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REVIEW

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A review of inverse problems for generalized elastic media: formulations, experiments, synthesis

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Abstract Starting from the seminal works of Toupin, Mindlin and Germain, a wide class of generalized elastic models have been proposed via the principle of virtual work, by postulating expressions of the elastic energy enriched by additional kinematic descriptors or by higher gradients of the placement. More recently, such models have been adopted to describe phenomena which are not consistent with the Cauchy-Born continuum, namely the size dependence of apparent elastic moduli observed for micro and nano-objects, wave dispersion, optical modes and band gaps in the dynamics of heterogeneous media. For those structures the mechanical response is affected by surface effects which are predominant with respect to the bulk, and the scale of the external actions interferes with the characteristic size of the heterogeneities. Generalized continua are very often referred to as media with microstructure although a rigorous deduction is lacking between the specific microstructural features and the constitutive equations. While in the forward modelling predictions of the observations are provided, the actual observations at multiple scales can be used inversely to integrate some lack of information about the model. In this review paper, generalized continua are investigated from the standpoint of inverse problems, focusing onto three topics, tightly connected and located at the border between multiscale modelling and the experimental assessment, namely: (i) parameter identification of generalized elastic models, including asymptotic methods and homogenization strategies; (ii) design of non-conventional tests, possibly integrated with full field measurements and advanced modelling; (iii) the synthesis of meta-materials, namely the identification of the microstructures which fit a target behaviour at the macroscale. The scientific literature on generalized elastic media, with the focus on the higher gradient models, is fathomed in search of questions and methods which are typical of inverse problems theory and issues related to parameter estimation, providing hints and perspectives for future research.

Keywords Generalized materials · Higher-order and higher grade elasticity · Principle of virtual work · Nonstandard boundary conditions · Homogenization · Asymptotic analysis · Inverse problems · Parameter identification · Non-conventional experiments

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1 Introduction

Although already G. Piola (died in 1850) considered elastic energies depending on higher gradients of the placement, see e.g. [1–3], in the last century [4] Toupin attempted to transfer concepts widely accepted in plate structural theory to the three dimensional elastic continuum, such as for instance that of distributed moments. Trying to describe the dispersive dynamic behavior observed in crystal lattices [5], Mindlin proposed a first strain gradient model (or, equivalently, including the second gradient of the placement), later enriched up to the second strain gradient to describe also the surface tension [6]. The shift of the resonant frequencies in crystals was investigated by Gurtin, who developed a continuum theory of elastic material surfaces [7]. Proposing a formulation of the second gradient elasticity under linearized displacements [8], Germain has indicated a clear and rigorous methodology centered on the principle of virtual work to deduce exotic external actions, consistent with assumed expressions of the inner work, and the governing equations in strong form. Starting from these seminal works, a wide class of generalized elastic models have been proposed via the principle of virtual work, by postulating expressions of the elastic energy enriched by additional kinematic descriptors or by higher gradients of the placement, referred to as higher order and higher grade models, respectively. Such relationships can be extended to include geometric nonlinearity and finite strain measures, with a Lagrangian and an Eulerian representation of the main field variables and of the generalized external actions, see [9–12]. The proposal of an enhanced continuum including a microstructural scale has implied the relaxation of the local action assumption and the introduction of the interactions with neighboring points into the constitutive equations.

During the last decades, more and more often such models have been used to interpret several phenomena concerning micro and nano-structures, such as (nano-)beams, wires, belt, films, frequently met in MEMS/NEMS applications [13,14]. For these bodies the surface to volume ratio becomes predominant although the thickness of the boundary region amounts to a few atomic layers, see [15–18]. Similar behavior can occur at the interphase of nano-inclusions or cavities within elastic composites [19,20]. The data gathered so far, which have revealed size-dependence of the apparent moduli for micro and nano-structures, dispersion of waves with optical modes and possible band gaps for the dynamic behaviour of architecture media, are not consistent with a Cauchy-Born continuum: researchers have been forced to explore concepts like characteristic lengths, boundary layers, nonstandard energies, attributing a key role to surface energy and to surface stress [21,22]. When it appeared clear that the generalized continua could contribute to develop applications of huge engineering interest, they became the subject of an extensive research [23]. Moreover, generalized elasticity has allowed one to obtain smooth closed-form solutions for diverse mechanical problems, regularizing the singularities found by conventional modelling, such as dislocations within an elastic medium [24–26].

In structural mechanics oriented to industrial applications, composite materials have had an enormous impact, causing a revolution also in the way of conceiving the structural design: their frequent use has required a homogeneous, macroscopic medium behaving equivalently from the mechanical standpoint, with apparent or effective properties. For this reason but also to study differential equations with highly oscillating coefficients, homogenization techniques have received considerable attention both in mechanics and in mathematics. During the last two decades, the new manufacturing processes, which allow for extremely fine realizations over a large range of scales, gave birth to the new research field of metamaterials (or architectured materials): prototypes can be realized rapidly, at low cost, directly from CAD drawings, thus reducing the gap among design, predictions and verification. More recently researchers realized that the homogenization of composites made of very highly contrasted materials could lead to exotic effective behaviors. Nowadays it is widely accepted that homogenization may distort the constitutive response of materials utilized at the microscale, and that a heterogeneous Cauchy continuum may generate a non Cauchy medium, for which the internal mechanical interactions cannot be exhaustively described by the Cauchy stress tensor alone, see e.g. [27,28]. Studying the effective properties of a heterogeneous linear medium with periodic fibers as reinforcement, markedly stiffer than the surrounding matrix (i.e. with a "high contrast"), [29] has proved that the macroscopic material is a second gradient material, i.e. a material whose energy depends on the second gradient of the displacement. As a further improvement, [30] proved that it is possible to generate a third gradient homogenized material using pantographic-type structures as elementary "bricks" at the microscale. From the theoretical standpoint, the discussion on higher grade models has been rapidly extended to the N-th order gradient, deducing recursively non-standard external actions, see [31]. In [32] several topologies of discrete lattices have been analysed, constituted of welded bars, with extensional, flexural, and torsional stiffness, computing in closed form for each one the energy of the equivalent homogenized medium. Moreover, in the presence of a structure with a periodic microstructure, if the condition of scale separation does not hold and a representative volume element cannot be properly specified, the homogenized, macroscopic response can be effectively described through a generalized elastic model, whose parameters however depend on the sample size.

Also the theory of inverse problems has grown enormously, with a celebrated, multidisciplinary attitude stemming from a hard core of theoretical methods and numerical strategies [33]. While the prediction of the observations is a forward problem, the use of the actual observations to infer some properties of a model, to overcome some lack of information about it, represents an inverse problem, see e.g. [34,35]. It may occur, in fact, that only a few experimental measurements are available, that they concern exclusively a part of the boundary (with underdetermined or overdetermined/redundant data), that the parameters of the model are unknown [36–38], or even that we do not known which model matches all the experimental evidences. Often these questions are related to the selection of a constitutive model, to its calibration and validation, on the basis of tests on a sample with assigned geometry. In other scenarios, it is the shape or the topology of the microstructure which is not prescribed a priori and must be identified, to fit some target. Alternatively, the boundary data in terms of tractions or displacements prescribed on a structure are unknown, being assumed a priori the mechanical properties of the body and available some observations on the deformed configuration. Inverse problems are rooted in the concept of information fusion, which strictu sensu deserves a probabilistic framework, see e.g. [39]. Inverse problems are usually ill-posed in the sense of Hadamard, especially when the percentage level of measurement noise is increased: the solution may not exist, may not be unique, may not depend continuously on data. Therefore it is extremely important making recourse to regularization strategies in the spirit of Tychhonoff, which enrich the available information trying to convexificate the discrepancy norm in the attempt to retrieve well-posedness. A crucial ingredient of any inverse problems is sensitivity analysis [40,41], i.e. the computation of the derivatives of the measurable quantities with respect to the model parameters to identify. On the basis of sensitivity analyses, it is possible to select the most informative measurable quantities to adopt as a basis for the identification of unknown parameters, to design non conventional experiments, to assess the influence of defects on the overall response [42]. Often, it may occur that several kind of information are available simultaneously, expressed though models acting at diverse scales and sharing information from different sources. The solutions of an inverse problem formulated at a given physical scale, may be predicted by a diverse (forward) model at a lower scale: further information can be provided as an a priori (or Bayesian) conjecture, by a hypothetical expert or on the basis of literature data, through the predictions of several models, or also on the basis of experimental measurements, in which often rough signals are mediated by mathematical models. Moreover, recourse can be made to multilevel or hierarchical formulations, having already a sound background in nonlinear programming.

This review, focusing on inverse problems for generalized elastic continua (in particular the higher grade media), will be articulated into three topics, closely related, at the border between multiscale modelling and experimental characterization. (i) Most of the models with generalized elasticity include a prohibitive number of parameters, several of which do not possess a clear engineering meaning and cannot be easily calibrated. Such a redundant number of parameters leads to ill-posedness, namely to lack of uniqueness and of robustness in the inverse procedures: even small perturbations of input data can lead to an unstable identification response. Furthermore, these models include generalized, nonstandard external actions, whose implementation and monitoring during truly experiments may result to be problematic. It is therefore urgent to investigate alternative formulations drastically reducing the number of parameters, see e.g. [25], also exploiting symmetry classes [43,44], and developing analytical and numerical micro–macro or discrete-to-continuum strategies, e.g. see [45]. As a peculiar feature, information on generalized stiffnesses can be provided by observations at different scales: at the macroscale, giving rise to standard identification strategies, as well as from lower scales, giving rise discrete-to-continuum approaches, for instance those resting on atomistic or ab initio simulations, or to micro–macro methods, exploiting asymptotic expansions and homogenization strategies. Mixed static-dynamic identification procedures can be investigated, including also dispersion curves, group and phase velocities as measurable quantities entering the discrepancy norm to minimize. It is worth emphasizing that finite element models of generalized materials require a higher continuity, consistent with the higher order gradients utilized, see e.g. [46–49]. Sensitivity analysis represents a sound basis to develop robust identification techniques, permitting to localize in space and time data with the highest information content, see [41,50]: in the present scenario, classical methods to compute sensitivities (Direct Differentiation, Adjoint State) must deal with the adopted homogenization procedure, as well as with the geometric nonlinearity. (ii) Non-conventional mechanical tests are being developed, apt to include also nonstandard boundary conditions and to activate deformation modes governed by higher order parameters. To this purpose, several tests can be carried out "in situ" coupled with advanced scanning equipment (such as X-ray computed microtomography, electron or atomic force microscopes) and enriched by full-field kinematic measurements: in particular, recourse can be made to Galerkin finite-element Digital Image Correlation (DIC) procedures (available in 2D, 2.5D and 3D-Volume formulations), see e.g. [51–54]. (iii) It is extremely important to specify which typologies of architectured materials give rise at the macroscale to generalized elastic continua: to this purpose, reference has been made to discrete lattices which admit not only different topologies but also diverse long-range interactions, see e.g. [55,56]. Stemming from fundamental theoretical results [29,30,32] and using the most recent methods of continualization and homogenization, the relationships between such microstructures and their macroscopic properties have been investigated. The ultimate aim is to design novel meta-materials whose homogenized response matches a macroscopic behavior assumed a priori as a target: this non-parametric identification problem is often referred to as synthesis of meta-materials [57,58].

The present review paper is organized as follows. Section 2 is devoted to the mathematical formulation of higher grade elastic models, specifying the governing parameters and their role. In Sect. 3 the main issues related to the parameter identification are outlined, including asymptotic methods and homogenization strategies. In Sect. 4 the dynamic response of architectured media is discussed. Non-conventional experiments useful for model calibration are presented in Sect. 5. Section 6 is devoted to the synthesis of meta-materials.

Notation

Recourse will be made to index, componentwise notation for the involved equations, although sometimes the relevant matrix or tensorial expressions will be reported, see [59]. Classical syntaxis of tensor algebra will be adopted, with the Einstein convention on the implicit sum of repeated indices: scalar product between vectors will be denoted equivalently as $\mathbf{a} \cdot \mathbf{b} = a_k b_k$; double dot product between two-rank tensors will be denoted as $\mathbf{A} : \mathbf{B} = A_{mn} B_{mn}$. We assume $\partial_m = \partial/\partial x^m$, but whenever possible we will use comma to indicate partial derivation of tensors, namely $\varepsilon_{ij,m} = \partial_m \varepsilon_{ij}$ and $\varepsilon_{ij,mn} = \partial_n \partial_m \varepsilon_{ij} = \partial_m \partial_n \varepsilon_{ij}$. It is worth remembering that the order of the mixed partial derivatives can be inverted due to the Schwarz theorem. Symmetrization of two indices will be seldom indicated by round brackets enclosing those indices, namely $u_{(i,j)} = 1/2\left(u_{i,j} + u_{j,i}\right)$, or $\varepsilon_{ij,(mn)}$. Gradient and Hessian operators of tensor $\boldsymbol{\varepsilon} \equiv \{\varepsilon_{ij}\}$ will be denoted also as $\nabla\boldsymbol{\varepsilon} = \partial_m \boldsymbol{\varepsilon} = \{\varepsilon_{ij,m}\}$ and $\nabla\nabla\boldsymbol{\varepsilon} = \nabla \otimes \nabla\, \boldsymbol{\varepsilon} = \partial_m \partial_n\, \boldsymbol{\varepsilon} = \{\varepsilon_{ij,mn}\}$. The Laplacian operator will be defined as $\nabla^2 = \nabla \cdot \nabla = \partial_m \partial_m = \|\nabla\|^2$, where $\nabla \cdot \mathbf{A}$ can be interpreted also as the divergence of the tensor $\mathbf{A}$ (i.e. $\nabla^2 = \text{div[grad]}$), and a formal analogy can be considered with the Euclidean norm of a vector $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. The bi- Laplacian will be denoted as $\nabla^4 = \nabla^2 \nabla^2 = \nabla \otimes \nabla : \nabla \otimes \nabla = \partial_m \partial_m \partial_n \partial_n = \|\nabla\|^2 \|\nabla\|^2$. The volume divergence of an n-rank tensor $\mathbf{A} = \{A_{i_1 \cdots i_n}\}$ will be denoted as $\text{div}[\mathbf{A}] = \{A_{i_1 \cdots i_{n-1} j, j}\}$. Symbol δ_{ij} will indicate the Kronecker second-order isotropic tensor, and ϵ_{ijk} the third order Levi-Civita symbol.

2 Higher gradient elasticity

For the readers' convenience, we will introduce the main formulations for generalized materials in a way favouring the simplest expressions, although corresponding to particular assumptions, passing progressively to their generalization. In this section we will consider models with only the displacement $\boldsymbol{u}$ as the kinematic descriptor. In a linearized approach, based on small displacements (small rotations) and small strains, the strain tensor $\boldsymbol{\varepsilon}^{(0)}$, its gradient $\boldsymbol{\varepsilon}^{(1)}$ and second gradient $\boldsymbol{\varepsilon}^{(2)}$ are defined in tensorial form as

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{(0)} = \frac{1}{2}\left[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right], \quad \boldsymbol{\varepsilon}^{(1)} = \nabla\boldsymbol{\varepsilon}^{(0)}, \quad \boldsymbol{\varepsilon}^{(2)} = \nabla\boldsymbol{\varepsilon}^{(1)} = \nabla\nabla\boldsymbol{\varepsilon}^{(0)}, \tag{1}$$

and in index form

$$\varepsilon_{ij} = \varepsilon_{ij}^{(0)} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right], \quad \varepsilon_{ijh}^{(1)} = \varepsilon_{ij,h}^{(0)}, \quad \varepsilon_{ijhk}^{(2)} = \varepsilon_{ijh,k}^{(1)} = \varepsilon_{ij,hk}^{(0)}, \tag{2}$$

where the symbol $\boldsymbol{\varepsilon}$ denotes the strain $\boldsymbol{\varepsilon}^{(0)}$ with two pedices (rank two tensor), the strain gradient $\boldsymbol{\varepsilon}^{(1)}$ and the gradient of the strain gradient $\boldsymbol{\varepsilon}^{(2)}$ indicate in turn rank three and rank four tensors. Besides, recourse will be made to the following properties and symmetries

$$\begin{aligned} &u_{i,jk} = \varepsilon_{ij,k} + \varepsilon_{ki,j} - \varepsilon_{jk,i}, \\ &\varepsilon_{ij} = \varepsilon_{ji}, \quad \varepsilon_{ijk}^{(1)} = \varepsilon_{jik}^{(1)}, \quad \varepsilon_{ijkl}^{(2)} = \varepsilon_{ijlk}^{(2)} = \varepsilon_{jikl}^{(2)}. \end{aligned} \tag{3}$$

It is worth noting that the first strain gradient corresponds to the second gradient of the displacement field, and the second strain gradient to the third gradient of the displacement. This terminology may be confusing for the reader.

2.1 Bi-Helmoltz regularization

In [25] a peculiar form of the second strain gradient elasticity has been considered, postulating for the stored energy density the expression

$$\mathcal{W} = \frac{1}{2} C_{ijkl}\, \varepsilon_{ij}\, \varepsilon_{kl} + \frac{1}{2} l_{s1}^2\, C_{ijkl}\, \varepsilon_{ij,m}\, \varepsilon_{kl,m} + \frac{1}{2} l_{s2}^4\, C_{ijkl}\, \varepsilon_{ij,mn}\, \varepsilon_{kl,mn}. \tag{4}$$

The above expression, quadratic in the strains and in their first and second gradients, is usually referred to as bi- Helmoltz, and rests on the form II of the Mindlin model [6]. Symbols l_{s1} and l_{s2} indicate the characteristic lengths which scale and make non dimensional the products of the strain partial derivatives: they represent the parameters governing the energy density besides those included in the conventional stiffness tensor C_{ijkl}. In absolute notation, the expression for the stored energy density is intricate because the symbols required for the operators on higher order tensors are not standard and they are therefore difficult to read. In fact, besides the first contribution, which represents the conventional strain energy including the tensor product $\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$ with four valences contracted with those of the stiffness tensor, in the other two nonlocal contributions there are tensor products of the strain first and second gradient by themselves, which have been contracted. In particular, the tensor product of the strain gradient (i.e. $\nabla\boldsymbol{\varepsilon}$) by itself is contracted as for the third and sixth indices, and it would results to be $\varepsilon_{ij,p}\, \varepsilon_{kl,m}\, \delta_{pm}$ or equivalently $\partial_p \boldsymbol{\varepsilon} \otimes \partial_m \boldsymbol{\varepsilon}\, \delta_{pm}$. Analogously the second nonlocal term, resting on the second gradient of the strain (i.e. $\nabla\nabla\boldsymbol{\varepsilon}$), would be expressed as $\varepsilon_{ij,pq}\, \varepsilon_{kl,mn}\, \delta_{pm}\delta_{qn}$ or equivalently $\partial_p\partial_q \boldsymbol{\varepsilon} \otimes \partial_m\partial_n \boldsymbol{\varepsilon}\ \delta_{pm}\delta_{qn}$. In the bi-Helmoltz formulation outlined above Eq. (4), all the products including the strains, the strain gradients, and the gradients of the strain gradient result to be, after contraction, four rank tensors, and the stiffness tensor C_{ijkl} is strictly sufficient to saturate all these valences: higher order generalized stiffness tensors are not required at all.

For the isotropic scenario, the fourth rank stiffness tensor can be decomposed additively into products of isotropic second- order Kronecker symbols [59],

$$C_{ijkl} = \lambda\ \delta_{ij}\delta_{kl} + \mu\ (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{5}$$

To ensure the energy be convex and nonnegative, the Lamé moduli and the characteristic lengths in Eqs. (4)–(5) must satisfy the conditions: $\mu \geq 0$, $(3\lambda + 2\mu) \geq 0$, $l_a^2 \geq 0$, $l_b^4 \geq 0$. The stored energy density already possesses all the constitutive properties of the material it is intended to describe. However, such properties are often more clearly expressed in terms of stress–strain relations. The strain measures are defined in Eq. (1) or in (2). By duality, the stress measures are the stress $\boldsymbol{\sigma}^{(0)}$, the double stress $\boldsymbol{\sigma}^{(1)}$ and the triple stress $\boldsymbol{\sigma}^{(2)}$ defined by differentiating the energy density with respect to the strain measures, namely

$$\boldsymbol{\sigma}^{(0)} = \frac{\partial\, \mathcal{W}}{\partial\, \boldsymbol{\varepsilon}^{(0)}}, \quad \boldsymbol{\sigma}^{(1)} = \frac{\partial\, \mathcal{W}}{\partial\, \boldsymbol{\varepsilon}^{(1)}}, \quad \boldsymbol{\sigma}^{(2)} = \frac{\partial\, \mathcal{W}}{\partial\, \boldsymbol{\varepsilon}^{(2)}}, \tag{6}$$

see also [60]. In index notation one finds

$$\begin{aligned} \sigma_{ij}^{(0)} &= \frac{\partial\, \mathcal{W}}{\partial\, \varepsilon_{ij}} = C_{ijmn}\, \varepsilon_{mn}, \\ \sigma_{ijk}^{(1)} &= \frac{\partial\, \mathcal{W}}{\partial\, \varepsilon_{ij,k}} = l_{s1}^2\, C_{ijmn}\, \varepsilon_{mn,k} = l_{s1}^2\, \sigma_{ij,k}, \\ \sigma_{ijkp}^{(2)} &= \frac{\partial\, \mathcal{W}}{\partial\, \varepsilon_{ij,kp}} = l_{s2}^4\, C_{ijmn}\, \varepsilon_{mn,kp} = l_{s2}^4\, \sigma_{ij,kp}, \end{aligned} \tag{7}$$

where the third equality along each row holds for homogeneous media, i.e. for media with constant fourth rank stiffness tensor C_{ijmn}. Symbol $\boldsymbol{\sigma}^{(0)}$ denotes the conventional Cauchy stress with two indices σ_{ij}, $\boldsymbol{\sigma}^{(1)}$ indicates the double stress with three indices $\sigma_{ijk}^{(1)}$, $\boldsymbol{\sigma}^{(2)}$ the triple stress with four indices $\sigma_{ijkl}^{(2)}$. The stress tensors of order higher than two are usually referred to as hyper-stresses. On the basis of the above considerations, one can write the stored energy density Eq. (4) in terms of (work-conjugate) generalized stress and strain components,

$$\begin{aligned} \mathcal{W} &= \frac{1}{2}\,\sigma_{kl}\,\varepsilon_{kl} + \frac{1}{2}\,\sigma^{(1)}_{klm}\,\varepsilon^{(1)}_{klm} + \frac{1}{2}\,\sigma^{(2)}_{klmn}\,\varepsilon^{(2)}_{klmn} \\ &= \frac{1}{2}\,\sigma_{kl}\,\varepsilon_{kl} + \frac{1}{2}\,l_{s1}^2\,\sigma_{kl,m}\,\varepsilon_{kl,m} + \frac{1}{2}\,l_{s2}^4\,\sigma_{kl,mn}\,\varepsilon_{kl,mn}, \end{aligned} \tag{8}$$

where the second row corresponds to a material with uniform stiffness. In [61] also the kinetic energy is introduced as the product of the linear momentum vector $\boldsymbol{\pi}$ and the velocity vector $\dot{\mathbf{u}}$,

$$\mathcal{T} = \frac{1}{2}\,\pi_i\,\dot{u}_i. \tag{9}$$

Hence, for a body occupying the volume $\Omega \subset \mathcal{R}^3$ the Lagrangian density $\mathcal{L}$ and the Action functional $\mathfrak{A}$ referred to the time interval $[t_1, t_2]$ can be expressed as

$$\mathcal{L} = \mathcal{T} - \mathcal{W}, \quad \mathfrak{A}(u_i(\mathbf{x})) = \int_{t_1}^{t_2}\int_{\Omega} \mathcal{L}(\mathbf{x}, t, \mathbf{u}, \dot{\mathbf{u}}, \mathbf{u}_{,i}, \mathbf{u}_{,ij}, \mathbf{u}_{,ijk})\,d\mathbf{x}\,dt. \tag{10}$$

The dynamic equilibrium configuration for our generalized elastic structure can be obtained by minimizing the Hamilton's functional $\mathfrak{A}$. By prescribing the first variation of the Action functional to vanish for any admissible variation $\delta\mathbf{u}$ of the displacement field $\mathbf{u}$ (see [62]), the Euler-Lagrange system of partial differential equations endowed by the boundary conditions are deduced as follows

$$\begin{aligned} \delta\mathfrak{A}(\mathbf{u}(\mathbf{x})) = \int_{t_1}^{t_2}\int_{\Omega} \delta\mathcal{L}\,d\mathbf{x}dt = \int_{t_1}^{t_2}\int_{\Omega} \Bigg\{ \frac{\partial\mathcal{L}}{\partial\mathbf{u}}\cdot\delta\mathbf{u} + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,i}}\cdot\delta\mathbf{u}_{,i} \\ + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\cdot\delta\mathbf{u}_{,ij} + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ijk}}\cdot\delta\mathbf{u}_{,ijk} + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,t}}\cdot\delta\mathbf{u}_{,t} \Bigg\}\,d\mathbf{x}\,dt = 0. \end{aligned} \tag{11}$$

being $i, j, k = 1, 2, 3$. In the above expression, the components of the displacement vector $\mathbf{u}$ and, consistently, of the derivative of the Lagrangian density $\mathcal{L}$ with respect to it are dealt with in vector notation, so that the standard symbol $\cdot$ can be utilized for their scalar product, whilst indices i,j and k are used exclusively to indicate the partial derivatives of the displacement vector involved in the integration by parts, with the sum convention: for instance, one has $\partial\mathcal{L}/\partial\mathbf{u}_{,ijk}\cdot\delta\mathbf{u}_{,ijk} = \partial\mathcal{L}/\partial\mathbf{u}_{,111}\cdot\delta\mathbf{u}_{,111} + \cdots$. The explicit dependence of the Lagrangian density on the displacement vector is not consistent with the objectivity principle, and the first addend at r.h.s is disregarded. In the variational approach, an irreducible representation of the integrands is sought, in which only variations of the displacement field and of its normal derivatives appear. As an example, the first addend of the second row of Eq. (11) can be transformed as follows

$$\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\cdot\delta\mathbf{u}_{,ij} = \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\cdot\delta\mathbf{u}_{,i}\right)_{,j} - \left[\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\right)_{,j}\cdot\delta\mathbf{u}\right]_{,i} + \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\right)_{,ji}\cdot\delta\mathbf{u}. \tag{12}$$

By applying the same strategy to the remaining addends in Eq. (11), through the fundamental lemma of the calculus of variations one obtains the following vector equation to hold within the strip $\Omega \times [t_1, t_2]$

$$-\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,t}}\right)_{,t} - \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,i}}\right)_{,i} + \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\right)_{,ij} - \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ijk}}\right)_{,ijk} = \mathbf{0}, \tag{13}$$

whilst one should meet $\forall\,\delta\mathbf{u}$

$$\begin{aligned} \int_{t_1}^{t_2}\int_{\Omega} \Bigg\{\Bigg[\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ijk}}\right)_{,kj} - \left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\right)_{,j} + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,i}}\Bigg]\cdot\delta\mathbf{u}\Bigg\}_{,i} \\ + \Bigg\{\Bigg[-\left(\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ijk}}\right)_{,k} + \frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ij}}\Bigg]\cdot\delta\mathbf{u}_{,i}\Bigg\}_{,j} + \Bigg\{\frac{\partial\mathcal{L}}{\partial\mathbf{u}_{,ijk}}\cdot\delta\mathbf{u}_{,ij}\Bigg\}_{,k}\,d\mathbf{x}\,dt = 0. \end{aligned} \tag{14}$$

The above integral leads to conditions over the differential boundaries of order up to three (namely over the outer surface, along the edges and at the wedges, see e.g. [43]), involving polynomials in the components of the Darboux basis vectors (normal, tangent and tangent normal) and the derivatives of the Lagrangian density with respect to the independent variables, to hold $\forall\, t \in [t_1, t_2]$. In fact, for the first addend in Eq. (14), by the divergence theorem one easily obtains an integral over the topological and differential boundary $\partial\Omega$, where the integrand projected along the normal to the surface is multiplied by $\delta\mathbf{u}$. For the other two addends, instead, one can easily notice that, after a first application of the divergence theorem and the transport to the boundary $\partial\Omega$, i.e. to the outer surface, the first and the second gradient of the virtual displacement vector still appear, and hence the contribution must be further reduced through a recursive application of the integration by parts and of the divergence theorem. It is worth emphasizing that, over the differential boundaries of order one and two, namely over $\partial\Omega$ and $\partial\partial\Omega$, an extended formulation of the divergence theorem is needed, which includes tangential projectors to the current manifold, see e.g. [9,10]. Moreover, to deduce in strong form the governing equation and to make explicit all the natural and essential boundary conditions, the assumption of isochronic motion is needed, namely $\delta\mathbf{u}|_{t_1} = \delta\mathbf{u}|_{t_2}$, or equivalently $\delta\mathbf{u}|_{t=t_1,t_2} = \mathbf{0}$, $\forall\, \mathbf{x} \in \Omega$, see [63].

It is worth mentioning that, following Eq. (13), through Eqs. (7)–(8) an effective stress tensor with rank two can be considered by computing the higher order divergence of the hyper-stresses and adding them with alternating signs as follows

$$\widehat{\sigma}_{ij} = \sigma_{ij} - \sigma^{(1)}_{ijk,\,k} + \sigma^{(2)}_{ijkp,\,kp}, \tag{15}$$

or by tensor notation

$$\widehat{\boldsymbol{\sigma}}_{ij} = \boldsymbol{\sigma}_{ij} - \operatorname{div}\left[\boldsymbol{\sigma}^{(1)}\right] + \operatorname{div}\left[\operatorname{div}\left[\boldsymbol{\sigma}^{(2)}\right]\right], \tag{16}$$

being div the standard divergence operator. In this way, the balance equation in terms of such an effective stress is formally unchanged with respect the one formulated by the Cauchy stress, namely $\widehat{\sigma}_{ij,\,j} + \mathcal{F}_i = 0$, being $\mathcal{F}_i$ the force volume density. Considering the above definition in the presence of uniform stiffness tensor, indicating the Laplace and the bi-Laplacian operators in turn by the symbols ∇^2 and ∇^4 (see the notation in Sect. 1), one finds in terms of the Cauchy stress $\boldsymbol{\sigma}^{(0)} = \boldsymbol{\sigma}$

$$\widehat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - l_{s1}^2\,\nabla^2\boldsymbol{\sigma} + l_{s2}^4\,\nabla^4\boldsymbol{\sigma} = \left(1 - l_{s1}^2\,\nabla^2 + l_{s2}^4\,\nabla^4\right)\boldsymbol{\sigma}. \tag{17}$$

This last expression is especially suggestive of the role played by the nonlocal contributions: they regularize the stress field by superimposing high order laplacians acting as smoothing filters at different scales, multiplied by powers of the characteristic lengths.

2.2 Second strain gradient formulation

For a center-symmetric material, one could introduce the following stored energy density, including higher rank generalized stiffness tensors,

$$\begin{aligned}\mathcal{W} = {} & \frac{1}{2}\,C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl} + \frac{1}{2}\,A_{ijklmn}\,\varepsilon_{ij,\,k}\;\varepsilon_{lm,n} + \frac{1}{2}\,B_{ijklmnpq}\,\varepsilon_{ij,\,kl}\,\varepsilon_{mn,\,pq} \\ & + E_{ijklmn}\,\varepsilon_{ij}\,\varepsilon_{kl,\,mn} + B^0_{ijkl}\,\varepsilon_{ij,\,kl}.\end{aligned} \tag{18}$$

With respect to the bi-Helmoltz formulation in Eq. (7), herein the parameters which govern the energy density are the independent components of the generalized stiffness tensors of increasing rank appearing above, contracted with the higher order rank tensor products $\boldsymbol{\varepsilon} \otimes \boldsymbol{\varepsilon}$, $\nabla\boldsymbol{\varepsilon} \otimes \nabla\boldsymbol{\varepsilon}$ and $\nabla\nabla\boldsymbol{\varepsilon} \otimes \nabla\nabla\boldsymbol{\varepsilon}$. If we neglect tensors E_{ijklmn} and B^0_{ijkl}, by equating the coefficients of the higher order strain gradients, i.e. $l_{s1}^2\, C_{ijkl}\delta_{pq} = A_{ijplmq}$ and $l_{s2}^4\, C_{ijkl}\delta_{pq}\delta_{mn} = B_{ijpmklqn}$, the characteristic length powers l_{s1}^2 and l_{s2}^4 appearing in Eq. (7) can be expressed as linear combinations of the higher order tensor independent components in Eq. (18). By differentiating the energy density, the generalized state equations are deduced,

$$\begin{aligned}\sigma_{ij} &= C_{ijmn}\,\varepsilon_{mn} + E_{ijklmn}\,\varepsilon_{kl,\,mn}, \\ \sigma^{(1)}_{ijk} &= A_{ijklmn}\,\varepsilon_{lm,\,n}, \\ \sigma^{(2)}_{ijkl} &= E_{mnijkl}\,\varepsilon_{mn} + B_{ijklmnpq}\,\varepsilon_{mn,\,pq} + B^0_{ijkl}.\end{aligned} \tag{19}$$

In a first approximation, for the kinetic energy density Eq. (9) one can assume as usual

$$\pi_\alpha = \frac{\partial \mathcal{T}}{\partial \dot{u}_\alpha} = \rho\, \dot{u}_\alpha. \tag{20}$$

A generalized expression for the kinetic energy will be discussed in Subsection 2.3. It is worth noting that, due to the central symmetry, contributions governed by tensors of odd rank are lacking. The first tensor, the stress tensor $\sigma_{(ij)}$, has 6 independent components over nominal $3^2 = 9$, where the round brackets indicate symmetrization with respect to the indices in argument. The double stress tensor $\sigma^{(1)}_{(ij)k}$ has $6 \times 3 = 18$ independent components over nominal $3^3 = 27$, and $\sigma^{(2)}_{(ij)(kp)}$ has $6 \times 6 = 15 + 15 + 6 = 36$ independent components over nominal $3^4 = 81$. Please note the difference with Mindlin stress $\tilde{\sigma}^{(2)}_{i(jkp)} = \partial \mathcal{W}/\partial u_{i,jkl}$, as remarked in [64]. From the mechanical standpoint, hyperstress $\sigma^{(1)}_{ijk}$ has the meaning of a force dipole (where index i indicates the direction of the forces, j the orientation of the level arm, k the orientation of the normal to the surface on which they act), whilst $\sigma^{(2)}_{ijkp}$ acts as a force quadrupole, i.e. a dipole of dipoles, or equivalently as a dipole of two moments. The last term in the triple stress $\sigma^{(2)}_{ijkl}$ is associated to the generation of new surfaces at the origin of the cohesive forces. Besides, for zero strain we have $\sigma^{(2)}_{ijkl} = B^0_{ijkl}$ so that B^0_{ijkl} is also called pre-triple stress.

For the general anisotropic second strain gradient case, the number of governing parameters is huge. The reason is that, as discussed above, the total number of independent variables amounts to 60: hence, the symmetric coefficient matrix of a quadratic form with this dimension includes $(60 \times 61)/2 = 1830$ coefficients. When considering the isotropic (first) strain gradient elasticity (i.e., by assuming $B_{ijklmnpq} = E_{ijklmn} = B^0_{ijkl} = 0$ in Eq. 19), the number of governing parameters is drastically reduced to 7, see [43,65]. In fact, by applying the additive decomposition of isotropic tensors with even rank to the sixth order tensor A_{ijklmn} (see [66]), considering also the symmetry conditions on the strain components Eq. (2), one obtains

$$\begin{aligned}
&C_{ijmn} = c_1\, \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) + c_2\, \delta_{ij}\delta_{mn},\\
&A_{ijklmn} = c_3\left(\delta_{ij}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jk}\delta_{lm} + \delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{jn}\delta_{lm}\right)\\
&+ c_4\, \delta_{ij}\delta_{kn}\delta_{ml} + c_5\left(\delta_{ik}\delta_{jl}\delta_{mn} + \delta_{im}\delta_{jk}\delta_{ln} + \delta_{ik}\delta_{jm}\delta_{ln} + \delta_{il}\delta_{jk}\delta_{mn}\right)\\
&+ c_6\left(\delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jl}\delta_{kn}\right)\\
&+ c_7\left(\delta_{il}\delta_{jn}\delta_{mk} + \delta_{im}\delta_{jn}\delta_{lk} + \delta_{in}\delta_{jl}\delta_{km} + \delta_{in}\delta_{jm}\delta_{kl}\right),
\end{aligned} \tag{21}$$

where $c_1 = \mu$ and $c_2 = \lambda$ are the two Lamé coefficients entering Eq. (5), and c_l $(l = 3, 4, 5, 6, 7)$ are five further constitutive coefficients governing the nonlocal response. Besides the well known conditions $\mu > 0$ and $3\lambda + 2\mu > 0$, the additional parameters much satisfy further inequalities to guarantee that the first strain gradient part of the quadratic form in Eq. (18) is positive defined: this algebraic condition is equivalent to assume that the stored energy at each point as a strictly convex function of its arguments. To deduce such inequalities, cumbersome manipulations are required to express the strain and strain gradient tensors in Voigt form and consistently the constitutive linear relationships Eq. (19) in a matrix form, before applying the Sylvester criterion, see [43]. The admissible set of parameters is specified by the following constraints

$$\begin{aligned}
&c_6 > 0,\ \frac{c_6}{2} < c_7 < c_6,\ 5\,c_4 + 4\,c_6 > 2\,c_7,\\
&c_5 > \frac{c_4(3\,c_6 + c_7) + 2(c_6^2 - 5\,c_3^2 - 6\,c_7\,c_3 - 2\,c_7^2 + c_6(2\,c_3 + c_7))}{4\,c_7 - 10\,c_4 - 8\,c_6}.
\end{aligned} \tag{22}$$

On the basis of Eq. (19), the momentum balance equation for a first strain gradient material can be expressed through the effective stress tensor (Eq. 16) as follows

$$\hat{\sigma}_{ij,\,j} + \mathcal{F}_i = C_{ijkl}\, \varepsilon_{kl,\,j} - A_{ijklmn}\, \varepsilon_{lm,\,nkj} + \mathcal{F}_i = 0. \tag{23}$$

Under the assumption of isotropic behaviour, expressing the strain gradients appearing above in terms of the displacement derivatives (Eq. 1-2) and developing all the products with the tensor components indicated in

Eq. (21), one can obtain the generalized counterpart of the Navier equations ($i = 1, 2, 3$)

$$\mu\, \nabla^2 \left(1 - \frac{l_a^2}{\mu}\, \nabla^2\right) u_i + (\lambda + \mu)\, \partial_i \left(1 - \frac{l_b^2}{\lambda + \mu}\, \nabla^2\right) u_{k,\,k} + \mathcal{F}_i = 0, \tag{24}$$

where the characteristic lengths result to be linear combinations of the nonlocal parameters, namely $l_a^2 = (c_5 + c_6 + c_7)$, and $l_b^2 = (4\,c_3 + c_4 + 3\,c_5 + c_6 + 3\,c_7)$. For the sake of clarity, in index notation one has $\partial_i\, \nabla^2\, u_{k,\,k} = u_{k,\,kmmi}$, and $\nabla^2\, \nabla^2\, u_i = u_{i,\,mmnn}$. In Eq. (24) the maximum derivation order for each displacement component u_i is four: by assuming the characteristic lengths l_a and l_b to vanish, one retrieves the Navier second-order equations for a Cauchy (isotropic) material.

It is worth emphasizing that the boundary conditions accompanying Equation (24) cannot be provided by just substituting the effective stress tensor into the Cauchy relationship for the boundary tractions. In fact, for a first strain gradient body an enriched set of boundary conditions is available, which include, besides the contact pressures depending not only on the normal but also on the local curvature, nonstandard forces over the outer surface working on the normal derivative of the displacement, and edge forces, as already discussed with reference to Eq. (14) deduced by a variational approach. All these conditions must be expressed consistently with the orientation of the piecewise regular faces and edges, of which the outer boundary of the body may consist, see e.g. [10].

The present linearized approach, based on small displacement (small rotations) and small strains, can be easily extended to a scenario with geometric nonlinearity, by using dual quantities defined in the Eulerian (or spatial) configurations, or, alternatively, variables defined in the Lagrangian (or material) configuration, such as Green-Lagrange-Saint-Venant tensor of finite strain and the Piola stress and hyper-stresses, see [11]. Moreover, as illustrated in [67], also the equilibrium configuration for a generalized material in a boundary value problem can be sought through minimizing a suitable functional.

2.3 Micro-inertia terms

As discussed in [68], also the kinetic energy can be enriched considering velocity gradients of higher grade, namely $\mathbf{v} = \{v_i\} = \{\dot{u}_i\}$, $\nabla \mathbf{v} = \partial_j \mathbf{v} = \{v_{i,\,j}\}$ and $\nabla\nabla\mathbf{v} = \partial_j\partial_k \mathbf{v} = \{v_{i,\,jk}\}$. In this way it is possible to describe at the macroscale the effect of lumped and distributed masses belonging to the microstructure. For instance, a generalized kinetic energy can be expressed as follows

$$\begin{aligned} \widetilde{\mathcal{T}} &= \widetilde{\mathcal{T}}(\mathbf{v}, \nabla\mathbf{v}, \nabla\nabla\mathbf{v}) \\ &= \frac{1}{2}\rho \left[\mathbf{v}\cdot\mathbf{v} + l_{d\,1}^2\, \nabla\mathbf{v} : \nabla\mathbf{v} + l_{d\,2}^4\, \nabla\nabla\mathbf{v} \,\vdots\, \nabla\nabla\mathbf{v}\right]. \end{aligned} \tag{25}$$

By differentiating the kinetic energy, the dual quantities (hyper momenta) are deduced

$$\begin{aligned} \boldsymbol{\pi} &= \frac{\partial \widetilde{\mathcal{T}}}{\partial \mathbf{v}} = \rho\, \mathbf{v}, \\ \boldsymbol{\pi}^{(1)} &= \frac{\partial \widetilde{\mathcal{T}}}{\partial \nabla\mathbf{v}} = \rho\, l_{d\,1}^2\, \nabla\mathbf{v} = l_{d\,1}^2\, \nabla\boldsymbol{\pi}, \\ \boldsymbol{\pi}^{(2)} &= \frac{\partial \widetilde{\mathcal{T}}}{\partial \nabla\nabla\mathbf{v}} = \rho\, l_{d\,2}^4\, \nabla\nabla\mathbf{v} = l_{d\,2}^4\, \nabla\nabla\boldsymbol{\pi}. \end{aligned} \tag{26}$$

where the second equalities along the second and third rows require a uniform material density. It is worth emphasizing that, to enrich the kinetic energy, herein two additional parameters $l_{d\,1}$ and $l_{d\,2}$ have been utilized, referred to as dynamic characteristic lengths (with subscript d), which scale the higher velocity gradients and make their contribution homogeneous with the conventional kinetic energy.

3 Calibration and validation of generalized elastic models

Simultaneously with the availability of powerful personal computers, several strategies have been proposed to estimate in a robust manner the governing parameters of advanced models on the basis of experimental data affected by uncertainty. In materials mechanics, the approach widely adopted to the experimental characterization of materials was initially based on mechanical tests the simplest possible, to be routinely carried out by not specialized personnel in standard conditions, conceived to generate uniaxial stress states within the sample. The aim was to allow for the estimation of a singular unknown parameter for each loading condition, through analytical, closed-form equations. With the progress of research, models have been proposed including an increasing number of parameters, several of which do not possess a clear engineering meaning. More and more often non-conventional (or non trivial) experiments are being designed on the basis of the predictions provided by advanced models, to generate within the specimen stress and strain states the closest possible to those expected under service and ultimate conditions, see [69]. Activating at once several deformation modes, the relevant governing parameters can be estimated simultaneously, as it occurs in real applications (on this subject see also Sect. 5). It is worth emphasizing that, when increasing the number of parameters governing a selected family of models, the minimum discrepancy between the model predictions and the experimental data certainly decreases at the beginning, but, exceeded a threshold, the same minimum error norm is expected to increase again. In fact, the optimal number of parameters or equivalently the "optimal" model within a certain class must be sought as a compromise among conflicting requirements of maximum information content and minimum representation, as indicated by Akaike's information criterion, e.g. see [70]. A redundant number of parameters frequently implies lack of uniqueness and of robustness in the identification procedure: this scenario worsens at increasing the level of data noise. For this review two typologies of inverse problems are considered of prominent interest, one parametric (α) and another one nonparametric (β). (α) The parametric inverse problems, i.e. which exhibit a finite number of unknowns, consist of identifying some parameters governing the macroscopic response within a selected class of models (for instance, the generalized stiffness tensors), being assigned a priori the topology and the constitutive law at the microscale, typically but not exclusively of Cauchy type. In this scenario, we must know in advance if the selected class of macroscopic relationship is adequate to describe the homogenized response of the specific material. (β) We assign a priori the macroscopic response within a class of generalized elastic models, and we intend to identify the microstructures giving rise through homogenization to the target behavior. This second scenario is commonly referred to as synthesis of meta-materials. Clearly, one could question whether a viable strategy exists to parametrize also problem (β), at least for practical purposes. It is worth noting that also the selection of a class of macroscopic models to be assumed in (α), carried out by a hypothetical expert, a priori with respect to the novel data sets, turns out to be nonparametric. The parameter identification problem, at point (α), will be dealt with in depth in the next subsections. The non-parametric inverse problem of the synthesis, at point (β), will be discussed in Sect. 6.

3.1 Parameter identification problems

On the basis of the above considerations and of the recent literature, parameter identification strategies are outlined at the next points (A)–(C). Let us adopt the superscripts $^{\mathrm{M}}$ and $^{\mathrm{m}}$ to indicate quantities entering in turn the macroscopic and the microscopic mechanical models. Let us indicate the unknown parameters entering the macroscopic, high gradient equations by the symbol $\mathbf{p}^{\mathrm{M}}$: as outlined above, for linear elastic models they represent the components of the generalized stiffness tensors. At the same time, such parameters can be predicted by (*forward*) homogenization techniques starting from information on the microstructure, concerning the elastic properties of the microscopic phases (usually but not necessarily of Cauchy type), and their topology. In the jargon of the inverse problems, we can say that they possess more privileged *measurement equations*, to be regarded as additional, qualified data. One can write

$$\mathbf{p}^{\mathrm{M}} = \mathcal{H}(C_{ijkl}^{\mathrm{m}\,(a)},\ \text{geometry}), \tag{27}$$

where symbol $\mathcal{H}$ denotes a homogenization operator, $C_{ijkl}^{\mathrm{m}\,(a)}$ indicate the microscopic stiffness matrix of the phase (a), whilst the information on the topology is generically coded under the label geometry. Of course, one could consider other predictions of the unknown parameters, for instance those resting on atomic interactions, utilized for ab initio simulations, or predictions provided by a different theory. On the other hand, if the constitutive behaviour of the equivalent macroscopic medium is governed by the parameters gathered in vector $\mathbf{p}^{\mathrm{M}}$, its response under a prescribed input can be expressed in terms of the measurable quantities $\mathbf{y}^{\mathrm{M}}$ as

$$\mathbf{y}^{\mathrm{M}} = \mathcal{M}(\mathbf{p}^{\mathrm{M}},\ \text{input}), \tag{28}$$

where $\mathcal{M}$ is referred to as forward operator (at the macroscale), and the label input denotes herein any non homogeneous boundary condition (natural or essential) prescribed to the material sample, in the experimental practice affected by uncertainty as well. It is worth emphasizing that non standard boundary conditions stem from energies depending on the higher order gradients of the placement. Hence, vector $\mathbf{y}^{\mathrm{M}}$ may include macroscopic displacements and their gradients or also (generalized) reaction forces which can be ideally measured during mechanical tests, technical difficulties aside.
(A) **Energy approaches**. As outlined in [45,71], to identify the parameters of the macroscopic medium an approach consists of assuming equal the elastic energy for the heterogeneous body and its macroscopic counterpart, and coincident the displacements over the outer border of the entire sample. This condition implies that the elastic energy is preserved when passing from the micro to the macroscale, as well as the sample outer boundary undergoes the same deformation according to both the descriptions, see also [72]. Let us denote by Ω the volume occupied by the body, being its boundary $\partial\Omega$, and let us consider small displacements and strains. Let us assume that the phases at the microscale are described by a standard Cauchy medium behaving linearly elastically, while the stored energy at the macroscale is a quadratic form of the generalized strains and of their gradients. Hence one can write

$$\int_{\Omega} \frac{1}{2}\,{}^{\mathrm{m}}C_{ijkl}\,{}^{\mathrm{m}}\varepsilon_{ij}\,{}^{\mathrm{m}}\varepsilon_{kl}\,d\Omega = \int_{\Omega} \left\{ \frac{1}{2}\,{}^{\mathrm{M}}C_{ijkl}\,{}^{\mathrm{M}}\varepsilon_{ij}\,{}^{\mathrm{M}}\varepsilon_{kl} + \frac{1}{2}\,{}^{\mathrm{M}}A_{ijklmn}\,{}^{\mathrm{M}}\varepsilon_{ij,k}\,{}^{\mathrm{M}}\varepsilon_{lm,n} + \ldots \right\} d\Omega, \tag{29}$$

and

$$\left.{}^{\mathrm{m}}u_i\right|_{\partial\Omega_u} = \left.{}^{\mathrm{M}}u_i\right|_{\partial\Omega_u}, \tag{30}$$

where apices m and M distinguish micro and macroscopic quantities, $\partial\Omega_u$ indicates the part of the boundary with prescribed displacements (namely, Dirichlet boundary conditions), ε_{ij} and $\varepsilon_{ij,k}$ indicate linearized strain fields and their partial derivatives (see Eqs. 1–2), ${}^{\mathrm{M}}C_{ijkl}$ and ${}^{\mathrm{M}}A_{ijklmn}$ denote generalized stiffness tensors (of rank four and six, respectively, see Eq. 18), constant over Ω since referred to the homogenized equivalent medium, which depend linearly on the unknown parameters gathered in vector $\mathbf{p}^{\mathrm{M}}$. For the sake of clarity, we have neglected in the expression of the energy the contributions coupling strains and their gradients. It is worth emphasizing that, in the present scenario, it would be too restrictive referring Eqs. (29)–(30) to the representative volume element (RVE), since the separation of scales does not necessarily hold and the macroscopic response varies depending on the ratio of the length scales.

If the topology and the elastic properties at the microscale are assumed to be known, by equating the inner energies (micro and macro) computed through the principle of virtual work under selected boundary conditions (the same at the micro and macroscale), parameters of the macroscopic model $\mathbf{p}^{\mathrm{M}}$ can be identified, see e.g. [45]. The integral contributions entering the inner work of the macroscopic continuum, playing the role of coefficients in the system of linear equations, can be computed through finite elements (see e.g. [73]), and the boundary conditions (shared by both the models) must be selected in such a way to achieve independent equations, but also in order to design feasible experiments. In the simplest formulations (e.g. bi-Helmoltz law described at Eq. 4, see e.g. [25]), only two characteristic lengths have to be estimated for the equivalent continuum, whilst the number of unknowns arises to seven in the case of isotropic, centro-symmetric materials, see e.g. [74]. Actually, one advantage of the approach specified by Eqs. (29)–(30) is the fact that it is not mandatory to involve the homogenization procedure, in any of its formulations: the fields ${}^{\mathrm{m}}\varepsilon_{ij}$ and ${}^{\mathrm{M}}\varepsilon_{ij}$ with ${}^{\mathrm{M}}\varepsilon_{ij,k}$ are independently computed by different models, sharing the same boundary conditions. Two drawbacks can be envisaged: firstly, the inner work at the microscale requires in general a detailed model, whose effort may result to be prohibitive; secondly, the macroscopic model includes generalized external actions ("exotic" N-forces) to be prescribed along the outer boundary (in the general scenario constituted of piecewise regular surfaces, with edges and wedges), which are not consistent with the microscopic continuum of Cauchy type (sustaining exclusively surface and volume density loading). Although in practice one can leave free such conditions over the boundary (for instance, the normal derivatives of the placement) or assume them to vanish, the consequences on the identification results must be critically and comparatively assessed. It is also convenient to properly design the sample geometry, in such a way to reduce possible inconsistencies along the boundaries between the micro and macro models, as discussed later. It is worth noting that Eqs. (29)–(30) can be easily extended to the case of geometric nonlinearity, once a suitable measure of finite strain is introduced, and to any nonlinear constitutive behaviour, although in other scenarios the dependence of r.h.s in Eq. (29) on the unknown parameters may turn out to be nonlinear. A few alternatives to finite element computations needed for this inverse strategy are considered at next points (A.1) and (A.2).

(A.1) If the selected boundary conditions correspond to practically feasible laboratory tests, truly experimental strain fields at the microscale, provided e.g. by full-field kinematic measurements, could be utilized to form the inner work integral at l.h.s of Eq. (29). In particular, Digital Image Correlation (DIC) techniques could provide displacements (over the surface or even within the bulk), and, through differentiation, the strain fields (see Section 4). This approach is similar in the spirit to the virtual field method based on the principle of virtual work and exploiting fictitious statically admissible fields over the same domain, see e.g. [75]. In 2D conditions recourse can be made also to digital interferometric methods which provide directly strain fields with a high resolution, although they require a more complex and less portable equipment and turn out to be extremely sensitive to vibrations, see e.g. [76]. (A.2) Even though the approach outlined herein represents an autonomous strategy suitable for forward analyses and for the design of non-conventional experiments, incidentally it could be utilized also within the inverse strategy at point (A), as an alternative way to form the macroscopic inner work and to deduce boundary displacements. In analogy with De Saint-Venant method for slender prisms, Ansatz can be proposed also for generalized elastic bodies, namely closed-form solutions in terms of the placement map, see [77,78]. For problems such as bending, classical kinematic solutions (i.e. those commonly utilized in first gradient theory) can be assigned within the body volume, and the relevant (hyper-)stress tensors are deduced from them according to the generalized constitutive equations (which include the unknown parameters). By enforcing the momentum balance for each scenario, the corresponding external actions are easily computed. The analytical relationships between the unknown parameters (e.g. generalized stiffness tensors and/or characteristic lengths) and these external actions, specify ideal (or Gedanken) experiments: among them, non-conventional tests can be carried out, selecting those configurations and measurements which turn out to be practically (and technologically) feasible. These macroscopic solution fields can be utilized to compute the inner work and provide the boundary conditions, needed to implement the numerical strategy outlined in (A). (B) **Discrete-to-continuum and micro–macro strategies**. Another class of strategies are commonly referred to as discrete-to-continuum methods: they provide (forward) predictions of the parameters governing the macroscopic behaviour as schematically indicated by Eq. (27), to be regarded at the same time as solutions of the inverse problem Eq. (28). Basically, these approaches exploit the available information on the internal lengths and on the elementary interactions often utilized in the atomistic or molecular descriptions of matter, upscaling them to the generalized continuum, see e.g. [79]. Owing to important theoretical contribution dated '60 s (see e.g. [80]), it is possible to correctly represent multiple-particle stationary systems, namely their ground state without making recourse to the Schrödinger equation. To describe fluctuations around the equilibrium state, the equations of motion are written for the generic particle depending on a harmonic approximation of the lattice potential energy, say V, whilst slowly varying displacements of these particles, say $u_j^{\mathbf{l}}$, are expanded in power series including as coefficients specific polynomial expressions of the lattice vector components. Such coefficients are equated to the second strain gradient equations expressed in terms of displacement derivatives and generalized higher order stiffness tensors. Resuming the method exposed in [79], at the location of the periodic lattice specified by vector $\mathbf{a}^{\mathbf{l}} = \sum_{i=1}^{3} l_i \mathbf{v}_i$, being $\mathbf{v}_i$ the primitive lattice vectors and l_i integers, one can write for small fluctuations around the reference state at $\mathbf{l} = \mathbf{0}$ the following Taylor's expansion

$$u_j^{\mathbf{l}} = u_j^{\mathbf{0}} + \frac{\partial u_j}{\partial x_k}\Big|_{\mathbf{x}=\mathbf{0}} a_k^{\mathbf{l}} + \frac{\partial^2 u_j}{\partial x_k \partial x_j}\Big|_{\mathbf{x}=\mathbf{0}} a_k^{\mathbf{l}} a_j^{\mathbf{l}} + \ldots, \tag{31}$$

and the equation of motion for the i-th atom becomes

$$m_i \ddot{u}_i^{\mathbf{0}} = -\sum_{\mathbf{l}} \mathbf{U}_{ij}^{(\mathbf{0},\mathbf{l})} u_j^{\mathbf{l}} = -\sum_{\mathbf{l}} \frac{\partial^2 V}{\partial x_i^{\mathbf{0}} \partial x_j^{\mathbf{l}}}\Big|_{\mathbf{x}=\mathbf{0}} u_j^{\mathbf{l}}. \tag{32}$$

The above sum is extended to all the crystals in the unit cell: m_i denotes the mass of the i-th atom of the crystal, symbol $\mathbf{U}_{ij}^{(\mathbf{0},\mathbf{l})}$ indicates the force exerted in the i-th direction on the atom at the location $\mathbf{x} = \mathbf{0}$ when the atom at the location $\mathbf{x} = \mathbf{l}$ undergoes a unit displacement along the direction j. The first gradient of the potential V vanishes in the equilibrium configuration. In [81] according to the first-principles density functional theory (DFT), Sutton-Chen interatomic potentials have been applied for the calculations of the second gradient parameters and of the characteristic lengths of cubic face centered (fcc) materials. Such potentials depend on the relative location of the atoms in the periodic lattice, say $r_j^{\alpha\beta} = r_j^{\beta} - r_j^{\alpha} = a_j^{\alpha\beta}$. By perturbing the equilibrium configuration, the relative displacements are expressed as a function of the strains and of their gradients,

$$u_i^{\alpha\beta} = \varepsilon_{ij}\, a_j^{\alpha\beta} + \frac{1}{2}\, \varepsilon_{ij,k}\, a_j^{\alpha\beta} a_k^{\alpha\beta} + \cdots . \tag{33}$$

Once that the analytical expression for the total energy density of a fcc lattice is provided, this energy density is set equal to the energy density expected in the framework of second gradient elasticity theory. By this strategy one takes into account both the short and the long range interactions among the atoms. A similar strategy has been utilized in [82] to characterize the surface elastic behaviour. Borrowing methods from statistical mechanics, in [83] mathematical equations have been provided relating the strain-gradient material constants to atomic displacement correlations in a molecular dynamics computational ensemble. Using the developed relations and atomistic calculations, the strain-gradient constants have been explicitly determined for some representative semiconductor, metallic, amorphous and polymeric materials. Dispersion curves provided by ab initio calculations have been compared to those deduced by strain gradient theory. Although the results appear to indicate that strain-gradient effects are negligible for the selected materials, the authors conjecture that they could play a major role in the presence of defects.

Analogously, micro–macro approaches can be utilized, but remaining both at a continuum level. In [84] e [85] unidimensional interactions have been postulated among particles of materials with a granular texture, governed by only two coefficients playing the role of microscopic stiffnesses (along the normal and the tangential directions). Under finite deformations the unknown parameters of generalized elasticity, namely the elements of the generalized stiffness matrices, have been deduced in closed-form by averaging such interactions over a representative volume, as it occurs in nonlocal approaches: analytical expressions are obtained, which include normal and tangential stiffnesses at the microscale besides a characteristic length. To illustrate the present approach, let us consider one grain, say r, and in a spherical neighborhood of it, at a distance L along the direction $\mathbf{n}^{rs}$, another grain, say s. The relative displacement of the two grains can be expressed via a truncated Taylor expansion of the placement as

$$u_i^{rs} = 2\, E_{ij}\, n_j^{rs} L + \frac{1}{2} L^2\, E_{ij,k}\, n_j^{rs} n_k^{rs}, \tag{34}$$

where E_{ij} and $E_{ij,k}$ denote the Green-Lagrange finite strain tensor and its gradient. The elemental stored energy density between such a pair of grains, say $\mathcal{W}^{rs}$, can be expressed via the normal component of the above defined relative displacement and the norm of the tangential displacement vector, denoted by symbol $\mathbf{u}_t^{rs}$,

$$\begin{aligned} &u_n^{rs} = \mathbf{u}^{rs} \cdot \mathbf{n}^{rs}, \quad \mathbf{u}_t^{rs} = \mathbf{u}^{rs} - (\mathbf{u}^{rs} \cdot \mathbf{n}^{rs})\mathbf{n}^{rs}, \\ &\mathcal{W}^{rs} = \frac{1}{2}\, k_n\, u_n^{rs\ 2} + \frac{1}{2}\, k_t \|\mathbf{u}_t^{rs}\|^2. \end{aligned} \tag{35}$$

Let us superimpose the contributions of all the particles in the neighborhood of grain r at a distance L (i.e. at varying s and $\mathbf{n}^{rs}$) by integrating the above elemental energy over the unit sphere S^2. In this way all the spatial directions can be covered starting from point r, i.e. by varying the components of the vector $\mathbf{n}^{rs} = \{n_i^{rs}\}$ ($i = 1, 2, 3$) constrained by the condition of unit norm $\|\mathbf{n}^{rs}\| = 1$. One obtains

$$\mathcal{E}^r = \int_{S^2} \left\{ \frac{1}{2}\, k_n u_n^{r\,2} + \frac{1}{2}\, k_t \|\mathbf{u}_t^r\|^2 \right\} dS. \tag{36}$$

As meaningful examples of the elastic and the higher order stiffnesses in $2D$, the following expressions have been obtained: $C_{1111} = \frac{L^2}{2}\left(\frac{3}{2\pi} k_n + \frac{4}{2\pi} k_t\right)$ and $C_{1122} = \frac{L^2}{8}\left(\frac{1}{2\pi} k_n - \frac{4}{2\pi} k_t\right)$, $A_{111111} = \frac{L^4}{256}\left(\frac{5}{2\pi} k_n + \frac{4}{2\pi} k_t\right)$ and $A_{221221} = \frac{L^4}{256}\left(\frac{1}{2\pi} k_n + \frac{4}{2\pi} k_t\right)$. This last approach, more general of first-principle or ab-initio methods which are limited to very small domains, could permit to retrieve very useful information on the unknown parameters, for those materials which can be regarded as granular at some scale.

By a similar strategy, in [86] data provided by a 3D detailed model at the microscale of a pantographic structure subjected to finite strains have been utilized to calibrate a reduced order, higher grade plate model at the macroscale, to improve the description of out of plane buckling. The discrepancy norm to minimize included the total energy and two diverse rotation angles, which are representative of the pantograph deformation.

As mentioned above, generalized elasticity can be utilized to predict the size dependence of apparent elastic properties at lower scales, for instance with reference to the bending stiffness of nano-beams, see e.g. [87]. Basically, this effect turns out to be predominant when the size of the sample becomes comparable to the microscopic length scale, and it will be discussed in Sect. 5 also from the experimental standpoint. Once the objective of the modelling and its scale have been clarified, the homogenization of the microstructure is not faced explicitly, although some authors assume the presence of a boundary layer, with properties different from those within the bulk (see e.g. [68,88]). In this scenario, generalized elasticity is dealt with as any constitutive law governed by several parameters to be calibrated. For instance, in the numerical study [89], the pull out test of a single bar embedded in a concrete matrix has been investigated: the aim was to calibrate the characteristic length of concrete regarded as a representative generalized material. Once restricted the three dimensional equations to a radial section, the authors have computed the sensitivity of the vertical displacement (in the axial direction, parallel to the fiber) with respect to the characteristic length. In particular, considering the plot of the axial displacement versus the radial coordinate, they have observed that the location of the inflection point is highly sensitive to the characteristic length. As a meaningful finding, the relationship between the inflection point and the characteristic length is found to be approximately linear. In [90] a two stage identification procedure has been applied to a micropolar granular material with elasto-plastic behavior. Firstly, the parameters governing the (non-polar) elasto-plastic response are identified, and, secondly, those governing the micropolar part of the constitutive model. While the first set of parameters are obtained through a simple inverse algorithm applied to the elasto-plastic constitutive relationship of non-polar solids, the second set of parameter has required a fully inverse computation in the sense of a back analysis of the underlying boundary-value problem.

(C) **Asymptotic methods and homogenization strategies**. Asymptotic analysis can be regarded as a constructive procedure to solve the (inverse) problem of identifying the homogenous continuum medium, macroscopically equivalent to an assigned microstructure which behaves linearly elastically. It is based on an Ansatz, the celebrated two-field asymptotic expansion in powers of the dimensionless coefficient $\epsilon = l/L$. Such a coefficient, assumed to be small, equals the ratio between the scale of the microscopic unit cell l, namely the characteristic size of the heterogeneities, and the wavelength of the loading L, comparable to the size of the body. Such an expansion permits to specify a relationship between displacements at the microscale, rapidly oscillating, depending on a characteristic size and on the microstructure, and the slowly varying macroscopic field with its derivatives, see e.g. [91–94]. By the asymptotic expansion the displacement field is expressed as a function of two variables representing two scales (see e.g. [95]), the macroscopic coordinate vector $\mathbf{x}$ and the microstructural coordinate $\boldsymbol{y} = \mathbf{x}/\epsilon$ acting as a zoom onto the unit cell, as follows

$$u_i^{\mathrm{m}}(\mathbf{x},\mathbf{y}) = u_i^{\mathrm{M}}(\mathbf{x}) + \sum_{\alpha=1}^{+\infty} \epsilon^\alpha \sum_{|\mathbf{q}|=\alpha} \psi^{(\alpha)}_{ir\,q_1\cdots q_\alpha}(\mathbf{y}) \, \frac{\partial^{|\mathbf{q}|}}{\partial \mathbf{x}^{\mathbf{q}}} u_r^{\mathrm{M}}(\mathbf{x}), \tag{37}$$

where: $\mathbf{q}$ denotes a multi-index, namely an ordered vector of α indices $(q_1 \cdots q_\alpha)$, each one assuming here values $q_i = 1, 2, 3$, with norm $|\mathbf{q}| = q_1 + \cdots + q_\alpha = \alpha$; the expression $\partial \mathbf{x}^{\mathbf{q}}$ has to be intended as the product $\partial x^{q_1} \cdots \partial x^{q_\alpha}$, where for assigned α all the permutations of indices have to be considered with the same norm; at varying α, the localization tensor functions $\psi^{(\alpha)}_{ir\,q_1\cdots q_\alpha}(\mathbf{y})$, with rank $\alpha+2$, are periodic over the cells and with a vanishing volume average. The above displacement field exhibits the same periodicity of the unit cell with respect to the variable $\mathbf{y}$, whilst it can be considered L periodic with respect to $\mathbf{x}$, being L the size of the structure. The origin of this asymptotic approach is rooted in the Taylor expansion of a system of equations depending on a scalar parameter, as discussed in [96]. The above Ansatz Eq. (37) conjectures that the micro-displacement field depends on both the gradients of the macroscopic displacement field $u_r^{\mathrm{M}}(\mathbf{x})$, and on the unknown perturbation functions $\psi^{(\alpha)}_{ir\,q_1\cdots q_\alpha}$ accounting for the effects of the heterogeneities. It establishes a relationship between the observable, measurable quantities of our body belonging to the microscale, where it behaves linearly elastically, and their counterparts at the macroscale. Superimposed to the slowly varying field $u_i^{\mathrm{M}}(\mathbf{x})$ there are a sequence of correction terms, which modulate locally the slowly varying gradients $\partial^{|\mathbf{q}|}/\partial \mathbf{x}^{\mathbf{q}}\, u_i^{\mathrm{M}}(\mathbf{x})$. The perturbation tensor functions are determined by solving non-homogeneous linearly elastic problems on the unit cell, with periodic boundary conditions (on tensor functions $\psi^{(\alpha)}_{ir\,q_1\cdots q_\alpha}(\mathbf{y})$) and body forces which depend on the geometrical and mechanical properties of the microstructure, see e.g. [95]. The asymptotic and variational-asymptotic techniques provide a mathematically rigorous tool of higher-order homogenization of periodic linear elastic heterogeneous materials, see e.g. [97].

To outline the main features of this method, let us define the volume average of a microscopic field over the unit cell as follows

$$\langle u_i^{\mathrm{m}}\left(\mathbf{x}, \mathbf{y}\right)\rangle_Y = \frac{1}{V_Y}\int_Y u_i\left(\mathbf{x}, \mathbf{y}\right)\, d\mathbf{y}, \tag{38}$$

being Y the region occupied by the unit cell and V_Y the measure of its volume. For the macroscopic displacement, on the basis of Eq. (37) one has $u_i^{\mathrm{M}} = \left\langle u_i^{\mathrm{m}}\left(\mathbf{x}, \mathbf{y}\right)\right\rangle_Y$. To differentiate the above two-argument function $u_i^{\epsilon}(\mathbf{x}) = u_i^{\mathrm{m}}(\mathbf{x}, \mathbf{x}/\epsilon)$ Eq. (37), on the basis of the chain rule we must operate as follows

$$\frac{\partial u_i^{\mathrm{m}}}{\partial x_j^{\epsilon}} = \frac{\partial u_i^{\mathrm{m}}}{\partial x_j} + \frac{1}{\epsilon}\frac{\partial u_i^{\mathrm{m}}}{\partial y_j}. \tag{39}$$

By introducing the microscopic strain field corresponding to the asymptotic expansion (Eq. 37) into the momentum balance, one finds

$$\frac{\partial}{\partial x_j^{\epsilon}}\left(C_{ijkl}^{\mathrm{m}}(\mathbf{y})\, u_{k,\,l}^{\mathrm{m}}\left(\mathbf{x}, \mathbf{y}\right)\right) + \mathcal{F}_i(\boldsymbol{x}) = 0, \tag{40}$$

being $\mathcal{F}_i$ the bulk force density. Several contributions are obtained multiplying diverse powers of ϵ: those corresponding to different levels of resolution must vanish singularly. The perturbation tensor functions $\psi_{ir\,q_1\cdots q_\alpha}^{(\alpha)}$, each one with zero average over the unit cell, are provided in a sequence as solutions of these recursive elastic problems, starting from that relevant to the lowest power of ϵ (which can also be a negative integer).

One drawback of the present approach is related the fact that the representation Eq. (37) truncated for a finite $\tilde{\alpha}$ may give rise to differential problems which are not elliptic, and this circumstance is the source of severe ill-conditioning when dealing with numerical methods such as finite elements. The approach is potentially rigorous but it holds for $\epsilon \to 0$: for intermediate value of the scale factors, the perturbative asymptotic approach may become a less accurate approximation to the actual solution. Strategies have been proposed to construct in a rigorous way "higher order" homogenised equations which are elliptic, providing solutions "closest" to the actual solution in a certain variational sense, see [98]. Asymptotic expansion can enter the microscopic inner work required at the above point (A) Eq. (29), providing micro–macro relationships for the stiffness tensors, see e.g. [99,100].

More recently, in [94] a homogenization procedure resting on the asymptotic expansion has been developed, with the aim to be consistent, namely: the higher order stiffness parameter are expected to vanish when the material becomes homogeneous and a Cauchy continuum is retrieved at the macroscale; moreover, the parameters governing the homogenized second gradient model are not affected by the number of unit cells included in the representative volume element. Let us indicate by $\mathbf{x}$ the macroscopic coordinates, with origin in the centroid of the representative volume element, and $\mathbf{y}$ the microscopic coordinates. Let us approximate the macroscopic displacement and its gradient by a Taylor expansion of initial point the origin $\mathbf{x} = \mathbf{0}$, truncated at the second order. In index notation, for the i-th macroscopic displacement one has

$$u_i^{\mathrm{M}}(\mathbf{x}) = u_i^{\mathrm{M}}|_{\mathbf{0}} + u_{i,j}^{\mathrm{M}}|_{\mathbf{0}}\ x_j + \frac{1}{2}u_{i,jk}^{\mathrm{M}}|_{\mathbf{0}}\ x_j x_k, \tag{41}$$

and for its p-th partial derivative

$$u_{i,p}^{\mathrm{M}}(\mathbf{x}) = u_{i,p}^{\mathrm{M}}|_{\mathbf{0}} + u_{i,pj}^{\mathrm{M}}|_{\mathbf{0}}\ x_j,\quad u_{i,pl}^{\mathrm{M}}(\mathbf{x}) = u_{i,pl}^{\mathrm{M}}|_{\mathbf{0}}. \tag{42}$$

Hence, by volume averaging through Eq. (38) one obtains

$$\left\langle u_{i,p}^{\mathrm{M}}(\mathbf{x})\right\rangle_Y = u_{i,p}^{\mathrm{M}}|_{\mathbf{0}},\quad \left\langle u_{i,pl}^{\mathrm{M}}(\mathbf{x})\right\rangle_Y = u_{i,lp}^{\mathrm{M}}|_{\mathbf{0}}. \tag{43}$$

The macroscopic energy density, corresponding to a second gradient material, is expressed as a quadratic form with respect to the averaged macroscopic strains and their gradients, as follows

$$\mathcal{W}^{\mathrm{M}} = \frac{1}{2}\left(C_{ijkl}\left\langle u_{i,j}^{\mathrm{M}}\right\rangle_Y \langle u_{k,l}^{\mathrm{M}}\rangle_Y + \left[C_{ijlm}\, I_{kn} + A_{ijklmn}\right]\left\langle u_{i,jk}^{\mathrm{M}}\right\rangle_Y \langle u_{l,mn}^{\mathrm{M}}\rangle_Y\right), \tag{44}$$

being $I_{kn} = < \mathbf{x}\otimes\mathbf{x} >_Y$ the second moment of the unit cell, divided by its volume.

On the other hand, the asymptotic expansion of the microscopic displacement field, truncated at the second order, results to be

$$u_i^{\mathrm{m}}(\mathbf{x},\mathbf{y}) \simeq u_i^{\mathrm{M}}(\mathbf{x}) + \epsilon\,\psi_{iab}^{(1)}(\mathbf{y})\,u_{a,b}^{\mathrm{M}}(\mathbf{x}) + \epsilon^2\,\psi_{iabc}^{(2)}(\mathbf{y})\,u_{a,bc}^{\mathrm{M}}(\mathbf{x}). \tag{45}$$

To compute the microscopic strains, one can differentiate the above truncated expansion with respect to the variable x_j^ϵ according to Eq. (39), obtaining

$$\begin{aligned}
u_{i,j}^{\mathrm{m}} &= \left(u_i^{\mathrm{m}}(\mathbf{x},\mathbf{y})\right)_{,\,x_j} + \frac{1}{\epsilon}\left(u_i^{\mathrm{m}}(\mathbf{x},\mathbf{y})\right)_{,\,y_j}\\
&= u_{i,\,j}^{\mathrm{M}}(\mathbf{x}) + \psi_{iab,\,j}^{(1)}(\mathbf{y})\,u_{a,\,b}^{\mathrm{M}}(\mathbf{x}) + \epsilon\,\psi_{iab}^{(1)}(\mathbf{y})\,u_{a,\,bj}^{\mathrm{M}}\\
&\quad + \epsilon\,\psi_{iabc,\,j}^{(2)}(\mathbf{y})\,u_{a,\,bc}^{\mathrm{M}}(\mathbf{x}) + \epsilon^2\,\psi_{iabc}^{(2)}(\mathbf{y})\,u_{a,\,bcj}^{\mathrm{M}}(\mathbf{x})\\
&= \left(\delta_{ai}\delta_{bj} + \psi_{iab,\,j}^{(1)}\right)u_{a,\,b}^{\mathrm{M}} + \epsilon\,u_{a,bc}^{\mathrm{M}}(\mathbf{x})\left(\psi_{iab}^{(1)}(\mathbf{y}) + \psi_{iabc,j}^{(2)}(\mathbf{y})\right) + \cdots\\
&= \left(\delta_{ai}\delta_{bj} + \psi_{abi,j}^{(1)}\right)\left(\left\langle u_{a,b}^{\mathrm{M}}\right\rangle + x_v\left\langle u_{a,bv}^{\mathrm{M}}\right\rangle\right)\\
&\quad + \epsilon\left\langle u_{a,bc}^{\mathrm{M}}\right\rangle(\mathbf{x})\left(\psi_{iab}^{(1)}(\mathbf{y}) + \psi_{iabc,j}^{(2)}(\mathbf{y})\right) + \cdots\\
&= L_{abij}\left\langle u_{a,b}^{\mathrm{M}}\right\rangle + \epsilon\,M_{abcij}\left\langle u_{a,bc}^{\mathrm{M}}\right\rangle + \cdots,
\end{aligned} \tag{46}$$

where tensors L_{abij}^{m} and M_{abcij}^{m} are defined as

$$\begin{aligned}
L_{abij}^{\mathrm{m}} &= \delta_{ai}\,\delta_{bj} + \frac{\partial\,\psi_{iab}^{(1)}}{\partial\,y_j},\\
M_{abcij}^{\mathrm{m}} &= x_c\left(\delta_{ai}\,\delta_{bj} + \frac{\partial\,\psi_{iab}^{(1)}}{\partial\,y_j}\right) + \left(\psi_{iab}^{(1)}\,\delta_{jc} + \frac{\partial\,\psi_{iabc}^{(2)}}{\partial\,y_j}\right).
\end{aligned} \tag{47}$$

After some manipulations, the macroscopic stiffness tensors entering a second gradient model can be provided through the following integrals over the representative volume element Y,

$$\begin{aligned}
C_{abcd}^{\mathrm{M}} &= \frac{1}{V_Y}\int_Y C_{ijkl}^{\mathrm{m}}\,L_{abij}^{\mathrm{m}}\,L_{cdkl}^{\mathrm{m}}\;dV_Y,\\
A_{abcdef}^{\mathrm{M}} &= \epsilon^2\left(\frac{1}{V_Y}\int_Y C_{ijkl}^{\mathrm{m}}\,M_{abcij}^{\mathrm{m}}\,M_{defkl}^{\mathrm{m}}\,dV_Y - C_{abef}^{\mathrm{M}}\int_{V_Y} y_c y_d\;dV_Y\right).
\end{aligned} \tag{48}$$

In the last equation a correction term was included, to enforce compatibility and reliability requirements for the strain gradient stiffness tensor. The above estimates of the macroscopic stiffness tensors have been deduced by assuming known a priori the geometry and the properties of the microstructure. The same relationships could be exploited within an inverse perspective, to identify the microstructures which fit a target homogenized response in terms of macroscopic generalized stiffness tensors.

In [100] the homogenization of a microstructured body has been investigated towards a second gradient equivalent medium. Let us denote by $\mathbf{y}$ the variable spanning the microstructure centred at the point $\mathbf{x}$ of the macroscale, in such a way that the relative position has zero average within the unit cell. The microscopic displacement field is assumed as follows

$$\begin{aligned}
\mathbf{u}(\mathbf{y})^{\mathrm{m}} &= \mathbf{u}^{\mathrm{hom}}(\mathbf{y}) + \widetilde{\mathbf{u}}(\mathbf{y})\\
&= \mathbf{E}(\mathbf{x})\,\mathbf{y} + \frac{1}{2}\,\mathbf{L}_K(\mathbf{x}) : (\mathbf{y}\otimes\mathbf{y}) + \widetilde{\mathbf{u}}(\mathbf{y}),\;\; \forall\,\mathbf{y}\in\, Y,
\end{aligned} \tag{49}$$

where the macroscopic, second gradient kinematic variables $(\mathbf{E},\mathbf{K})$ are evaluated at the point $\mathbf{x}$, and the fluctuation $\widetilde{\mathbf{u}}(\mathbf{y})$ is periodic on the unit cell Y. A passage from Mindlin I to Mindlin II formulations is carried out by the authors through a sixth rank tensor T_{ijklmn}, in such a way that $L_{K\,ijk} = T_{ijklmn}\,K_{lmn}$ gathers components of the tensor $\mathbf{K}$ rearranged and linearly combined. It is worth noting that, in the case of a Cauchy continuum, only the affine contribution depending on the strain tensor $\mathbf{E}$ is considered, and $\mathbf{L}_K(\mathbf{x}) = \mathbf{K} = \mathbf{0}$. The equivalence of energy implies

$$\left\langle \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\varepsilon} \right\rangle_Y = \frac{1}{2}\left(\boldsymbol{\Sigma} : \mathbf{E} + \mathbf{S} : \mathbf{K}\right), \tag{50}$$

being $\mathbf{K} = \mathbf{E}_{,x} = \{E_{ij,k}\}$ and S_{ijk} its work conjugate hyperstress tensor. The expression $\mathbf{S} = \int_Y \boldsymbol{\sigma} \otimes \mathbf{y}\, dV_Y$ strictly holds if the micro-fluctuation field $\tilde{\mathbf{u}}(\mathbf{y})$ is periodic. This assumption, together with that of antiperiodic tractions over the boundary of the representative volume element, is typical of the scenario in which the macroscopic length scale L exceeds significantly the length l of the microstructure, namely $L \gg l$, and the separation of scale is met: the challenging issue would be that of predicting also the intermediate situations. Now let us make reference to a representative volume element constituted of one or several unit cells. Preliminarily, is worth emphasizing that $\mathbf{U} = \langle \mathbf{u} \rangle_Y$, $\mathbf{E} = \langle \mathbf{u}_{,\mathbf{y}} \rangle_Y$ and $\mathbf{K}(\mathbf{x}) = \mathbf{E}_{,\mathbf{x}}(\mathbf{x})$. Herein we intend to compute the homogenized stiffness tensor of the second gradient medium at the macroscale, specifying the constitutive relationship

$$\begin{pmatrix} \boldsymbol{\Sigma} \\ \mathbf{S} \end{pmatrix} = \begin{pmatrix} \mathbf{C}^{\mathrm{M}} & \mathbf{B}^{\mathrm{M}} \\ \mathbf{B}^{\mathrm{M}\,T} & \mathbf{A}^{\mathrm{M}} \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{K} \end{pmatrix}. \tag{51}$$

For centro-symmetric media the coupling tensor $\mathbf{B}$ vanishes. As a meaningful result [100], combining Eqs. (49)–(50), for prescribed macroscopic strain $\mathbf{E}$ and its gradient $\mathbf{K}$, the energy density of the strain gradient continuum, say $\mathcal{W}^{\mathrm{M}}$, equals the elastic energy of the microscopic representative volume, in turn expressed as the minimum of the microscopic energy in the unit cell Y over the set of periodic micro-displacement fluctuations $\tilde{u}_i$,

$$\mathcal{W}^{\mathrm{M}}(\mathbf{E}, \mathbf{K}) = \arg\min_{\tilde{\mathbf{u}}} \left\{ \int_Y \frac{1}{2}\mathbf{C}^{\mathrm{m}} : \left(\mathbf{E} + \mathbf{L}_K \mathbf{y} + \tilde{\mathbf{u}}_{,\mathbf{y}}\right) : \left(\mathbf{E} + \mathbf{L}_K \mathbf{y} + \tilde{\mathbf{u}}_{,\mathbf{y}}\right) dV_Y \right\}, \tag{52}$$

where the microscopic strain field in the integral is obtained by differentiating Eq. (49). On the basis of this result, suitable localization tensors can be computed, obtaining

$$\begin{aligned} \tilde{u}_i(\mathbf{x}, \mathbf{y}) &= H^E_{ikl}(\mathbf{y}) E_{kl}(\mathbf{x}) + H^K_{iklm}(\mathbf{y}) (L_K)_{klm}(\mathbf{x}), \\ \tilde{\varepsilon}_{ij}(\mathbf{x}, \mathbf{y}) &= \tilde{u}_{(i,j)} = \mathcal{A}^E_{ijkl}(\mathbf{y}) E_{kl}(\mathbf{x}) + \mathcal{A}^K_{ijklm}(\mathbf{y})\, K_{klm}(\mathbf{x}), \end{aligned} \tag{53}$$

being $\mathcal{A}^E_{ijkl} = \mathbb{1}_{ijkl} + H^E_{ijk,l}$, $\mathcal{A}^K_{ijklm} = \mathbb{1}_{ijkl}\, y_m + H^K_{ijkl,m}$, and $\mathbb{1}$ denotes the fourth order identity tensor. By making use of the generalized expression of Hill's energy equivalence Eq. (50), one finally obtains the macroscopic stiffness tensors entering Eq. (51),

$$\begin{aligned} C^{\mathrm{M}}_{ijkl} &= \left\langle \mathcal{A}^E_{pqij}\, C^{\mathrm{m}}_{pqrs}\, \mathcal{A}^E_{rskl} \right\rangle_Y, \quad B^{\mathrm{M}}_{ijklm} = \left\langle \mathcal{A}^E_{pqij}\, C^{\mathrm{m}}_{pqrs}\, \mathcal{A}^K_{rsklm} \right\rangle_Y, \\ B^{\mathrm{M}}_{lmijk} &= \left\langle \mathcal{A}^K_{pqijk}\, C^{\mathrm{m}}_{pqrs}\, \mathcal{A}^E_{rslm} \right\rangle_Y, \quad A^{\mathrm{M}}_{ijklmt} = \left\langle \mathcal{A}^K_{pqijk}\, C^{\mathrm{m}}_{pqrs}\, \mathcal{A}^K_{rslmt} \right\rangle_Y. \end{aligned} \tag{54}$$

Such estimates turn out to be independent of the number of unit cells included in the representative volume element, and for homogeneous composites they must provide vanishing strain gradient moduli. In another contribution [101], the same author have used the unfolding method as an alternative homogenization scheme for the closed form identification of the strain-gradient elastic moduli of composite materials, discussed at the light of the method of oscillating functions earlier proposed by Luc Tartar [102].

To compute the effective properties of a periodic composite when assuming conventional Cauchy equations (both for the phases at the microscale and the equivalent medium at the macroscale), affine displacements have been usually prescribed along the frontier of the representative volume element, endowed with periodicity conditions on the displacement field. In computational second-order homogenization methods, see e.g. [103–106], the microscopic displacement is expressed as the sum of a polynomial form with respect to the local coordinates, involving macroscopic strains and their gradients as coefficients, and of an unknown fluctuation. Most of the approaches include a quadratic Ansatz, and the additive micro-displacement fluctuation field is usually assumed to be periodic, as in Eq. (49). Such a higher order homogenization scheme is at the basis of the so-called FE^2 numerical methods: in these strategies, the constitutive model of the equivalent continuum at each (integration) point of a structure is replaced by a boundary value problem over the unit cell of the underlying heterogeneous material. Although computationally expensive, these methods can deal with nonlinear problems without defining explicitly the constitutive relationship for the continuum at the macroscale. It is worth emphasizing that, for the generalized continua, boundary conditions must be apt to generate non vanishing strain gradients.

As discussed in [107], let us imagine to prescribe a quadratic displacement field to all the material points of a representative volume element Y centered in $\mathbf{x}$, namely

$$\mathbf{u}^{\mathrm{m}}(\mathbf{y}) = \mathbf{E}\,\mathbf{y} + \frac{1}{2}\mathbf{D} : (\mathbf{y} \otimes \mathbf{y}), \quad u_i^{\mathrm{m}}(\mathbf{y}) = E_{ij}\, y_j + \frac{1}{2} D_{ijk}\, y_j y_k, \;\; \forall \mathbf{y} \in Y, \tag{55}$$

being $\mathbf{E}$ and $\mathbf{D}$ in turn a second and a third rank tensor, with suitable symmetries. By differentiating and averaging Eq. (55) over the unit cell Y, one would obtain

$$\left\langle \mathbf{u}^{\mathrm{m}}_{,\mathbf{y}} \right\rangle_Y = \mathbf{E}, \;\; \left\langle \mathbf{u}^{\mathrm{m}}_{,\mathbf{yy}} \right\rangle_Y = \mathbf{D}. \tag{56}$$

However, by requiring that each point of the unit cell obeys to the displacement field in Eq. (55), a response excessively stiff is expected: hence, let us restrict such a condition to the boundary of the representative volume element, namely assumed to hold $\forall\, \mathbf{y} \in \partial Y$. Through the Gauss theorem one finds

$$\left\langle \mathbf{u}^{\mathrm{m}}_{,\mathbf{y}} \right\rangle_Y = \frac{1}{V_Y} \int_{\partial Y} \mathbf{u}^{\mathrm{m}} \otimes \mathbf{n} \;\, d\partial V_Y = \mathbf{E}, \;\; \left\langle \mathbf{u}^{\mathrm{m}}_{,\mathbf{yy}} \right\rangle_Y = \frac{1}{V_Y} \int_{\partial Y} \mathbf{u}^{\mathrm{m}}_{,\mathbf{y}} \otimes \mathbf{n} \;\, d\partial Y \neq \mathbf{D}. \tag{57}$$

Hence, the first relation in Eq. (57) involving the average of the gradient remains correct, i.e. it is possible to control the average strain inside the representative element by simply prescribing quadratic displacement over its boundary. Unfortunately, the second condition including the average of the Hessian is in general not valid: the average second gradient of the microscopic displacement cannot be directly controlled by the second gradient of a quadratic field. At this point let us try to perturb the quadratic condition Eq. (55) adding a fluctuation field, say v_i, namely

$$u_i^{\mathrm{m}} = E_{ij}\, y_j + \frac{1}{2} D_{ijk}\, y_j y_k + v_i, \;\; \forall\, \mathbf{y} \in \partial Y. \tag{58}$$

Actually, if a strain gradient is applied there is no reason to require that the fluctuation field is periodic over the boundary (as well as antiperiodic the relevant tractions). Moreover, by the Hill-Mandel equivalence of energy one obtains

$$\begin{aligned} \langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle_Y &= \langle \boldsymbol{\sigma} \rangle_Y : \boldsymbol{E} + \langle \boldsymbol{\sigma} \otimes \mathbf{y} \rangle_Y \,\vdots\, \boldsymbol{D} + \left\langle \boldsymbol{\sigma} : \boldsymbol{v}_{,\mathbf{x}} \right\rangle_Y \\ &= \mathbf{E} : \mathbf{C}^{\mathrm{M}} : \mathbf{E} + \mathbf{K} \,\vdots\, \mathbf{A}^{\mathrm{M}} \,\vdots\, \mathbf{K}. \end{aligned} \tag{59}$$

Only if the fluctuation v_i is periodic, the last addend along the first row of Eq. (59) vanishes, i.e. $\left\langle \boldsymbol{\sigma} : \boldsymbol{v}_{,\mathbf{x}} \right\rangle_Y = 0$, the second conditions in Eq. (56) holds, and one is entitled to accept the equalities $\mathbf{K} = \left\langle \mathbf{u}_{,\mathbf{yy}} \right\rangle_Y = \mathbf{D}$ together with the expression $\mathbf{S} = \langle \boldsymbol{\sigma} \otimes \mathbf{y} \rangle_Y$ for the hyperstress. For peculiar symmetries of the unit cell, under specific loading conditions, the contribution of the fluctuation terms may become negligible, and anti-periodicity conditions of tractions at opposite locations of the unit cell may be verified. An extremely interesting approach which generalizes second-order homogenization to quasi periodic composites with grading directions (namely, to functionally graded materials) has been developed in [108].

3.1.1 Discrete lattice systems

To homogenization purposes, the discrete systems constituted of slender elements (such as truss or beams) variously constrained, referred to as lattice models, deserve a very special attention, see e.g. [109]. The reticulated topology, like a porous material, favours the efficient design by periodic modules, and permits the realization by additive manufacturing techniques. Lattice structures represent a reference composite material with high contrast among the phases, since the material heterogeneity is mainly geometrical due to the presence of voids. Such a feature acts as a strong regularization in the synthesis problem, permitting a drastic reduction of the dimensions in the design space, as it will be highlighted in Sect. 6. Moreover, mechanical interactions of different kinds (short and long range) can be easily implemented in the discrete lattice, with a suggestive visualization through connection springs of different lengths.

Continualization methods of periodic lattices rely on the idea of replacing discrete equations of motion with continuum type differential equations, to compute the static and dynamic effective properties at a continuous level, see e.g. [110,111]. These approaches rest on two fundamental steps: (i) replacing discrete kinematics with field variables that vary continuously in space; (ii) expressing the microscopic degrees of freedom through a truncated Taylor expansion versus the selected continuous variables. In [112] the strain gradient elastic properties of architectured materials (exemplified by hexagons and square geometries) have been evaluated in closed form versus the unit cell lattice microstructural parameters, in particular the edge length of the hexagon and of the square unit cells. The shape of the unit cell was varied in a self-similar manner, in such a way that the volume fraction remained constant (leaving unchanged the Cauchy response). In particular shape sensitivity of the macroscopic inner energy has been evaluated with respect to perturbations of the area of the unit cell. For further details on the topological derivatives, see also [113].

For discrete lattice systems, a heuristic homogenization approach "à la Piola" has provided very interesting results, see e.g. [114,115]. By selecting peculiar points of the microscopic structure, kinematic variables are introduced for the equivalent continuum matching the microscopic fields at those specific locations. After introducing continuum counterparts of the microscopic properties, letting the characteristic size tend to zero, a continuous Lagrangian density function is obtained, and by a variational approach the equations governing the homogenized continuum are deduced, endowed with the relevant boundary conditions. In [116,117] a second-gradient elastic material has been identified as the homogeneous solid equivalent to a periodic planar lattice characterized by a hexagonal unit cell. This cell is constituted of three different linearly elastic bars ordered in such a way that the hexagonal symmetry is preserved and hinged at each node, so that the lattice bars are subject to pure axial strain while bending is prevented. Closed form-expressions for the identified non-local constitutive parameters have been provided by prescribing the elastic energy equivalence between the lattice and the continuum solid, under remote displacement conditions having a dominant quadratic component. For a discrete lattice, as developed in [32], it is possible to compute analytically the relevant energies of the homogenized continuum: analyzing a wide number of configurations, a sort of library of microstructures can be realized, classified on the basis of their macroscopic behaviors. The millennium conjecture to be proved is that, due to the linear background, when a macroscopic high grade behavior is assigned as a target, the microstructures whose homogenized response matches this datum can be constructed by a sequence of basic operations, suitably combining elementary structures selected from a library. A discussion on this topic can be found in the Sect. 6.

4 Dynamic behaviour

The elastodynamic response of architectured materials cannot be described by an equivalent Cauchy-Born continuum at the macroscale, since this classical approach is dramatically not able to describe the frequency spectrum of waves propagating through such media, see [5,118]. In particular, the presence of optical modes and possibly of band gaps (i.e. of stopping frequencies) is not consistent with Cauchy mechanics at the macroscale: on the other hand, the use of Cauchy modelling for the single phases at the microscale would be prohibitive in most of the scenarios. As well known, wave propagation in microstructured materials depends on the ratio between the wavelength and a characteristic size of the microstructure [119–122]. Indeed, when this ratio decreases (i.e. when the wavelength approaches this characteristic size) quantities such as phase and group velocities [123] deviate significantly from their low frequency/long wavelength values. This phenomenon is referred to as dispersion of waves, see e.g. [124–127]: in other terms, classical elastodynamics properties, such as the effective elastic moduli and the effective mass density, must be regarded as frequency varying parameters. Wave propagation in heterogeneous media has to be characterized by the following, multiple features: wave dispersion, since the phase (and group) velocity are not constant but depend on the frequency; directivity, i.e. the initial polarization direction of waves can be distorted depending on the frequency; the presence of optical branches, i.e. some modes may possess cut-off frequencies. Classical Cauchy elasticity has to be regarded as a theory of the linear approximation of the acoustic branches: this theory is usually sufficiently accurate to describe the behavior of homogeneous or slightly heterogeneous anisotropic media, see [128]. However, when increasing the frequency or the wavenumber, this theory fails to capture almost all the key features, see e.g. [129,130]. For instance, contrarily to classical elasticity, wave propagation in hexagonal (chiral or achiral) lattices (for instance the honeycomb structures) becomes anisotropic as the frequency increases. Analogous considerations hold for the wave transmission and reflection in the presence of interfaces [131]. Moreover, since strain-gradient elasticity is dispersive, the phase and the group velocities have to be treated as different quantities.

In the present framework of generalized elasticity, under small displacement and strain approximation, the kinetic energy and the elastic energy densities can be defined as follows

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}\,\pi_i\,v_i + \frac{1}{2}\,\pi^{(1)}_{ij}\,v_{i,j}, \\ \mathcal{W} &= \frac{1}{2}\,\sigma_{ij}\,\varepsilon_{ij} + \frac{1}{2}\,\sigma^{(1)}_{ijk}\,\varepsilon^{(1)}_{ijk}, \end{aligned} \tag{60}$$

where symbols have the following meaning: $\mathbf{v} = \{v_i\} = \{\dot{u}_i\}$ and $\nabla\mathbf{v} = \{v_{i,j}\} = \{\dot{u}_{i,j}\}$ denote the velocity vector and its spatial gradient; $\boldsymbol{\varepsilon} = \{\varepsilon_{ij}\} = \{u_{(i,j)}\}$ and $\boldsymbol{\varepsilon}^{(1)} = \{\varepsilon_{ij,k}\} = \{u_{(i,j)k}\}$ the strain tensor and its gradient; $\boldsymbol{\pi} = \{\pi_i\}$ and $\boldsymbol{\pi}^{(1)} = \{\pi^{(1)}_{ij}\} = \{\pi_{i,j}\}$ the momentum and the hyper-momentum, consistently with Eqs. (25)–(26); $\boldsymbol{\sigma} = \{\sigma_{ij}\}$ and $\boldsymbol{\sigma}^{(1)} = \{\sigma_{ij,k}\}$ the Cauchy stress tensor and the hyperstress tensor. Gathering into a unique vector all the kinematic quantities, and the static (dual) ones as well, the following compact notation can be used for the first strain gradient elasticity

$$\begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi}^{(1)} \\ \boldsymbol{\sigma} \\ \boldsymbol{\sigma}^{(1)} \end{pmatrix} = \begin{pmatrix} \rho\mathbf{I} & \mathbf{K} & \mathbf{0} & \mathbf{0} \\ \mathbf{K}^T & \mathbf{J} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{M} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}^T & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{v} \\ \nabla\mathbf{v} \\ \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^{(1)} \end{pmatrix}. \tag{61}$$

Symbol $\rho\,\mathbf{I} = \rho\,\delta_{ij}$ indicates the macroscopic mass density, $\mathbf{J} = \{J_{ijkl}\}$ the fourth order (micro-) inertia tensor, $\mathbf{K} = \{K_{ijk}\}$ the coupling tensor between the velocity vector and its gradient; as for the elastic energy, $\mathbf{C} = \{C_{ijkl}\}$ denotes the fourth order stiffness tensor, $\mathbf{A} = \{A_{ijklmn}\}$ the sixth order stiffness tensor, and $\mathbf{M} = \{M_{ijklm}\}$ the fifth order coupling tensor between stress and hyperstress tensors. Space and time are assumed to be decoupled. By prescribing the stationarity of the action functional, introducing the effective stress tensor (see also Eqs. 15-16)as $\hat{\sigma}_{ij} = \sigma_{ij} - \sigma^{(1)}_{ijk,k}$, the linear momentum equation can be written as follows

$$\sigma_{ij,j} - \sigma^{(1)}_{ijk,kj} = \hat{\sigma}_{ij,j} = \dot{\pi}_i - \dot{\pi}^{(1)}_{ij,j}. \tag{62}$$

By using Eq. (61), one obtains

$$\begin{aligned} &C_{ijkl}\,\varepsilon_{kl,j} + M_{ijklm}\,\varepsilon^{(1)}_{klm,j} - M_{lmijk}\,\varepsilon_{lm,kj} - A_{ijklmn}\,\varepsilon^{(1)}_{lmn,kj} \\ &= \rho\,\delta_{ij}\dot{v}_j + K_{ijk}\dot{v}_{j,k} - K_{kij}\,\dot{v}_{k,j} - J_{ijkl}\dot{v}_{k,lj}. \end{aligned} \tag{63}$$

Finally, in terms of displacement components and their derivatives one can write

$$C_{ijkl}\,u_{k,lj} + M^{\sharp}_{ijklm}\,u_{k,lmj} - A_{ijklmn}\,u_{l,mnkj} = \rho\,\ddot{u}_i + K^{\sharp}_{ipq}\ddot{u}_{q,p} - J_{ipqr}\ddot{u}_{q,rp}, \tag{64}$$

being $M^{\sharp}_{ijklm} = M_{ijklm} - M_{lmijk}$ and $K^{\sharp}_{ipq} = K_{ipq} - K_{pqi}$. Odd rank tensors $M^{\sharp}_{ijklm}$ and $K^{\sharp}_{ipq}$ vanish for common materials, although M_{ijklm} and K_{ipq} are not null. As discussed in [65,132], there exists 14 symmetry classes for a second gradient material. In the seminal study [133] this model was utilized to interpret results of ballistic experiments by acoustic phonons of crystalline GaAs samples.

Useful considerations to characterize architectured materials can be provided by studying the propagation of bulk plane waves. To this purpose a complex representation is used, namely $u_i = U_i\,\mathcal{V}\,\exp(\imath\{\omega t - \mathbf{k}\cdot\mathbf{x}\})$ where $\imath$ is the imaginary unit ($\imath^2 = -1$), $\mathcal{V}$ is a complex amplitude (generating a phase shift besides affecting the wave intensity), U_i is responsible for the wave polarization, ω is the angular velocity, and $\mathbf{k} = k\,\boldsymbol{\xi}$ the wave vector (being $\|\boldsymbol{\xi}\| = 1$), $\mathbf{x}$ the position vector and t the time variable. Actually, different definitions of velocity are possible when dealing with waves. In the present context, one must distinguish the group and the phase velocity: the former is defined as $\mathbf{v}^p = \frac{\omega}{k}\,\boldsymbol{\xi}$, and corresponds to a secant quantity, the speed of the wavefront of single harmonic wave oscillations for a wave propagating toward the direction $\boldsymbol{\xi}$; the latter is defined as $\mathbf{v}^g = \frac{\partial\omega}{\partial\mathbf{k}}$, and corresponds to a tangent speed. For the sake of clarity, let us assume $M^{\sharp}_{ijklm}$ and $K^{\sharp}_{ipq}$ in Eq. (64) to vanish. Substituting the above Ansatz into the balance equation, the so called acoustic equation is deduced (sign $-$ was simplified at both the sides)

$$Q_{is}\,U_s = \omega^2\,\left(\rho\,\mathbb{1}_{is} + J_{ipsr}k_r k_p\right)\,U_s. \tag{65}$$

The tensor at l.h.s., referred to as generalized acoustical gyrotropic tensor, is responsible of the acoustic activity of a second gradient medium. It has the expression

$$Q_{is} = \left\{ C_{ijsl} k_j k_l + A_{ijpsmn} \, k_j k_p k_m k_n \right\}. \tag{66}$$

For each choice of the wavevector $\mathbf{k}$, the linear system in Eq. (65) represents an eigenvalue problem, governed by symmetric and positive defined coefficient matrices, see e.g. [134]. The classical definition of acoustic tensor is retrieved when $A_{ijklmn} \to 0$, or also $\omega \to 0$ (namely, for low frequencies). Several experimental techniques rest on the analysis of the wave velocities in a medium, generated for instance through a photon-phonon mechanism. By inverse analysis, knowing the phonon velocity, one can retrieve the elastic properties, possibly anisotropic, of the medium, see Sect. 5.3.3. This strategy can be extremely effective in the case of films or coatings, as discussed in [18].

Moreover, even in the framework of generalized elasticity the capabilities of a model to predict the elastodynamics features vary in a remarkable manner, and, unfortunately, increase with the complexity of the formulation, see e.g. [23]. A higher order approximation of the kinetic energy is usually required, as in Eqs. (25)–(26), including novel contributions regarded as micro-inertia and additional characteristic lengths, see [68, 135]. As discussed in [120], a class of gradient elasticity theories suitable for dynamics applications involves mixed spatial- temporal derivatives by incorporating higher-order contributions both in the stiffness terms and in the inertia terms. While inclusion of the Laplacian of the strain in the constitutive relation is essential to remove singularities from the elastic strain field, the acceleration gradients (micro-inertia terms) are of paramount importance to capture the dispersive character of the wave propagation. These considerations can lead to select constitutive equations like the following

$$\sigma_{ij} = C_{ijkl} \left(\varepsilon_{kl} - l_a^2 \, \varepsilon_{kl,\,nn} + l_b^2 \, \ddot{\varepsilon}_{kl} - l_c^4 \, \ddot{\varepsilon}_{kl,\,pp} \right), \tag{67}$$

which, besides the independent elastic moduli entering tensor C_{ijkl}, are governed by three parameters, l_a^2 (with dimension [L^2]) multiplying the strain laplacian, l_b^2 (with dimension[T^2]) and l_c^4 (with dimension [T^{2}L^2]) multiplying in turn the second time derivative of the strain and of the strain laplacian. These parameters make reference to characteristic lengths, as those met in the formulation of bi-Helmoltz elastic medium Eq. (4), but also to internal time scales, as the strain were expanded in series with respect to the time variable also. The present parameters l_b and l_c are not dimensionally consistent with the lengths l_{d1} and l_{d2} entering Eq. (25), which were defined starting from the velocity field. If we consider the linear momentum balance, in the form $\sigma_{ij,j} - \rho \ddot{u}_i = 0$, assuming homogeneous elastic moduli and rearranging terms one has

$$C_{ijkl} \left(u_{k,lj} - l_a^2 \, u_{k,\,lnnj} \right) = \rho \left(\ddot{u}_i - l_b^2 \frac{1}{\rho} C_{ijkl} \ddot{u}_{k,\,lj} + l_c^4 \frac{1}{\rho} \, C_{ijkl} \ddot{u}_{k,\,lppj} \right). \tag{68}$$

To interpret such a model, a mechanical scheme has been proposed in which at the macroscale both lumped masses and distributed masses are present. The distributed masses, representing the macroscopic counterpart of microscopic lumped masses, are responsible for the higher order inertia contributions. For the experimental validation, the authors made reference to a well-documented experimental campaign, concerning ultrasonic wave propagation in fresh and hardened concrete as well as in mortar specimens [136].

Alternative approaches have been proposed to investigate the dynamic response of generalized materials. In [137] a two-scale approach has been developed based on the perturbation of peculiar standing waves, with high frequencies and long wavelengths, allowing one to identify the Bloch spectra through a differential eigenvalue problem. In [95] an asymptotic expansion approach has been proposed, through a sequence of elastic problems including also dynamic effects proportional to the mass of the phases. The micro-fluctuation functions turn out to depend on the angular frequency and on the wave-vector, and provide a sufficiently accurate approximation of the lowest (acoustic) branch of the dispersion curves obtained by the Floquet-Bloch theory.

A guidance distinction can be made between higher order and higher grade continua: the former, where an enriched set of descriptors is available, as in the micromorphic model, are able to detect optical branches; the latter, where only higher gradients of the placement are considered, can describe dispersion and directivity but not optical modes. It is worth noting that the strain- gradient elasticity, belonging to the latter class of models, can be retrieved as a low frequency, long wavelength approximation of the micromorphic kinematics. In conclusion, generalized continua can provide a good description of the local dynamics of the architectured continuum for a window of wavelengths, beyond which they will fail to predict the actual dynamics. The optimal elastic model must be selected as a compromise between conflicting requirements of simplicity, in particular low number of parameters, and prediction accuracy. To develop a robust identification method for generalized materials it is crucial to process both static and dynamic data, adopting mixed static-dynamic strategies. In particular, for a truly quantitative assessment the dispersion curves, the phase and the group velocities should be included as measurable quantities within the discrepancy norm to minimize, considering for instance as a target the values provided by Floquet-Bloch analyses.

In [135] a 3D printed pantographic metamaterial has been subjected to forced oscillations for a range of frequencies, and the parameters governing the homogenized second gradient model have been calibrated. The parameter estimation has been decomposed into two stages. Since the deformation and kinetic energies are independent, the corresponding governing parameters are estimated separately. For the former, quasi-static loading condition makes kinetic energy negligible. For the latter, loading close to the eigenfrequencies let the inertial terms dominate. For a given frequency of the forced oscillation far from the eigenfrequencies, the objective function ζ to minimize includes the squared L_2 norm of the residuals between the displacements at suitable nodes of the finite element discretization (orderly gathered within a unique vector $\mathbf{u}^{\text{comp}}(\mathbf{p})$), depending on the unknown parameter vector $\mathbf{p}$, and their experimental counterparts provided by a 2D Digital Image Correlation procedure (2D DIC),

$$\begin{aligned} \widehat{\mathbf{p}}^{\text{M}} &= \arg\min_{\mathbf{p}^{\text{M}}\in\mathcal{K}} \zeta(\mathbf{p}^{\text{M}}) \\ \zeta(\mathbf{p}^{\text{M}}) &= \|\mathbf{u}^{\text{exp}} - \mathbf{u}^{\text{comp}}(\mathbf{p}^{\text{M}})\|^2, \end{aligned} \tag{69}$$

where the admissible set $\mathcal{K}$ takes into account possible constraints on the parameters. Moreover, through a 2.5 DIC system, based on pictures provided by two synchronized cameras, also the out of plane displacement of the sheets was monitored. Closely to the eigenfrequency this displacement component turned out to be markedly large, hence the plane modelling resulted to be inadequate. However, at increasing the frequency also the residuals of in plane displacements increased significantly, since the measured profiles were markedly different from the model predictions: this circumstance is probably due to the absence of higher order (micro-)inertia contributions in the model utilized by the authors. For a critical assessment of the macroscopic second gradient models, wave propagation in a planar pantographic lattice was also simulated by means of a microscopic model including generalized Euler-Bernoulli beams, see [138]. In [139], to assess the properties of anisotropic media by ultrasonic bulk and guided waves, the authors have proposed to compare the experimental and computed dispersion curves by the matching functional

$$\zeta(\mathbf{p}^{\text{M}}) = \sum_i \left(k_i^{\text{exp}} - k_i^{\text{comp}}(\mathbf{p}^{\text{M}})\right)^2, \tag{70}$$

where k_i indicates the wavenumber corresponding to the i-th frequency. In [140] a novel particle-based second gradient material has been proposed, for which a symmetric, positive defined acoustic tensor was deduced, predicting dispersion of waves (for a comparison, consider Eqs. 65-66). For the identification of the relevant generalized stiffness tensors (with any class of symmetry), the objective function to minimize, resting on $N \times N'$ pairs of experimental measurements $(k_r, \boldsymbol{\xi}_s)$, being k_r the r-th wavenumber and $\boldsymbol{\xi}_s$ the s-th unit wave-vector, has been selected as follows

$$\zeta(\mathbf{p}^{\text{M}}) = \sum_r \sum_s \left\{ \det\left[\widetilde{Q}_{ik}^{rs}(\mathbf{p}^{\text{M}}) - \rho \left(v^{rs}\right)^2 \delta_{ik} \right] \right\}^2, \tag{71}$$

being $v^{rs} = v(k_r, \boldsymbol{\xi}_s)$ the (r, s) phase velocity, $\widetilde{Q}_{ik}^{rs} = \widetilde{Q}_{ik}(\mathbf{p}^{\text{M}}, k_r, \boldsymbol{\xi}_s)$ the acoustic tensor of the proposed model dependent on both the generalized elastic parameters and on the considered (r, s) datum, δ_{ik} the Kronecker symbol.

5 Non-conventional experiments and integrated modelling

The design of novel experimental configurations and loading conditions, referred to as non-conventional tests, represents a cutting-edge topic, aiming at improving the calibration and validation stages of advanced constitutive models. One of the features of such non-conventional assessment is the choice of suitable loading conditions expected to activate deformation modes governed by selected parameters of an advanced mechanical model. In particular, when dealing with architectured materials behaving at the macroscale as generalized elastic media, a crucial point concerns the reliable implementation (or the measurement) of a suitable history of generalized external actions (such as surface, edge and wedge forces working on the placement or on its normal derivatives) and/or of their dual kinematic quantities, see e.g. [141,142]. The standard experiments for the mechanical characterization of material samples can be modified by considering generalized boundary conditions, to be prescribed or, in a dual fashion, monitored, see e.g. [143]. For instance, one can assign over the boundary of the sample not only the displacements but also their higher-order normal derivatives: from a dual perspective, exotic N-forces can be applied onto a higher order differential boundary of the sample. Also the sample geometry can be modified consistently: for instance, by increasing the thickness of a rectangular sheet sample closely to one edge, one could expect to reduce significantly the derivative of the displacement vector normally to it, as carried out in [135]. Also the micro and nano-indentation, which exhibits well known size effect, can be investigated at the light of high gradient elasticity, see [144].

5.1 Full-field kinematic measurements

In order to characterize the experimental response of architectured materials under static and dynamic loading conditions, besides the conventional point sensors full-field measurements are being utilized more and more frequently. In particular, Digital Image Correlation (DIC) techniques can process (sequences of) digital images monitoring the deformation of a material sample, detecting the displacement field by which the image texture is passively advected. Such displacements represent the solution of an inverse problem referred to as *motion estimation*, resting on the minimization of a suitable matching functional between pairs of digital images. By adopting a Galerkin, finite element discretization of the displacement field, the problem from infinite dimensional becomes finite-dimensional, having an unknowns the nodal displacements of the kinematic interpolation. It is worth emphasizing that the digital images are intrinsically multiscale, and one can easily switch from a fine to a coarse description (and vice versa) simply by averaging (or interpolating) sets of pixels (or voxels): an image pyramid can be generated convoluting the original pictures with a suitable smoothing kernel, and the analysis can be repeated recursively starting from the highest to the lowest level. Such a multiscale strategy plays a regularization role in the presence of a problem severely ill-posed, which admits numerous secondary minima at increasing the data noise. Secondly, when full-field kinematic data are available, one can focus the attention on a smaller subdomain, neglecting overall boundary conditions prescribed to the entire sample: hence, a local virtual tests can be specified, due to the availability of displacement histories over the border of that subdomain, see [145]. In particular, one can consider: (i) 2D DIC, processing a sequence of bidimensional pictures acquired by a single camera, and providing plane displacements over flat surfaces; (ii) 3D-Volume DIC, processing a sequence of three dimensional digital images provided e.g. by X-ray computed micro-tomography (i.e. tomographic reconstructions), providing three dimensional displacements within the bulk of an opaque sample, see [146]; (iii) 2.5 DIC (or stereo DIC), processing bidimensional pictures acquired by at least two synchronized cameras, and providing three dimensional displacement vector over a curved, opaque surface. When selecting a finite element approach, DIC measurements are expressed as nodal variables governing suitable shape functions, and can be easily compared with the output of a finite element mechanical simulation sharing the same discretization. Isogeometric approaches can be utilized for finite element based DIC, allowing for a higher order continuity of the displacement field, needed for the subsequent forward and inverse analyses, see e.g. [147–150]. Pictures can be acquired by a conventional or by a high-speed camera, possibly equipped with zoom objectives, being clear that for the optical monitoring the spatial resolution is however limited by the wavelength of the natural light. On the contrary, if digital images are provided by means of advanced scanning equipment with an "in-situ" apparatus, the resolution can be orders of magnitude higher, see [151–154]. Although these image-based approaches are not new, see e.g. [155], challenging issues are still being investigating, such as large displacements (feeding convergence by simulations), automatic segmentation and tracking of diverse features, and, most of all, novel strategies to integrate and regularize measurements with forward and inverse analyses. Both static and dynamic tests can be monitored by DIC, see e.g. [156]: however, space/time sampling must be effective to reduce as much as possible the storage of data without losing information and preventing convergence. In this respect, sensitivity analysis by advanced finite element models can be extremely beneficial, in order to localize data more rich of information. DIC-estimated displacements can be regularized by including as a penalty the a priori information relevant to a mechanical model, in practice projecting the solution onto the space spanned by the model equations (in turn depending on the unknown parameters), thus reducing also the spurious oscillations effected by noise.

5.1.1 Hierarchical approaches

On the basis of the above considerations, especially for static tests on architectured materials at the lower scales the state of the art suggests to synergistically combine diverse ingredients: an existing equipment for advanced scanning (SEM, TEM, AFM, X-ray microCT, etc.), coupled with an "in situ" miniaturized loading apparatus, provides conventional data (e.g. reaction force and prescribed displacement history) enriched by digital images; such images can be processed, mostly off-line, by DIC procedures to obtain displacements; the conventional data and the displacement fields thus detected can be used to feed the forward simulations, playing also a regularization role, and to estimate the unknown parameters through identification procedures. Therefore a sequence of inverse problems is exploited: one problem (inner or follower) rests on digital images, and provides as output displacement (and strain) fields; a second problem (outer or leader) uses the above DIC-detected displacements to obtain estimates of the material parameters by minimizing a suitable discrepancy norm and to drive the simulation as Dirichlet conditions along the boundary. A bilevel programming problem can be formulated,

$$\begin{aligned} &\hat{\mathbf{p}} = \arg\min_{\mathbf{p}\in\mathcal{K}} \|\mathbf{u}^{\text{exp}}_{\mathring{\Omega}} - \mathbf{u}^{\text{comp}}(\mathbf{p}, \mathbf{u}^{\text{exp}}_{\partial\Omega})\|^2 \\ &\text{subject to :} \\ &\mathbf{u}^{\text{exp}}_{\Omega} = \arg\min_{\mathbf{u}} \int_{\Omega} (f(\mathbf{x}) - g(\mathbf{x}+\mathbf{u}))^2 \, d\Omega, \end{aligned} \tag{72}$$

where Ω is the problem domain and the Region-Of-Interest (ROI), f and g denote the elemental pair of digital images acquired at different instants (the former referred to as undeformed or reference image, the latter as deformed one), $\mathbf{u}^{\text{exp}}_{\Omega}$ the displacement field minimizing the matching functional provided by DIC, partitioned in data belonging to the interior domain $\mathring{\Omega}$, entering the discrepancy norm as measurable quantities, and those along the boundary $\partial\Omega$, driving the forward operator as boundary conditions prescribed deterministically. The forward operator $\mathbf{u}^{\text{comp}}(\mathbf{p}, \mathbf{u}^{\text{exp}}_{\partial\Omega})$ is herein regarded as a function of two arguments, the parameters to identify, and the (Dirichlet) boundary conditions. In fact, it is also possible to generalize the formulation including the actual boundary conditions as unknowns, expected to be close but not coincident with those experimentally measured, affected by noise, with smoothing purposes, see e.g. [157,158]. It is worth noting that, when using X-ray microtomography (and 3D-Volume DIC), images f and g are a pair of three dimensional tomographic reconstructions, and hence in turn they represent the output of an inverse problem, based on a set of 2D radiographies. Hence, if useful for regularization purposes, a further inner level could be made explicit in the programming problem formulation Eq. (72), see e.g. [159]. At the macroscale (see e.g. [53,160] for the pantographic-type samples), such an integrated methodology turns out to be less complex: non-conventional tests under static or dynamic loading can be carried out by an ordinary mechanical testing machine, monitored by one or multiple cameras, and then processed for parameter identification [158]. Tests at micro and nano-scale under dynamic loading still represent a challenging issue.

5.2 Models for micro and nano-beams

The experimental information on structures at the lower scales, such as micro and nano-beams, wires or plates, exhibit a significant industrial interest: these elements play the role of sensors and actuators as well as of probe microscopy systems especially for MEMS and NEMS devices, and an extreme attention has been paid to understanding their mechanical behavior, see e.g. [14]. Deformation modes of micro or nano-beams, constituted not only of the traditional silicon-based materials but also of metals and polymers, are being extensively investigated using strain gradient theories and surface elasticity. For alternative nonlocal approaches, see e.g. [161–166].

In [167] it was stated that the percentage reduction of elastic stiffness in nanosized beams with respect to the Cauchy continuum predictions, namely $(D - D_c)/D_c$, scales as $\alpha\frac{S}{Eh}$, where S is the surface elastic modulus (to be clarified in the subsection 5.3.2), E is the bulk Young modulus, coefficient α depends on the specific geometry, h denotes a dimension typical of the structure in point. Assuming as characteristic length the ratio $h_0 = S/E$, the bulk modulus is dominating when $h \gg h_0$ and the size effect is negligible, whilst surface effects become predominant when $h \leq h_0$. The bending stiffness $D_c = EI_x$ deduced from Bernoulli-Navier assumption (with a Cauchy material) and governing the relation between moment M_x and curvature χ_x, must be corrected through entering a surface dependent elastic modulus S and a second order inertia H relevant to the border of the cross section, namely $D = (EI_x + SH)$.

Several attempts have been made to develop an advanced strain gradient beam model governed by only a few parameters apt to describe size effects. In [168] a Bernoulli-Euler micro-beam, based on a simplified version of the first strain gradient model proposed by Ru and Aifantis [169], has been used to carry out static and dynamic analyses. In [87] an alternative first strain gradient formulation has been proposed, governed by only three scale lengths besides the two Lamé moduli. The elastic energy density for the beam material has been postulated as follows

$$\mathcal{W} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} + p_i\,\gamma_i + m_{ij}^S\,\chi_{ij}^S + \tau_{ijk}^{(1)}\,\eta_{ijk}^{(1)}. \tag{73}$$

The above energy includes, as mutually independent deformation metrics, besides the components of linearized strain tensor $\varepsilon_{ij} = u_{(i,j)}$, the gradient of the volumetric deformation $\gamma_i = \partial_i\,\varepsilon_{kk}$ or dilatational gradient, the deviatoric stretch gradient tensor $\eta_{ijk}^{(1)}$ and the symmetric rotation gradient tensor χ_{ij}^S, defined as

$$\begin{aligned}
\chi_{ij}^S &= \frac{1}{4}\,\left(\epsilon_{ipq}\,\partial_p\varepsilon_{qj} + \epsilon_{jpq}\,\partial_p\varepsilon_{qi}\right),\\
\eta_{ijk}^{\mathrm{S}} &= \frac{1}{3}\,\left(u_{i,\,jk} + u_{k,\,ij} + u_{j,\,ki}\right),\\
\eta_{ijk}^{(1)} &= \eta_{ijk}^{\mathrm{S}} - \frac{1}{5}\,\left(\delta_{ij}\,\eta_{mmk}^{\mathrm{S}} + \delta_{ki}\,\eta_{mmj}^{\mathrm{S}} + \delta_{jk}\,\eta_{mmi}^{\mathrm{S}}\right),
\end{aligned} \tag{74}$$

being as usual δ_{ij} and ϵ_{ijk} the Kronecker and Levi-Civita symbols. In a redundant way superscript $(\cdot)^S$ emphasizes the symmetrization of the tensor in argument. In particular, it is worth noting that: η_{ijk}^{S} is totally symmetric with respect to any permutation of its indices; $\eta_{ijk}^{(1)}$ is a traceless tensor, since all its trace vectors vanish, having $\delta_{ij}\,\eta_{ijk}^{(1)} = \delta_{jk}\,\eta_{ijk}^{(1)} = \delta_{ki}\,\eta_{ijk}^{(1)} = 0$; χ_{ij}^S is the symmetrization of the curl of the strain, equal in turn to the gradient (transpose) of the rotation vector, which is the axial vector of the small rotation tensor, namely $\vartheta_i = 1/2\,\epsilon_{ijk}\,u_{k,j}$. Besides the isotropically elastic relationship between the linearized strain and the Cauchy stress, governed by the Lamé moduli λ and μ, the generalized dual variables are defined as follows

$$p_i = 2\,\mu\,l_0^2\,\gamma_i,\quad \tau_{ijk}^{(1)} = 2\,\mu\,l_1^2\,\eta_{ijk}^{(1)},\quad m_{ij}^S = 2\,\mu\,l_2^2\,\chi_{ij}^S, \tag{75}$$

where symbols l_0, l_1 and l_2 denote the three characteristic lengths. The elastic energy density is assumed to be independent of the anti-symmetric rotation gradient tensor, see also [170], whilst the kinetic energy density is defined conventionally as $\mathcal{T} = 1/2\,\rho\,\dot{w}^2$. The Bernoulli-Navier kinematics rests on the well known relationships $u_x = -z\,w^{(\mathrm{I})}$, $u_y = w(x)$, $u_z = 0$, being the beam oriented as the x-axis and $w(I)$ the first derivative of the transversal displacement. By prescribing the stationarity of the Hamilton action functional, the following differential equations is deduced for the unidimensional beam, to hold $\forall(x,t) \in [0, L] \times [t_1, t_2]$,

$$S\,w^{(\mathrm{IV})} - K\,w^{(\mathrm{VI})} + \rho A\ddot{w} + q = 0, \tag{76}$$

where $w(x)$ is the beam deflection, $w^{(\mathrm{K})}$ (with K Roman numeral) denotes its K-th order spatial derivative with respect to the coordinate x, and $\ddot{w}$ its second order time derivative. Furthermore, ρ and A indicate the material density and the area of the cross section, q the distributed transversal loading, symbols S and K denote generalized structural stiffnesses, dependent on the cross section geometry, on the elastic moduli and on the characteristic lengths. The sixth-order ordinary differential equation (76) is endowed by the boundary conditions which must hold at the beam ends $\forall t \in [t_1, t_2]$, namely the essential conditions involving the deflection and its derivative up to the second order, i.e. $\delta w(0) = 0$, $\delta w^{(\mathrm{I})}(0) = 0$ and $\delta w^{(\mathrm{II})}(0) = 0$ (and their analogs at $x = L$), or the natural ones including the generalized actions at $x = 0$, namely

$$\begin{aligned}
&\Big\{V(0) - \Big[K\,w^{(\mathrm{V})}(0) - S\,w^{(\mathrm{III})}(0)\Big]\Big\} = 0,\\
&\Big\{M(0) - \Big[-K\,w^{(\mathrm{II})}(0) + S\,w^{(\mathrm{III})}(0)\Big]\Big\} = 0,\\
&\Big\{M^h(0) - K\,w^{(\mathrm{III})}(0)\Big\} = 0,
\end{aligned} \tag{77}$$

and their analogs at $x = L$. Symbols V and M denote as usual the conventional shear action and the bending moment, whilst M^h indicates a higher order moment. In addition, the condition $\rho A \dot{w}\, \delta w \Big|_{t_1}^{t_2} = 0$ must hold $\forall x \in [0, L]$.

In [81] an analytical formulation has been provided to describe the bending of a Bernoulli-Euler nano-beam in the framework of the Mindlin second strain gradient theory. After some manipulations, the stored energy of the beam assumes the form

$$\mathcal{E} = \frac{1}{2} \int_0^L \left\{ I_1\, w^{(\mathrm{II})\,2} + I_2\, w^{(\mathrm{III})\,2} + I_3\, w^{(\mathrm{IV})\,2} + I_4\, w^{(\mathrm{II})} w^{(\mathrm{IV})} \right\} dx, \tag{78}$$

where coefficients I_i $(i = 1, 4)$ are linear combinations of the coefficient of Mindlin second gradient model, see Eq. (21). Exclusively the distributed transversal loading q has been considered by the authors as external action, besides the concentrated shear forces and moments acting at the beam ends. By a variational approach the governing equation is deduced,

$$I_1\, w^{(\mathrm{III})} + (I_4 - I_2)\; w^{(\mathrm{VI})} + I_3\, w^{(\mathrm{VIII})} - q = 0, \tag{79}$$

endowed by the boundary conditions which must hold at the beam ends $\forall t \in [t_1, t_2]$, namely the essential conditions involving the deflection and its derivative up to the third order, i.e. $\delta w(0) = 0$, $\delta w^{(\mathrm{I})}(0) = 0$, $\delta w^{(\mathrm{II})}(0) = 0$ and $\delta w^{(\mathrm{III})}(0) = 0$ (and their analogs at $x = L$), or the natural ones including the generalized actions at $x = 0$ and $x = L$. For brevity we report exclusively the condition dual of $\delta w(0)$,

$$-\, I_1 w^{(\mathrm{III})}(0) - \left[\frac{1}{2} I_4 - I_2 \right] w^{(\mathrm{V})}(0) - I_3 w^{(\mathrm{VII})}(0) - V(0) = 0, \tag{80}$$

being V the shear action. It is worth noting that the authors have not considered concentrated generalized actions at the beam ends besides M and V, which in principle would be consistent with the postulated energy including irreducible terms working on $w^{(\mathrm{III})}$ and $w^{(\mathrm{IV})}$.

A Timoshenko beam model has been developed in [171]. As usual, the beam kinematics is governed by the variables $\psi(x)$, $w(x)$ and $\beta(x) = w(x)' - \psi$, which denote the bending rotation, the beam deflection and the shear slip, respectively. The kinetic energy density is expressed as superposition of two contributions related to the translation and rotation of the cross section, namely $\mathcal{T} = 1/2\, \rho\, A\, \dot{w}^2 + 1/2\, \rho\, I\, \dot{\psi}^2$. By selecting the same first gradient model for the beam material specified in Eqs. (73)–(75), through the Hamilton's principle the following system of differential equations is deduced, $\forall (x, t) \in [0, L] \times [t_1, t_2]$

$$\begin{aligned} &(k_3 + k_4)\, w^{(\mathrm{IV})} + (k_3 - 2k_4)\, \psi^{(\mathrm{III})} - k_5\, (w^{(\mathrm{II})} - \phi^{(\mathrm{I})}) + \rho A \ddot{w} - q = 0 \\ &\rho I \psi^{(\mathrm{II})} - k_3\, w^{(\mathrm{IV})} - (k_3 - 2k_4)\, w^{(\mathrm{III})} + (k_2 + k_3 + 4k_4) \psi^{(\mathrm{II})} - k_5 (w^{(\mathrm{I})} - \psi) = 0. \end{aligned} \tag{81}$$

The coefficients k_i, entering the system equations and herein not reported for brevity, depend on the Lamé modulus μ, on the characteristic lengths and on the geometry of the beam cross section. The boundary conditions over the space-time domain are: $\rho A \dot{w} \delta w \Big|_{t_1}^{t_2} = 0$ and $\rho I \dot{\psi} \delta \psi \Big|_{t_1}^{t_2} = 0$, which must hold $\forall x \in [0, L]$, whilst $\forall t \in [t_1, t_2]$ one has as essential conditions $\delta w(0) = 0$, $\delta w^{(\mathrm{I})}(0) = 0$, $\delta \psi(0) = 0$, $\delta \psi^{(\mathrm{I})}(0) = 0$, and the analogs at $x = L$, or the natural ones including the generalized actions at $x = 0$ and $x = L$, of which for brevity we report exclusively the one dual of $\delta w(0)$,

$$V + (k_3 + k_4)\, w^{(\mathrm{III})} + (k_3 - 2k_4)\, \psi^{(\mathrm{II})} - k_5\, (w^{(\mathrm{I})} - \psi) = 0, \tag{82}$$

being V the shear action. Static bending and free vibration of a simply supported beam have been investigated by the present model, predicting size effect at decreasing the characteristic lengths in terms of increased rigidity and natural frequencies.

On the basis of the literature contributions outlined above, it appears clear that the implementation of higher grade elasticity models in conventional beams naturally leads to governing equations of higher order as for the spatial derivatives, with respect to conventional linear elastic behaviour: when adopting the Euler-Bernoulli assumptions, equations up to the eighth order are deduced, whilst in the case of Timoshenko kinematics a system of two coupled linear differential equations is found, each one not exceeding the fourth order. The higher order of the differential equations is consistent with an enriched variety of admissible mechanical behaviours, non predictable before by the conventional elastica, although causality issues are often neglected

[172]. Natural boundary conditions include non trivial differential expressions which lead to define generalized actions, not always made explicit by the authors. Suitable formulations allow one to consider, besides the Lamé moduli, a limited number of parameters or characteristic lengths, apt to introduce size dependence. Similar enriched models have been developed for nanoplates, see e.g. [173,174].

5.3 Experimental assessment at lower scales

The experimental tests carried out at the micro and nanoscale exhibit a huge complexity and high levels of uncertainty: they usually imply to face inverse problems, since most of the quantities of interest are not directly observable, and, due to numerous sources of error, cross correlation of several data is often carried out making recourse to mathematical models.

An emblematic case is the estimation of the effective elastic modulus of nanostructures, which indeed represents a bottleneck for the predictability and reproducibility of their performance. Methods utilized so far suffer of severe inaccuracies, as discussed e.g. in [175]. For instance, bending tests on nanobelts have been carried out by an Atomic Force Microscope (AFM), which has adequate force and displacement resolution, but lacks of in situ real-time imaging capabilities: it is difficult to accurately detect the contact area and the actual distance of the force from the clamping region of the beam. Moreover, the methodology based on the bending test works effectively for intermediate values of elastic modulus, but it is not suitable for beams very flexible or extremely stiff. Also the AFM-based nanoindentation technique is sensitive to substrate effects for small diameter nanowires. The assessment of resonant frequencies of a nanobeam under bending or in extension mode exhibits poor repeatability due to the mass changes caused by surface contamination, the presence of oxide, thin films or absorbed particles: the extra mass causes a decrease of the natural frequencies, leading to underestimate the elastic modulus. Moreover, beam models so far utilized to fit data were borrowed from macroscale applications and turned out to be poor.

5.3.1 Pull-in instability

To avoid some of the aforementioned issues, a strategy based on electrostatic pull-in instability has been proposed for the estimation of the apparent Young's modulus. The advantages of the pull-in mechanisms rest on its sharp behaviour, at increasing the loading distributed along the length of the beam. This strategy has met the emerging need for MEMS process nondestructive monitoring and material property measurement at the wafer level, during both development and manufacturing. When a voltage is applied between the substrate and the beam (for instance in the cantilever configuration), the electrostatic pressure deflects the cantilever. By increasing the applied voltage beyond a critical value, called pull-in voltage, a stable equilibrium position of the cantilever ceases to exist and the cantilever snaps, deflecting downward, toward the underlying fixed ground plane. The pull-in behavior depends on the electrostatic attraction generated by the applied voltage, on the elasticity of the beam and on its configuration. For a doubly clamped configuration, the nonlinear ordinary differential equation governing the pull-in instability is expressed as (see [176])

$$\tilde{E}\, I\, w^{(\mathrm{IV})} \simeq -\alpha \frac{\epsilon_0\, b\, V^2}{2\, w^2} \left(1 + 0.65\, \frac{w}{b}\right), \tag{83}$$

where $w = w(x)$ denotes the vertical displacement of the beam aligned with the horizontal axis, which describes also the distance between the current position of each point and the substrate, $I = \frac{bt^3}{12}$ denotes the cross section inertia, being t the beam thickness and b the out of plane dimension, $\tilde{E}$ indicate the apparent Young's modulus, ϵ_0 is the permittivity of vacuum, V indicates the applied voltage, α a coefficient dependent on the geometry. It is worth noting that the quantities entering Eq. (83) do not depend on the mass of the beam or on its density. Besides the doubly clamped (or fixed-fixed) configuration, this methodology is being mainly applied to cantilever beams and to clamped circular diaphragms. When implemented as a MEMS test chip or drop-in pattern, this method is referred to as "M-Test". In the industrial practice, the robust estimation of the pull-in voltage is based upon two intermediate parameters, products of material properties and geometric coefficients, referred to as $S = \tilde{\sigma} t g_0$ and $B = \tilde{E} t^3 g_0^3$, being $\tilde{\sigma}$ the effective residual stress, g_0 the initial gap between the undeformed beam and the substrate.

5.3.2 Surface and bulk properties

During the last two decades, the accurate assessment of the elastic size effect for micro and nano-structures has required a significant effort to improve both the experimental techniques and the mechanical modelling. For instance, in [177] the bending behavior of micro-beams of single crystalline silicon and silicon nitride has been monitored by combining two diverse and independent devices: AFM has been used to measure the deflection, whilst local strains have been detected by Raman effect. To provide the reader with an order of magnitude, in this experimental campaign beams had a thickness ranging from a few micrometers to 200 nanometers, with a thickness to length ratio of less than 1/20, loaded by forces ranging from 0.1 to 200 μN: observed deflections were in the order of 0.05 up to 35 μm. To interpret experimental data, the authors made use of an Euler-Bernoulli beam, with a couple stress material model, governed by a unique characteristic length besides the isotropic elastic moduli, see e.g. [178].

Another testing strategy is based on the analysis of the dynamic response of a nanowire (NW) cantilever when subjected to an alternating electrostatic field of prescribed frequency inside a Transmission Electron Microscope (TEM). Since the electric charge oscillates at the frequency of the applied voltage, mechanical resonance is induced when the applied frequency matches the natural vibration frequency, from which the Young's modulus of the nanowire is calculated, see e.g. [179]. However, the results available in the literature exhibit a large scatter. In [180] an experimental-computational strategy has been proposed to accurately quantify size effects on the Young's modulus of ZnO (zinc oxide) nanowires. A MEMS-based miniaturized testing system was integrated within a TEM to carry out uniaxial tensile tests, i.e. *in situ*. In the MEMS-based system, load was applied using a thermal actuator on one side of the testing stage, and the displacement was measured using a differential capacitive sensor on the other side. By this approach the Young's modulus of oriented ZnO nanowires has been assessed as a function of the wire diameter. When the diameter passed from 80 to 20 nm, an increase of the Young's modulus from 140 to 160 GPa was found: for wires larger than 100 nm, the values of Young's modulus remained stably around 140 [GPa], to be considered hence as the bulk value. Molecular dynamics simulations for nanowires with diameters between 5 and 20 nm have been carried out to complement the experimental findings. This numerical-experimental approach has provided a fundamental understanding of the observed elasticity size effect, resolving an existing controversy.

In [87] the presence of a stiff layer closely to the surface of a nanobeam has been investigated, in order to quantity possible effects on the apparent Young modulus. A simple scheme was suggested for a rectangular cross section with overall height h: let E_1 denote the Young's modulus of the bulk, and E_2 the surface modulus of the material belonging to the upper and lower layers, each one having a thickness t (layers closely the vertical sides are neglected). By estimating a uniform apparent modulus E under tensile loading, the percentage error with respect to the actual bulk value E_1 can be roughly quantified as

$$e_t = \frac{E - E_1}{E_1} = \frac{2t}{h}\left(\frac{E_2}{E_1} - 1\right). \tag{84}$$

An analogous formula can be deduced for the bending stiffness,

$$e_b = \frac{D - D_1}{D_1} = \frac{E_2 - E_1}{E_1}\left(1 - \left[1 - \frac{2t}{h}\right]^3\right), \tag{85}$$

where D is the bending stiffness resulting from a uniform distribution of the apparent Young's modulus (as the surface layers were negligible). At varying e_t, e_b did not exceed 40% when $2t/h \to 0$. Since the size effect observed for a beam under tension (with almost no strain gradients) resulted to be significantly lower with respect to that observed under bending (when strain gradients are present), the authors have stated that the doubling of the normalized stiffness does not depend mainly on the presence of stiffer outer layers closely to the surface, but should be attributed to strain gradient effects. As a matter of fact, the authors completely neglected the presence of a surface stress within the outer layers, and considered exclusively surface elasticity (with $E_2 > E_1$) as it were necessarily positive. An experimental campaign was carried out on epoxy microbeams with thickness ranging from 20 to 200 micrometers and lengths from 10 to 50 times the thickness, subjected to cantilever bending by a nanoloading apparatus: a beam model, resting on the first gradient formulation outlined in Eqs. (73)–(75), has shown a satisfactory agreement with these data.

In [181] similar deviations in stiffness of polypropylene micro-cantilevers have been observed under bending. Polypropylene micro-cantilevers, fabricated via injection molding resulting in a non-homogeneous microstructure due to their semi-crystalline nature, had the length and the width of a few hundreds micrometers, the thickness being approximately 20 micrometers. A nanoindenter was used to bend the samples, measuring the corresponding deflection. The authors have proposed a micropolar elastic model governed by only one additional material parameter (besides the isotropic elastic moduli) to provide consistent predictions of the size effects. The stiffness values, estimated by this micropolar model on the basis of experimental data, resulted at least four times higher than the values provided by the conventional beam theory.

The difference of mechanical response between the bulk and the boundary wall of such nano-objects has led to the so called core-shell model, according to which different properties are attributed to these parts, as outlined in [182]. Within a surface layer, the total stress tensor, herein with dimensions of a force per unit length, can be regarded as the superposition of two contributions, namely

$$\tau_{\alpha\beta} = \sigma_{\alpha\beta} + \mathbb{S}_{\alpha\beta\gamma\eta}\, \varepsilon_{\gamma\eta} \qquad (\alpha, \beta, \gamma, \eta = 1, 2), \tag{86}$$

the former addend at r.h.s being referred to as surface stress (or residual surface stress), independent of the strain level, the latter as surface elasticity. At each point of a surface, which is a bidimensional manifold, one should ideally consider 2D surface strain and stress tensors, being $\mathbb{S}$ a 2D stiffness tensor with a consistent four order rank: for an isotropic surface, however, these tensors are spherical (i.e. $\sigma_{\alpha\beta} = \varsigma\, \delta_{\alpha\beta}$), and a scalar representation is usually met. It is worth emphasizing that both the surface stress and the surface modulus can be either positive or negative. The estimation of these surface properties in micro or nano-beam on the basis of their response to selected loading conditions represents a further inverse problem. Static tests such as bending or nanoindentation are certainly able to detect a variation of the mechanical properties, but according to the state of the art they cannot distinguish the contribution of surface stress from that of surface elasticity (see also [183]). Both the surface elasticity and the surface stress, originated by physical mechanisms referred to as surface relaxation and reconstruction [184,185], can result in changes of the resonant frequencies of a micro/nano-structure. In the dynamics of such nanostructures, a shift of exclusively one resonant frequency can be generated by infinite combinations of surface elasticity and surface stress. However, as observed in [182], if the shifts of two or more resonant frequencies are considered simultaneously, both surface elasticity and surface stress can be retrieved uniquely. To interpret these results, two beam models have been critically and comparatively considered, both based on the Euler-Bernoulli kinematics. The so called concentrated force model considers a beam with bending stiffness $\tilde{D}$, at the free end of which a fictitious concentrated loading $F = \sigma A$ is applied (violating the boundary conditions), with equation

$$\tilde{D}\, w^{(\mathrm{IV})} - \sigma A\, w^{(\mathrm{II})} + m\, \ddot{w} = 0. \tag{87}$$

On the contrary, the distributed loading model includes a distributed load tangential to the beam, equal to $\frac{\sigma A}{L}$, and is governed by the equation

$$\tilde{D}\, w^{(\mathrm{IV})} - \frac{\sigma A}{L}(L - x)\, w^{(\mathrm{II})} + \frac{\sigma A}{L}\, w^{(\mathrm{I})} + m\, \ddot{w} = 0, \tag{88}$$

endowed by four boundary conditions. The bending stiffness $\tilde{D}$ is expressed as the sum of a conventional term, $\tilde{E} I$ (being $\tilde{E}$ the effective Young's modulus), and of another contribution depending on the surface stiffness and on the geometry of the cross section, see [186]. On the basis of the experimental results, the surface stress turned out to be highly sensitive to the Poisson's ratio in cantilever plates, in contrast to doubly-clamped beams. Furthermore, the beam response in the doubly-clamped configuration resulted to be strongly sensitive to the surface stress in comparison to cantilever plates. This superior performance increases with the aspect ratio of the beam, namely with the length divided by the width. The local curvature near the clamped end seemed to be responsible of these different behaviour.

5.3.3 Ultrasonic and spectroscopy techniques

As discussed in [187,188], several measurement techniques have been developed based on vibrational and ultrasonic processes of matter which involve only reversible strains, hence resulting able to detect in an indirect way the elastic properties of a sample. Until truly supramolecular scales are reached, the elastic continuum paradigm remains appropriate for the description and the analysis of ultrasonic regimes: at a lower size the same concept of acoustic excitations as well as that of strain field begins to lose significance. Some of these techniques are based on optical excitation and/or on the detection of ultrasonic waves, and operate either in the time domain or in the frequency domain. To the present purposes it is worth emphasizing that most of these techniques exploit inverse formulations, although in the engineering practice drastic simplifying assumptions (e.g. that of isotropic behavior) allow one to rapidly achieve straightforward solutions with an approximation more than satisfactory. Among the techniques based on vibrational excitations, those using optical methods can be effectively applied to small objects, for which the mechanical contact with actuators or sensors becomes critical. Most of these techniques rely on the interaction between optical electromagnetic fields and mechanical acoustic fields, like the acoustic phonons generated by a laser pulse in a crystalline structure: such an acousto-optic (or photoelastic) coupling mechanism has being applied since a long time for the characterization of bulk samples and of supported films (see also [18]). If the outer surface of the sample is planar, a laser beam focused on a spot, sized up to few tens micrometers, generates in a few instants (typically the pulse has a pico or femtosecond duration) strains and a temperature field almost uniform: after that, a plane wave is launched in the direction perpendicular to the surface. The arrival of the echo(es) at the surface is thus detected, the delay providing a measure of the velocity of the acoustic waves for propagating through the medium in the direction normal to the surface. Such a velocity provides information about the elastic properties of the film, see [189]. The shift in the frequency of the peaks related to the longitudinal and transversal deformation modes provides the corresponding velocities v_L and v_T, through the relationships $v_L = \Delta f_L \lambda/(2\,n\, \sin[\theta/2])$ and $v_T = \Delta f_T \lambda/(2\,n\, \sin[\theta/2])$, being Δf_L and Δf_T the frequency shifts in the longitudinal and shear modes, n the refractive index of the medium, λ the wavelength of the laser light, θ the angle of the backscattering configuration. For a Cauchy material, such velocities in turn are related to the elastic stiffness of the matter usually assumed to be isotropic, namely $C_{1111} = \rho\, v_L^2$ and $C_{1212} = \rho\, v_T^2$. For a higher order material, the eigenvalue problem Eq. (65) must be considered. Of course, the characterization of the elastic parameters by ultrasonics waves can be correlated to the results of other mechanical tests, such as for instance nano-indentation, see e.g. [190].

6 Nonparametric synthesis problems

In the recent literature the expression *synthesis* of meta-materials, also referred to as *inverse design*, usually indicates the design process of a periodic microstructure whose homogenized response (at the macroscale) fits a prescribed, target performance. Macroscopic (apparent or effective) properties, stemming from the selected micro-architecture, turn out to be not trivial and usually not achievable by common materials, from the quantitative or qualitative standpoint: this circumstance motivates the enormous industrial interest in this topic, see e.g. [191]. The synthesis of meta-materials represents a highly complex inverse problem, for which a general strategy still does not exists and multidisciplinary, heuristic approaches are often utilized: the unknown is the geometry (or the architecture), namely a continuous distribution of matter for one or more phases within a representative volume element, and the relevant elastic properties. In other words, in the synthesis problem we intend to identify the inverse of the map from the microstructure to a target property at the macroscale: in particular, herein the focus is on architectured continua behaving as higher grade materials at the macroscale, being usually the microscopic phases of Cauchy type. The depictive expression *inverse homogenization* is seldom used in the literature: in fact, the inversion of the homogenization operator is attempted, regarded as a function also of the shape and the topology of the microstructure and not only of the phase mechanical properties with the relevant stress/strain fields [192,193]. By symbols, following the expression used for the *forward* homogenization Eq. (27), one can write

$$\left[\, C^{\mathrm{m}\,(a)}_{ijkl},\ \text{geometry} \,\right] = \mathcal{H}^{-1}\left(\mathbf{p}^{\mathrm{M}}\right). \tag{89}$$

Unfortunately, one of the main features of this inverse problem is the non uniqueness of its solutions. Therefore, Eq. (89) must be formulated in weak sense as a nonlinear programming problem in which the homogenized stiffness tensors, depending on the unknown microstructure, are the closest possible to their counterparts assumed as a target (in a suitable matrix norm). Moreover, the minimizing sequence provided by the numerical procedures is expected to strongly depend on the initialization and on the underlying parametrization. This also means that, if uniqueness is required, additional conditions should be added as constraints. In Eq. (89) the presence as unknowns of both the microscopic stiffness of the phase (a) and its topology, under the label geometry, is particularly suggestive, since these information are in a sense correlated: from the mechanical standpoint, a prescribed geometry, for instance the presence of a hole in the unit cell, can be approximated by a sound material with a vanishing stiffness occupying the same region, and vice versa.

The enormous advances in additive manufacturing and 3D printing, using a variety of compounds, is offering an extraordinary context for the rapid and low cost realization of prototypes directly from CAD design and for their optimization, see [194]. This circumstance, endowed by the progress in computational methods such as machine learning and multi-objective topology optimization, are expected to favor the development of novel metamaterials, see [195–198]. Moreover, there is a general trend to move from the tailoring of single properties toward the realization of multiple functionalities, where the distinction between materials and devices is increasingly blurred. In this Section we will emphasize some issues which are strictly related with the synthesis of novel metamaterials, namely: the choice of an adequate parametrization or data pattern analysis (in the jargon of machine learning); the search for regularization provisions to drastically reduce the high dimensionality of the design problem, e.g. by investigating discrete lattice models and continualization methods; the implementation of a physically based recursivity in design, exploiting the principle of virtual work and passing through the realization of prototypes and their experimental characterization.

In [199] a novel structural optimization method has been proposed, based on periodic surface modeling, to simultaneously optimize the shape and the topology of a meta-material, drastically reducing the dimension of the design space. Such an approach was originally utilized to design porous structures, easily modeled as implicit surfaces or introducing distance functions. The structures generated by the implicit surface models have smooth surface boundaries, but they are rather difficult to obtain through topology optimization methods resting on a finite element discretization. With this approach, the shape and topology optimization of the micro-structures can be formulated as a unique mixed-integer problem, restricting the number of admissible choices to only a discrete number. In [200] an inverse design framework has been introduced for the precise tailoring of nonlinear mechanical responses in periodic microstructures. Values of a nonlinear stress-strain relationship under finite strain are prescribed as a target at the macroscale, and the microscopic material distribution inside the representative volume element is sought by minimizing the discrepancy between the homogenized stresses assumed as a target and those computed by a displacement-driven finite element model, function of a continuous material density field. The representative volume element, with periodic boundary conditions, is however constituted, besides the voids, of a unique phase with uniform properties assumed a priori.

More recently, data-driven machine learning strategies are being used to foster the design process, see e.g. [201]. The core idea underlying data-driven design is to extract meaningful patterns from data that are difficult to manage since high dimensional, or for which a physical model is not yet available, and to incorporate them into the design process. Data-driven design consists of three main stages, namely data acquisition, unit cell level learning and optimization by machine learning, multiscale system design. A digital database, freely accessible to the engineering and academic community, would certainly foster the progress in this field. In [202] a methodology inspired from diffusion models for video denoising is proposed, to generate by machine learning geometries of representative elements behaving according to target mechanical nonlinear behaviour. Video diffusion generative models, trained on full-field data of periodic cellular structures obtained through a stochastic process, turn out to successfully predict their homogenized nonlinear deformation and stress response under compression in the large strain regime. This proposal avoids the common strategy of directly learning a map which links a given microstructure to a specific property: the system is trained to mimic full-field stress and strain distribution, provided by finite element simulations on the unit cell.

It is worth emphasizing that the machine learning techniques (in their diverse typologies) cannot be conceived as a kind of magical box, carrying out autonomously the whole task to seek for the optimal micro-architecture. Of course, algorithms and codes for topology and shape optimization represent an important help for this class of inverse problems. However, their solutions remain generally confined within a specific family of microstructures, on which the algorithm has been trained: the algorithm can operate small perturbations around a reference architecture, starting from a suitable initialization. The actual problem is how to pass from one family of architectures to another one completely different, and in this respect the question recalls the original distinction between parameter and non parameter identification (with ideally infinite dimensions), and that of local minima in optimization. In any case, the engineering investigation can be guided by a theoretical conjecture or Ansatz (in the spirit of De Saint-Venant and of Rayleigh-Ritz), to be possibly considered as a priori Bayesian information. Besides the general strategies suitable for a wide range of metamaterials, in the present context microstructures are sought for which the homogenized medium behaves as a higher grade material.

Only recently researchers have realized that the homogenization of composites made of very highly contrasted materials could lead to exotic effective behaviors. Although the single phases obey to the Cauchy postulates, the mechanical response of such composites at the macroscale may not enter the framework of Cauchy continuum, since the internal mechanical interactions are no more exhaustively described by a Cauchy double rank stress tensor, and higher order hyper-stress tensors are required. Some important theoretical results have corroborated this statement and opened the way to new achievements. In [203] the topological closure of the set of elasticity functionals was investigated. Any quadratic functional, sufficiently regular, can be approximated by a sequence of the linear elastic energies of suitable composite materials: in particular. those with a high contrast among the constituents provide the most exotic behaviour. Energies depending on the second gradient of the displacement or even nonlocal energies can be obtained. However, these rigorous mathematical proofs, based on the notion of weak convergence of functionals (such as Mosco or Γ-convergence) do not provide any constructive result. In [56] a rigorous proof was provided of the fact that the homogenized response of a microscopic lattice, strongly inhomogeneous, converges to the behaviour of a higher gradient material. For this typology of periodic systems, when the size of the representative volume element tends to zero some of their physical properties are diverging, while some others are vanishing in this limit. In [32,204] discrete lattices constituted of welded bars, with axial, flexural and torsional stiffness, have been utilized to generate homogenized generalized materials. For these lattice structures, closed form expressions of the homogenized energy is obtained by straightforward matrix algebra. It is worth emphasizing that, contrarily of what one could expect, the flexural stiffness is not responsible of the second gradient contribution: on the contrary, main contributions are due to extensional stiffness and to the peculiar topology of the unit cell. The preliminary choice of confining the synthesis to discrete systems constituted exclusively of 1D truss and beams, variously constrained, namely to a discrete lattice surrounded by voids, certainly represents an *a priori* regularization for the problem at hand. In this case, also the constraints and their location as well as the range of the interactions become a part of the problem solution. Analogous considerations hold for discrete systems constituted of 2D bricks such as plates and shells.

For truss structures of pantographic types, namely constituted of systems of trusses connected by ideal hinges referred to as pivots, in [30] Γ-convergence has been proved to higher grade materials. Pantographic structures have become the reference microstructure for the study of second gradient materials in statics and dynamics [138,205], both in the form of sheets and beams, see e.g. [114,206–211], from the theoretical [212], numerical and experimental standpoints [213–215]. These studies have been extended to woven fabrics made of two families of inextensible fibres modelled at the macroscale by a second gradient model, straight [216–220] or curvilinear [221,222], and also to hyperbolic paraboloidal (hypar) shells constituted of elastic nets of flexible fibers [223].

Besides the pantographs, it is legitimate to ask how general can be the homogenized response of discrete lattices, and under which typologies of microscopic interactions an assigned target behaviour at the macroscale can be retrieved. In [224] the dynamic response of discrete lattices with different typologies of interactions has been investigated. Firstly, a simple neighbor interaction has been considered, with distributed springs at the macroscale, connecting concentrate masses at the macroscale as well as lumped masses at the microscale. Microstructural effects on stiffness have been taken into account by keeping the quadratic terms of Taylor expansions in the continualization process. Inertia microstructural effects have been implemented with the use of springs that possess a uniformly distributed mass. Secondly, a two neighbour interaction of long range kind has been included. The homogenized response of the two schemes naturally leads to Mindlin first and second strain gradient models, respectively.

In [225] conditions have been investigated under which an exact correspondence exists between gradient elasticity equations for a continuum, and finite differences equations for a discrete lattice with nearest-neighbor and next nearest neighbor interactions. Generally, standard finite differences equations does not represent exactly the discrete analog of partial derivatives, since the solution of a finite difference equation does not coincide with the solution of a partial differential equation (on a discrete grid) and the standard finite differences do not satisfy the same algebraic relations as the operators of continuous differentiation. Therefore, the author has proposed a novel difference operator, named T-operator, permitting to achieve such a coincidence in exact sense, namely without the subjective or a priori choice of neglecting terms with an order higher than k. Moreover he noticed that such T-operators implement long range interactions of power type which are well known in mathematical physics and in practice correspond to interatomic interactions. The correspondence between the discrete (lattice) theory and the continuum theory lies mainly in the fact that the operations admissible for these theories should obey the same laws. The standard discrete (lattice) models do not always reflect the physical reality: they will more adequately describe elastic media, when these models will include more accurately long-range forces that represent electromagnetic interactions of a long-range nature, for instance those decreasing as $1/n^{\alpha}$. In [226] the same author has highlighted the enormous capabilities of the lattice models also in view of implementing nonlocal or fractional constitutive relationships.

In [227,228] a general approach to the synthesis of metamaterials has been outlined, articulated into the following, orientative steps. (a) Firstly, the target behaviour at the macroscale, selected *a priori*, must be clearly identified, and suitably modelled from the mathematical standpoint by Lagrangians and, if needed, by dissipative potentials. (b) Secondly, the microstructural model can be linked to the macroscale by the principle of virtual work. (c) To this purpose a hierarchical approach must be exploited: in the most general scenario we expect that a meta-material includes modular architectures at different length scales (see e.g. [229]). The selected microstructure must be accurately modelled to predict, for instance, the stress and strain states and possible irreversible phenomena due to the production process. These stages must be repeated several times, to implement the necessary improvements. (d) The synthesis scheme, obtained as above and available in digital form, must be transformed into a prototype, for instance through 3D printing techniques. (e) The experimental characterization of a prototype, for instance by X-ray computed microtomography, is extremely important since it can reveal *a posteriori* possible discrepancies, as for the geometry and the presence of voids, cracks or inhomogeneities, of the final product with respect to its digital model, depending on the selected printing technology. Similarly, other tests can provide information on the final mechanical properties. Iterative cycles of prediction-synthesis-characterization, points (a)–(e), provide a feedback to improve the predictions and favour the discovery of novel materials, giving rise simultaneously to the design of new target functionalities, see [230]. In particular, CAD models, suitably parametrized, see e.g. [231], can be imported into finite element codes, providing sensitivities with respect to the design variables and allowing an automatic optimization of the topology [232]. For instance, in [233] the relaxed micromorphic model, endowed by a well posed system of boundary conditions, has provided important information to design new acoustic metamaterials made up of a periodic arrangement of unit cells. The authors presented a metastructure combining elemental bricks of different type/shape and bricks of classical Cauchy materials in such a way that most of the elastic energy, for instance generated by a source of vibrations, is concentrated in specific regions, thus permitting substitution of a few components and the possible re-use in case of deterioration or damage.

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