Bigg) + \sum_{b\in\tilde{\mathbb{M}}_a} \pi_{ab}^{(p_a)} \Bigg(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) \Bigg) - \eta'_a P_a^{(p_a,\boldsymbol{q}_a)}(0) \Bigg] x(k_n).$$

Then, $\mathsf{E}\{\mathcal{L}''_{b,0}(x(k_{n+1}))\}|_{x(k_n),R_n=a} - \mathcal{L}''_{a,0}(x(k_n)) \leq -(1/\eta_a)\lambda_{11}\|x(k_n)\|^2 \leq -\lambda_{11}\|x(k_n)\|^2$ is ensured by (31), where $\lambda_{11} \triangleq \min_{a\in\mathbb{M},p_1\in\mathbb{P}_1,\ldots,p_M\in\mathbb{P}_M,\boldsymbol{q}_1\in\mathbb{Q}'_1,\ldots,\boldsymbol{q}_M\in\mathbb{Q}'_M} \big\{\lambda_{\min}\big(\eta'_a P_a^{(p_a,\boldsymbol{q}_a)}(0) - \sum_{b\in\bar{\mathbb{M}}_a} \pi_{ab}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau)\big) - \sum_{b\in\tilde{\mathbb{M}}_a} \pi_{ab}^{(p_a)}\big(\sum_{\tau\in\bar{\mathbb{S}}_{ab}} f_{ab}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) + \sum_{\tau\in\tilde{\mathbb{S}}_{ab}} f_{ab}^{(q_{ab})}(\tau) D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau)\big)\big)\big\}$. This confirms the satisfaction of (8), and completes the proof.

F. Proof of Theorem 6

Define the matrix variables $W_a^{(p_a,\boldsymbol{q}_a)}(0,\varsigma) \triangleq X^{-\top} U_a^{(p_a,\boldsymbol{q}_a)}(0,\varsigma) X^{-1}$, $\mathfrak{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath) \triangleq X^{-\top} \mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath) X^{-1}$ and $\mathfrak{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\bar{\tau}_a) \triangleq X^{-\top} \mathfrak{U}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\bar{\tau}_a) X^{-1}$, where $\mathfrak{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\jmath) \triangleq \sum_{\imath=\jmath+1}^{\bar{\tau}_a} \mathcal{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\imath,\jmath)$, $a\in\mathbb{M}$, $\varsigma\in\mathbb{I}_{[0,\bar{\tau}_a]}$, $\jmath\in\mathbb{I}_{[0,\bar{\tau}_a-1]}$, $(p_a,\boldsymbol{q}_a)\in\mathbb{P}_a\times\mathbb{Q}'_a$, $(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)\in\mathbb{P}'\times\mathbb{Q}''\times\mathbb{Q}'_a$. As an analog of the proof of Theorem 4, one obtains from (33) that $\forall a\in\mathbb{M}$, $\forall(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)\in\mathbb{P}'\times\mathbb{Q}''\times\mathbb{Q}'_a$,

$$\sum_{\tau=1}^{\bar{\tau}_a} \left(\prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon) \right)^\top \mathcal{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\tau,\tau) \prod_{\varepsilon=0}^{\tau-1} \mathcal{A}_a(\varepsilon) - \sum_{\tau=1}^{\bar{\tau}_a} \mathcal{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\tau,0) \prec \mathbf{0} \tag{70}$$

and from (34), one has

$$\sum_{\tau=1}^{\bar{\tau}_a} \mathcal{W}_a^{(\boldsymbol{p},\mathbf{q},\boldsymbol{q}_a)}(\tau,0) - W_a^{(p_a,\boldsymbol{q}_a)}(0,0) \prec \mathbf{0}. \tag{71}$$

Now, we set $W_a^{(p_a,\boldsymbol{q}_a)}(0,0) \triangleq P_a^{(p_a,\boldsymbol{q}_a)}(0)$ for all $a,b\in\mathbb{M}$ $(a\neq b)$, $(p_a,\boldsymbol{q}_a)\in\mathbb{P}_a\times\mathbb{Q}'_a$, and combine (70) and (71). The following inequality can be satisfied:

$$\sum_{b\in\mathbb{M}} \sum_{\tau=1}^{\bar{\tau}_a} \frac{\theta_{ab}^{(p_a,q_{ab})}(\tau)}{\eta'_a} D_{ab}^{(p_b,\boldsymbol{q}_b)}(\tau) - P_a^{(p_a,\boldsymbol{q}_a)}(0) \prec \mathbf{0}$$

which is equivalent to (31) according to the definition of $\theta_{ab}^{(p_a,q_{ab})}(\tau)$ in (32).

The rest of this proof follows the proof of Theorem 5.

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