



Stabilization of loosely coupled schemes for 0D–3D fluid–structure interaction problems with application to cardiovascular modelling

Francesco Regazzoni¹

Received: 28 February 2022 / Revised: 20 August 2024 / Accepted: 23 December 2024 /

Published online: 25 January 2025

© The Author(s) 2025

Abstract

In this paper we analyze the numerical oscillations affecting loosely coupled schemes for hybrid-dimensional 0D–3D fluid–structure interaction (FSI) problems, which arise e.g. in the field of cardiovascular modeling, and we propose a novel stabilized scheme that cures this issue. We study several loosely coupled schemes, including the Dirichlet–Neumann (DN) and Neumann–Dirichlet (ND) schemes. In the first one, the 0D fluid model prescribes the pressure to the 3D structural mechanics model and receives the flow. In the second one, on the contrary, the fluid model receives the pressure and prescribes the flow. The terms DN and ND, employed in the FSI literature, are borrowed from domain decomposition methods, although here a single iteration is performed before moving on to the next time step (that is, the coupling is treated explicitly). Should the fluid be enclosed in a cavity, the DN scheme is affected by non-physical oscillations whose origin lies in the balloon dilemma, for which we provide an algebraic interpretation. Moreover, we show that also the ND scheme can be unstable for a range of parameter choices. Surprisingly, increasing either the viscous dissipation or the inertia of the structure favours the onset of oscillations and, for certain parameter choices, the ND is unconditionally unstable. In the presence of inertial terms, by reducing the time step size below a certain threshold, the amplitude of the numerical oscillations is even amplified. We provide an explanation for these facts and establish sharp stability bounds on the time step size. Our analysis extends to Robin–Robin schemes, based on linear combinations of the conditions of pressure continuity and either volume or flux continuity. While appropriate choices of Robin coefficients can achieve numerical stability, tuning these coefficients can be challenging in practice. To address these issues, we propose a numerically consistent stabilization term for the Neumann–Dirichlet scheme, inspired by physical insight on the onset of oscillations. We prove that our proposed stabilized scheme is absolutely stable for any choice of time step size. Notably, the proposed scheme does not require parameter tuning.

✉ Francesco Regazzoni
francesco.regazzoni@polimi.it

¹ MOX - Department of Mathematics, Politecnico di Milano, P.zza Leonardo da Vinci 32, 20133 Milano, Italy

These results are verified by several numerical tests. Finally, we apply the proposed stabilized scheme to an important problem in cardiac electromechanics, namely the coupling between a 3D cardiac model and a closed-loop lumped-parameter model of blood circulation. In this setting, our proposed scheme successfully removes the non-physical oscillations that would otherwise affect the numerical solution.

Mathematics Subject Classification 65N

1 Introduction

The numerical solution of fluid–structure interaction (FSI) problems presents many challenges [2, 17, 39, 56]. In this work, we focus on loosely coupled schemes, which alternate the resolution of the fluid and solid subproblems with an explicit coupling, that is by performing a single iteration before moving on to the subsequent time step. Loosely coupled schemes offer a significantly lower computational cost with respect to fully coupled schemes, that feature an implicit coupling between the two subproblems. However, they are often accompanied by numerical instability issues [14, 19, 22, 31]. The numerical stability of loosely coupled schemes has been investigated in [11] and in subsequent works [4, 16, 20, 35]. In these research works, however, either the flow rate of the incoming fluid or its pressure is considered to be assigned. This framework therefore does not include cases of practical interest, such as the interaction between a cardiac chamber and blood. In this case, in fact, the blood flow entering and leaving the chamber is not known *a priori*, but should be related to the blood pressure by means of a suitable mathematical model describing the dynamics of the heart valves [30, 53] (see Fig. 1). This can lead to numerical instabilities when using loosely coupled schemes. As a matter of fact, non-physical oscillations may occur even with simplified fluid models, such as 0D models (lumped-parameter models) and by neglecting fluid inertia [3, 30, 40, 53]. These issues are still not well understood, and, to the best of our knowledge, to date there is no mathematical explanation for the onset of oscillations and no estimates that can predict for which parameter values and time step sizes they occur. Moreover, the only known cure to date is to use fully coupled schemes (that is, adopting an implicit coupling) instead of loosely coupled ones.

In this paper we analyze and propose a cure for these numerical oscillations. To this end, we consider the cardiac case as a model problem with a great applicative interest. This test case comprises several features that are challenging from the numerical standpoint. Indeed, when the heart valves are closed (during the so-called isovolumetric phases of the heartbeat), the problem translates into that of an incompressible fluid enclosed in a cavity. Instead, when the outflow valves open, we have a rapid flow ejection. Furthermore, this problem presents some peculiarities, related to the fact that the structure does not move in a purely passive way, but might present an active stress. This, as we shall see, has implications for numerical stability, and can be a further cause of non-physical oscillations. As for the fluid, we will limit ourselves to a 0D description, which is typically adopted for simulations of cardiac electromechanics [3, 30, 40, 53], and which in itself poses a wide range of numerical challenges.

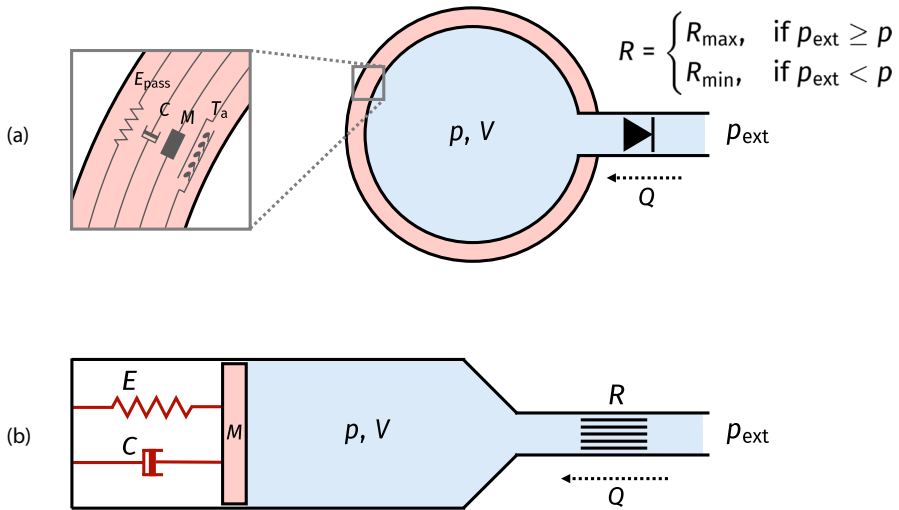


Fig. 1 Representation of two model problems that fall within the scope of this paper. The problem (a) consists of a cardiac chamber, represented as an idealized spherical cavity (in red), interacting with blood (in blue), whose state is characterized by the pressure p and the volume V . The dynamics of the chamber accounts for stiffness (E_{pass}), inertia (M), viscosity (C) and the presence of an active tension (T_a). The chamber is connected to an external region of pressure p_{ext} through a valve, characterized by a blood flow resistance R , which is very large when $p_{\text{ext}} \geq p$ (closed valve), very small when $p_{\text{ext}} < p$ (open valve). As we will show, this problem can be formally reduced to the piston problem (b), in which a fluid exerts a pressure p against a piston characterized by stiffness (E), inertia (M) and viscous damping (C). The fluid flow is determined as $Q = (p_{\text{ext}} - p)/R$, where R is the fluid flow resistance. More details will be provided below (colour figure online)

At the continuous level, the 3D structure model and the 0D fluid model are coupled by imposing the continuity of fluxes and pressures [43, 53]. At the numerical level, instead, several coupling schemes between the 3D and the 0D sub-models are possible. Monolithic fully coupled approaches consist in solving the two subproblems simultaneously by means of a Newton or Quasi-Newton nonlinear solver [21, 30]. This requires at each iteration the calculation of the Jacobian matrix (or an approximation of it), which contains the derivatives of the residual with respect to the unknowns. However, assembling the blocks of the Jacobian matrix associated with the fluid model might involve a significant implementation effort, especially for fluid models with a large number of unknowns (see e.g. [1, 53]). Partitioned fully coupled schemes, instead, envisage alternating between the solution of the structural and fluid subproblems until convergence [35, 38]. These schemes, however, are characterized by a significant computational cost, as the two subproblems need to be solved several times for each time step. An attractive alternative to fully coupled schemes is represented by loosely coupled schemes, that alternate the solution of the fluid model and of the structure mechanics model, with a single iteration per time step. This allows for a much easier implementation than monolithic fully coupled schemes and yields the possibility to reuse pre-existing (even black-box) solvers for either the 3D structure model or the 0D

fluid model. Moreover, compared to partitioned fully coupled schemes, loosely coupled schemes feature a significantly lower computational cost. However, as mentioned, loosely coupled schemes may suffer from numerical instability, which is manifested in non-physical oscillations in the solution.

In this paper we will first focus on two loosely coupled schemes. Within the first one, the structural model provides the flows and receives the pressures from the fluid model; within the second scheme, instead, the structural model provides the pressures and receives the flows. These schemes are known in the literature, respectively, as Dirichlet-Neumann and Neumann-Dirichlet scheme [4, 11, 31, 35]. These names are borrowed, with a little abuse of notation, from the field of domain decomposition methods (see e.g. [46, 58]). However, it is important to note that, while in domain decomposition methods one iterates between the two models until convergence, with loosely coupled schemes one advances to the next time step after a single iteration between the two models. Although with a 0D fluid there is no physical boundary, we will use the terms Dirichlet-Neumann and Neumann-Dirichlet by analogy to the 3D case. Indeed, in the 0D case, the Dirichlet condition translates into imposing fluid flow, while the Neumann condition into imposing pressure.

Although the Dirichlet-Neumann scheme requires a simpler implementation than Neumann-Dirichlet's (pressures are assigned as boundary conditions to the structure mechanics model and fluxes are post-processed from the displacement), it is characterized by serious numerical instability issues when an incompressible fluid is enclosed in a cavity (this is the case, e.g., of isovolumetric phases of the heartbeat [32]). This issue, known as balloon dilemma, is explained by the impossibility to satisfy the incompressibility constraint of the fluid contained in the cavity if the pressure is assigned [5, 27]. As a workaround, it is possible to iterate the resolution of the structure mechanics problem, updating the pressure based on an estimate of the cavity compliance (i.e. $\partial V / \partial p$, where p and V are the cavity pressure and volume, respectively), in order to satisfy the volumetric constraint [23, 32]. However, a poorly accurate estimate of the compliance could lead to a very slow convergence or even to failure of the solver.

To avoid the balloon dilemma, the pressures acting inside the cavities should not be decoupled from the displacement. This requirement can be accommodated in the framework of the Neumann-Dirichlet loosely coupled approach, in which the structure mechanics model is solved under the constraint that the volume contained in the cavity coincides with that predicted, at the previous iteration, by the fluid model. Then, the obtained pressure is provided to the fluid model, to advance to the next time iteration. The pressure in the cavity can be interpreted in this context as a Lagrange multiplier that ensures that the volume constraint is satisfied [3, 43, 53].

The Neumann-Dirichlet scheme solves the balloon dilemma and prevents the spurious oscillations that would otherwise occur for cavity problems when using the Dirichlet-Neumann scheme. However, it is possible that in presence of a rapid fluid flow (e.g. in cardiac case, when at least one valve is open) non-physical oscillations still occur, even when using the Neumann-Dirichlet scheme. In Fig. 2 we display an example of oscillations obtained when applying the Neumann-Dirichlet scheme to a multiscale model of cardiac electromechanics of the left ventricle (the model is described in detail in Sect. 5.2). As we will show, these oscillations may depend on

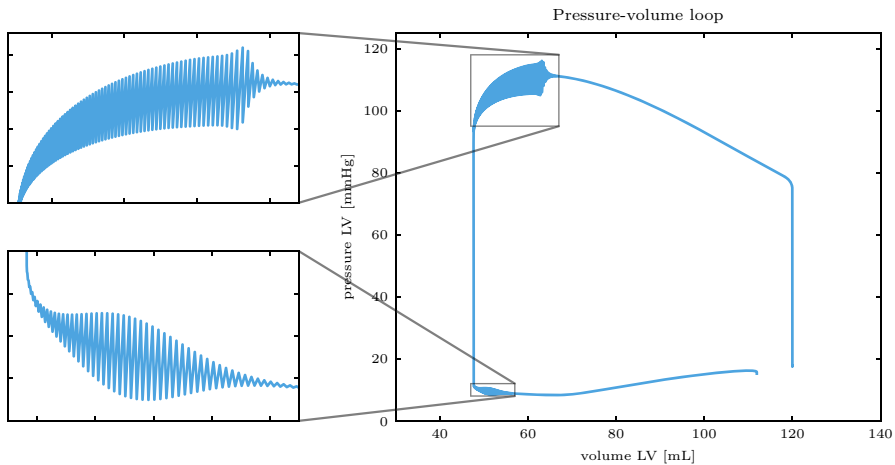


Fig. 2 Non-physical oscillations, obtained when employing the Neumann–Dirichlet scheme to numerically approximate the left ventricle model of cardiac electromechanics described in Sect. 5.2. The graph shows the simulation results in the pressure–volume phase plane. The onset of oscillations is shown in two enlargements (left)

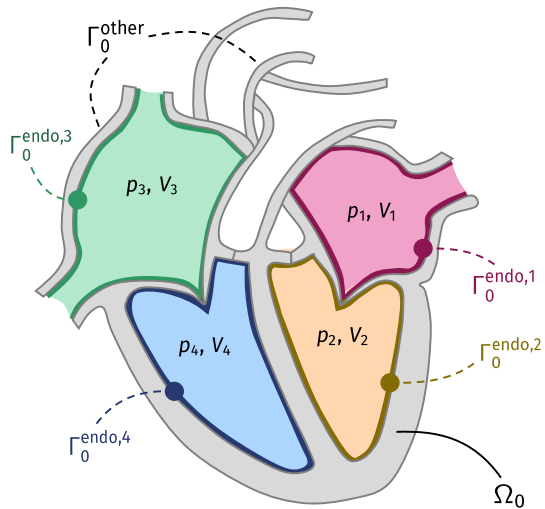
the time step used, on the model parameters (in particular the stiffness, viscosity and inertia associated with the structure) and on the force generation model in use.

A more general approach for developing loosely coupled schemes in FSI problems involves taking linear combinations of the continuity conditions for flow and stress, resulting in schemes based on Robin-type transmission conditions [4, 9–11, 18, 26]. The Dirichlet–Neumann and Neumann–Dirichlet schemes can in fact be seen as particular cases of the Robin–Robin scheme, as well as other schemes obtained by combining different types of transmission conditions. Among these, the Robin–Neumann scheme is particularly popular for hemodynamic problems, even though its stability and accuracy strongly depend on the value of the Robin coefficient, which can be challenging to tune in practice [8, 25, 26]. In this paper, we extend the Robin–Robin scheme, previously developed for 3D–3D problems, to hybrid-dimensional 0D–3D FSI problems, and we investigate its stability within the context of the problems considered in this work.

This paper has a twofold objective. On the one hand, it aims to provide a theoretical basis that explains the occurrence of the above-mentioned oscillatory phenomena and that can predict for which parameter values and for which time step range the numerical scheme is stable. On the other hand, it proposes a stabilized scheme that cures these non-physical oscillations, while retaining the advantages of loosely coupled schemes and without requiring parameter tuning.

The paper is organized as follows. First, in Sect. 2, we introduce the mathematical models and the numerical schemes considered in this paper. Then, in Sect. 3 we study the numerical stability of the schemes introduced above, by focusing on a minimal model of mechanics that, thanks to its simplicity, allows to shed light on the mechanisms underlying the onset of numerical oscillations. In Sect. 4 we propose a cure

Fig. 3 Representation of the domain and boundary conditions of problem (2) with four cavities ($N = 4$), corresponding to the two atria and the two ventricles



for the observed instability issues, by means of the introduction of a numerically consistent stabilization term. The stabilized scheme is first introduced for the minimal model, and later extended to the general case (including 3D mechanics models), and the stability of the scheme is proved under very general hypotheses on the mechanics model. Then, in Sect. 5, we present and discuss several numerical results obtained, on the one hand, with a model FSI problem and, on the other hand, with a multiscale and multiphysics model of cardiac electromechanics. Finally, we draw our conclusions and final remarks in Sect. 6.

2 Mathematical models and numerical schemes

In this section we introduce a generic model of cardiac mechanics and blood dynamics, encompassing many of the models available in the literature [3, 30, 40, 53] and the associated numerical schemes that will be addressed in the sequel of this paper.

2.1 Mathematical models

We consider a computational domain Ω_0 , representing the region of space occupied by the muscle tissue at rest (see Fig. 3). The domain Ω_0 can represent a 4-chamber whole heart, or a part of it, composed of $N \leq 4$ chambers. To each point of Ω_0 , we associate a unit vector $\mathbf{f}_0$, denoting the local direction of cardiac fibers. Let us denote by $\mathbf{d}(\mathbf{x}, t)$ the displacement of the material point $\mathbf{x}$ at time t (we have $\mathbf{d}: \Omega_0 \times [0, +\infty) \rightarrow \mathbb{R}^3$). We then introduce the variables $p_1(t), p_2(t), \dots, p_N(t)$, which represent the (constant in space) blood pressures contained in the N heart chambers. Similarly, we denote by $\Gamma_0^{\text{endo},i} \subset \partial\Omega_0$, for $i = 1, \dots, N$, the domain boundary associated with the i -th chamber (endocardial cavity). Since the cardiac tissue is an active material, we denote by $T_a(\mathbf{x}, t)$ the local active tension that acts in the fiber direction. The active tension can

be *a priori* imposed [3], or provided by means of a suitable cardiomyocyte model [36, 50, 55]. Finally, to model the passive mechanical behavior of the tissue, we consider an hyperelastic model, associated with the strain energy density functional $\mathcal{W}$. Hence, the Piola-Kirchhoff stress tensor reads

$$\mathbf{P}(\mathbf{d}, T_a) = \frac{\partial \mathcal{W}(\mathbf{F})}{\partial \mathbf{F}} + T_a \frac{\mathbf{F}\mathbf{f}_0 \otimes \mathbf{f}_0}{\sqrt{\mathcal{I}_{4f}}}, \quad (1)$$

where the invariant $\mathcal{I}_{4f} = \mathbf{F}\mathbf{f}_0 \cdot \mathbf{F}\mathbf{f}_0$ is a measure of the tissue stretch along the fiber direction. We remark that the scaling factor $\sqrt{\mathcal{I}_{4f}}$ guarantees that T_a corresponds to the active force per unit area (in reference coordinates) along the fibers direction, thus allowing to link microscale models of active force generation with the macroscale (see e.g. [48, Sec. 7.2.2] and [44, 60]). In conclusion, the dynamics of the displacement field $\mathbf{d}$ is described by the following equation, expressing the balance of momentum for the continuum [42]:

$$\left\{ \begin{array}{ll} \rho \frac{\partial^2 \mathbf{d}}{\partial t^2} - \nabla \cdot \mathbf{P}(\mathbf{d}, T_a) = \mathbf{0}, & \text{in } \Omega_0 \times (0, +\infty), \\ \mathbf{P}(\mathbf{d}, T_a)\mathbf{N} = -p_i \mathbf{J}\mathbf{F}^{-T}\mathbf{N} & \text{on } \Gamma_0^{\text{endo},i} \times (0, +\infty), \text{ for } i = 1, \dots, N, \\ + \text{boundary conditions} & \text{on } \Gamma_0^{\text{other}} \times (0, +\infty), \\ \mathbf{d} = \mathbf{d}_0, \quad \frac{\partial \mathbf{d}}{\partial t} = \mathbf{0}, & \text{in } \Omega_0 \times \{0\}, \end{array} \right. \quad (2)$$

where $\mathbf{N}$ denotes the outer unit normal vector and where suitable boundary conditions are assigned on the remaining portion of the boundary $\Gamma_0^{\text{other}} = \partial\Omega_0 \setminus \bigcup_{i=1}^N \Gamma_0^{\text{endo},i}$. Typically, spring-dashpot boundary conditions are considered, to account for the presence of the pericardium (see e.g. [53, 60]).

For the sake of brevity, in what follows we will write Eq. (2) in the following compact form, by denoting by $\mathbf{p}(t) = (p_1(t), p_2(t), \dots, p_N(t))$ the vector of the N pressures:

$$\left\{ \begin{array}{ll} \mathcal{L}(\ddot{\mathbf{d}}(t), \dot{\mathbf{d}}(t), \mathbf{d}(t), \mathbf{p}(t), t) = \mathbf{0} & t > 0, \\ \mathbf{d}(0) = \mathbf{d}_0, \quad \dot{\mathbf{d}}(0) = \mathbf{0}, \end{array} \right. \quad (3)$$

In the structure mechanics model (3), henceforth called model $\mathcal{S}$, by $\mathcal{L}$ we denote a differential operator that encodes both the PDE and the associated boundary conditions. Moreover, $\dot{\mathbf{d}}$ and $\ddot{\mathbf{d}}$ denote the first and second time derivatives. These terms are associated with viscosity and inertia, respectively. In particular, viscous effects can be due either to the boundary conditions or to the material constitutive model. Some authors neglect these terms, i.e. consider a quasi-static formulation, which can be obtained as a special case of (3) [44]. In what follows we will investigate the effects of neglecting inertia, viscosity or both in the structure mechanics model, in terms of numerical stability of the discrete scheme.

To close the model $\mathcal{S}$, we couple it with a lumped-parameter model of fluid dynamics (denoted as model $\mathcal{F}$), describing the vessels surrounding the heart, or even the whole

circulatory system. We consider an ODE model in the form of

$$\begin{cases} \dot{\mathbf{c}}(t) = \mathbf{H}(\mathbf{c}(t), \mathbf{p}(t), t) & t > 0, \\ \mathbf{c}(0) = \mathbf{c}_0, \end{cases} \quad (4)$$

where $\mathbf{c}(t)$ is the state vector of the model $\mathcal{F}$ (including pressures, volume and fluxes of the circulatory compartments represented within the model). For concrete examples of (4), see [7, 30, 53].

To couple the models $\mathcal{S}$ and $\mathcal{F}$, we consider the following pairing condition, that enforces the correspondence of the cardiac chamber volumes represented within the two models:

$$\mathbf{V}_{\mathcal{S}}(\mathbf{d}(t)) = \mathbf{V}_{\mathcal{F}}(\mathbf{c}(t)) \quad t > 0, \quad (5)$$

where $\mathbf{V}_{\mathcal{S}} = (V_{\mathcal{S}}^1, V_{\mathcal{S}}^2, \dots, V_{\mathcal{S}}^N)$ and $\mathbf{V}_{\mathcal{F}} = (V_{\mathcal{F}}^1, V_{\mathcal{F}}^2, \dots, V_{\mathcal{F}}^N)$ are two vectors collecting the blood volumes associate with the N cardiac chambers, as predicted by the $\mathcal{S}$ and $\mathcal{F}$ models respectively. Hence, the coupled FSI model $\mathcal{F}$ – $\mathcal{S}$ reads

$$\begin{cases} \mathcal{L}(\ddot{\mathbf{d}}(t), \dot{\mathbf{d}}(t), \mathbf{d}(t), \mathbf{p}(t), t) = \mathbf{0} & t > 0, \\ \dot{\mathbf{c}}(t) = \mathbf{H}(\mathbf{c}(t), \mathbf{p}(t), t) & t > 0, \\ \mathbf{V}_{\mathcal{S}}(\mathbf{d}(t)) = \mathbf{V}_{\mathcal{F}}(\mathbf{c}(t)) & t > 0, \\ \mathbf{d}(0) = \mathbf{d}_0, \dot{\mathbf{d}}(0) = \mathbf{0}, \mathbf{c}(0) = \mathbf{c}_0. \end{cases} \quad (6)$$

We remark that the blood pressures $\mathbf{p}(t)$ can be interpreted in this context as Lagrange multipliers that enforce the volume compatibility between the models $\mathcal{S}$ and $\mathcal{F}$ at each time $t > 0$. At each time step, indeed, the structural model leads to a saddle-point problem in the variables $(\mathbf{d}, \mathbf{p})$. Details on the numerical treatment of this problem are reported in “Appendix D”.

2.2 Numerical schemes

We now deal with the numerical approximation of the FSI model $\mathcal{F}$ – $\mathcal{S}$. Since our goal is to investigate the numerical stability properties of the time discretization scheme, we do not focus on any particular space discretization technique. For example, we observe that Eq. (6) can be interpreted as a system of ODEs resulting from the semi-discretization in space of the continuous model, using, e.g., the Finite Element Method [13]. Furthermore, for simplicity, we consider a first-order finite difference scheme to advance the individual models in time.

We consider a uniform sampling of the time interval with time step $\Delta t > 0$. Specifically, we define $t_k = k \Delta t$, and we denote with superscript (k) the approximation of a given variable at the k -th time iteration (e.g., we have $\mathbf{d}^{(k)} \approx \mathbf{d}(t_k)$).

Fully coupled scheme

One solution approach is to address at each time step the system:

$$\begin{cases} \mathcal{L} \left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}, t_{k+1} \right) = \mathbf{0} & k = 0, 1, \dots \\ \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(\star)}, \mathbf{p}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \mathbf{V}_S(\mathbf{d}^{(k+1)}) = \mathbf{V}_F(\mathbf{c}^{(k+1)}) & k = 0, 1, \dots \end{cases} \quad (7)$$

with the following initial conditions (we shall henceforth consider these conditions to be understood)

$$\mathbf{d}^{(0)} = \mathbf{d}^{(-1)} = \mathbf{d}_0, \quad \mathbf{c}^{(0)} = \mathbf{c}_0. \quad (8)$$

Concerning the time advance of the two sub-models $\mathcal{S}$ and $\mathcal{F}$, we consider a fully implicit scheme for the former (due to its stiffness [24]), while for the latter we consider either an implicit scheme (setting $(\star) = (k+1)$) or an explicit one (setting $(\star) = (k)$).

ND scheme: Neumann (fluid)–Dirichlet (solid)

An alternative approach is to solve the two models in a loosely coupled way. A possible instance is:

$$\begin{cases} \mathcal{L} \left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}, t_{k+1} \right) = \mathbf{0} & k = 0, 1, \dots \\ \mathbf{V}_S(\mathbf{d}^{(k+1)}) = \mathbf{V}_F(\mathbf{c}^{(k)}) & k = 0, 1, \dots \\ \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(\star)}, \mathbf{p}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \end{cases} \quad (9)$$

At each time step we first update the variable $\mathbf{d}^{(k+1)}$, under the volume constraint $\mathbf{V}_S(\mathbf{d}^{(k+1)}) = \mathbf{V}_F(\mathbf{c}^{(k)})$. For this condition to be satisfied, the multiplier $\mathbf{p}^{(k+1)}$ must necessarily be determined simultaneously with $\mathbf{d}^{(k+1)}$. Successively, we update the variable $\mathbf{c}^{(k+1)}$ by employing the pressure vector $\mathbf{p}^{(k+1)}$ determined as above.

We denote the scheme as *Neumann–Dirichlet (ND)* scheme, as the fluid variable $\mathbf{c}$ is updated by assigning the pressure (which, in this context, can be regarded as a stress on the fluid), while the solid variable $\mathbf{d}$ is updated by assigning the volume (which amounts to prescribe the displacement).

DN scheme: Dirichlet (fluid)–Neumann (solid)

The dual of the ND scheme of Eq. (9) reads

$$\begin{cases} \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(*)}, \mathbf{p}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) = \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k)}) & k = 0, 1, \dots \\ \mathcal{L}\left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}, t_{k+1}\right) = \mathbf{0} & k = 0, 1, \dots \end{cases} \quad (10)$$

Differently to the ND scheme, here the pressures $\mathbf{p}^{(k+1)}$ are determined simultaneously with the variable $\mathbf{c}^{(k+1)}$ of model $\mathcal{F}$, and are then passed to the model $\mathcal{S}$. For this reason, we call this scheme Dirichlet–Neumann (DN), since the fluid subproblem is solved with assigned volume (that is by controlling the displacement), while the structural subproblem is solved with assigned stress at the boundary.

RR schemes: Robin (fluid)–Robin (solid)

To introduce loosely coupled schemes based on Robin-type transmission conditions in the current setting, distinct sets of variables must be employed to track the pressures associated with the fluid and with the structure model, say $\mathbf{p}_{\mathcal{F}}$ and $\mathbf{p}_{\mathcal{S}}$. Then, by taking a linear combination between the volume compatibility condition (5) and the condition $\mathbf{p}_{\mathcal{F}}(t) = \mathbf{p}_{\mathcal{S}}(t)$, we get the following loosely coupled Robin–Robin scheme:

$$\begin{cases} \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(*)}, \mathbf{p}_{\mathcal{F}}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) + \beta_{\mathcal{F}} \mathbf{p}_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k)}) + \beta_{\mathcal{F}} \mathbf{p}_{\mathcal{S}}^{(k)} & k = 0, 1, \dots \\ \mathcal{L}\left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}_{\mathcal{S}}^{(k+1)}, t_{k+1}\right) = \mathbf{0} & k = 0, 1, \dots \\ \alpha_{\mathcal{S}} \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k+1)}) + \beta_{\mathcal{S}} \mathbf{p}_{\mathcal{S}}^{(k+1)} = \alpha_{\mathcal{S}} \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) + \beta_{\mathcal{S}} \mathbf{p}_{\mathcal{F}}^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (11)$$

where $\alpha_{\mathcal{F}}$, $\beta_{\mathcal{F}}$, $\alpha_{\mathcal{S}}$ and $\beta_{\mathcal{S}}$ are real parameters. For consistency with the continuous problem of Eq. (6), the two conditions must be linearly independent, that is we require $\alpha_{\mathcal{F}}\beta_{\mathcal{S}} \neq \alpha_{\mathcal{S}}\beta_{\mathcal{F}}$. We notice that in (11) the update order can be reversed, solving the $\mathcal{S}$ subproblem first and then the $\mathcal{F}$ subproblem, leading in practice simply to a shift of one position of the indices (see “Appendix A”).

For simplicity, we characterize the scheme using the parameters $\gamma_{\mathcal{F}} = \alpha_{\mathcal{F}}/\beta_{\mathcal{F}}$ and $\gamma_{\mathcal{S}} = \alpha_{\mathcal{S}}/\beta_{\mathcal{S}}$, taking values in $[-\infty, +\infty]$. The choice $\gamma_* = 0$ (with $*$ = $\mathcal{F}$, $\mathcal{S}$) corresponds to a Neumann transmission condition, while the choice $\gamma_* = \pm\infty$ (regardless of sign) corresponds to a Dirichlet transmission condition. Indeed, both the ND scheme (9) and the DN scheme (10) can be recovered as particular cases of the

scheme (11), obtained by setting, respectively, $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}}) = (0, \pm\infty)$ and $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}}) = (\pm\infty, 0)$ (please notice that, for the ND scheme, it is convenient to consider the version of the V-RR scheme reported in (48)).

We also consider an alternative scheme, wherein the volume compatibility condition (5) is replaced by a condition on the fluxes, namely on the time derivative of volumes, discretized by means of finite differences:

$$\begin{cases} \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(*)}, \mathbf{p}_{\mathcal{F}}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} \frac{\mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) - \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k)})}{\Delta t} + \beta_{\mathcal{F}} \mathbf{p}_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} \frac{\mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k)}) - \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k-1)})}{\Delta t} + \beta_{\mathcal{F}} \mathbf{p}_{\mathcal{S}}^{(k)} & k = 0, 1, \dots \\ \mathcal{L}\left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}_{\mathcal{S}}^{(k+1)}, t_{k+1}\right) = \mathbf{0} & k = 0, 1, \dots \\ \alpha_{\mathcal{S}} \frac{\mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k+1)}) - \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k)})}{\Delta t} + \beta_{\mathcal{S}} \mathbf{p}_{\mathcal{S}}^{(k+1)} = \alpha_{\mathcal{S}} \frac{\mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) - \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k)})}{\Delta t} + \beta_{\mathcal{S}} \mathbf{p}_{\mathcal{F}}^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (12)$$

As they are based on the continuity of volumes and of fluxes, respectively, we denote Eq. (11) by V-RR scheme and Eq. (12) by Q-RR scheme. We remark that, by setting $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}}) = (0, \pm\infty)$ and $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}}) = (\pm\infty, 0)$, both the V-RR and the Q-RR reduce to the ND and DN schemes. However, when non-degenerate Robin transmission conditions are used – meaning that either $\gamma_{\mathcal{F}}$ or $\gamma_{\mathcal{S}}$ is nonzero and finite – the two schemes yield different results and, as we will later demonstrate, may exhibit different stability and accuracy properties.

3 Stability analysis

In this section, we study the numerical stability of the schemes presented in Sect. 2.2. With this goal, we introduce a minimal model, that is a simplified model of cardiac mechanics that features a minimal degree of complexity, but sufficient to reproduce a variety of behaviors exhibited by more comprehensive models. The aim of the minimal model is to gain insight on the mechanisms behind the onset of oscillations, which will be leveraged to design a stable scheme in Sect. 4. Nonetheless, it is important to emphasize that the scope of this paper extends beyond the minimal model presented here; in fact, in Sect. 4.3, the main results will be extended to the general case of the model of Eq. (2).

3.1 A minimal model of cardiac mechanics

To define the minimal model $\mathcal{F}$ – $\mathcal{S}$, we consider a zero-dimensional description of a single cardiac chamber, depicted in Fig. 1a. More precisely, we consider the following model of cardiac mechanics (model $\mathcal{S}$), where $V_{\mathcal{S}}(t)$ denotes the chamber volume and represents the unique state variable

$$\begin{cases} M\ddot{V}_S(t) + C\dot{V}_S(t) + p_{\text{pass}}(V_S(t)) + A T_a(t) = p(t) & t > 0, \\ V_S(0) = V_0, \dot{V}_S(0) = 0. \end{cases} \quad (13)$$

The model is obtained from the 3D model described in Sect. 2.1 by means of suitable simplifying assumptions (see “Appendix B” for the detailed derivation). This model accounts for both inertia and viscous damping, through the coefficients $M \geq 0$ and $C \geq 0$, respectively. The coefficient $A \geq 0$ is associated with the active force generated by the muscle. Finally $p_{\text{pass}}(V_S)$ represents the steady-state pressure corresponding to a given volume V_S when the chamber is fully relaxed (i.e. $T_a = 0$). It thus corresponds to the end-diastolic (ED) pressure-volume relationship, also known as Klotz’s curve [33]. The function p_{pass} should be constitutively prescribed and it represents the 0D fingerprint of the 3D hyperelastic constitutive law.

Concerning the model $\mathcal{F}$, we connect the cardiac chamber to a valve and we consider a prescribed external pressure $p_{\text{ext}}(t)$. Thus, the chamber volume $V_{\mathcal{F}}(t)$ evolves according to following simple fluid model (model $\mathcal{F}$):

$$\begin{cases} \dot{V}_{\mathcal{F}}(t) = Q_{\text{valve}}(p_{\text{ext}}(t) - p(t)) & t > 0, \\ V_{\mathcal{F}}(0) = V_0, \end{cases} \quad (14)$$

where Q_{valve} is given by either $Q_{\text{valve}}^{\text{out}}$ or $Q_{\text{valve}}^{\text{in}}$:

$$Q_{\text{valve}}^{\text{out}}(\Delta p) = \begin{cases} \frac{\Delta p}{R_{\min}}, & \text{if } \Delta p < 0, \\ \frac{\Delta p}{R_{\max}}, & \text{if } \Delta p \geq 0, \end{cases}, \quad Q_{\text{valve}}^{\text{in}}(\Delta p) = \begin{cases} \frac{\Delta p}{R_{\max}}, & \text{if } \Delta p < 0, \\ \frac{\Delta p}{R_{\min}}, & \text{if } \Delta p \geq 0, \end{cases} \quad (15)$$

where $R_{\max} \gg 1$ and $R_{\min} \ll 1$ represent the hemodynamic resistances corresponding to a closed and open valve, respectively. The function $Q_{\text{valve}}^{\text{out}}$ (respectively, $Q_{\text{valve}}^{\text{in}}$) describes an outflow (respectively, inflow) valve. Indeed, the valve is open in case of negative (respectively, positive) pressure gradient and vice versa.

Hence, the coupled minimal model $\mathcal{F}$ – $\mathcal{S}$ reads

$$\begin{cases} M\ddot{V}_S(t) + C\dot{V}_S(t) + p_{\text{pass}}(V_S(t)) + A T_a(t) = p(t) & t > 0, \\ \dot{V}_{\mathcal{F}}(t) = Q_{\text{valve}}(p_{\text{ext}}(t) - p(t)) & t > 0, \\ V_S(t) = V_{\mathcal{F}}(t) & t > 0, \\ V_S(0) = V_0, \dot{V}_S(0) = 0, V_{\mathcal{F}}(0) = V_0, \end{cases} \quad (16)$$

which represents a particular case of Eq. (6), provided we adopt the following identification for the (one-dimensional) state variables $\mathbf{d}(t) = [V_S(t)]$ and $\mathbf{c}(t) = [V_{\mathcal{F}}(t)]$. A further abstraction of the cardiac mechanics minimal model (16) is the piston problem in Fig. 1b. This is formally obtained from the model (16), by employing the linearly elastic law $p_{\text{pass}}(V_S) = E(V_S - V_0)$, where $E > 0$ is known as elastance and V_0 is

the resting volume, by neglecting active force and by assuming a constant fluid flow resistance R .

The ND scheme applied to the minimal model (16) reads

$$\begin{cases} M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a(t_{k+1}) = p^{(k+1)} & k = 0, 1, \dots \\ V_S^{(k+1)} = V_{\mathcal{F}}^{(k)} & k = 0, 1, \dots \\ \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p^{(k+1)}) & k = 0, 1, \dots \end{cases} \quad (17)$$

with initial conditions

$$V_S^{(0)} = V_S^{(-1)} = V_{\mathcal{F}}^{(0)} = V_0.$$

Conversely, the DN scheme reads (with the same initial conditions as above)

$$\begin{cases} \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p^{(k+1)}) & k = 0, 1, \dots \\ V_{\mathcal{F}}^{(k+1)} = V_S^{(k)} & k = 0, 1, \dots \\ M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a(t_{k+1}) = p^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (18)$$

The V-RR scheme is given by

$$\begin{cases} \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p_{\mathcal{F}}^{(k+1)}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} V_{\mathcal{F}}^{(k+1)} + \beta_{\mathcal{F}} p_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} V_S^{(k)} + \beta_{\mathcal{F}} p_S^{(k)} & k = 0, 1, \dots \\ M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a(t_{k+1}) = p_S^{(k+1)} & k = 0, 1, \dots \\ \alpha_S V_S^{(k+1)} + \beta_S p_S^{(k+1)} = \alpha_S V_{\mathcal{F}}^{(k+1)} + \beta_S p_{\mathcal{F}}^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (19)$$

with same initial condition as above, supplemented by an initial condition for the pressure $p_S^{(0)} = p_0$. Similarly, the Q-RR scheme reads as follows (with the same initial condition as the V-RR scheme):

$$\begin{cases} \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p_{\mathcal{F}}^{(k+1)}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} + \beta_{\mathcal{F}} p_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} \frac{V_S^{(k)} - V_S^{(k-1)}}{\Delta t} + \beta_{\mathcal{F}} p_S^{(k)} & k = 0, 1, \dots \\ M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a(t_{k+1}) = p_S^{(k+1)} & k = 0, 1, \dots \\ \alpha_S \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + \beta_S p_S^{(k+1)} = \alpha_S \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} + \beta_S p_{\mathcal{F}}^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (20)$$

3.2 Definitions and lemmas

Before studying the numerical stability of the schemes (17)–(20), we recall the definition of zero-stability and absolute stability of a numerical scheme [45]. The definitions refer to the following abstract problem, where $\psi^{(k)} \in \mathbb{R}^d$ denotes the state variable at time t_k :

$$\begin{cases} \psi^{(k)} = \Phi(\psi^{(k-1)}, t_k, \Delta t) & k = 1, 2, \dots \\ \psi^{(0)} = \psi_0, \end{cases} \quad (21)$$

We notice that both the ND and DN schemes are particular cases of Eq. (21), as we show in Sect. 3.3. We then consider the perturbed problem:

$$\begin{cases} \tilde{\psi}^{(k)} = \Phi(\tilde{\psi}^{(k-1)}, t_k, \Delta t) + \Delta t \, \eta^{(k)} & k = 1, 2, \dots \\ \tilde{\psi}^{(0)} = \psi_0 + \eta^{(0)}, \end{cases} \quad (22)$$

where $\eta^{(k)}$ denotes a suitable perturbation. We consider the following definitions, adapted from [45] to account for the second order nature of the problem at hand.

Definition 1 (zero-stability) Let us consider a finite time $T > 0$ and a uniform subdivision of the time interval $(0, T)$ into N subintervals, i.e. $\Delta t = T/N$. The numerical scheme (21) is said *zero-stable* if

$$\exists \Delta t_0 > 0, \exists C > 0 : \quad \sup_{k=0, \dots, N} |\tilde{\psi}^{(k)} - \psi^{(k)}| \leq C(T) \sup_{k=0, \dots, N} |\eta^{(k)}| \quad \forall \Delta t \in (0, \Delta t_0],$$

where $\psi^{(k)}$ and $\tilde{\psi}^{(k)}$ are solutions to problems (21) and (22), respectively.

Definition 2 (absolute stability) Let us consider a given time step size $\Delta t > 0$. We consider the solution of the numerical problem for $t_k \rightarrow +\infty$. The numerical scheme (21) is said *absolutely stable* in correspondence to Δt if

$$\exists C > 0 : \quad \lim_{k \rightarrow +\infty} |\tilde{\psi}^{(k)} - \psi^{(k)}| \leq C \sup_{k \geq 0} |\eta^{(k)}|,$$

where $\psi^{(k)}$ and $\tilde{\psi}^{(k)}$ are solutions to problems (21) and (22), respectively. Moreover, we say that the method is *unconditionally absolutely stable* if it is absolutely stable for any $\Delta t > 0$.

The zero-stability property of a scheme guarantees that its solution does not blow up on a finite time horizon when $\Delta t \rightarrow 0^+$. This in particular entails that the noise introduced by round-off errors is not amplified. The absolute stability property deals instead with the behavior of the solution for $t \rightarrow +\infty$. We remark that unconditional absolute stability implies zero-stability [45].

The stability properties of a numerical scheme in the form (21) can be linked with the spectral properties of the Jacobian matrix of the iteration function Φ . Specifically,

they are linked with its spectral radius (i.e. the modulus of the largest eigenvalue), denoted by $\rho(\cdot)$. We recall the following result, proved in [52]:

Lemma 1 (Prop. 1 of [52]) *Let us consider the numerical scheme (21) with $\Phi: \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ differentiable with respect to its first argument. If the condition*

$$\exists \Delta t_0 > 0, \exists \alpha \in \mathbb{R}: \quad \rho \left(\frac{\partial \Phi(\psi, t, \Delta t)}{\partial \psi} \right) \leq 1 + \alpha \Delta t \quad \forall \psi \in \mathbb{R}^d, t \geq 0, \Delta t \in (0, \Delta t_0] \quad (23)$$

holds true, then the scheme (21) is zero-stable. Moreover, if for a given Δt the condition

$$\exists \rho_0 < 1: \quad \rho \left(\frac{\partial \Phi(\psi, t, \Delta t)}{\partial \psi} \right) \leq \rho_0 \quad \forall \psi \in \mathbb{R}^d, t \geq 0, \quad (24)$$

holds true, then the scheme is absolutely stable in correspondence to Δt .

We have then the following result.

Lemma 2 *Let us suppose that the iteration function Φ in Eq. (21) is assigned implicitly by*

$$\mathcal{H}(\psi^{(k)}, \psi^{(k-1)}, t_k, \Delta t) = \mathbf{0} \quad k = 1, 2, \dots \quad (25)$$

where $\mathcal{H}: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^d$ is continuously differentiable in its first argument and differentiable with respect to the second one. We define, for $\psi_2, \psi_1 \in \mathbb{R}^d, t > 0, \Delta t > 0$:

$$\begin{aligned} \mathbf{A}(\psi_2, \psi_1, t, \Delta t) &:= \frac{\partial \mathcal{H}(\psi_2, \psi_1, t, \Delta t)}{\partial \psi_2} \in \mathbb{R}^{d \times d}, \\ \mathbf{B}(\psi_2, \psi_1, t, \Delta t) &:= \frac{\partial \mathcal{H}(\psi_2, \psi_1, t, \Delta t)}{\partial \psi_1} \in \mathbb{R}^{d \times d}. \end{aligned} \quad (26)$$

Then, if $\mathbf{A}(\psi_2, \psi_1, t, \Delta t)$ is non-singular, the Jacobian matrix of the iteration function is given by

$$\frac{\partial \Phi(\psi_1, t, \Delta t)}{\partial \psi_1} = -\mathbf{A}^{-1}(\psi_2, \psi_1, t, \Delta t) \mathbf{B}(\psi_2, \psi_1, t, \Delta t)$$

where $\psi_2 = \Phi(\psi_1, t, \Delta t)$.

Proof This is a direct consequence of the implicit function theorem (see e.g. [34]). $\square$

Thanks to Lemmas 1 and 2, both zero-stability and absolute stability of a numerical scheme assigned in implicit form (such as the ND and DN schemes of Eqs. (9) and (10)) can be linked with the spectral properties of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$. In particular, a spectral radius strictly lower than 1 entails absolute stability (and hence zero-stability).

Table 1 Reference parameters of the minimal model of Eq. (13). The numerical values are obtained by means of a 0D reduction of a realistic and anatomically detailed 3D cardiac mechanics model. The procedure is detailed in “Appendix B”

Parameter	Measure unit	Value
$\bar{E}_{\text{pass}}$	mmHg mL ⁻¹	0.09
$\bar{E}_{\text{act}}$	mmHg mL ⁻¹	36
$\bar{C}$	mmHg mL ⁻¹ ms	24
$\bar{M}$	mmHg mL ⁻¹ ms ²	8.6
R_{max}	mmHg mL ⁻¹ s	$7.5 \cdot 10^4$
R_{min}	mmHg mL ⁻¹ s	$7.5 \cdot 10^{-3}$

3.3 Spectral analysis

We now study, on the basis of the results of Sect. 3.2, the numerical stability of the ND, DN, V-RR and Q-RR schemes of Eqs. (17)–(20). We observe that the ND and DN schemes can be recast in the abstract form (21), provided that we define, for $k \geq 0$, the state vector $\psi^{(k)} := (V_S^{(k)}, V_S^{(k-1)}, V_{\mathcal{F}}^{(k)}, p^{(k)})^T$. The matrices **A** and **B**, for the ND and DN schemes, are respectively given by

$$\mathbf{A}_{\text{ND}} = \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} & -\frac{M}{\Delta t^2} & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Delta t^{-1} & R^{-1} \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{B}_{\text{ND}} = \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) & \frac{M}{\Delta t^2} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix};$$

$$\mathbf{A}_{\text{DN}} = \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} & -\frac{M}{\Delta t^2} & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \Delta t^{-1} & R^{-1} \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \mathbf{B}_{\text{DN}} = \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) & \frac{M}{\Delta t^2} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (27)$$

We have defined

$$E(V_S) = E_{\text{pass}}(V_S) = p'_{\text{pass}}(V_S)$$

as the elastance of the cardiac chamber, while R denotes either R_{min} or R_{max} , depending on whether the valve is open or closed. As E and R depend on the current state, also **A** depends on the state. Nonetheless, we will leave this dependence as understood.

Similarly, to recast the V-RR and Q-RR schemes in the abstract form (21), we introduce the state vector $\psi^{(k)} := (V_S^{(k)}, V_S^{(k-1)}, V_{\mathcal{F}}^{(k)}, V_{\mathcal{F}}^{(k-1)}, p_S^{(k)}, p_{\mathcal{F}}^{(k)})^T$. The corresponding matrices are reported in “Appendix A”. In “Appendix A” we also show that, as expected, the matrices associated with the ND and DN schemes can be recovered from those associated with Robin–Robin schemes, by suitable choosing the Robin coefficients.

We now numerically study the spectrum of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$, for different parameters values. As reference values, we consider those given in Tab. 1, which correspond to realistic values for the mechanics of a human left ventricle (see

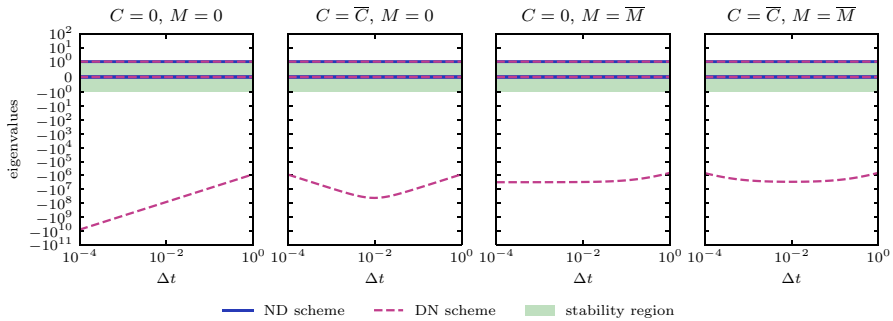


Fig. 4 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and DN schemes in the closed valve case ($R = R_{\max}$, $E = \bar{E}_{\text{pass}}$). From left to right: quasi-static case, only viscosity enabled, only inertia enabled, both viscosity and inertia enabled

“Appendix B”). In particular, we investigate the effect of inertia ($M \neq 0$) and viscous damping ($C \neq 0$), compared to a quasi-static formulation of the structure mechanics problem ($M = C = 0$). First, in Sect. 3.3.1, we consider the case of closed valve ($R = R_{\max}$); then, in Sect. 3.3.2, that of open valve ($R = R_{\min}$). We start by addressing the ND and DN scheme, and successively extend our analysis to the more general case of RR schemes.

3.3.1 Closed valve

Let us consider the case of a closed valve (which corresponds to $R = R_{\max}$). In Fig. 4 we show the dependence of the eigenvalues of $-\mathbf{A}^{-1}\mathbf{B}$ on the time step size Δt , in the quasi-static case and when adding inertia and/or viscosity. We observe that in all the cases, the ND scheme has all eigenvalues with modulus less than one and it is therefore numerically stable. The DN scheme, instead, has a very large negative eigenvalue ($\mu < -10^5$). Adding inertia and viscosity slightly lowers its magnitude, which however is still very high.

The instability of the DN scheme when the valve is closed is due to the balloon dilemma [5, 27], that is the impossibility of satisfying the incompressibility constraint when the pressure is assigned. As a matter of fact, by considering for simplicity the quasi-static case ($C = M = 0$), the eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ read

$$\mu_{\pm} = \frac{R_{\max}}{2E\Delta t} \left(\pm \sqrt{1 + \frac{4E\Delta t}{R_{\max}}} - 1 \right).$$

When we approach the incompressible limit (i.e. for $R_{\max} \rightarrow +\infty$), $\mu_{+} \rightarrow 1^{-}$ and $\mu_{-} \rightarrow -\infty$. Hence, the more the valve resists blood flow, the greater the magnitude of the numerical oscillations will be, because of the eigenvalue $\mu_{-} \rightarrow -\infty$. This provides in fact an algebraic interpretation of the balloon dilemma.

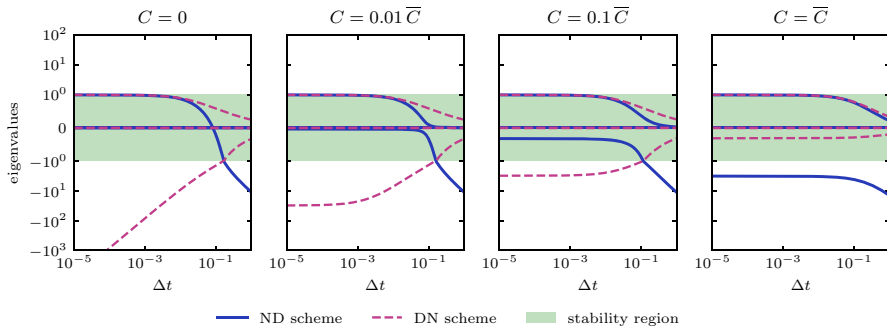


Fig. 5 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and DN schemes in the open valve case and without inertia ($R = R_{\min}$, $E = \bar{E}_{\text{pass}}$, $M = 0$), for increasing viscosity (see titles)

3.3.2 Open valve

Let us now consider the case of an open valve (corresponding to $R = R_{\min}$). In the case of quasi-static mechanics, as shown in the left-most plot of Fig. 5, the ND scheme is absolutely stable for small Δt , and unstable for large Δt , while the DN scheme has opposite behavior. More precisely, there exists a critical Δt that acts as threshold of absolute stability for the two schemes ND and DN (for the parameter values here considered, it is approximately $\Delta t = 0.1\text{s}$). Moreover, as the zero-stability property of a scheme deals with its behavior for $\Delta t \rightarrow 0^+$ (see Def. 1), the ND scheme, unlike the DN scheme, is zero-stable.

We now study the effect of inertia and viscosity. In Fig. 5 we consider the spectrum in the case $M = 0$ and for increasing viscosity. Similarly, in Fig. 6, we set $C = 0$ and we consider increasing values of M (increasing inertia). The presence of either viscosity or inertia is able to stabilize the DN scheme, whereas it worsens the stability properties of the ND scheme. More precisely, both viscosity and inertia introduce in the spectrum associated with the ND scheme an eigenvalue with negative real part, which, for some values of Δt , has modulus greater than 1 thus making the ND scheme unstable. Moreover, Fig. 5 shows that there is a critical viscosity value, above which the ND scheme is never stable. Instead, inertia makes the ND scheme unstable for low values of Δt (see Fig. 6). Therefore, when inertia is present, the scheme turns out to be absolutely stable only in an interval of values of Δt , which gets narrower and narrower as M increases. We notice that, since for $\Delta t \rightarrow 0^+$ inertia always yields an eigenvalue of modulus greater than 1, the presence of inertia makes the ND scheme to be non zero-stable.

Finally, in Fig. 7 we show the combined effects of viscosity and inertia. This is in line with the results obtained above: the presence of viscosity and inertia improves the stability of the DN scheme and worsens that of the ND scheme.

3.3.3 The effect of active stress

In the minimal model (13), the amount of active stress $T_a(t)$ is prescribed. However, very often in the literature, the active stress is computed by means of suitable models

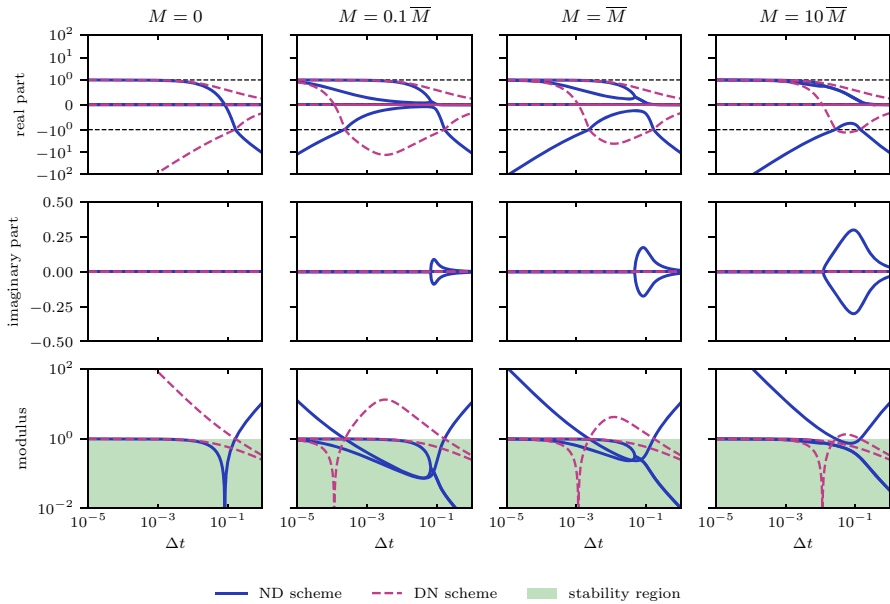


Fig. 6 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and DN schemes in the open valve case and without viscosity ($R = R_{\min}$, $E = \bar{E}_{\text{pass}}$, $C = 0$), for increasing inertia (see titles)

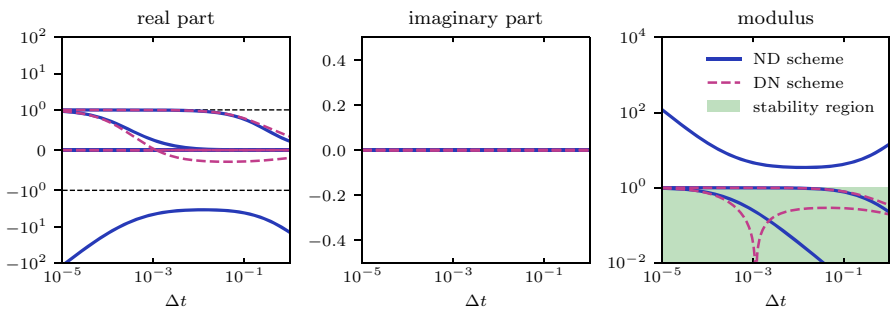


Fig. 7 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and DN schemes in the open valve case, in presence of both viscosity and inertia ($R = R_{\min}$, $E = \bar{E}_{\text{pass}}$, $C = \bar{C}$, $M = \bar{M}$)

describing the subcellular activity of cardiomyocytes [36, 50, 55]. These models are typically characterized by a feedback from the myocardial mechanics model: when the tissue rapidly contracts, the force generated decreases [49]. For this reason, we now investigate the effect that this feedback may have on the stability analysis conducted above.

At the macroscopic level, the effect of the feedback acting from tissue mechanics to force generation is known as force-velocity relationship and was first observed by the Nobel laureate Archibald V. Hill in 1938 [29]. Its origin lies in fact at the microscopic level. The active force is generated by a very large number (approximately 10^8 per mm^3 of tissue) of crossbridges, that is bounded actin-myosin pairs. Each myosin

protein, due to its elasticity, acts as a microscopic spring, thus producing the active force observable at the macroscopic level. The presence of these actin-myosin bonds not only generates a net force in the direction of the cardiac fibres, but also has the effect of opposing rapid changes in fibre length (whence the force-velocity relationship observed by A. V. Hill). Their action therefore results in an increased stiffness of the tissue.

As a matter of fact, the active stiffness (that is the amount of stiffness associated with attached crossbridges) can be computed from the active force generation at hand. Let us consider a generic force generation model, in the following form (that encompasses most of the models available in literature, see e.g. [36, 37, 41, 50, 54])

$$\begin{cases} \dot{\mathbf{r}}(t) = \mathbf{h}(\mathbf{r}(t), c(t), \lambda(t), \dot{\lambda}(t)), & t > 0, \\ T_a(t) = g(\mathbf{r}(t)), & t > 0, \\ \mathbf{r}(0) = \mathbf{r}_0, \end{cases} \quad (28)$$

where $\mathbf{r}(t)$ is the state variables vector, $c(t)$ is the intracellular calcium concentration (an input of the model, often denoted by $[\text{Ca}^{2+}]_i$), while $\lambda = \sqrt{I_{4f}} - 1$ represents the tissue strain in the fiber direction. Finally, $\mathbf{h}$ and g are suitable functions. As shown in [52], the active stiffness (henceforth denoted as K_a) is given by

$$K_a = \frac{\partial \dot{T}_a}{\partial \dot{\lambda}} = \nabla_{\mathbf{r}} g \cdot \frac{\partial \mathbf{h}}{\partial \dot{\lambda}}. \quad (29)$$

To study the effect of using a force generation model as in Eq. (28) on the stability of the numerical schemes considered in this paper, we proceed as follows. First, we consider the coupled model obtained by pairing the minimal model of Eq. (16) with Eq. (28). Then, we study the spectrum of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ associated with the numerical discretization of this coupled model and we apply Lemma 1. In “Appendix C” we report the detailed calculations in this regard and we here report only the final result. We show that adding an active mechanics model has the equivalent effect of increasing the passive elastance E_{pass} . More precisely, we obtain matrices equivalent to Eq. (27), but where the passive elastance E_{pass} is replaced by an augmented elastance given by $E = E_{\text{pass}} + E_{\text{act}}$. As a matter of fact, the active elastance E_{act} is proportional to the active stiffness K_a . This result is not surprising, but on the contrary it is in line with the microscopic interpretation of the force-velocity relationship mentioned above, whereby the latter results from an increase in microscopic stiffness due to attached crossbridges.

The above remarks also makes it possible to apply the results of Sect. 3.3 to the case of using force generation models that take into account the force-velocity relationship, provided that we consider a suitably augmented elastance. In fact, at the peak of systole, active elastance E_{act} is much larger than passive elastance E_{pass} (up to 400 times greater, see “Appendix C”). This implies that using a force generation model with force-velocity relationship, rather than using a prescribed force transient, might substantially impact the numerical stability of the scheme in use.

In Fig. 8 we study the eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for both the

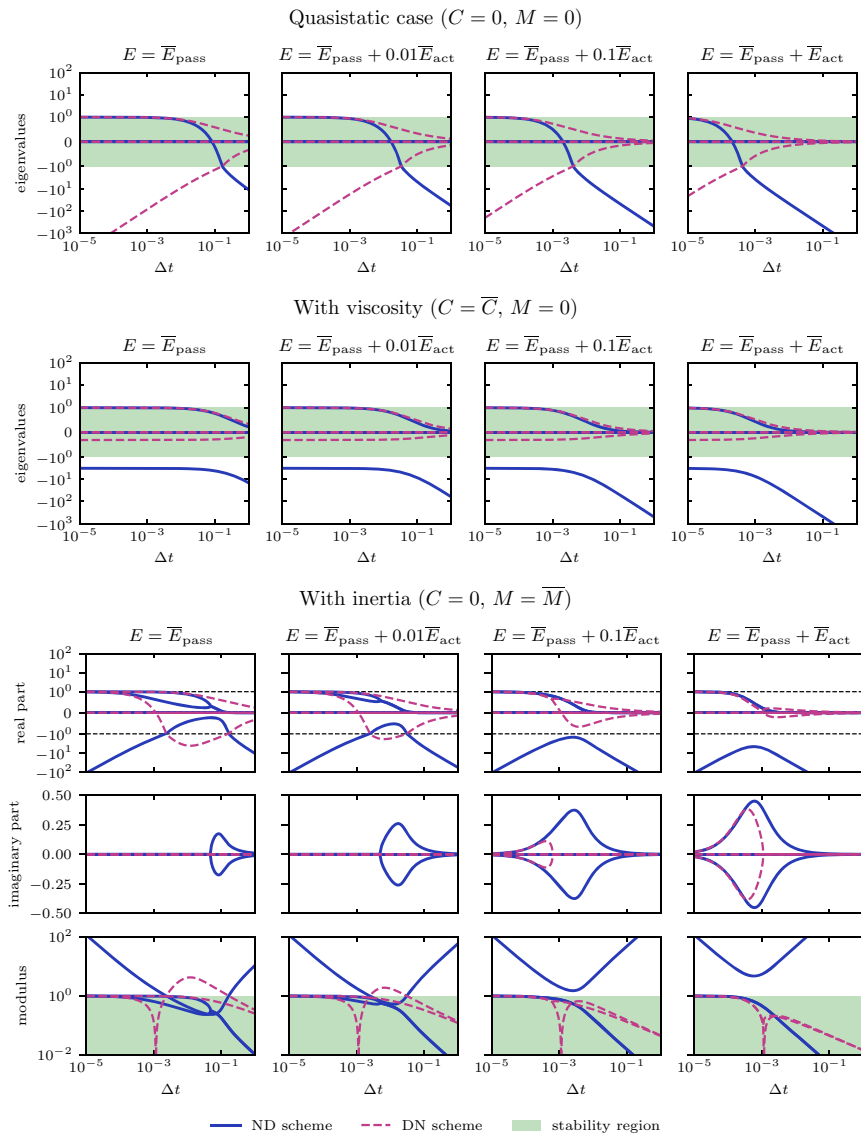


Fig. 8 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and DN schemes in the open valve case ($R = R_{\min}$), for increasing elastance (see titles)

ND and DN schemes for increasing values of the total elastance E . Specifically, we move from the case when the tissue is fully deactivated ($E = \bar{E}_{\text{pass}}$) to the case of full activation ($E = \bar{E}_{\text{pass}} + \bar{E}_{\text{act}}$). Intermediate values correspond to a partial activation of the tissue. The results show that the presence of an active stress better the stability properties of the DN scheme, while it worsen those of the ND scheme. Let us consider for instance the quasi-static case (first row of Fig. 8). When the tissue

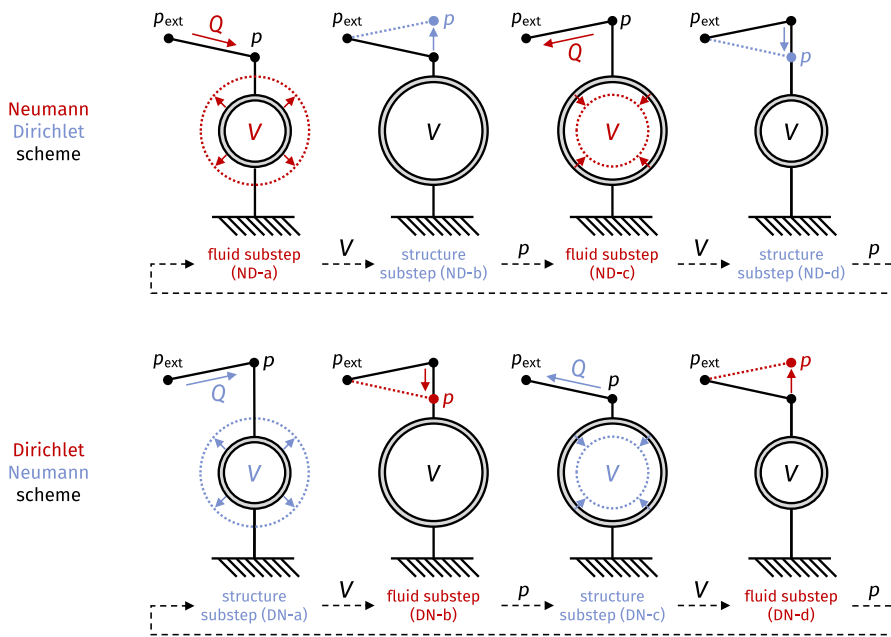


Fig. 9 Schematic representation of the oscillations occurring with the ND scheme (top) and with the DN scheme (bottom). The substeps associated with the solution of the structure mechanics problem (model $\mathcal{S}$) are shown in blue, while those related to the fluid problem (model $\mathcal{F}$) are shown in red. The chamber is represented as a balloon, with a volume V associated with it. The height of the bullet above the balloon symbolizes the associated pressure, while the bullet on the left represents the external pressure. When the latter is higher, there is a positive pressure gradient between the outer and inner regions of the chamber; when it is lower, the pressure gradient is negative

if not activated, the ND scheme is absolutely stable if and only if $\Delta t < \overline{\Delta t}$, where the stability threshold is approximately $\overline{\Delta t} \simeq 2 \cdot 10^{-1} \text{s}$. When the tissue is fully activated, however, the stability threshold is much stricter ($\overline{\Delta t} \simeq 5 \cdot 10^{-4} \text{s}$).

3.4 Analysis of the non-physical oscillations in ND and DN schemes

The spectral analysis of Sect. 3.3 allowed us to identify the ranges of parameters and of Δt for which the ND and DN schemes are either stable or unstable. We now analyze, in light of the results of Sect. 3.3, the non-physical oscillations that can occur using the two ND and DN schemes, when the stability conditions are not met. The occurrence of these oscillations is represented schematically in Fig. 9.

Let us first consider the ND scheme (first row of Fig. 9). At the first step (ND-a), since the external pressure (p_{ext}) is higher than the chamber pressure (p), the fluid model predicts a blood flow Q towards the chamber and a consequent increase in volume V . At the next step (ND-b), the structure mechanics model receives as an input the volume calculated at step ND-a and generates, due to the increase in volume, an increase in pressure. If the time step length Δt is sufficiently large, the pressure increase is such that p is now greater than p_{ext} , thus generating a backflow at the fluid

substep ND-c, which leads to a decrease in volume. At step ND-d the pressure must necessarily drop, thus returning to the state represented in step ND-a and the cycle is repeated. The onset of oscillations is therefore related to the amplitude of pressure jumps occurring in ND-b and ND-d. If these are large enough to reverse the sign of the pressure gradient, nonphysical oscillations occur. It is clear that a large pressure jump is favored by a large value of the elastance E . This is in line with the results in Sect. 3.3. Moreover, when viscosity or inertia are included into the model, the cardiac chamber opposes rapid changes in volume more strongly. Therefore, in ND-b and ND-d, where a volume change is imposed, the larger the inertia or viscosity, the larger the pressure jump with which the model responds. This is also in line with the findings of Sect. 3.3, predicting that, when using the ND scheme, the presence of viscosity or inertia makes it more likely that oscillations will occur.

Conversely, in the DN scheme (second row of Fig. 9), the pressure is determined by the fluid model, which receives volume changes as input from the structure mechanics model. In this case, therefore, large elastance (E), viscosity (C) or stiffness (M) hamper the non-physical oscillations. In fact, each of these factors causes the structure mechanics model to respond with smaller volume changes to a given pressure change, thus preventing numerical instabilities from being generated. This is also consistent with the results of the spectral analysis of Sect. 3.3, whereby the DN scheme enjoys better stability properties when either E , C or M are large.

3.5 Robin–Robin schemes

We now extend our analysis to the V-RR and Q-RR schemes. For brevity, we directly address the general case that includes both viscosity and inertia.

As is typical for Robin–Robin type schemes, numerical stability is highly sensitive to the choice of the parameters $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ [4, 8, 25]. To examine this, we leverage the results from Sect. 3 to study the stability of the V-RR and Q-RR schemes as a function of these parameters. Specifically, we plot in the $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}})$ plane the maximum spectral radius obtained for each scheme, considering the cases $R = R_{\min}$ and $R = R_{\max}$, allowing us to evaluate stability across different regimes (open and closed valve conditions).

The results are summarized in Fig. 10. For each of the four cases considered, the central graph displays a region of the $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}})$ plane on a logarithmic scale, covering the ranges $[-10^3, -10^{-3}]$ and $[10^{-3}, 10^3]$ for both parameters. The boxes on the sides show the spectral radius when either $\gamma_{\mathcal{F}}$ or $\gamma_{\mathcal{S}}$ takes values of $\pm\infty$. Consistent with previous observations, schemes with parameters set to either $-\infty$ or $+\infty$ are in practice equivalent, resulting in identical spectral radii. It is important to note that the cases where both $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ are $\pm\infty$, or where $\gamma_{\mathcal{F}} = \gamma_{\mathcal{S}}$ (corresponding to the bisector of the plane), are not physically meaningful because the transmission conditions are not linearly independent, leading to a loss of consistency with the physical problem. This representation also includes the ND scheme (centered in the horizontal outer boxes, corresponding to $\gamma_{\mathcal{F}} = 0$ and $\gamma_{\mathcal{S}} = \pm\infty$) and the DN scheme (centered in the vertical outer boxes, corresponding to $\gamma_{\mathcal{F}} = \pm\infty$ and $\gamma_{\mathcal{S}} = 0$). Additionally, the

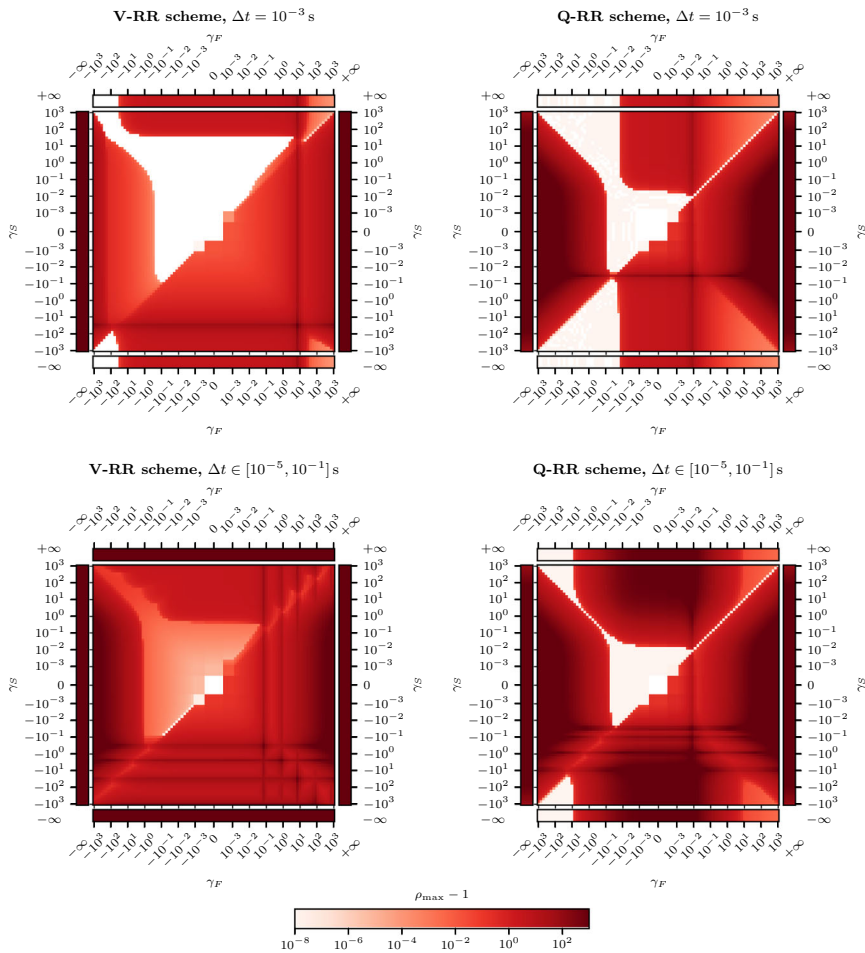


Fig. 10 Spectral radius analysis for the V-RR (left) and Q-RR schemes (right), with fixed Δt (top, $\Delta t = 10^{-3}$ s) and variable Δt (bottom, $\Delta t \in [10^{-5}, 10^{-1}]$ s). For each case, the spectral radii are computed for $R = R_{\min}$ and $R = R_{\max}$, and the maximum value ($\rho_{\max}$) across different R and Δt (in the variable Δt case) is considered. Finally, we display the quantity $\rho_{\max} - 1$ on a logarithmic scale. In each of the four cases, the central box shows the spectral radius as a function of (γ_F, γ_S) within the range of $[-10^3, 10^3]^2$. The left and right outer boxes correspond to $\gamma_F = -\infty$ and $\gamma_F = +\infty$, respectively, while the top and bottom outer boxes correspond to $\gamma_S = \pm\infty$. For the V-RR scheme, the coefficients γ_F and γ_S are given in mmHg mL $^{-1}$, while for the Q-RR scheme they are expressed in mmHg mL $^{-1}$ s

Robin–Neumann (RN) scheme corresponds to the horizontal axis $\gamma_S = 0$, while the Neumann–Robin (NR) scheme corresponds to the vertical axis $\gamma_F = 0$.

In the upper left of Fig. 10, we analyze the V-RR scheme with a fixed time step size (specifically, $\Delta t = 10^{-3}$ s). As expected from previous analyses, both the ND and DN schemes exhibit eigenvalues with a modulus greater than 1: for the ND scheme, this occurs when the valve is open, and for the DN scheme, when the valve is closed. However, there are parameter values for γ_F and γ_S where the maximum spectral

radius does not substantially exceeds the critical threshold. Although round-off errors may cause the computed spectral radius to slightly exceed one, this does not lead to oscillatory or blow-up behavior in practice, as it will be confirmed by the numerical tests in Sect. 5.1.2.

In the lower left of Fig. 10, instead of fixing the time step size Δt , we explore a wide range of Δt , and report the maximum spectral radius computed. This analysis tests whether suitable choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ can yield an absolutely stable scheme. However, the results show that, even for choices of Robin parameters for which the scheme can be stable with a fixed Δt , when Δt varies at least one eigenvalue exits the unit disk in the complex plane, leading to numerical instabilities.

On the right side of Fig. 10, we repeat the analysis for the Q-RR scheme, considering both fixed and variable Δt . In this case as well, although the ND and DN schemes present a spectral radius strictly greater than one for some parameter choices, certain choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ ensure that the spectral radius does not substantially violate the stability condition. Interestingly, when Δt varies, the instability region expands but does not cover the entire plane, leaving some choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ for which the scheme remains free from numerical oscillations regardless of the choice of Δt .

Finally, it is important to note that this analysis focuses on stability alone and does not consider the accuracy of the methods, which may also be affected by the choice of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$. We will address this aspect in Sect. 5.1.2.

4 Stabilized scheme

In this section, we propose a numerical scheme that cures the numerical instabilities analyzed in Sect. 3. First, in Sect. 4.1, we present the rationale behind its derivation. Then, we study its stability properties using the tools introduced in Sect. 3.2, by considering the minimal model (Sect. 4.2). Finally, in Sect. 4.3, we generalize the result for a generic structure model.

4.1 Derivation of the stabilized scheme (ND-stab)

The analysis of Sect. 3 revealed that the DN scheme might feature eigenvalues with a very large modulus (this is related to the balloon dilemma, see Sect. 3.3.1). In comparison, the worst case of the ND scheme has less severe instabilities issues. For this reason, we focus on the latter scheme, and we aim to find a stabilization term that cures the numerical oscillations that might occur.

Following the analysis of Sect. 3.4, the non-physical oscillations are related to the structure mechanics substep, which updates the pressure while being unaware of the effect the pressure update will have on the volume change within the next fluid substep. A too large change in pressure might indeed reverse the flux through the valve. To cure this issue, we propose to incorporate, within the structure mechanics substep, a prediction of the volume variation due to the net blood flux that will be generated

within the next fluid substep. Specifically, within (17), we replace the equation

$$V_S^{(k+1)} = V_{\mathcal{F}}^{(k)} \quad (30)$$

by the equation

$$V_S^{(k+1)} = V_{\mathcal{F}}^{(k)} + \Delta t \, Q_{\text{stab}}(p^{(k+1)}, t_{k+1}). \quad (31)$$

having defined

$$Q_{\text{stab}}(p, t) = Q_{\text{valve}}(p_{\text{ext}}(t) - p).$$

The additional term $\Delta t \, Q_{\text{stab}}$ plays the role of a stabilization term. This term is numerically consistent: indeed, by replacing in (31), the exact solution of (16), the perturbation with respect to the third line of (16) tends to zero when $\Delta t \rightarrow 0^+$ (see e.g. [47, Chap 2]). The stabilized ND scheme (henceforth denoted by ND-stab scheme), then reads

$$\begin{cases} M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A \, T_a(t_{k+1}) = p^{(k+1)} & k = 0, 1, \dots \\ V_S^{(k+1)} = V_{\mathcal{F}}^{(k)} + \Delta t \, Q_{\text{stab}}(p^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p^{(k+1)}) & k = 0, 1, \dots \end{cases} \quad (32)$$

We remark that the ND-stab scheme preserves the loosely coupled structure of the ND scheme, as the newly introduced term does not involve the fluid state variable $V_{\mathcal{F}}^{(k+1)}$.

4.2 Stability analysis of the ND-stab scheme

We now study the numerical stability of the ND-stab scheme of Eq. (32), based on the lemmas of Sect. 3.2. With reference to Lemma 2, the $\mathbf{B}$ matrix of the ND-stab scheme coincides with that of the ND scheme (see Eq. (27)), while the $\mathbf{A}$ matrix changes as follows

$$\mathbf{A}_{\text{ND-stab}} = \mathbf{A}_{\text{ND}} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Delta t}{R} \end{pmatrix}, \quad \mathbf{B}_{\text{ND-stab}} = \mathbf{B}_{\text{ND}}. \quad (33)$$

We have the following result.

Proposition 1 *The ND-stab scheme of Eq. (32) is unconditionally absolutely stable (and hence zero-stable) for any physically meaningful choice of parameters (that is $E \geq E_0$ for some $E_0 > 0$, $R > 0$, $C \geq 0$, $M \geq 0$).*

Proof The non-zero eigenvalues of the Jacobian matrix $-\mathbf{A}_{\text{ND-stab}}^{-1} \mathbf{B}_{\text{ND-stab}}$ read (they can be obtained through symbolic computations)

$$\begin{aligned}
 \mu_{\pm} &= \frac{2M + (C + R)\Delta t \pm \Delta t \sqrt{(C + R)^2 - 4EM}}{2(M + (C + R)\Delta t + E\Delta t^2)} \\
 &= \frac{2M + \hat{C}\Delta t \pm \Delta t \sqrt{\hat{C}^2 - 4EM}}{2(M + \hat{C}\Delta t + E\Delta t^2)}
 \end{aligned} \quad (34)$$

where we have defined $\hat{C} = C + R$. We consider two scenarios, depending on the sign of $\hat{C}^2 - 4EM$. Let us first suppose that $\hat{C}^2 \geq 4EM$. Then, we have

$$0 \leq \hat{C} \pm \sqrt{\hat{C}^2 - 4EM} \leq 2\hat{C},$$

which entails

$$0 \leq \frac{M}{M + \hat{C}\Delta t + E\Delta t^2} \leq \mu_{\pm} \leq \frac{M + \hat{C}\Delta t}{M + \hat{C}\Delta t + E\Delta t^2} < 1.$$

Consider now the case when $\hat{C}^2 < 4EM$. We have:

$$\mu_{\pm} = \frac{2M + \hat{C}\Delta t \pm i\Delta t \sqrt{4EM - \hat{C}^2}}{2(M + \hat{C}\Delta t + E\Delta t^2)},$$

which entails

$$\begin{aligned}
 |\mu_{\pm}|^2 &= \frac{(2M + \hat{C}\Delta t)^2 + \Delta t^2(4EM - \hat{C}^2)}{4(M + \hat{C}\Delta t + E\Delta t^2)^2} \\
 &= \frac{M^2 + M\hat{C}\Delta t + EM\Delta t^2}{M^2 + \hat{C}^2\Delta t^2 + E^2\Delta t^4 + 2M\hat{C}\Delta t + 2ME\Delta t^2 + 2\hat{C}E\Delta t^3} < 1.
 \end{aligned}$$

The eigenvalues of the Jacobian matrix $-\mathbf{A}_{\text{ND-stab}}^{-1}\mathbf{B}_{\text{ND-stab}}$ are thus bounded in modulus by a constant strictly lower than 1. The thesis follows from Lemmas 1 and 2. $\square$

In order to numerically verify the result of Proposition 1, in Fig. 11 we show the trend with respect to Δt of the spectrum of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and ND-stab scheme, under different conditions (quasi-static case, with viscosity, with inertia and finally with both viscosity and inertia). The results show that, as expected, the ND-stab scheme has eigenvalues all less than one in modulus.

4.3 ND-stab scheme for a generic structure model

The ND-stab scheme, despite being derived for the minimal model of Eq. (32), can be generalized for a much wider class of models of FSI problems. We now show that the stability result of Proposition 1 can be extended to the generic structure mechanics model of Eq. (3), while still considering the minimal fluid model (14). Interestingly,

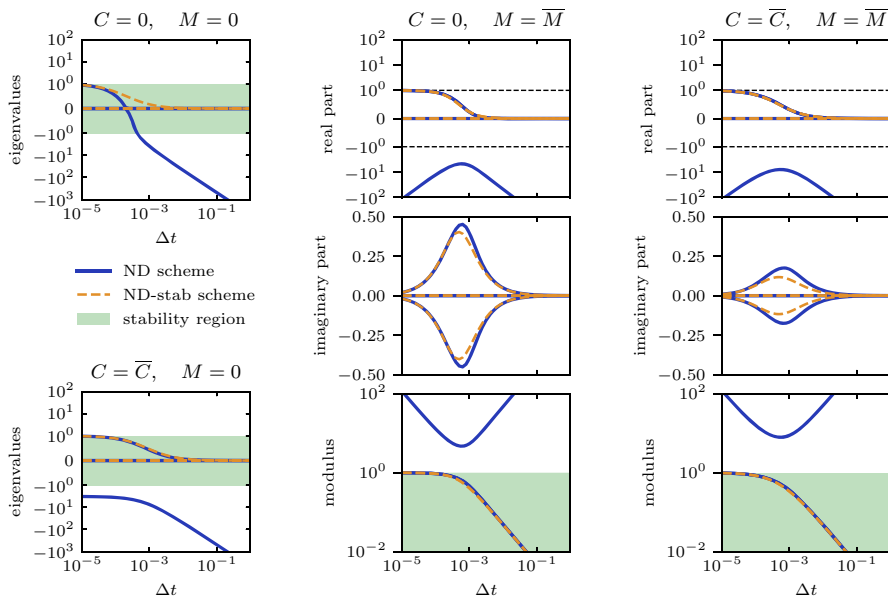


Fig. 11 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND and ND-stab schemes in the open valve case ($R = R_{\min}$)

in the proof we will formally reduce the generic model to the minimal one. Let us then consider the following FSI problem

$$\begin{cases} \mathcal{L}(\ddot{\mathbf{d}}(t), \dot{\mathbf{d}}(t), \mathbf{d}(t), p(t), t) = 0 & t > 0, \\ \dot{V}_{\mathcal{F}}(t) = Q_{\text{valve}}(p_{\text{ext}}(t) - p(t)) & t > 0, \\ V_{\mathcal{S}}(\mathbf{d}(t)) = V_{\mathcal{F}}(t) & t > 0, \\ \mathbf{d}(0) = \mathbf{d}_0, \dot{\mathbf{d}}(0) = \mathbf{0}, V_{\mathcal{F}}(0) = V_0, \end{cases} \quad (35)$$

and the associated ND-stab scheme:

$$\begin{cases} \mathcal{L}\left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, p^{(k+1)}, t_{k+1}\right) = 0 & k = 0, 1, \dots \\ V_{\mathcal{S}}(\mathbf{d}^{(k+1)}) = V_{\mathcal{F}}^{(k)} + \Delta t Q_{\text{stab}}(p^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p^{(k+1)}) & k = 0, 1, \dots \end{cases} \quad (36)$$

We remark that problem (35) might arise from the Finite Element space discretization of virtually any nonlinear elastic or viscoelastic structure model. We then consider the following physically motivated assumptions.

(A1) Let us define the mass matrix $\mathcal{M}$ and the damping matrix $\mathcal{C}$ as follows

$$\mathcal{M}(\ddot{\mathbf{d}}, \dot{\mathbf{d}}, \mathbf{d}, p, t) := \frac{\partial \mathcal{L}}{\partial \ddot{\mathbf{d}}}, \quad \mathcal{C}(\ddot{\mathbf{d}}, \dot{\mathbf{d}}, \mathbf{d}, p, t) := \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{d}}}.$$

We assume that both $\mathcal{M}$ and $\mathcal{C}$ has real nonnegative eigenvalues, bounded from above by $\lambda_{\mathcal{M}}^{\max} < +\infty$ and by $\lambda_{\mathcal{C}}^{\max} < +\infty$, respectively, uniformly with respect to the arguments. This assumption encompasses the case when inertia and/or damping are not included in the model ($\mathcal{M} = \mathcal{C} = \mathbb{O}$, the zero tensor).

(A2) Let us define the stiffness matrix

$$\mathcal{E}(\ddot{\mathbf{d}}, \dot{\mathbf{d}}, \mathbf{d}, p, t) := \frac{\partial \mathcal{L}}{\partial \mathbf{d}}. \quad (37)$$

We assume that $\mathcal{E}$ has real eigenvalues, bounded from below by $\lambda_{\mathcal{E}}^{\min} > 0$, uniformly with respect to its arguments.

This assumption arises from thermodynamical requirements (ellipticity of the stiffness operator).

(A3) For physical consistency, we need the following relationship to hold true for any choice of $\ddot{\mathbf{d}}, \dot{\mathbf{d}}, \mathbf{d}, p, t$:

$$\frac{\partial \mathcal{L}}{\partial p} = -\frac{\partial V_{\mathcal{S}}}{\partial \mathbf{d}}.$$

This requirement derives from the principle of momentum conservation (see [51, 53]). In Proposition 2, we show that this holds true if, e.g., the boundary conditions of Eq. (2) are employed.

(A4) Let us define

$$R(p, t) := -\left(\frac{\partial Q_{\text{valve}}(p_{\text{ext}}(t) - p)}{\partial p}\right)^{-1}. \quad (38)$$

We assume that $0 < R(p, t) < R_{\max}$ for some $R_{\max} < +\infty$, uniformly with respect to p and t . This amounts to say that the valve has finite resistance, coherently with the valve model of Eq. (15).

The following proposition proves that Assumption (A3) holds true in case the boundary conditions of (2) are employed.

Proposition 2 *Suppose that the chamber volume is enclosed in the surface Γ_0^{endo} . Then, both Problem (2) and its Finite Element discretization satisfy Assumption (A3).*

Proof As in Problem (35), we consider $N = 1$ cavities. The weak formulation of Problem (2) reads:

$$\int_{\Omega_0} \rho \frac{\partial^2 \mathbf{d}}{\partial t^2} \cdot \mathbf{v} dV_0 + \int_{\Omega_0} \mathbf{P}(\mathbf{d}, T_a) : \nabla \mathbf{v} dV_0 + \int_{\Gamma_0^{\text{endo}}} p \mathbf{J} \mathbf{F}^{-T} \mathbf{N} \cdot \mathbf{v} dS_0 = 0 \quad \forall \mathbf{v} \in V := [H^1(\Omega_0)]^3.$$

Notice that additional terms could be present for other portions of the boundary, but this would not affect the result. It follows that, for any $\mathbf{v} \in V$:

$$\left\langle \frac{\partial \mathcal{L}}{\partial p}, \mathbf{v} \right\rangle = \int_{\Gamma_0^{\text{endo}}} J \mathbf{F}^{-T} \mathbf{N} \cdot \mathbf{v} \, dS_0,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V and its dual V^* .

Let us denote by Ω_0^f the volume enclosed into Γ_0^{endo} . Moreover, we denote by $\Gamma^{\text{endo}}(\mathbf{d})$ the result of deforming the surface Γ_0^{endo} according to the displacement field $\mathbf{d}$, and by $\Omega^f(\mathbf{d})$ the enclosed volume. We denote by $\mathbf{n}$ the unit vector, normal to the surface $\Gamma^{\text{endo}}(\mathbf{d})$ in the current configuration, directed from the solid to the fluid domain. The fluid volume function then reads:

$$V_S(\mathbf{d}) = \int_{\Omega^f(\mathbf{d})} 1 \, dV_t = \int_{\Omega_0^f} J \, dV_0.$$

By Reynolds transport theorem [28], the Gateaux derivative of the volume reads, for any $\mathbf{v} \in V$

$$\left\langle \frac{\partial V_S}{\partial \mathbf{d}}, \mathbf{v} \right\rangle = - \int_{\Gamma^{\text{endo}}(\mathbf{d})} \mathbf{v} \cdot \mathbf{n} \, dS_t = - \int_{\Gamma_0^{\text{endo}}} J \mathbf{F}^{-T} \mathbf{N} \cdot \mathbf{v} \, dS_0,$$

where we have computed the pull-back in the reference configuration. This proves Assumption (A3). To prove the discrete case, it is sufficient to take as $\mathbf{v}$ the elements of the basis of the Finite Element space. $\square$

We are now ready to prove the main result of this work, which concerns the numerical stability of the ND-stab scheme for a generic (including 3D) structure model.

Theorem 1 *Let us consider the 0D–3D problem (35), with assumptions (A1)–(A2)–(A3)–(A4).*

The ND-stab scheme of Eq. (36) is unconditionally absolutely stable (and hence zero-stable).

Proof The scheme (36) belongs to the abstract family (21), provided that we define the state vector $\psi^{(k)} := (\mathbf{d}^{(k)}, \mathbf{d}^{(k-1)}, V_{\mathcal{F}}^{(k)}, p^{(k)})^T$. With reference to Lemmas 1 and 2, we have

$$\mathbf{A} = \begin{pmatrix} \mathcal{E} + \frac{\mathcal{M}}{\Delta t^2} + \frac{\mathcal{C}}{\Delta t} & -\frac{\mathcal{M}}{\Delta t^2} & \mathbf{0} & -\mathbf{r} \\ \mathbb{O} & \mathbb{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & \Delta t^{-1} & R^{-1} \\ \mathbf{r}^T & \mathbf{0}^T & 0 & \frac{\Delta t}{R} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -\left(\frac{\mathcal{M}}{\Delta t^2} + \frac{\mathcal{C}}{\Delta t}\right) & \frac{\mathcal{M}}{\Delta t^2} & \mathbf{0} & \mathbf{0} \\ -\mathbb{I} & \mathbb{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^T & \mathbf{0}^T & -\Delta t^{-1} & 0 \\ \mathbf{0}^T & \mathbf{0}^T & -1 & 0 \end{pmatrix}. \quad (39)$$

where we have defined $\mathbf{r} := \frac{\partial V_S}{\partial \mathbf{d}}$ and where the dependence on the arguments is understood. We denote by $\mathbb{I}$ and $\mathbb{O}$ the identity and zero matrices, respectively. We

consider the generalized eigenvalues problem $\mu \mathbf{A}\psi + \mathbf{B}\psi = \mathbf{0}$, for $\psi \neq \mathbf{0}$ and we look for non-zero eigenvalues $\mu \neq 0$. By writing the eigenvector as $\psi = (\hat{\mathbf{d}}, \mathbf{d}, V_{\mathcal{F}}, p)^T$, each eigenvalue-eigenvector pair solves

$$\frac{\mu^2 - 2\mu + 1}{\Delta t^2} \mathcal{M} \mathbf{d} + \frac{\mu^2 - \mu}{\Delta t} \mathcal{C} \mathbf{d} + \mathcal{E} \mathbf{d} = \mu p \mathbf{r} \quad (40a)$$

$$\frac{\mu - 1}{\Delta t} V_{\mathcal{F}} = -\frac{\mu p}{R} \quad (40b)$$

$$\mu^2 \mathbf{r}^T \mathbf{d} = V_{\mathcal{F}} - \frac{\Delta t}{R} \mu p \quad (40c)$$

We first consider the case when $\mathbf{d} = \mathbf{0}$. From (40b) and (40c), it follows $V_{\mathcal{F}} = 0$ and hence, by using again (40b), $p = 0$. Thus we get $\psi = \mathbf{0}$ that is not a valid eigenvector. Therefore, from now on we assume $\mathbf{d} \neq \mathbf{0}$. From (40b) and (40c) it follows $V_{\mathcal{F}} = \mu \mathbf{r}^T \mathbf{d}$ and hence $p = \frac{R}{\Delta t} (1 - \mu) \mathbf{r}^T \mathbf{d}$. Replacing this into (40a), we have:

$$\left(\frac{\mu^2 - 2\mu + 1}{\Delta t^2} \mathcal{M} + \frac{\mu^2 - \mu}{\Delta t} (\mathcal{C} + R \mathbf{r} \mathbf{r}^T) + \mathcal{E} \right) \mathbf{d} = \mathbf{0}. \quad (41)$$

We left multiply (41) by the Hermitian transpose of $\mathbf{d}$, thus getting

$$\frac{\mu^2 - 2\mu + 1}{\Delta t^2} \hat{M} + \frac{\mu^2 - \mu}{\Delta t} \hat{C} + \hat{E} = 0, \quad (42)$$

where we have defined

$$\hat{M} = \frac{\mathbf{d}^H \mathcal{M} \mathbf{d}}{\mathbf{d}^H \mathbf{d}}, \quad \hat{C} = \frac{\mathbf{d}^H (\mathcal{C} + R \mathbf{r} \mathbf{r}^T) \mathbf{d}}{\mathbf{d}^H \mathbf{d}}, \quad \hat{E} = \frac{\mathbf{d}^H \mathcal{E} \mathbf{d}}{\mathbf{d}^H \mathbf{d}}$$

Assumptions (A1) and (A4) entail $0 \leq \hat{M} \leq \lambda_{\mathcal{M}}^{\max}$ and $0 \leq \hat{C} \leq \lambda_{\mathcal{C}}^{\max} + R_{\max} \|\mathbf{r}\|^2$. Moreover, by assumption (A2) we have $\hat{E} \leq \lambda_{\mathcal{E}}^{\min}$. From Eq. (42) it follows that two eigenvalues are possible, according to

$$\mu_{\pm} = \frac{2\hat{M} + \hat{C}\Delta t \pm \Delta t \sqrt{\hat{C}^2 - 4\hat{E}\hat{M}}}{2(\hat{M} + \hat{C}\Delta t + \hat{E}\Delta t^2)}.$$

We have thus formally recovered the minimal model of Proposition 1 (see Eq. (34)). Therefore, by proceeding as in the proof of Proposition 1, we have

$$|\mu_{\pm}| \leq \frac{\hat{M} + \hat{C}\Delta t}{\hat{M} + \hat{C}\Delta t + \hat{E}\Delta t^2}, \quad (43)$$

if $\hat{C}^2 \geq 4\hat{E}\hat{M}$; otherwise, we have

$$|\mu_{\pm}|^2 = \frac{\hat{M}^2 + \hat{M}\hat{C}\Delta t + \hat{E}\hat{M}\Delta t^2}{\hat{M}^2 + \hat{C}^2\Delta t^2 + \hat{E}^2\Delta t^4 + 2\hat{M}\hat{C}\Delta t + 2\hat{M}\hat{E}\Delta t^2 + 2\hat{C}\hat{E}\Delta t^3}. \quad (44)$$

In both cases, the eigenvalues have modulus strictly lower than one. This proves zero-stability. To prove absolute stability, instead, we need to show that the eigenvalues modulus is bounded by a constant strictly lower than 1, uniformly with respect to the state ψ . This is proved by the fact that both the right-hand sides of Eqs. (43) and (44) are non-increasing in $\hat{E}$, which is bounded from below by $\lambda_{\mathcal{E}}^{\min}$, and by the fact that they are continuous in $(\hat{M}, \hat{C})$, which belongs to the compact set $[0, \lambda_{\mathcal{M}}^{\max}] \times [0, \lambda_{\mathcal{C}}^{\max} + R_{\max}\|\mathbf{r}\|^2]$. $\square$

Finally, we show how the ND-stab scheme, that we proposed on the minimal model (14), can be generalized to more complex fluid models. The vector collecting the net blood fluxes associated with the N , as predicted by the fluid model, is given by

$$\frac{d\mathbf{V}_{\mathcal{F}}(\mathbf{c}(t))}{dt} = \frac{\partial \mathbf{V}_{\mathcal{F}}(\mathbf{c}(t))}{\partial \mathbf{c}} \mathbf{H}(\mathbf{c}(t), \mathbf{p}(t), t).$$

Therefore, we define the flux function

$$\mathbf{Q}_{\text{stab}}(\mathbf{p}, \mathbf{c}, t) = \frac{\partial \mathbf{V}_{\mathcal{F}}(\mathbf{c})}{\partial \mathbf{c}} \mathbf{H}(\mathbf{c}, \mathbf{p}, t).$$

The ND-stab scheme then reads

$$\begin{cases} \mathcal{L}\left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}, t_{k+1}\right) = 0 & k = 0, 1, \dots \\ \mathbf{V}_{\mathcal{S}}(\mathbf{d}^{(k+1)}) = \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k)}) + \Delta t \mathbf{Q}_{\text{stab}}(\mathbf{p}^{(k+1)}, \mathbf{c}^{(k)}, t_{k+1}) & k = 0, 1, \dots \\ \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(*)}, \mathbf{p}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \end{cases} \quad (45)$$

For this case we do not provide a theoretical result, but will show numerical evidence of the effectiveness of the stabilization term in Sect. 5.

We notice that, unlike for the ND scheme of Eq. (9), for the ND-stab scheme the pressures $\mathbf{p}^{(k+1)}$ also account as unknowns in the second equation of the structure mechanics substep of the ND-stab scheme of Eq. (45). The numerical resolution of the structure mechanics substep can be efficiently performed through the Newton method. In “Appendix D” we report an efficient algorithm, based on a Schur complement reduction, for solving the linear system associated with each Newton iteration.

5 Results

In this section, we present and discuss some numerical tests aimed at verifying the theoretical results presented in this paper and the effectiveness of the proposed method. First, in Sect. 5.1, we consider the minimal model introduced in Sect. 3.1. Then, in Sect. 5.2, we consider the application of the ND-stab scheme to a multi-scale left ventricular model of cardiac electromechanics.

5.1 Minimal model

We consider the minimal model of Eq. (16) (see also Fig. 1a). To demonstrate the generality of the results of Sects. 3 and 4, that is valid for both linear and nonlinear models, we consider a nonlinear law for the passive constitutive behavior of the cardiac chamber. Specifically, we consider the Klotz's law [33]:

$$p_{\text{pass}}(V) = A_n \left(\frac{V - V_0^{\text{klotz}}}{V_{30}^{\text{klotz}} - V_0^{\text{klotz}}} \right)^{B_n}, \quad (46)$$

with $V_0^{\text{klotz}} = 10\text{mL}$, $V_{30}^{\text{klotz}} = 170\text{mL}$, $A_n = 28.2\text{mmHg}$ and $B_n = 2.79$. We consider the parameters reported in Tab. 1 with no active tension ($T_a(t) \equiv 0$). Moreover, we consider the case of an inflow cardiac valve (i.e. $Q_{\text{valve}} = Q_{\text{valve}}^{\text{in}}$). We run the simulation until time $T = 0.2\text{s}$, starting with a volume $V_0 = 120\text{mL}$ and imposing the following external pressure transient:

$$p_{\text{ext}}(t) = p_{\min} + (p_{\max} - p_{\min}) \sin \left(\frac{\pi t}{T} \right),$$

with $p_{\min} = 5\text{mmHg}$ and $p_{\max} = 20\text{mmHg}$.

In this test case, the pressure inside the cardiac chamber at time $t = 0$ is $p(0) = 10\text{mmHg}$. The valve is therefore closed (negative pressure gradient: $p_{\text{ext}}(t) - p(t) < 0$) and the chamber volume is almost constant in time. Then, approximately at $t = 0.03\text{s}$, the external pressure $p_{\text{ext}}(t)$ exceeds the internal pressure $p(t)$. Consequently, the valve opens and blood begins to flow into the chamber, causing an increase in fluid volume. Finally, the sign of the pressure gradient reverses again and the valve closes, causing the volume to become constant in time.

In order to numerically verify the results of Sects. 3 and 4, we perform several numerical simulations of the model described above, for different time step sizes and by comparing the results obtained with the different schemes. In each test, we superimpose the obtained numerical solutions with a reference solution, obtained by means of a fully coupled scheme and with a very small time step size ($\Delta t = 10^{-6}\text{s}$). We also report the values of relative errors, with respect to the reference solution, associated with the different schemes, both for the pressure and the volume variables. The errors are computed as follows, where y represents either the pressure or the volume:

$$e_y = \sqrt{\frac{\sum_{k=0}^N \left(y^{(k)} - y_{\text{ref}}^{(k)} \right)^2}{\sum_{k=0}^N \left(y_{\text{ref}}^{(k)} - \bar{y}_{\text{ref}} \right)^2}},$$

where the numerical solution $y^{(k)}$ is interpolated on the same time discretization of $y_{\text{ref}}^{(k)}$ and where the time-average solution $\bar{y}_{\text{ref}} = (N + 1)^{-1} \sum_{k=0}^N y_{\text{ref}}^{(k)}$ is subtracted in the denominator, to prevent artificial underestimation of the error. This adjustment is essential when the solution features a range of variability much smaller than its average value, as in the case of the volume.

In Sect. 5.1.1, we compare the ND scheme, commonly used in the context of cardiac electromechanical 0D–3D simulations [3, 43, 53], with the proposed ND-stab scheme. Then, in Sect. 5.1.2, we consider the more general framework of RR schemes, in comparison with the ND-stab scheme.

5.1.1 Numerical simulations (ND versus ND-stab scheme)

First, we consider the case when viscosity is present ($C \neq 0$), while inertia is neglected ($M = 0$). The analysis of Sect. 3 predicts that the ND scheme is never absolutely stable for $C = \bar{C}$ (see Fig. 5). This is verified by our numerical results. In fact, the solution of the ND scheme exhibits nonphysical pressure oscillations throughout the duration of the open valve phase (see first two columns of Fig. 12). Moreover, as expected, when the valve is closed the oscillations are not present. Coherently with the predictions, these oscillations are not cured even by decreasing the time step size. Instead, the amplitude of the oscillations remains constant (this is due to the fact that the negative eigenvalue has a horizontal asymptote for $\Delta t \rightarrow 0^+$, see Fig. 5). On the other hand, by decreasing the viscosity coefficient (in particular for $C = 0.1 \bar{C}$) the ND scheme becomes absolutely stable for any choice of Δt (see Fig. 5). Our results also confirm this prediction (see e.g. the third column of Fig. 12 for the case $\Delta t = 5 \cdot 10^{-3}$ s). On the other hand, in all the considered cases, our proposed ND-stab scheme does not present numerical oscillations and the solution is close to the reference one.

The predictions of Sect. 3 are also verified in the case with inertia and without viscosity ($M \neq 0$, $C = 0$), as shown in Fig. 13. Indeed, according to the stability analysis considered in Fig. 6, for $\Delta t = 5 \cdot 10^{-3}$ s the ND scheme is absolutely stable. On the other hand, when $\Delta t \rightarrow 0^+$ numerical oscillations show up, with increasing amplitude (the eigenvalue responsible for the oscillations indeed blows up when $\Delta t \rightarrow 0^+$). Also in this case, the ND-stab successfully cures the nonphysical oscillations.

Finally, in Fig. 14 we consider the case when both viscosity and inertia are included in the model. Also in this case, the ND scheme exhibits the nonphysical oscillations predicted by the spectral analysis considered in Fig. 7, while the ND-stab scheme yields a stable solution, that is close to the reference one.

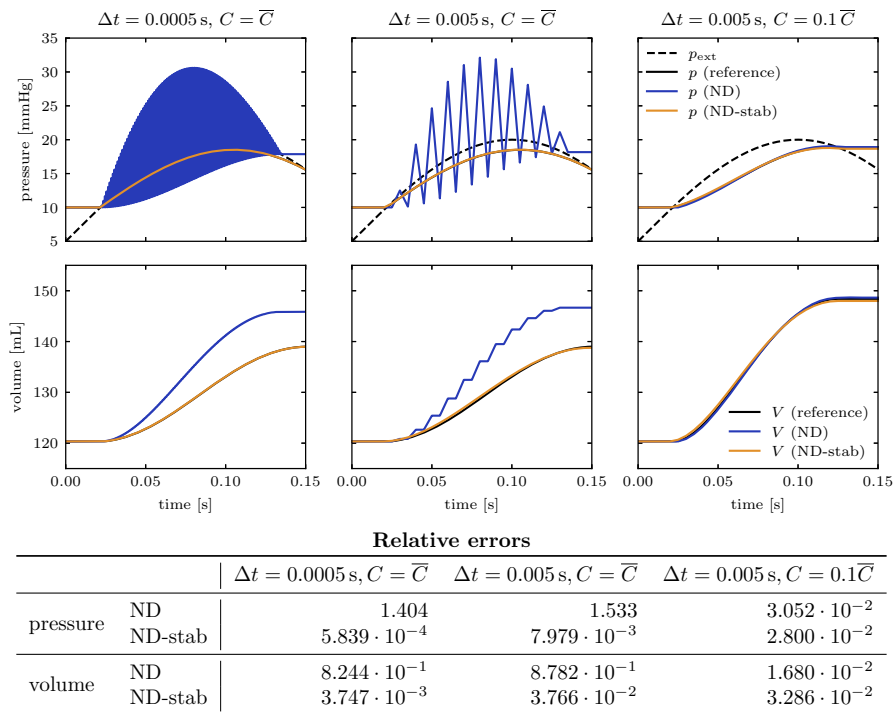


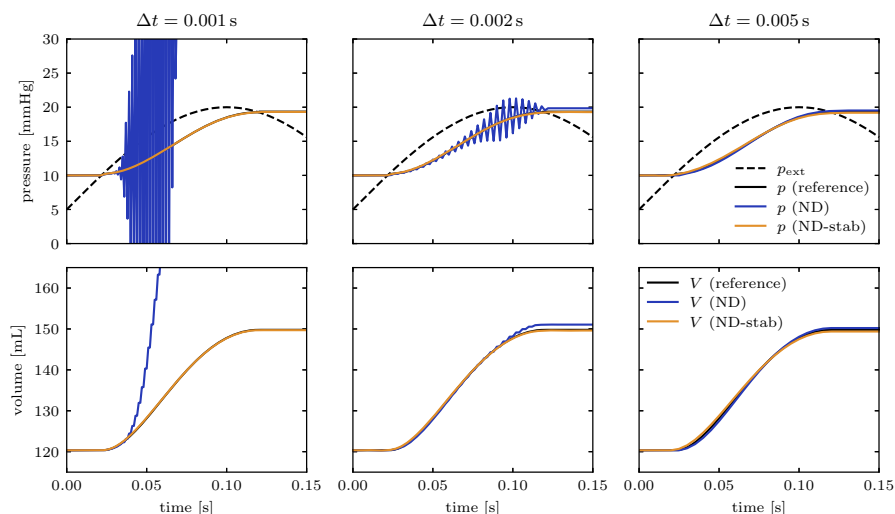
Fig. 12 Results of numerical simulations of the model of Sect. 5.1, in the case when viscosity is present ($C \neq 0$), while inertia is neglected ($M = 0$). The values of C and Δt are reported in the titles. We compare the results obtained with the ND scheme, the ND-stab scheme and the reference solution. In the table we report the relative errors, with respect to the reference solution, associated with the ND and ND-stab schemes

5.1.2 Numerical simulations (RR schemes versus ND-stab scheme)

We now consider schemes based on Robin–Robin transmission conditions. Since for these schemes, the variables describing volume and pressure for the two models, $\mathcal{F}$ and $\mathcal{S}$, can exhibit significantly different values during the simulations, we report both $p_{\mathcal{F}}$ and $p_{\mathcal{S}}$, as well as $V_{\mathcal{F}}$ and $V_{\mathcal{S}}$, in the figures of this section, both in the graphs and in the error calculations.

We first consider the V-RR scheme with a fixed time step size ($\Delta t = 10^{-3} \text{ s}$). From the analysis of Sect. 3.5, we expect that the scheme will be unstable for some choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$, while for other choices it will be free from numerical oscillations. The numerical simulations conducted with the model described at the beginning of Sect. 5.1 confirm these expectations. For brevity, we only present the results obtained for the case $\gamma_{\mathcal{F}} = 0$ and $\gamma_{\mathcal{S}} \in \{1, 10, 100\} \text{ mmHg mL}^{-1}$, corresponding to a Neumann–Robin scheme (V-NR). This choice, particularly the intermediate case, has been chosen through trial-and-error by varying both Robin coefficients and identifying those that yield the most accurate results.

As shown in Fig. 15, the V-NR scheme produces accurate results without numerical



Relative errors		$\Delta t = 0.001$ s	$\Delta t = 0.002$ s	$\Delta t = 0.005$ s
pressure	ND	4.417	$1.843 \cdot 10^{-1}$	$3.034 \cdot 10^{-2}$
	ND-stab	$7.453 \cdot 10^{-3}$	$1.460 \cdot 10^{-2}$	$3.553 \cdot 10^{-2}$
volume	ND	2.490	$5.968 \cdot 10^{-2}$	$3.149 \cdot 10^{-2}$
	ND-stab	$6.598 \cdot 10^{-3}$	$1.310 \cdot 10^{-2}$	$3.274 \cdot 10^{-2}$

Fig. 13 Results of numerical simulations of the model of Sect. 5.1, in the case when inertia is present ($M = \bar{M}$), while viscosity is neglected ($C = 0$). The values of Δt are reported in the titles. We compare the results obtained with the ND scheme, the ND-stab scheme and the reference solution. In the table we report the relative errors, with respect to the reference solution, associated with the ND and ND-stab schemes

instabilities for $\gamma_S = 10 \text{ mmHg mL}^{-1}$, while for $\gamma_S = 100 \text{ mmHg mL}^{-1}$ the scheme becomes unstable, consistently with the spectral analysis in Fig. 10. Lowering γ_S to 1 mmHg mL^{-1} , though it does not result in severe instabilities as in the $\gamma_S = 100 \text{ mmHg mL}^{-1}$ case, yields a less accurate solution. In fact, even with a fixed Δt , the V-RR scheme requires careful selection of γ_F and γ_S because, even in regions where the spectral radius does not significantly exceed 1, the numerical solution may still be inaccurate. The ND-stab scheme, as predicted by theory, shows no numerical instabilities in any of the cases considered. Quantitatively, as shown in the table in Fig. 15, it is also more accurate than the V-NR scheme in all cases considered.

We now examine the behavior of the V-RR scheme as $\Delta t \rightarrow 0$. We focus on the case with $\gamma_F = 0$ and $\gamma_S = 10 \text{ mmHg mL}^{-1}$ (V-RN scheme), which produced the most accurate results among those considered earlier. As illustrated in Fig. 16, halving Δt results in a proportional reduction in errors for both the V-RN and ND-stab schemes. However, upon further reducing Δt , while the ND-stab scheme continues to improve in accuracy, the V-RN scheme becomes unstable, leading to a diverging solution. This behavior aligns with the analysis in Sect. 3.5, which indicates that the V-RR scheme lacks unconditional stability.

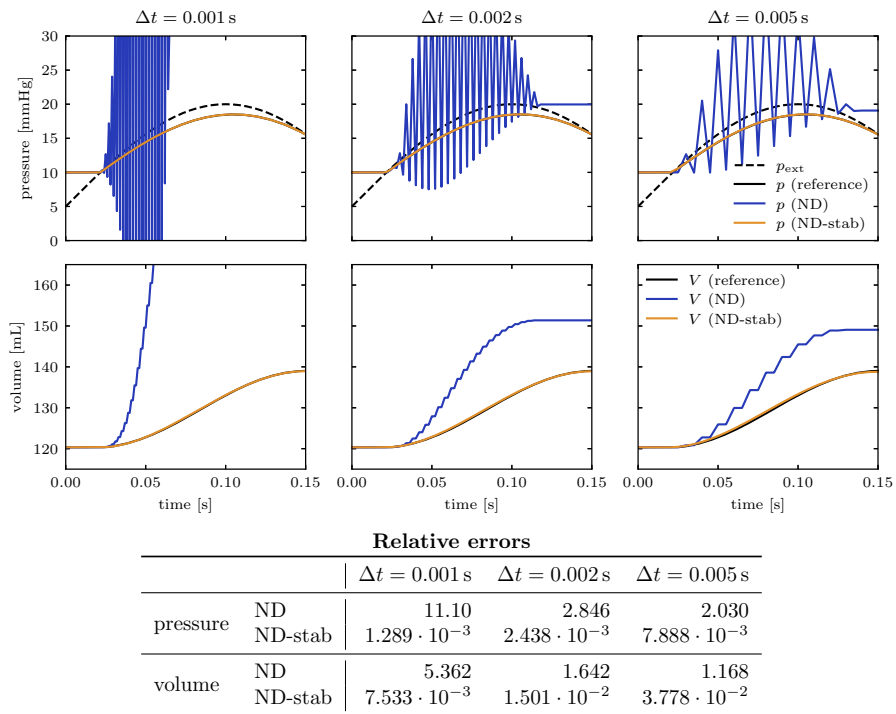
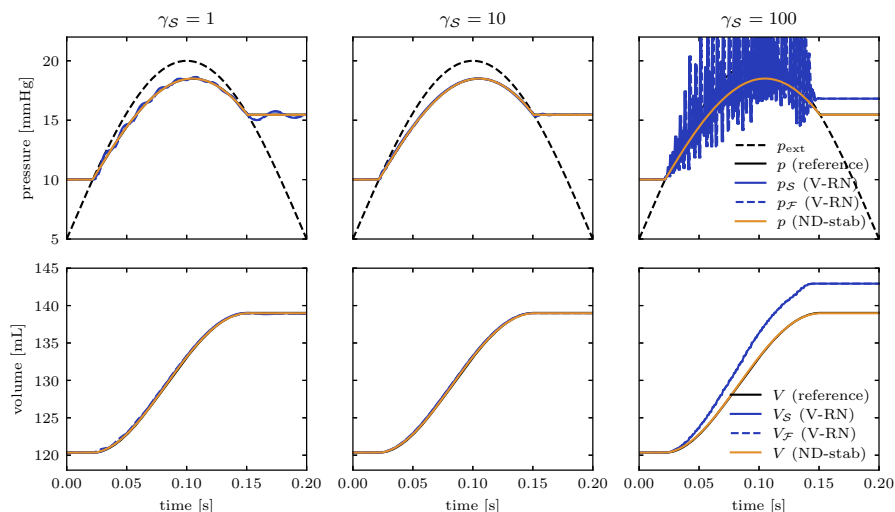


Fig. 14 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$). The values of Δt are reported in the titles. We compare the results obtained with the ND scheme, the ND-stab scheme and the reference solution. In the table we report the relative errors, with respect to the reference solution, associated with the ND and ND-stab schemes

Let us now focus on the Q-RR scheme. Based on the analysis in Sect. 3.5, we expect that with appropriate choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$, the scheme should remain numerically stable as $\Delta t \rightarrow 0$, leading to a convergent behavior. Numerical simulations confirm this expectation. For brevity, we report here results for a single value of Δt , examining how the solution depends on different selections of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$. Specifically, we evaluate three cases: Q-RN, Q-NR, and Q-RR with both $\gamma_{\mathcal{F}} \neq 0$ and $\gamma_{\mathcal{S}} \neq 0$, shown in Figs. 17, 18, and 19, respectively. The findings align with the spectral analysis in Fig. 10: none of the cases exhibit numerical oscillations, except for the scenario where $\gamma_{\mathcal{F}} = 0$ and $\gamma_{\mathcal{S}} = 0.05 \text{ mmHg mL}^{-1} \text{ s}$. This particular choice indeed corresponds to a spectral radius of the Jacobian matrix that is significantly greater than 1.

The results also underscore that, even within the stability region, certain choices of $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ can produce inaccurate solutions. This is noticeable both in the error tables and even visually in the graphs. Notably, when $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ converge toward each other, the two transmission conditions tend toward the situation where they are linearly dependent ($\gamma_{\mathcal{F}} = \gamma_{\mathcal{S}}$), which leads to a significant loss of accuracy. Moreover, although not shown here for brevity, we observe that the other two regions of the $(\gamma_{\mathcal{F}}, \gamma_{\mathcal{S}})$ plane identified as stable in Fig. 10, that is when $\gamma_{\mathcal{F}}$ and $\gamma_{\mathcal{S}}$ are both large in magnitude with $\gamma_{\mathcal{F}}$ negative, also result in inaccurate solutions. This is not



		Relative errors		
		$\gamma_S = 1$	$\gamma_S = 10$	$\gamma_S = 100$
pressure	V-RN (p_F)	$6.405 \cdot 10^{-2}$	$1.164 \cdot 10^{-2}$	$8.915 \cdot 10^{-1}$
	V-RN (p_S)	$6.728 \cdot 10^{-2}$	$2.689 \cdot 10^{-2}$	$8.959 \cdot 10^{-1}$
	ND-stab	$3.324 \cdot 10^{-3}$	$3.324 \cdot 10^{-3}$	$3.324 \cdot 10^{-3}$
volume	V-RN (V_F)	$2.740 \cdot 10^{-2}$	$2.060 \cdot 10^{-2}$	$3.871 \cdot 10^{-1}$
	V-RN (V_S)	$2.111 \cdot 10^{-2}$	$2.004 \cdot 10^{-2}$	$3.871 \cdot 10^{-1}$
	ND-stab	$6.778 \cdot 10^{-3}$	$6.778 \cdot 10^{-3}$	$6.778 \cdot 10^{-3}$

Fig. 15 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$), with $\Delta t = 0.001$ s. We compare the results obtained with the V-RN scheme ($\gamma_F = 0$, γ_S reported in the titles), the ND-stab scheme and the reference solution. The coefficients γ_S are expressed in mmHg mL⁻¹. In the table we report the relative errors, with respect to the reference solution

surprising, as this configuration approaches a Dirichlet–Dirichlet scheme, which fails to ensure pressure continuity. In summary, as highlighted in Fig. 10, achieving higher accuracy requires operating near the boundary of the numerical instability region, making the selection of γ_F and γ_S particularly delicate. This poses a challenge in practical applications, where the detailed analysis conducted here may not be feasible. This underscores an advantage of the ND-stab scheme, which eliminates the need for parameter tuning.

As a matter of fact, in all the cases here considered, the ND-stab scheme yields a stable solution, with an accuracy that is larger or comparable with that of the Q-RR scheme.

5.1.3 Convergence test

The ND-stab scheme introduces, in the ND scheme, a numerically consistent stabilization term of first order (see Sect. 4.1), $\mathcal{O}(\Delta t)$, when $\Delta t \rightarrow 0^+$. Since the numerical

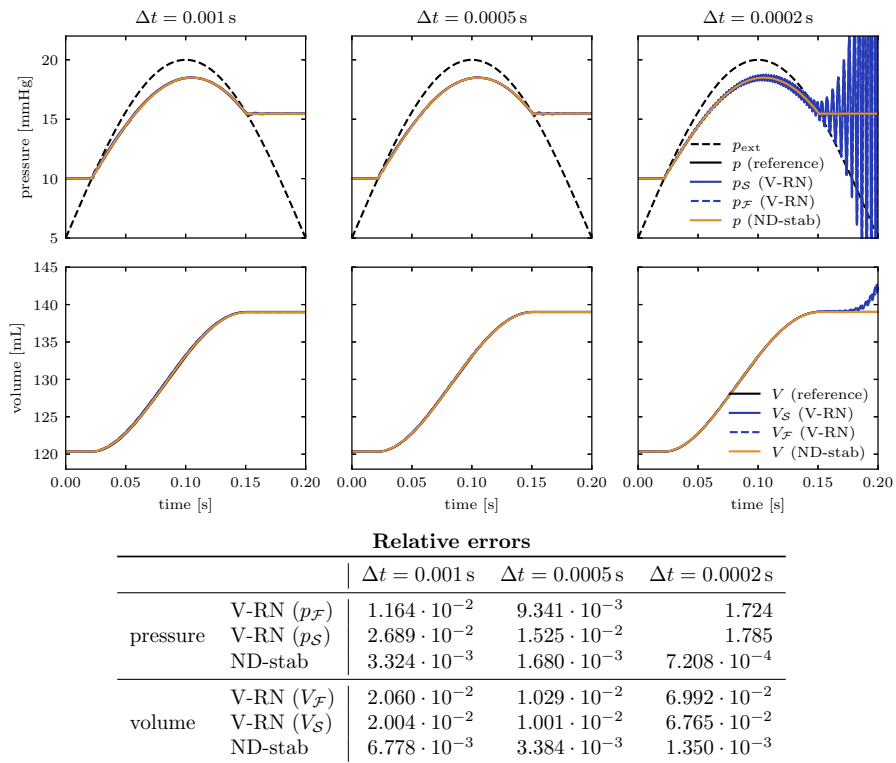


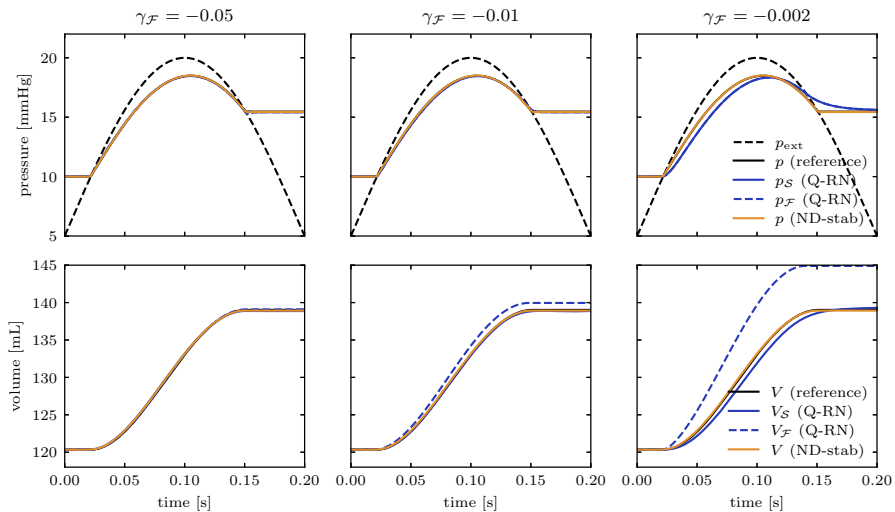
Fig. 16 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$). We compare the results obtained with the V-RN scheme ($\gamma_F = 0$, $\gamma_S = 10 \text{ mmHg mL}^{-1}$), the ND-stab scheme and the reference solution. The values of Δt are reported in the titles. In the table we report the relative errors, with respect to the reference solution

scheme of time advancement is also of first order, we expect the ND-stab scheme to be convergent of order 1 with respect to Δt (see e.g. [47]).

To numerically verify what said above, we solve the model of Sect. 5.1 by the ND-stab scheme, progressively halving the time step size Δt , from 2^{-5} s to 2^{-14} s. We then plot the trend of the errors obtained against a reference solution, achieved via a fully coupled scheme for $\Delta t = 2^{-20}$ s $\simeq 10^{-6}$ s. For simplicity, we only consider the case with both viscosity and inertia. In Fig. 20 we show that both the $L^2(0, T)$ and the $L^\infty(0, T)$ errors follow the expected first order trend with respect to Δt , both for the pressure and for the volume solution.

5.2 Multiscale cardiac electromechanics

We now apply the ND-stab scheme to a multiscale simulation of cardiac electromechanics. We employ the cardiac model proposed in [53], that we briefly recall in Sects. 5.2.1 and 5.2.2. Still, we remark that the ND-stab scheme proposed in this



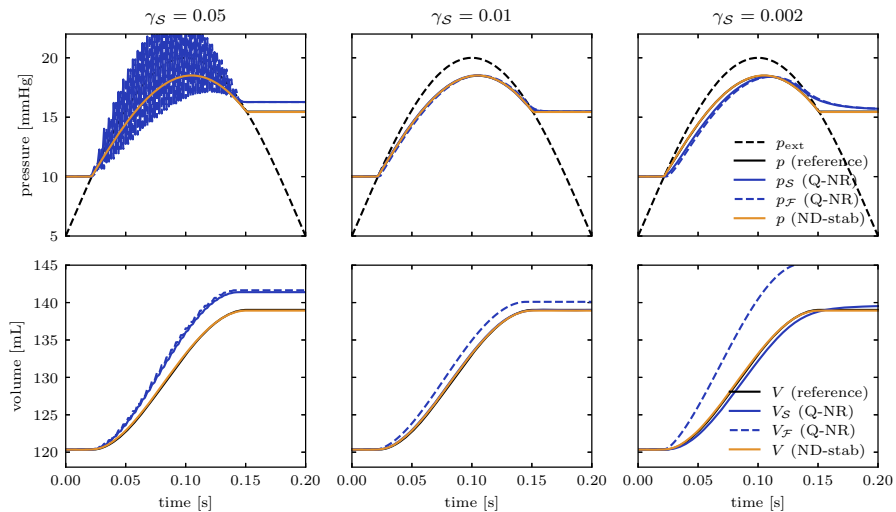
Relative errors		$\gamma_{\mathcal{F}} = -0.05$	$\gamma_{\mathcal{F}} = -0.01$	$\gamma_{\mathcal{F}} = -0.002$
pressure	Q-RN ($p_{\mathcal{F}}$)	$1.358 \cdot 10^{-2}$	$2.548 \cdot 10^{-2}$	$1.789 \cdot 10^{-1}$
	Q-RN ($p_{\mathcal{S}}$)	$1.358 \cdot 10^{-2}$	$2.548 \cdot 10^{-2}$	$1.789 \cdot 10^{-1}$
	ND-stab	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$
volume	Q-RN ($V_{\mathcal{F}}$)	$6.640 \cdot 10^{-3}$	$1.178 \cdot 10^{-1}$	$6.834 \cdot 10^{-1}$
	Q-RN ($V_{\mathcal{S}}$)	$1.709 \cdot 10^{-2}$	$1.464 \cdot 10^{-2}$	$9.166 \cdot 10^{-2}$
	ND-stab	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$

Fig. 17 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$), with $\Delta t = 0.002$ s. We compare the results obtained with the Q-RN scheme ($\gamma_{\mathcal{F}}$ reported in the titles, $\gamma_{\mathcal{S}} = 0$), the ND-stab scheme and the reference solution. The coefficients $\gamma_{\mathcal{F}}$ are expressed in $\text{mmHg mL}^{-1}\text{s}$. In the table we report the relative errors, with respect to the reference solution

paper can be applied to any coupled mechanics-hemodynamics model in the form of Eq. (6).

5.2.1 The electromechanical model

We consider a realistic left-ventricular geometry, processed from the Zygote 3D heart model [61] by exploiting the semi-automatic meshing tool proposed in [15] (see Fig. 21b). The geometry is further processed through the algorithm proposed in [53] to recover the unloaded reference configuration. To generate the cardiac fibers, we adopt the Bayer-Blake-Plank-Trayanova algorithm [6]. Cardiac muscle contraction is driven by an electrical signal that propagates throughout the myocardium, thereby causing muscle tissue to contract. To model these processes, we adopt the monodomain equation [12], the *ten Tusscher-Panfilov* ionic model [57] and the RDQ20-MF model [50] of microscale force generation. Tissue mechanics is described through Eq. (2), where for the hyperelastic energy functional $\mathcal{W}$ we adopt the quasi-incompressible exponential constitutive law of [59]. To model the interaction of the ventricle with



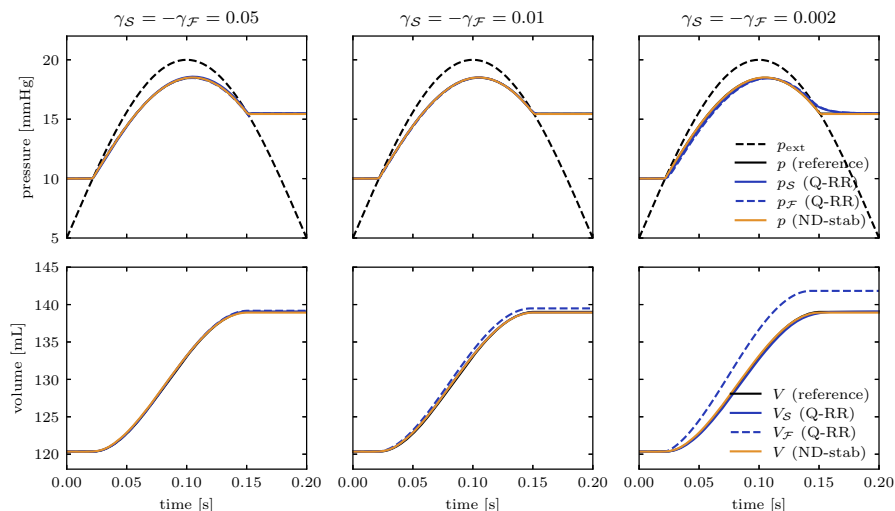
Relative errors		$\gamma_S = 0.05$	$\gamma_S = 0.01$	$\gamma_S = 0.002$
pressure	Q-NR (p_F)	$5.812 \cdot 10^{-1}$	$4.975 \cdot 10^{-2}$	$2.113 \cdot 10^{-1}$
	Q-NR (p_S)	$5.965 \cdot 10^{-1}$	$1.714 \cdot 10^{-2}$	$1.617 \cdot 10^{-1}$
	ND-stab	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$
volume	Q-NR (V_F)	$3.212 \cdot 10^{-1}$	$1.727 \cdot 10^{-1}$	$7.473 \cdot 10^{-1}$
	Q-NR (V_S)	$2.849 \cdot 10^{-1}$	$1.861 \cdot 10^{-2}$	$7.175 \cdot 10^{-2}$
	ND-stab	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$

Fig. 18 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$), with $\Delta t = 0.002$ s. We compare the results obtained with the Q-NR scheme ($\gamma_F = 0$, γ_S reported in the titles), the ND-stab scheme and the reference solution. The coefficients γ_S are expressed in mmHg mL⁻¹s. In the table we report the relative errors, with respect to the reference solution

the pericardium, we employ spring-damper boundary conditions, while to model the interaction with the part of the myocardium above the artificial ventricular base, we adopt energy-consistent boundary conditions [51]. More precisely, we complement Eq. (2) with the following boundary conditions

$$\left\{ \begin{array}{l} \mathbf{P}(\mathbf{d}, T_a) \mathbf{N} = \left[K_{\perp}^{\text{epi}} (\mathbf{N} \otimes \mathbf{N}) + K_{\parallel}^{\text{epi}} (\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) \right] \mathbf{d} \\ \quad + \left[C_{\perp}^{\text{epi}} (\mathbf{N} \otimes \mathbf{N}) + C_{\parallel}^{\text{epi}} (\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) \right] \frac{\partial \mathbf{d}}{\partial t}, \quad \text{on } \Gamma_0^{\text{epi}} \times (0, T), \\ \mathbf{P}(\mathbf{d}, T_a) \mathbf{N} = p |J \mathbf{F}^{-T} \mathbf{N}| \frac{\int_{\Gamma_0^{\text{endo}}} J \mathbf{F}^{-T} \mathbf{N} d\Gamma_0}{\int_{\Gamma_0^{\text{base}}} |J \mathbf{F}^{-T} \mathbf{N}| d\Gamma_0}, \quad \text{on } \Gamma_0^{\text{base}} \times (0, T), \end{array} \right. \quad (47)$$

where the constants $K_{\perp}^{\text{epi}}$, $K_{\parallel}^{\text{epi}}$, $C_{\perp}^{\text{epi}}$, $C_{\parallel}^{\text{epi}} \in \mathbb{R}^+$ are associated with the stiffness and the viscosity of the tissue surrounding the epicardium in the normal or tangential directions, respectively, and where we denote by p the left ventricle blood pressure.



Relative errors

		$\gamma_S = -\gamma_F = 0.05$	$\gamma_S = -\gamma_F = 0.01$	$\gamma_S = -\gamma_F = 0.002$
pressure	Q-RR (p_F)	$1.264 \cdot 10^{-2}$	$1.589 \cdot 10^{-2}$	$9.523 \cdot 10^{-2}$
	Q-RR (p_S)	$2.275 \cdot 10^{-2}$	$7.960 \cdot 10^{-3}$	$6.979 \cdot 10^{-2}$
	ND-stab	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$	$6.548 \cdot 10^{-3}$
volume	Q-RR (V_F)	$1.296 \cdot 10^{-2}$	$7.303 \cdot 10^{-2}$	$3.575 \cdot 10^{-1}$
	Q-RR (V_S)	$8.988 \cdot 10^{-3}$	$1.510 \cdot 10^{-2}$	$2.833 \cdot 10^{-2}$
	ND-stab	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$	$1.349 \cdot 10^{-2}$

Fig. 19 Results of numerical simulations of the model of Sect. 5.1, in the case when both inertia and viscosity are present ($C = \bar{C}$, $M = \bar{M}$), with $\Delta t = 0.002$ s. We compare the results obtained with the Q-RR scheme (γ_F and γ_S reported in the titles), the ND-stab scheme and the reference solution. The coefficients γ_F and γ_S are expressed in $\text{mmHg mL}^{-1}\text{s}$. In the table we report the relative errors, with respect to the reference solution

To model blood circulation (model $\mathcal{F}$), we adopt the closed-loop lumped-parameter model of [53]. This model is based on a compartmental description of the systemic and pulmonary, arterial and venous circulatory networks (modeled as resistance-inductance-capacitance circuits), the four cardiac chambers (modeled by means of time-varying elastance models) and the four valves (modeled as diodes, as in Eq. 15). The coupled 0D–3D FSI model is represented in Fig. 21a, while in Fig. 21c, the numerically obtained tissue displacement is displayed at selected time instants. More details on the employed mathematical models and their coupling can be found in [53].

5.2.2 Numerical approximation

For the numerical approximation of the multiscale left ventricular model (model $\mathcal{S}$), we employ bilinear Finite Elements defined on two nested hexahedral meshes (a finer one for the electrophysiological variables and a coarser one for the structure mechanics variables). For the time integration of this multiphysics model we employ the staggered approach proposed in [53], that envisages – at each time step – a sequential update of

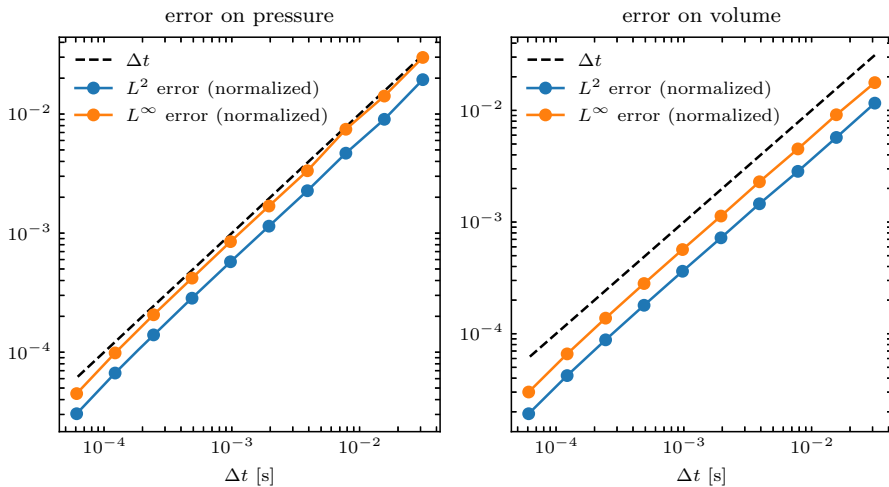


Fig. 20 Error trend on the numerical solution obtained by means of the ND-stab scheme for the model of Sect. 5.1

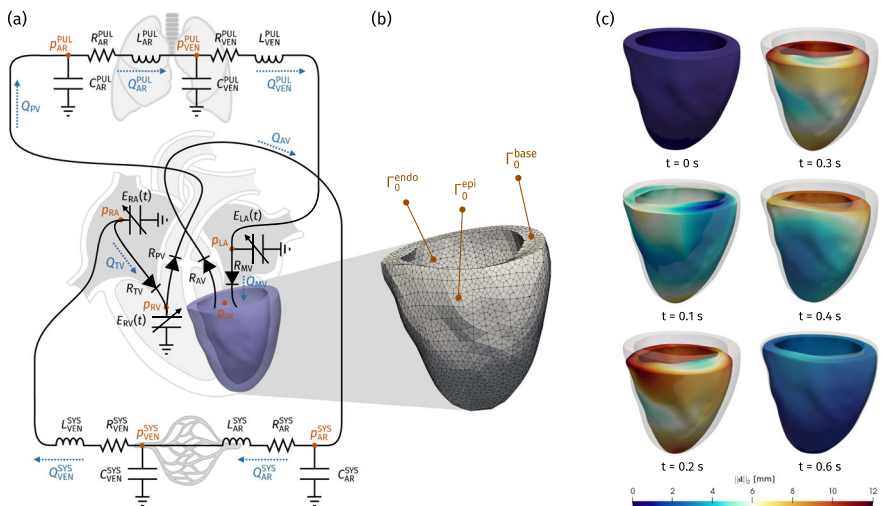


Fig. 21 The multiscale coupled mechanics-hemodynamics model of Sect. 5.2 (a); computational mesh and boundary tags (b); snapshots of the numerical solution at selected time instants showing the tissue displacement (c)

the ionic variables, the electrical potential, the activation variables and finally the tissue displacement. We adopt a time step size of $5 \cdot 10^{-4}$ s. To update the electrophysiological variables, we employ a 5 times finer time step size. For the time integration of the circulation model (model $\mathcal{F}$), we employ a Forward Euler scheme.

Concerning the coupling between the model $\mathcal{S}$ and the model $\mathcal{F}$, we compare the ND scheme with the ND-stab scheme (see Eqs. (9) and (45), respectively).

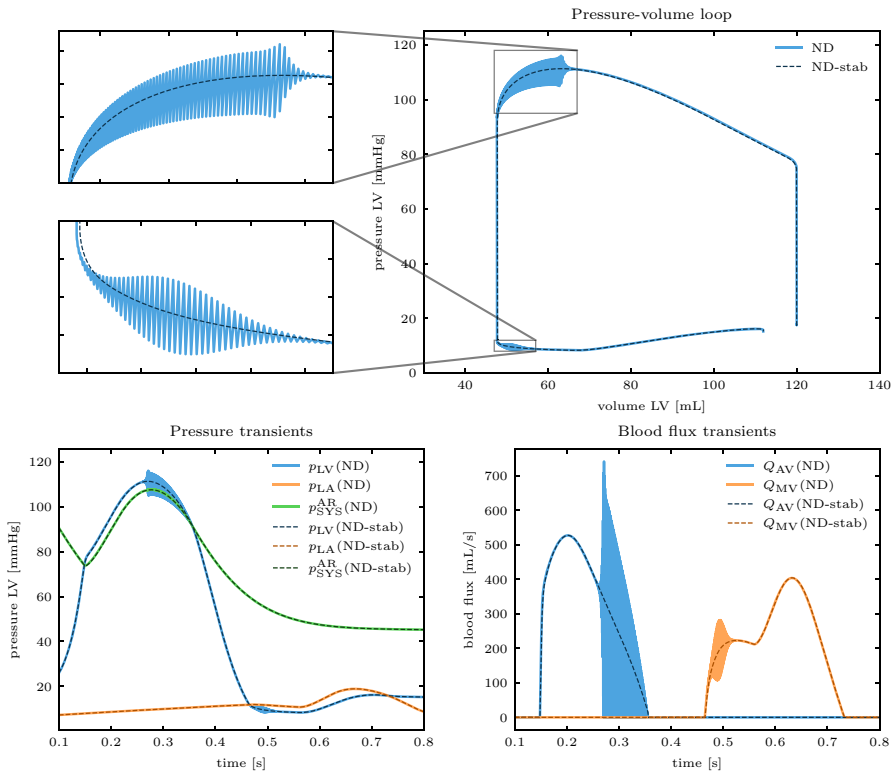


Fig. 22 Numerical solution of pressures, volume and blood fluxes of the multiscale 0D–3D model of cardiac electromechanics of Sect. 5.2: comparison between ND and ND-stab schemes. Top: pressure volume loop. Bottom left: pressure transients associated with the left ventricle (p_{LV}), the left atrium (p_{LA}) and the systemic arterial circulation (p_{AR}^{SYS}). Bottom right: blood fluxes across the aortic valve (Q_{AV}) and the mitral valve (Q_{MV})

5.2.3 Comparison of ND and ND-stab schemes

In Fig. 22 we show the transients of pressures, volumes, and blood fluxes obtained using the ND and ND-stab schemes for the numerical approximation of the multiscale model of cardiac electromechanics and hemodynamics described in Sect. 5.2.1.

On the one hand, we notice that the solution obtained using the ND scheme presents numerical oscillations in two phases of the heartbeat. Both of these episodes occur when at least one valve is open (in one case the aortic valve, in the other case the mitral valve). This is in line with the results of Sect. 3, which show that the ND scheme features instability issues mainly when the resistance between the chamber and the external circulation is low, i.e. when one valve is open.

On the other hand, the stabilization term introduced by the ND-stab scheme proposed in this paper is able to successfully remove the non-physical oscillations that would otherwise affect the volume and pressure transients, as well as the tissue displacement and the computed stress.

6 Conclusions

In this paper we investigated the numerical stability of loosely coupled schemes for 0D–3D FSI problems. We focused on a problem of great applicative interest, that is the interaction between 3D models of cardiac mechanics and 0D models of blood dynamics. We considered several schemes. First, we focused on two schemes commonly used for 0D–3D FSI problems, namely the Dirichlet-Neumann (DN) scheme, in which pressures are updated simultaneously with the variables of the fluid model, and the Neumann-Dirichlet (ND) scheme, in which pressures are determined simultaneously with the displacement of the structure mechanics model. Then, we extended our analysis to schemes based on Robin-type boundary conditions, relying on linear combinations of the pressure continuity conditions, on one hand, and of either volume (V-RR scheme) or flux (Q-RR scheme) continuity, on the other hand. Notably, both the V-RR and the Q-RR schemes, for suitable choices of the Robin coefficients, reduce to the DN and the ND schemes.

We provided analytical results on the numerical stability of the considered schemes. We showed that the DN scheme suffers from severe instability issues in case of an incompressible fluid enclosed in a cavity (e.g., during isovolumetric phases of an heartbeat). This is related to the so-called balloon dilemma [5, 27], for which we provided an algebraic interpretation in terms of an eigenvalue of the Jacobian matrix that tends to $-\infty$ when the closed valve resistance increases, thus approaching the incompressible limit. The ND scheme, instead, is numerically stable when the valves are closed but might exhibit nonphysical oscillations when at least one valve is open (that is, with a non negligible fluid flow). The onset of these numerical oscillations is associated with the following features of the model at hand.

- *Stiffness of the structure.* A stiff tissue (equivalently, a large elastance) promotes the onset of oscillations. For parameters values typical of cardiac models, however, stiffness per se imposes very mild limits on Δt (nearly $\Delta t < 10^{-1}$ s).
- *Viscosity.* When the structure mechanics model features a large energy dissipation (either due to viscous boundary conditions or to a visco-elastic structure mechanics model), the ND scheme is numerically unstable. In this case, reducing Δt does not cure the numerical instabilities.
- *Inertia.* Similarly to viscosity, a large inertia makes the ND scheme unstable. Moreover, when inertial effects are included in the structure mechanics model, for small enough Δt the ND scheme features an eigenvalue lower than -1 and the scheme is numerically unstable. In conclusion, the ND scheme is never zero-stable when inertia is present and consequently is never convergent.
- *Active stress model with velocity dependence.* When dealing with active materials, and specifically when using a cardiac force generation model that accounts for fiber shortening velocity feedback on active force, the ND scheme may suffer from numerical instabilities. In particular, when the tissue is highly activated (typically in systole), crossbridge stiffness at the microscopic level has the effect, at the macroscopic level, of an increase in the effective stiffness (the increase can be up to 400-fold). This brings us back to the first point in this list, but with much more severe limitations on Δt .

The analysis revealed that, while the V-RR scheme can achieve stability with appropriate choices of the Robin coefficients, it ultimately becomes unstable as $\Delta t \rightarrow 0$ for realistic physical parameter values. This means that the V-RR scheme is not zero-stable and therefore not convergent. The Q-RR scheme, instead, with suitable selections of the Robin coefficients, features zero-stability and convergence. However, in both schemes, selecting the correct Robin coefficients is a delicate task. Even when the scheme is stable, in fact, accuracy may be limited. Furthermore, the parameter values that ensure good accuracy are often close to the stability threshold, making the practical selection of parameters particularly challenging, as is commonly observed in the literature [4, 8].

We then introduced a numerically consistent stabilization term that is able to prevent the onset of numerical oscillations in the ND scheme. We have shown that the stabilized ND scheme (ND-stab scheme) is absolutely stable for any choice of Δt and of model parameters. Remarkably, unlike RR schemes, the proposed scheme is free from parameters that need to be tuned, a significant advantage in practical contexts. From the implementation point of view, this stabilization term requires only a small modification with respect to the ND scheme. In addition, we proposed an effective algorithm, based on a Schur complement reduction, for solving the linear system that emerges at each Newton iteration when advancing the structure mechanics model. Our scheme requires solving $N + 1$ linear systems (N being the number of cavities considered in the model), where the matrix is the Jacobian of the standalone structure mechanics model, thus allowing reusing pre-existing solvers and preconditioners.

Our numerical simulations confirmed the theoretical results mentioned above. Specifically, we showed simulations obtained with a minimal FSI model, performing numerical tests under different conditions and verifying the expected order of convergence with respect to Δt . Then, we showed a simulation obtained with a multiscale model of cardiac electromechanics in a left ventricle, coupled with a lumped-parameter closed-loop model of the circulatory system. All the cases described above confirm that the stabilization term proposed in this paper is successful in removing the nonphysical oscillations, for any choice of parameters and for any value of Δt .

Appendix A Additional details on Robin–Robin schemes

In this appendix, we report additional details regarding the V-RR and the Q-RR schemes. First, for the reader convenience, we report here the versions of the V-RR scheme with reversed update order with respect to Eq. (11), consisting in solving, at each step, the $\mathcal{S}$ model prior to the $\mathcal{F}$ model:

$$\begin{cases} \mathcal{L} \left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}_S^{(k+1)}, t_{k+1} \right) = \mathbf{0} & k = 0, 1, \dots \\ \alpha_S \mathbf{V}_S(\mathbf{d}^{(k+1)}) + \beta_S \mathbf{p}_S^{(k+1)} = \alpha_S \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k)}) + \beta_S \mathbf{p}_{\mathcal{F}}^{(k)} & k = 0, 1, \dots \\ \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(*)}, \mathbf{p}_{\mathcal{F}}^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} \mathbf{V}_{\mathcal{F}}(\mathbf{c}^{(k+1)}) + \beta_{\mathcal{F}} \mathbf{p}_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} \mathbf{V}_S(\mathbf{d}^{(k+1)}) + \beta_{\mathcal{F}} \mathbf{p}_S^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (48)$$

Similarly, the Q-RR scheme with reversed order with respect to Eq. (12) reads:

$$\begin{cases} \mathcal{L} \left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}_S^{(k+1)}, t_{k+1} \right) = \mathbf{0} & k = 0, 1, \dots \\ \alpha_S \frac{\mathbf{V}_S(\mathbf{d}^{(k+1)}) - \mathbf{V}_S(\mathbf{d}^{(k)})}{\Delta t} + \beta_S \mathbf{p}_S^{(k+1)} = \alpha_S \frac{\mathbf{V}_F(\mathbf{c}^{(k)}) - \mathbf{V}_F(\mathbf{c}^{(k-1)})}{\Delta t} + \beta_S \mathbf{p}_F^{(k)} & k = 0, 1, \dots \\ \frac{\mathbf{c}^{(k+1)} - \mathbf{c}^{(k)}}{\Delta t} = \mathbf{H}(\mathbf{c}^{(k)}, \mathbf{p}_F^{(k+1)}, t_{k+1}) & k = 0, 1, \dots \\ \alpha_F \frac{\mathbf{V}_F(\mathbf{c}^{(k+1)}) - \mathbf{V}_F(\mathbf{c}^{(k)})}{\Delta t} + \beta_F \mathbf{p}_F^{(k+1)} = \alpha_F \frac{\mathbf{V}_S(\mathbf{d}^{(k+1)}) - \mathbf{V}_S(\mathbf{d}^{(k)})}{\Delta t} + \beta_F \mathbf{p}_S^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (49)$$

Then, complementing Sect. 3.1 we report the matrices $\mathbf{A}_{V-RR}$ and $\mathbf{B}_{V-RR}$, associated with the V-RR scheme of Eq. (19), as well as the matrices $\mathbf{A}_{Q-RR}$ and $\mathbf{B}_{Q-RR}$, associated with the Q-RR scheme of Eq. (20).

$$\begin{aligned} \mathbf{A}_{V-RR} &= \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} - \frac{M}{\Delta t^2} & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta t^{-1} & 0 & 0 & R^{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \alpha_S & 0 & -\alpha_S & 0 & \beta_S & -\beta_S \\ 0 & 0 & \alpha_F & 0 & 0 & \beta_F \end{pmatrix}; \\ \mathbf{B}_{V-RR} &= \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) \frac{M}{\Delta t^2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha_F & 0 & 0 & 0 & -\beta_F & 0 \end{pmatrix}; \\ \mathbf{A}_{Q-RR} &= \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} - \frac{M}{\Delta t^2} & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta t^{-1} & 0 & 0 & R^{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{\alpha_S}{\Delta t} & -\frac{\alpha_S}{\Delta t} & -\frac{\alpha_S}{\Delta t} & \frac{\alpha_S}{\Delta t} & \beta_S & -\beta_S \\ 0 & 0 & \frac{\alpha_F}{\Delta t} & -\frac{\alpha_F}{\Delta t} & 0 & \beta_F \end{pmatrix}; \\ \mathbf{B}_{Q-RR} &= \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) \frac{M}{\Delta t^2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\alpha_F}{\Delta t} & \frac{\alpha_F}{\Delta t} & 0 & 0 & -\beta_F & 0 \end{pmatrix}. \end{aligned} \quad (50)$$

It is interesting to show that, by condensing suitable variables, we can recover the matrices of Eq. (27), associated with the ND and DN schemes, from the matrices of Eq. (50). Let us first consider the scheme of Eq. (19). The variable p_S can be eliminated together with the last line, by exploiting the relationship

$$p_S^{(k)} = p_{\mathcal{F}}^{(k)} + \frac{\alpha_S}{\beta_S} \left(V_{\mathcal{F}}^{(k)} - V_S^{(k)} \right).$$

In this way, we can recast the scheme in the abstract form (21), by considering state vector $\psi^{(k)} := (V_S^{(k)}, V_S^{(k-1)}, V_{\mathcal{F}}^{(k)}, p_{\mathcal{F}}^{(k)})^T$, thus obtaining the matrices:

$$\begin{aligned} \mathbf{A}_{\text{DN}} &= \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} + \frac{\alpha_S}{\beta_S} - \frac{M}{\Delta t^2} - \frac{\alpha_S}{\beta_S} & -1 & & \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & \Delta t^{-1} & R^{-1} \\ & 0 & 0 & \alpha_{\mathcal{F}} & \beta_{\mathcal{F}} \end{pmatrix}; \\ \mathbf{B}_{\text{DN}} &= \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) & \frac{M}{\Delta t^2} & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} & 0 \\ \frac{\beta_{\mathcal{F}}\alpha_S}{\beta_S} - \alpha_{\mathcal{F}} & 0 & -\frac{\beta_{\mathcal{F}}\alpha_S}{\beta_S} & -\beta_{\mathcal{F}} \end{pmatrix}. \end{aligned} \quad (51)$$

It is now apparent that, by setting $(\alpha_{\mathcal{F}}, \beta_{\mathcal{F}}) = (1, 0)$ and $(\alpha_S, \beta_S) = (0, 1)$, we recover the matrices $\mathbf{A}_{\text{DN}}$ and $\mathbf{B}_{\text{DN}}$.

On the other hand, to recover the matrices associated with the ND scheme, it is convenient to consider the version of the V-RR scheme with switched order between the S and $\mathcal{F}$ updates, which reads:

$$\begin{cases} M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a(t_{k+1}) = p_S^{(k+1)} & k = 0, 1, \dots \\ \alpha_S V_S^{(k+1)} + \beta_S p_S^{(k+1)} = \alpha_S V_{\mathcal{F}}^{(k)} + \beta_S p_{\mathcal{F}}^{(k)} & k = 0, 1, \dots \\ \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p_{\mathcal{F}}^{(k+1)}) & k = 0, 1, \dots \\ \alpha_{\mathcal{F}} V_{\mathcal{F}}^{(k+1)} + \beta_{\mathcal{F}} p_{\mathcal{F}}^{(k+1)} = \alpha_{\mathcal{F}} V_S^{(k+1)} + \beta_{\mathcal{F}} p_S^{(k+1)} & k = 0, 1, \dots \end{cases} \quad (52)$$

Now, we eliminate the variable $p_{\mathcal{F}}$ through the relationship

$$p_{\mathcal{F}}^{(k)} = p_S^{(k)} + \frac{\alpha_{\mathcal{F}}}{\beta_{\mathcal{F}}} \left(V_S^{(k)} - V_{\mathcal{F}}^{(k)} \right).$$

thus getting the following matrices, associated with the state vector $\psi^{(k)} := (V_S^{(k)}, V_S^{(k-1)}, V_{\mathcal{F}}^{(k)}, p_S^{(k)})^T$:

$$\mathbf{A}_{\text{ND}} = \begin{pmatrix} E + \frac{M}{\Delta t^2} + \frac{C}{\Delta t} - \frac{M}{\Delta t^2} & 0 & -1 \\ 0 & 1 & 0 \\ R^{-1} \frac{\alpha_{\mathcal{F}}}{\beta_{\mathcal{F}}} & 0 & \Delta t^{-1} - R^{-1} \frac{\alpha_{\mathcal{F}}}{\beta_{\mathcal{F}}} \\ \alpha_{\mathcal{S}} & 0 & 0 \end{pmatrix};$$

$$\mathbf{B}_{\text{ND}} = \begin{pmatrix} -\left(\frac{M}{\Delta t^2} + \frac{C}{\Delta t}\right) \frac{M}{\Delta t^2} & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\Delta t^{-1} \\ -\frac{\beta_{\mathcal{S}} \alpha_{\mathcal{F}}}{\beta_{\mathcal{F}}} & 0 & \frac{\beta_{\mathcal{S}} \alpha_{\mathcal{F}}}{\beta_{\mathcal{F}}} - \alpha_{\mathcal{S}} - \beta_{\mathcal{S}} \end{pmatrix};$$
(53)

As expected, by setting $(\alpha_{\mathcal{S}}, \beta_{\mathcal{S}}) = (1, 0)$ and $(\alpha_{\mathcal{F}}, \beta_{\mathcal{F}}) = (0, 1)$, we obtain the matrices $\mathbf{A}_{\text{ND}}$ and $\mathbf{B}_{\text{ND}}$.

Appendix B Derivation of the minimal model

In this appendix, we present the derivation of the minimal model of left ventricle mechanics of Eq. (13), starting from the 3D model of Eqs. (2) and (47). More precisely, the derivation of the minimal model is based on the energy balance stemming from the 3D model. As demonstrated in [51, 53] – by multiplying the first equation of Eq. (2) by $\frac{\partial \mathbf{d}}{\partial t}$, integrating over Ω_0 and accounting for the boundary conditions – we have:

$$\frac{d}{dt} \mathcal{K}(t) + \frac{d}{dt} \mathcal{E}(t) = \Pi^{\text{act}}(t) + \Pi^{\text{diss}}(t) + \Pi^{\text{press}}(t), \quad (54)$$

that is a balance between the rate of change of the total kinetic ($\mathcal{K}$) and potential ($\mathcal{E}$) energy and the power associated with active forces (Π^{act}), viscous dissipation (Π^{diss}) and blood pressure (Π^{press}), defined as

$$\begin{aligned} \mathcal{E}(t) &= \int_{\Omega_0} \mathcal{W}(\mathbf{F}) d\mathbf{x} + \frac{1}{2} \int_{\Gamma_0^{\text{epi}}} \left[K_{\perp}^{\text{epi}} |\mathbf{d} \cdot \mathbf{N}|^2 + K_{\parallel}^{\text{epi}} |(\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) \mathbf{d}|^2 \right] d\Gamma_0, \\ \mathcal{K}(t) &= \frac{1}{2} \int_{\Omega_0} \rho \left| \frac{\partial \mathbf{d}}{\partial t} \right|^2 d\mathbf{x}, \quad \Pi^{\text{act}}(t) = - \int_{\Omega_0} T_a \frac{\mathbf{F} \mathbf{f}_0 \otimes \mathbf{f}_0}{\sqrt{\mathcal{I}_4 f}} : \nabla \left(\frac{\partial \mathbf{d}}{\partial t} \right) d\mathbf{x}, \\ \Pi^{\text{diss}}(t) &= - \int_{\Gamma_0^{\text{epi}}} \left[C_{\perp}^{\text{epi}} \left| \frac{\partial \mathbf{d}}{\partial t} \cdot \mathbf{N} \right|^2 + C_{\parallel}^{\text{epi}} \left| (\mathbf{I} - \mathbf{N} \otimes \mathbf{N}) \frac{\partial \mathbf{d}}{\partial t} \right|^2 \right] d\Gamma_0, \\ \Pi^{\text{press}}(t) &= p(t) \frac{d}{dt} V(\mathbf{d}(t)). \end{aligned}$$

where p and V represent the pressure and volume of the blood contained in the left ventricle.

To derive a lumped-parameter model for the left ventricle dynamics, we consider the volume V as a state variable, and we express the terms of Eq. (54) as functions of V and its time derivatives, making suitable approximations where needed.

First, by assuming that the potential energy is function of the state variable V , we have $\mathcal{E}(t) = \tilde{\mathcal{E}}(V(t))$ for some function $\tilde{\mathcal{E}}$. We accordingly define the steady-state pressure corresponding to a non activated tissue as $p_{\text{pass}}(V) := \tilde{\mathcal{E}}'(V)$.

Then, we assume that the inertial and damping effects are mainly associated with the tissue motion in radial direction. We then consider a displacement of the form of $\mathbf{d}(\mathbf{x}, t) = (r(t) - r_0)\mathbf{e}_\rho(\mathbf{x})$, where $\mathbf{e}_\rho$ is the radial unit vector, having set without loss of generality the origin in the barycenter of Ω_0 . It follows:

$$\mathcal{K}(t) \simeq \frac{1}{2} \left[\int_{\Omega_0} \rho \, d\mathbf{x} \right] \dot{r}^2(t) = \frac{1}{2} M_{\text{LV}} \dot{r}^2(t), \quad \Pi^{\text{diss}}(t) \simeq -C_{\perp}^{\text{epi}} |\Gamma_0^{\text{epi}}| \dot{r}^2(t) \quad (55)$$

where M_{LV} denotes the left ventricle mass, $|\Gamma_0^{\text{epi}}|$ is the epicardial surface area and where we assumed that $\mathbf{N} \simeq \mathbf{e}_\rho$ on the epicardium. By dimensional analysis, we assume that the ventricle volume and radius are linked by a relationship of the type of $V_{\text{LV}}(t) = \alpha r^3(t)$, for some constant α . For a half-spherical cavity, e.g., we have $\alpha = \frac{2}{3}\pi$. It follows

$$\begin{aligned} \dot{r}^2(t) &= \frac{1}{9} \alpha^{-\frac{2}{3}} V^{-\frac{4}{3}}(t) \dot{V}^2(t) \\ \frac{d}{dt} \left(\dot{r}^2(t) \right) &= \frac{2}{9} \alpha^{-\frac{2}{3}} V^{-\frac{4}{3}}(t) \left[\ddot{V}(t) - \frac{2}{9} V^{-1}(t) \dot{V}^2(t) \right] \dot{V}(t) \\ &\simeq \frac{2}{9} \alpha^{-\frac{2}{3}} V^{-\frac{4}{3}}(t) \ddot{V}(t) \dot{V}(t) \end{aligned} \quad (56)$$

where we have neglected the term $\dot{V}^2(t)$, under the hypothesis of small velocities.

Concerning the active force power term Π^{act} we notice that, thanks to the identity (see [52])

$$\frac{\mathbf{F}\mathbf{f}_0 \otimes \mathbf{f}_0}{\sqrt{\mathcal{I}_{4f}}} = \frac{\partial \lambda}{\partial \mathbf{F}},$$

and assuming an homogeneous active tension and fibers elongation, we have

$$\Pi^{\text{act}}(t) = - \int_{\Omega_0} T_a \dot{\lambda} \, d\mathbf{x} \simeq -|\Omega_0| T_a(t) \dot{\lambda}(t), \quad (57)$$

where $|\Omega_0|$ denotes the volume of the reference domain. Finally, we need to link the fibers elongation λ to the state variable V . We thus assume that $\lambda = \psi(V)$, for some increasing function ψ .

By replacing Eqs. (55)–(57) into Eq. (54), we get

$$\begin{aligned} \frac{1}{9} M_{\text{LV}} \alpha^{-\frac{2}{3}} V^{-\frac{4}{3}}(t) \ddot{V}(t) \dot{V}(t) + p_{\text{pass}}(V(t)) \dot{V}(t) + |\Omega_0| T_a(t) \psi'(V(t)) \dot{V}(t) \\ + \frac{1}{9} C_{\perp}^{\text{epi}} |\Gamma_0^{\text{epi}}| \alpha^{-\frac{2}{3}} V^{-\frac{4}{3}}(t) \dot{V}^2(t) - p(t) \dot{V}(t) = 0, \end{aligned}$$

which entails

$$M(V)\ddot{V}(t) + C(V)\dot{V}(t) + p_{\text{pass}}(V(t)) + A(V)T_a(t) = p(t),$$

having defined

$$M(V) = \frac{1}{9}M_{\text{LV}}\alpha^{-\frac{2}{3}}V^{-\frac{4}{3}}(t), \quad C(V) = \frac{1}{9}C_{\perp}^{\text{epi}}|\Gamma_0^{\text{epi}}|\alpha^{-\frac{2}{3}}V^{-\frac{4}{3}}(t), \quad A(V) = |\Omega_0|\psi'(V(t))$$

as the equivalent mass, viscosity and active stress factor. Finally, for simplicity we assume the factors M , C and A as constants by setting $M = M(\bar{V})$, $C = C(\bar{V})$ and $A = A(\bar{V})$, for a reference volume $\bar{V}$. The minimal model thus reads:

$$M\ddot{V}(t) + C\dot{V}(t) + p_{\text{pass}}(V(t)) + AT_a(t) = p(t). \quad (58)$$

To provide realistic values of the coefficients entering in Eq. (58), we consider the 3D electromechanical model, based on the Zygote heart [61], described in Sect. 5.2.1. Specifically, we set $|\Omega_0| = 100\text{mL}$, $|\Gamma_0^{\text{epi}}| = 140\text{cm}^2$, $M_{\text{LV}} = 0.1\text{kg}$ and $C_{\perp}^{\text{epi}} = 2 \cdot 10^4 \text{Pa s m}^{-1}$. To link the chamber radius and its volume, we assume a half-sphere approximation ($\alpha = \frac{2}{3}\pi$). From simulation data, we estimate the derivative of ψ as $\psi' = 3.5 \cdot 10^{-3} \text{mL}^{-1}$. To define p_{pass} , we fit the end-diastolic pressure-volume relationship through the Klotz's curve [33] of Eq. (46). Finally, to compute the coefficients we set $\bar{V} = 120\text{mL}$. The resulting coefficients are reported in Tab. 1.

Appendix C The effect of active stress on numerical stability

We here provide details on the discussion of Sect. 3.3.3, regarding the role played by active stress in the stability analysis of Sect. 3. As a minimal model of active force generation, we consider the generalized Huxley model of [49], that reads

$$\begin{cases} \dot{K}_a(t) = -r K_a(t) + \mu_K P(t), & t > 0, \\ \dot{T}_a(t) = -r T_a(t) + \mu_T P(t) + \dot{\lambda}(t) K_a(t), & t > 0, \\ K_a(0) = K_{a,0}, \quad T_a(0) = T_{a,0}, \end{cases} \quad (59)$$

where r , μ_K and μ_T are suitable constants and $P(t)$ represents the fraction of permissive regulatory units at time t . The so-called permissivity $P(t)$ is in fact the chemical input of the model. Specifically, we have $P(t) = 0$ when the tissue is relaxed, $P(t) > 0$ when activated. As shown in [49], model (59) exhibits the main experimental features of force generation in sarcomeres and can be regarded as a linearization of more complex models. To link the microscopic to the macroscopic level, coherently with the

derivation “Appendix B”, we set $\lambda(t) = \psi(V(t))$. Let us consider the ND scheme, that reads

$$\left\{ \begin{array}{l} M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) + A T_a^{(k+1)} = p^{(k+1)} \quad k = 0, 1, \dots \\ \frac{K_a^{(k+1)} - K_a^{(k)}}{\Delta t} = -r K_a^{(k+1)} + \mu_K P(t_{k+1}), \quad k = 0, 1, \dots \\ \frac{T_a^{(k+1)} - T_a^{(k)}}{\Delta t} = -r T_a^{(k+1)} + \mu_T P(t_{k+1}) + \frac{\psi(V_S^{(k+1)}) - \psi(V_S^{(k)})}{\Delta t} K_a^{(k+1)}, \quad k = 0, 1, \dots \\ V_S^{(k+1)} = V_{\mathcal{F}}^{(k)} \quad k = 0, 1, \dots \\ \frac{V_{\mathcal{F}}^{(k+1)} - V_{\mathcal{F}}^{(k)}}{\Delta t} = Q_{\text{valve}}(p_{\text{ext}}(t_{k+1}) - p^{(k+1)}) \quad k = 0, 1, \dots \end{array} \right. \quad (60)$$

The updated active stress then reads

$$T_a^{(k+1)} = \frac{1}{1 + r \Delta t} \left[T_a^{(k)} + K_a^{(k+1)} \left(\psi(V_S^{(k+1)}) - \psi(V_S^{(k)}) \right) + \Delta t \mu_T P(t_{k+1}) \right]. \quad (61)$$

Thus, for $\Delta t \rightarrow 0^+$

$$T_a^{(k+1)} \approx T_a^{(k)} + K_a^{(k+1)} \left(\psi(V_S^{(k+1)}) - \psi(V_S^{(k)}) \right) \quad (62)$$

which entails (since $K_a^{(k)}$ varies slowly) that

$$T_a^{(k+1)} \approx \bar{T}_a + K_a^{(k+1)} \psi(V_S^{(k+1)}), \quad (63)$$

Replacing Eq. (63) into Eq. (60), we get:

$$\begin{aligned} M \frac{V_S^{(k+1)} - 2V_S^{(k)} + V_S^{(k-1)}}{\Delta t^2} + C \frac{V_S^{(k+1)} - V_S^{(k)}}{\Delta t} + p_{\text{pass}}(V_S^{(k+1)}) \\ + A K_a^{(k+1)} \psi(V_S^{(k+1)}) + A \bar{T}_a = p^{(k+1)} \end{aligned} \quad (64)$$

Therefore, we formally recover the case of prescribed active stress (see Eq. (17)), provided that we replace the purely passive pressure function $p_{\text{pass}}(V_S)$ with

$$p_{\text{pass+act}}(V_S) = p_{\text{pass}}(V_S) + A K_a \psi(V_S). \quad (65)$$

The first term of Eq. (65) accounts for the passive elastance, while the second one accounts for the active elastance, due to microscale crossbridge links. In conclusion, for $\Delta t \rightarrow 0^+$, the stability analysis of Sect. 3.3 is still valid, provided that we redefine the elastance as $E = E_{\text{pass}} + E_{\text{act}}$, where

$$E_{\text{pass}}(V_S) = p'_{\text{pass}}(V_S); \quad E_{\text{act}}(V_S) = A K_a \psi'(V_S). \quad (66)$$

By using the parameters of the original model (see [49]), we get $E_{\text{act}} = 36 \text{ mmHg mL}^{-1}$.

We now check whether the two additional eigenvalues associated with the dynamics of K_a and T_a can introduce instabilities in the numerical solution or not. From the second equation, it follows

$$K_a^{(k+1)} = \frac{1}{1 + r\Delta t} \left[K_a^{(k)} + \Delta t \mu_K P(t_{k+1}) \right]. \quad (67)$$

Thus, the active stiffness dynamics is associated with the eigenvalue

$$\mu = (1 + r\Delta t)^{-1} < 1, \quad (68)$$

that cannot be responsible of numerical instabilities, being strictly lower than one.

To study the eigenvalue associated with the dynamics of T_a , instead, we consider the following approximate equation, obtained by neglecting high-order effects in the mechanics equilibrium and by subtracting the equation for two consecutive time steps:

$$\begin{aligned} p^{(k+1)} - p^{(k)} &\approx \left(p_{\text{pass}}(V_S^{(k+1)}) - p_{\text{pass}}(V_S^{(k)}) \right) + A \left(T_a^{(k+1)} - T_a^{(k)} \right) \\ &\approx E_{\text{pass}}(V_S^{(k+1)}) \left(V_S^{(k+1)} - V_S^{(k)} \right) + A \left(T_a^{(k+1)} - T_a^{(k)} \right) \\ &\approx \frac{E_{\text{pass}}(V_S^{(k+1)})}{\psi'(V_S^{(k+1)})} \left(\psi(V_S^{(k+1)}) - \psi(V_S^{(k)}) \right) + A \left(T_a^{(k+1)} - T_a^{(k)} \right). \end{aligned} \quad (69)$$

Therefore, replacing (69) into Eq. (60), we have

$$\begin{aligned} (1 + r\Delta t) T_a^{(k+1)} &= T_a^{(k)} + K_a^{(k+1)} \left(\psi(V_S^{(k+1)}) - \psi(V_S^{(k)}) \right) + \Delta t \mu_T P(t_{k+1}) \\ &\approx T_a^{(k)} + \frac{K_a^{(k+1)} \psi'(V_S^{(k+1)})}{E_{\text{pass}}(V_S^{(k+1)})} \\ &\quad \times \left(\left(p^{(k+1)} - p^{(k)} \right) - A \left(T_a^{(k+1)} - T_a^{(k)} \right) \right) \\ &\quad + \Delta t \mu_T P(t_{k+1}), \end{aligned} \quad (70)$$

which entails that the eigenvalue associated with the dynamics of T_a is approximated by

$$\mu = \frac{1 + \frac{E_{\text{act}}}{E_{\text{pass}}}}{1 + r\Delta t + \frac{E_{\text{act}}}{E_{\text{pass}}}} < 1, \quad (71)$$

which cannot lead to numerical instabilities either.

To validate the results of the asymptotic analysis of the current appendix, we numerically compute the eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ associated with the ND scheme when the model $\mathcal{F}$ – $\mathcal{S}$ is coupled with the activation model (59) (that is scheme

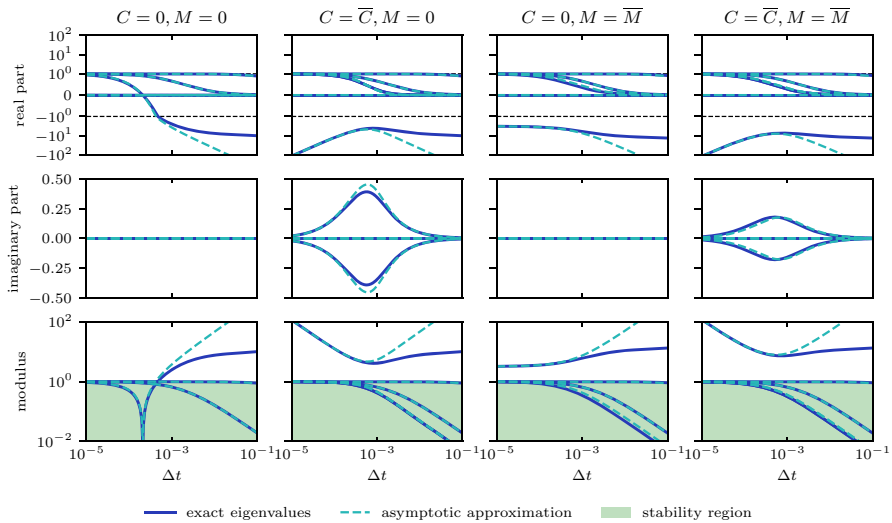


Fig. 23 Eigenvalues of the Jacobian matrix $-\mathbf{A}^{-1}\mathbf{B}$ for the ND scheme in the open valve case: comparison of the exact eigenvalues and their approximation obtained through the asymptotic analysis of “Appendix C”

(60)) under different conditions (quasi-static case, with viscosity, inertia or both) and for different values of Δt . Then, we compare the results with the eigenvalues obtained when the ND scheme is applied to the $\mathcal{F}$ – $\mathcal{S}$ alone (that is scheme (17)), with an increased elastance ($E = E_{\text{pass}} + E_{\text{act}}$). To the latter set eigenvalues, we also append the estimated eigenvalues of Eqs. (68) and (71). The results, shown in Fig. 23, show that, as expected, for small values of Δt the approximation is very accurate. Moreover, the eigenvalues obtained through the asymptotic approximation provide an accurate estimate of the stability thresholds.

Appendix D Algorithm for the resolution of the structure mechanics substep of the ND-stab scheme

We present an algorithm for the numerical resolution of the nonlinear system of equations arising from the structure mechanics substep of the ND-stab scheme of Eq. (45). To ease the notation, we write the nonlinear system at hand as

$$\begin{cases} \mathbf{r}_d^{(k)}(\mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}) = \mathbf{0}, \\ \mathbf{r}_p^{(k)}(\mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}) = \mathbf{0}, \end{cases} \quad (72)$$

having defined the residuals

$$\begin{aligned} \mathbf{r}_d^{(k)}(\mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}) &= \mathcal{L} \left(\frac{\mathbf{d}^{(k+1)} - 2\mathbf{d}^{(k)} + \mathbf{d}^{(k-1)}}{\Delta t^2}, \frac{\mathbf{d}^{(k+1)} - \mathbf{d}^{(k)}}{\Delta t}, \mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}, t_{k+1} \right), \\ \mathbf{r}_p^{(k)}(\mathbf{d}^{(k+1)}, \mathbf{p}^{(k+1)}) &= \mathbf{V}_S(\mathbf{d}^{(k+1)}) - \mathbf{V}_F(\mathbf{c}^{(k)}) - \Delta t \mathbf{Q}_{\text{stab}}(\mathbf{p}^{(k+1)}, \mathbf{c}^{(k)}, t_{k+1}). \end{aligned}$$

We solve the nonlinear system (72) by means the Newton method. Specifically, to update the state variables $(\mathbf{d}^{(k)}, \mathbf{p}^{(k)})$ from iteration k to $k + 1$, we set $\mathbf{d}^0 = \mathbf{d}^{(k)}$, $\mathbf{p}^0 = \mathbf{p}^{(k)}$ and then we iteratively update according to $\mathbf{d}^{j+1} = \mathbf{d}^j + \Delta \mathbf{d}^j$, $\mathbf{p}^{j+1} = \mathbf{p}^j + \Delta \mathbf{p}^j$, where the increments $\Delta \mathbf{d}^j$ and $\Delta \mathbf{p}^j$ are solution of the linear system

$$\begin{pmatrix} \mathbf{J}_{\mathbf{d}\mathbf{d}}^j & \mathbf{J}_{\mathbf{d}\mathbf{p}}^j \\ \mathbf{J}_{\mathbf{p}\mathbf{d}}^j & \mathbf{J}_{\mathbf{p}\mathbf{p}}^j \end{pmatrix} \begin{pmatrix} \Delta \mathbf{d}^j \\ \Delta \mathbf{p}^j \end{pmatrix} = - \begin{pmatrix} \mathbf{r}_{\mathbf{d}}^{(k)}(\mathbf{d}^j, \mathbf{p}^j) \\ \mathbf{r}_{\mathbf{p}}^{(k)}(\mathbf{d}^j, \mathbf{p}^j) \end{pmatrix}. \quad (73)$$

Finally, we set $\mathbf{d}^{(k+1)} = \mathbf{d}^j$, $\mathbf{p}^{(k+1)} = \mathbf{p}^j$ where j is the Newton iteration after a suitable convergence criterion is attained.

Concerning the numerical solution of the linear system (73), we adopt a Schur complement reduction, as proposed in [48, 53]. We notice that, denoting by N the number of cavities in the mechanical model, we can split the matrices and vectors as

$$\begin{aligned} \mathbf{J}_{\mathbf{d}\mathbf{p}}^j &= \begin{pmatrix} | & & | \\ \mathbf{j}_{\mathbf{d}1} & \dots & \mathbf{j}_{\mathbf{d}N} \\ | & & | \end{pmatrix}, \quad \Delta \mathbf{p}^j = \begin{pmatrix} \Delta p_1 \\ \vdots \\ \Delta p_N \end{pmatrix}, \\ \mathbf{J}_{\mathbf{p}\mathbf{d}}^j &= \begin{pmatrix} -\mathbf{j}_{\mathbf{1d}}^T & - \\ \vdots & \\ -\mathbf{j}_{\mathbf{Nd}}^T & - \end{pmatrix}, \quad \mathbf{r}_{\mathbf{p}}^{(k)}(\mathbf{d}^j, \mathbf{p}^j) = \begin{pmatrix} \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_N \end{pmatrix}. \end{aligned}$$

At each Newton iteration we first solve the following $N + 1$ linear system associated with the Jacobian matrix of the structure mechanics problem:

$$\mathbf{J}_{\mathbf{d}\mathbf{d}}^j \mathbf{w} = \mathbf{r}_{\mathbf{d}}^{(k)}(\mathbf{d}^j, \mathbf{p}^j), \quad \mathbf{J}_{\mathbf{d}\mathbf{d}}^j \mathbf{w}_i = \mathbf{j}_{\mathbf{d}i} \quad \text{for } i = 1, \dots, N.$$

Then, we recover the pressure increments by solving the $N \times N$ linear system (we recall that $N \leq 4$):

$$\left[\mathbf{J}_{\mathbf{p}\mathbf{p}}^j - \begin{pmatrix} -\mathbf{j}_{\mathbf{1d}}^T & - \\ \vdots & \\ -\mathbf{j}_{\mathbf{Nd}}^T & - \end{pmatrix} \begin{pmatrix} | & & | \\ \mathbf{w}_1 & \dots & \mathbf{w}_N \\ | & & | \end{pmatrix} \right] \Delta \mathbf{p}^j = \begin{pmatrix} \mathbf{r}_1 - \mathbf{j}_{\mathbf{1d}} \cdot \mathbf{w} \\ \vdots \\ \mathbf{r}_N - \mathbf{j}_{\mathbf{Nd}} \cdot \mathbf{w} \end{pmatrix}$$

Finally, the displacement increment is computed as:

$$\Delta \mathbf{d}^j = -\mathbf{w} - \sum_{i=2}^N \mathbf{w}_i \Delta p_i.$$

Acknowledgements The author acknowledges the grant Dipartimento di Eccellenza 2023–2027 (Dipartimento di Matematica, Politecnico di Milano). The author has received support from the project PRIN2022, MUR, funded by the European Union (Grant P2022N5ZNP). The author is a member of GNCS, “Gruppo Nazionale per il Calcolo Scientifico” (National Group for Scientific Computing) of INdAM (Istituto Nazionale di Alta Matematica).

Funding Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Arts, T., Delhaas, T., Bovendeerd, P., Verbeek, X., Prinzen, F.W.: Adaptation to mechanical load determines shape and properties of heart and circulation: the circadapt model. *Am. J. Physiol.-Heart Circ. Physiol.* **288**(4), H1943–H1954 (2005)
2. Astorino, M., Grandmont, C.: Convergence analysis of a projection semi-implicit coupling scheme for fluid-structure interaction problems. *Numer. Math.* **116**(4), 721–767 (2010)
3. Augustin, C.M., Gsell, M.A., Karabelas, E., Willemen, E., Prinzen, F.W., Lumens, J., Vigmond, E.J., Plank, G.: A computationally efficient physiologically comprehensive 3D–0D closed-loop model of the heart and circulation. *Comput. Methods Appl. Mech. Eng.* **386**, 114092 (2021)
4. Badia, S., Nobile, F., Vergara, C.: Fluid-structure partitioned procedures based on robin transmission conditions. *J. Comput. Phys.* **227**(14), 7027–7051 (2008)
5. Bao, Y., Donev, A., Griffith, B.E.: An immersed boundary method with divergence-free velocity interpolation. *J. Comput. Phys.* **347**, 183–206 (2017)
6. Bayer, J.D., Blake, R.C., Plank, G., Trayanova, N.A.: A novel rule-based algorithm for assigning myocardial fiber orientation to computational heart models. *Ann. Biomed. Eng.* **40**(10), 2243–2254 (2012)
7. Blanco, P.J., Feijóo, R.A.: A 3d–1d–0d computational model for the entire cardiovascular system. *Comput. Mech.* **24**, 5887–5911 (2010)
8. Bucelli, M., Gabriel, M.G., Quarteroni, A., Gigante, G., Vergara, C.: A stable loosely-coupled scheme for cardiac electro-fluid-structure interaction. *J. Comput. Phys.* **490**, 112326 (2023)
9. Bukač, M., Čanić, S., Glowinski, R., Tambača, J., Quaini, A.: Fluid-structure interaction in blood flow capturing non-zero longitudinal structure displacement. *J. Comput. Phys.* **235**, 515–541 (2013)
10. Burman, E., Fernández, M.A.: Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility. *Comput. Methods Appl. Mech. Eng.* **198**(5–8), 766–784 (2009)
11. Causin, P., Gerbeau, J.-F., Nobile, F.: Added-mass effect in the design of partitioned algorithms for fluid-structure problems. *Comput. Methods Appl. Mech. Eng.* **194**(42–44), 4506–4527 (2005)
12. Colli Franzone, P., Pavarino, L., Scacchi, S.: *Mathematical Cardiac Electrophysiology*, vol. 13. Springer, Berlin (2014)
13. Dedè, L., Quarteroni, A., Regazzoni, F.: Mathematical and numerical models for the cardiac electromechanical function. *Atti della Accademia Nazionale dei Lincei, Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei - Matematica e Applicazioni* **32**(2), 233–272 (2021)
14. Deparis, S., Fernández, M.A., Formaggia, L.: Acceleration of a fixed point algorithm for fluid-structure interaction using transpiration conditions. *ESAIM Math. Model. Numer. Anal.* **37**(4), 601–616 (2003)
15. Fedele, M., Quarteroni, A.: Polygonal surface processing and mesh generation tools for the numerical simulation of the cardiac function. *Int. J. Numer. Methods Biomed. Eng.* **37**, e3435 (2021)
16. Fernández, M.A.: Incremental displacement-correction schemes for incompressible fluid–structure interaction. *Numer. Math.* **123**(1), 21–65 (2013)
17. Fernández, M.A., Mullaert, J.: Convergence and error analysis for a class of splitting schemes in incompressible fluid–structure interaction. *IMA J. Numer. Anal.* **36**(4), 1748–1782 (2016)
18. Fernández, M.A., Mullaert, J., Vidrascu, M.: Explicit Robin–Neumann schemes for the coupling of incompressible fluids with thin-walled structures. *Comput. Methods Appl. Mech. Eng.* **267**, 566–593 (2013)

19. Formaggia, L., Moura, A., Nobile, F.: On the stability of the coupling of 3d and 1d fluid–structure interaction models for blood flow simulations. *ESAIM Math. Model. Numer. Anal.* **41**(4), 743–769 (2007)
20. Förster, C., Wall, W.A., Ramm, E.: Artificial added mass instabilities in sequential staggered coupling of nonlinear structures and incompressible viscous flows. *Comput. Methods Appl. Mech. Eng.* **196**(7), 1278–1293 (2007)
21. Gerach, T., Schuler, S., Fröhlich, J., Lindner, L., Kovacheva, E., Moss, R., Wülfers, E.M., Seemann, G., Wieners, C., Loewe, A.: Electro-mechanical whole-heart digital twins: a fully coupled multi-physics approach. *Mathematics* **9**(11), 1247 (2021)
22. Gerardo-Giorda, L., Nobile, F., Vergara, C.: Analysis and optimization of robin-robin partitioned procedures in fluid–structure interaction problems. *SIAM J. Numer. Anal.* **48**(6), 2091–2116 (2010)
23. Gerbi, A.: Numerical Approximation of Cardiac Electro-fluid-mechanical Models: Coupling Strategies for Large-scale Simulation. Ph.D. thesis, Ecole Polytechnique Fédérale de Lausanne (2018)
24. Gerbi, A., Dedè, L., Quarteroni, A.: A monolithic algorithm for the simulation of cardiac electromechanics in the human left ventricle. *Math. Eng.* **1**(1), 1–37 (2018)
25. Gigante, G., Vergara, C.: On the choice of interface parameters in robin-robin loosely coupled schemes for fluid–structure interaction. *Fluids* **6**(6), 213 (2021)
26. Gigante, G., Vergara, C.: On the stability of a loosely-coupled scheme based on a robin interface condition for fluid–structure interaction. *Comput. Math. Appl.* **96**, 109–119 (2021)
27. Griffith, B.E., Patankar, N.A.: Immersed methods for fluid–structure interaction. *Annu. Rev. Fluid Mech.* **52**, 421–448 (2020)
28. Gurtin, M.E.: *An Introduction to Continuum Mechanics*. Academic Press, New York (1982)
29. Hill, A.: The heat of shortening and the dynamic constants of muscle. *Proc. R. Soc. Lond. B Biol. Sci.* **126**(843), 136–195 (1938)
30. Hirschvogel, M., Bassilious, M., Jagschies, L., et al.: A monolithic 3d–0d coupled closed-loop model of the heart and the vascular system: experiment-based parameter estimation for patient-specific cardiac mechanics. *Int. J. Numer. Methods Biomed. Eng.* **33**(8), e2842 (2017)
31. Hou, G., Wang, J., Layton, A.: Numerical methods for fluid–structure interaction-a review. *Commun. Comput. Phys.* **12**(2), 337–377 (2012)
32. Kerckhoffs, R.C., Neal, M.L., Gu, Q., Bassingthwaite, J.B., Omens, J.H., McCulloch, A.D.: Coupling of a 3d finite element model of cardiac ventricular mechanics to lumped systems models of the systemic and pulmonic circulation. *Ann. Biomed. Eng.* **35**(1), 1–18 (2007)
33. Klotz, S., Hay, I., Dickstein, M.L., Yi, G.-H., Wang, J., Maurer, M.S., Kass, D.A., Burkhoff, D.: Single-beat estimation of end-diastolic pressure-volume relationship: a novel method with potential for noninvasive application. *Am. J. Physiol.-Heart Circ. Physiol.* **291**(1), H403–H412 (2006)
34. Krantz, S.G., Parks, H.R.: *The Implicit Function Theorem: History, Theory, and Applications*. Springer, Berlin (2012)
35. Küttler, U., Wall, W.A.: Fixed-point fluid–structure interaction solvers with dynamic relaxation. *Comput. Mech.* **43**(1), 61–72 (2008)
36. Land, S., Niederer, S.A., Aronsen, J.M., Espe, E.K., Zhang, L., Louch, W.E., Sjaastad, I., Sejersted, O.M., Smith, N.P.: An analysis of deformation-dependent electromechanical coupling in the mouse heart. *J. Physiol.* **590**(18), 4553–4569 (2012)
37. Land, S., Park-Holohan, S., Smith, N., dos Remedios, C., Kentish, J., Niederer, S.: A model of cardiac contraction based on novel measurements of tension development in human cardiomyocytes. *J. Mol. Cell. Cardiol.* **106**, 68–83 (2017)
38. Landajuela, M., Vidrascu, M., Chapelle, D., Fernández, M.A.: Coupling schemes for the FSI forward prediction challenge: comparative study and validation. *Int. J. Numer. Methods Biomed. Eng.* **33**(4), e2813 (2017)
39. Lefrançois, E., Boufflet, J.-P.: An introduction to fluid-structure interaction: application to the piston problem. *SIAM Rev.* **52**(4), 747–767 (2010)
40. Levrero-Florencio, F., Margara, F., Zacur, E., Bueno-Orovio, A., Wang, Z., Santiago, A., Aguado-Sierra, J., Houzeaux, G., Grau, V., Kay, D., et al.: Sensitivity analysis of a strongly-coupled human-based electromechanical cardiac model: effect of mechanical parameters on physiologically relevant biomarkers. *Comput. Methods Appl. Mech. Eng.* **361**, 112762 (2020)
41. Niederer, S.A., Hunter, P.J., Smith, N.P.: A quantitative analysis of cardiac myocyte relaxation: a simulation study. *Biophys. J.* **90**(5), 1697–1722 (2006)
42. Ogden, R.W.: *Non-Linear Elastic Deformations*. Courier Corporation, North Chelmsford (1997)

43. Piersanti, R., Regazzoni, F., Salvador, M., Corno, A. F., Dede, L., Vergara, C., Quarteroni, A.: 3D–0D closed-loop model for the simulation of cardiac biventricular electromechanics. *Comput. Methods Appl. Mech. Eng.* (accepted) (2021)
44. Quarteroni, A., Dede', L., Manzoni, A., Vergara, C.: *Mathematical modelling of the human cardiovascular system: data, numerical approximation, clinical applications*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge (2019)
45. Quarteroni, A., Sacco, R., Saleri, F.: *Numerical Mathematics*, vol. 37. Springer, Berlin (2010)
46. Quarteroni, A., Valli, A.: *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, Oxford (1999)
47. Quarteroni, A., Valli, A.: *Numerical Approximation of Partial Differential Equations*, vol. 23. Springer, Berlin (2008)
48. Regazzoni, F.: *Mathematical modeling and Machine Learning for the numerical simulation of cardiac electromechanics*. Ph.D. thesis, Politecnico di Milano (2020)
49. Regazzoni, F., Dedè, L., Quarteroni, A.: Active force generation in cardiac muscle cells: mathematical modeling and numerical simulation of the actin-myosin interaction. *Vietnam J. Math.* 1–32 (2020)
50. Regazzoni, F., Dedè, L., Quarteroni, A.: Biophysically detailed mathematical models of multiscale cardiac active mechanics. *PLoS Comput. Biol.* **16**(10), e1008294 (2020)
51. Regazzoni, F., Dedè, L., Quarteroni, A.: Machine learning of multiscale active force generation models for the efficient simulation of cardiac electromechanics. *Comput. Methods Appl. Mech. Eng.* **370**, 113268 (2020)
52. Regazzoni, F., Quarteroni, A.: An oscillation-free fully partitioned scheme for the numerical modeling of cardiac active mechanics. *Comput. Methods Appl. Mech. Eng.* **373**, 113506 (2021)
53. Regazzoni, F., Salvador, M., Africa, P., Fedele, M., Dedè, L., Quarteroni, A.: A cardiac electromechanical model coupled with a lumped-parameter model for closed-loop blood circulation. *J. Comput. Phys.* **457**, 111083 (2022)
54. Rice, J., Wang, F., Bers, D., de Tombe, P.: Approximate model of cooperative activation and crossbridge cycling in cardiac muscle using ordinary differential equations. *Biophys. J.* **95**(5), 2368–2390 (2008)
55. Rice, J., Winslow, R., Hunter, W.: Comparison of putative cooperative mechanisms in cardiac muscle: length dependence and dynamic responses. *Am. J. Physiol.-Heart Circ. Physiol.* **276**(5), H1734–H1754 (1999)
56. Tallec, P.L., Mani, S.: Numerical analysis of a linearised fluid–structure interaction problem. *Numer. Math.* **87**(2), 317–354 (2000)
57. Ten Tusscher, K.H., Panfilov, A.V.: Alternans and spiral breakup in a human ventricular tissue model. *Am. J. Physiol.-Heart Circ. Physiol.* **291**(3), H1088–H1100 (2006)
58. Toselli, A., Widlund, O.: *Domain Decomposition Methods-Algorithms and Theory*, vol. 34. Springer, Berlin (2004)
59. Usyk, T.P., LeGrice, I.J., McCulloch, A.D.: Computational model of three-dimensional cardiac electromechanics. *Comput. Vis. Sci.* **4**(4), 249–257 (2002)
60. Washio, T., Yoneda, K., Okada, J., Kariya, T., Sugiura, S., Hisada, T.: Ventricular fiber optimization utilizing the branching structure. *Int. J. Numer. Methods Biomed. Eng.* e02753 (2015)
61. Zygote Media Group Inc. Zygote solid 3D heart generation II development report. Technical report. (2014)