A Discrete-Time Integral Sliding Mode Control Law for Systems with Matched and Unmatched Disturbances

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***Abstract*— This letter proposes a discrete-time integral sliding mode (DT-ISM) control strategy for linear time-invariant systems subject to matched and unmatched disturbances. The DT-ISM strategy is defined based on a discrete-time model of the system obtained from its continuous-time counterpart, providing numerical procedures to determine the sets in which the disturbances are contained, starting from the corresponding sets in the continuous-time domain. The DT-ISM law is based on disturbance estimation to ideally steer the sliding variable to zero in one discrete step, and achieves a quasi-DT-ISM in the presence of bounded estimation errors. The effectiveness of the proposed control law is tested in simulation combined with a robust model predictive control law.**

***Index Terms*— Integral sliding mode control, discrete-time systems, uncertain systems, linear systems.**

I. Introduction

Sliding mode control laws are typically formulated as discontinuous control laws in continuous time, and guarantee perfect rejection of matched disturbances, once a suitably designed sliding variable is steered to zero in finite time [1]. Integral sliding mode (ISM) controllers are instead designed to ideally make the system dynamics coincide with the nominal dynamics of the closed-loop system obtained by applying a given stabilizing control law [2]. This is possible when only matched disturbances are present, while the presence of unmatched disturbances only allows the ISM controller to reduce the effect of the uncertainties on the closed-loop system [3]–[5]. The ISM control input is applied summed to the stabilizing control input, and the resulting sliding variable is equal to zero from the beginning [2].

Discrete-time sliding mode control laws directly assume the impossibility of an infinite-frequency switching of the discontinuous input [6, Chap. 9], and have been formulated following different approaches (see, e.g., [7]–[12]). In particular, discrete-time approaches were defined for ISM (henceforth, DT-ISM) control laws as well. Specifically, in [13], a DT-ISM approach was proposed for systems with matched disturbances and linear stabilizing control laws, such as those obtainable via pole placement or linear quadratic regulator (LQR); the same approach was extended from regulation to output tracking in [14]. A similar approach, also considering matched disturbances and linear stabilizing control laws, was proposed in [15], [16], making use of a reference model. Linear stabilizing control laws were also the topic of the DT-ISM control approach discussed in [17], in which the disturbance term was allowed to contain unmatched components as well, and the control input was designed as a discontinuous law that did not make use of disturbance estimates (which were instead used in [13]–[16]). In [18], a discontinuous DT-ISM controller for systems with matched disturbances was proposed, together with a stabilizing control law designed via model predictive control (MPC). The same approach was extended to systems with unmatched disturbances in [19]. Finally, the authors of [20] proposed a DT-ISM control law designed using an MPC-based approach.

In all mentioned works on DT-ISM control, the bounds on the disturbance terms were always given in terms of 2-norms, apart from [18], [20], where the infinity-norm was utilized, and [19], where the disturbances were subject to box constraints. Moreover, if we indicate the value of the sliding variable at the discrete step k as σ_k (which in DT-ISM control formulations must be equal to zero from the beginning, i.e., $\sigma_0 = 0$), the DT-ISM equivalent control at step k was calculated by posing $\sigma_{k+1} = 0$ in [13]–[16], and, as an alternative approach, by posing $\sigma_{k+1} - \sigma_k = 0$ in [17]–[20].

This letter proposes a DT-ISM control law for systems with matched and unmatched disturbances, assumed to be contained in generic convex and compact sets. The system equations and the information on the disturbance term are initially given in continuous time, and their equivalent discrete-time representation is obtained via exact discretization and

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numerical methods, respectively. Compared to [13], in which the discrete-time system was also obtained starting from its continuous-time formulation, rather than studying how the discrete-time disturbance is influenced by the choice of the sampling time as in [13], we determine the exact expression of the discrete-time disturbance set starting from its continuous-time expression. Differently from [18], [19], which also accounted for bounds on the control inputs, in our work we consider generic closed convex sets for both disturbances and inputs (as opposed to, for instance, infinity norms). Also, the proposed strategy makes use of a disturbance estimate to ideally steer $\sigma_{k+1} = 0$ (contrary to [19]), which directly allows one to obtain a quasi-sliding mode in the presence of persistent disturbances. As a result, the disturbance term to be handled by the stabilizing law is, under certain conditions, reduced. It is worth highlighting that the exact definition of the disturbance sets is instrumental for applying any nonlinear stabilizing control law, in particular in constrained control problems, thus avoiding the imposition of excessively conservative bounds.

Notation: Given two integers $n_i \leq n_f$, we define $\mathbb{N}_{[n_i,n_f]} \triangleq \{n_i, n_i+1, ..., n_f\}$, e.g., $\mathbb{N}_{[2,4]} \triangleq \{2,3,4\}$. Given a set $\mathcal{Y} \subseteq \mathbb{R}^n$, we indicate its interior as $\text{int}(\mathcal{Y})$. Given a matrix $M \in \mathbb{R}^{m\times n}$, then $M\mathcal{Y} \triangleq \{z \in \mathbb{R}^m \ : \ z = My, \ y \in \mathcal{Y}\}$. The Minkowski sum of two sets $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathbb{R}^n$ is $\mathcal{Y}_1 \oplus \mathcal{Y}_2 \triangleq \{y_1 + y_2 \ : \ y_1 \in \mathcal{Y}_1, y_2 \in \mathcal{Y}_2\}$ and their Pontryagin difference is $\mathcal{Y}_1 \ominus \mathcal{Y}_2 \triangleq \{z \in \mathbb{R}^n : z + y \in \mathcal{Y}_1, \ \forall y \in \mathcal{Y}_2\}$. Given a vector $a \in \mathbb{R}^n$, $\|a\|$ denotes its Euclidean norm.

II. Recalling continuous-time ISM

In this section, we briefly summarize the main concepts of continuous-time ISM control, which will be used as a reference to define the proposed DT-ISM control formulation. Consider the fully controllable continuous-time linear time-invariant (LTI) system

$$\dot{x}(t) = A_c x(t) + B_c u(t) + w(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ (with $m \leq n$) are the state (fully available for feedback) and control vectors, respectively, whereas $w \in \mathbb{R}^n$ is the disturbance term. The following assumption holds:

Assumption 1: The disturbance $w(t)$ in system (1) is bounded as

$$w(t) \in \mathcal{W}_c, \ t \in \mathbb{R}_{\geq 0}. \tag{2}$$

where $\mathcal{W}_c$ is a convex and compact set. □

Following the classical formulation of [2] adapted to the specific case of LTI systems, the control variable is split into two components, and in particular $u(t) = u_0(t) + u_1(t)$, where $u_0(t)$ is defined by a stabilizing control law, whereas $u_1(t)$ is the ISM control input. This input has the aim of zeroing the following sliding variable:

$$s(t) \triangleq s_0(t) + z(t) \in \mathbb{R}^m \tag{3}$$

where $s_0(t) \triangleq Cx(t)$, with C defined such that the product CB_c is invertible. Also, the time evolution of $z(t)$ is defined by

$$\dot{z}(t) = -C\left(A_c x(t) + B_c u_0(t)\right), \ \text{with } z(0) = -s_0(0). \tag{4}$$

For the theoretical development of this letter in the following section, it is important to notice that (3) and (4) imply

$$\dot{s}(t) = C\dot{x}(t) + \dot{z}(t) = C(B_c u_1(t) + w(t)), \tag{5}$$

and that $s(0) = 0$. The continuous-time ISM control input $u_1(t)$ is defined as a discontinuous function with an amplitude sufficiently large to dominate the effect of the disturbance $w(t)$. It has the aim of maintaining $\dot{s}(t) \equiv 0$ and thus, as $s(0) = 0$, $s(t) \equiv 0$, for $t \geq 0$ (see, e.g., [2], [4]). The effect of $u_1(t)$ partially compensates the disturbance $w(t)$ [4], and one can substitute a new term $d(t) \triangleq (I - B_c(CB_c)^{-1}C)w(t)$ to $B_c u_1(t) + w(t)$ in (1). This leads to obtaining a new system dynamics

$$\dot{x}(t) = A_c x(t) + B_c u_0(t) + d(t), \tag{6}$$

which is used to design the control law $u_0(t)$.

III. DT-ISM formulation

A. System and sliding variable dynamics

If one can sample the state $x(t)$ and consequently redefine the control input only at given sampling times, a discrete-time formulation of the above continuous-time ISM law has to be introduced. Defining $k \in \mathbb{N}_{\geq 0}$ as the discrete step and $T \in \mathbb{R}_{>0}$ as the sampling interval, we assume that $u_0(t)$ and $u_1(t)$ remain constant in $[kT, kT+T)$, and indicate their values in this time interval as $u_{0,k}$ and $u_{1,k}$, respectively.

To provide a discrete-time version of the ISM formulation, we first perform an exact discretization of system (1). After defining $x_k \triangleq x(kT)$, $u_k = u(kT)$, $A \triangleq e^{A_c T}$, $B \triangleq \int_0^T e^{A_c\tau} d\tau B_c$, and

$$w_k \triangleq \int_0^T e^{A_c\tau} w(kT + T - \tau) d\tau, \tag{7}$$

one obtains

$$x_{k+1} = Ax_k + Bu_k + w_k. \tag{8}$$

Notice that w_k is not the sampled version of $w(t)$, but rather represents the effect that $w(t)$ has on the evolution of the state in one sampling interval.

In continuous-time ISM, the ideal aim of $u_1(t)$ would be the perfect compensation of the effect of the disturbance $w(t)$ on the evolution of the state; as this cannot be done in general, then $s(t)$ is defined in an m-dimensional space (based on matrix C), and $u_1(t)$ aims at compensating the effect of $w(t)$ in this m-dimensional space. In discrete time, the variation of the state value in one sampling interval given by the combined effect of $u_{1,k}$ and w_k is simply given by $Bu_{1,k} + w_k$. Analogously to the continuous-time case, one would want to achieve $Bu_{1,k} + w_k = 0$, but in general this would require $m = n$. Therefore, in this letter, we aim at compensating the effect of the disturbance in the space defined by C, i.e., to ideally impose $C(Bu_{1,k} + w_k) = 0$. For this purpose, we define a new sliding variable $\sigma \in \mathbb{R}^m$, initialized at $\sigma(0) = 0$, the dynamics of which is given by

$$\sigma_{k+1} = \sigma_k + C(Bu_{1,k} + w_k). \tag{9}$$

The DT-ISM control law will then be defined to ideally obtain a DT-ISM, defined as follows:

Definition 1: A *DT-ISM* is said to take place for system (8), (9) if $\sigma_k = 0$ for all $k \geq 0$. □

B. Definition of the ISM control law

In this work we employ the discrete-time sliding mode interpretation used by Utkin and co-authors for example in [6], [10] and used in the DT-ISM literature in [13]–[16], in which the sliding mode control variable is aimed at steering the sliding variable to zero in one discrete step. As σ_{k+1} also depends on w_k, the knowledge of w_k would be required to be able to define $u_{1,k}$ so as to obtain $\sigma_{k+1} = 0$. As w_k is unknown, an estimate $\hat{w}_k$ has to be used instead. This can be obtained for instance setting $\hat{w}_k \triangleq w_{k-1}$ in case of slowly-varying disturbances, or obtained via linear interpolation from previous values or via disturbance observers. Then, the actual disturbance value at step k can be obtained a-posteriori at step $k+1$ as $w_k = x_{k+1} - Ax_k - Bu_k$, as we assume full state feedback without measurement noise. The following result holds:

Proposition 1: Given system (8), (9), if, for all $k \in \mathbb{R}_{\geq 0}$,

$$u_{1,k} = -(CB)^{-1}C\left(\tilde{w}_{k-1} + \hat{w}_k\right) \tag{10}$$

where $\tilde{w}_k \triangleq w_k - \hat{w}_k$ and $w_{-1} = \tilde{w}_{-1} = 0$, then, the value of the sliding variable for $k \in \mathbb{N}_{\geq 0}$ will be

$$\sigma_k = C\tilde{w}_{k-1}. \tag{11}$$

Proof: At step $k = 0$, based on $\hat{w}_k$, the estimated value of σ_1 is $\hat{\sigma}_1 = \sigma_0 + CBu_{1,0} + C\hat{w}_0$. As $\sigma_0 = 0$, the value of $u_{1,0}$ that achieves $\hat{\sigma}_1 = 0$ is $u_{1,0} = -(CB)^{-1}C\hat{w}_0$, leading, through (9), to $\sigma_1 = C\tilde{w}_0$. At the next discrete step, the estimated value of σ_2 is $\hat{\sigma}_2 = \sigma_1 + CBu_{1,1} + C\hat{w}_1$, and the value of $u_{1,1}$ that achieves $\hat{\sigma}_2 = 0$ is $u_{1,1} = -(CB)^{-1}C\left(\tilde{w}_0 + \hat{w}_1\right)$, leading to $\sigma_2 = \sigma_1 - CB(CB)^{-1}C\left(\tilde{w}_0 + \hat{w}_1\right) + Cw_1 = C\tilde{w}_1$. The same result, shifted in time, will be obtained for the following discrete steps, and therefore the proposition is proven by induction. ■

C. Sets related to disturbance and control inputs

One of the contributions of this work is the explicit definition of all the disturbance and input sets related to DT-ISM starting from continuous-time information. We start by determining a set $\mathcal{W}$ such that $w_k \in \mathcal{W}$ for all $k \in \mathbb{N}_{\geq 0}$. As $w(t) \in \mathcal{W}_c$ according to Assumption 1 and w_k can be obtained from $w(t)$ on the basis of (7), then $\mathcal{W}$ can be interpreted as the set of all states $v \in \mathbb{R}^n$ of system

$$\dot{v}(t) = A_c v(t) + \omega(t) \tag{12}$$

that can be reached in a time interval T, given $v(0) = 0$ and $\omega(t) \in \mathcal{W}_c$ for $t \in [0, T]$. In other terms $\mathcal{W}$ is the so-called *reachability set* (see, e.g., [21, Def. 6.1]). As reported in [21, Theorem 6.3], the fact that $\mathcal{W}_c$ is convex and compact (Assumption 1) implies that also $\mathcal{W}$ is convex and compact. Different numerical methods exist able to determine an over-approximation of $\mathcal{W}$, depending on the type of set that defines $\mathcal{W}_c$: ellipsoid [22], polytope [23], zonotope [24] or support function [25]. The interested reader is referred to [26] for a detailed overview of the topic.

Two more sets related to the disturbances have to be introduced. The first is the set $\hat{\mathcal{W}} \subset \mathbb{R}^n$, defined such that $\hat{w}_k \in \hat{\mathcal{W}}$ for all $k \in \mathbb{N}_{\geq 0}$. In order to be able to obtain estimates that cover the whole range of possible disturbance values, surely one needs $\hat{\mathcal{W}} \supseteq \mathcal{W}$.

The second set is $\tilde{\mathcal{W}} \subset \mathbb{R}^n$, defined such that $\tilde{w}_k \in \tilde{\mathcal{W}}$ for all $k \in \mathbb{N}_{\geq 0}$. The definition of this set depends on the specific disturbance estimation technique that is employed.

In several practical applications, it is reasonable to assume the presence of bounds on the control input u_k, i.e., to assume that $u_k \in \mathcal{U}$ for all $k \in \mathbb{R}_{\geq 0}$, with $\mathcal{U}$ being a convex and compact set. Given the expression of $u_{1,k}$ in (10), surely $u_{1,k} \in \mathcal{U}_1$, with

$$\mathcal{U}_1 \triangleq -(CB)^{-1}C\left(\tilde{\mathcal{W}} \oplus \hat{\mathcal{W}}\right). \tag{13}$$

As a consequence, the stabilizing control law will be constrained as $u_{0,k} \in \mathcal{U}_0$, with $\mathcal{U}_0 \triangleq \mathcal{U} \ominus \mathcal{U}_1$.

As a relevant example of calculation of $\hat{\mathcal{W}}$ and $\tilde{\mathcal{W}}$, consider the case $\hat{w}_k \triangleq w_{k-1}$ (which is the most commonly used approach in DT-ISM works). In this case, $\hat{\mathcal{W}} = \mathcal{W}$. Also, as $\tilde{w}_k = w_k - w_{k-1}$, $\tilde{\mathcal{W}}$ depends on the maximum variation that w_k can attain in one discrete step. More precisely, the following result can be proved:

Proposition 2: If $\hat{w}_k \triangleq w_{k-1}$, and $\dot{w}(t) \in \dot{\mathcal{W}}_c$, with $\dot{\mathcal{W}}_c$ convex and compact, then $\tilde{\mathcal{W}}$ is the reachable set in T seconds from the origin of system (12) with $\omega(t) \in T\dot{\mathcal{W}}_c$.

Proof: By definition of w_k in (7), it follows that

$$\begin{aligned} & w_{k+1} - w_k \\ &= \int_0^T e^{A_c\tau}\left(w(kT + 2T - \tau) - w(kT + T - \tau)\right)d\tau. \end{aligned} \tag{14}$$

The term $w(kT+2T-\tau) - w(kT+T-\tau)$ can be equivalently expressed as $\int_{kT+T-\tau}^{kT+2T-\tau} \dot{w}(t)dt$. If the average value of $\dot{w}(t)$ in $t \in [t, t+T]$ for a fixed T is introduced as

$$\bar{\dot{w}}(t) \triangleq \frac{1}{T}\int_t^{t+T} \dot{w}(t)dt, \tag{15}$$

which exists finite as $\dot{w}(t) \in \dot{\mathcal{W}}_c$, then

$$w_{k+1} - w_k = \int_0^T e^{A_c\tau} T\bar{\dot{w}}(kT + T - \tau)d\tau. \tag{16}$$

Based on known result of convex analysis (see, e.g., [27, Ch. 2]), given a function $\psi(t) : \mathbb{R}_{[t_1,t_2]} \to \mathbb{R}_{\geq 0}$ with $\int_{t_1}^{t_2} \psi(t)dt = 1$, then

$$\int_{t_1}^{t_2} \psi(t)\dot{w}(t)dt \in \mathrm{co}\left(\mathcal{C}_{[t_1,t_2]}\right), \tag{17}$$

where $\mathrm{co}\left(\mathcal{C}_{[t_1,t_2]}\right)$ is the convex hull of the curve $\mathcal{C}_{[t_1,t_2]}$ that $\dot{w}(t)$ described in the time interval $[t_1, t_2]$. As a particular case, with reference to (15), we take $t_1 = kT + T - \tau$, $t_2 = kT + 2T - \tau$ and $\psi(t) = T^{-1}$, which satisfies $\int_{kT+T-\tau}^{kT+2T-\tau} \psi(t)dt = 1$, and determine that

$$\bar{\dot{w}}(kT + T - \tau) \in \mathrm{co}\left(\mathcal{C}_{[kT+T-\tau,kT+2T-\tau]}\right). \tag{18}$$

As $\mathcal{C}_{[kT+T-\tau,kT+2T-\tau]} \subset \dot{\mathcal{W}}_c$ and $\dot{\mathcal{W}}_c$ is convex by assumption, then $\mathrm{co}\left(\mathcal{C}_{[kT+T-\tau,kT+2T-\tau]}\right) \subseteq \dot{\mathcal{W}}_c$, and thus

$\bar{w}(kT+T-\tau) \in \dot{\mathcal{W}}_c$. Based on this result, (16) can be restated, by posing $\omega(t) \triangleq T\bar{w}(t)$, as

$$w_{k+1} - w_k = \int_0^T e^{A_c\tau}\omega(kT+T-\tau)d\tau, \qquad (19)$$

with $\omega(kT+T-\tau) \in T\dot{\mathcal{W}}_c$. As a consequence, set $\tilde{\mathcal{W}}$ can be calculated as the reachable set in T seconds from the origin of system (12) with $\omega(t) \in T\dot{\mathcal{W}}_c$. ■

D. Achievement of quasi-ISM

From the expression of σ_k in (11), it is clear that an ideal DT-ISM as stated in Definition 1 can be achieved only if the disturbance estimate is perfectly accurate, i.e., $\tilde{w}_k = 0$, for all $k \in \mathbb{N}_{\geq 0}$. In such case, indeed, (11) can be written as $\sigma_k = 0$. In general, one can attain a quasi-DT-ISM, defined as follows:

Definition 2: A *quasi-DT-ISM* takes place for system (8), (9) if there exists $\epsilon \in \mathbb{R}_{\geq 0}$ such that $\|\sigma_k\| \leq \epsilon$ for all $k \geq 0$. The value of ϵ depends on specific characteristics of the disturbance and/or of its estimate, and converges to zero under ideal conditions. □

From (11), it yields

$$\|\sigma_k\| \leq \epsilon \triangleq \max_{\tilde{w} \in \tilde{\mathcal{W}}} \|C\tilde{w}\| \qquad (20)$$

for $k \in \mathbb{N}_{\geq 1}$. In this case, the width of the band in which σ_k is confined depends on the quality of the disturbance estimation. In the ideal case $\hat{w}_k = w_k$ for $k \geq 0$, the quasi-DT-ISM turns into a DT-ISM, as $\epsilon = 0$.

E. Closed-loop state dynamics

The next step consists of evaluating how the closed-loop system dynamics can be written once $u_{1,k}$ in (10) is applied to system (8). Specifically, one has

$$x_{k+1} = Ax_k + Bu_{0,k} - B\,(CB)^{-1}\,C\,(\tilde{w}_{k-1} + \hat{w}_k) + w_k. \quad (21)$$

Therefore, applying the proposed DT-ISM law, the original disturbance w_k is substituted by a new disturbance $d_k \triangleq -B\,(CB)^{-1}\,C\,(\tilde{w}_{k-1} + \hat{w}_k) + w_k$. After substituting $\hat{w}_k = w_k - \tilde{w}_k$, one obtains $d_k = (I - B\,(CB)^{-1}\,C)w_k + B\,(CB)^{-1}\,C(\tilde{w}_k - \tilde{w}_{k-1})$, and thus the set $\mathcal{D}$ to which d_k belongs is given by

$$\mathcal{D} = (I - B\,(CB)^{-1}\,C)\mathcal{W} \oplus B\,(CB)^{-1}\,C(\tilde{\mathcal{W}} \oplus -\tilde{\mathcal{W}}). \quad (22)$$

It would not be convenient to use the proposed DT-ISM law unless $\mathcal{D}$ defines a disturbance that is "smaller" than that defined by $\mathcal{W}$. A rigorous way to define the concept of "smaller disturbance" can be for instance that $\max_{d_k \in \mathcal{D}} \|d_k\| < \max_{w_k \in \mathcal{W}} \|w_k\|$. The achievement of this goal depends on the choice of C, and on the given realizations of $\hat{\mathcal{W}}$ and $\tilde{\mathcal{W}}$. As the proposed ISM law is based on disturbance estimation, a poor estimation (and consequently a large set $\tilde{\mathcal{W}}$) might lead to failing this goal. On the other hand, if the estimation strategy is nearly perfect and thus $\tilde{\mathcal{W}} \simeq \{0\}$, then $\mathcal{D} \simeq (I - B\,(CB)^{-1}\,C)\hat{\mathcal{W}}$. In such a case, [4, Prop. 2] guarantees us that $\max_{d_k \in \mathcal{D}} \|d_k\|$ is minimized by choosing $C = B'$.

IV. Case study

As a case study, we consider a continuous-time system in form (1), with $n = 2$ states and $m = 1$ input, defined by

$$A_c = \begin{bmatrix} 1.8232 & 1.1552 \\ 0 & -5.1083 \end{bmatrix}, \; B_c = \begin{bmatrix} -0.6091 \\ 12.7706 \end{bmatrix}. \qquad (23)$$

The disturbance $w(t)$ is assumed to be contained in $\mathcal{W}_c$, defined as $\mathcal{W}_c \triangleq [-0.004, 0.004] \times [-0.5731, 0.5731]$, while its derivative is assumed to be contained in $\dot{\mathcal{W}}_c$, defined as $\dot{\mathcal{W}}_c \triangleq [-0.004, 0.004] \times [-0.5731, 0.5731]$. The system is discretized with a sampling interval $T = 0.1$ s. The obtained system in form (8), which coincides with the benchmark problem introduced in [9], is defined by

$$A = \begin{bmatrix} 1.2 & 0.1 \\ 0 & 0.6 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \qquad (24)$$

A low-dimensional single-input system is considered for the sake of simplicity, but the calculation of $u_{1,k}$ in (10) can be carried out inexpensively on standard hardware for relatively large values of n and $m \leq n$, as it only consists of multiplying the constant matrix $-(CB)^{-1}C \in \mathbb{R}^{m \times n}$, which is calculated a-priori, by vector $(\tilde{w}_{k-1} + \hat{w}_k) \in \mathbb{R}^n$. As a preliminary step, one has to obtain $\tilde{w}_{k-1}$ and $\hat{w}_k$, but this is also achieved via few matrix-vector multiplications, at least when $\hat{w}_k \triangleq w_{k-1}$. Indeed, in this case $\hat{w}_k \triangleq x_k - Ax_{k-1} - Bu_{k-1}$ and $\tilde{w}_{k-1} \triangleq w_{k-1} - \hat{w}_{k-1}$.

Regarding the disturbance terms in discrete time, $\mathcal{W}$ is obtained as a polytope using the CORA Toolbox for Matlab [28], finding a reachable set as detailed in Section III-C. The matrix C that defines the sliding variable is chosen as $C = B'$, while the disturbance is estimated as $\hat{w}_k = w_{k-1}$. As a consequence, $\hat{\mathcal{W}} = \mathcal{W}$, while the set $\tilde{\mathcal{W}}$ containing the estimation error is obtained as a polytope using the CORA Toolbox, following the theory reported in Proposition 2. The

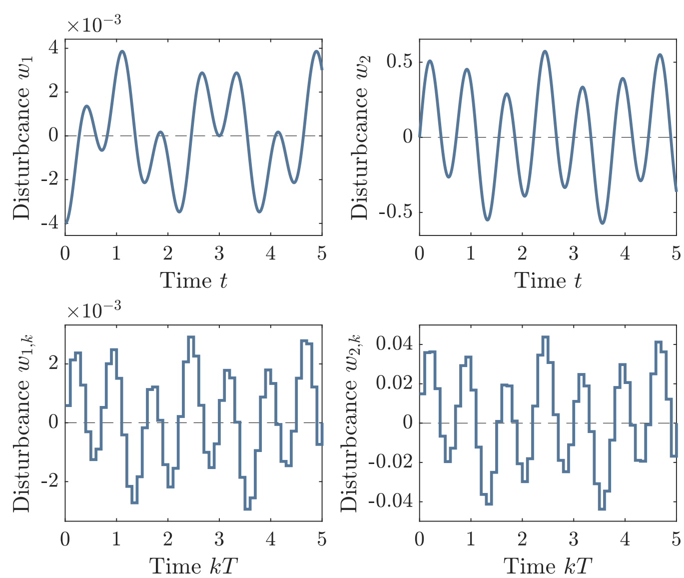

Fig. 1. Time evolution of the disturbance terms $\boldsymbol{w}$ affecting the continuous-time system (on the top) and the corresponding disturbance $\boldsymbol{w_k}$ in discrete time (on the bottom).

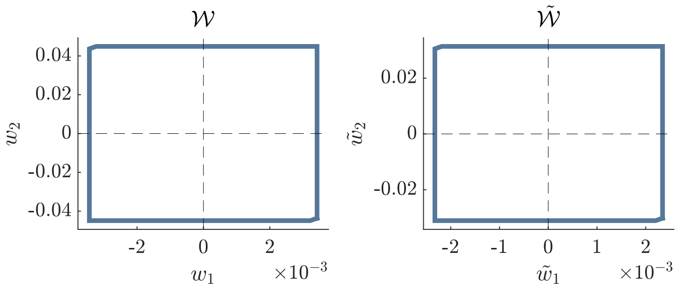

Fig. 2. Disturbance sets in discrete time, $\mathcal{W}$ and $\tilde{\mathcal{W}}$.

time evolution of continuous-time and corresponding discrete-time disturbances are illustrated in Fig. 1, whereas sets $\mathcal{W}$ and $\tilde{\mathcal{W}}$ can be seen in Fig. 2. A quasi-DT-ISM can be guaranteed with $\epsilon = 0.0314$, derived as detailed in Section III-D. The resulting set containing the DT-ISM control input is $\mathcal{U}_1 = [-0.0763, 0.076]$, while the resulting disturbance set is $\mathcal{D} = [-0.0034, 0.0034] \times [-0.0624, 0.0624]$.

The stabilizing control law is chosen as a robust MPC law based on the approach of [29]. It is assumed that input and state constraints are present in the form $u \in \mathcal{U}$ and $x \in \mathcal{X}$, where $\mathcal{U} = [-1, 1]$ and $\mathcal{X} = [-0.5, 0.5] \times [-1, 1]$. As the DT-ISM law is applied together with the MPC law, the amplitude available to the MPC controller is $\mathcal{U}_0 = [-0.9237, 0.924]$, while the disturbance that the robust MPC law has to account for is contained in $\mathcal{D}$ as defined above. To design the MPC law, we first define an unconstrained stabilizing control law Kx_k (with $K = [-4.5028 \;\; -0.9561]$) via infinite-horizon LQR, setting $Q = 100 \cdot I$ and $R = 1$. The set of initial conditions $\mathcal{X}_f$, such that by applying the LQR law the robust satisfaction of input and state constraints is guaranteed, is

$$\begin{aligned}\mathcal{X}_f \triangleq \{x_0 \in \mathbb{R}^n \;:\; & x_{k+1} = (A + BK)x_k + d_k \in \mathcal{X} \\ & \wedge \; Kx_k \in \mathcal{U}_0, \; \forall d_k \in \mathcal{D}, \; \forall t \in \mathbb{N}_{\geq 0}\}\end{aligned} \tag{25}$$

and is obtained via linear programming by applying [30, Alg. 6.1]. The robust MPC strategy [29] relies on defining suitable tightened sets such that, if the nominal evolution of inputs and states lies in these sets, then the actual evolution of inputs and states lies in the original constraints. To do that, one has to calculate the set of states reachable in i discrete steps by system $x_{k+1} = (A + BK)x_k + d_k$, assuming $x_0 = 0$ and any $d_k \in \mathcal{D}$ for $k \in \mathbb{N}_{[0,i-1]}$ [21], which is defined via Minkowski sum as follows:

$$\mathcal{R}_i = \bigoplus_{k=0}^{i} (A + BK)^k \mathcal{D}. \tag{26}$$

To determine $u_{0,k}$, a sequence of $N = 5$ instances of a vector $c^* \in \mathbb{R}^m$ (namely $\mathbf{c}^* \triangleq \begin{bmatrix} c_0^* & c_1^* & \dots & c_{N-1}^* \end{bmatrix}$) is determined at each step k by solving the following optimal control problem (OCP):

$$\mathbf{c}^* = \arg\min_{\mathbf{c}} \sum_{i=0}^{N-1} c_i' c_i \tag{27a}$$

$$\text{subj. to } x_{i+1} = (A + BK)x_i + Bc_i, \; i \in \mathbb{N}_{[0,N-1]} \tag{27b}$$

$$Kx_i + c_i \in \mathcal{U}_0 \ominus K\mathcal{R}_i, \; i \in \mathbb{N}_{[0,N-1]} \tag{27c}$$

$$x_i \in \mathcal{X} \ominus \mathcal{R}_i, \; i \in \mathbb{N}_{[0,N-1]} \tag{27d}$$

$$x_N \in \mathcal{X}_f \ominus \mathcal{R}_N \tag{27e}$$

where $\mathbf{c}$ represents a general sequence of vectors of which $\mathbf{c}^*$ is the optimal realization. After the OCP (27) is solved, the MPC control input is defined as $u_{0,k} = Kx_k + c_{0,k}^*$, where $c_{0,k}^*$ is the value of c_0^* from the OCP solution at step k. At the next discrete step, x_{k+1} is measured and a new OCP with the same *prediction horizon* N is solved again.

By applying the DT-ISM law together with the MPC law, the resulting control action results in being

$$u_k = Kx_k + c_{0,k}^* - (B'B)^{-1}B'(\tilde{w}_{k-1} + \hat{w}_k). \tag{28}$$

Constraint (27c) in the OCP (27) guarantees that $u_{0,k} \in \mathcal{U}_0$, and thus, we can guarantee that $u_k \in \mathcal{U}$ as $u_{1,k} \in \mathcal{U}_1$. Furthermore, based on the theory in [29] (to which the reader is referred for a more detailed description of the MPC law, which could not be inserted here due to space limitation) guarantees that $x_k \in \mathcal{X}$ and x_k converges to the *minimal robust positively invariant set* for system $x_{k+1} = (A + BK)x_k + w_k$, i.e., $\mathcal{R}_\infty \triangleq \lim_{i\to\infty} \mathcal{R}_i$, which is a polytope [21, Prop. 6.9]. The set $\mathcal{R}_\infty$ and the sets $\mathcal{R}_i$ defined in (26) were calculated using the Multi-Parametric Toolbox 3.0 (MPT3) for MATLAB [31].

As compared to applying the MPC law to the original system with full disturbance w_k, the presence of the DT-ISM term allows one to reduce the size of sets $\mathcal{R}_i$ in the OCP (27). This enables the convergence to a smaller set $\mathcal{R}_\infty$ and enlarges the domain of attraction $\mathcal{X}_0$, i.e., the set of states from which an OCP solution exists, as long as a sufficiently large control amplitude is allowed within $\mathcal{U}$ for both the MPC and the DT-ISM terms. Fig. 3 shows the time evolution of the states fulfilling the constraints, the sliding variable confined in the layer of size ϵ and the input evolution with its MPC and DT-ISM components inside the bounds. Fig. 4 shows instead the phase portrait with domain of attraction $\mathcal{X}_0$, the set $\mathcal{X}_f$, terminal set $\mathcal{X}_f \ominus \mathcal{R}_N$, and the minimal robust positively invariant set $\mathcal{R}_\infty$. The state trajectory never violates the constraints and, as expected, the state then enters $\mathcal{X}_f \ominus \mathcal{R}_N$, and finally converges to the bounded set $\mathcal{R}_\infty$.

V. Conclusions

This paper contributes to the literature of DT-ISM control by proposing a novel control law that takes into account the explicit and exact definition of the disturbance sets in discrete time, starting from their continuous-time formulation. Such a definition is indeed instrumental to apply any nonlinear (in particular constrained) control law playing the role of stabilizing control component. Stability results on the closed-loop dynamics are discussed and finally assessed in simulation on a benchmark example.

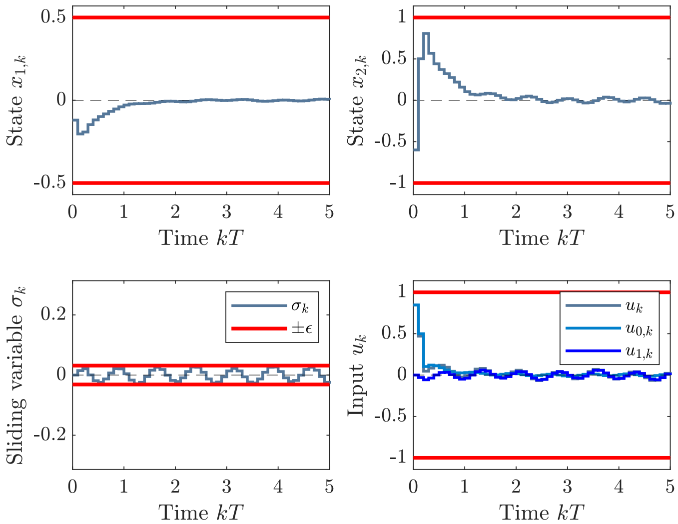

Fig. 3. MPC with DT-ISM control, from top left, clockwise. Time evolution of the state $\boldsymbol{x_{1,k}}$ (solid blue line) and bounds $\boldsymbol{x_{1_{\min}} = -0.5}$ and $\boldsymbol{x_{1_{\max}} = 0.5}$ (solid red line). Time evolution of the state $\boldsymbol{x_{2,k}}$ (solid blue line) and bounds $\boldsymbol{x_{2_{\min}} = -1}$ and $\boldsymbol{x_{2_{\max}} = 1}$ (solid red line). Time evolution of the input $\boldsymbol{u_k = u_{0,k} + u_{1,k}}$ (solid blue line) and bounds $\boldsymbol{u_{\min} = -1}$ and $\boldsymbol{u_{\max} = 1}$ (solid red line). Time evolution of the sliding variable $\boldsymbol{\sigma_k}$ (solid blue line) and bound $\boldsymbol{\epsilon = 0.0314}$ (solid red line).

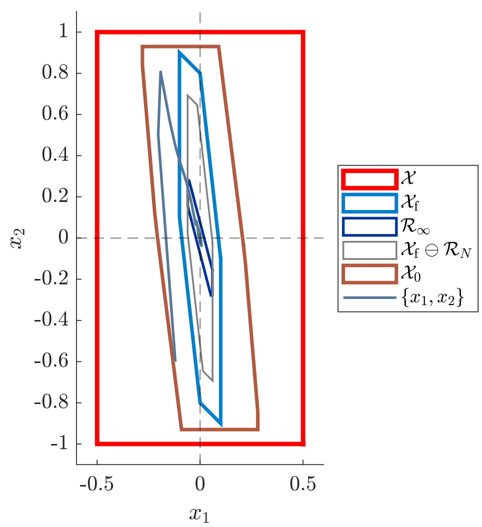

Fig. 4. MPC with DT-ISM control, state-space portrait. Box constraint set $\boldsymbol{\mathcal{X}}$ (red line), domain of attraction $\boldsymbol{\mathcal{X}_0}$ (brown line), set $\boldsymbol{\mathcal{X}_f}$ (cyan line), terminal set $\boldsymbol{\mathcal{X}_f \ominus \mathcal{R}_N}$ (gray line), minimal robust positively invariant set $\boldsymbol{\mathcal{R}_\infty}$ (dark-blue line), state trajectory (blue line).

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