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Vacuum Polarization with Zero-Range Potentials on a Hyperplane

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Abstract: The quantum vacuum fluctuations of a neutral scalar field induced by background zero-range potentials concentrated on a flat hyperplane of co-dimension 1 in $(d + 1)$ -dimensional Minkowski spacetime are investigated. Perfectly reflecting and semitransparent surfaces are both taken into account, making reference to the most general local, homogeneous and isotropic boundary conditions compatible with the unitarity of the quantum field theory. The renormalized vacuum polarization is computed for both zero and non-zero mass of the field, implementing a local version of the zeta regularization technique. The asymptotic behaviors of the vacuum polarization for small and large distances from the hyperplane are determined to leading order. It is shown that boundary divergences are softened in the specific case of a pure Dirac delta potential.

Keywords: Casimir effect; vacuum polarization; zeta regularization; zero-range interactions; boundary divergences

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1. Introduction

Casimir physics deals with the ubiquitous London–Van der Waals dispersion forces, arising from the spontaneous polarization of neutral atoms and molecules, in a regime where retardation effects are not negligible. Accordingly, the resulting Casimir forces between macroscopic bodies are truly quantum and relativistic in nature.

In a pioneering work dating back to 1948 [1], following a suggestion of Bohr, Hendrik Casimir made a groundbreaking theoretical prediction: two parallel, neutral conducting plates would experience a mutually attractive force $F = \hbar c \frac{\pi^2}{240} \Sigma / a^4$ (a and Σ denoting, respectively, the distance between the plates and their surface area), due to a variation of the electromagnetic quantum vacuum energy induced by the presence of the plates themselves. This astonishing result indicates that a detailed microscopic description of the plates constituents is actually unnecessary for the computation of the previously mentioned dispersion forces, at least to leading order. Indeed, it is sufficient to consider effective models where relativistic quantum fields are influenced by classical boundaries, external potentials, or even curved or topologically non-trivial background geometries. Building on this crucial feature, the study of the Casimir effect has nowadays become a well-established and extremely active line of research, both on the theoretical and on the experimental side. Here we content ourselves with mentioning the classical essays [2–7], making also reference to the vast literature cited therein.

Assuming that quantum fields are confined by perfectly reflecting boundaries is a strong idealization: understandably, no real material is going to behave as a perfect conductor in any frequency range of the electromagnetic field. It comes as no surprise that a price must be paid for this simplification. As first pointed out by Deutsch and Candelas in

1979 [8], renormalized expectation values of local observables, such as the vacuum energy density, generically diverge in a non-integrable way as the boundary is approached. This leads inevitably to the emergence of anomalies in the computation of the associated global observables (see also [9–11]). Similar issues appear even if the confinement of the quantum field is produced by a smooth external potential diverging at infinity [12]. On the contrary, no pathologies are expected to occur when the external potential is regular and vanishes rapidly enough at large distances.

An intermediate regime between smooth confining potentials and hard boundaries can be realized through singular zero-range potentials. Their mathematical description ultimately amounts to prescribing suitable boundary conditions for the quantum field on sets of small co-dimension (1, 2 or 3), where the distributional potentials are supposed to be concentrated. At the same time, such singular potentials can often be interpreted as limits (in resolvent sense) of sharply peaked, regular potentials. More technical details on these subjects can be found, e.g., in [13–17]. Nowadays, a quite rich literature is available regarding the analysis of Casimir-type settings with external zero-range potentials. The Casimir effect in presence of surface Dirac delta potentials, interpreted as semi-transparent walls responsible for a partial confinement of the quantum field, was first addressed by Mamaev and Trunov [18] and later examined in various configurations by several authors [19–26]. More recently, considerable attention was devoted to the study of renormalized vacuum expectations of global observables (such as the total energy) in presence of generalized zero-range interactions concentrated on sets of co-dimension 1, corresponding to mixtures of $\delta - \delta'$ potentials [27–32]. Before proceeding, let us also mention that various models with point impurities, modelled via distributional potentials concentrated on sets of co-dimension 3, were analyzed in [33–41].

The present work studies the vacuum fluctuations of a canonically quantized, neutral scalar field in $(d + 1)$ -dimensional Minkowski spacetime (with $d \geq 1$) in the presence of a flat hyperplane of co-dimension 1. Both the massive and massless theories are considered. The presence of the hyperplane is described in terms of boundary conditions for the field and its normal derivative. It is worth remarking that all local, homogeneous and isotropic boundary conditions compatible with the unitarity of the quantum field theory are taken into account. Of course, two qualitatively different scenarios are allowed. The first one corresponds to a perfectly reflecting plane, yielding a total confinement of the field on either of the half-spaces that it separates; this setting is naturally portrayed in terms of classical boundary conditions of Dirichlet, Neumann or Robin type. The second one refers to a semitransparent plane, which can be tunnelled through by the quantum field; this situation is described making reference to generalized δ - δ' potentials concentrated on the plane.

The main object of investigation is the vacuum polarization, namely the renormalized expectation value of the field squared at any spacetime point. This is computed implementing the ζ -regularization technique in the formulation outlined in [42] (see also [43–45]), which allows to derive explicit integral representations in all cases of interest. These representations are then employed to determine the asymptotic behavior of the vacuum polarization close to the hyperplane and far away from it. In this connection, the primary purpose is to inspect the presence of boundary divergences. For a perfectly reflecting hyperplane, it is found that the vacuum polarization always diverges near the plane (logarithmically for $d = 1$ and with a power law for $d \geq 2$, with respect to the distance from the plane); notably, the leading order term in the asymptotic expansion is always independent of the parameters describing specific boundary conditions. Similar divergences also occur for a semitransparent plane; however in this case the leading order asymptotics depend explicitly on the parameters appearing in the characterization of the boundary conditions. To say more, the leading order divergent contribution is absent for a specific choice of the parameters, corresponding to a pure Dirac delta potential. Some motivations explaining why this very model plays a somehow distinguished role are presented.

The paper is organized as follows. Section 2 provides an overview of the local zeta regularization framework described in [42]. In Section 3 the renormalized vacuum po-

larization for a scalar field in presence of a perfectly reflecting plane is analyzed. The analogous observable in the case of a semitransparent plane is examined in Section 4. In both Sections 3 and 4 the case of a massive field is first considered, and the corresponding massless theory is subsequently addressed by a limiting procedure. Finally, Appendix A presents a self-contained derivation of the heat kernel on the half-line for generic Robin boundary conditions at the origin, a tool used in the computations of Section 3.

2. General Theory

The purpose of this section is to present a brief and self-contained summary of some general techniques extracted from [42], to be systematically employed in the sequel.

2.1. Quantum Field Theory and the Fundamental Operator

We work in natural units of measure ($c = 1, \hbar = 1$) and identify $(d + 1)$ -dimensional Minkowski spacetime with \mathbb{R}^{d+1} using a set of inertial coordinates $(x^\mu)_{\mu=0,1,\dots,d} \equiv (t, \mathbf{x})$, such that the Minkowski metric has components $(\eta_{\mu\nu}) = \text{diag}\{-1, 1, \dots, 1\}$.

Making reference to the standard formalism of canonical quantization, we describe a neutral scalar field living on a spatial domain $\Omega \subset \mathbb{R}^d$ as an operator-valued distribution $\hat{\phi} : (t, \mathbf{x}) \in \mathbb{R} \times \Omega \mapsto \hat{\phi}(t, \mathbf{x}) \in \mathcal{L}_{sa}(\mathfrak{F})$. Here \mathfrak{F} is the bosonic Fock space constructed on the single-particle Hilbert space $L^2(\Omega)$ of square-integrable functions, and $\mathcal{L}_{sa}(\mathfrak{F})$ is the set of unbounded self-adjoint operators on it. We denote with $|0\rangle \in \mathfrak{F}$ the corresponding vacuum state (not to be confused with the true Minkowskian vacuum) and assume that the dynamics is determined by a generalized Klein–Gordon equation of the form

$$(\partial_{tt} + \mathcal{A})\hat{\phi} = 0,$$

where $\mathcal{A} : \text{dom}(\mathcal{A}) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ is a non-negative and self-adjoint operator on the single-particle Hilbert space. The non-negativity of \mathcal{A} is in fact an indispensable requirement for a well-behaved quantum field theory, free of pernicious instabilities. In typical applications, \mathcal{A} is a Schrödinger-type differential operator on the spatial domain Ω , possibly including a static external potential $V : \Omega \rightarrow \mathbb{R}$, i.e.,

$$\mathcal{A} = -\Delta + V.$$

Correspondingly, whenever the spatial domain has a boundary $\partial\Omega$ it is essential to specify suitable conditions on it. We understand these boundary conditions to be encoded in the definition of the operator self-adjointness domain $\text{dom}(\mathcal{A})$. It goes without saying that the class of admissible potentials and boundary conditions is restricted by the fundamental hypotheses of self-adjointness and non-negativity for \mathcal{A} .

The configurations analyzed in the present work regard a scalar field influenced solely by the presence of an either perfectly reflecting or semitransparent hyperplane π which, without loss of generality, can be parametrized as

$$\pi = \{\mathbf{x} \equiv (x_1, \dots, x_d) \in \mathbb{R}^d \mid x_1 = 0\}. \tag{1}$$

As already mentioned in the Introduction, the coupling between the field and this hyperplane can always be described in terms of suitable boundary conditions for the field and its normal derivative on π . Accordingly, in all cases we shall characterize the fundamental operator \mathcal{A} as a self-adjoint extension of the closable symmetric operator $(-\Delta + m^2) \upharpoonright C_c^\infty(\mathbb{R}^d \setminus \pi)$ on $L^2(\mathbb{R}^d \setminus \pi) \equiv L^2(\mathbb{R}^d)$ (here $m = \text{const.} \geq 0$ indicates the mass of the field). We refer to Section 2.3 for more details.

2.2. ζ -Regularization and Renormalization

As well known, a quantum field theory of the type outlined in the preceding subsection is typically plagued by ultraviolet divergences. A viable way to regularize these divergences is the ζ -regularization approach, originally proposed by Dowker and Critch-

ley [46], Hawking [47] and Wald [48]. We employ this approach, following the systematic formulation provided in [42]. Assuming for now that \mathcal{A} is strictly positive (with this we mean that the spectrum $\sigma(\mathcal{A})$ is contained in $[\varepsilon, +\infty)$ for some $\varepsilon > 0$; in the sequel we will indicate how to relax this condition), we firstly introduce the ζ -smeared field operator

$$\hat{\phi}^u := (\mathcal{A}/\kappa^2)^{-u/4} \hat{\phi};$$

here $u \in \mathbb{C}$ is the regularizing parameter and $\kappa > 0$ is a mass scale factor, included for dimensional reasons. Notice that the initial non-regularized theory is formally recovered setting $u = 0$.

Next, we consider the ζ -regularized vacuum polarization at any spacetime point $(t, \mathbf{x}) \in \mathbb{R} \times \Omega$, that is the regularized 2-point function at equal times evaluated along the space diagonal $\{\mathbf{x}, \mathbf{y} \in \Omega \mid \mathbf{y} = \mathbf{x}\}$:

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle \equiv \langle 0 | \hat{\phi}^u(t, \mathbf{x}) \hat{\phi}^u(t, \mathbf{y}) | 0 \rangle \Big|_{\mathbf{y}=\mathbf{x}}.$$

This quantity can be expressed in terms of the integral kernel associated to a suitable complex power of the fundamental operator \mathcal{A} ; more precisely, we have [42] (Equation (2.26))

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle = \frac{\kappa^u}{2} \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}}. \tag{2}$$

Notice that the expression on the right-hand side of (2) does not depend on the time coordinate $t \in \mathbb{R}$, as expected for static configurations like the ones we are considering.

On very general grounds it can be shown that the function $(\mathbf{x}, \mathbf{y}) \mapsto \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x}, \mathbf{y})$ belongs to $C^j(\Omega \times \Omega)$ for any $u \in \mathbb{C}$ and $j \in \{1, 2, 3, \dots\}$ such that $\text{Re } u > d - 1 + j$. Especially, let us remark that the said function is regular along the diagonal $\{\mathbf{y} = \mathbf{x}\}$ for $\text{Re } u$ large enough. Furthermore, for any fixed $\mathbf{x}, \mathbf{y} \in \Omega$ (even for $\mathbf{y} = \mathbf{x}$), the map $u \mapsto \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x}, \mathbf{y})$ is analytic in the complex strip $\{u \in \mathbb{C} \mid \text{Re } u > d - 1\}$ and possesses a meromorphic extension to the whole complex plane \mathbb{C} with at most simple pole singularities [42,49–51]. In light of these results, we proceed to define the *renormalized vacuum polarization* at $(t, \mathbf{x}) \in \mathbb{R} \times \Omega$ as

$$\langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren} := RP \Big|_{u=0} \langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle. \tag{3}$$

Here and in the following we denote with $RP \Big|_{u=0}$ the regular part of the Laurent expansion near $u = 0$. (For any complex-valued meromorphic function f defined in a complex neighbourhood of $u = 0$, making reference to its Laurent expansion $f(u) = \sum_{\ell=-\infty}^{+\infty} f_\ell u^\ell$ we define the regular part as $(RP f)(u) = \sum_{\ell=0}^{+\infty} f_\ell u^\ell$, which yields in particular $RP \Big|_{u=0} f = f_0$). Notably, if no pole arises at $u = 0$, Equation (3) simply amounts to evaluating the analytic continuation at this point and ultraviolet renormalization is attained with no need to subtract divergent quantities; on the contrary, when the meromorphic extension has a pole at $u = 0$, Equation (3) matches a minimal subtraction prescription [42,48,52].

Before we proceed, let us point out that a modification of the above construction is required whenever the fundamental operator \mathcal{A} is non-negative but not strictly positive, namely, when its spectrum contains a right neighbourhood of 0. In this case an infrared cut-off must be added in advance, and ultimately removed after renormalization of ultraviolet divergences. For example, one can replace \mathcal{A} with $\mathcal{A} + m^2$ ($m > 0$) and compute the limit $m \rightarrow 0^+$ at last, after analytic continuation at $u = 0$. Concerning the present work, this modification plays a key role when the field is massless ($m = 0$); in this case the renormalized vacuum polarization is determined as the zero mass limit ($m \rightarrow 0^+$) of the analogous quantity in the massive theory, namely,

$$\langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren}^{(massless)} := \lim_{m \rightarrow 0^+} \langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren}^{(massive)}. \tag{4}$$

2.3. Factorized Configurations

Let us now restrict the attention to product configurations with

$$\Omega = \Omega_1 \times \mathbb{R}^{d-1}, \quad \mathcal{A} = \mathcal{A}_1 \otimes \mathbf{1}_{d-1} + \mathbf{1}_1 \otimes (-\Delta_{d-1}),$$

where $\Omega_1 \subset \mathbb{R}$ is any open interval, \mathcal{A}_1 is a positive self-adjoint operator on $L^2(\Omega_1)$, and $-\Delta_{d-1}$ indicates the free Laplacian on \mathbb{R}^{d-1} with $\text{dom}(-\Delta_{d-1}) = H^2(\mathbb{R}^{d-1}) \subset L^2(\mathbb{R}^{d-1})$. It is implied that $\text{dom}(\mathcal{A}) = \overline{\text{dom}(\mathcal{A}_1) \otimes H^2(\mathbb{R}^{d-1})} \subset L^2(\Omega) \equiv L^2(\Omega_1) \otimes L^2(\mathbb{R}^{d-1})$.

Under these circumstances, everything is determined upon factorization by the reduced operator \mathcal{A}_1 acting on the 1-dimensional spatial domain Ω_1 . In particular, let us highlight that for any $\tau > 0$ the heat kernel $e^{-\tau\mathcal{A}}(\mathbf{x}, \mathbf{y})$ and the reduced analogue $e^{-\tau\mathcal{A}_1}(x_1, y_1)$ fulfil

$$e^{-\tau\mathcal{A}}(\mathbf{x}, \mathbf{y}) = e^{-\tau\mathcal{A}_1}(x_1, y_1) e^{\tau\Delta_{d-1}}(\mathbf{x}_{d-1}, \mathbf{y}_{d-1}),$$

where we put $\mathbf{x}_{d-1} \equiv (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$ and denoted with $e^{\tau\Delta_{d-1}}(\mathbf{x}_{d-1}, \mathbf{y}_{d-1})$ the free heat kernel in \mathbb{R}^{d-1} , that is

$$e^{\tau\Delta_{d-1}}(\mathbf{x}_{d-1}, \mathbf{y}_{d-1}) = \frac{1}{(4\pi\tau)^{\frac{d-1}{2}}} e^{-\frac{|\mathbf{x}_{d-1} - \mathbf{y}_{d-1}|^2}{4\tau}}.$$

Taking into account the above considerations, from the basic Mellin-type identity

$$\mathcal{A}^{-s}(\mathbf{x}, \mathbf{y}) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} e^{-\tau\mathcal{A}}(\mathbf{x}, \mathbf{y}),$$

we infer by direct evaluation

$$\mathcal{A}^{-s}(\mathbf{x}, \mathbf{y}) \Big|_{\mathbf{y}=\mathbf{x}} = \frac{1}{(4\pi)^{\frac{d-1}{2}} \Gamma(s)} \int_0^\infty d\tau \tau^{s-\frac{d+1}{2}} e^{-\tau\mathcal{A}_1}(x_1, y_1) \Big|_{y_1=x_1}.$$

Together with Equation (2), the latter relation allows us to derive the following representation formula for the ζ -regularized vacuum polarization, valid for $u \in \mathbb{C}$ with $\text{Re } u > d - 1$:

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle = \frac{\kappa^u}{2(4\pi)^{\frac{d-1}{2}} \Gamma(\frac{u+1}{2})} \int_0^\infty d\tau \tau^{\frac{u-d}{2}} e^{-\tau\mathcal{A}_1}(x_1, y_1) \Big|_{y_1=x_1}. \tag{5}$$

Let us now return to the configurations portrayed at the end of Section 2.1, involving a scalar field restrained by the presence of a hyperplane π . Whenever the effective interaction between the field and π is isotropic and homogeneous along the hyperplane, these configurations exhibit the factorization property discussed above. More precisely, referring to the parametrization (1) of π , under the hypotheses just mentioned it is natural to consider the reduced domain $\Omega_1 = \mathbb{R} \setminus \{0\}$ and to characterize the associated operator \mathcal{A}_1 as a self-adjoint extension of the symmetric operator $(-\partial_{x_1 x_1} + m^2) \upharpoonright C_c^\infty(\mathbb{R} \setminus \{0\})$. We recall that the domains of these self-adjoint extensions are indeed restrictions of the maximal domain $H^2(\mathbb{R} \setminus \{0\}) \equiv H^2(\mathbb{R}_-) \oplus H^2(\mathbb{R}_+)$ (where $\mathbb{R}_+ \equiv (0, +\infty)$ and $\mathbb{R}_- \equiv (-\infty, 0)$), determined by suitable boundary conditions at the gap point $x_1 = 0$. As a matter of fact, the models discussed in the upcoming Sections 3 and 4 encompass all admissible self-adjoint realizations of the reduced operator $-\partial_{x_1 x_1} + m^2$ on $\mathbb{R} \setminus \{0\}$, respecting the basic requirement of positivity. Notice however that this scheme does not reproduce the entire class of (positive) self-adjoint realizations of the full operator $-\Delta + m^2$ on $\mathbb{R}^d \setminus \pi$, since non-homogeneous and non-local self-adjoint realizations are being omitted. Let us mention a pair of examples which are not covered by our analysis: on one hand, local but non-homogeneous boundary conditions appear in the description of δ -potentials supported on π with non-constant coupling coefficients, formally corresponding to operators like $-\Delta + m^2 + \alpha(\mathbf{x}_{d-1}) \delta_\pi$ (with $\mathbf{x}_{d-1} \equiv (x_2, \dots, x_d) \in \mathbb{R}^{d-1} \simeq \pi$); on the other hand, homogeneous but non-local boundary conditions are used to characterize operators like $-\Delta + m^2 + \alpha(-\Delta_{d-1}) \delta_\pi$, with $\alpha(-\Delta_{d-1})$ a suitable self-adjoint operator on $L^2(\pi)$ defined by functional calculus [30].

In the following sections, after providing a precise definition of the reduced operator \mathcal{A}_1 under analysis and an explicit expression for the associated heat kernel $e^{-\tau \mathcal{A}_1}(x_1, y_1)$, we proceed to construct the analytic continuation of the map $u \mapsto \langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle$ to the whole complex plane starting from the representation formula (5). The renormalized vacuum fluctuation is ultimately computed following the prescriptions (3) and (4).

3. Perfectly Reflecting Plane

In this section we analyze the admissible scenarios where the hyperplane π behaves as a perfectly reflecting surface, providing a total decoupling of the two half-spaces which it separates. To this purpose, taking into account the general arguments presented in the preceding Section 2 and making reference to [14] (Thm. 3.2.3), we consider the family of reduced operators labelled as follows by the pairs $b_+, b_- \in \mathbb{R} \cup \{+\infty\}$ (more precisely, one should refer to the elements $\mathbf{h}^\pm = (h_0^\pm, h_1^\pm)$ of the real projective space \mathbf{P}_1 ; with respect to this, we are setting $b_\pm = \pm (h_1^\pm / h_0^\pm)$, intending $b_\pm = +\infty$ if $h_0^\pm = 0$):

$$\begin{aligned} \text{dom}(\mathcal{A}_1) &:= \{ \psi \in H^2(\mathbb{R} \setminus \{0\}) \mid \mp \psi'(0^\pm) + b_\pm \psi(0^\pm) = 0 \}, \\ \mathcal{A}_1 \psi &= (-\partial_{x_1 x_1} + m^2) \psi \quad \text{in } \mathbb{R} \setminus \{0\}. \end{aligned} \tag{6}$$

The above definition entails classical boundary conditions of Neumann, Dirichlet or Robin type, chosen independently on the two sides of the gap point $x_1 = 0$ (*viz.*, on the two sides of the hyperplane π); especially, Neumann conditions are retrieved for $b_\pm = 0$ and Dirichlet ones for $b_\pm = +\infty$. The \pm signs in Equation (6) are fixed so as to comply with the canonical Robin form $(\partial_{\mathbf{n}} \psi + b_\pm \psi)|_{x_1=0^\pm} = 0$, where \mathbf{n} denotes the unit outer normal. Let us also emphasize that, with our units of measure, the parameters b_\pm are dimensionally equivalent to a mass. In passing, we point out that the Casimir effect for Robin boundary conditions was previously analyzed in [53–56].

For any $b_+, b_- \in \mathbb{R} \cup \{+\infty\}$ the spectrum of the reduced operator \mathcal{A}_1 comprises an invariant purely absolutely continuous part and at most two isolated eigenvalues below the continuous threshold, depending on the sign of b_\pm . More precisely, we have

$$\begin{aligned} \sigma(\mathcal{A}_1) &= \sigma_{ac}(\mathcal{A}_1) \cup \sigma_p(\mathcal{A}_1), \quad \sigma_{ac}(\mathcal{A}_1) = [m^2, +\infty); \\ \sigma_p(\mathcal{A}_1) &= \begin{cases} \emptyset & \text{for } b_+, b_- \in [0, +\infty) \cup \{+\infty\}, \\ \{m^2 - b_+^2\} & \text{for } b_+ \in (-\infty, 0), b_- \in [0, +\infty) \cup \{+\infty\}, \\ \{m^2 - b_-^2\} & \text{for } b_+ \in [0, +\infty) \cup \{+\infty\}, b_- \in (-\infty, 0), \\ \{m^2 - b_+^2, m^2 - b_-^2\} & \text{for } b_+, b_- \in (-\infty, 0). \end{cases} \end{aligned}$$

This makes evident that the required positivity of \mathcal{A}_1 is ensured if and only if

$$b_+, b_- \in (-m, +\infty) \cup \{+\infty\}, \quad m > 0, \tag{7}$$

two conditions which we assume to be fulfilled until the end of this section.

The heat kernel associated to the aforementioned reduced operator \mathcal{A}_1 is given by (see Appendix A; $\theta(\cdot)$ is the Heaviside step function)

$$\begin{aligned} e^{-\tau \mathcal{A}_1}(x_1, y_1) &= \\ &= \frac{e^{-m^2 \tau}}{\sqrt{4\pi\tau}} \left[\theta(x_1) \theta(y_1) \left(e^{-\frac{|x_1 - y_1|^2}{4\tau}} + e^{-\frac{|x_1 + y_1|^2}{4\tau}} - 2b_+ \int_0^\infty dw e^{-b_+ w - \frac{(w + x_1 + y_1)^2}{4\tau}} \right) \right. \\ &\quad \left. + \theta(-x_1) \theta(-y_1) \left(e^{-\frac{|x_1 - y_1|^2}{4\tau}} + e^{-\frac{|x_1 + y_1|^2}{4\tau}} - 2b_- \int_0^\infty dw e^{-b_- w - \frac{(w - x_1 - y_1)^2}{4\tau}} \right) \right]. \end{aligned} \tag{8}$$

It can be checked by direct inspection of Equation (8) that the reduced heat kernel along the diagonal $y_1 = x_1$ fulfils, for $\tau \rightarrow 0^+$,

$$e^{-\tau \mathcal{A}_1}(x_1, y_1) \Big|_{y_1=x_1} = \frac{e^{-m^2 \tau}}{\sqrt{4\pi\tau}} \left[1 + \mathcal{O}\left(e^{-\frac{|x_1|^2}{\tau}}\right) \right] = \frac{1}{\sqrt{\tau}} \left[\sum_{n=0}^{+\infty} \frac{(-1)^n m^{2n}}{\sqrt{4\pi} n!} \tau^n + \mathcal{O}\left(e^{-\frac{|x_1|^2}{\tau}}\right) \right].$$

This makes evident that, for any $x_1 \neq 0$, the local heat kernel coefficients coincide with those of a massive scalar field propagating freely, with no hyperplane to restrict its motion. In the case of zero mass the said coefficients all vanish but the first one (equal to 1). As usual, the boundary terms in the “small-time” asymptotic expansion of the heat kernel are exponentially suppressed with the distance from the boundary. The presence of boundaries (such as the hyperplane π) is taken into account more efficiently by the $\tau \rightarrow 0^+$ expansion of the heat trace $\text{Tr}(e^{-\tau A_1})$; however, the heat trace cannot be employed for the computation of local observables, such as the vacuum polarization considered here. To say more, in the present setting it appears that the heat trace coefficients do not coincide with the integrals over the line \mathbb{R} of the local analogues in the heat kernel expansion.

Inserting the above expression (8) into Equation (5), we obtain the following integral representation of the ζ -regularized vacuum polarization:

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle = \frac{\kappa^u}{2(4\pi)^{d/2} \Gamma(\frac{u+1}{2})} \int_0^\infty d\tau \tau^{\frac{u-d-1}{2}} e^{-m^2\tau} \times \tag{9}$$

$$\left[1 + e^{-\frac{(x_1)^2}{\tau}} - 2b_+ \theta(x_1) \int_0^\infty dw e^{-b_+w - \frac{(w+2x_1)^2}{4\tau}} - 2b_- \theta(-x_1) \int_0^\infty dw e^{-b_-w - \frac{(w-2x_1)^2}{4\tau}} \right].$$

In accordance with the general theory outlined in Section 2, it can be checked by direct inspection that the above representation (9) makes sense for

$$u \in \mathbb{C} \quad \text{with} \quad \text{Re } u > d - 1, \tag{10}$$

a condition needed especially to ensure the convergence of the integral w.r.t. the variable τ for $\tau \rightarrow 0^+$. Besides, the expression of the right-hand side of Equation (9) is an analytic function of u inside the semi-infinite complex strip identified by Equation (10).

In order to determine the meromorphic extensions of the map $u \mapsto \langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle$ to the whole complex plane, let us firstly mention a couple of identities involving the Euler Gamma function Γ and the modified Bessel function of second kind K_ν (see, respectively, ref. [57] (Equation (5.9.1)) and [57] (Equation (10.32.10))):

$$\int_0^\infty d\tau \tau^{\nu-1} e^{-m^2\tau} = m^{-2\nu} \Gamma(\nu), \quad \text{for all } m > 0, \nu \in \mathbb{C} \text{ with } \text{Re } \nu > 0; \tag{11}$$

$$\int_0^\infty d\tau \tau^{\nu-1} e^{-m^2\tau - \frac{p^2}{\tau}} = 2^{\nu+1} p^{2\nu} \mathfrak{K}_{-\nu}(2mp), \quad \text{for all } m, p > 0, \nu \in \mathbb{C}, \tag{12}$$

where, for later convenience, we introduced the functions (for $\nu \in \mathbb{C}$)

$$\mathfrak{K}_\nu : (0, +\infty) \rightarrow \mathbb{C}, \quad \mathfrak{K}_\nu(w) := w^\nu K_\nu(w). \tag{13}$$

Let us now return to Equation (9) and notice that the integration order therein can be exchanged by Fubini’s theorem, for any u as in Equation (10). Then, using the previous identities (11) and (12), by a few additional manipulations and the change of integration variable $w = 2|x_1|v$, we obtain

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle = \frac{m^{d-1}(\kappa/m)^u \Gamma(\frac{u-d+1}{2})}{2^{d+1}\pi^{d/2} \Gamma(\frac{u+1}{2})} + \frac{2^{\frac{u-3d+1}{2}} (\kappa|x_1|)^u}{\pi^{d/2} \Gamma(\frac{u+1}{2}) |x_1|^{d-1}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|)$$

$$- \frac{\theta(x_1) b_+ 2^{\frac{u-3d+5}{2}} (\kappa|x_1|)^u}{\pi^{d/2} \Gamma(\frac{u+1}{2}) |x_1|^{d-2}} \int_0^\infty dv \frac{e^{-2b_+|x_1|v}}{(v+1)^{d-1-u}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1))$$

$$- \frac{\theta(-x_1) b_- 2^{\frac{u-3d+5}{2}} (\kappa|x_1|)^u}{\pi^{d/2} \Gamma(\frac{u+1}{2}) |x_1|^{d-2}} \int_0^\infty dv \frac{e^{-2b_-|x_1|v}}{(v+1)^{d-1-u}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1)). \tag{14}$$

Recall that the reciprocal of the Gamma function is analytic on the whole complex plane. Conversely, the Gamma function appearing in the numerator of the first term is a

meromorphic function of u , with simple poles where its argument is equal to a non-positive integer, i.e.,

$$u = d - 1 - 2\ell, \quad \text{with } \ell \in \{0, 1, 2, \dots\}. \tag{15}$$

On the other hand, from basic features of the modified Bessel function K_ν we infer that the function \mathfrak{K}_ν introduced in Equation (13) fulfils the following: for any fixed $w > 0$, the map $\nu \mapsto \mathfrak{K}_\nu(w)$ is analytic on the whole complex plane [57] (§10.25(ii)); for any fixed $\nu \in \mathbb{C}$, the map $w \in (0, +\infty) \mapsto \mathfrak{K}_\nu(w)$ is analytic, continuous up to $w = 0$ and decaying with exponential speed for $w \rightarrow +\infty$ [57] (§10.31 and Equations (10.25.2), (10.27.4) and (10.40.2)). In particular, let us briefly comment on the integrals appearing in Equation (14). Since the integrand functions therein are continuous at $w = 0$, the lower extreme of integration is never problematic. On the other side, from [57] (10.40.2) we infer

$$\begin{aligned} & e^{-2b_\pm|x_1|v} (v+1)^{u-d+1} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1)) \\ &= \sqrt{\frac{\pi}{2}} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2-u}{2}} e^{-2(b_\pm+m)|x_1|v} v^{\frac{u-d}{2}} (1+o(1)) \quad \text{for } w \rightarrow +\infty, \end{aligned}$$

which shows that the condition $b_\pm > -m$ established in Equation (7) is in fact indispensable to grant the convergence of the said integrals in Equation (14).

In light of the above considerations, Equation (14) does in fact provide the meromorphic extension of the ζ -regularized vacuum polarization $\langle 0 | (\hat{\phi}^\mu(t, \mathbf{x}))^2 | 0 \rangle$ to the whole complex plane, with isolated simple pole singularities at the points indicated in Equation (15). We can then proceed to compute the renormalized vacuum polarization, implementing the general prescription (3). In this regard, special attention must be paid to the first term on the right-hand side of Equation (14), since it presents a pole at $u = 0$ when the space dimension d is odd. More precisely, using some basic properties of the Gamma function [57] (§5) and indicating with $H_\ell := \sum_{j=1}^\ell \frac{1}{j}$ the ℓ -th harmonic number for $\ell \in \{0, 1, 2, \dots\}$ ($H_0 \equiv 0$ by convention), we deduce

$$\frac{(\kappa/m)^u \Gamma(\frac{u-d+1}{2})}{\Gamma(\frac{u+1}{2})} = \begin{cases} \frac{(-1)^{\frac{d}{2}} \sqrt{\pi}}{\Gamma(\frac{d+1}{2})} + \mathcal{O}(u) & \text{for } d \text{ even,} \\ \frac{(-1)^{\frac{d-1}{2}}}{\sqrt{\pi} \Gamma(\frac{d+1}{2})} \left[\frac{2}{u} + H_{\frac{d-1}{2}} + 2 \log\left(\frac{2\kappa}{m}\right) \right] + \mathcal{O}(u) & \text{for } d \text{ odd.} \end{cases}$$

Noting that all other terms in Equation (14) are regular at $u = 0$, from (3) we obtain

$$\langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren} = \langle 0 | \hat{\phi}^2 | 0 \rangle_{ren}^{(free)} + \langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)}; \tag{16}$$

$$\langle 0 | \hat{\phi}^2 | 0 \rangle_{ren}^{(free)} = \begin{cases} \frac{(-1)^{\frac{d}{2}} \pi m^{d-1}}{(4\pi)^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2})} & \text{for } d \text{ even,} \\ \frac{(-1)^{\frac{d-1}{2}} m^{d-1}}{(4\pi)^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2})} \left[H_{\frac{d-1}{2}} + 2 \log\left(\frac{2\kappa}{m}\right) \right] & \text{for } d \text{ odd;} \end{cases} \tag{17}$$

$$\begin{aligned} \langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)} &:= \frac{1}{2^{\frac{3d-1}{2}} \pi^{\frac{d+1}{2}} |x_1|^{d-1}} \left[\mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|) \right. \\ &\quad - \theta(x_1) 4b_+|x_1| \int_0^\infty dv \frac{e^{-2b_+|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\ &\quad \left. - \theta(-x_1) 4b_-|x_1| \int_0^\infty dv \frac{e^{-2b_-|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \right]. \end{aligned} \tag{18}$$

It is worth remarking that $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ is in fact a constant which depends solely on the mass m of the field and, possibly, on the renormalization mass parameter κ (if the space dimension d is odd). In particular, it does not depend on the coordinate x_1 , namely, the distance from the hyperplane π , nor on the parameters b_{\pm} defining the boundary conditions on π . For these reasons it is natural to regard $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ as a pure free-theory contribution (which explains the choice of the superscript). In contrast, $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ is a contribution which truly accounts for the presence of the hyperplane and for the boundary conditions on it.

Owing to the above considerations, one might be tempted to discard the free-theory term $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ and regard $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ as the only physically relevant contribution to the vacuum polarization. Despite being tenable, this standpoint actually suffers from a drawback. Indeed, let us anticipate that $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ plays a key role in the cancellation of some infrared divergences which would otherwise affect the massless theory in space dimension $d = 1$ (see the subsequent Section 3.3.1). Therefore, we reject the standpoint sketched above and regard the sum $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ defined in Equation (16) as the true physically sensible observable.

3.1. Neumann and Dirichlet Conditions

We already mentioned in the comments below Equation (6) that Neumann and Dirichlet boundary conditions correspond to $b_{\pm} = 0$ and $b_{\pm} = +\infty$, respectively. Of course, the free-theory contribution $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ remains unchanged in both cases, so let us focus on the term $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$.

In the case of Neumann conditions where $b_{\pm} = 0$, it appears that the expressions in the second and third line of Equation (18) vanish identically. Regarding the case of Dirichlet conditions, the limits $b_{\pm} \rightarrow +\infty$ can be easily computed as follows, making the change of integration variable $z = 2b_+|x_1|v$ and using the dominated convergence theorem:

$$\begin{aligned} & \lim_{b_{\pm} \rightarrow +\infty} \left[4b_{\pm}|x_1| \int_0^{\infty} dv \frac{e^{-2b_{\pm}|x_1|v}}{(v+1)^{d-1}} \Re_{\frac{d-1}{2}}(2m|x_1|(v+1)) \right] \\ &= \lim_{b_{\pm} \rightarrow +\infty} \left[2 \int_0^{\infty} dz \frac{e^{-z}}{\left(\frac{z}{2b_+|x_1|} + 1\right)^{d-1}} \Re_{\frac{d-1}{2}}\left(2m|x_1|\left(\frac{z}{2b_+|x_1|} + 1\right)\right) \right] \\ &= 2 \Re_{\frac{d-1}{2}}(2m|x_1|) \int_0^{\infty} dz e^{-z} = 2 \Re_{\frac{d-1}{2}}(2m|x_1|). \end{aligned}$$

Summarizing, Equation (18) reduces to

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = \pm \frac{\Re_{\frac{d-1}{2}}(2m|x_1|)}{2^{\frac{3d-1}{2}} \pi^{\frac{d+1}{2}} |x_1|^{d-1}}, \tag{19}$$

for Neumann (+) and Dirichlet (-) boundary conditions, respectively.

3.2. Asymptotics for $x_1 \rightarrow 0^{\pm}$ and $x_1 \rightarrow \pm\infty$

Hereafter we investigate the behavior of the renormalized vacuum polarization $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ close to the hyperplane π and far way from it. For brevity we only present the leading order asymptotics, although a refinement of the arguments outlined below would actually permit to derive asymptotic expansions at any order.

Before proceeding, let us stress once more that $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ does not depend on the coordinate x_1 ; thus, it is sufficient to analyze the term $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ (see Equation (18)).

3.2.1. The Limit $x_1 \rightarrow 0^\pm$

Let us first notice that the functions \mathfrak{K}_ν defined in Equation (13) have the following asymptotic expansions, which can be easily derived from [57] [Equations (10.31.1) and (10.30.2)] (here and in the sequel $\gamma_{EM} = 0.57721\dots$ indicates the Euler–Mascheroni constant):

$$\begin{aligned} \mathfrak{K}_0(w) &= -\log(w/2) + \gamma_{EM} + \mathcal{O}(w^2 \log w), \quad \text{for } w \rightarrow 0^+; \quad (20) \\ \mathfrak{K}_\nu(w) &= 2^{\nu-1} \Gamma(\nu) + \mathcal{O}(w^{\min\{2+\nu, 2\nu\}} \log w), \quad \text{for } w \rightarrow 0^+ \text{ and } \nu > 0. \quad (21) \end{aligned}$$

Next, consider the integrals appearing in the second and third lines of Equation (18). For any finite $b_\pm \in (-m, +\infty)$ (cf. Equation (7)), via the change of variable $z = 2m|x_1|(v + 1)$ we get

$$\begin{aligned} 4b_\pm|x_1| \int_0^\infty dv \frac{e^{-2b_\pm|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\ = \frac{2b_\pm}{m} (2m|x_1|)^{d-1} e^{2b_\pm|x_1|} \int_{2m|x_1|}^\infty dz \frac{e^{-\frac{b_\pm}{m}z}}{z^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(z) \end{aligned}$$

Writing $\int_{2m|x_1|}^\infty = \int_{2m|x_1|}^{z_0} + \int_{z_0}^\infty$ for some $z_0 > 0$ fixed arbitrarily and replacing the integrand inside $\int_{2m|x_1|}^{z_0}$ with its Taylor expansion at $z = 0$ (recall, especially, Equations (20) and (21)), by a few additional computations we deduce, for $|x_1| \rightarrow 0^+$,

$$4b_\pm|x_1| \int_0^\infty dv \frac{e^{-2b_\pm|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) = \begin{cases} \mathcal{O}(1) & \text{for } d = 1, \\ \mathcal{O}(|x_1| \log |x_1|) & \text{for } d = 2, \\ \mathcal{O}(|x_1|) & \text{for } d \geq 3. \end{cases}$$

Summing up, the above arguments allow us to infer that, in the limit $x_1 \rightarrow 0^\pm$,

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = \begin{cases} -\frac{1}{2\pi} \log(m|x_1|) + \mathcal{O}(1) & \text{for } d = 1, \\ \frac{1}{8\pi|x_1|} + \mathcal{O}(\log|x_1|) & \text{for } d = 2, \\ \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}|x_1|^{d-1}} [1 + \mathcal{O}(|x_1|)] & \text{for } d \geq 3. \end{cases} \quad (22)$$

It is remarkable that the above leading order expansions do not depend on the parameters b_\pm , describing the boundary conditions. In particular, the same results remain valid for Neumann conditions, corresponding to $b_\pm = 0$. On the contrary, a separate analysis is required for Dirichlet conditions, which is formally recovered for $b_\pm \rightarrow +\infty$ (a limit which clearly does not commute with $x_1 \rightarrow 0^\pm$); in this case, starting from Equation (19) and using again Equations (20) and (21) one can derive asymptotic expansions which coincide with those reported in Equation (22), except for the opposite overall sign.

The qualitative behavior described above is likely related to the fact that the leading surface singularity of the heat trace does not depend on the parameters of the specific Robin conditions used for describing the hyperplane in the case of perfect reflection (I thank one of the anonymous referees for indicating this connection).

3.2.2. The Limit $x_1 \rightarrow \pm\infty$

It is a well known fact that local observables of Casimir type for massive fields are typically suppressed with exponential rate in the regime of large distances from the boundaries. In the sequel we provide quantitative estimates for $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$, confirming this general expectation. To this purpose, let us first point out that the functions \mathfrak{K}_ν fulfil (see Equation (13) and [57] [Equation (10.40.2)])

$$\mathfrak{K}_\nu(w) = \sqrt{\frac{\pi}{2}} e^{-w} w^{\nu-1/2} \left(1 + \mathcal{O}(1/w)\right), \quad \text{for } w \rightarrow +\infty \text{ and } \nu \geq 0. \quad (23)$$

Consider now the integral expressions in Equation (18). Using the above relation and making the change of variable $z = |x_1|v$, for $|x_1| \rightarrow +\infty$ we deduce

$$\begin{aligned} & 4b_\pm |x_1| \int_0^\infty dv \frac{e^{-2b_\pm |x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\ &= 2\sqrt{2\pi} b_\pm e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \int_0^\infty dz e^{-2(b_\pm+m)z} \left(1 + \mathcal{O}(z/|x_1|)\right) \\ &= \frac{\sqrt{2\pi} b_\pm}{b_\pm + m} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \left(1 + \mathcal{O}(1/|x_1|)\right). \end{aligned}$$

In view of the above results, from Equation (18) we infer

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = \frac{m^{\frac{d-2}{2}}}{2(4\pi)^{d/2}} \left(\frac{m-b_\pm}{m+b_\pm}\right) \frac{e^{-2m|x_1|}}{|x_1|^{d/2}} \left[1 + \mathcal{O}\left(\frac{1}{|x_1|}\right)\right] \quad \text{for } x_1 \rightarrow \pm\infty. \quad (24)$$

The case of Dirichlet boundary conditions can be alternatively addressed taking the limit $b_\pm \rightarrow +\infty$ in Equation (24), or starting from Equation (19) and using again Equation (23):

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = -\frac{m^{\frac{d-2}{2}}}{2(4\pi)^{d/2}} \frac{e^{-2m|x_1|}}{|x_1|^{d/2}} \left[1 + \mathcal{O}\left(\frac{1}{|x_1|}\right)\right] \quad \text{for } x_1 \rightarrow \pm\infty.$$

In any case $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ approaches the constant free-theory value $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ with exponential speed. This behavior is in a way reminiscent of scalar fields localization on branes modelled by domain walls in cosmological scenarios, where matter fields are allegedly confined to lower-dimensional defects while gravity is free to propagate in a bulk higher-dimensional spacetime. Such models are believed to be a viable alternative to the Kaluza–Klein compactification argument for studying various open problems in Standard Model physics and cosmology [58–61] (see also [62]).

3.3. Vacuum Polarization for a Massless Field

Let us now address the case of a massless field, fulfilling generic boundary conditions of the form written in Equation (6). In this context the hypothesis (7) entails

$$b_+, b_- \in [0, +\infty) \cup \{+\infty\},$$

and under this condition we can implement the general arguments reported in Section 2. Especially, let us recall that the renormalized vacuum polarization for a massless field is obtained as the zero-mass limit of the analogous quantity for a massive field, see Equation (4). In the sequel we discuss separately the cases with space dimension $d = 1$ and $d \geq 2$, for both technical and physical reasons.

3.3.1. Space Dimension $d = 1$

This case deserves a separate analysis, due to the emergence of some delicate infrared features. As a matter of fact, both $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ and $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ diverge in the limit $m \rightarrow 0^+$; however, their sum $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ remains finite, except when the boundary conditions are of Neumann type.

To account for the above claims, let us firstly notice that Equation (17) yields (for $d = 1$)

$$\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)} = \frac{1}{2\pi} \log\left(\frac{2\kappa}{m}\right), \quad (25)$$

which is patently divergent in the limit $m \rightarrow 0^+$.

Now consider the term $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$. For $b_+ = b_- = 0$ (similar results can be derived also if only one of b_+, b_- is equal to zero), namely in the case of Neumann conditions, from Equations (19) and (20) we readily infer (for fixed $x_1 \neq 0$)

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = -\frac{1}{2\pi} \log(m|x_1|) + \mathcal{O}(1), \quad \text{for } m \rightarrow 0^+,$$

which, together with Equations (16) and (25), implies in turn

$$\lim_{m \rightarrow 0^+} \langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren} = \lim_{m \rightarrow 0^+} \left[\frac{1}{2\pi} \log\left(\frac{\kappa}{m^2|x_1|}\right) + \mathcal{O}(1) \right] = +\infty.$$

This is nothing but an unavoidable manifestation of the infrared divergences which typically affect massless theories in low space dimension. Taking notice of this fact, in the remainder of this subsection we restrict the attention to

$$b_+, b_- \in (0, +\infty).$$

With this requirement, using Equation (20) and noting that the incomplete Gamma function $\Gamma(a, z)$ fulfils (here we make the change of integration variable $z = 2b_{\pm}|x_1|(v + 1)$ and use [57] (Equation (8.2.2))

$$\begin{aligned} 2b_{\pm}|x_1| \int_0^{\infty} dv e^{-2b_{\pm}|x_1|v} \log(v + 1) &= -\left(e^{-2b_{\pm}|x_1|v} \log(v + 1) \right)_0^{+\infty} + \int_0^{\infty} dv \frac{e^{-2b_{\pm}|x_1|v}}{v + 1} \\ &= e^{2b_{\pm}|x_1|} \int_{2b_{\pm}|x_1|}^{\infty} dz \frac{e^{-z}}{z} = e^{2b_{\pm}|x_1|} \Gamma(0, 2b_{\pm}|x_1|), \end{aligned}$$

from Equation (18) we deduce the following for $m \rightarrow 0^+$:

$$\begin{aligned} \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} &= \frac{1}{2\pi} \left[-\log(m|x_1|) + \gamma_{EM} \right. \\ &\quad - \theta(x_1) 4b_+|x_1| \int_0^{\infty} dv e^{-2b_+|x_1|v} \left(-\log(m|x_1|) + \gamma_{EM} - \log(v + 1) \right) \\ &\quad \left. - \theta(-x_1) 4b_-|x_1| \int_0^{\infty} dv e^{-2b_-|x_1|v} \left(-\log(m|x_1|) + \gamma_{EM} - \log(v + 1) \right) \right] \\ &\quad + \mathcal{O}\left((m|x_1|)^2 \log(m|x_1|) \right) \\ &= \frac{1}{2\pi} \left[\log(m|x_1|) - \gamma_{EM} + 2\theta(x_1) e^{2b_+|x_1|} \Gamma(0, 2b_+|x_1|) \right. \\ &\quad \left. + 2\theta(-x_1) e^{2b_-|x_1|} \Gamma(0, 2b_-|x_1|) \right] + \mathcal{O}\left((m|x_1|)^2 \log(m|x_1|) \right). \end{aligned}$$

From here and from Equations (4) and (25), we finally obtain

$$\begin{aligned} \langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} &= \lim_{m \rightarrow 0^+} \left[\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)} + \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} \right] \\ &= \frac{1}{2\pi} \left[\log(2\kappa|x_1|) - \gamma_{EM} + 2\theta(x_1) e^{2b_+|x_1|} \Gamma(0, 2b_+|x_1|) \right. \\ &\quad \left. + 2\theta(-x_1) e^{2b_-|x_1|} \Gamma(0, 2b_-|x_1|) \right]. \end{aligned} \tag{26}$$

The case of Dirichlet boundary conditions is retrieved taking the limit $b_{\pm} \rightarrow +\infty$ and noting that the incomplete Gamma function fulfils $\lim_{w \rightarrow +\infty} e^w \Gamma(0, w) = 0$ (see [57] (Equation (8.11.2))), which gives

$$\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} = \frac{1}{2\pi} \left[\log(2\kappa|x_1|) - \gamma_{EM} \right].$$

For any $b_+, b_- \in (0, +\infty)$, the asymptotic behavior of $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)}$ for small and large distances from the point $\pi \equiv \{x_1 = 0\}$ can be easily derived from the explicit expression (26), using the known series expansions for the incomplete Gamma function (see [57] (Equations (8.7.6) and (8.11.2))). More precisely, to leading order we have

$$\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} = \begin{cases} -\frac{1}{2\pi} \log(\kappa|x_1|) + \mathcal{O}(1) & \text{for } x_1 \rightarrow 0^\pm, \\ \frac{1}{2\pi} \log(\kappa|x_1|) + \mathcal{O}(1) & \text{for } x_1 \rightarrow \pm\infty. \end{cases}$$

3.3.2. Space Dimension $d \geq 2$

In this case it can be easily checked that the free-theory contribution $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ vanishes in the limit $m \rightarrow 0^+$ (see Equation (17)). Bearing this in mind, let us focus on the term $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$. Recalling the asymptotic relation (21) for \mathfrak{K}_ν , by dominated convergence from Equation (18) we infer

$$\lim_{m \rightarrow 0^+} \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 - \theta(x_1) 4b_+|x_1| \int_0^\infty dv \frac{e^{-2b_+|x_1|v}}{(v+1)^{d-1}} - \theta(-x_1) 4b_-|x_1| \int_0^\infty dv \frac{e^{-2b_-|x_1|v}}{(v+1)^{d-1}} \right].$$

To say more, via the change of variable $z = 2b_\pm|x_1|(v+1)$, the above integrals can be expressed in terms of incomplete Gamma functions $\Gamma(a, z)$ (see [57] (Equation (8.2.2))). Summing up, we ultimately obtain

$$\begin{aligned} \langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} &= \lim_{m \rightarrow 0^+} \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} \\ &= \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 - 2\theta(x_1) (2b_+|x_1|)^{d-1} e^{2b_+|x_1|} \Gamma(2-d, 2b_+|x_1|) \right. \\ &\quad \left. - 2\theta(-x_1) (2b_-|x_1|)^{d-1} e^{2b_-|x_1|} \Gamma(2-d, 2b_-|x_1|) \right]. \end{aligned} \tag{27}$$

The renormalized vacuum polarization for a massless field subject to Neumann or Dirichlet boundary conditions can be deduced from the above result evaluating the limits $b_\pm \rightarrow 0^+$ or $b_\pm \rightarrow +\infty$, respectively. To be more precise, taking into account that $\lim_{w \rightarrow 0^+} w^{1-a} e^{w} \Gamma(a, w) = 0$ and $\lim_{w \rightarrow +\infty} w^{1-a} e^{w} \Gamma(a, w) = 1$ (see [57] (Equations (8.7.6) and (8.11.2))), for Neumann (+) and Dirichlet (-) conditions we get

$$\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} = \pm \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}}.$$

The same result can be alternatively derived using (21) to compute the limit $m \rightarrow 0^+$ of Equation (19). Additionally in this case, for any $b_+, b_- \in (0, +\infty)$ the behavior of $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)}$ for $x_1 \rightarrow 0^\pm$ and $x_1 \rightarrow \pm\infty$ can be inferred from Equation (27) using the corresponding expansions for the incomplete Gamma function (see [57] (Equations (8.7.6) and (8.11.2))). To leading order, we have

$$\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} = \begin{cases} \frac{1}{8\pi|x_1|} + \mathcal{O}(\log|x_1|) & \text{for } d = 2, x_1 \rightarrow 0^\pm, \\ \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}|x_1|^{d-1}} [1 + \mathcal{O}(|x_1|)] & \text{for } d \geq 3, x_1 \rightarrow 0^\pm, \\ -\frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}|x_1|^{d-1}} [1 + \mathcal{O}(1/|x_1|)] & \text{for } x_1 \rightarrow \pm\infty. \end{cases}$$

4. Semitransparent Plane

Let us now examine configurations where the hyperplane π can be regarded as a semitransparent surface. In this connection, recalling the general arguments of Section 2 and referring again to [14] (Thm. 3.2.3), we consider the family of reduced operators labelled as follows by the elements of the unitary group $U(2)$:

$$\begin{aligned} \text{dom}(\mathcal{A}_1) &:= \left\{ \psi \in H^2(\mathbb{R} \setminus \{0\}) \mid \begin{pmatrix} \psi(0^+) \\ \psi'(0^+) \end{pmatrix} = \omega \begin{pmatrix} \alpha & \beta \\ \gamma & \zeta \end{pmatrix} \begin{pmatrix} \psi(0^-) \\ \psi'(0^-) \end{pmatrix} \right\}, \\ \mathcal{A}_1 \psi &= (-\partial_{x_1 x_1} + m^2) \psi \quad \text{in } \mathbb{R} \setminus \{0\}, \end{aligned} \tag{28}$$

where

$$\omega \in \mathbb{C} \text{ with } |\omega| = 1 \quad \text{and} \quad \alpha, \beta, \gamma, \zeta \in \mathbb{R} \text{ with } \alpha\zeta - \beta\gamma = 1. \tag{29}$$

Two distinguished one-parameter subfamilies are respectively obtained for either $\beta = 0, \gamma \in \mathbb{R}, \omega = \alpha = \zeta = 1$ or $\beta \in \mathbb{R}, \gamma = 0, \omega = \alpha = \zeta = 1$. These formally correspond to reduced operators of the form $\mathcal{A}_1 = -\partial_{x_1 x_1} + m^2 + \gamma \delta$ or $\mathcal{A}_1 = -\partial_{x_1 x_1} + m^2 + \beta \delta'$, containing the well-known distributional delta and delta-prime potentials. The other admissible choices of parameters formally correspond to mixtures of delta and delta-prime potentials concentrated at $x_1 = 0$ (see [14,63] (§3.2.4)). Let us further remark that for $\beta = \gamma = 0$ and $\omega = \alpha = \zeta = 1$ the reduced operator \mathcal{A}_1 is just the free Laplacian on the line; this case corresponds to a configuration where the quantum field does not interact with the plane π . It is worth noting that an equivalent characterization of the operator \mathcal{A}_1 defined in Equation (28) can be obtained using the general approach of [15].

For any choice of the parameters $\omega, \alpha, \beta, \gamma, \zeta$ compatible with Equation (29), the spectrum of the reduced operator \mathcal{A}_1 possesses an invariant purely absolutely continuous part; in addition to this, at most two isolated eigenvalues can appear. To be more precise, from [64] (Equation (2.13)) we infer

$$\begin{aligned} \sigma(\mathcal{A}_1) &= \sigma_{ac}(\mathcal{A}_1) \cup \sigma_p(\mathcal{A}_1), \quad \sigma_{ac}(\mathcal{A}_1) = [m^2, +\infty), \\ \sigma_p(\mathcal{A}_1) &= \begin{cases} \left\{ m^2 - \frac{\gamma^2}{(\alpha + \zeta)^2} \right\} & \text{for } \beta = 0, \gamma/(\alpha + \zeta) < 0, \\ \left\{ m^2 - \Lambda_-^2 \right\} & \text{for } \beta \neq 0, \Lambda_- < 0 \leq \Lambda_+, \\ \left\{ m^2 - \Lambda_-^2, m^2 - \Lambda_+^2 \right\} & \text{for } \beta \neq 0, \Lambda_- < \Lambda_+ < 0, \\ \emptyset & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$\Lambda_\pm := \frac{\alpha + \zeta}{2\beta} \pm \frac{\sqrt{(\alpha - \zeta)^2 + 4}}{2|\beta|}, \quad \text{for } \beta \neq 0. \tag{30}$$

Notice that for $\beta = 0$ we have $\alpha\zeta = 1$ (see Equation (31)), which grants $\alpha + \zeta \neq 0$; on the other hand the constants Λ_\pm defined in Equation (30) are well defined and finite for any $\beta \neq 0$. From the above results, by a few elementary considerations we deduce that \mathcal{A}_1 is positive if and only if one of the following two alternatives occurs, for $m > 0$:

$$\beta = 0 \text{ and } \gamma/(\alpha + \zeta) > -m \quad \text{or} \quad \beta \neq 0 \text{ and } \Lambda_+ > \Lambda_- > -m. \tag{31}$$

We shall henceforth assume the parameters $\omega, \alpha, \beta, \gamma, \zeta$ to fulfil the latter Equation (31), in addition to the conditions previously stated in Equation (29).

To proceed, let us recall that the heat kernel associated to the reduced operator \mathcal{A}_1 for $m = 0$ was formerly computed in [64]; taking into account that the addition of a mass term only produces the overall multiplicative factor $e^{-\tau m^2}$, from [64] (Equation (3.4)) (see also Equations (2.12) and (3.2) of the cited reference) we infer

$$e^{-\tau \mathcal{A}_1}(x_1, y_1) = \frac{e^{-m^2 \tau - \frac{|x_1 - y_1|^2}{4\tau}}}{\sqrt{4\pi\tau}} \begin{cases} L(x_1, y_1) e^{-\frac{(|x_1| + |y_1|)^2}{4\tau}} - \frac{\gamma}{\alpha + \zeta} (1 + L(x_1, y_1)) \int_0^\infty dw e^{-\frac{\gamma}{\alpha + \zeta} w - \frac{(w + |x_1| + |y_1|)^2}{4\tau}} & \text{for } \beta = 0, \\ \text{sgn}(x_1 y_1) e^{-\frac{(|x_1| + |y_1|)^2}{4\tau}} + \int_0^\infty dw (M_+(x_1, y_1) e^{-\Lambda_+ w} - M_-(x_1, y_1) e^{-\Lambda_- w}) e^{-\frac{(w + |x_1| + |y_1|)^2}{4\tau}} & \text{for } \beta \neq 0, \end{cases} \quad (32)$$

where we introduced the notations ($\theta(\cdot)$ is the Heaviside step function and $\text{sgn}(\cdot)$ is the sign function)

$$L(x_1, y_1) := \frac{\alpha - \zeta}{\alpha + \zeta} \text{sgn}(x_1) \theta(x_1 y_1) - \left(1 - \frac{2(\text{Re } \omega + \text{sgn}(x_1) i \text{Im } \omega)}{\alpha + \zeta}\right) \theta(-x_1 y_1),$$

$$M_\pm(x_1, y_1) := \frac{\text{sgn}(\beta)}{\sqrt{(\alpha - \zeta)^2 + 4}} \left[\theta(x_1 y_1) \left((\alpha + \zeta) \Lambda_\pm - 2\gamma - \frac{(\alpha - \zeta) \Lambda_\pm}{2} \text{sgn}(x_1) \right) - \theta(-x_1 y_1) \Lambda_\pm \left(\frac{\alpha + \zeta}{2} + \text{Re } \omega + \text{sgn}(x_1) i \text{Im } \omega \right) \right].$$

Notice in particular that, for any $x_1 \in \mathbb{R} \setminus \{0\}$, we have

$$L(x_1) \equiv L(x_1, x_1) = \frac{\alpha - \zeta}{\alpha + \zeta} \text{sgn}(x_1), \quad (33)$$

$$M_\pm(x_1) \equiv M_\pm(x_1, x_1) = \frac{\text{sgn}(\beta)}{\sqrt{(\alpha - \zeta)^2 + 4}} \left((\alpha + \zeta) \Lambda_\pm - 2\gamma - \frac{(\alpha - \zeta) \Lambda_\pm}{2} \text{sgn}(x_1) \right).$$

Let us remark that considerations similar to those reported below Equation (8) remain valid for the heat kernel in Equation (32).

Substituting the above expression for $e^{-\tau \mathcal{A}_1}(x_1, y_1)$ into Equation (5), we obtain the following integral representation for the ζ -regularized vacuum polarization:

$$\langle 0 | (\hat{\phi}^\mu(t, \mathbf{x}))^2 | 0 \rangle = \frac{\kappa^\mu}{2(4\pi)^{d/2} \Gamma(\frac{u+1}{2})} \int_0^\infty d\tau \tau^{\frac{u-d-1}{2}} e^{-m^2 \tau} \times \begin{cases} 1 + L(x_1) e^{-\frac{(x_1)^2}{\tau}} - \frac{\gamma}{\alpha + \zeta} (1 + L(x_1)) \int_0^\infty dw e^{-\frac{\gamma}{\alpha + \zeta} w - \frac{(w+2|x_1|)^2}{4\tau}} & \text{for } \beta = 0, \\ 1 + e^{-\frac{(x_1)^2}{\tau}} + \int_0^\infty dw (M_+(x_1) e^{-\Lambda_+ w} - M_-(x_1) e^{-\Lambda_- w}) e^{-\frac{(w+2|x_1|)^2}{4\tau}} & \text{for } \beta \neq 0. \end{cases} \quad (34)$$

Regarding this representation, one can make considerations analogous to those reported below Equation (9). Especially, it can be checked by direct inspection that the integrals on the right-hand side of (34) are convergent and define an analytic function of u in the complex strip $\text{Re } u > d - 1$, in agreement with the general theory.

Now, let us proceed to determine the analytic continuation of the map $u \mapsto \langle 0 | (\hat{\phi}^\mu(t, \mathbf{x}))^2 | 0 \rangle$. Using once more the identities (11) (12) introduced in the previous Section 3 and making again the change of integration variable $w = 2|x_1|v$, we obtain

$$\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle = \frac{m^{d-1} (\kappa/m)^u \Gamma\left(\frac{u-d+1}{2}\right)}{2^{d+1} \pi^{d/2} \Gamma\left(\frac{u+1}{2}\right)} + \frac{2^{\frac{u-3d+1}{2}} (\kappa|x_1|)^u}{\pi^{d/2} \Gamma\left(\frac{u+1}{2}\right) |x_1|^{d-1}} \times$$

$$\times \begin{cases} L(x_1) \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|) \\ - (1 + L(x_1)) \frac{2\gamma|x_1|}{\alpha + \zeta} \int_0^\infty dv \frac{e^{-\frac{2\gamma|x_1|}{\alpha+\zeta}v}}{(v+1)^{d-1-u}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1)), & \text{for } \beta = 0, \\ \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|) + 2M_+(x_1)|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_+|x_1|v}}{(v+1)^{d-1-u}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1)) \\ - 2M_-(x_1)|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_-|x_1|v}}{(v+1)^{d-1-u}} \mathfrak{K}_{\frac{d-1-u}{2}}(2m|x_1|(v+1)), & \text{for } \beta \neq 0, \end{cases} \quad (35)$$

where $\mathfrak{K}_\nu(w) = w^\nu K_\nu(w)$ are the functions defined in Equation (13). The same considerations reported below Equation (14) apply to the present context. As a result, Equation (35) yields the meromorphic extension of $\langle 0 | (\hat{\phi}^u(t, \mathbf{x}))^2 | 0 \rangle$ to the whole complex plane, with isolated simple pole singularities at

$$u = d - 1 - 2\ell, \quad \text{with } \ell \in \{0, 1, 2, \dots\}.$$

Special attention must be paid to the fact that the first addendum on the right-hand side of Equation (35) has a pole at $u = 0$ if the space dimension d is odd. On the contrary, all other terms in Equation (35) are analytic at $u = 0$. Taking these facts into account, we can proceed to compute the renormalized vacuum polarization using the general prescription (3):

$$\langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren} = \langle 0 | \hat{\phi}^2 | 0 \rangle_{ren}^{(free)} + \langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)}; \quad (36)$$

$$\langle 0 | \hat{\phi}^2 | 0 \rangle_{ren}^{(free)} = \begin{cases} \frac{(-1)^{\frac{d}{2}} \pi m^{d-1}}{(4\pi)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)} & \text{for } d \text{ even,} \\ \frac{(-1)^{\frac{d-1}{2}} m^{d-1}}{(4\pi)^{\frac{d+1}{2}} \Gamma\left(\frac{d+1}{2}\right)} \left[H_{\frac{d-1}{2}} + 2 \log\left(\frac{2\kappa}{m}\right) \right] & \text{for } d \text{ odd;} \end{cases} \quad (37)$$

$$\langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)} =$$

$$\begin{cases} \frac{1}{2^{\frac{3d-1}{2}} \pi^{\frac{d+1}{2}} |x_1|^{d-1}} \left[L(x_1) \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|) \right. \\ \left. - (1 + L(x_1)) \frac{2\gamma|x_1|}{\alpha + \zeta} \int_0^\infty dv \frac{e^{-\frac{2\gamma|x_1|}{\alpha+\zeta}v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \right], & \text{for } \beta = 0, \\ \frac{1}{2^{\frac{3d-1}{2}} \pi^{\frac{d+1}{2}} |x_1|^{d-1}} \left[\mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|) \right. \\ \left. + 2M_+(x_1)|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_+|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \right. \\ \left. - 2M_-(x_1)|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_-|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \right], & \text{for } \beta \neq 0. \end{cases} \quad (38)$$

Before moving on, let us remark that $\langle 0 | \hat{\phi}^2 | 0 \rangle_{ren}^{(free)}$ is exactly the same free-theory contribution arising in the case of a perfectly reflecting plane (cf. Equation (17)), while $\langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)}$ contains all the information related to the semitransparent hyperplane π . As expected, from Equation (38) it can be readily deduced that $\langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)} = 0$ when $\beta = \gamma = 0$ and $\omega = \alpha = \zeta = 1$, namely when the quantum field is not affected by the presence of the hyperplane π .

4.1. Asymptotics for $x_1 \rightarrow 0^\pm$ and $x_1 \rightarrow \pm\infty$

In order to determine the asymptotic behavior of the renormalized vacuum polarization $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ for small and large distances from the plane π , we retrace the same arguments already described in the previous Section 3.2 for the case of a perfectly reflecting plane. Additionally, in the present situation we just provide a leading order analysis, focusing primarily on the non-constant term $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ of Equation (38).

4.1.1. The Limit $x_1 \rightarrow 0^\pm$

First of all, recall the asymptotic expansions (20) and (21) for the functions \mathfrak{K}_ν . Taking these into account, regarding the integral expressions in Equation (38) we infer the following for $|x_1| \rightarrow 0$ (cf. Section 3.2.1):

$$\begin{aligned} & \frac{2\gamma|x_1|}{\alpha+\zeta} \int_0^\infty dv \frac{e^{-\frac{2\gamma|x_1|}{\alpha+\zeta}v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\ &= \frac{\gamma(2m|x_1|)^{d-1}}{(\alpha+d)m} e^{\frac{2\gamma|x_1|}{\alpha+d}} \int_{2m|x_1|}^\infty dz \frac{e^{-\frac{\gamma}{(\alpha+\zeta)m}z}}{z^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(z) = \begin{cases} \mathcal{O}(1) & \text{for } d=1, \\ \mathcal{O}(|x_1| \log|x_1|) & \text{for } d=2, \\ \mathcal{O}(|x_1|) & \text{for } d \geq 3; \end{cases} \\ & 2|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_\pm|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\ &= \frac{(2m|x_1|)^{d-1}}{m} e^{2\Lambda_\pm|x_1|} \int_{2m|x_1|}^\infty dz \frac{e^{-\frac{\Lambda_\pm}{m}z}}{z^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(z) = \begin{cases} \mathcal{O}(1) & \text{for } d=1, \\ \mathcal{O}(|x_1| \log|x_1|) & \text{for } d=2, \\ \mathcal{O}(|x_1|) & \text{for } d \geq 3. \end{cases} \end{aligned}$$

Taking the above estimates into account, from Equation (38) we infer for $x_1 \rightarrow 0^\pm$

$$\begin{aligned} & \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} \\ &= \begin{cases} -\frac{\text{sgn}(x_1)}{2\pi} \left(\frac{\alpha-\zeta}{\alpha+\zeta}\right) \log(m|x_1|) + \mathcal{O}(1) & \text{for } d=1 \text{ and } \beta=0, \\ -\frac{1}{2\pi} \log(m|x_1|) + \mathcal{O}(1) & \text{for } d=1 \text{ and } \beta \neq 0, \\ \frac{\text{sgn}(x_1)}{8\pi|x_1|} \left(\frac{\alpha-\zeta}{\alpha+\zeta}\right) + \mathcal{O}(\log|x_1|) & \text{for } d=2 \text{ and } \beta=0, \\ \frac{1}{8\pi|x_1|} + \mathcal{O}(\log|x_1|) & \text{for } d=2 \text{ and } \beta \neq 0, \\ \frac{\text{sgn}(x_1)\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}|x_1|^{d-1}} \left(\frac{\alpha-\zeta}{\alpha+\zeta}\right) [1 + \mathcal{O}(|x_1|)] & \text{for } d \geq 3 \text{ and } \beta=0, \\ \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}}|x_1|^{d-1}} [1 + \mathcal{O}(|x_1|)] & \text{for } d \geq 3 \text{ and } \beta \neq 0. \end{cases} \end{aligned} \tag{39}$$

Let us briefly comment the above results. Comparing Equations (22) and (39), it appears that $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ presents the same kind of divergence near the plane π , whether the latter be perfectly reflecting or semitransparent; as a matter of fact, the leading order terms in (22) and (39) exactly coincide for any $d \geq 1$ when $\beta \neq 0$.

On the other side, the expansions in Equation (39) call the attention to two subfamilies, parametrized by

$$\beta = 0, \quad \gamma \in \mathbb{R}, \quad \alpha = \zeta = \pm 1. \tag{40}$$

In these cases the leading order contribution vanishes identically (for any $d \geq 1$), implying that the divergence of the renormalized vacuum polarization $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ near the hyperplane π is somehow softened. While the occurrence of this phenomenon appears to be accidental for $\alpha = \zeta = -1$, some intuition can be gained instead regarding

the case with $\alpha = \zeta = +1$. We already mentioned that the subfamily with $\beta = 0, \gamma \in \mathbb{R}$ and $\alpha = \zeta = 1$ describes a delta-type potential concentrated on the hyperplane π . This is actually the “less singular” distributional potential amid the ones associated to the boundary conditions written in Equation (28). There are at least three interdependent ways to understand the latter claim:

- (i) Except for the pure delta case, all distributional potentials mentioned below Equation (28) comprise at least one derivative of the Dirac delta function (see [14,63] (§3.2.4)). It is therefore evident that, as distributions, they are more singular than the Dirac delta function itself.
- (ii) In the case of a delta potential, the field is required to be continuous across the plane π where the potential is concentrated. More precisely, in this case the functions belonging to the domain of the reduced operator \mathcal{A}_1 are continuous at $x_1 = 0$, namely $\psi(0^+) = \psi(0^-) = \psi(0)$, with discontinuous first derivative fulfilling the jump condition $\psi'(0^+) - \psi'(0^-) = \gamma \psi(0)$. In contrast, in all other cases the field exhibits a discontinuity, meaning that $\psi(0^+) \neq \psi(0^-)$.
- (iii) It is well-known that a delta potential concentrated on a surface of co-dimension 1 (such as the hyperplane π) can be approximated (in resolvent sense) by regular short-range interactions [13] (§I.3.2). In light of this, delta potentials can be reasonably regarded as a crossing point between smooth background potentials and classical hard-wall boundaries. Given that renormalized Casimir observables present no singularity when the external potentials are smooth, it is not entirely surprising that the boundary behavior is less singular in the case of delta potentials. The above line of thinking does not apply in the case of non-pure delta potentials, since the approximation of the latters by regular potentials is far more problematic [65].

On top of the above considerations, it is worth noting that the absence of the leading order divergence for $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ near the hyperplane in the case of a pure delta potential could be related to a phenomenon regarding the heat trace for the same model, namely, the vanishing of the first non-trivial coefficient in its small time asymptotic expansion [66,67] (I thank again one of the anonymous referees for indicating this connection).

Let us finally recognize that, whenever the leading order terms in the asymptotic expansions (39) vanish, the study of the sub-leading contributions becomes crucial. A detailed investigation of this subject is deferred to future works.

4.1.2. The Limit $x_1 \rightarrow \pm\infty$

In this paragraph we proceed to examine the behavior of the renormalized vacuum polarization in the regime of large distances from the hyperplane π . Recalling once more that the term $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ is constant, we restrict our analysis to the expression $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$.

Firstly, recall the asymptotic expansion (23) for the functions \mathfrak{K}_ν . Then, making the change of variable $z = |x_1|v$, we derive the following expansions of the integral expressions in Equation (38) for $|x_1| \rightarrow +\infty$:

$$\begin{aligned}
 & \frac{2\gamma|x_1|}{\alpha+\zeta} \int_0^\infty dv \frac{e^{-\frac{2\gamma|x_1|}{\alpha+\zeta}v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\
 &= \frac{\sqrt{2\pi}\gamma}{\alpha+\zeta} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \int_0^\infty dz e^{-2(\frac{\gamma}{\alpha+\zeta}+m)z} \left(1 + \mathcal{O}(z/|x_1|)\right) \\
 &= \sqrt{\frac{\pi}{2}} \frac{\frac{\gamma}{\alpha+\zeta}}{\frac{\gamma}{\alpha+\zeta}+m} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \left(1 + \mathcal{O}(1/|x_1|)\right); \\
 & 2|x_1| \int_0^\infty dv \frac{e^{-2\Lambda_\pm|x_1|v}}{(v+1)^{d-1}} \mathfrak{K}_{\frac{d-1}{2}}(2m|x_1|(v+1)) \\
 &= \sqrt{2\pi} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \int_0^\infty dz e^{-2(\Lambda_\pm+m)z} \left(1 + \mathcal{O}(z/|x_1|)\right) \\
 &= \sqrt{\frac{\pi}{2}} \frac{1}{\Lambda_\pm+m} e^{-2m|x_1|} (2m|x_1|)^{\frac{d-2}{2}} \left(1 + \mathcal{O}(1/|x_1|)\right).
 \end{aligned}$$

From here and from Equation (38), in the limit $x_1 \rightarrow \pm\infty$ we infer

$$\begin{aligned}
 \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} &= \frac{m^{\frac{d-2}{2}}}{2(4\pi)^{d/2}} \frac{e^{-2m|x_1|}}{|x_1|^{d/2}} \times \\
 & \begin{cases} \frac{(\alpha-\zeta)m \operatorname{sgn}(x_1) - \gamma}{(\alpha+\zeta)m + \gamma} \left[1 + \mathcal{O}\left(\frac{1}{|x_1|}\right)\right] & \text{for } \beta = 0, \\ \frac{6\gamma + 2\beta m^2 + 4(\alpha+\zeta)m - (\alpha-\zeta)m \operatorname{sgn}(x_1)}{2(\gamma + \beta m^2 + (\alpha+\zeta)m)} \left[1 + \mathcal{O}\left(\frac{1}{|x_1|}\right)\right] & \text{for } \beta \neq 0. \end{cases} \quad (41)
 \end{aligned}$$

The above relations show that also in the case of a semitransparent plane the renormalized expectation $\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)}$ decays exponentially fast far away from the hyperplane π . In other words, the difference between the full vacuum polarization $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}$ and the constant free-theory term $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ becomes exponentially small.

4.2. Vacuum Polarization for a Massless Field

We now examine the renormalized vacuum polarization for a massless field in presence of a semitransparent hyperplane. Making reference to Equations (28) and (31), for a sensible quantum field theory we must require either

$$\beta = 0 \text{ and } \gamma/(\alpha + \zeta) \geq 0 \quad \text{or} \quad \beta \neq 0 \text{ and } \Lambda_+ > \Lambda_- \geq 0. \quad (42)$$

In accordance with the general arguments of Section 2 (see, especially, Equation (4)), we proceed to determine the renormalized observable of interest evaluating the zero-mass limit $m \rightarrow 0^+$ of the analogous quantity in the massive theory.

Similarly to the configuration with a perfectly reflecting surface, the cases with space dimension $d = 1$ and $d \geq 2$ need to be analyzed separately.

4.2.1. Space Dimension $d = 1$

A careful analysis is demanded for this specific model, due to the emergence of the same infrared pathologies already discussed in Section 3.3.1. Indeed, let us recall that the renormalized vacuum polarization comprises a free theory contribution which is divergent in the limit $m \rightarrow 0^+$ (see Equation (25)):

$$\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)} = \frac{1}{2\pi} \log\left(\frac{2\kappa}{m}\right).$$

On the other hand, by arguments similar to those described in Section 3.3.1, from Equation (18) we obtain the following asymptotic expansions for $m \rightarrow 0^+$, involving the incomplete Gamma function $\Gamma(a, z)$:

$$\langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} = \begin{cases} \frac{1}{2\pi} \left[\log(m|x_1|) - \gamma_{EM} + (1 + L(x_1)) e^{\frac{2\gamma|x_1|}{\alpha+\zeta}} \Gamma\left(0, \frac{2\gamma|x_1|}{\alpha+\zeta}\right) \right] \\ \quad + \mathcal{O}\left((m|x_1|)^2 \log(m|x_1|)\right) & \text{for } \beta = 0, \\ \frac{1}{2\pi} \left[-3 \log(m|x_1|) + 3\gamma_{EM} \right. \\ \quad \left. - \frac{M_+(x_1)}{\Lambda_+} e^{2\Lambda_+|x_1|} \Gamma(0, 2\Lambda_+|x_1|) + \frac{M_-(x_1)}{\Lambda_-} e^{2\Lambda_-|x_1|} \Gamma(0, 2\Lambda_-|x_1|) \right] \\ \quad + \mathcal{O}\left((m|x_1|)^2 \log(m|x_1|)\right) & \text{for } \beta \neq 0. \end{cases} \tag{43}$$

Here we also used the following basic identity, which can be easily deduced from Equations (30) and (33):

$$1 + \frac{M_+(x_1)}{\Lambda_+} - \frac{M_-(x_1)}{\Lambda_-} = 3.$$

For $\beta \neq 0$, the above relations together with Equation (36) make evident that

$$\lim_{m \rightarrow 0^+} \langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren} = \lim_{m \rightarrow 0^+} \left[\frac{1}{2\pi} \log\left(\frac{2\kappa}{m^4|x_1|^3}\right) + \mathcal{O}(1) \right] = +\infty,$$

indicating that in this case the renormalized expectation $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{massless}$ is irredeemably divergent in the infrared for a massless field. Before we proceed, let us point out a connection between this result and the similar conclusions drawn in Section 3.3.1 for the case of Neumann conditions on the hyperplane π : Neumann conditions are formally recovered in the present scenario as the limit case where $\beta \rightarrow \infty$, with $\gamma = 0, \omega = \alpha = \zeta = 1$.

Let us henceforth assume

$$\beta = 0.$$

Under this condition, from the above results and from Equation (4) we readily infer

$$\begin{aligned} \langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} &= \lim_{m \rightarrow 0^+} \left[\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)} + \langle 0|\hat{\phi}^2(x_1)|0\rangle_{ren}^{(plane)} \right] \\ &= \frac{1}{2\pi} \left[\log(2\kappa|x_1|) - \gamma_{EM} + \left(1 + \frac{\alpha - \zeta}{\alpha + \zeta} \operatorname{sgn}(x_1) \right) e^{\frac{2\gamma|x_1|}{\alpha+\zeta}} \Gamma\left(0, \frac{2\gamma|x_1|}{\alpha+\zeta}\right) \right]. \end{aligned} \tag{44}$$

Using again known properties of the incomplete Gamma function, it is easy to derive the following leading order asymptotic expansions of the the above expression:

$$\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)} = \begin{cases} \mp \frac{1}{2\pi} \left(\frac{\alpha - \zeta}{\alpha + \zeta} \right) \log\left(\frac{\gamma|x_1|}{\alpha + \zeta}\right) + \mathcal{O}(1) & \text{for } x_1 \rightarrow 0^\pm, \\ \frac{1}{2\pi} \log(\kappa|x_1|) + \mathcal{O}(1) & \text{for } x_1 \rightarrow \pm\infty. \end{cases}$$

It is remarkable that the renormalized vacuum polarization $\langle 0|\hat{\phi}^2(t, \mathbf{x})|0\rangle_{ren}^{(massless)}$ remains finite in the limit $x_1 \rightarrow 0^\pm$ when $\beta = 0, \gamma \in \mathbb{R}$ and $\alpha = \zeta = \pm 1$. Recall that the very same phenomenon occurs in the case of a massive field (see Section 4.1.1 and the comments reported therein).

4.2.2. Space Dimension $d \geq 2$

Recall that for $d \geq 2$ the free-theory contribution $\langle 0|\hat{\phi}^2|0\rangle_{ren}^{(free)}$ vanishes in the limit $m \rightarrow 0^+$ (see Equation (17)). Taking this into account, by arguments analogous to those described in Section 3.3.2, from Equation (38) we get

$$\begin{aligned}
 \langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren}^{(massless)} &= \lim_{m \rightarrow 0^+} \langle 0 | \hat{\phi}^2(x_1) | 0 \rangle_{ren}^{(plane)} \\
 &= \begin{cases} \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[L(x_1) - (1 + L(x_1)) e^{\frac{2\gamma|x_1|}{\alpha+\zeta}} \left(\frac{2\gamma|x_1|}{\alpha+\zeta} \right)^{d-1} \Gamma\left(2-d, \frac{2\gamma|x_1|}{\alpha+\zeta}\right) \right] & \text{for } \beta = 0, \\ \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 + \frac{M_+(x_1)}{\Lambda_+} e^{2\Lambda_+|x_1|} (2\Lambda_+|x_1|)^{d-1} \Gamma(2-d, 2\Lambda_+|x_1|) \right. \\ \qquad \qquad \qquad \left. - \frac{M_-(x_1)}{\Lambda_-} e^{2\Lambda_-|x_1|} (2\Lambda_-|x_1|)^{d-1} \Gamma(2-d, 2\Lambda_-|x_1|) \right] & \text{for } \beta \neq 0. \end{cases} \tag{45}
 \end{aligned}$$

Using once more the known expansions of the incomplete Gamma function for small and large values of the argument, we derive the following leading order asymptotics:

$$\begin{aligned}
 \langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren}^{(massless)} &= \begin{cases} \pm \left(\frac{\alpha - \zeta}{\alpha + \zeta} \right) \frac{1}{8\pi |x_1|} + \mathcal{O}(\log |x_1|) & \text{for } \beta = 0, d = 2 \text{ and } x_1 \rightarrow 0^\pm, \\ \pm \left(\frac{\alpha - \zeta}{\alpha + \zeta} \right) \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 + \mathcal{O}(|x_1|) \right] & \text{for } \beta = 0, d \geq 3 \text{ and } x_1 \rightarrow 0^\pm, \\ - \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 + \mathcal{O}(1/|x_1|) \right] & \text{for } \beta = 0, d \geq 2 \text{ and } x_1 \rightarrow \pm\infty; \\ \frac{1}{8\pi |x_1|} + \mathcal{O}(\log |x_1|) & \text{for } \beta \neq 0, d = 2 \text{ and } x_1 \rightarrow 0^\pm, \\ \frac{\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 + \mathcal{O}(|x_1|) \right] & \text{for } \beta \neq 0, d \geq 3 \text{ and } x_1 \rightarrow 0^\pm, \\ \frac{3\Gamma(\frac{d-1}{2})}{(4\pi)^{\frac{d+1}{2}} |x_1|^{d-1}} \left[1 + \mathcal{O}(1/|x_1|) \right] & \text{for } \beta \neq 0, d \geq 2 \text{ and } x_1 \rightarrow \pm\infty. \end{cases} \tag{46}
 \end{aligned}$$

Notice that also in this case the local divergences of the renormalized polarization $\langle 0 | \hat{\phi}^2(t, \mathbf{x}) | 0 \rangle_{ren}^{(massless)}$ near the hyperplane π are softened for $\beta = 0, \gamma \in \mathbb{R}$ and $a = d = \pm 1$; again, we refer to the analysis of Section 4.1.1.

5. Conclusions

In this paper we investigated the quantum vacuum fluctuations of a scalar field in presence of a flat hyperplane of co-dimension 1 in $(d + 1)$ -dimensional Minkowski spacetime. We analyzed this configuration assuming the hyperplane to be either a perfectly reflecting surface or, in alternative, a semitransparent one. To this purpose, we made reference to the most general, local and homogeneous boundary conditions for the field which are compatible with the unitarity of the quantum theory.

For both the qualitatively distinct scenarios mentioned above, we firstly derived an explicit expression for the renormalized vacuum polarization in the case of a massive field. This result was obtained using known expressions of the corresponding heat kernels and implementing a local version of the zeta regularization approach, developed in previous works. We subsequently discussed the same configurations for a massless field by a natural limiting procedure, paying due attention to some infrared pathologies arising in space dimension $d = 1$.

In all cases, we determined the asymptotic behavior (to leading order) of the vacuum polarization for both large and small distances from the hyperplane. In this regard, we distinguished two contributions: one is ascribable to the free theory, being a constant determined solely by the mass of the field and by the space dimension; the other truly accounts for the presence of the hyperplane, given that it depends on the distance from the hyperplane itself and on the parameters fixing the boundary conditions. We showed that the latter hyperplane contribution decays exponentially (resp. polynomially) with the distance from the hyperplane for a massive (resp. massless) field, regardless of the specific boundary conditions. Having said that, the main object of interest were in fact

the divergences occurring near the hyperplane. In this connection, we found that the leading order divergence of the vacuum polarization is actually independent of the specific boundary conditions for a perfectly reflecting surface. On the contrary, in the case of a semitransparent hyperplane, described by a generic mixture of δ - δ' potentials, the leading order in the asymptotic expansion for small distances of the vacuum polarization does depend on the parameters fixing the boundary conditions. Notably, the boundary divergence is softened (meaning that the coefficient of the leading order divergence vanishes identically) in the case of a pure δ -type potential.

Finally, let us mention that the study of other local observables, such as the energy density and the pressure components of the stress-energy tensor, for the same models considered in this paper are currently under investigation and will be discussed in a future work.

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Appendix A. The Heat Kernel for a Perfectly Reflecting Plane

In this appendix we report more details about the derivation of Equation (8), regarding the heat kernel $e^{-\tau \mathcal{A}_1}(x_1, y_1)$ associated to the operator \mathcal{A}_1 defined in Equation (6). For the sake of presentation, hereafter we restrict the attention to the massless case, fixing

$$m = 0. \tag{A1}$$

Of course this implies no loss of generality, since the heat kernel for $m > 0$ can always be recovered by an elementary shift:

$$e^{-\tau \mathcal{A}_1}(x_1, y_1) \Big|_{m>0} = e^{-m^2 \tau} \left(e^{-\tau \mathcal{A}_1}(x_1, y_1) \Big|_{m=0} \right). \tag{A2}$$

As far as the present analysis is concerned, it is further convenient to refer to the decomposition $L^2(\mathbb{R}) \equiv L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$ (where $\mathbb{R}_+ \equiv (0, +\infty)$ and $\mathbb{R}_- \equiv (-\infty, 0)$) and to consider the equivalent characterization of \mathcal{A}_1 given by (cf. (6))

$$\begin{aligned} \mathcal{A}_1 &= A_+ \oplus A_-, \\ \text{dom}(A_\pm) &:= \{ \psi_\pm \in H^2(\mathbb{R}_\pm) \mid \mp \psi'_\pm(0^\pm) + b_\pm \psi_\pm(0^\pm) = 0 \}, \\ A_\pm \psi_\pm &= -\psi''_\pm \quad \text{in } \mathbb{R}_\pm. \end{aligned} \tag{A3}$$

Against this background, the heat kernel can be expressed as

$$e^{-\tau \mathcal{A}_1}(x_1, y_1) = \begin{cases} e^{-\tau A_+}(x_1, y_1) & \text{for } x_1, y_1 > 0, \\ e^{-\tau A_-}(x_1, y_1) & \text{for } x_1, y_1 < 0. \end{cases} \tag{A4}$$

Furthermore, the integral kernels $e^{-\tau A_+}(x_1, y_1)$ on \mathbb{R}_+ and $e^{-\tau A_-}(x_1, y_1)$ on \mathbb{R}_- are related as follows by an elementary reflection argument:

$$e^{-\tau A_-}(x_1, y_1) = e^{-\tau A_+}(-x_1, -y_1) \Big|_{b_+ = b_-} \quad \text{for } x_1, y_1 < 0. \tag{A5}$$

Before proceeding, let us point out that an explicit expression for the heat kernel on the half-line with Robin boundary conditions, namely $e^{-\tau A_+}(x_1, y_1)$, was previously derived in [68] for $b_+ > 0$, via an elegant technique allowing to translate Robin problems into Dirichlet ones and vice versa. However, a few variations of this approach are required when $b_+ < 0$, in order to include the discrete spectrum contribution arising in this case.

Here we prefer to present a computation of $e^{-\tau A_+}(x_1, y_1)$ for any $b_+ \in \mathbb{R}$ (including $b = +\infty$ as a limit case), starting from its eigenfunction expansion (an approach in fact also hinted at in [68]). To this purpose, we firstly notice that the spectrum of the operator A_+ defined in Equation (A3) is

$$\begin{aligned} \sigma(A_+) &= \sigma_{ac}(A_+) \cup \sigma_p(A_+); \\ \sigma_{ac}(A_+) &= [0, +\infty), \quad \sigma_p(A_+) = \begin{cases} \emptyset & \text{for } b_+ \in [0, +\infty) \cup \{+\infty\}, \\ \{-b_+^2\} & \text{for } b_+ \in (-\infty, 0). \end{cases} \end{aligned}$$

Accordingly, we refer to the Hilbert space decomposition $L^2(\mathbb{R}_+) = L^2_{ac}(\mathbb{R}_+) \oplus L^2_p(\mathbb{R}_+)$, involving the subspace $L^2_{ac}(\mathbb{R}_+)$ of absolute continuity for A_+ and its orthogonal complement $L^2_p(\mathbb{R}_+)$. A complete set of generalized eigenfunctions $\{\psi_k\}_{k \in (0, +\infty)}$ spanning $L^2_{ac}(\mathbb{R}_+)$ and the normalized eigenfunction $\psi_{b_+} \in L^2(\mathbb{R}_+)$ associated to the possible negative eigenvalue are given by [68] (Equation (3.15))

$$\begin{aligned} \psi_k(x_1) &:= \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{k^2 + b_+^2}} \left(k \cos(kx_1) + b_+ \sin(kx_1) \right) \quad (k > 0); \\ \psi_{b_+}(x_1) &:= \sqrt{2|b_+|} e^{-|b_+|x_1} \quad \text{for } b_+ < 0. \end{aligned}$$

In light of the above considerations, we obtain the eigenfunction expansion (for $\tau > 0$ and $x_1, y_1 > 0$)

$$\begin{aligned} e^{-\tau A_+}(x_1, y_1) &= \int_0^\infty dk e^{-\tau k^2} \psi_k(x_1) \psi_k(y_1) + \theta(-b_+) e^{-\tau(-b_+^2)} \psi_{b_+}(x_1) \psi_{b_+}(y_1) \\ &= \frac{2}{\pi} \int_0^\infty dk e^{-\tau k^2} \frac{1}{k^2 + b_+^2} \left(k \cos(kx_1) + b_+ \sin(kx_1) \right) \left(k \cos(ky_1) + b_+ \sin(ky_1) \right) \\ &\quad + \theta(-b_+) 2|b_+| e^{\tau b_+^2 - |b_+|(x_1 + y_1)}. \end{aligned}$$

By simple trigonometric identities and using [69] (Equations (3.954)), from here it follows

$$\begin{aligned}
 e^{-\tau A_+}(x_1, y_1) &= \frac{1}{\pi} \int_0^\infty dk e^{-\tau k^2} \left[\cos(k(x_1 - y_1)) + \cos(k(x_1 + y_1)) \right. \\
 &\quad \left. - \frac{2b_+^2}{k^2 + b_+^2} \cos(k(x_1 + y_1)) + \frac{2kb_+}{k^2 + b_+^2} \sin(k(x_1 + y_1)) \right] \\
 &\quad + \theta(-b_+) 2|b_+| e^{\tau b_+^2 - |b_+|(x_1 + y_1)}. \\
 &= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} + \theta(-b_+) 2|b_+| e^{\tau b_+^2 - |b_+|(x_1 + y_1)} \\
 &\quad - \frac{|b_+|}{2} e^{\tau b_+^2} \left[2 \cosh(|b_+|(x_1 + y_1)) - e^{-|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} - \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right. \\
 &\quad \left. - e^{|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} + \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right] \\
 &\quad - \frac{b_+}{2} e^{\tau b_+^2} \left[2 \sinh(|b_+|(x_1 + y_1)) + e^{-|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} - \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right. \\
 &\quad \left. - e^{|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} + \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right].
 \end{aligned}$$

where $\operatorname{erf}(\cdot)$ indicates the usual error function [57] (§7). Performing a few elementary manipulations and using the basic identity $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ [57] (Equation (7.4.1)), we obtain

$$\begin{aligned}
 e^{-\tau A_+}(x_1, y_1) &= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} - b_+ e^{\tau b_+^2 + b_+(x_1 + y_1)} \\
 &\quad + b_+ e^{\tau b_+^2} \left[-\theta(-b_+) e^{-|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} - \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right. \\
 &\quad \left. + \theta(b_+) e^{|b_+|(x_1 + y_1)} \operatorname{erf}\left(|b_+|\sqrt{\tau} + \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right] \\
 &= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} \\
 &\quad - b_+ e^{\tau b_+^2 + b_+(x_1 + y_1)} \left[1 - \operatorname{erf}\left(b_+\sqrt{\tau} + \frac{x_1 + y_1}{2\sqrt{\tau}}\right) \right].
 \end{aligned}$$

On account of [57] (Equation (7.2.2)) and via a basic change of the integration variable, from the latter result we get

$$\begin{aligned}
 e^{-\tau A_+}(x_1, y_1) &= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} \\
 &\quad - \frac{2b_+}{\sqrt{\pi}} e^{\tau b_+^2 + b_+(x_1 + y_1)} \int_{b_+\sqrt{\tau} + \frac{x_1 + y_1}{2\sqrt{\tau}}}^\infty dz e^{-z^2} \\
 &= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} - \frac{b_+}{\sqrt{\pi\tau}} \int_0^\infty dw e^{-b_+w - \frac{(w + x_1 + y_1)^2}{4\tau}}.
 \end{aligned} \tag{A6}$$

Together with the preceding Equations (A2), (A4) and (A5), this suffices to prove Equation (8) in the main text.

Let us finally comment on the limit case $b_+ = +\infty$. Setting $p := b_+w$ (with $b_+ > 0$ finite) in the final identity of Equation (A6), we get

$$e^{-\tau A_+}(x_1, y_1) = \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 - y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1 + y_1|^2}{4\tau}} - \frac{1}{\sqrt{\pi\tau}} \int_0^\infty dp e^{-p - \frac{(p/b_+ + x_1 + y_1)^2}{4\tau}}.$$

Starting from here, by the dominated convergence theorem we obtain

$$\begin{aligned}
& \lim_{b_+ \rightarrow +\infty} e^{-\tau A_+}(x_1, y_1) \\
&= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1-y_1|^2}{4\tau}} + \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1+y_1|^2}{4\tau}} - \frac{2}{\sqrt{4\pi\tau}} e^{-\frac{|x_1+y_1|^2}{4\tau}} \int_0^\infty dp e^{-p} \\
&= \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1-y_1|^2}{4\tau}} - \frac{1}{\sqrt{4\pi\tau}} e^{-\frac{|x_1+y_1|^2}{4\tau}},
\end{aligned}$$

which reproduces as expected the heat kernel on the half-line for Dirichlet boundary conditions (cf. the corresponding considerations reported in Section 3).

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