



The semiclassical limit on a star-graph with Kirchhoff conditions

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Abstract

We consider the dynamics of a quantum particle of mass m on a n -edges star-graph with Hamiltonian $H_K = -(2m)^{-1}\hbar^2\Delta$ and Kirchhoff conditions in the vertex. We describe the semiclassical limit of the quantum evolution of an initial state supported on one of the edges and close to a Gaussian coherent state. We define the limiting classical dynamics through a Liouville operator on the graph, obtained by means of Kreĭn's theory of singular perturbations of self-adjoint operators. For the same class of initial states, we study the semiclassical limit of the wave and scattering operators for the couple (H_K, H_D^\oplus) , where H_D^\oplus is the Hamiltonian with Dirichlet conditions in the vertex.

Keywords Semiclassical dynamics · Quantum graphs · Coherent states · Scattering theory

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1 Introduction

Aim of this work is to provide the semiclassical dynamics and scattering for an approximate coherent state propagating freely on a star-graph, in the presence of Kirchhoff conditions in the vertex.

Since the pioneering work of Kottos and Smilansky [17], having in mind applications to quantum chaos, the semiclassical limit of quantum graphs is often understood as the study of the distribution of eigenvalues (or resonances, see [18]) of self-adjoint realizations of $-(2m)^{-1}\hbar^2\Delta$ on the graph.

To the best of our knowledge, a first study of the semiclassical limit for quantum dynamics on graphs is due to Barra and Gaspard [2] (see also [3], where the limiting classical model is comprehensively discussed). In this case, the semiclassical limit is understood in terms of the convergence of a Wigner-like function for graphs when \hbar (the reduced Planck constant) goes to zero.

Inspired by the work of Hagedorn [12], instead, we look directly at the dynamics of the wave-function, for a class of initial states which are close to Gaussian coherent states supported on one of the edges of the graph.

Closely related to our work is a series of papers by Chernyshev and Shafarevich [6, 8, 9] in which the authors study the $\hbar \rightarrow 0$ limit of Gaussian wave packets propagating on graphs. Their main interest is the asymptotic growth (for large times) of the number of wave packets propagating on the graph. The main tool for the analysis is the complex WKB method by Maslov (see [19]). We also point out the work [7], by the same authors, in which they study the small \hbar asymptotics of the eigenvalues of Schrödinger operators on quantum graphs (with Kirchhoff conditions in the vertices and in the presence of potential terms).

In our previous work [5] we studied the semiclassical limit in the presence of a singular potential. Specifically, we considered the operator H_α , which is the quantum Hamiltonian in $L^2(\mathbb{R})$ formally written as $H_\alpha = -\frac{\hbar^2}{2m}\Delta + \alpha\delta_0$, where m is the mass of the particle, δ_0 is the Dirac-delta distribution centered in $x = 0$, and α is a real constant measuring the strength of the potential. Given a Gaussian coherent state on the real line of the form

$$\Psi_{\sigma,\xi}^{\hbar}(x) := \frac{1}{(2\pi\hbar)^{1/4}\sqrt{\sigma}} \exp\left(-\frac{1}{4\hbar\sigma_0\sigma}(x-q)^2 + \frac{i}{\hbar}p(x-q)\right) \quad \text{for } x \in \mathbb{R}, \quad (1.1)$$

with $\sigma \in \mathbb{C}$, $\text{Re } \sigma = \sigma_0 > 0$ and $\xi \equiv (q, p) \in \mathbb{R}^2$, we studied the limit $\hbar \rightarrow 0$ of $e^{-i\frac{t}{\hbar}H_\alpha}\Psi_{\sigma_0,\xi}^{\hbar}$.

To this aim we reasoned as follows. For fixed $x \in \mathbb{R}$, consider the classical wave function defined by

$$\phi_{\sigma,x}^{\hbar} : \mathbb{R}^2 \rightarrow \mathbb{C}, \quad \phi_{\sigma,x}^{\hbar}(\xi) := \Psi_{\sigma,\xi}^{\hbar}(x).$$

Consider the vector field $X_0(q, p) = (p/m, 0)$ associated with the free classical Hamiltonian $h_0(q, p) = p^2/(2m)$ (q is the position and p the momentum of the

classical particle of mass m), and the Liouville operator

$$\text{dom}(L) := C_0^\infty(\mathbb{R}^2), \quad L := -i X_0 \cdot \nabla = -i \frac{p}{m} \frac{\partial}{\partial q}.$$

Set

$$\xi_t := \left(q + \frac{pt}{m}, p \right), \quad A_t := \frac{p^2 t}{2m}, \quad \sigma_t := \sigma_0 + \frac{it}{2m\sigma_0},$$

where ξ_t is the solution of the free Hamilton equations, A_t is the (free) classical action, and σ_t takes into account the spreading of the wave function.

If the dynamics is free (i.e., $\alpha = 0$), one has the identity

$$(e^{-i \frac{t}{\hbar} H_0} \Psi_{\sigma_0, \xi}^\hbar)(x) = e^{\frac{i}{\hbar} A_t} \Psi_{\sigma_t, \xi_t}^\hbar(x).$$

The latter can be rewritten as

$$(e^{-i \frac{t}{\hbar} H_0} \Psi_{\sigma_0, \xi}^\hbar)(x) = e^{\frac{i}{\hbar} A_t} (e^{itL_0} \phi_{\sigma_t, x}^\hbar)(\xi), \tag{1.2}$$

where e^{itL_0} is the realization in $L^\infty(\mathbb{R}^2)$ of the strongly continuous (in $L^2(\mathbb{R}^2)$) group of evolution generated by the self-adjoint operator $L_0 = \overline{L}$; explicitly, one has $e^{itL_0} f(\xi) = f(\xi_t)$.

Since H_α is a self-adjoint extension of $H_0^\circ := H_0 \upharpoonright C_c^\infty(\mathbb{R} \setminus \{0\})$, mimicking the identity (1.2), we compared $(e^{-i \frac{t}{\hbar} H_\alpha} \Psi_{\sigma_0, \xi}^\hbar)(x)$ with $e^{\frac{i}{\hbar} A_t} (e^{itL_\beta} \phi_{\sigma_t, x}^\hbar)(\xi)$, with L_β a self-adjoint extension of $L_0^\circ := L_0 \upharpoonright C_c^\infty(\mathcal{M}_0)$, $\mathcal{M}_0 := \mathbb{R}^2 \setminus \{(0, p) \mid p \in \mathbb{R}\}$. Here, β is a real constant which parameterizes the self-adjoint extension, and it turns out that the optimal choice is $\beta = 2\alpha/\hbar$ (see [5] for the details).

In the same spirit, in the present work, we study the small \hbar asymptotic of $e^{-i \frac{t}{\hbar} H_K} \Xi_{\sigma_0, \xi}^\hbar$ where H_K is the quantum Hamiltonian defined as the self-adjoint realization of $-(2m)^{-1} \hbar^2 \Delta$ on the star-graph with Kirchhoff conditions in the vertex, and $\Xi_{\sigma, \xi}^\hbar$ resembles a coherent state concentrated on one edge of the graph (see Sect. 1.3 below for the precise definition).

The paper is structured as follows. In the remaining sections of the introduction we give the main definitions and results. Sections 2 and 3 contain a detailed description of the quantum and semiclassical dynamics on the star-graph respectively. In Sect. 4 we give the proofs of Theorems 1.4 and 1.6. Section 5 contains some additional remarks and comments. The paper is concluded by an ‘‘Appendix A’’ in which we present a proof of a technical result, namely an explicit formula for the wave operators for the pair (Dirichlet Laplacian, Neumann Laplacian) on the half-line.

1.1 Quantum dynamics on the star-graph

By star-graph we mean a non-compact graph, with n edges (or leads) and one vertex. Each edge can be identified with a half-line, the origins of the half-lines coincide and identify the only vertex of the graph.

We recall that the Hilbert space associated with the star-graph is $L^2(\mathcal{G}) \equiv \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+)$, with the natural scalar product and norm; in particular, for the L^2 -norm we use the notation

$$\|\Psi\|_{L^2(\mathcal{G})} \equiv \left(\int_{\mathcal{G}} dx |\Psi(x)|^2 \right)^{1/2} := \left(\sum_{\ell=1}^n \int_0^\infty dx |\Psi_\ell(x)|^2 \right)^{1/2}.$$

If $\Psi \in L^2(\mathcal{G})$, $\Psi_\ell \in L^2(\mathbb{R}_+)$ is its ℓ -th component with respect to the decomposition $\bigoplus_{\ell=1}^n L^2(\mathbb{R}_+)$. In a similar way one can define the associated Sobolev spaces; in particular, we set $H^2(\mathcal{G}) \equiv \bigoplus_{\ell=1}^n H^2(\mathbb{R}_+)$, with the natural scalar product and norm.

We are primarily interested in the semiclassical limit of the quantum dynamics generated by the Kirchhoff Laplacian on the star-graph, which is the operator

$$\begin{aligned} \text{dom}(H_K) &:= \left\{ \Psi \in H^2(\mathcal{G}) \mid \psi_1(0) = \dots = \psi_n(0), \sum_{\ell=1}^n \psi'_\ell(0) = 0 \right\}, \quad (1.3) \\ H_K \Psi &:= -\frac{\hbar^2}{2m} \Psi''; \quad (1.4) \end{aligned}$$

here Ψ'' denotes the element of $L^2(\mathcal{G})$ with components Ψ''_ℓ , and $\Psi(0)$ (resp., $\Psi'(0)$) the vector in \mathbb{C}^n with components $\psi_\ell(0)$ (resp., $\psi'_\ell(0)$). Functions in $\text{dom}(H_K)$ are said to satisfy Kirchhoff (also called Neumann, or standard, or natural) boundary conditions.

In the analysis of the semiclassical limit of the wave and scattering operators, we will have to fix a reference dynamics on the star-graph. To this aim we will consider the operator H_D^\oplus [see also the equivalent definition in Eqs. (2.5)–(2.6) below]

$$\begin{aligned} \text{dom}(H_D^\oplus) &:= \left\{ \Psi \in H^2(\mathcal{G}) \mid \psi_1(0) = \dots = \psi_n(0) = 0 \right\}, \\ H_D^\oplus \Psi &:= -\frac{\hbar^2}{2m} \Psi''; \end{aligned}$$

we remark that H_D^\oplus can be understood as the direct sum of n copies of the Dirichlet Hamiltonian on the half-line (see Sect. 2.2 below for further details).

We recall that the quantum wave operators and the corresponding scattering operator on $L^2(\mathcal{G})$, are defined by

$$\Omega^\pm := \text{s-lim}_{t \rightarrow \pm\infty} e^{i\frac{t}{\hbar} H_K} e^{-i\frac{t}{\hbar} H_D^\oplus}, \quad (1.5)$$

$$S := (\Omega^+)^* \Omega^-. \quad (1.6)$$

These operators can be computed explicitly (see Proposition 2.2 and Remark 2.3 below), and component-wisely for $\ell = 1, \dots, n$ they read as follows:

$$(\Omega^\pm \Psi)_\ell = \sum_{\ell'=1}^n \left(\delta_{\ell,\ell'} - \frac{1}{n} (1 \mp \mathcal{F}_c^* \mathcal{F}_s) \right) \Psi_{\ell'},$$

where $\delta_{\ell,\ell'}$ is the Kronecker delta, \mathcal{F}_s and \mathcal{F}_c are the Fourier-sine and Fourier-cosine transforms respectively [see Eqs. (2.11) and (2.12)];

$$(S \Psi)_\ell = \sum_{\ell'=1}^n \left(\delta_{\ell,\ell'} - \frac{2}{n} \right) \Psi_{\ell'}. \tag{1.7}$$

1.2 Semiclassical dynamics on a star-graph

The generator of the semiclassical dynamics on the star-graph is obtained as a self-adjoint realization of the differential operator $-i \frac{p}{m} \frac{\partial}{\partial q}$ in $\oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$, $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$. To recover it we will make use of the method to classify the singular perturbations of self-adjoint operators developed by one of us in [20] (see also [21]).

To do so, the first step is to identify a simple dynamics on the star-graph, more precisely its generator. We shall consider classical particles moving on the edges of the graph with elastic collision at the vertex.

We start by considering the dynamics of a classical particle on the half-line with elastic collision at the origin. We obtain its generator as a limiting case from our previous work [5] and denote it by L_D . We postpone the precise definition of L_D to Sect. 3.1. Here we just note few facts.

$L_D : \text{dom}(L_D) \subset L^2(\mathbb{R}_+ \times \mathbb{R}) \rightarrow L^2(\mathbb{R}_+ \times \mathbb{R})$ is self-adjoint and acts on elements of its domain as

$$(L_D f)(q, p) = -i \frac{p}{m} \frac{\partial f}{\partial q}(q, p) \quad \text{for } (q, p) \in \mathbb{R}_+ \times \mathbb{R}.$$

For all $t \in \mathbb{R}$, the action of the unitary evolution group associated with it is explicitly given by

$$(e^{itL_D} f)(q, p) = \begin{cases} f(q + \frac{pt}{m}, p) & \text{if } q + \frac{pt}{m} > 0, \\ -f(-q - \frac{pt}{m}, -p) & \text{if } q + \frac{pt}{m} < 0. \end{cases} \tag{1.8}$$

The (trivial) classical dynamics of a particle on the star-graph with elastic collision at the vertex can be defined in the following way. Denote by f a function of the form

$$f \in \oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R}), \quad f(q, p) \equiv \begin{pmatrix} f_1(q, p) \\ \vdots \\ f_n(q, p) \end{pmatrix}.$$

If $\|f\|_{\oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})} = 1$, $|f_\ell(q, p)|^2 dq dp$ can be interpreted as the probability of finding a particle on the ℓ -th edge of the graph, with position in the interval $[q, q + dq]$ and momentum in the interval $[p, p + dp]$.

Define the operator $L_D^\oplus := \oplus_{\ell=1}^n L_D$; the associated dynamics is generated by the unitary group $e^{iL_D^\oplus t} = \oplus_{\ell=1}^n e^{iL_D t}$, and it is trivial in the sense that it can be fully understood in terms of the dynamics on the half-line described above.

We consider the map

$$(\gamma_+^\oplus f)_\ell(p) := \lim_{q \rightarrow 0^+} f_\ell(q, p) \quad \text{for } \ell = 1, \dots, n,$$

defined on sufficiently smooth functions (we refer to Sect. 3 for the details). This map can be extended to a continuous one on $\text{dom}(L_D^\oplus)$. The operator $L_D^\oplus \upharpoonright \ker(\gamma_+^\oplus)$ is symmetric; in Theorem 3.3, by using the approach developed in [20,21] we identify a family of self-adjoint extensions. Among those we select the one that turns out to be useful to study the semiclassical limit of $\exp(-iH_K t/\hbar)$ and denote it by L_K .

We postpone the precise definition of L_K to Sect. 3.3, see, in particular, Remark 3.4. Here we just give component-wisely the formula for the associated unitary group, for $\ell = 1, \dots, n$ and for all $t \in \mathbb{R}$:

$$(e^{itL_K} f)_\ell(q, p) = \begin{cases} f_\ell(q + \frac{pt}{m}, p) & \text{if } q + \frac{pt}{m} > 0, \\ \sum_{\ell'=1}^n \left(\frac{2}{n} - \delta_{\ell, \ell'}\right) f_{\ell'}\left(-q - \frac{pt}{m}, -p\right) & \text{if } q + \frac{pt}{m} < 0. \end{cases} \tag{1.9}$$

We define the classical wave operators and the corresponding scattering operator on $\oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$ by

$$\Omega_{cl}^\pm := \lim_{t \rightarrow \pm\infty} e^{itL_D^\oplus} e^{-itL_K} \tag{1.10}$$

and

$$S_{cl} := (\Omega_{cl}^+)^* \Omega_{cl}^- . \tag{1.11}$$

These operators can be computed explicitly (see Proposition 3.8 below), and component-wisely they read as follows for $\ell = 1, \dots, n$ (θ is the Heaviside step function):

$$\begin{aligned} (\Omega_{cl}^\pm f)_\ell &= \sum_{\ell'=1}^n \left(\delta_{\ell, \ell'} - \frac{2}{n} \theta(\mp p) \right) f_{\ell'} ; \\ (S_{cl} f)_\ell &= \sum_{\ell'=1}^n \left(\delta_{\ell, \ell'} - \frac{2}{n} \right) f_{\ell'} . \end{aligned} \tag{1.12}$$

1.3 Truncated coherent states on the star-graph

In general, there is no natural definition of a coherent state on a star-graph, neither there is a unique way to extend coherent states through the vertex. Since we are interested in

initial states concentrated on one edge of the graph, we introduce the following class of initial states. We denote by $\tilde{\Psi}_{\sigma,\xi}^{\hbar}$ the unnormalized restriction of $\Psi_{\sigma,\xi}^{\hbar}$ [see Eq. (1.1)] to \mathbb{R}_+ , namely,

$$\tilde{\Psi}_{\sigma,\xi}^{\hbar} \in L^2(\mathbb{R}_+), \quad \tilde{\Psi}_{\sigma,\xi}^{\hbar}(x) = \Psi_{\sigma,\xi}^{\hbar}(x) \quad \text{for } x > 0. \tag{1.13}$$

On the graph we consider the quantum states defined as

Definition 1.1 (*Quantum states*) Let $\sigma \in \mathbb{C}$, with $\text{Re } \sigma = \sigma_0 > 0$, and $\xi = (q, p) \in \mathbb{R}_+ \times \mathbb{R}$; consider any normalized function $\Xi_{\sigma,\xi}^{\hbar} \in L^2(\mathbb{R}_+)$, such that

$$\|\Xi_{\sigma,\xi}^{\hbar} - \tilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \leq C_0 e^{-\varepsilon \frac{q^2}{\hbar|\sigma|^2}} \quad \text{for some } C_0, \varepsilon > 0. \tag{1.14}$$

We are primarily interested in quantum states on the star-graph of the form

$$\Xi_{\sigma,\xi}^{\hbar} \in L^2(\mathcal{G}), \quad \Xi_{\sigma,\xi}^{\hbar} \equiv \begin{pmatrix} \Xi_{\sigma,\xi}^{\hbar} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Remark 1.2 In Definition 1.1 we assume that the constants C_0 and ε do not depend on \hbar, ξ or σ . In what follows, whenever we refer to a state of the form $\Xi_{\sigma,\xi}^{\hbar}$ the constants C_0 and ε are the ones given in Definition 1.1.

One could choose the quantum states $\Xi_{\sigma,\xi}^{\hbar}$ (and the corresponding classical states given below) in a different way. As a matter of fact, for any choice of $\Xi_{\sigma,\xi}^{\hbar}$ the terms $C_0 \exp(-\varepsilon q^2/(\hbar\sigma_0^2))$ and $C_0 \exp(-\varepsilon (q + pt/m)^2/(\hbar|\sigma_t|^2))$ in the bounds in Theorems 1.4 and 1.6 would be replaced by $\|\Xi_{\sigma_0,\xi}^{\hbar} - \tilde{\Psi}_{\sigma_0,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}$ and $\|\Xi_{\sigma_t,\xi_t}^{\hbar} - \tilde{\Psi}_{\sigma_t,\xi_t}^{\hbar}\|_{L^2(\mathbb{R}_+)}$ respectively.

Correspondingly, we will consider the family of classical states

Definition 1.3 (*Classical states*) For any σ, ξ , and $\Xi_{\sigma,\xi}^{\hbar}$ as in Definition 1.1, consider the function $\Sigma_{\sigma,x}^{\hbar} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$\Sigma_{\sigma,x}^{\hbar}(\xi) := \Xi_{\sigma,\xi}^{\hbar}(x). \tag{1.15}$$

We will make use of the family of classical states on the star-graph given by $\Sigma_{\sigma,x}^{\hbar} \equiv (\Sigma_{\sigma,x}^{\hbar}, 0, \dots, 0)^T$. In general the functions $\Sigma_{\sigma,x}^{\hbar}$ do not belong to $L^2(\mathbb{R}_+ \times \mathbb{R})$ but we will always assume that they are in $L^\infty(\mathbb{R}_+ \times \mathbb{R})$ (see Lemma 4.2, and Examples 4.3 and 4.4 below). We remark that the classical operators $e^{-itL_K}, \Omega_{cl}^\pm$, and S_{cl} can be naturally extended to $L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

1.4 Main results

Our first result concerns the semiclassical limit of the dynamics.

Theorem 1.4 *Let $\sigma_0 > 0$, $\xi = (q, p) \in \mathbb{R}_+ \times \mathbb{R}$ and consider any initial state of the form $\Xi_{\sigma_0, \xi}^{\hbar}$, together with its classical analogue $\Sigma_{\sigma_0, x}^{\hbar}$. Then, for all $t \in \mathbb{R}$ there holds*

$$\begin{aligned} & \left(\int_{\mathcal{G}} dx \left| e^{-i \frac{t}{\hbar} H_K} \Xi_{\sigma_0, \xi}^{\hbar}(x) - e^{\frac{i}{\hbar} A t} (e^{i t L_K} \Sigma_{\sigma_t, x}^{\hbar})(\xi) \right|^2 \right)^{1/2} \\ & \leq C_0 e^{-\varepsilon \frac{q^2}{\hbar \sigma_0^2}} + 2C_0 e^{-\varepsilon \frac{(q+pt/m)^2}{\hbar |\sigma_t|^2}} + \sqrt{2} e^{-\frac{q^2}{4\hbar \sigma_0^2}}. \end{aligned} \tag{1.16}$$

Remark 1.5 Let $t_{coll}(\xi) := -mq/p$ be the classical collision time. Whenever $|t - t_{coll}| \leq \eta m \sigma_0 \sqrt{\hbar}/|p|$ for some positive constant η the second term on the right-hand side of Eq. (1.16) is larger than $2C_0 e^{-\varepsilon \eta^2}$.

In the second part of our analysis we study the semiclassical limit of the wave operators and of the scattering operator.

Theorem 1.6 *Let $\sigma_0 > 0$, $\xi = (q, p) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ and consider any state of the form $\Xi_{\sigma_0, \xi}^{\hbar}$, together with its classical analogue $\Sigma_{\sigma_0, x}^{\hbar}$. Then, there hold*

$$\begin{aligned} & \left(\int_{\mathcal{G}} dx \left| \Omega^{\pm} \Xi_{\sigma_0, \xi}^{\hbar}(x) - (\Omega_{cl}^{\pm} \Sigma_{\sigma_0, x}^{\hbar})(\xi) \right|^2 \right)^{1/2} \\ & \leq \sqrt{\frac{2}{n}} \left(\sqrt{2} C_0 e^{-\varepsilon \frac{q^2}{\hbar \sigma_0^2}} + e^{-\frac{q^2}{4\hbar \sigma_0^2}} + e^{-\frac{\sigma_0^2 p^2}{\hbar}} \right), \end{aligned} \tag{1.17}$$

and

$$S \Xi_{\sigma_0, \xi}^{\hbar}(x) = (S_{cl} \Sigma_{\sigma_0, x}^{\hbar})(\xi). \tag{1.18}$$

Identity (1.18) is an immediate consequence of Eqs. (1.7) and (1.12), and of the definitions of $\Xi_{\sigma_0, \xi}^{\hbar}$ and $\Sigma_{\sigma_0, x}^{\hbar}$.

Remark 1.7 Equation (1.17) makes evident that $\Omega^{\pm} \Xi_{\sigma_0, \xi}^{\hbar}$ and $(\Omega_{cl}^{\pm} \Sigma_{\sigma_0, (\cdot)}^{\hbar})(\xi)$ are exponentially close (with respect to the natural topology of $L^2(\mathcal{G})$) in the semiclassical limit $\hbar \sigma_0^2 / q^2, \hbar / \sigma_0^2 p^2 \rightarrow 0^+$ for any $\xi = (q, p)$ with $q > 0$ and $p \neq 0$.

As a matter of fact, it can be proved that the relation (1.17) remains valid also for $p = 0$ if one puts $(\Omega_{cl}^{\pm} \Sigma_{\sigma_0, x}^{\hbar})(q, 0) = \Sigma_{\sigma_0, x}^{\hbar}(q, 0)$; the latter position appears to be reasonable and is indeed compatible with the computations reported in the proof of Proposition 3.8. Nonetheless, since $\exp(-\sigma_0^2 p^2 / \hbar) = 1$ in this case, the resulting upper bound is of limited interest for what concerns the semi-classical limit. To say more, for $p = 0$ and $\hbar \sigma_0^2 / q^2$ (or C_0) small enough, by a variation of the arguments described in the proof of Theorem 1.6 one can derive the lower bound

$$\left(\int_{\mathcal{G}} dx \left| \Omega^{\pm} \Xi_{\sigma_0, (q, 0)}^{\hbar}(x) - (\Omega_{cl}^{\pm} \Sigma_{\sigma_0, x}^{\hbar})(q, 0) \right|^2 \right)^{1/2}$$

$$\geq \sqrt{\frac{2}{n}} \left(1 - e^{-\frac{q^2}{4\hbar\sigma_0^2}} - \sqrt{2} C_0 e^{-\varepsilon \frac{q^2}{\hbar\sigma_0^2}} \right).$$

This shows that, as might be expected, the classical scattering theory does not provide a good approximation for the quantum analogue when $p = 0$. On the contrary, notice that Eq. (1.16) ensures a significant control of the error for the dynamics at any finite time $t \in \mathbb{R}$ even for $p = 0$.

2 The quantum theory

2.1 Dirichlet dynamics on the half-line

Let us first consider the free quantum Hamiltonian for a quantum particle of mass m on the whole real line, defined as usual by

$$H_0 : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad H_0 \psi := -\frac{\hbar^2}{2m} \psi'',$$

together with the associated free unitary group $U_t^0 := e^{-i\frac{t}{\hbar}H_0}$ ($t \in \mathbb{R}$). Correspondingly, let us recall that for any $\psi \in L^2(\mathbb{R})$ we have

$$(U_t^0 \psi)(x) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_{\mathbb{R}} dy e^{i\frac{m}{2\hbar} \frac{(x-y)^2}{t}} \psi(y). \tag{2.1}$$

Let us further introduce the Dirichlet Hamiltonian on the half-line \mathbb{R}_+ , defined as usual by

$$\text{dom}(H_D) := H_0^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+), \quad H_D \psi := -\frac{\hbar^2}{2m} \psi'',$$

and refer to the associated unitary group $U_t^D := e^{-i\frac{t}{\hbar}H_D}$ ($t \in \mathbb{R}$). As well known, the latter operator can be expressed as

$$U_t^D = \mathcal{U}_t^- - \mathcal{U}_t^+, \tag{2.2}$$

where, in view of the identity (2.1), we introduced the bounded operators on $L^2(\mathbb{R}_+)$ defined as follows for $\psi \in L^2(\mathbb{R}_+)$ and $x \in \mathbb{R}_+$:

$$(\mathcal{U}_t^\pm \psi)(x) := \sqrt{\frac{m}{2\pi i \hbar t}} \int_0^\infty dy e^{i\frac{m}{2\hbar} \frac{(x\pm y)^2}{t}} \psi(y). \tag{2.3}$$

Remark 2.1 Let us consider the bounded operator

$$\Theta : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}), \quad (\Theta \psi)(x) = \begin{cases} \psi(x) & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

together with its adjoint

$$\Theta^* : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}_+), \quad (\Theta^* \psi)(x) = \psi(x) \quad (x \in \mathbb{R}_+).$$

Namely, Θ gives the extension by zero to the whole real line \mathbb{R} of any function on \mathbb{R}_+ , while Θ^* is the restriction to \mathbb{R}_+ of any function on \mathbb{R} . Note that Θ is an isometry. In fact, $\Theta^* \Theta$ is the identity on $L^2(\mathbb{R}_+)$ and $\Theta \Theta^*$ is an orthogonal projector (but not the identity) on $L^2(\mathbb{R})$; more precisely, we have (θ is the Heaviside step function)

$$\begin{aligned} \Theta^* \Theta &= \mathbf{1} \quad \text{on } L^2(\mathbb{R}_+), \\ \Theta \Theta^* : L^2(\mathbb{R}) &\rightarrow L^2(\mathbb{R}), \quad (\Theta \Theta^* \psi)(x) = \theta(x) \psi(x) \quad (x \in \mathbb{R}). \end{aligned}$$

To proceed let us consider the parity operator

$$P : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (P \psi)(x) = \psi(-x) \quad (x \in \mathbb{R}).$$

Of course P is a unitary, self-adjoint involution which commutes with the free Hamiltonian H_0 , i.e.,

$$H_0 P = P H_0.$$

Furthermore it can be checked by direct inspection that

$$\text{ran}(P \Theta) = \ker(\Theta^*)$$

Using the bounded linear maps introduced above, one can express the operators defined in Eq. (2.3) as follows:

$$\mathcal{U}_t^- = \Theta^* U_t^0 \Theta, \quad \mathcal{U}_t^+ = \Theta^* P U_t^0 \Theta = \Theta^* U_t^0 P \Theta. \tag{2.4}$$

Recalling that $(U_t^0)^* = U_{-t}^0$, the above relation allow us to infer

$$(\mathcal{U}_t^-)^* = \mathcal{U}_{-t}^-, \quad (\mathcal{U}_t^+)^* = \mathcal{U}_{-t}^+.$$

Let us finally point out that, on account of the obvious operator norms $\|\Theta\| = \|\Theta^*\| = 1$, $\|U_t^0\| = 1$ and $\|P\| = 1$, from Eq. (2.4) it readily follows

$$\|\mathcal{U}_t^\pm\| \leq 1.$$

2.2 Dirichlet and Kirchhoff dynamics on the star-graph

Let us now introduce the quantum Hamiltonian on the graph \mathcal{G} , corresponding to Dirichlet boundary conditions at the vertex. This coincides with the direct sum of n copies of the Dirichlet Hamiltonian H_D on the half-line \mathbb{R}_+ , namely:

$$\text{dom}(H_D^\oplus) := \oplus_{\ell=1}^n \text{dom}(H_D) = \left(\oplus_{\ell=1}^n H_0^1(\mathbb{R}_+) \right) \cap H^2(\mathcal{G}), \tag{2.5}$$

$$\begin{aligned}
 H_D^\oplus &:= \oplus_{\ell=1}^n H_D : \text{dom}(H_D^\oplus) \subset L^2(\mathcal{G}) \rightarrow L^2(\mathcal{G}), \\
 H_D^\oplus \Psi &:= -\frac{\hbar^2}{2m} \begin{pmatrix} \Psi_1'' \\ \vdots \\ \Psi_n'' \end{pmatrix}.
 \end{aligned}
 \tag{2.6}$$

In view of the identity (2.2), it can be readily inferred that the corresponding unitary group $e^{-i\frac{t}{\hbar}H_D^\oplus}$ ($t \in \mathbb{R}$) can be expressed as

$$e^{-i\frac{t}{\hbar}H_D^\oplus} = \oplus_{\ell=1}^n \mathcal{U}_t^- - \oplus_{\ell=1}^n \mathcal{U}_t^+,
 \tag{2.7}$$

where \mathcal{U}_t^\pm is defined as in Eq. (2.3).

To proceed let us consider the Kirchhoff Hamiltonian on the graph \mathcal{G} . This is defined as in Eqs. (1.3)–(1.4). In what follows we denote by \mathbb{S} the $n \times n$ matrix with components

$$(\mathbb{S})_{\ell,\ell'} := \delta_{\ell,\ell'} - \frac{2}{n}, \quad \text{for } \ell, \ell' = 1, \dots, n.
 \tag{2.8}$$

By a slight abuse of notation we use the same symbol to denote the operator in $L^2(\mathcal{G})$ defined by

$$(\mathbb{S} \Psi)_\ell := \sum_{\ell'=1}^n (\mathbb{S})_{\ell,\ell'} \Psi_{\ell'} \quad \Psi \in L^2(\mathcal{G}).$$

By arguments similar to those given in the proof of [1, Thm. 2.1] (cf. also [11] and [15, Eq. (7.1)]) we get

$$e^{-i\frac{t}{\hbar}H_K} = \oplus_{\ell=1}^n \mathcal{U}_t^- - \mathbb{S} \oplus_{\ell=1}^n \mathcal{U}_t^+.
 \tag{2.9}$$

2.3 The quantum wave operators and scattering operator

Let us consider the wave operators and the corresponding scattering operator on $L^2(\mathcal{G})$ respectively defined in Eqs. (1.5) and (1.6).

Since H_K has purely absolutely continuous spectrum $\sigma(H_K) = [0, \infty)$, we have that Ω^\pm are unitary on the whole Hilbert space $L^2(\mathcal{G})$, i.e.,

$$(\Omega^\pm)^* \Omega^\pm = \mathbf{1},$$

which in turn ensures¹

$$\|\Omega^\pm\| = 1.
 \tag{2.10}$$

¹ Of course, the same identity (2.10) can be derived straightforwardly from the fact that Ω^\pm are defined as strong limits of unitary operators.

Let us define the unitary operators $\mathcal{F}_s : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$ and $\mathcal{F}_c : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$:

$$(\mathcal{F}_s \psi)(k) := -\frac{2i}{\sqrt{2\pi}} \int_0^\infty dx \sin(kx) \psi(x) \quad (k \in \mathbb{R}_+); \quad (2.11)$$

$$(\mathcal{F}_c \psi)(k) := \frac{2}{\sqrt{2\pi}} \int_0^\infty dx \cos(kx) \psi(x) \quad (k \in \mathbb{R}_+). \quad (2.12)$$

The wave operators can be computed explicitly. To this aim one could use the results from Weder [22] (see also references therein), with some modifications, since in [22] the reference dynamics is given by the Hamiltonian with Neumann boundary conditions. For the sake of completeness, we prefer to give an explicit derivation of the result, obtained by taking the limit $t \rightarrow \pm\infty$ on the unitary groups. We remark that in [22] the formulae are obtained by using the Jost functions.

We have the following explicit formulae for the wave operators:

Proposition 2.2 *The quantum wave operators can be expressed as*

$$\Omega^\pm = \frac{1}{2} \oplus_{\ell=1}^n (\mathbf{1} \pm \mathcal{F}_c^* \mathcal{F}_s) + \frac{1}{2} \mathbb{S} \oplus_{\ell=1}^n (\mathbf{1} \mp \mathcal{F}_c^* \mathcal{F}_s). \quad (2.13)$$

Proof By Eqs. (2.7) and (2.9) we easily obtain the identity

$$\Omega^\pm = s\text{-}\lim_{t \rightarrow \pm\infty} \left(\oplus_{\ell=1}^n (\mathcal{U}_{-t}^- \mathcal{U}_t^- - \mathcal{U}_{-t}^- \mathcal{U}_t^+) + \mathbb{S} \oplus_{\ell=1}^n (\mathcal{U}_{-t}^+ \mathcal{U}_t^+ - \mathcal{U}_{-t}^+ \mathcal{U}_t^-) \right). \quad (2.14)$$

Let us find more convenient expressions for the operators on the right-hand side. Let $\psi \in L^2(\mathbb{R}_+)$ and define

$$\psi_e(x) := \begin{cases} \psi(x) & \text{if } x > 0 \\ \psi(-x) & \text{if } x < 0 \end{cases}; \quad \psi_o(x) := \begin{cases} \psi(x) & \text{if } x > 0 \\ -\psi(-x) & \text{if } x < 0 \end{cases}.$$

In view of Remark (2.1), we have: $\Theta \psi = \frac{\psi_e + \psi_o}{2}$, $\Theta^*(\psi_e + \psi_o) = 2\psi$, and $\Theta^*(\psi_e - \psi_o) = 0$. Hence, see Eq. (2.4),

$$\begin{aligned} \mathcal{U}_{-t}^- \mathcal{U}_t^- \psi &= \Theta^* U_{-t}^0 \Theta \Theta^* U_t^0 \Theta \psi \\ &= \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_e)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right. \\ &\quad \left. + U_{-t}^0 (\Theta^* U_t^0 \psi_o)_o \right] \\ &\quad (\text{since } U_t^0 \psi_e \text{ is even and } U_t^0 \psi_o \text{ is odd, } (\Theta^* U_t^0 \psi_e)_e = U_t^0 \psi_e, (\Theta^* U_t^0 \psi_o)_o \\ &\quad = U_t^0 \psi_o, \text{ and we get}) \\ &= \frac{1}{4} \Theta^* \left[\psi_e + U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o + \psi_o \right] \end{aligned}$$

$$= \frac{1}{2} \psi + \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right].$$

On the other hand,

$$\begin{aligned} \mathcal{U}_{-t}^- \mathcal{U}_t^+ \psi &= \Theta^* U_{-t}^0 \Theta \Theta^* U_t^0 P \Theta \psi \\ &\quad (\text{Since } P \Theta \psi = P(\psi_e + \psi_o)/2 = (\psi_e - \psi_o)/2 \text{ we get}) \\ &= \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_e)_e - U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right. \\ &\quad \left. - U_{-t}^0 (\Theta^* U_t^0 \psi_o)_o \right] \\ &= \frac{1}{4} \Theta^* \left[\psi_e - U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o - \psi_o \right] \\ &= \frac{1}{4} \Theta^* \left[-U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right]. \end{aligned}$$

Hence

$$(\mathcal{U}_{-t}^- \mathcal{U}_t^- - \mathcal{U}_{-t}^- \mathcal{U}_t^+) \psi = \frac{1}{2} \psi + \frac{1}{2} \Theta^* U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e.$$

Recall that U_t^D is the unitary group generated by the Dirichlet Laplacian on the half-line; its integral kernel is given by

$$U_t^D(x, y) = U_t^0(x - y) - U_t^0(x + y) \quad \text{for } x, y \in \mathbb{R}_+.$$

Moreover, let U_t^N be the unitary group generated by the Neumann Laplacian on the half-line; its integral kernel is given by

$$U_t^N(x, y) = U_t^0(x - y) + U_t^0(x + y) \quad \text{for } x, y \in \mathbb{R}_+.$$

Note that, for $x \in \mathbb{R}_+$,

$$\begin{aligned} (\Theta^* U_t^0 \psi_o)(x) &= \int_{\mathbb{R}} dy U_t^0(x - y) \psi_o(y) \\ &= \int_0^\infty dy (U_t^0(x - y) - U_t^0(x + y)) \psi(y) = (U_t^D \psi)(x), \end{aligned}$$

similarly

$$(\Theta^* U_t^0 \psi_e)(x) = \int_0^\infty dy (U_t^0(x - y) + U_t^0(x + y)) \psi(y) = (U_t^N \psi)(x).$$

Hence,

$$\Theta^* U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e = \Theta^* U_{-t}^0 (U_t^D \psi)_e = U_{-t}^N U_t^D \psi,$$

and

$$(\mathcal{U}_{-t}^- \mathcal{U}_t^- - \mathcal{U}_{-t}^- \mathcal{U}_t^+) \psi = \frac{1}{2} \psi + \frac{1}{2} U_{-t}^N U_t^D \psi. \tag{2.15}$$

A similar computation gives

$$\begin{aligned} \mathcal{U}_{-t}^+ \mathcal{U}_t^+ \psi &= \Theta^* U_{-t}^0 P \Theta \Theta^* U_t^0 P \Theta \psi \\ &= \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_e)_e - U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e - U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right. \\ &\quad \left. + U_{-t}^0 (\Theta^* U_t^0 \psi_o)_o \right] \\ &= \frac{1}{2} \psi - \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right], \end{aligned}$$

and

$$\begin{aligned} \mathcal{U}_{-t}^+ \mathcal{U}_t^- \psi &= \Theta^* U_{-t}^0 P \Theta \Theta^* U_t^0 \Theta \psi \\ &= \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_e)_e + U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e - U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right. \\ &\quad \left. - U_{-t}^0 (\Theta^* U_t^0 \psi_o)_o \right] \\ &= \frac{1}{4} \Theta^* \left[U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e - U_{-t}^0 (\Theta^* U_t^0 \psi_e)_o \right]. \end{aligned}$$

Hence

$$(\mathcal{U}_{-t}^+ \mathcal{U}_t^+ - \mathcal{U}_{-t}^+ \mathcal{U}_t^-) \psi = \frac{1}{2} \psi - \frac{1}{2} \Theta^* U_{-t}^0 (\Theta^* U_t^0 \psi_o)_e = \frac{1}{2} \psi - \frac{1}{2} U_{-t}^N U_t^D \psi. \tag{2.16}$$

To compute the wave operator we have to evaluate the limits $s\text{-}\lim_{t \rightarrow \pm\infty} U_{-t}^N U_t^D$; the latter give the wave operators Ω_{ND}^\pm for the pair (Dirichlet Laplacian, Neumann Laplacian) on the half-line, which are computed in Proposition A.1 and equal $\pm \mathcal{F}_c^* \mathcal{F}_s$. This, together with Eqs. (2.14), (2.15), and (2.16) concludes the proof of identity (2.13).

Remark 2.3 Note that the wave operators do not depend on \hbar . Moreover the scattering operator is given by²

$$S = (\Omega^+)^* \Omega^- = \mathbb{S}, \tag{2.17}$$

the same formula is written component-wisely in Eq. (1.7). The matrix \mathbb{S} given here equals $-\sigma^{(v)}$, where $\sigma^{(v)}$ is the scattering matrix at the vertex, as defined in [4, Def. 2.1.1.] (see also [4, Ex. 2.1.7, p.41] and [16, Ex. 2.4]). The minus sign is due to the fact that as reference Hamiltonian we chose Dirichlet boundary conditions, instead of Neumann boundary conditions, see also [16].

² The last identity in Eq. (2.17) can be derived by a simple computation starting from Eq. (2.13), recalling that \mathcal{F}_s and \mathcal{F}_c are unitary operators, and noting that $\mathbb{S}^* = \mathbb{S}$, $\mathbb{S}^2 = 1$.

3 The semiclassical theory

3.1 Classical dynamics on the half-line with elastic collision at the origin

We start by recalling some basic definitions and results from [5]. Let $X_0(q, p) = (p/m, 0)$ be the vector field associated with the free (classical) Hamiltonian of a particle of mass m and consider the differential operator

$$L : \mathcal{S}'(\mathbb{R}^2) \rightarrow \mathcal{S}'(\mathbb{R}^2), \quad Lf := -i X_0 \cdot \nabla f$$

in the space of tempered distributions $\mathcal{S}'(\mathbb{R}^2)$.

We denote by

$$L_0 : \text{dom}(L_0) \subseteq L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2), \quad (L_0 f)(q, p) = -i \frac{p}{m} \frac{\partial f}{\partial q}(q, p),$$

$$\text{dom}(L_0) := \{f \in L^2(\mathbb{R}^2) \mid Lf \in L^2(\mathbb{R}^2)\},$$

the maximal realization of L in $L^2(\mathbb{R}^2)$. Posing $R_z^0 := (L_0 - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$ one has

$$(R_z^0 f)(q, p) = \text{sgn}(\text{Im}z) \frac{i m}{|p|} \int_{\mathbb{R}} dq' \theta((q - q') p \text{Im}z) e^{imz(q-q')/p} f(q', p). \quad (3.1)$$

Moreover the action of the (free) unitary group e^{itL_0} ($t \in \mathbb{R}$) is given by

$$(e^{itL_0} f)(q, p) = f\left(q + \frac{pt}{m}, p\right).$$

For any $f \in \mathcal{S}(\mathbb{R}^2)$ we define the map $(\gamma f)(p) := f(0, p)$. For a comparison with the results in [5, Sec. 2], recall that the map γ can be equivalently defined as $(\gamma f)(p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \tilde{f}(k, p)$ where $\tilde{f}(k, p)$ is the Fourier transform of $f(q, p)$ in the variable q . By [5, Lem. 2.1], the map γ extends to a bounded operator $\gamma : \text{dom}(L_0) \rightarrow L^2(\mathbb{R}, |p| dp)$, where $\text{dom}(L_0) \subset L^2(\mathbb{R}^2)$ is endowed with the graph norm. Hence, for any $z \in \mathbb{C} \setminus \mathbb{R}$ the operator $\gamma R_z^0 : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}, |p| dp)$ is bounded, and so is its adjoint (in \bar{z}):

$$G_z : L^2(\mathbb{R}, |p|^{-1} dp) \rightarrow L^2(\mathbb{R}^2), \quad G_z := (\gamma R_z^0)^*$$

(here $L^2(\mathbb{R}, |p|^{-1} dp)$ and $L^2(\mathbb{R}, |p| dp)$ are considered as a dual couple with respect to the duality induced by the scalar product in $L^2(\mathbb{R})$). An explicit calculation gives

$$(G_z u)(q, p) = \theta(q p \text{Im}z) \text{sgn}(\text{Im}z) \frac{i m}{|p|} e^{imzq/p} u(p).$$

Next we consider the classical motion of a point particle of mass m on the whole real line, with elastic collision at the origin. The generator of the dynamics, denoted

by L_∞ , is obtained as a limiting case, for $\beta \rightarrow \infty$, of the operator L_β defined in [5]. To this aim we set

$$\Lambda_z^\infty : L^2(\mathbb{R}, |p|dp) \rightarrow L^2(\mathbb{R}, |p|^{-1}dp), \quad (\Lambda_z^\infty u)(p) := i \operatorname{sgn}(\operatorname{Im}z) \frac{2|p|}{m} u(p).$$

In addition, let us consider the projector on even functions (here either $\rho(p) = |p|$ or $\rho(p) = |p|^{-1}$)

$$\Pi_{ev} : L^2(\mathbb{R}, \rho dp) \rightarrow L^2(\mathbb{R}, \rho dp), \quad (\Pi_{ev} f)(p) := \frac{1}{2} (f(p) + f(-p)).$$

By [5, Thm. 2.2], here employed with $\beta \rightarrow \infty$, the operator L_∞ is defined by

$$\begin{aligned} \operatorname{dom}(L_\infty) &:= \{f \in L^2(\mathbb{R}^2) \mid f = f_z + G_z \Lambda_z^\infty \Pi_{ev} \gamma f_z, f_z \in \operatorname{dom}(L_0)\}, \\ (L_\infty - z)f &= (L_0 - z)f_z, \end{aligned}$$

for all $z \in \mathbb{C} \setminus \mathbb{R}$. The associated resolvent operator $R_z^\infty := (L_\infty - z)^{-1}$ ($z \in \mathbb{C} \setminus \mathbb{R}$) can be expressed as follows, in terms of the free resolvent R_z^0 and of the trace operator γ :

$$(R_z^\infty f)(q, p) = (R_z^0 f)(q, p) - \theta(qp \operatorname{Im}z) e^{imzq/p} [(\gamma R_z^0 f)(p) + (\gamma R_z^0 f)(-p)]. \tag{3.2}$$

More explicitly, we have

$$\begin{aligned} (R_z^\infty f)(q, p) &= \operatorname{sgn}(\operatorname{Im}z) \frac{im}{|p|} \int_{\mathbb{R}} dq' \left[\theta((q - q') p \operatorname{Im}z) e^{imz(q - q')/p} f(q', p) \right. \\ &\quad - \theta(qp \operatorname{Im}z) \left(\theta(-q' p \operatorname{Im}z) e^{imz(q - q')/p} f(q', p) \right. \\ &\quad \left. \left. + \theta(q' p \operatorname{Im}z) e^{imz(q + q')/p} f(q', -p) \right) \right]. \end{aligned} \tag{3.3}$$

Correspondingly, let us recall that [5, Prop. 2.4] gives, for all $t \in \mathbb{R}$,

$$\begin{aligned} (e^{itL_\infty} f)(q, p) &= (e^{itL_0} f)(q, p) \\ &\quad - \theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \left[(e^{itL_0} f)(q, p) + (e^{itL_0} f)(-q, -p) \right] \end{aligned} \tag{3.4}$$

(here, for a comparison with [5], we used $(e^{itL_0} f(-\cdot, -\cdot))(q, p) = (e^{itL_0} f)(-q, -p)$).

To proceed, let us introduce the lateral traces defined by

$$\begin{aligned} (\gamma_+(R_z^\infty f))(p) &:= (\gamma R_z^0 f)(p) - \theta(p \operatorname{Im}z) [(\gamma R_z^0 f)(p) + (\gamma R_z^0 f)(-p)] \\ &= \theta(-p \operatorname{Im}z) (\gamma R_z^0 f)(p) - \theta(p \operatorname{Im}z) (\gamma R_z^0 f)(-p), \end{aligned} \tag{3.5}$$

$$\begin{aligned} (\gamma_-(R_z^\infty f))(p) &:= (\gamma R_z^0 f)(p) - \theta(-p \operatorname{Im}z) [(\gamma R_z^0 f)(p) + (\gamma R_z^0 f)(-p)] \\ &= \theta(p \operatorname{Im}z) (\gamma R_z^0 f)(p) - \theta(-p \operatorname{Im}z) (\gamma R_z^0 f)(-p) \end{aligned} \tag{3.6}$$

(here we used the trivial identity $\theta(-s) = 1 - \theta(s)$). Clearly, $\gamma_{\pm}(R_z^{\infty} f)$ are odd functions and, using again [5, Lem. 2.1],

$$\gamma_{\pm} : \text{dom}(L_{\infty}) \rightarrow L_{\text{odd}}^2(\mathbb{R}, |p|dp) \quad (3.7)$$

is a bounded operator. We remark that the action of γ_{\pm} can be understood as $(\gamma_{\pm}(R_z^{\infty} f))(p) = \lim_{q \rightarrow 0^{\pm}} (R_z^{\infty} f)(q, p)$.

For the subsequent developments it is convenient to express the free resolvent R_z^0 in terms of R_z^{∞} . More precisely, we have the following explicit characterization.

Lemma 3.1 *For any $z \in \mathbb{C} \setminus \mathbb{R}$ and for any $f \in L^2(\mathbb{R}^2)$, there holds*

$$\begin{aligned} (R_z^0 f)(q, p) &= (R_z^{\infty} f)(q, p) \\ &\quad - \theta(qp \text{Im} z) \text{sgn}(q) e^{imzq/p} \left[(\gamma_+ R_z^{\infty} f)(p) - (\gamma_- R_z^{\infty} f)(p) \right]. \end{aligned}$$

Proof From Eqs. (3.5) and (3.6) we readily infer that

$$\begin{aligned} \theta(-p \text{Im} z) (\gamma_+ R_z^{\infty} f)(p) &= \theta(-p \text{Im} z) (\gamma R_z^0 f)(p), \\ \theta(p \text{Im} z) (\gamma_- R_z^{\infty} f)(p) &= \theta(p \text{Im} z) (\gamma R_z^0 f)(p). \end{aligned}$$

The above relations imply, in turn,

$$(\gamma R_z^0 f)(p) = \theta(-p \text{Im} z) (\gamma_+ R_z^{\infty} f)(p) + \theta(p \text{Im} z) (\gamma_- R_z^{\infty} f)(p),$$

and, since $\gamma_{\pm} R_z^{\infty} f$ are odd functions,

$$(\gamma R_z^0 f)(-p) = -\theta(p \text{Im} z) (\gamma_+ R_z^{\infty} f)(p) - \theta(-p \text{Im} z) (\gamma_- R_z^{\infty} f)(p).$$

Substituting the latter identities into Eq. (3.2) and noting that

$$\theta(qp \text{Im} z) = (\theta(q) + \theta(-q)) \theta(qp \text{Im} z) = \theta(q) \theta(p \text{Im} z) + \theta(-q) \theta(-p \text{Im} z),$$

and

$$\text{sgn}(q) \theta(qp \text{Im} z) = \theta(q) \theta(p \text{Im} z) - \theta(-q) \theta(-p \text{Im} z),$$

we obtain

$$\begin{aligned} &(R_z^{\infty} f)(q, p) \\ &= (R_z^0 f)(q, p) - e^{imzq/p} \left[\theta(q) \theta(p \text{Im} z) (\gamma_- R_z^{\infty} f)(p) \right. \\ &\quad \left. - \theta(q) \theta(p \text{Im} z) (\gamma_+ R_z^{\infty} f)(p) \right. \\ &\quad \left. + \theta(-q) \theta(-p \text{Im} z) (\gamma_+ R_z^{\infty} f)(p) - \theta(-q) \theta(-p \text{Im} z) (\gamma_- R_z^{\infty} f)(p) \right] \end{aligned}$$

$$= (R_z^0 f)(q, p) + \theta(qp \operatorname{Im} z) \operatorname{sgn}(q) e^{imzq/p} \left[(\gamma_+ R_z^\infty f)(p) - (\gamma_- R_z^\infty f)(p) \right],$$

which suffices to infer the thesis.

Similarly, for the unitary operator describing the dynamics we have

Lemma 3.2 *For any $t \in \mathbb{R}$ and for any $f \in L^2(\mathbb{R}^2)$, there holds*

$$\begin{aligned} (e^{itL_0} f)(q, p) &= (e^{itL_\infty} f)(q, p) \\ &\quad - \theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \left[(e^{itL_\infty} f)(q, p) + (e^{itL_\infty} f)(-q, -p) \right]. \end{aligned}$$

Proof First note that the identity in Eq. (3.4) entails

$$\begin{aligned} (e^{itL_\infty} f)(-q, -p) &= (e^{itL_0} f)(-q, -p) \\ &\quad - \theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \left[(e^{itL_0} f)(q, p) + (e^{itL_0} f)(-q, -p) \right]. \end{aligned}$$

From the above relation and the previously cited equation, we derive

$$\begin{aligned} &(e^{itL_\infty} f)(q, p) + (e^{itL_\infty} f)(-q, -p) \\ &= \left(1 - 2 \theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \right) \left[(e^{itL_0} f)(q, p) + (e^{itL_0} f)(-q, -p) \right], \end{aligned}$$

which in turn implies

$$\begin{aligned} &\theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \left[(e^{itL_\infty} f)(q, p) + (e^{itL_\infty} f)(-q, -p) \right] \\ &= - \theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) \left[(e^{itL_0} f)(q, p) + (e^{itL_0} f)(-q, -p) \right]. \end{aligned}$$

The thesis follows upon substitution of the above identity into Eq. (3.4).

Let us now consider the natural decomposition³

$$\begin{aligned} L^2(\mathbb{R}^2) &\equiv L^2((\mathbb{R}_- \cup \mathbb{R}_+) \times \mathbb{R}, dqdp) \\ &= (L^2(\mathbb{R}_-, dq) \oplus L^2(\mathbb{R}_+, dq)) \otimes L^2(\mathbb{R}, dp) \\ &= (L^2(\mathbb{R}_-, dq) \otimes L^2(\mathbb{R}, dp)) \oplus (L^2(\mathbb{R}_+, dq) \otimes L^2(\mathbb{R}, dp)) \\ &= L^2(\mathbb{R}_- \times \mathbb{R}) \oplus L^2(\mathbb{R}_+ \times \mathbb{R}), \end{aligned} \tag{3.8}$$

³ All the equalities in Eq. (3.8) must be understood as isomorphisms of Hilbert spaces (see, e.g., [13, p. 85]).

and notice that both the subspaces $L^2(\mathbb{R}_- \times \mathbb{R}) \equiv L^2(\mathbb{R}_- \times \mathbb{R}) \oplus \{0\}$ and $L^2(\mathbb{R}_+ \times \mathbb{R}) \equiv \{0\} \oplus L^2(\mathbb{R}_+ \times \mathbb{R})$ are left invariant by the resolvent R_z^∞ , this is evident from Eq. (3.3). Taking this into account, we introduce the bounded operator

$$R_z^D : L^2(\mathbb{R}_+ \times \mathbb{R}) \rightarrow L^2(\mathbb{R}_+ \times \mathbb{R}), \quad (R_z^D f)(q, p) := (R_z^\infty(0 \oplus f))(q, p). \quad (3.9)$$

By direct computations, from Eq. (3.3) (here employed with $q > 0$) we get

$$\begin{aligned} (R_z^D f)(q, p) &= \operatorname{sgn}(\operatorname{Im}z) \frac{i m}{|p|} \int_0^\infty dq' \left[\theta((q - q')p \operatorname{Im}z) e^{imz(q-q')/p} f(q', p) \right. \\ &\quad \left. - \theta(p \operatorname{Im}z) e^{imz(q+q')/p} f(q', -p) \right] \\ &= \operatorname{sgn}(\operatorname{Im}z) \frac{i m}{|p|} e^{imzq/p} \left[\theta(p \operatorname{Im}z) \left(\int_0^q dq' e^{-imzq'/p} f(q', p) \right) \right. \\ &\quad \left. - \int_0^\infty dq' e^{imzq'/p} f(q', -p) \right) \\ &\quad \left. + \theta(-p \operatorname{Im}z) \int_q^\infty dq' e^{-imzq'/p} f(q', p) \right]. \end{aligned}$$

We denote by L_D the self-adjoint operator in $L^2(\mathbb{R}_+ \times \mathbb{R})$ having R_z^D as resolvent, so that

$$\operatorname{dom}(L_D) = \{ f \in L^2(\mathbb{R}_+ \times \mathbb{R}) \mid (0 \oplus f) \in \operatorname{dom}(L_\infty) \}, \quad L_D f := L_\infty(0 \oplus f). \quad (3.10)$$

For all $(q, p) \in \mathbb{R}_+ \times \mathbb{R}, t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$, from the above definition and from Eq. (3.4) we get⁴

$$\begin{aligned} (e^{itL_D} f)(q, p) &= (e^{itL_\infty}(0 \oplus f))(q, p) \\ &= \theta\left(q + \frac{pt}{m}\right) (e^{itL_0}(0 \oplus f))(q, p) \\ &\quad - \theta\left(-q - \frac{pt}{m}\right) (e^{itL_0}(0 \oplus f))(-q, -p), \end{aligned} \quad (3.11)$$

which describes the motion of a classical particle on the half-line \mathbb{R}_+ with elastic collision at $q = 0$. Let us also mention that, in view of the basic identity

$$(e^{itL_0}(0 \oplus f))(q, p) = \theta\left(q + \frac{pt}{m}\right) (e^{itL_0}(0 \oplus f))(q, p), \quad (3.12)$$

⁴ Note that for $q > 0$ we have $\theta(-tqp) \theta\left(\frac{|pt|}{m} - |q|\right) = \theta(-tp) \theta\left(-\frac{pt}{m} - q\right) = \theta\left(-q - \frac{pt}{m}\right) = 1 - \theta\left(q + \frac{pt}{m}\right)$.

the above relation (3.11) is equivalent to

$$(e^{itL_D} f)(q, p) = (e^{itL_0}(0 \oplus f))(q, p) - (e^{itL_0}(0 \oplus f))(-q, -p). \quad (3.13)$$

Another equivalent (and more explicit) formula for the action of the unitary group e^{itL_D} is the one given in Eq. (1.8).

Finally, from Lemmata 3.1 and 3.2 (here employed with $q > 0$) we derive, respectively,

$$\begin{aligned} (R_z^0(0 \oplus f))(q, p) &= (R_z^D f)(q, p) - \theta(p \operatorname{Im} z) e^{imzq/p} (\gamma_+ R_z^D f)(p), \\ (e^{itL_0}(0 \oplus f))(q, p) &= \theta\left(q + \frac{pt}{m}\right) (e^{itL_D} f)(q, p) \\ &\quad - \theta\left(-q - \frac{pt}{m}\right) (e^{itL_D} f)(-q, -p). \end{aligned} \quad (3.14)$$

3.2 Classical dynamics on the graph with total reflection at the vertex

Let us now consider the “classical” Hilbert space $\oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$ and indicate any of its elements with the vector notation

$$f \in \oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R}), \quad f(q, p) \equiv \begin{pmatrix} f_1(q, p) \\ \vdots \\ f_n(q, p) \end{pmatrix}.$$

Let L_D be defined according to Eq. (3.10), and consider the classical dynamics on the star-graph \mathcal{G} with total elastic collision in the vertex; this is described by the self-adjoint operator

$$L_D^\oplus := \oplus_{\ell=1}^n L_D.$$

The associated resolvent and time evolution operators are respectively given by

$$\begin{aligned} R_z^{D\oplus} &:= (L_D^\oplus - z)^{-1} = \oplus_{\ell=1}^n R_z^D, \\ e^{itL_D^\oplus} &= \oplus_{\ell=1}^n e^{itL_D}. \end{aligned}$$

Explicitly, for $\ell = 1, \dots, n$ and $t \in \mathbb{R}$, from Eq. (1.8) we derive

$$(e^{itL_D^\oplus} f)_\ell(q, p) = \begin{cases} f_\ell\left(q + \frac{pt}{m}, p\right) & \text{if } q + \frac{pt}{m} > 0, \\ -f_\ell\left(-q - \frac{pt}{m}, -p\right) & \text{if } q + \frac{pt}{m} < 0. \end{cases}$$

3.3 Singular perturbations of the classical dynamics on the graph

Let us consider the restriction to $\text{dom}(L_D)$ of the trace map γ_+ introduced in Eq. (3.7); this defines a bounded operator

$$\gamma_+ : \text{dom}(L_D) \rightarrow L^2_{\text{odd}}(\mathbb{R}, |p|dp).$$

We use the above map to define a trace operator on the graph:

$$\gamma^\oplus_+ := \bigoplus_{\ell=1}^n \gamma_+ : \bigoplus_{\ell=1}^n \text{dom}(L_D) \rightarrow \bigoplus_{\ell=1}^n L^2_{\text{odd}}(\mathbb{R}, |p|dp).$$

In what follows we use the technique developed by one of us in [20,21] to characterize all the self-adjoint extensions of the symmetric operator $L_D^\oplus \upharpoonright \ker(\gamma^\oplus_+)$ (see Theorem 3.3 below). Among those we select the one that turns out to be useful to study the semiclassical limit of $\exp(-iH_{Kt}/\hbar)$, see Remark 3.4.

To proceed, we introduce the operator

$$\begin{aligned} \check{G}_z^\oplus &:= \gamma^\oplus_+ R_z^{D\oplus} \equiv \bigoplus_{\ell=1}^n \check{G}_z^+ : \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R}) \rightarrow \bigoplus_{\ell=1}^n L^2_{\text{odd}}(\mathbb{R}, |p|dp), \\ \check{G}_z^+ &:= \gamma_+ R_z^D, \end{aligned}$$

and its adjoint with complex conjugate parameter,

$$\begin{aligned} G_z^\oplus &:= (\check{G}_z^\oplus)^* \equiv \bigoplus_{\ell=1}^n G_z^+ : \bigoplus_{\ell=1}^n L^2_{\text{odd}}(\mathbb{R}, |p|dp) \rightarrow \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R}), \\ G_z^+ &:= (\check{G}_z^+)^*. \end{aligned}$$

Note that by Eq. (3.1) one has

$$(\gamma R_z^0(0 \oplus f))(p) = \text{sgn}(\text{Im}z) \frac{im}{|p|} \int_0^\infty dq' \theta(-p\text{Im}z) e^{-imzq'/p} f(q', p).$$

From the latter identity, together with Eqs. (3.9) and (3.5), we derive

$$\begin{aligned} (\check{G}_z^+ f)(p) &= (\gamma_+ R_z^\infty(0 \oplus f))(p) \\ &= \theta(-p\text{Im}z) (\gamma R_z^0(0 \oplus f))(p) - \theta(p\text{Im}z) (\gamma R_z^0(0 \oplus f))(-p), \\ &= \text{sgn}(\text{Im}z) \frac{im}{|p|} \int_0^\infty dq' \left(\theta(-p\text{Im}z) e^{-imzq'/p} f(q', p) \right. \\ &\quad \left. - \theta(p\text{Im}z) e^{imzq'/p} f(q', -p) \right). \end{aligned} \tag{3.15}$$

In view of the latter expression, for all $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and any $\phi \in L^2_{\text{odd}}(\mathbb{R}, |p|dp)$ we have

$$\int_{\mathbb{R}_+ \times \mathbb{R}} dqdp \overline{(G_z^+ \phi)(q, p)} f(q, p) = \int_{\mathbb{R}} dp \overline{\phi(p)} (\check{G}_z^+ f)(p)$$

$$\begin{aligned}
 &= \int_{\mathbb{R}} dp \overline{\phi(p)} \left(\operatorname{sgn}(\operatorname{Im} \bar{z}) \frac{im}{|p|} \int_0^\infty dq \left(\theta(-p \operatorname{Im} \bar{z}) e^{-im \bar{z} q/p} f(q, p) \right. \right. \\
 &\quad \left. \left. - \theta(p \operatorname{Im} \bar{z}) e^{im \bar{z} q/p} f(q, -p) \right) \right) \\
 &= \int_{\mathbb{R}_+ \times \mathbb{R}} dq dp \overline{(\phi(p) - \phi(-p))} \theta(p \operatorname{Im} z) \operatorname{sgn}(\operatorname{Im} z) \left(-\frac{im}{|p|} \right) e^{-im \bar{z} q/p} f(q, p) \\
 &= \int_{\mathbb{R}_+ \times \mathbb{R}} dq dp \overline{\theta(p \operatorname{Im} z) \operatorname{sgn}(\operatorname{Im} z) \frac{2im}{|p|} e^{im z q/p} \phi(p)} f(q, p),
 \end{aligned}$$

which proves that

$$(G_z^+ \phi)(q, p) = 2 g_z(q, p) \phi(p), \quad g_z(q, p) := \theta(p \operatorname{Im} z) \operatorname{sgn}(\operatorname{Im} z) \frac{im}{|p|} e^{im z q/p}. \tag{3.16}$$

On account of the identities

$$(\gamma_+(g_z - g_w))(p) = \frac{im}{2|p|} (\operatorname{sgn}(\operatorname{Im} z) - \operatorname{sgn}(\operatorname{Im} w)),$$

which can be easily checked by a direct calculation (see also [5, p. 7]), and

$$\gamma_+(G_z^+ - G_w^+) = (z - w) (G_w^+)^* G_z^+,$$

which is consequence of the first resolvent identity (see [20, Lem. 2.1], paying attention to the different sign convention in the definition of the resolvent), we have that the linear map

$$\begin{aligned}
 \operatorname{dom}(M_z^+) &:= L_{\text{odd}}^2(\mathbb{R}, |p|^{-1} dp) \cap L_{\text{odd}}^2(\mathbb{R}, |p| dp), \\
 M_z^+ &: \operatorname{dom}(M_z^+) \subset L_{\text{odd}}^2(\mathbb{R}, |p|^{-1} dp) \rightarrow L_{\text{odd}}^2(\mathbb{R}, |p| dp), \\
 (M_z^+ \phi)(p) &:= 2 m_z^\infty(p) \phi(p), \quad m_z^\infty(p) := -\operatorname{sgn}(\operatorname{Im} z) \frac{im}{2|p|}, \tag{3.17}
 \end{aligned}$$

satisfies the identities

$$(M_z^+)^* = M_{\bar{z}}^+, \quad M_z^+ - M_w^+ = (w - z)(G_w^+)^* G_z^+.$$

Hence, setting

$$\begin{aligned}
 M_z^\oplus &: \bigoplus_{\ell=1}^n \operatorname{dom}(M_z^+) \subset \bigoplus_{\ell=1}^n L_{\text{odd}}^2(\mathbb{R}, |p|^{-1} dp) \rightarrow \bigoplus_{\ell=1}^n L_{\text{odd}}^2(\mathbb{R}, |p| dp), \\
 M_z^\oplus &:= \bigoplus_{\ell=1}^n M_z^+,
 \end{aligned}$$

one gets the identities

$$(M_z^\oplus)^* = M_{\bar{z}}^\oplus, \quad M_z^\oplus - M_w^\oplus = (w - z)(G_w^\oplus)^* G_z^\oplus.$$

To proceed, let us consider any orthogonal projector $\Pi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ and any self-adjoint operator $\mathbb{B} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, represented by the matrices with components $(\Pi)_{\ell, \ell'}$ and $(\mathbb{B})_{\ell, \ell'}$ respectively. By a slight abuse of notation we use the same symbols to denote the corresponding operators on vector valued functions; e.g., for $\mathbf{f} = \bigoplus_{\ell=1}^n f_\ell \in \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$, one has $\Pi \mathbf{f} \in \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$ with components $(\ell = 1, \dots, n)$

$$(\Pi \mathbf{f})_\ell = \sum_{\ell'=1}^n (\Pi)_{\ell \ell'} f_{\ell'},$$

or, for $\bigoplus_{\ell=1}^n \phi_\ell \in \bigoplus_{\ell=1}^n L^2_{odd}(\mathbb{R}, \rho dp)$, one has $\Pi(\bigoplus_{\ell=1}^n \phi_\ell) := \bigoplus_{\ell=1}^n \left(\sum_{j=1}^n (\Pi)_{\ell j} \phi_j \right)$, and similarly for \mathbb{B} . Then, by [20, Thm. 2.1] here employed with $\tau := \Pi \gamma_+^\oplus$, we obtain the following

Theorem 3.3 *Let $z \in \mathbb{C} \setminus \mathbb{R}$. Assume that $\Pi M_z^\oplus = M_z^\oplus \Pi$ and $\Pi \mathbb{B} = \mathbb{B} \Pi$. Then, the linear bounded operator*

$$R_z^{\Pi, \mathbb{B}} := R_z^{D^\oplus} + G_z^\oplus \Pi (\mathbb{B} + M_z^\oplus)^{-1} \Pi \check{G}_z^\oplus \tag{3.18}$$

is the resolvent of a self-adjoint extension $L_{\Pi, \mathbb{B}}$ of the densely defined, closed symmetric operator $L_D^\oplus \upharpoonright \ker(\gamma_+^\oplus)$. Such an extension acts on its domain

$$\begin{aligned} \text{dom}(L_{\Pi, \mathbb{B}}) := \{ & \mathbf{f} \in \bigoplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R}) \mid \\ & \mathbf{f} = \mathbf{f}_z + G_z^\oplus \Pi (\mathbb{B} + M_z^\oplus)^{-1} \Pi \gamma_+^\oplus \mathbf{f}_z, \mathbf{f}_z \in \text{dom}(L_D^\oplus) \} \end{aligned}$$

according to

$$(L_{\Pi, \mathbb{B}} - z) \mathbf{f} = (L_D^\oplus - z) \mathbf{f}_z. \tag{3.19}$$

Remark 3.4 We use the notation L_K (where K stands for Kirchhoff) to denote the self-adjoint extension corresponding to the choices

$$\mathbb{B} = 0, \quad \Pi = \frac{1}{n} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}. \tag{3.20}$$

We denote the associated resolvent operator with $R_z^K := (L_K - z)^{-1}$.

In the sequel, we proceed to determine the unitary evolution associated with the above choices by means of arguments analogous to those described in the proof of [5, Prop. 2.4].

Proposition 3.5 For all $f \in \oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$ and for all $t \in \mathbb{R}$ there holds

$$(e^{itL_K} f)(q, p) = \begin{pmatrix} (e^{itL_0}(0 \oplus f_1))(q, p) \\ \vdots \\ (e^{itL_0}(0 \oplus f_n))(q, p) \end{pmatrix} - \mathbb{S} \begin{pmatrix} (e^{itL_0}(0 \oplus f_1))(-q, -p) \\ \vdots \\ (e^{itL_0}(0 \oplus f_n))(-q, -p) \end{pmatrix} \quad (3.21)$$

where $\mathbb{S} := \mathbf{1} - 2 \Pi$ was already defined in relation with the quantum scattering operator, see Eq. (2.8).

Proof Throughout the whole proof we work component-wisely, denoting with $\ell \in \{1, \dots, n\}$ a fixed index. Let us first remark that the resolvent (3.18), with \mathbb{B}, Π as in Eq. (3.20), acts on any element $f \in \oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$ according to

$$\begin{aligned} (R_z^K f)_\ell(q, p) &= (R_z^D f_\ell)(q, p) + (2g_z(q, p)) \sum_{j'=1}^n \frac{(\Pi)_{\ell j'}}{2m_z^\infty(p)} \sum_{j=1}^n (\Pi)_{j' j} (\gamma_+ R_z^D f_j)(p) \\ &= (R_z^D f_\ell)(q, p) + \frac{g_z(q, p)}{m_z^\infty(p)} \frac{1}{n} \sum_{j=1}^n (\gamma_+ R_z^D f_j)(p). \end{aligned}$$

From the above relation we derive the following, recalling the explicit expressions for $g_z(q, p)$ and $m_z^\infty(p)$ given in Eqs. (3.16) and (3.17), as well as Eq. (3.15) for $\gamma_+ R_z^D f = \check{G}_z^+ f$:

$$\begin{aligned} (R_z^K f)_\ell(q, p) &= (R_z^D f_\ell)(q, p) \\ &+ \theta(p \operatorname{Im} z) \operatorname{sgn}(\operatorname{Im} z) \frac{2im}{n|p|} \sum_{j=1}^n \int_0^\infty dq' e^{imz(q+q')/p} f_j(q', -p). \end{aligned}$$

We now proceed to compute the unitary operator e^{-itL_K} ($t \in \mathbb{R}$) by inverse Laplace transform, using the above representation for the resolvent R_z^K . Let us first assume $t > 0$; then, for any $c > 0$ and $f \in \operatorname{dom}(L_K)$, we get (see [10, Ch. III, Cor. 5.15])

$$\begin{aligned} (e^{-itL_K} f)_\ell &= \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r+ic}^{r+ic} dz e^{-izt} (R_z^K f)_\ell \\ &= \frac{e^{ct}}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r}^r dk e^{-itk} (R_{k+ic}^K f)_\ell. \end{aligned}$$

On the one hand, recalling Eq. (3.13) we have

$$\begin{aligned} \frac{e^{ct}}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r}^r dk e^{-itk} (R_{k+ic}^D f_\ell)(q, p) &= (e^{-itL_D} f_\ell)(q, p) \\ &= (e^{-itL_0}(0 \oplus f_\ell))(q, p) - (e^{-itL_0}(0 \oplus f_\ell))(-q, -p). \end{aligned}$$

On the other hand, noting that $\theta(p\text{Im}(k+ic)) = \theta(pc) = \theta(p)$ and $\text{sgn}(\text{Im}(k+ic)) = \text{sgn}(c) = +1$ for $c > 0$, by computations similar to those reported in the proof of [5, Prop. 2.4] we get⁵

$$\begin{aligned} & \frac{e^{ct}}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r}^r dk e^{-itk} \theta(p\text{Im}(k+ic)) \text{sgn}(\text{Im}(k+ic)) \frac{2im}{n|p|} \\ & \sum_{j=1}^n \int_0^\infty dq' e^{im(k+ic)(q+q')/p} f_j(q', -p) \\ & = \theta(p) \frac{2m}{n|p|} \frac{e^{c(t-mq/p)}}{2\pi} \sum_{j=1}^n \lim_{r \rightarrow \infty} \int_{-r}^r dk e^{i(mq/p-t)k} \\ & \int_{\mathbb{R}} dq' e^{im(k+ic)q'/p} \theta(q') f_j(q', -p) \\ & = \theta(p) \frac{2m}{n|p|} e^{c(t-mq/p)} \\ & \sum_{j=1}^n \mathfrak{F}^{-1} \left(\mathfrak{F}(\theta(\cdot) f_j(\cdot, -p))(-m(*+ic)/p) \right) (mq/p-t) \\ & = \theta(p) \frac{2}{n} \sum_{j=1}^n \theta\left(-q + \frac{pt}{m}\right) f_j\left(-q + \frac{pt}{m}, -p\right) \\ & = \frac{2}{n} \sum_{j=1}^n (e^{-itL_0}(0 \oplus f_j))(-q, -p). \end{aligned}$$

Summing up, the above relations imply

$$\begin{aligned} (e^{-itL_K} f)_\ell & = (e^{-itL_0}(0 \oplus f_\ell))(q, p) - (e^{-itL_0}(0 \oplus f_\ell))(-q, -p) \\ & \quad + \frac{2}{n} \sum_{j=1}^n (e^{-itL_0}(0 \oplus f_j))(-q, -p). \end{aligned} \tag{3.22}$$

For $t < 0$ one can perform similar computations, starting from the following identity where $c > 0$:

$$\begin{aligned} (e^{-itL_K} f)_\ell & = -\frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r-ic}^{r-ic} dz e^{-izt} (R_z^K f)_\ell \\ & = -\frac{e^{-ct}}{2\pi i} \lim_{r \rightarrow \infty} \int_{-r}^r dk e^{-itk} (R_{k-ic}^K f)_\ell. \end{aligned}$$

⁵ Especially, recall the following basic identity regarding the unitary Fourier transform \mathfrak{F} and its inverse \mathfrak{F}^{-1} :

$$\mathfrak{F}^{-1} \left(\mathfrak{F}h(a(*+ic)) \right) (q) = \frac{e^{cq}}{|a|} h(q/a) \quad \text{for } a \in \mathbb{R} \setminus \{0\}, c > 0, q \in \mathbb{R},$$

which holds true whenever $\frac{e^c}{|a|} h(\cdot/a) \in L^2(\mathbb{R})$. In addition, keep in mind the relation written in Eq. (3.12).

We omit the related details for brevity. In the end, one obtains exactly the same expression as in Eq. (3.22), which with the trivial replacement $t \rightarrow -t$ proves Eq. (3.21).

Remark 3.6 By Eq. (3.12) we infer that the action of the unitary group e^{itL_K} is explicitly given, component-wisely, by Eq. (1.9).

Remark 3.7 Recalling the explicit form of Π (see Eq. (3.20)) we obtain

$$\mathbb{S} = \frac{1}{n} \begin{pmatrix} n-2 & -2 & \cdots & -2 \\ -2 & n-2 & \cdots & -2 \\ \vdots & \vdots & \ddots & \vdots \\ -2 & -2 & \cdots & n-2 \end{pmatrix}, \quad \mathbf{1} - \Pi = \frac{1}{n} \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}. \quad (3.23)$$

In particular, for a star-graph with three edges ($n = 3$) we have

$$\mathbb{S} = \frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix},$$

whence $\mathbb{S} = -\mathbb{M}$ with respect to the notation used in [1].

3.4 The semiclassical wave operators and scattering operator

Let us consider the wave operators and the corresponding scattering operator on $\oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$, respectively defined by Eqs. (1.10) and (1.11). The following proposition provides explicit expressions for these operators.

Proposition 3.8 *The limits in Eq. (1.10) exist point-wisely for any $\xi \equiv (q, p) \in \mathbb{R}_+ \times (\mathbb{R} \setminus \{0\})$ and in $L^2(\mathbb{R}_+ \times \mathbb{R})$ for any $f \in \oplus_{\ell=1}^n L^2(\mathbb{R}_+ \times \mathbb{R})$; moreover, there holds*

$$(\Omega_{cl}^\pm f)(q, p) = [\mathbf{1} - \theta(\mp p) 2 \Pi] f(q, p) = [\theta(\pm p) \mathbf{1} + \theta(\mp p) \mathbb{S}] f(q, p). \quad (3.24)$$

Furthermore, the scattering operator is given by

$$S_{cl} = \mathbf{1} - 2 \Pi = \mathbb{S}. \quad (3.25)$$

Proof First of all let us point out that for all $t \in \mathbb{R}$ and any $\ell \in \{1, \dots, n\}$, Eqs. (3.13) and (3.21) give, respectively,

$$\begin{aligned} (e^{itL_D} f)(q, p) &= (e^{itL_0}(0 \oplus f))(q, p) - (e^{itL_0}(0 \oplus f))(-q, -p) \\ &= (0 \oplus f)\left(q + \frac{pt}{m}, p\right) - (0 \oplus f)\left(-q - \frac{pt}{m}, -p\right) \\ &= \theta\left(q + \frac{pt}{m}\right) f\left(q + \frac{pt}{m}, p\right) - \theta\left(-q - \frac{pt}{m}\right) f\left(-q - \frac{pt}{m}, -p\right); \\ (e^{-itL_K} f)_\ell(q, p) &= (e^{-itL_0}(0 \oplus f_\ell))(q, p) + \sum_{j=1}^n (2\Pi - \mathbf{1})_{\ell j} (e^{-itL_0}(0 \oplus f_j))(-q, -p) \end{aligned}$$

$$\begin{aligned}
 &= (0 \oplus f_\ell)\left(q - \frac{pt}{m}, p\right) + \sum_{j=1}^n (2\Pi - \mathbf{1})_{\ell_j} (0 \oplus f_j)\left(-q + \frac{pt}{m}, -p\right) \\
 &= \theta\left(q - \frac{pt}{m}\right) f_\ell\left(q - \frac{pt}{m}, p\right) \\
 &\quad + \sum_{j=1}^n (2\Pi - \mathbf{1})_{\ell_j} \theta\left(-q + \frac{pt}{m}\right) f_j\left(-q + \frac{pt}{m}, -p\right).
 \end{aligned}$$

In view of the above relations, recalling that $q > 0$, by direct computations we obtain

$$\begin{aligned}
 (e^{itL_D^\oplus} e^{-itL_K} f)_\ell(q, p) &= (e^{itL_D} (e^{-itL_K} f)_\ell)(q, p) \\
 &= \theta\left(q + \frac{pt}{m}\right) (e^{-itL_K} f)_\ell\left(q + \frac{pt}{m}, p\right) \\
 &\quad - \theta\left(-q - \frac{pt}{m}\right) (e^{-itL_K} f)_\ell\left(-q - \frac{pt}{m}, -p\right) \\
 &= \theta\left(q + \frac{pt}{m}\right) f_\ell(q, p) - \theta\left(-q - \frac{pt}{m}\right) \sum_{j=1}^n (2\Pi - \mathbf{1})_{\ell_j} f_j(q, p) \\
 &= f_\ell(q, p) - \theta\left(-q - \frac{pt}{m}\right) \sum_{j=1}^n 2 \pi_{\ell_j} f_j(q, p),
 \end{aligned}$$

which gives

$$(e^{itL_D^\oplus} e^{-itL_K} f)(q, p) = f(q, p) - \theta\left(-q - \frac{pt}{m}\right) (2\Pi f)(q, p).$$

Then Eq. (3.24) follows noting that $\theta\left(-q - \frac{pt}{m}\right) \rightarrow \theta(\mp p)$ for $t \rightarrow \pm\infty$.

Next, since Π is symmetric, by elementary arguments we get

$$((\Omega_{cl}^\pm)^* f)(q, p) = [\mathbf{1} - \theta(\mp p) 2\Pi] f(q, p).$$

On account of the above identity we obtain

$$\begin{aligned}
 (S_{cl} f)(q, p) &= ((\Omega_{cl}^+)^* \Omega_{cl}^- f)(q, p) = [\mathbf{1} - \theta(-p) 2\Pi] (\Omega_{cl}^- f)(q, p) \\
 &= [\mathbf{1} - \theta(-p) 2\Pi] [\mathbf{1} - \theta(p) 2\Pi] f(q, p) = [\mathbf{1} - 2\Pi] f(q, p),
 \end{aligned}$$

which proves Eq. (3.25).

4 Comparison of the semiclassical and quantum theories

Recall the definition of coherent states in $L^2(\mathbb{R})$ given in Eqs. (1.1), and (1.2) describing their free evolution.

Additionally, notice that the action of the operators e^{-itL_K} , Ω_{cl}^\pm , and S_{cl} can be extended in a natural way to $\bigoplus_{\ell=1}^n L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

For later reference let us point out the following auxiliary result, regarding the functions $\tilde{\Psi}_{\sigma,\xi}^{\hbar}$ defined in Eq. (1.13), and the operators \mathcal{U}_t^\pm defined in Eq. (2.3).

Lemma 4.1 *For all $\xi = (q, p) \in \mathbb{R}_+ \times \mathbb{R}$ and for any $t \in \mathbb{R}$ there holds*

$$\| \mathcal{U}_t^\pm \tilde{\Psi}_{\sigma_0,\xi}^{\hbar} - e^{\frac{i}{\hbar} A t} \tilde{\Psi}_{\sigma_t, \mp \xi_t}^{\hbar} \|_{L^2(\mathbb{R}_+)} \leq \frac{1}{\sqrt{2}} e^{-\frac{q^2}{4\hbar\sigma_0^2}}.$$

Proof Let us first remark that, on account of the considerations reported in Remark 2.1, $\tilde{\Psi}_{\sigma_0,\xi}^{\hbar} = \Theta^* \Psi_{\sigma_0,\xi}^{\hbar}$ and for any $x > 0$ we have the chain of identities

$$\begin{aligned} (\mathcal{U}_t^\pm \tilde{\Psi}_{\sigma_0,\xi}^{\hbar})(x) &= (U_t^0 \theta \Psi_{\sigma_0,\xi}^{\hbar})(\mp x) = (U_t^0 \Psi_{\sigma_0,\xi}^{\hbar})(\mp x) + E_t^{\hbar}(\mp x) \\ &= e^{\frac{i}{\hbar} A t} \Psi_{\sigma_t, \xi_t}^{\hbar}(\mp x) + E_t^{\hbar}(\mp x) \\ &= e^{\frac{i}{\hbar} A t} \Psi_{\sigma_t, \mp \xi_t}^{\hbar}(x) + E_t^{\hbar}(\mp x) = e^{\frac{i}{\hbar} A t} \tilde{\Psi}_{\sigma_t, \mp \xi_t}^{\hbar}(x) + E_t^{\hbar}(\mp x), \end{aligned}$$

where we put $E_t^{\hbar}(x) := -(U_t^0 (1 - \theta) \Psi_{\sigma_0,\xi}^{\hbar})(x)$ for convenience of notation and used the trivial identity $\Psi_{\sigma,\xi}^{\hbar}(-x) = \Psi_{\sigma,-\xi}^{\hbar}(x)$.

On the other hand, via an explicit calculation involving the trivial inequality $e^{-\eta(a+b)^2} \leq e^{-\eta(a^2+b^2)}$ for $\eta, a, b \geq 0$ we obtain

$$\begin{aligned} \|E_t^{\hbar}(\mp \cdot)\|_{L^2(\mathbb{R}_+)}^2 &\leq \|E_t^{\hbar}\|_{L^2(\mathbb{R})}^2 = \|U_t^0 (1 - \theta) \Psi_{\sigma_0,\xi}^{\hbar}\|_{L^2(\mathbb{R})}^2 = \|(1 - \theta) \Psi_{\sigma_0,\xi}^{\hbar}\|_{L^2(\mathbb{R})}^2 \\ &= \int_{-\infty}^0 dx |\Psi_{\sigma_0,\xi}^{\hbar}(x)|^2 = \frac{1}{\sqrt{2\pi\hbar\sigma_0}} \int_0^\infty dx e^{-\frac{(x+q)^2}{2\hbar\sigma_0^2}} \leq \frac{1}{2} e^{-\frac{q^2}{2\hbar\sigma_0^2}}, \end{aligned} \tag{4.1}$$

which proves the thesis in view of the previous arguments.

To proceed let us point out the forthcoming lemma which characterizes a large class of functions satisfying the condition in Definition 1.1.

Lemma 4.2 *Let $\eta > 0$, $\xi = (q, p) \in \mathbb{R}_+ \times \mathbb{R}$ and consider a family of functions $\chi_{q,\eta} \in L^\infty(\mathbb{R}_+)$, uniformly bounded in $L^\infty(\mathbb{R}_+)$ with respect to η, q and such that*

$$\chi_{q,\eta}(x) = 1 \quad \text{for } |x - q| < \eta q. \tag{4.2}$$

Then, the functions $\Xi_{\sigma,\xi}^{\hbar} \in L^2(\mathbb{R}_+)$ defined by

$$\Xi_{\sigma,\xi}^{\hbar}(x) := \frac{1}{\|\chi_{q,\eta} \tilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}} \chi_{q,\eta}(x) \tilde{\Psi}_{\sigma,\xi}^{\hbar}(x) \quad (x > 0) \tag{4.3}$$

fulfill the condition (1.14) with $\varepsilon < \min\{1/4, \eta^2/8\}$ and C_0 depending only on $\|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)}$.

Proof Let us first remark that the states $\Xi_{\sigma,\xi}^{\hbar}$ defined in Eq. (4.3) have unit norm in $L^2(\mathbb{R}_+)$ by construction; moreover, again from Eq. (4.3) it follows that $\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar} = \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \Xi_{\sigma,\xi}^{\hbar}$. Taking these facts into account, we have

$$\begin{aligned} \|\Xi_{\sigma,\xi}^{\hbar} - \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} &\leq \|\Xi_{\sigma,\xi}^{\hbar} - \chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} + \|(\chi_{q,\eta} - 1) \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \\ &= |1 - \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}| \|\Xi_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \\ &\quad + \|(\chi_{q,\eta} - 1) \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \\ &= |1 - \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}| + \|(\chi_{q,\eta} - 1) \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

On one hand, recalling the definition (1.13) of $\widetilde{\Psi}_{\sigma,\xi}^{\hbar}$ and that $\|\Psi_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R})} = 1$, using the basic inequality $(a - b)^2 \leq |a^2 - b^2|$ for $a, b > 0$ we get

$$\begin{aligned} |1 - \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}|^2 &\leq \left| \|\Psi_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R})}^2 - \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 \right| \\ &= \left| \int_{\mathbb{R}} dx |\Psi_{\sigma,\xi}^{\hbar}(x)|^2 - \int_0^{\infty} dx |\chi_{q,\eta}(x) \Psi_{\sigma,\xi}^{\hbar}(x)|^2 \right| \\ &\leq \int_{\mathbb{R}} dx |1 - \theta(x) |\chi_{q,\eta}(x)|^2| |\Psi_{\sigma,\xi}^{\hbar}(x)|^2 \\ &\leq \int_{\mathbb{R}} dx (1 - \theta(x)) |\Psi_{\sigma,\xi}^{\hbar}(x)|^2 + \int_{\mathbb{R}} dx \theta(x) |1 - |\chi_{q,\eta}(x)||^2 |\Psi_{\sigma,\xi}^{\hbar}(x)|^2. \end{aligned}$$

Recalling the hypothesis (4.2), the explicit expression (1.1) for $\Psi_{\sigma,\xi}^{\hbar}$, and that we are assuming $q > 0$, from the above results we derive

$$\begin{aligned} |1 - \|\chi_{q,\eta} \widetilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}|^2 &\leq \int_{-\infty}^0 dx |\Psi_{\sigma,\xi}^{\hbar}(x)|^2 \\ &\quad + \int_{\mathbb{R}_+ \cap \{|x-q| > \eta\}} dx |1 - |\chi_{q,\eta}(x)||^2 |\Psi_{\sigma,\xi}^{\hbar}(x)|^2 \\ &\leq \frac{1}{\sqrt{2\pi\hbar}|\sigma|} \left[\int_0^{\infty} dx e^{-\frac{(x+q)^2}{2\hbar|\sigma|^2}} \right. \\ &\quad \left. + (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)})^2 \int_{\{|x-q| > \eta\}} dx e^{-\frac{(x-q)^2}{2\hbar|\sigma|^2}} \right] \\ &\leq \frac{1}{\sqrt{2\pi\hbar}|\sigma|} \left[e^{-\frac{q^2}{2\hbar|\sigma|^2}} \int_0^{\infty} dx e^{-\frac{x^2}{2\hbar|\sigma|^2}} \right. \\ &\quad \left. + (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)})^2 e^{-\frac{\eta^2 q^2}{4\hbar|\sigma|^2}} \int_{\mathbb{R}} dx e^{-\frac{(x-q)^2}{4\hbar|\sigma|^2}} \right] \\ &\leq \frac{1}{2} e^{-\frac{q^2}{2\hbar|\sigma|^2}} + \sqrt{2} (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)})^2 e^{-\frac{\eta^2 q^2}{4\hbar|\sigma|^2}}. \end{aligned}$$

On the other hand, by arguments similar to those employed above we get

$$\begin{aligned} \|(\chi_{q,\eta} - 1) \tilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 &= \int_0^\infty dx \left| \chi_{q,\eta}(x) - 1 \right|^2 \left| \Psi_{\sigma,\xi}^{\hbar}(x) \right|^2 \\ &\leq (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)})^2 \frac{1}{\sqrt{2\pi\hbar}|\sigma|} \int_{\{|x-q|>\eta q\}} dx e^{-\frac{(x-q)^2}{2\hbar|\sigma|^2}} \\ &\leq \sqrt{2} (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)})^2 e^{-\frac{\eta^2 q^2}{4\hbar|\sigma|^2}}. \end{aligned}$$

Summing up, the previous results and the basic relation $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b > 0$ imply

$$\begin{aligned} \|\Xi_{\sigma,\xi}^{\hbar} - \tilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} &\leq \frac{1}{\sqrt{2}} e^{-\frac{q^2}{4\hbar|\sigma|^2}} + 2^{5/4} (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)}) e^{-\frac{\eta^2 q^2}{8\hbar|\sigma|^2}} \\ &\leq \left(\frac{1}{\sqrt{2}} + 2^{5/4} (1 + \|\chi_{q,\eta}\|_{L^\infty(\mathbb{R}_+)}) \right) e^{-\varepsilon \frac{q^2}{\hbar|\sigma|^2}}, \end{aligned}$$

which suffices to infer the thesis on account of the uniform boundedness of $\chi_{q,\eta}$.

Example 4.3 For $\eta \in (0, 1]$, consider the sharp cut-off functions

$$\chi_{q,\eta}(x) = \begin{cases} 0 & \text{if } x \leq (1 - \eta)q, \\ 1 & \text{if } x > (1 - \eta)q, \end{cases}$$

which clearly satisfy the hypothesis of Lemma 4.2. The corresponding elements $\Xi_{\sigma,\xi}^{\hbar} \in L^2(\mathbb{R}_+)$ defined according to Eq. (4.3) consist of normalized truncations of the coherent state $\Psi_{\sigma,\xi}^{\hbar}$ and fulfill the condition (1.14) as a consequence.

It is worth noting that for $\eta = 1$ we have $\chi_{q,\eta} \equiv 1$ on \mathbb{R}_+ , so that the associated function $\Xi_{\sigma,\xi}^{\hbar}$ is just the re-normalization of the bare truncation $\tilde{\Psi}_{\sigma,\xi}^{\hbar}$ introduced in Eq. (1.13), i.e.,

$$\Xi_{\sigma,\xi}^{\hbar} = \tilde{\Psi}_{\sigma,\xi}^{\hbar} / \|\tilde{\Psi}_{\sigma,\xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}.$$

Example 4.4 For $\eta \in (0, 1/2)$, consider the smooth functions on \mathbb{R}_+ such that

$$\chi_{q,\eta}(x) = \begin{cases} 0 & \text{for } |x - q| > (1 - \eta)q, \\ 1 & \text{for } |x - q| < \eta q, \end{cases} \quad \left| \chi_{q,\eta}(x) \right| \leq 1.$$

Again, the assumptions of Lemma 4.2 are certainly verified and the related functions $\Xi_{\sigma,\xi}^{\hbar}$ have compact support in \mathbb{R}_+ , besides satisfying the bound (1.14).

In addition to states fulfilling the requirements of Definitions 1.1 and 1.3, our arguments will often involve the unnormalized element

$$\Psi_{\sigma_0, \xi}^{\hbar} \in L^2(\mathcal{G}), \quad \Psi_{\sigma_0, \xi}^{\hbar} \equiv \begin{pmatrix} \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.4}$$

along with its classical counterpart

$$\Phi_{\sigma_0, x}^{\hbar} \in L^2(\mathcal{G}), \quad \Phi_{\sigma_0, x}^{\hbar} \equiv \begin{pmatrix} \tilde{\Phi}_{\sigma_0, x}^{\hbar} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.5}$$

with $\tilde{\Psi}_{\sigma_0, \xi}^{\hbar}$ defined as in Eq. (1.13) and

$$\tilde{\Phi}_{\sigma_0, x}^{\hbar}(\xi) := \tilde{\Psi}_{\sigma_0, \xi}^{\hbar}(x).$$

4.1 Comparing the dynamics. Proof of Theorem 1.4

Let $\Psi_{\sigma_0, \xi}^{\hbar}$ and $\Phi_{\sigma_0, (\cdot)}^{\hbar}(\xi)$ be, respectively, as in Eqs. (4.4) and (4.5), and note that from the triangular inequality it follows

$$\begin{aligned} & \| e^{-i \frac{t}{\hbar} H_K} \Xi_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{it L_K} \Sigma_{\sigma_t, (\cdot)}^{\hbar})(\xi) \|_{L^2(\mathcal{G})} \\ & \leq \| e^{-i \frac{t}{\hbar} H_K} \Xi_{\sigma_0, \xi}^{\hbar} - e^{-i \frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} \|_{L^2(\mathcal{G})} \\ & \quad + \| e^{-i \frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{it L_K} \Phi_{\sigma_t, (\cdot)}^{\hbar})(\xi) \|_{L^2(\mathcal{G})} \\ & \quad + \| e^{\frac{i}{\hbar} A_t} (e^{it L_K} \Phi_{\sigma_t, (\cdot)}^{\hbar})(\xi) - e^{\frac{i}{\hbar} A_t} (e^{it L_K} \Sigma_{\sigma_t, (\cdot)}^{\hbar})(\xi) \|_{L^2(\mathcal{G})}. \end{aligned} \tag{4.6}$$

Regarding the first term on the right-hand side of Eq. (4.6), by the unitarity of $e^{-i \frac{t}{\hbar} H_K}$ and the condition (1.14) we infer

$$\begin{aligned} \| e^{-i \frac{t}{\hbar} H_K} \Xi_{\sigma_0, \xi}^{\hbar} - e^{-i \frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} \|_{L^2(\mathcal{G})} &= \| \Xi_{\sigma_0, \xi}^{\hbar} - \Psi_{\sigma_0, \xi}^{\hbar} \|_{L^2(\mathcal{G})} \\ &= \| \Xi_{\sigma_0, \xi}^{\hbar} - \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} \|_{L^2(\mathbb{R}_+)} \leq C_0 e^{-\varepsilon \frac{q^2}{\hbar \sigma_0^2}}. \end{aligned}$$

As for the second term in Eq. (4.6), note that Eqs. (2.9) and (3.21) give

$$e^{-i \frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{it L_K} \Phi_{\sigma_t, (\cdot)}^{\hbar})(\xi)$$

$$= \begin{pmatrix} (\mathcal{U}_t^- \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{itL_0} (0 \oplus \tilde{\Phi}_{\sigma_t, (\cdot)}^{\hbar}))) (\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \mathbb{S} \begin{pmatrix} (\mathcal{U}_t^+ \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{itL_0} (0 \oplus \tilde{\Phi}_{\sigma_t, (\cdot)}^{\hbar}))) (-\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $(e^{itL_0} (0 \oplus \tilde{\Phi}_{\sigma_t, x}^{\hbar})) (\pm \xi) = \tilde{\Phi}_{\sigma_t, x}^{\hbar} (\pm \xi_t) = \tilde{\Psi}_{\sigma_t, \pm \xi_t}^{\hbar} (x)$ for $x \in \mathbb{R}_+$, from the above identity and from Lemma 4.1 we deduce⁶

$$\begin{aligned} & \| e^{-i \frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} (e^{itL_K} \Phi_{\sigma_t, (\cdot)}^{\hbar}) (\xi) \|_{L^2(\mathcal{G})}^2 \\ & \leq 2 \| \mathcal{U}_t^- \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} \tilde{\Psi}_{\sigma_t, \xi_t}^{\hbar} \|_{L^2(\mathbb{R}_+)}^2 \\ & \quad + 2 \sum_{\ell=1}^n |\mathbb{S}_{\ell 1}|^2 \| \mathcal{U}_t^+ \tilde{\Psi}_{\sigma_0, \xi}^{\hbar} - e^{\frac{i}{\hbar} A_t} \tilde{\Psi}_{\sigma_t, -\xi_t}^{\hbar} \|_{L^2(\mathbb{R}_+)}^2 \leq 2 e^{-\frac{q^2}{2\hbar\sigma_0^2}}. \end{aligned}$$

Let us finally consider the third term in Eq. (4.6). Recalling again the identity (3.21), we obtain

$$\begin{aligned} & (e^{itL_K} \Phi_{\sigma_t, (\cdot)}^{\hbar}) (\xi) - (e^{itL_K} \Sigma_{\sigma_t, (\cdot)}^{\hbar}) (\xi) \\ & = \begin{pmatrix} (e^{itL_0} (0 \oplus \tilde{\Phi}_{\sigma_t, (\cdot)}^{\hbar})) (\xi) - (e^{itL_0} (0 \oplus \tilde{\Sigma}_{\sigma_t, (\cdot)}^{\hbar})) (\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \\ & \quad - \mathbb{S} \begin{pmatrix} (e^{itL_0} (0 \oplus \tilde{\Phi}_{\sigma_t, (\cdot)}^{\hbar})) (-\xi) - (e^{itL_0} (0 \oplus \tilde{\Sigma}_{\sigma_t, (\cdot)}^{\hbar})) (-\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \end{aligned}$$

From the above identity, by arguments similar to those employed previously we get

$$\| (e^{itL_K} \Phi_{\sigma_t, (\cdot)}^{\hbar}) (\xi) - (e^{itL_K} \Sigma_{\sigma_t, (\cdot)}^{\hbar}) (\xi) \|_{L^2(\mathcal{G})}^2$$

⁶ Note also that, on account of Eq. (3.23), we have

$$1 + \sum_{\ell=1}^n |\mathbb{S}_{\ell 1}|^2 = 1 + |\mathbb{S}_{11}|^2 + \sum_{\ell=2}^n |\mathbb{S}_{\ell 1}|^2 = 1 + \left(\frac{n-2}{n}\right)^2 + (n-1) \left(\frac{2}{n}\right)^2 = 2.$$

$$\begin{aligned} &\leq 2 \|\tilde{\Psi}_{\sigma_t, \xi_t}^{\hbar} - \Xi_{\sigma_t, \xi_t}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + 2 \sum_{\ell=1}^n |\mathbb{S}_{\ell 1}|^2 \|\tilde{\Psi}_{\sigma_t, -\xi_t}^{\hbar} - \Xi_{\sigma_t, -\xi_t}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 \leq 4C_0^2 e^{-2\varepsilon \frac{q_t^2}{h|\sigma_t|^2}}. \end{aligned}$$

Summing up, the above bounds imply the thesis.

4.2 Comparing the wave and scattering operators. Proof of Theorem 1.6

Note the identity $\mathbb{S}(1, 0, \dots, 0)^T = (1 - 2/n, -2/n, \dots, -2/n)^T$ and recall the expression of Ω_{cl}^{\pm} given in Eq. (2.13). Then, by simple computations we get ($\ell = 1, \dots, n$)

$$(\Omega_{cl}^{\pm} \Xi_{\sigma_0, \xi}^{\hbar})_{\ell} = \begin{cases} \left(1 - \frac{1}{n} (1 \mp \mathcal{F}_c^* \mathcal{F}_s)\right) \Xi_{\sigma_0, \xi}^{\hbar} & \text{if } \ell = 1, \\ -\frac{1}{n} (1 \mp \mathcal{F}_c^* \mathcal{F}_s) \Xi_{\sigma_0, \xi}^{\hbar} & \text{if } \ell \neq 1. \end{cases}$$

Additionally, recalling the expression of Ω_{cl}^{\pm} given in Eq. (3.24), we get

$$(\Omega_{cl}^{\pm} \Sigma_{\sigma_0, x}^{\hbar})_{\ell}(\xi) = \begin{cases} \left(1 - \theta(\mp p) \frac{2}{n}\right) \Sigma_{\sigma_0, x}^{\hbar}(\xi) & \text{if } \ell = 1, \\ -\theta(\mp p) \frac{2}{n} \Sigma_{\sigma_0, x}^{\hbar}(\xi) & \text{if } \ell \neq 1. \end{cases}$$

In view of these results together with the identity (1.15) we derive

$$\begin{aligned} \|\Omega_{cl}^{\pm} \Xi_{\sigma_0, \xi}^{\hbar} - (\Omega_{cl}^{\pm} \Sigma_{\sigma_0, (\cdot)}^{\hbar})(\xi)\|_{L^2(\mathcal{G})} &= \frac{1}{\sqrt{n}} \|(1 \mp \mathcal{F}_c^* \mathcal{F}_s - 2\theta(\mp p)) \Xi_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \\ &= \frac{1}{\sqrt{n}} \|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s) \Xi_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

By the bound $\|(1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s\| \leq |1 - 2\theta(\mp p)|\|\mathcal{F}_c\| + \|\mathcal{F}_s\| \leq 2$ and by Eq. (1.14), we infer

$$\begin{aligned} &\|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s) \Xi_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \\ &\leq 2C_0 e^{-\varepsilon \frac{q_0^2}{h\sigma_0^2}} + \|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s) \tilde{\Psi}_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

In what follows we prove the following upper bound which concludes the proof of the theorem

$$\|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s) \tilde{\Psi}_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)} \leq \sqrt{2} \left(e^{-\frac{\sigma_0^2 p^2}{\hbar}} + e^{-\frac{q_0^2}{4h\sigma_0^2}} \right). \tag{4.7}$$

We start with the identity

$$\begin{aligned} & \|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s)\tilde{\Psi}_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 \\ &= \frac{2}{\pi} \int_0^\infty dk \left| \int_0^\infty dx \left((1 - 2\theta(\mp p)) \cos(kx) \pm i \sin(kx) \right) \Psi_{\sigma_0, \xi}^{\hbar}(x) \right|^2. \end{aligned}$$

Considering separately the cases $p > 0$ and $p < 0$ for the two possible choices of the signs, it is easy to convince oneself that

$$\begin{aligned} & \|((1 - 2\theta(\mp p))\mathcal{F}_c \mp \mathcal{F}_s)\tilde{\Psi}_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R}_+)}^2 \\ &= \frac{2}{\pi} \begin{cases} \int_0^\infty dk \left| \int_0^\infty dx e^{ikx} \Psi_{\sigma_0, \xi}^{\hbar}(x) \right|^2 & \text{if } p > 0, \\ \int_0^\infty dk \left| \int_0^\infty dx e^{-ikx} \Psi_{\sigma_0, \xi}^{\hbar}(x) \right|^2 & \text{if } p < 0. \end{cases} \end{aligned}$$

Recall that the Fourier transform of $\Psi_{\sigma_0, \xi}^{\hbar}$ is given by

$$(\mathcal{F}\Psi_{\sigma_0, \xi}^{\hbar})(k) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx e^{-ikx} \Psi_{\sigma_0, \xi}^{\hbar}(x) = \sqrt{\sigma_0} \left(\frac{2\hbar}{\pi}\right)^{1/4} e^{-\hbar\sigma_0^2(k-p/\hbar)^2 - ikq}.$$

Let us assume $p > 0$, we have the chain of identities/inequalities

$$\begin{aligned} \left(\frac{2}{\pi} \int_0^\infty dk \left| \int_0^\infty dx e^{ikx} \Psi_{\sigma_0, \xi}^{\hbar}(x) \right|^2\right)^{1/2} &= 2 \|(\mathcal{F}\theta \Psi_{\sigma_0, \xi}^{\hbar})(-\cdot)\|_{L^2(\mathbb{R}_+)} \\ &\leq 2 \|(\mathcal{F}\Psi_{\sigma_0, \xi}^{\hbar})(-\cdot)\|_{L^2(\mathbb{R}_+)} \\ &\quad + 2 \|(\mathcal{F}(1 - \theta)\Psi_{\sigma_0, \xi}^{\hbar})(-\cdot)\|_{L^2(\mathbb{R}_+)}. \end{aligned}$$

Reasoning like for the bound in Eq. (4.1), we obtain

$$\|(\mathcal{F}\Psi_{\sigma_0, \xi}^{\hbar})(-\cdot)\|_{L^2(\mathbb{R}_+)} = \left(\int_0^\infty dk \left|(\mathcal{F}\Psi_{\sigma_0, \xi}^{\hbar})(-k)\right|^2\right)^{1/2} \leq \frac{1}{\sqrt{2}} e^{-\frac{\sigma_0^2 p^2}{\hbar}}$$

and

$$\begin{aligned} \|(\mathcal{F}(1 - \theta)\Psi_{\sigma_0, \xi}^{\hbar})(-\cdot)\|_{L^2(\mathbb{R}_+)} &\leq \|(\mathcal{F}(1 - \theta)\Psi_{\sigma_0, \xi}^{\hbar})\|_{L^2(\mathbb{R})} \\ &= \|(1 - \theta)\Psi_{\sigma_0, \xi}^{\hbar}\|_{L^2(\mathbb{R})} \\ &= \left(\int_{-\infty}^0 dx \left|\Psi_{\sigma_0, \xi}^{\hbar}(x)\right|^2\right)^{1/2} \leq \frac{1}{\sqrt{2}} e^{-\frac{q^2}{4\hbar\sigma_0^2}}, \end{aligned}$$

which conclude the proof of the bound (4.7) for $p > 0$. The proof of the bound for $p < 0$ is identical and we omit it.

Identity (1.18) follows immediately from Eqs. (1.7) and (1.12).

5 Final remarks

5.1 A comparison with different approaches to the definition of a classical dynamics on the graph

Our approach to the semiclassical limit was inspired by Hagedorn’s work [12].

In general, a coherent state (on the real-line) is the wave function $\psi^{\hbar} : \mathbb{R} \rightarrow \mathbb{C}$ defined as

$$\psi^{\hbar}(x) = \psi^{\hbar}(\sigma, \check{\sigma}, q, p; x) := \frac{1}{(2\pi\hbar)^{1/4}\sqrt{\sigma}} \exp\left(-\frac{\check{\sigma}}{4\hbar\sigma}(x-q)^2 + \frac{i}{\hbar}p(x-q)\right),$$

with $(p, q) \in \mathbb{R}^2$ and $\sigma, \check{\sigma} \in \mathbb{C} \setminus \{0\}$ such that $\text{Re}(\check{\sigma}\sigma^{-1}) = |\sigma|^{-2} > 0$.

In his seminal paper [12], Hagedorn provides the semiclassical evolution of a coherent state in the presence of a regular (at least $C^2(\mathbb{R})$) interaction potential V . By one of the main results in [12], the quantum evolution of the coherent state ψ^{\hbar} is close (with respect to the $L^2(\mathbb{R})$ -norm and for \hbar small enough) to the wave function $x \mapsto e^{\frac{i}{\hbar}A_t} \psi^{\hbar}(\sigma_t, \check{\sigma}_t, q_t, p_t; x)$, where the pair (q_t, p_t) is the solution of the Hamilton equations

$$\begin{cases} \dot{q}_t = \frac{1}{m} p_t, \\ \dot{p}_t = -V'(q_t), \\ (q_0, p_0) = (q, p), \end{cases}$$

the pair $(\sigma_t, \check{\sigma}_t)$ is given by

$$\sigma_t = \frac{\partial q_t}{\partial q} \sigma + \frac{i}{2} \frac{\partial q_t}{\partial p} \check{\sigma}, \quad \check{\sigma}_t = -2i \frac{\partial p_t}{\partial q} \sigma + \frac{\partial p_t}{\partial p} \check{\sigma}$$

and A_t is the classical action

$$A_t = \int_0^t ds \left(\frac{p_s^2}{2m} - V(q_s) \right).$$

In our notation, one can associate with the quantum state ψ^{\hbar} the phase space function $(q, p) \mapsto (\phi^{\hbar}(\sigma, \check{\sigma}, x))(q, p) := \psi^{\hbar}(\sigma, \check{\sigma}, q, p; x)$. By this correspondence, one gets $\psi^{\hbar}(\sigma_t, \check{\sigma}_t, q_t, p_t; x) = (e^{itL_V} \phi^{\hbar}(\sigma_t, \check{\sigma}_t, x))(q, p)$, where L_V denotes the Liouville operator associated with the vector field of the classical Hamiltonian $\frac{p^2}{2m} + V(q)$; this is analogous to $e^{itL_K} \Sigma_{\sigma_t, x}^{\hbar}(\xi)$, $\xi \equiv (q, p)$.

We remark that, unlike the case of a quantum particle in the presence of a regular potential, in general there is no trajectory of a classical particle which describes the

semiclassical limit of a quantum evolution of the form $e^{-i\frac{t}{\hbar}H_K} \Xi_{\sigma_0, \xi}^{\hbar}$. As a consequence, the semiclassical dynamics is not described by the Hamilton equations. One way to overcome this difficulty is to assign a probability to every possible path on the graph. Typically, the probability of a certain path is postulated, and given in terms of the square modulus of the quantum transition (or stability) amplitudes (see, e.g., [3, Sec. II.A] or [4, Sec. 6.1]). For a star-graph the latter coincide with the elements of the (vertex) scattering matrix, defined for generic boundary conditions, e.g., in [16, Thm. 2.1] or [4, Lem. 2.1.3]. For Kirchhoff boundary conditions the elements of the scattering matrix are given by $\frac{2}{n} - \delta_{\ell, \ell'}$, $\ell, \ell' = 1, \dots, n$, see [3, Eq. (1)] (for the star-graph $C_{bb'} = 1$), and [4, Ex. 2.1.7, p. 41]. This is the approach used (for compact graphs) by Kottos and Smilansky in [17] and in several other works, see, e.g., [3], the review [14], and the monograph [4]. We have already noted that, up to a sign, the coefficients $\frac{2}{n} - \delta_{\ell, \ell'}$ coincide with the elements of the matrix \mathbb{S} identifying both the classical and quantum scattering operators.

Compared to this approach, we followed a different train of thought, starting from the trivial dynamics of a classical particle on the half-line with elastic collision at the origin and making use of a Kreĭn-type resolvent formula (see [20,21]) to find a suitable singular perturbation of the Liouville operator associated with such a trivial dynamics. We remark that, in a similar way, one can reconstruct the Hamiltonian H_K starting from the Hamiltonian of a quantum particle on the half-line with Dirichlet boundary conditions at the origin.

We defined the generator of the trivial dynamics on the half-line through Eq. (1.8). Note that if $f \in \text{dom}(L_D)$, then $f_t = e^{-itL_D} f$ satisfies the Liouville equation

$$i \frac{\partial}{\partial t} f_t = L_D f,$$

but the action of the group can be extended in a natural way to any bounded function.

Since the evolution is unitary in $L^2(\mathbb{R}_+ \times \mathbb{R})$, if $\|f\|_{L^2(\mathbb{R}_+ \times \mathbb{R})} = 1$, we can interpret

$$\rho_t(q, p) := |(e^{-itL_D} f)(q, p)|^2$$

as a probability density in the phase space $\mathbb{R}_+ \times \mathbb{R}$. Setting $\rho(q, p) := |f(q, p)|^2$, for all $t \in \mathbb{R}$ one has that

$$\rho_t(q, p) = \begin{cases} \rho\left(q - \frac{pt}{m}, p\right) & \text{if } q - \frac{pt}{m} > 0 \\ \rho\left(-q + \frac{pt}{m}, -p\right) & \text{if } q - \frac{pt}{m} < 0, \end{cases} \tag{5.1}$$

and it satisfies the equation

$$i \frac{\partial}{\partial t} \rho_t = -i X_0 \cdot \nabla \rho_t,$$

for all $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $q - \frac{pt}{m} \neq 0$.

We remark that, assuming elastic collision at the origin, a classical particle moving on the half-line follows a simple, though discontinuous, trajectory in the phase space: at any time $t \in \mathbb{R}$ any initial state $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$ is mapped to

$$\varphi_t(q, p) := \begin{cases} \left(q + \frac{pt}{m}, p \right) & \text{if } q + \frac{pt}{m} > 0, \\ \left(-q - \frac{pt}{m}, -p \right) & \text{if } q + \frac{pt}{m} < 0. \end{cases} \quad (5.2)$$

Hence, given a density $\rho : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ in the phase space, one has the identity $\rho_t(q, p) = \rho(\varphi_{-t}(q, p))$ [see Eqs. (5.1) and (5.2)]. In this sense, the group e^{-itL_D} given in Eq. (1.8) describes a classical particle on the half-line. The function $f_t := e^{-itL_D} f$ should be interpreted as a classical wave function, with associated probability density function in the phase space given by $\rho_t(q, p) := |f_t(q, p)|^2$.

On a graph this interpretation fails when the generator of the dynamics is e^{-itL_K} . In particular, from Eq. (1.9), for all $t \in \mathbb{R}$ it follows

$$|(e^{-itL_K} f)_\ell(q, p)|^2 = \begin{cases} \left| f_\ell\left(q - \frac{pt}{m}, p\right) \right|^2 & \text{if } q - \frac{pt}{m} > 0, \\ \left| \sum_{\ell'} |(\mathbb{S})_{\ell, \ell'}| f_{\ell'}\left(-q + \frac{pt}{m}, -p\right) \right|^2 & \text{if } q - \frac{pt}{m} < 0, \end{cases}$$

but the density $\rho_{\ell, t}(q, p) = |(e^{-itL_K} f)_\ell(q, p)|^2$ cannot be understood in terms of a trajectory of a classical particle since in this case there is no phase space flow φ_t such that $\rho_t(q, p) = \rho(\varphi_{-t}(q, p))$. Also, such a density does not coincide with the time evolution of the initial density $|f|^2$ as prescribed by Barra and Gaspard, see [3, Eq. (10)]. The latter, adapted to our setting and notation, and taking into account the fact that we are considering a non-compact graph, would give (for all $t \in \mathbb{R}$)

$$\rho_{\ell, t}^{BG}(q, p) = \begin{cases} \left| f_\ell\left(q - \frac{pt}{m}, p\right) \right|^2 & \text{if } q - \frac{pt}{m} > 0, \\ \sum_{\ell'} |(\mathbb{S})_{\ell, \ell'}|^2 \left| f_{\ell'}\left(-q + \frac{pt}{m}, -p\right) \right|^2 & \text{if } q - \frac{pt}{m} < 0. \end{cases}$$

Additionally, we remark that the initial state $\Sigma_{\sigma, x}^{\hbar}(\xi)$ does not define a probability density by the relation $\rho_\ell(\xi) = |(\Sigma_{\sigma, x}^{\hbar}(\xi))_\ell|^2$, because, in general, it does not belong to $L^2(\mathbb{R}_+ \times \mathbb{R})$.

Nevertheless, the phase space evolution of the approximated coherent states $\Sigma_{\sigma, x}^{\hbar}(\xi)$ induced by the Liouville operator L_K turns out to be a useful tool for the investigation of the semiclassical limit of the quantum evolution on the graph.

5.2 Coherent states on a star-graph with an even number of edges

Obviously, by superposition, one could consider initial states of the form

$$\Xi_{(\sigma_1, \xi_1), \dots, (\sigma_n, \xi_n)}^{\hbar} \equiv \begin{pmatrix} \Xi_{\sigma_1, \xi_1}^{\hbar} \\ \vdots \\ \Xi_{\sigma_n, \xi_n}^{\hbar} \end{pmatrix},$$

with $\sigma_\ell > 0$ and $\xi_\ell = (q_\ell, p_\ell) \in \mathbb{R}_+ \times \mathbb{R}$, and results similar to the ones stated in Theorems 1.4 and 1.6 hold true (with additive error terms). For n even it is possible to construct states that propagate exactly like coherent states on the real line. Suppose n even and consider a state $\Psi_{\sigma, \xi}^{\hbar}$ defined component-wisely by

$$(\Psi_{\sigma, \xi}^{\hbar})_\ell := \begin{cases} \Psi_{\sigma, \xi}^{\hbar} & \text{if } \ell = 1, \dots, n/2, \\ \Psi_{\sigma, -\xi}^{\hbar} & \text{if } \ell = n/2 + 1, \dots, n. \end{cases}$$

It is easy to check that such states belong to the domain of H_K . Next, consider the state $e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar}$. Taking the time derivative component by component one has

$$i\hbar \frac{\partial}{\partial t} e^{i\hbar A_t} (\Psi_{\sigma_t, \xi_t}^{\hbar})_\ell = -\frac{\hbar^2}{2m} e^{i\hbar A_t} (\Psi_{\sigma_t, \xi_t}^{\hbar})''_\ell,$$

by the definition of coherent states, see Eq. (1.1). Since $e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar} \in \text{dom}(H_K)$ the latter is equivalent to

$$i\hbar \frac{\partial}{\partial t} e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar} = H_K e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar}.$$

Moreover $e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar} |_{t=0} = \Psi_{\sigma_0, \xi}^{\hbar}$. Hence, $e^{-i\frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} = e^{i\hbar A_t} \Psi_{\sigma_t, \xi_t}^{\hbar}$. On the other hand, define the classical state $\Phi_{\sigma, x}^{\hbar}$, component-wisely, by $(\Phi_{\sigma, x}^{\hbar})_\ell(\xi) := (\Psi_{\sigma, \xi}^{\hbar})_\ell(x)$. Noting the identity $\sum_{\ell'=1}^n (\frac{2}{n} - \delta_{\ell, \ell'}) (\Phi_{\sigma, x}^{\hbar})_{\ell'}(-\xi) = (\Phi_{\sigma, x}^{\hbar})_\ell(\xi)$, and by using Eq. (1.9), it is easy to infer the identity $(e^{itL_K} \Phi_{\sigma_t, (\cdot)}^{\hbar})(\xi) = \Phi_{\sigma_t, (\cdot)}^{\hbar}(\xi_t) = \Psi_{\sigma_t, \xi_t}^{\hbar}$, so that

$$e^{-i\frac{t}{\hbar} H_K} \Psi_{\sigma_0, \xi}^{\hbar} = e^{i\hbar A_t} (e^{itL_K} \Phi_{\sigma_t, (\cdot)}^{\hbar})(\xi),$$

which is equivalent to (1.2). In this sense, up to a normalization factor $\sqrt{2/n}$, states of the form $\Psi_{\sigma, \xi}^{\hbar}$ are coherent states on a star-graph with an even number of edges.

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A Appendix: Wave operators for the pair (Dirichlet Laplacian, Neumann Laplacian) on the half-line

Proposition A.1 Let Ω_{ND}^\pm be the wave operators for the pair (Dirichlet Laplacian, Neumann Laplacian) in $L^2(\mathbb{R}_+)$, defined by

$$\Omega_{ND}^\pm := s\text{-}\lim_{t \rightarrow \pm\infty} U_{-t}^N U_t^D .$$

There holds true:

$$\Omega_{ND}^\pm = \pm \mathcal{F}_c^* \mathcal{F}_s .$$

Proof We use a density argument. Let $\psi \in L^2(\mathbb{R}_+)$. For any $\varepsilon > 0$ there exists $\varphi \in C_0^\infty(\mathbb{R}_+)$ such that $\|\psi - \varphi\|_{L^2(\mathbb{R}_+)} \leq \varepsilon/4$. Recalling the trivial bounds $\|U_t^{D/N}\| \leq 1$ and $\|\mathcal{F}_{s/c}\| \leq 1$, we infer

$$\|(U_{-t}^N U_t^D \mp \mathcal{F}_c^* \mathcal{F}_s) \psi\|_{L^2(\mathbb{R}_+)} \leq \frac{\varepsilon}{2} + \|(U_{-t}^N U_t^D \mp \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)} .$$

Hence, it is enough to prove that

$$\lim_{t \rightarrow \pm\infty} \|(U_{-t}^N U_t^D \mp \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)} = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}_+) . \tag{A.1}$$

Note that

$$\begin{aligned} \|(U_{-t}^N U_t^D \mp \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)} &= \|(U_t^D \mp U_t^N \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)} \\ &= \|(U_t^D \mathcal{F}_s^* \mp U_t^N \mathcal{F}_c^*) \mathcal{F}_s \varphi\|_{L^2(\mathbb{R}_+)} . \end{aligned}$$

Moreover, for all $t \in \mathbb{R}$ the following identities hold true:

$$(U_t^D \psi)(x) = \frac{2i}{\sqrt{2\pi}} \int_0^\infty dk e^{-ik^2 t} \sin(kx) (\mathcal{F}_s \psi)(k) ;$$

$$(U_t^N \psi)(x) = \frac{2}{\sqrt{2\pi}} \int_0^\infty dk e^{-ik^2 t} \cos(kx) (\mathcal{F}_c \psi)(k).$$

Hence,

$$\begin{aligned} (U_t^D \mathcal{F}_s^* \psi)(x) &= \frac{2i}{\sqrt{2\pi}} \int_0^\infty dk e^{-ik^2 t} \sin(kx) \psi(k), \\ (U_t^N \mathcal{F}_c^* \psi)(x) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty dk e^{-ik^2 t} \cos(kx) \psi(k), \end{aligned}$$

which give

$$\begin{aligned} ((U_t^D \mathcal{F}_s^* \mp U_t^N \mathcal{F}_c^*) \mathcal{F}_s \varphi)(x) &= \frac{2}{\sqrt{2\pi}} \int_0^\infty dk (i \sin(kx) \mp \cos(kx)) e^{-ik^2 t} (\mathcal{F}_s \varphi)(k) \\ &= \mp \frac{2}{\sqrt{2\pi}} \int_0^\infty dk e^{\mp ikx - ik^2 t} (\mathcal{F}_s \varphi)(k). \end{aligned}$$

We have obtained the following explicit formula for the quantity we are interested in

$$\|(U_{-t}^N U_t^D \mp \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)}^2 = \frac{2}{\pi} \int_0^\infty dx \left| \int_0^\infty dk e^{\mp ikx - ik^2 t} (\mathcal{F}_s \varphi)(k) \right|^2.$$

We note that, to prove the statement for Ω_{ND}^+ we have to study the limit $t \rightarrow +\infty$ of

$$\|(U_{-t}^N U_t^D - \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)}^2 = \frac{2}{\pi} \int_0^\infty dx \left| \int_0^\infty dk e^{-ikx - ik^2 t} (\mathcal{F}_s \varphi)(k) \right|^2, \tag{A.2}$$

while, to prove the statement for Ω_{ND}^- we have to study the limit $t \rightarrow -\infty$ of

$$\|(U_{-t}^N U_t^D + \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)}^2 = \frac{2}{\pi} \int_0^\infty dx \left| \int_0^\infty dk e^{ikx + ik^2 |t|} (\mathcal{F}_s \varphi)(k) \right|^2.$$

In what follows we focus the attention on the limit $t \rightarrow +\infty$. The other limit is obtained with trivial modifications. Hence, from now on we assume $t > 0$. From Eq. (A.2), changing variables $k \rightarrow \eta = k t^{1/2}$ and $x \rightarrow y = x/t^{1/2}$, we obtain

$$\begin{aligned} \|(U_{-t}^N U_t^D - \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)}^2 &= \frac{2}{\pi t^{1/2}} \int_0^\infty dy \left| \int_0^\infty d\eta e^{-i\eta y - i\eta^2} (\mathcal{F}_s \varphi)(\eta/t^{1/2}) \right|^2 \\ &= \frac{2}{\pi t^{1/2}} \int_0^1 dy |F(y, t)|^2 + \frac{2}{\pi t^{1/2}} \int_1^\infty dy |F(y, t)|^2, \end{aligned}$$

with

$$F(y, t) := \int_0^\infty d\eta e^{-i\eta y - i\eta^2} (\mathcal{F}_s \varphi)(\eta/t^{1/2}).$$

For any $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\mathcal{F}_s \varphi$ belongs to $C^\infty(\mathbb{R}_+)$, it decays at infinity faster than any polynomial in $1/k$, $(\mathcal{F}_s \varphi)(0) = 0$, moreover

$$|(\mathcal{F}_s \varphi)(k)| \leq \frac{2k}{\sqrt{2\pi}} \int_0^\infty dx x |\varphi(x)|.$$

Hence,

$$\int_0^\infty dk \frac{1}{k^\delta} |\mathcal{F}_s \varphi(k)| < \infty \quad \text{for all } \delta < 2. \tag{A.3}$$

Additionally, for any $\varphi \in C_0^\infty(\mathbb{R}_+)$, $\mathcal{F}_c \varphi$ belongs to $C^\infty(\mathbb{R}_+)$, it decays at infinity faster than any polynomial in $1/k$, and $|(\mathcal{F}_c \varphi)(k)| \leq \frac{2}{\sqrt{2\pi}} \int_0^\infty dx |\varphi(x)|$. Hence,

$$\int_0^\infty dk \frac{1}{k^\delta} |\mathcal{F}_c \varphi(k)| < \infty \quad \text{for all } \delta < 1. \tag{A.4}$$

Starting from the identity $e^{-i\eta y - i\eta^2} = \frac{i}{y+2\eta} \frac{d}{d\eta} e^{-i\eta y - i\eta^2}$, by integration by parts, we obtain

$$F(y, t) = F_1(y, t) + F_2(y, t),$$

with

$$F_1(y, t) := \int_0^\infty d\eta e^{-i\eta y - i\eta^2} \frac{2i}{(y+2\eta)^2} (\mathcal{F}_s \varphi)(\eta/t^{1/2})$$

and

$$F_2(y, t) := \frac{1}{t^{1/2}} \int_0^\infty d\eta e^{-i\eta y - i\eta^2} \frac{1}{i(y+2\eta)} (\mathcal{F}_c((\cdot)\varphi))(\eta/t^{1/2}).$$

Since y and η are both positive, from the trivial inequality $\frac{1}{(y+2\eta)^a} \leq \frac{1}{y^b (2\eta)^c}$ for all $y, \eta > 0$ and for all $a, b, c > 0$ such that $a = b + c$, we deduce that

$$\begin{aligned} |F_1(y, t)| &\leq \frac{C}{y^{1/4}} \int_0^\infty d\eta \frac{1}{\eta^{7/4}} |(\mathcal{F}_s \varphi)(\eta/t^{1/2})| \\ &= \frac{C}{t^{3/8} y^{1/4}} \int_0^\infty d\xi \frac{1}{\xi^{7/4}} |(\mathcal{F}_s \varphi)(\xi)| = \frac{C}{t^{3/8} y^{1/4}}; \end{aligned}$$

here and in the following C denotes a generic positive constant whose value may depend only on integrals of the sine (or cosine) Fourier transform of φ (or $(\cdot)\varphi$), as in Eqs. (A.3) and (A.4); the value of C may change from line to line and precise values for the constants can be obtained. In a similar way we infer,

$$|F_2(y, t)| \leq \frac{C}{t^{1/2} y^{1/4}} \int_0^\infty d\eta \frac{1}{\eta^{3/4}} |(\mathcal{F}_c((\cdot)\varphi))(\eta/t^{1/2})|$$

$$= \frac{C}{t^{3/8} y^{1/4}} \int_0^\infty d\xi \frac{1}{\xi^{3/4}} |(\mathcal{F}_c((\cdot)\varphi))(\xi)| = \frac{C}{t^{3/8} y^{1/4}}.$$

Hence,

$$\frac{2}{\pi t^{1/2}} \int_0^1 dy |F(y, t)|^2 \leq \frac{C}{t^{5/4}} \int_0^1 dy \frac{1}{y^{1/2}} \leq \frac{C}{t^{5/4}}.$$

On the other hand, for $y > 1$,

$$|F_1(y, t)| \leq \frac{C}{y} \int_0^\infty d\eta \frac{1}{\eta} |(\mathcal{F}_s \varphi)(\eta/t^{1/2})| = \frac{C}{y} \int_0^\infty d\xi \frac{1}{\xi} |(\mathcal{F}_s \varphi)(\xi)| = \frac{C}{y}$$

and

$$|F_2(y, t)| \leq \frac{C}{t^{1/2} y} \int_0^\infty d\eta |(\mathcal{F}_c((\cdot)\varphi))(\eta/t^{1/2})| = \frac{C}{y} \int_0^\infty d\xi |(\mathcal{F}_c((\cdot)\varphi))(\xi)| = \frac{C}{y}.$$

So, we obtain

$$\frac{2}{\pi t^{1/2}} \int_1^\infty dy |F(y, t)|^2 \leq \frac{C}{t^{1/2}} \int_1^\infty dy \frac{1}{y^2} \leq \frac{C}{t^{1/2}}.$$

In this way we have proved that, for all $\varphi \in C_0^\infty(\mathbb{R}_+)$ there exists a constant C such that

$$\|(U_{-t}^N U_t^D - \mathcal{F}_c^* \mathcal{F}_s) \varphi\|_{L^2(\mathbb{R}_+)} \leq \frac{C}{t^{1/4}} \quad \text{for all } t > 1,$$

and the latter claim gives the limit in Eq. (A.1) for $t \rightarrow +\infty$.

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