



Quantitative isoperimetric inequalities for classical capillarity problems

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Abstract

We consider capillarity functionals which measure the perimeter of sets contained in a Euclidean half-space assigning a constant weight $\lambda \in (-1, 1)$ to the portion of the boundary that touches the boundary of the half-space. Depending on λ , sets that minimize this capillarity perimeter among those with fixed volume are known to be suitable truncated balls lying on the boundary of the half-space. We first give a new proof based on an ABP-type technique of the sharp isoperimetric inequality for this class of capillarity problems. Next we prove two quantitative versions of the inequality: a first sharp inequality estimates the Fraenkel asymmetry of a competitor with respect to the optimal bubbles in terms of the energy deficit; a second inequality estimates a notion of asymmetry for the part of the boundary of a competitor that touches the boundary of the half-space in terms of the energy deficit. After a symmetrization procedure, the quantitative inequalities follow from a novel combination of a quantitative ABP method with a selection-type argument.

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1 Introduction

The classical isoperimetric problem in the Euclidean space \mathbb{R}^n , for $n \geq 2$, aims at minimizing the $(n - 1)$ -dimensional area of boundaries of sets having fixed finite volume. More precisely, given $v > 0$, one aims to characterize minimizers to the problem

$$\inf \{ P(E) : E \subset \mathbb{R}^n, |E| = v \}, \tag{1.1}$$

where $P(E)$ denotes the perimeter of E (see Sect. 2.1) and $|E|$ the Lebesgue measure of E . It is well-known that balls (uniquely) minimize (1.1), cf. [18] or [47, Chapter 14], and this is encoded in the classical isoperimetric inequality

$$P(E) \geq n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}}, \tag{1.2}$$

where ω_n denotes the measure of the unit ball in \mathbb{R}^n . To prove a *quantitative* version of (1.2) means to estimate the distance of a competitor from the set of minimizers in terms of the energy deficit of the competitor with respect to the infimum of the problem. The first quantitative isoperimetric inequality for (1.1) with sharp exponents was proved in [29], and it reads

$$\alpha(E)^2 \leq C(n)D(E), \tag{1.3}$$

where $\alpha(E)$ and $D(E)$ are respectively the Fraenkel asymmetry and the isoperimetric deficit of E , i.e.,

$$\alpha(E) := \inf \left\{ \frac{E \Delta B(|E|, x)}{|E|} : x \in \mathbb{R}^n \right\} \quad D(E) := \frac{P(E) - P(B(|E|))}{P(B(|E|))},$$

where $B(v, x)$ denotes the ball in \mathbb{R}^n with volume v centered at x , for $v > 0$ and $x \in \mathbb{R}^n$, and $B(v) := B(v, 0)$. The inequality (1.3) improves the previous non-sharp inequality proved in [38], after [24, 39]. We also mention [16, 20, 21, 27, 40] for further quantitative isoperimetric inequalities for possibly anisotropic perimeters, [5, 11, 13] for quantitative isoperimetric inequalities on manifolds, and [4, 6, 15, 17, 28] about weighted quantitative isoperimetric inequalities.

In this paper we prove quantitative isoperimetric inequalities for the following classical capillarity problems. If E is a measurable set in the half-space $\{x_n > 0\} \subset \mathbb{R}^n$ and $\lambda \in (-1, 1)$, we define the weighted perimeter functional

$$P_\lambda(E) := P(E, \{x_n > 0\}) - \lambda \mathcal{H}^{n-1}(\partial^* E \cap \{x_n = 0\}),$$

where \mathcal{H}^k , for $k \geq 0$, denotes the k -dimensional Hausdorff measure in \mathbb{R}^n , and $\partial^* E$ denotes the reduced boundary of E (see Sect. 2.1). Interpreting the perimeter as a measure of the surface tension of a liquid drop, the constant λ basically represent the relative adhesion coefficient between a liquid drop and the solid walls of the container given by $\{x_n > 0\}$.

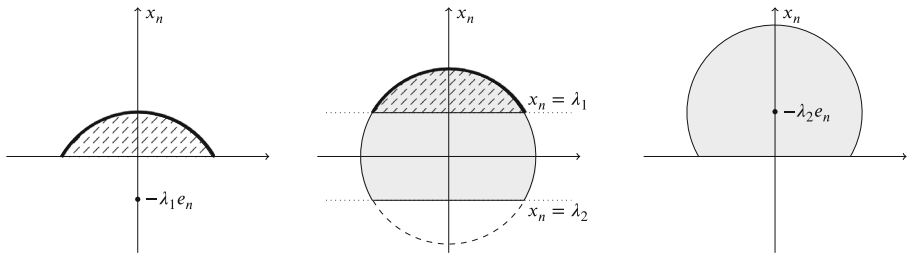


Fig. 1 Left: $B^{\lambda_1}(|B^{\lambda_1}|)$ for some $\lambda_1 > 0$. Middle: diagonally striped set = B^{λ_1} , gray set = B^{λ_2} , for some $\lambda_1 > 0, \lambda_2 < 0$. Right: $B^{\lambda_2}(|B^{\lambda_2}|)$ for some $\lambda_2 < 0$

If $v > 0$, we consider the isoperimetric capillarity problem

$$\inf \{P_\lambda(E) : E \subset \{x_n > 0\}, |E| = v\}. \tag{1.4}$$

Minimizers for (1.4), below called isoperimetric sets for (1.4), are given by suitably truncated balls lying on the boundary of the half-space. More precisely, if $B^\lambda = \{x \in B_1(0) \subset \mathbb{R}^n : \langle x, e_n \rangle > \lambda\}$, and for $v > 0$ we set

$$B^\lambda(v) := \frac{v^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}} (B^\lambda - \lambda e_n)$$

then minimizers for (1.4) are sets of the form

$$B^\lambda(v, x) := B^\lambda(v) + x, \tag{1.5}$$

for $x \in \{x_n = 0\}$, see [47, Theorem 19.21] and Fig. 1.

The first variational results regarding capillarity problems go back to works by Giusti, Gonzalez, Massari and Tamanini who established existence, symmetry and regularity results for the isotropic sessile drop problem, where an additional potential energy representing gravity is added to the minimization of P_λ (see [32–37]; see also [22] where uniqueness results for the symmetric sessile drop were established). We refer to [23] and [47, Chapters 19, 20] for a more complete treatment regarding classical results. More recently, in [3] the shape of liquid drops and crystals, resting on a horizontal surface and under the influence of gravity, are described in the anisotropic setting. The shape and the fine regularity of volume constrained minimizers of weighted perimeters like P_λ , where the weight on the interface touching the boundary of the container may be nonconstant and where an additional potential term is present, are addressed in [12, 19, 48]. In [19] the perimeter functional measuring the area of the interface that does not touch the container is also possibly anisotropic. Recently, the isoperimetric problem for the relative perimeter of sets contained in the complement of a convex set had been addressed in [14], where a sharp isoperimetric inequality is established, and in [30], where the rigidity of the inequality is addressed in the generality of measurable sets. Extensions of [14] to higher codimension have been considered in [43, 45].

The minimality of sets $B^\lambda(v, x)$ for (1.4) comes with an isoperimetric inequality for P_λ , see Theorem 3.3 below. In order to prove a quantitative isoperimetric inequality for (1.4), we define the corresponding Fraenkel asymmetry and isoperimetric deficit by setting

$$\alpha_\lambda(E) := \inf \left\{ \frac{|E \Delta B^\lambda(v, x)|}{v} : x \in \{x_n = 0\} \right\}, \quad D_\lambda(E) := \frac{P_\lambda(E) - P_\lambda(B^\lambda(v))}{P_\lambda(B^\lambda(v))},$$

for any $E \subset \{x_n > 0\}$ with volume $|E| = v$. The infimum defining the asymmetry is, in fact, a minimum.

The first main result of the paper is the following

Theorem 1.1 *Let $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$. There exists a constant $c_{\text{iso}} = c_{\text{iso}}(n, \lambda) > 0$ such that for any measurable set $E \subset \mathbb{R}^n \cap \{x_n > 0\}$ with finite measure there holds*

$$\alpha_\lambda(E)^2 \leq c_{\text{iso}} D_\lambda(E). \tag{1.6}$$

As for the classical (1.2), perturbing the boundary of an optimal bubble only inside the container $\{x_n > 0\}$, it is possible to check that exponents in (1.6) are sharp.

In the context of these capillarity problems it is also spontaneous to consider a notion of asymmetry for the part of the boundary of a set that touches the half-plane $\{x_n = 0\}$. For a measurable set $E \subset \{x_n > 0\}$, we define

$$\beta_\lambda(E) := \inf \left\{ \frac{\mathcal{H}^{n-1}(\partial^* E \cap \{x_n = 0\}) \Delta \partial^* B^\lambda(|E|, x) \cap \{x_n = 0\}}{\mathcal{H}^{n-1}(\partial^* B^\lambda(|E|, x) \cap \{x_n = 0\})} : x \in \{x_n = 0\} \right\}.$$

The previous quantity measures the asymmetry of the set $\partial^* E \cap \{x_n = 0\}$ with respect to $(n - 1)$ -dimensional balls in $\{x_n = 0\}$ having volume equal to the one of the trace of the optimal bubble corresponding to the volume of E . We establish the following second quantitative isoperimetric inequality, that provides a quantitative estimate on β_λ .

Theorem 1.2 *Let $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$. There exists a constant $c'_{\text{iso}} = c'_{\text{iso}}(n, \lambda) > 0$ such that for any measurable set $E \subset \mathbb{R}^n \cap \{x_n > 0\}$ with finite measure there holds*

$$\beta_\lambda(E) \leq c'_{\text{iso}} \max \left\{ D_\lambda(E), D_\lambda(E)^{\frac{1}{2n}} \right\}. \tag{1.7}$$

Strategy of the proof and comments. Observing that, roughly speaking, the minimization problem (1.4) is symmetric with respect to the first $n - 1$ axes, it is possible to adapt arguments in the spirit of [29] to see that, in order to prove Theorem 1.1, it is sufficient to prove (1.6) in the class of Schwarz-symmetric sets, see Corollary 4.9. Here a set E is said to be Schwarz-symmetric (Definition 2.1) if the intersection $E \cap \{x_n = t\}$ is an $(n - 1)$ -dimensional ball in $\{x_n = t\}$ centered at $(0, t)$, for any t . However, we point out that it seems not possible to push the strategy of [29] to the very end to prove Theorem 1.1. Indeed the arguments in [29] require to Schwarz-symmetrize a competitor with respect to a preferred axis depending on the competitor, while in our case it is only possible to symmetrize with respect to the n -th axis. One finds an analogous obstruction also in a possible adaptation of the proof via symmetrization revised in [46] (see also [26]); in [46] the quantitative isoperimetric inequality for Schwarz-symmetric sets is eventually obtained performing a quantitative version of Gromov’s proof [49] of the isoperimetric inequality, but again after having symmetrized a competitor with respect to a convenient axis.

The proof of (1.6) in the class of Schwarz-symmetric sets is achieved here with a new combination of the so-called selection principle [1, 16] with an Alexandrov–Bakelman–Pucci-type technique in the spirit of [17].

In the recent [17], the authors prove sharp quantitative isoperimetric inequalities for a class of isoperimetric problems in cones where volume and perimeter are weighted in terms of a function satisfying suitable homogeneity and concavity properties. The proof in [17] stems from the fact that the isoperimetric inequality for the corresponding problem was proved in

[10] by a so-called ABP argument. The methods that go under the name of ABP techniques were originally employed to derive regularity estimates for second order elliptic equations [31, Chapter 9] and they were applied to give a new direct proof of the classical isoperimetric inequality in [7, 8] (see [9] for a detailed account on the method). More precisely, for the classical isoperimetric problem, if $E \subset \mathbb{R}^n$ is a smooth connected open set, one would consider a solution u to

$$\begin{cases} \Delta u = \frac{P(E)}{|E|} & \text{on } E, \\ \partial_\nu u = 1 & \text{on } \partial E, \end{cases} \tag{1.8}$$

where $\partial_\nu u$ denotes outward normal derivative. It is immediate to check that $\nabla u(E') \supset B_1(0)$ where $E' := \{x \in E : \nabla^2 u(x) \geq 0\}$, hence the area formula together with the arithmetic–geometric mean inequality readily imply the Euclidean isoperimetric inequality, indeed

$$\omega_n = |B_1(0)| \leq \int_{E'} \det \nabla^2 u \leq \int_E \left(\frac{\Delta u}{n}\right)^n = \frac{P(E)^n}{n^n |E|^{n-1}}.$$

Now the rough idea is that a control on the energy deficit should control the "asymmetry" of the solution u with respect to the solution corresponding to the optimal shape $B_1(0)$, that is the radially symmetric parabola $|x|^2/2$. In fact, this is achieved in [17] by controlling the asymmetry of a *coupling* function which is defined as a suitable convex envelope of u . Adapting arguments from [21], in [17] the authors then show that it is possible to employ trace-type theorems to estimate the asymmetry of a competitor set in terms of the asymmetry of the coupling function, which is in turn estimated by the energy deficit.

In Theorem 3.3 below we will give an ABP proof of the isoperimetric inequality for the problem (1.4) by analyzing an elliptic problem analogous to (1.8), see (3.3). We are then in position to consider a coupling function as done in [17] and we can quantitatively estimate its asymmetry, which will be achieved in Proposition 5.4. Moreover, Schwarz-symmetric sets that are sufficiently small C^1 -perturbations (in the sense of Definition 5.10) of an optimal bubble (1.5) readily verify the needed trace-type inequalities that relate asymmetry of the competitor with the asymmetry of the coupling. Hence this establishes the quantitative inequality (1.6) for Schwarz-symmetric C^1 -perturbations of optimal bubbles, see Corollary 5.11. Observe that, in our setting, isoperimetric sets are just Lipschitz-regular and a set E is C^1 -close to an optimal bubble if just the relative boundary $\partial E \cap \{x_n > 0\}$ is close to the relative boundary of an optimal bubble as C^1 -hypersurfaces with boundary.

Once (1.6) is proved for C^1 -perturbations of optimal bubbles (Corollary 5.11), we want apply a selection-type argument in the spirit of [1, 16] in the class of Schwarz-symmetric sets in order to extend the validity of the quantitative inequality to the whole class of Schwarz-symmetric sets. In this way we also avoid the implementation of further technical results that in [17, 21] allow to reduce to just consider sets that enjoy the required trace-type inequalities.

Roughly speaking, in a selection-type argument one argues by contradiction assuming existence of sets contradicting the quantitative isoperimetric inequality and one uses such sets to define an auxiliary minimization problem, cf. (5.22). Minimizers to the previous problem still contradict the quantitative isoperimetric inequality, but at the same time they are shown to be small C^1 -perturbations of some isoperimetric set, contradicting the inequality already proved for sets given by small perturbations of optimal ones.

In our case, we will prove that minimizers E to the auxiliary minimization problem are C^1 -perturbations of optimal bubbles up to the boundary of the half-space $\{x_n > 0\}$ as a consequence of the classical interior regularity of (Λ, r_0) -minimizers of the perimeter (Definition A.1), see [51] and [47, Chapter 26], together with a simple variational argument

that allows us to propagate the regularity up to the boundary of the half-space $\{x_n > 0\}$. This essentially follows from the fact that a Schwarz-symmetric local (Λ, r_0) -minimizer E in $\{x_n > 0\}$ is locally of class C^1 and has bounded mean curvature (in a generalized sense, see Lemma A.2); hence a uniform bound on the whole second fundamental form on a portion of boundary $\partial E \cap \{0 < x_n < \varepsilon\}$ follows just by showing that the set $\partial E \cap \{0 < x_n < \varepsilon\}$ is far from the axis of revolution of E (see Lemma A.4), and the latter holds in the proof of Theorem 1.1 by an energy estimate holding for minimizers to the auxiliary minimization problems.

We stress that in our case it is not clear how to apply a Fuglede-type argument [16, 24] to prove the quantitative inequality for sets given by small C^1 -perturbations of optimal ones. Indeed, the classical Fuglede’s method relies on the precise knowledge of the eigenvalues of the Laplace–Beltrami operator, which is not available for the operator on spherical caps corresponding to optimal bubbles (1.5) for generic $\lambda \in (-1, 1)$. Moreover, observe that in our case it is not possible to globally parametrize C^1 -close boundaries one on the other as normal graphs in general, introducing a further nontrivial technical difficulty in the implementation of a Fuglede-type argument.

Once Theorem 1.1 is proved, for the proof of Theorem 1.2 we argue as follows. First we establish a quantitative inequality that estimates the Hausdorff distance between the relative boundary in $\{x_n > 0\}$ of a competitor E and the relative boundary of some bubble in terms of the Fraenkel asymmetry of E , under the assumption that E is a so-called (K, r_0) -quasiminimal set, see Definition 6.2 and Lemma 6.4. This is achieved since quasiminimal sets enjoy uniform density estimates at boundary points, see Theorem 6.3. Exploiting Theorem 1.1, the previous quantitative inequality yields an inequality of the form (1.7) in the class of quasiminimal sets. Eventually, Theorem 1.2 follows by applying again a selection-type argument where now β_λ plays the role of the Fraenkel asymmetry.

Organization. In Sect. 2 we collect definitions and facts on sets of finite perimeter and we prove some preliminary results on the capillarity functional. In Sect. 3 we establish the sharp isoperimetric inequality for P_λ via ABP method and we prove preliminary facts on the Fraenkel asymmetry and on the deficit. Section 4 is devoted to a series of technical results that allow to reduce the analysis to Schwarz-symmetric sets. In Sect. 5 we complete the proof of Theorem 1.1 by proving (1.6) on Schwarz-symmetric sets. Finally in Sect. 6 we prove Theorem 1.2. In Appendix A we recall some facts about (Λ, r_0) -minimizers and we recall a formula for the mean curvature of axially symmetric hypersurfaces.

Addendum. After this work was completed, we became aware of the independent [42], where the authors exploit the quantitative isoperimetric inequality for anisotropic perimeters provided in [21] to give a result analogous to Theorem 1.1 in the anisotropic setting.

2 Preliminaries

From now on and for the rest of the paper it is assumed that $\lambda \in (-1, 1)$ and $n \in \mathbb{N}$ with $n \geq 2$ are fixed.

List of symbols. Throughout the paper we shall adopt the following notation.

- $|\cdot|$ denotes Lebesgue measure in \mathbb{R}^n .
- $B_r := B_r(0) \subset \mathbb{R}^n$ for $r > 0$, $B := B_1$.
- $B^\lambda = \{x \in B : \langle x, e_n \rangle > \lambda\}$.
- $B^\lambda(v) := \frac{v^{\frac{1}{n}}}{|B^\lambda|^{\frac{1}{n}}}(B^\lambda - \lambda e_n)$, for any $v > 0$.

- $B^\lambda(v, x) := B^\lambda(v) + x$, for any $x \in \{x_n = 0\}$. In particular $B^\lambda(v) = B^\lambda(v, 0)$.
- $c(n, \lambda), C(n, \lambda)$ denote strictly positive constants, depending on n, λ only, that may change from line to line.
- $d_{\mathcal{H}}$ denotes Hausdorff distance in \mathbb{R}^n .
- $H := \{x_n \leq 0\}$.
- \mathcal{H}^d denotes d -dimensional Hausdorff measure in \mathbb{R}^n , for $d \geq 0$.
- $P_\lambda(B^\lambda) := P_\lambda(B^\lambda(|B^\lambda|, x)) = P(B, \{x_n > \lambda\}) - \lambda \mathcal{H}^{n-1}(B \cap \{x_n = \lambda\})$.
- $Q_r := [-r, r]^n \subset \mathbb{R}^n$, for any $r > 0$.
- $r_\lambda := \min \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\}, R_\lambda := \max \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\}$.

2.1 Sets of finite perimeter

We recall basic definitions and properties regarding sets of finite perimeter, referring to [2, 47] for a complete treatment on the subject. The perimeter of a measurable set $E \subset \mathbb{R}^n$ in an open set $A \subset \mathbb{R}^n$ is defined by

$$P(E, A) := \sup \left\{ \int_E \operatorname{div} T(x) \, dx : T \in C_c^1(A; \mathbb{R}^n), \|T\|_\infty \leq 1 \right\}. \tag{2.1}$$

Denoting $P(E) := P(E, \mathbb{R}^n)$, we say that E is a set of finite perimeter if $P(E) < +\infty$. In such a case, the characteristic function χ_E has a distributional gradient $D\chi_E$ that is a vector-valued Radon measure on \mathbb{R}^n such that

$$\int_E \operatorname{div} T(x) \, dx = - \int_{\mathbb{R}^n} T \, dD\chi_E, \quad \forall T \in C_c^1(\mathbb{R}^n; \mathbb{R}^n).$$

It can be proved that the set function $P(E, \cdot)$ defined in (2.1) is the restriction of a nonnegative Borel measure to open sets. The measure $P(E, \cdot)$ coincides with the total variation $|D\chi_E|$ of the distributional gradient, and it is concentrated on the reduced boundary

$$\partial^* E := \left\{ x \in \operatorname{spt}|D\chi_E| : \exists v^E(x) := - \lim_{r \rightarrow 0} \frac{D\chi_E(B_r(x))}{|D\chi_E(B_r(x))|} \text{ with } |v^E(x)| = 1 \right\}.$$

Introducing the sets of density $t \in [0, 1]$ points for E defined by

$$E^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{r \rightarrow 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = t \right\},$$

we have that the reduced boundary coincides both with $\mathbb{R}^n \setminus (E^{(1)} \cup E^{(0)})$ and with the set $E^{(1/2)}$ up to \mathcal{H}^{n-1} -negligible sets. The vector v^E is called the generalized outer normal of E . Moreover $P(E, \cdot) = \mathcal{H}^{n-1} \llcorner \partial^* E$, and the distributional gradient can be written as $D\chi_E = -v^E \mathcal{H}^{n-1} \llcorner \partial^* E$.

We conclude by recalling the definition of Schwarz symmetrization and of Schwarz-symmetric set.

Definition 2.1 Let $E \subset \mathbb{R}^n$ be a Borel set. Then its Schwarz symmetrization (with respect to the n -th axis) is the set

$$E^* := \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \omega_{n-1}|x'|^{n-1} < \mathcal{H}^{n-1}(E \cap \{x_n = t\}), t \in \mathbb{R}\}.$$

A Borel set $E \subset \mathbb{R}^n$ is said to be Schwarz-symmetric if it coincides with its Schwarz symmetrization, up to negligible sets.

2.2 Preliminary results on the capillarity functional

In this section we prove some preparatory results on the functional P_λ . From now on, H denotes the closed half-space $H := \{x_n \leq 0\}$.

Remark 2.2 Let $E \subset \mathbb{R}^n \setminus H$ be a measurable set. We observe that

$$P_\lambda(E) = \int_{\partial^* E \setminus H} 1 - \lambda \langle e_n, \nu^E \rangle \, d\mathcal{H}^{n-1},$$

where ν^E is the generalized outer normal to E . In particular, since $|\lambda| < 1$, we have that $P_\lambda(E) \geq 0$. The previous identity follows from the divergence theorem, indeed

$$0 = \int_E \operatorname{div} e_n \, dx = -\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \int_{\partial^* E \setminus H} \langle e_n, \nu^E \rangle \, d\mathcal{H}^{n-1}.$$

We now aim at proving an approximation result for sets of finite perimeter contained in $\mathbb{R}^n \setminus H$ by sequences of sets having smooth boundary relative in $\mathbb{R}^n \setminus H$. We will employ the following lemma.

Lemma 2.3 *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. For any $t \geq 0$ define the diffeomorphism $\varphi_t : \{x_n \geq 0\} \rightarrow \{x_n \geq 0\}$ given by $\varphi_t(x', x_n) := (x', x_n(1 + te^{-|x'|^2}))$, where we wrote $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ for any $x \in \{x_n \geq 0\} \subset \mathbb{R}^n$. Then*

(1) *for at most countably many $t \in [0, +\infty)$ there holds*

$$\mathcal{H}^{n-1} \left(\left\{ x \in \partial^*(\varphi_t(E)) \setminus H : \nu^{\varphi_t(E)}(x) = \pm e_n \right\} \right) > 0;$$

(2) *for at most countably many $\nu \in \mathbb{S}^{n-1}$ there holds*

$$\mathcal{H}^{n-1} \left(\left\{ x \in \partial^* E : \nu^E(x) = \pm \nu \right\} \right) > 0.$$

Proof We begin by proving (1). Define $f_t : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ by $f_t(x') := 1 + te^{-|x'|^2}$, and let

$$A_t := \left\{ (x', x_n) \in \partial^* E \cap \mathbb{R}^n \setminus H \text{ with } x' \in \mathbb{R}^{n-1} \setminus \{0\} : \nu^E(x', x_n) \text{ is proportional to } (x_n \nabla f_t(x'), f_t(x')) \right\}.$$

We claim that, if $s, r > 0$, with $s \neq r$, then $A_r \cap A_s = \emptyset$. Indeed, if $A_r \cap A_s \neq \emptyset$, there exist $(\bar{x}', \bar{x}_n) \in \partial^* E \cap \mathbb{R}^n \setminus H$ and $\bar{\alpha} \in \mathbb{R} \setminus \{0\}$ such that

$$(\bar{x}_n \nabla f_r(\bar{x}'), f_r(\bar{x}')) = \bar{\alpha} (\bar{x}_n \nabla f_s(\bar{x}'), f_s(\bar{x}')).$$

Since $x_n > 0$ and $x' \neq 0$, then $\bar{x}_n \nabla f_r(\bar{x}') = \bar{\alpha} \bar{x}_n \nabla f_s(\bar{x}')$ implies $r = \bar{\alpha} s$. Thus $f_r(\bar{x}') = \bar{\alpha} f_s(\bar{x}')$ implies $1 = \bar{\alpha}$, which in turn implies $r = s$, contradiction.

For $k \in \mathbb{N}_{\geq 1}$ let $M_k := \{t > 0 : \mathcal{H}^{n-1}(A_t) > 1/k\}$. Since $A_r \cap A_s = \emptyset$ for $r \neq s$ and since $P(E) < +\infty$, it follows that M_k is finite for any $k \geq 1$, and thus $\{t > 0 : \mathcal{H}^{n-1}(A_t) > 0\} = \bigcup_k M_k$ is countable.

Since the differential of φ_t is represented by the matrix

$$d\varphi_t(x', x_n) = \begin{pmatrix} \operatorname{Id}_{\mathbb{R}^{n-1}} & 0 \\ x_n \nabla f_t(x') & f_t(x') \end{pmatrix},$$

requiring that $v^{\varphi_t(E)}(\varphi_t(x)) = \pm e_n$ for some $x = (x', x_n)$ means that

$$v^E(x', x_n) \text{ is proportional to } (x_n \nabla f_t(x'), f_t(x')),$$

see [47, Proposition 17.1] and [2, Proposition 2.88]. Therefore for any $t \notin \cup_k M_k$ there holds

$$\mathcal{H}^{n-1} \left(\left\{ x \in \partial^*(\varphi_t(E)) \setminus H : v^{\varphi_t(E)}(x) = \pm e_n \right\} \right) = 0.$$

The proof of (2) is analogous, noticing that $N_k := \{v \in \mathbb{S}^{n-1} : \mathcal{H}^{n-1}(\{x \in \partial^*E : v^E = v\}) > 1/k\}$ is finite for any $k \in \mathbb{N}_{\geq 1}$, hence $\cup_k N_k$ is at most countable. □

Lemma 2.4 (Approximation with regular sets) *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. Then there exists a sequence of sets $E_i \subset \mathbb{R}^n \setminus H$ such that*

- (1) E_i is a bounded set such that $\partial E_i \setminus \partial H$ is a smooth hypersurface (possibly with smooth boundary) such that either $\partial E_i \cap \partial H = \emptyset$ or $\partial E_i \setminus \partial H$ intersects ∂H orthogonally;
- (2) $E_i \rightarrow E$ in L^1 , $P(E_i, \mathbb{R}^n \setminus H) \rightarrow P(E, \mathbb{R}^n \setminus H)$, $\mathcal{H}^{n-1}(\partial E_i \cap \partial H) \rightarrow \mathcal{H}^{n-1}(\partial^*E \cap \partial H)$, as $i \rightarrow +\infty$.
In particular, $P(E_i) \rightarrow P(E)$ and $P_\lambda(E_i) \rightarrow P_\lambda(E)$ as $i \rightarrow +\infty$, for any $\lambda \in (-1, 1)$;
- (3) $\mathcal{H}^{n-1}(\{x \in \partial^*E_i : v^{E_i}(x) = \pm e_j\}) = 0$ for any $j = 1, \dots, n - 1$;
- (4) $\mathcal{H}^{n-1}(\{x \in \partial^*E_i \setminus H : v^{E_i}(x) = \pm e_n\}) = 0$.

Proof Since $P(E), |E| < +\infty$, by a diagonal argument we can assume without loss of generality that E is bounded.

Step 1. We first construct a sequence $F_i \subset \mathbb{R}^n \setminus H$ such that 1. and 2. hold with F_i in place of E_i . Let us denote by F the union of E with the reflection of E with respect to the hyperplane $\{x_n = 0\}$. There exists a sequence of smooth sets $\tilde{F}_i \subset \mathbb{R}^n$, symmetric with respect to $\{x_n = 0\}$, such that they converge to F in $L^1(\mathbb{R}^n)$ and $P(\tilde{F}_i) \rightarrow P(F)$ (see, e.g., [2, Theorem 3.42]). The fact that \tilde{F}_i is symmetric with respect to $\{x_n = 0\}$ follows as \tilde{F}_i can be obtained as superlevel set of a convolution of χ_F with a symmetric mollifier. In particular, if $\partial \tilde{F}_i \cap \partial H \neq \emptyset$, then $\partial \tilde{F}_i$ intersects H orthogonally, and thus $\partial \tilde{F}_i \cap \partial H$ is a smooth $(n - 2)$ -dimensional manifold. Let $F_i := \tilde{F}_i \setminus H$. Then $F_i \rightarrow E$ in L^1 and

$$P(F_i, \mathbb{R}^n \setminus H) = \frac{1}{2}P(\tilde{F}_i) \rightarrow \frac{1}{2}P(F) = P(E, \mathbb{R}^n \setminus H).$$

Let us define the function

$$f : \mathbb{R}^n \rightarrow [0, +\infty) \\ f(v) = |v| - \lambda \langle e_n, v \rangle.$$

Note that f is continuous; moreover $f(tv) = tf(v)$ for any $t \geq 0$ and f is convex. Let us set

$$\mu_i := v^{F_i} \mathcal{H}^{n-1} \llcorner (\partial^*F_i \cap (\mathbb{R}^n \setminus H)) \\ \mu := v^E \mathcal{H}^{n-1} \llcorner (\partial^*E \cap (\mathbb{R}^n \setminus H)).$$

Since $v^{F_i} \mathcal{H}^{n-1} \llcorner \partial^*F_i \rightarrow v^E \mathcal{H}^{n-1} \llcorner \partial^*E$ weakly* in \mathbb{R}^n , then $\mu_i \rightarrow \mu$ weakly* in $\mathbb{R}^n \setminus H$. Since we already know that $|\mu_i|(\mathbb{R}^n \setminus H) \rightarrow |\mu|(\mathbb{R}^n \setminus H)$, by Reshetnyak continuity theorem (see, e.g., [2, Theorem 2.39]) we deduce $P_\lambda(F_i) = \int f(\mu_i/|\mu_i|) d|\mu_i| \rightarrow \int f(\mu/|\mu|) d|\mu| = P_\lambda(E)$.

Step 2. Let us consider the flow $\varphi_t : \mathbb{R}^n \setminus H \rightarrow \mathbb{R}^n \setminus H$ such that

$$\varphi_t(x', x_n) := (x', x_n(1 + te^{-|x'|^2})),$$

for $t \geq 0$. By Lemma 2.3 for a.e. t we have that $\varphi_t(F_i)$ satisfies (4). Moreover, for any $t \geq 0$ there exists a sequence of rotations $\mathcal{R}_{j,t} : (x', x_n) \mapsto (R_{j,t}(x'), x_n)$ along the n -th axis converging to the identity such that

$$\mathcal{H}^{n-1}(\{x \in \partial^*(\mathcal{R}_{j,t}(\varphi_t(F_i))) : \nu^{\mathcal{R}_{j,t}(\varphi_t(F_i))}(x) = \pm e_l\}) = 0$$

for all $l = 1, \dots, n - 1$. By a diagonal argument, since $\varphi_t(F_i)$ maintains orthogonal intersection with ∂H , one extract the desired sequence E_i . □

Corollary 2.5 *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure. Then*

$$P_\lambda(E) \geq \frac{1 - \lambda}{2} P(E).$$

Proof We can assume that E is as in the conclusion of Lemma 2.4. The claim readily follows from the fact that the orthogonal projection on ∂H is 1-Lipschitz and surjective from $\partial E \cap (\mathbb{R}^n \setminus H)$ onto $\partial E \cap \partial H$, recalling that 1-Lipschitz maps do not increase the Hausdorff measure (see, e.g., [2, Proposition 2.49]). □

3 Isoperimetric inequality for P_λ , Fraenkel asymmetry and deficit

3.1 Isoperimetric inequality via ABP method

In this section we give a proof of the isoperimetric inequality for the capillarity functional P_λ exploiting an ABP method. We start by computing the capillarity perimeter of bubbles $B^\lambda(v)$.

Lemma 3.1 *There holds $P_\lambda(B^\lambda(v)) = n|B^\lambda|^{\frac{1}{n}} v^{\frac{n-1}{n}}$, for any $v \geq 0$ and $\lambda \in (-1, 1)$.*

Proof By scale invariance, it is sufficient to prove that $P(B^\lambda, \{x_n > \lambda\}) - \lambda \mathcal{H}^{n-1}(\partial B^\lambda \cap \{x_n = \lambda\})$ is equal to $n|B^\lambda|$. Indeed, let $u(x) = \frac{1}{2}|x|^2$. Then

$$\begin{aligned} n|B^\lambda| &= \int_{B^\lambda} \Delta u = P(B^\lambda, \{x_n > \lambda\}) + \int_{\partial B^\lambda \cap \{x_n = \lambda\}} \langle -e_n, x \rangle \\ &= P(B^\lambda, \{x_n > \lambda\}) - \lambda \mathcal{H}^{n-1}(\partial B^\lambda \cap \{x_n = \lambda\}). \end{aligned}$$

□

Remark 3.2 Let $E \subset \mathbb{R}^n \setminus H$ be a connected open set such that $\partial E \setminus H$ is a smooth hypersurface with boundary that intersects ∂H orthogonally. Then the Neumann problem

$$\begin{cases} \Delta u = \frac{P_\lambda(E)}{|E|} & \text{in } E, \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial E \setminus \partial H, \\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on } \partial E \cap \partial H, \end{cases} \tag{3.1}$$

has a solution $u \in C^1(\overline{E}) \cap C^\infty(E)$.

Indeed, existence of a weak solution of (3.1) follows by classical arguments exploiting the Riesz representation theorem. By [50, Proposition 3.6] there exists $\gamma > 0$ such that every weak solution is in $C^{0,\gamma}(E)$. Hence we can apply [44, Theorem 1] to the equivalent problem

$$\begin{cases} \Delta u - u = \frac{P_\lambda(E)}{|E|} - u =: f & \text{in } E, \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial E \setminus \partial H, \\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on } \partial E \cap \partial H \end{cases}$$

getting that a weak solution is in fact $C^1(\bar{E})$.

Theorem 3.3 (Isoperimetric inequality for P_λ) *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| \in (0, +\infty)$. Then*

$$\frac{P_\lambda(E)}{|E|^{\frac{n-1}{n}}} \geq \frac{P_\lambda(B^\lambda)}{|B^\lambda|^{\frac{n-1}{n}}} = n|B^\lambda|^{\frac{1}{n}}. \tag{3.2}$$

Moreover, equality occurs in (3.2) if and only if $E = B^\lambda(|E|)$ up to a translation and up to negligible sets.

Proof We just give a proof of the inequality (3.2) here, referring to [47, Theorem 19.21] for an alternative proof comprising the characterization of minimizers.

By the standard isoperimetric inequality, we can assume that $\mathcal{H}^{n-1}(\partial^*E \cap \partial H) > 0$. By Lemma 2.4, we can further assume that E is a bounded set such that $\partial E \setminus \partial H$ is smooth and intersects ∂H orthogonally.

Let us further assume for the moment that E is connected. Let u be the solution of the Neumann problem

$$\begin{cases} \Delta u = \frac{P_\lambda(E)}{|E|} & \text{in } E, \\ \frac{\partial u}{\partial \nu} = 1 & \text{on } \partial E \setminus \partial H, \\ \frac{\partial u}{\partial \nu} = -\lambda & \text{on } \partial E \cap \partial H, \end{cases} \tag{3.3}$$

where $\partial u/\partial \nu$ denotes the outer normal derivative of u on ∂E . Observe that such a solution exists and $u \in C^1(\bar{E}) \cap C^\infty(E)$ (see Remark 3.2). We consider the ‘‘lower contact set’’ of u defined by

$$\Gamma_u := \{x \in E : u(y) \geq u(x) + \langle \nabla u(x), y - x \rangle \text{ for all } y \in \bar{E}\}. \tag{3.4}$$

We claim that

$$B^\lambda \subset \nabla u(\Gamma_u). \tag{3.5}$$

To show (3.5), take any $p \in B^\lambda$. Let $x \in \bar{E}$ be a point such that

$$\min_{y \in \bar{E}} \{u(y) - \langle p, y \rangle\} = u(x) - \langle p, x \rangle.$$

If $x \in \partial E \setminus \partial H$ then the exterior normal derivative of $u(y) - \langle p, y \rangle$ at x would be nonpositive and hence $(\partial u/\partial \nu)(x) \leq |p| < 1$, a contradiction with (3.3). Similarly, if $x \in \partial E \cap \partial H$ then $(\partial u/\partial \nu)(x) \leq \langle p, -e_n \rangle < -\lambda$, a contradiction with (3.3). It follows that $x \in E$ and, therefore, that x is an interior minimum of the function $u(y) - \langle p, y \rangle$ over \bar{E} . In particular $p = \nabla u(x)$ and $x \in \Gamma_u$, hence Claim (3.5) is now proved.

From (3.5), since $u \in C^\infty(E)$ and $\Gamma_u \subset E$, we can apply the area formula on ∇u to deduce

$$|B^\lambda| \leq |\nabla u(\Gamma_u)| = \int_{\nabla u(\Gamma_u)} dp \leq \int_{\Gamma_u} |\det \nabla^2 u(x)| dx. \tag{3.6}$$

Since points $x \in \Gamma_u$ are interior minima for $y \mapsto u(y) - \langle \nabla u(x), y \rangle$, then $\nabla^2 u(x)$ is positively semi-definite. Hence by the arithmetic–geometric mean inequality

$$|\det \nabla^2 u| = \det \nabla^2 u \leq \left(\frac{\Delta u}{n}\right)^n \quad \text{in } \Gamma_u.$$

Hence

$$|B^\lambda| \leq \int_{\Gamma_u} \det \nabla^2 u \, dx \leq \int_{\Gamma_u} \left(\frac{\Delta u}{n}\right)^n \, dx \leq \int_E \left(\frac{\Delta u}{n}\right)^n \, dx,$$

since $\Delta u \equiv P_\lambda(E)/|E|$. Plugging in the value of Δu , the claimed inequality follows.

It remains to consider the case when E is not connected, hence when E is a disjoint union of finitely many bounded sets E_i , for $i = 1, \dots, k$, such that $\partial E_i \setminus \partial H$ is smooth and intersects ∂H orthogonally. Summing over $i = 1, \dots, k$ the isoperimetric inequality that we just proved for P_λ on each component E_i , and exploiting the subadditivity of $t \mapsto t^{\frac{n-1}{n}}$, the final inequality follows. \square

3.2 Asymmetry and deficit

We now define the Fraenkel asymmetry with respect to optimal bubbles $B^\lambda(v, x)$ and the deficit corresponding to the functional P_λ , proving some preliminary properties on these quantities.

Definition 3.4 (Fraenkel asymmetry) Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with measure $|E| = v \in (0, +\infty)$. We define

$$\alpha_\lambda(E) := \inf \left\{ \frac{|E \Delta B^\lambda(v, x)|}{v} : x \in \{x_n = 0\} \right\}.$$

It is readily checked that the Fraenkel asymmetry of E is a minimum.

Definition 3.5 (Isoperimetric deficit) Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with measure $|E| = v \in (0, +\infty)$. We define

$$D_\lambda(E) := \frac{P_\lambda(E) - P_\lambda(B^\lambda(v))}{P_\lambda(B^\lambda(v))}.$$

Lemma 3.6 *There exists $\bar{c} = \bar{c}(n, \lambda) > 0$ such that if a Borel set $E \subset \mathbb{R}^n \setminus H$ satisfies $D_\lambda(E) < \bar{c}$ then $P(E, \partial H) > 0$.*

Proof If E is a Borel set such that $D_\lambda(E) < \bar{c}$ and $P(E, \partial H) = 0$, then the standard isoperimetric inequality together with Lemma 3.1 imply

$$n\omega_n^{\frac{1}{n}} |E|^{\frac{n-1}{n}} \leq P(E) = P_\lambda(E) < (1 + \bar{c})P_\lambda(B^\lambda(|E|)) = n|B^\lambda|^{\frac{1}{n}}(1 + \bar{c})|E|^{\frac{n-1}{n}}.$$

Since $|E| > 0$ for \bar{c} small enough, we get a contradiction if \bar{c} is sufficiently small. \square

Lemma 3.7 *If $\{E_i\}_{i \in \mathbb{N}}$ and E are sets of finite perimeter in $\mathbb{R}^n \setminus H$ with finite measure such that $E_i \rightarrow E$ in L^1_{loc} . Then*

$$\liminf_{i \rightarrow +\infty} P_\lambda(E_i) \geq P_\lambda(E), \quad \liminf_{i \rightarrow +\infty} D_\lambda(E_i) \geq D_\lambda(E).$$

Proof Exploiting Theorem 2.2 and Reshetnyak lower semicontinuity theorem (see, e.g., [2, Theorem 2.38]), one easily checks that P_λ is lower semicontinuous with respect to L^1_{loc} convergence (see the proof of Lemma 2.4). \square

Lemma 3.8 *If $\{E_i\}_{i \in \mathbb{N}}$ and E are sets of finite perimeter in $\mathbb{R}^n \setminus H$ with finite measure such that $|E| > 0$ and such that $E_i \rightarrow E$ in $L^1(\mathbb{R}^n \setminus H)$. Then*

$$\lim_{i \rightarrow +\infty} \alpha_\lambda(E_i) = \alpha_\lambda(E)$$

Proof The claim is a standard exercise that follows from the fact that the L^1 -convergence holds on the whole $\mathbb{R}^n \setminus H$. □

Corollary 3.9 *If $\{E_i\}_{i \in \mathbb{N}}$ are sets of finite perimeter in $\mathbb{R}^n \setminus H$ such that $|E_i \setminus K| = 0$ for any i for some compact set $K \subset \mathbb{R}^n$, and if*

$$\sup_{i \in \mathbb{N}} |E_i| + P_\lambda(E_i) < \infty,$$

then there exists E of finite perimeter in $\mathbb{R}^n \setminus H$ and $i_k \rightarrow \infty$ as $k \rightarrow \infty$ such that

$$E_{i_k} \xrightarrow{L^1} E \quad \liminf_k P_\lambda(E_{i_k}) \geq P_\lambda(E).$$

Proof By Corollary 2.5 we have that $P_\lambda(E_i) \geq \frac{1-\lambda}{2} P(E)$. Then $\sup_{i \in \mathbb{N}} P(E_i) < \infty$. Hence by classical precompactness of sets of finite perimeter (see, e.g., [47, Theorem 12.26]) and recalling Lemma 3.7, the claim follows. □

4 Reduction to bounded symmetric sets

In the following arguments, in order to prove Theorem 1.1, we will repeatedly reduce ourselves to consider sets E having isoperimetric deficit smaller than some chosen constant. This reduction is always possible.

Indeed, let $\delta > 0$ be some positive constant; if E is a set of finite perimeter such that $D_\lambda(E) \geq \delta$, since $\alpha_\lambda(E) \leq 2$, we immediately get

$$\alpha_\lambda^2(E) \leq \frac{4}{\delta} \delta \leq \frac{4}{\delta} D_\lambda(E).$$

Therefore, if Theorem 1.1 is proved on sets with deficit $\leq \delta$, then it is proved for any set. Hence,

within this section we will assume that $D_\lambda(E) < \bar{c}$ for any competitor E involved, where \bar{c} is given by Lemma 3.6.

In particular $P(E, \partial H) > 0$.

4.1 Reduction to bounded sets

In this section we prove that, in order to prove Theorem 1.1, it is sufficient to prove the quantitative isoperimetric inequality (1.6) among suitably uniformly bounded sets.

From now on, we shall denote $Q_l := [-l, l]^n \subset \mathbb{R}^n$. We start by proving an estimate on the area of horizontal slices of a set in terms of its deficit.

Lemma 4.1 *Let $E \subset \mathbb{R}^n \setminus H$ be a bounded set of finite perimeter such that $\partial E \cap \mathbb{R}^n \setminus H$ is a smooth hypersurface (possibly with smooth boundary) with $|E| = |B^\lambda|$ and such that $\mathcal{H}^{n-1}(\{x \in \partial^* E \setminus H : \nu^E(x) = \pm e_n\}) = 0$. Then*

$$\mathcal{H}^{n-1}(E \cap \{x_n = t\}) \geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} \left(1 - \frac{|E \cap \{x_n < t\}|}{|B^\lambda|} \right)^{\frac{n-1}{n}} - 1 - D_\lambda(E) \right), \tag{4.1}$$

for every $t > 0$. In particular

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} - 1 - D_\lambda(E) \right). \tag{4.2}$$

Moreover, if $F \subset \mathbb{R}^n \setminus H$ is a set of finite perimeter with $|F| = |B^\lambda|$, then (4.1) holds with F in place of E for almost every $t > 0$, and (4.2) holds with F in place of E .

The estimates given by Lemma 4.1 are clearly nontrivial only when the deficit is sufficiently small. On the other hand, if the deficit $D_\lambda(E)$ is sufficiently small, since $\omega_n/|B^\lambda| > 1$, (4.1) and (4.2) essentially yield a quantitative version of Lemma 3.6, nontrivial also for slices $\mathcal{H}^{n-1}(E \cap \{x_n = t\})$ with $t > 0$ as long as $|E \cap \{x_n < t\}|$ is small.

Proof of Lemma 4.1 Let $v_E(t) := \mathcal{H}^{n-1}(E \cap \{x_n = t\})$ for any $t > 0$, and let $g(t) := |E \cap \{x_n < t\}|/|B^\lambda|$. By the standard isoperimetric inequality we have that

$$\begin{aligned} P(E, \{x_n > t\}) + v_E(t) &= P(E \cap \{x_n > t\}) \geq n\omega_n^{\frac{1}{n}} |E \cap \{x_n > t\}|^{\frac{n-1}{n}} \\ &= P_\lambda(B^\lambda) \left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - g(t))^{\frac{n-1}{n}}, \end{aligned} \tag{4.3}$$

for any $t > 0$. Moreover, for any $t > 0$, we observe that for any $x' \in \partial^* E \cap \partial H$, the halfline $[0, t] \ni x_n \mapsto (x', x_n)$ either intersects $\partial^* E \cap \{0 < x_n \leq t\}$ or it intersects $E \cap \{x_n = t\}$. Therefore

$$P(E, \{0 < x_n \leq t\}) + v_E(t) \geq \mathcal{H}^{n-1}(\partial^* E \cap \partial H), \tag{4.4}$$

for any $t > 0$. Hence we conclude that

$$\begin{aligned} P_\lambda(B^\lambda)(1 + D_\lambda(E)) &= P_\lambda(E) \\ &= P(E, \{x_n > t\}) + P(E, \{0 < x_n \leq t\}) - \lambda \mathcal{H}^{n-1}(\partial^* E \cap \partial H) \\ &\stackrel{(4.4)}{\geq} P(E, \{x_n > t\}) - v_E(t) \\ &\stackrel{(4.3)}{\geq} P_\lambda(B^\lambda) \left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - g(t))^{\frac{n-1}{n}} - 2v_E(t), \end{aligned}$$

for any $t > 0$, which yields (4.1). By [47, Theorem 18.11] (see also [29, Theorem 6.1]), the function v_E belongs to $W^{1,1}(0, +\infty)$, thus (4.2) follows by letting $t \rightarrow 0^+$ in (4.1).

Now if $F \subset \mathbb{R}^n \setminus H$ is as in the assumptions, let E_i be given by Lemma 2.4 applied to F , and let $\tilde{E}_i := (|B^\lambda|/|E_i|)^{\frac{1}{n}} E_i$. Hence the inequality (4.2) and the right hand side of (4.1) applied with $E = \tilde{E}_i$ pass to the limit as $i \rightarrow \infty$. Moreover

$$\begin{aligned} |\tilde{E}_i \Delta E| &= \int_0^{+\infty} \mathcal{H}^{n-1}(\tilde{E}_i \Delta E \cap \{x_n = t\}) dt \\ &\geq \int_0^{+\infty} \left| \mathcal{H}^{n-1}(\tilde{E}_i \cap \{x_n = t\}) - \mathcal{H}^{n-1}(E \cap \{x_n = t\}) \right| dt. \end{aligned}$$

Since $|\tilde{E}_i \Delta E| \rightarrow 0$, the left hand side of (4.1) passes to the limit as well as $i \rightarrow \infty$, for a.e. $t > 0$. □

We are ready to prove the claimed reduction to bounded sets. The proof follows the line of [29, Lemma 5.1], essentially truncating a competitor with coordinate slabs having estimated width. To give a bound for the truncation in the n -th direction will need to modify the argument in [29, Lemma 5.1] and we will exploit Lemma 4.1.

Lemma 4.2 (Reduction to bounded sets) *There exist $l = l(n, \lambda) > 0$ and $C_1 = C_1(n, \lambda) > 0$ such that, if $E \subset \mathbb{R}^n \setminus H$ is a set of finite perimeter with $|E| \in (0, +\infty)$, then there exists a set of finite perimeter $E' \subset Q_l \cap (\mathbb{R}^n \setminus H)$ such that $|E'| = |B^\lambda|$ and*

$$\alpha_\lambda(E) \leq \alpha_\lambda(E') + C_1 D_\lambda(E), \quad D_\lambda(E') \leq C_1 D_\lambda(E). \tag{4.5}$$

Proof By scale-invariance of the asymmetry and of the deficit, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. First of all we observe that we may prove the claim assuming that $\partial E \cap \mathbb{R}^n \setminus H$ is smooth and

$$\mathcal{H}^{n-1}(\{x \in \partial^* E \cap \mathbb{R}^n \setminus H : \nu^E(x) = \pm e_i\}) = 0 \tag{4.6}$$

for all $i = 1, \dots, n$. Indeed, if E is a generic set of finite perimeter, then by Lemma 2.4 there exists a sequence of smooth sets $\{E_i\}_{i \in \mathbb{N}}$ converging to E such that (4.6) holds. If we know that the claim holds for E_i , we get the existence of $E'_i \subset Q_l \cap (\mathbb{R}^n \setminus H)$ such that (4.5) holds with E, E' replaced by E_i, E'_i . Hence we can apply Corollary 3.9 on the sequence E'_i , and by Lemmas 3.7 and 3.8 the inequalities (4.5) pass to the limit.

Without loss of generality, we can further assume that

$$D_\lambda(E) < (2^{1/n} - 1)/4. \tag{4.7}$$

Let us consider the axis x_1 first. Thanks to (4.6), by [47, Theorem 18.11] (see also [29, Theorem 6.1]) we deduce that

$$v_E(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (t, x') \in E\}) \quad \text{for } t \in \mathbb{R}$$

belongs to $W^{1,1}(\mathbb{R})$, hence we may assume that v_E is continuous. Setting

$$E_t^- := \{x \in E : x_1 < t\}$$

and

$$P_\lambda(E, \{x_1 < t\}) := P(E, \{x \in \mathbb{R}^n \setminus H : x_1 < t\}) - \lambda \mathcal{H}^{n-1}(\{x \in \partial^* E \cap \partial H : x_1 < t\})$$

for all $t \in \mathbb{R}$, by smoothness of E we have that

$$P_\lambda(E_t^-) = P_\lambda(E, \{x_1 < t\}) + v_E(t), \quad P_\lambda(E \setminus E_t^-) = P_\lambda(E, \{x_1 > t\}) + v_E(t), \tag{4.8}$$

where $P_\lambda(E, \{x_1 > t\})$ is defined analogously. Let us now define the function $g : \mathbb{R} \rightarrow [0, +\infty)$ given by

$$g(t) := \frac{|E_t^-|}{|B^\lambda|}.$$

Hence g is a nondecreasing C^1 function with $g'(t) = v_E(t)/|B^\lambda|$. Let $-\infty \leq a < b \leq +\infty$ be such that $\{t : 0 < g(t) < 1\} = (a, b)$. If $t \in (a, b)$, then by (3.2) we have

$$P_\lambda(E_t^-) \geq g(t)^{\frac{n-1}{n}} P_\lambda(B^\lambda).$$

Similarly,

$$P_\lambda(E \setminus E_t^-) \geq (1 - g(t))^{\frac{n-1}{n}} P_\lambda(B^\lambda).$$

Therefore, from (4.6) and (4.8) we get that

$$P_\lambda(E) + 2v_E(t) \geq P_\lambda(B^\lambda) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} \right)$$

for all $t \in (a, b)$. Since by definition we have $P_\lambda(E) = P_\lambda(B^\lambda)(1 + D_\lambda(E))$, we obtain

$$v_E(t) \geq \frac{1}{2} P_\lambda(B^\lambda) \left(g(t)^{\frac{n-1}{n}} + (1 - g(t))^{\frac{n-1}{n}} - 1 - D_\lambda(E) \right). \tag{4.9}$$

Let us now define the concave function

$$\psi : [0, 1] \rightarrow [0, +\infty) \quad \psi(t) := t^{\frac{n-1}{n}} + (1 - t)^{\frac{n-1}{n}} - 1.$$

Note that $\psi(0) = \psi(1) = 0$ and ψ achieves its maximum at $\psi(1/2) = 2^{1/n} - 1$, hence by concavity

$$\psi(t) = \psi \left(2t \frac{1}{2} + 0 \right) \geq 2t\psi(1/2) + 0 = 2(2^{1/n} - 1)t \quad \forall t \in \left[0, \frac{1}{2} \right]. \tag{4.10}$$

Recall that by (4.7) there holds $2D_\lambda(E) < \psi(1/2)$. Let $a < t_1 < t_2 < b$ be such that $g(t_1) = 1 - g(t_2)$ and $\psi(g(t_1)) = \psi(g(t_2)) = 2D_\lambda(E)$. Then

$$\psi(g(t)) \geq 2D_\lambda(E) \quad \forall t \in (t_1, t_2) \tag{4.11}$$

and, by (4.10),

$$g(t_1) = 1 - g(t_2) \leq \frac{D_\lambda(E)}{2^{1/n} - 1}. \tag{4.12}$$

For any $t_1 \leq t \leq t_2$ we have

$$\begin{aligned} v_E(t) &\stackrel{(4.9)}{\geq} \frac{1}{2} P_\lambda(B^\lambda) (\psi(g(t)) - D_\lambda(E)) \\ &= \frac{1}{4} P_\lambda(B^\lambda) \psi(g(t)) + \frac{1}{4} P_\lambda(B^\lambda) (\psi(g(t)) - 2D_\lambda(E)) \\ &\stackrel{(4.11)}{\geq} \frac{n|B^\lambda|}{4} \psi(g(t)). \end{aligned} \tag{4.13}$$

Since $v_E(t) = |B^\lambda|g'(t)$, we have

$$t_2 - t_1 \stackrel{(4.13)}{\leq} \frac{4}{n} \int_{t_1}^{t_2} \frac{g'(t)}{\psi(g(t))} dt = \frac{4}{n} \int_{g(t_1)}^{g(t_2)} \frac{1}{\psi(s)} ds \leq \frac{4}{n} \int_0^1 \frac{1}{\psi(s)} ds =: \alpha, \tag{4.14}$$

for some $\alpha = \alpha(n) > 0$.

Let

$$\begin{aligned} \tau_1 &= \max \left\{ t \in (a, t_1] : v_E(t) \leq \frac{n|B^\lambda|D_\lambda(E)}{2} \right\}, \\ \tau_2 &= \min \left\{ t \in [t_2, b) : v_E(t) \leq \frac{n|B^\lambda|D_\lambda(E)}{2} \right\}. \end{aligned}$$

Note that τ_1 and τ_2 are well defined since v_E is continuous and $v_E(t) \rightarrow 0$ as $t \rightarrow a$ or $t \rightarrow b$; moreover, by (4.11) and (4.13), $v_E(\tau_1) = v_E(\tau_2) = \frac{n|B^\lambda|D_\lambda(E)}{2}$. Moreover, from (4.12) and by definition of τ_1 , we have

$$t_1 - \tau_1 \leq \frac{2}{n|B^\lambda|D_\lambda(E)} \int_{\tau_1}^{t_1} v_E(t) dt = \frac{2}{nD_\lambda(E)} \int_{\tau_1}^{t_1} g'(t) dt \leq \frac{2g(t_1)}{nD_\lambda(E)} \leq \frac{2}{n(2^{1/n} - 1)},$$

and an analogous estimate holds for $\tau_2 - t_2$.

We consider the truncation $\tilde{E} := E \cap \{x : \tau_1 < x_1 < \tau_2\}$. From the above estimate and (4.14), we have that $\tau_2 - \tau_1 < \beta$ for some $\beta = \beta(n) > 0$. Moreover by (4.8) and (4.12), by the definition of τ_1, τ_2 , and since $P(E, \{x_1 < \tau_1, x_1 > \tau_2\}) \geq \lambda \mathcal{H}^{n-1}(\partial^* E \cap \partial H \cap \{x_1 < \tau_1, x_1 > \tau_2\})$ (see the proof of Corollary 2.5) we can estimate

$$|\tilde{E}| \geq |B^\lambda| \left(1 - 2 \frac{D_\lambda(E)}{2^{1/n} - 1}\right), \quad P_\lambda(\tilde{E}) \leq P_\lambda(E) + n|B^\lambda|D_\lambda(E). \tag{4.15}$$

We finally define

$$\sigma := \left(\frac{|B^\lambda|}{|\tilde{E}|}\right)^{1/n}, \quad E' := \sigma \tilde{E}.$$

Clearly, $|E'| = |B^\lambda|$ and by (4.15) we get that E' is contained in a strip $\{\tau'_1 < x_1 < \tau'_2\}$, with $\tau'_2 - \tau'_1 \leq \sigma(\tau_2 - \tau_1) \leq l'$, where $l' = l'(n, \lambda) > 0$. Let us now show that E' satisfies (4.5) for a suitable constant $C_1 = C_1(n, \lambda) > 0$ that may change from line to line. To this aim, since we are assuming $D_\lambda(E)$ small by (4.7), from (4.15) we get that $1 \leq \sigma \leq 1 + C_0 D_\lambda(E)$, with $C_0 = C_0(n)$. Thus, from (4.15) and (4.7), we get

$$\begin{aligned} P_\lambda(E') &= \sigma^{n-1} P_\lambda(\tilde{E}) \leq \sigma^{n-1} (P_\lambda(E) + n|B^\lambda|D_\lambda(E)) \\ &= \sigma^{n-1} P_\lambda(B^\lambda)(1 + 2D_\lambda(E)) \leq P_\lambda(B^\lambda)(1 + C_1 D_\lambda(E)). \end{aligned}$$

Hence, the second inequality in (4.5) follows. To prove the first inequality, let us denote by $B^\lambda(|B^\lambda|, p)$, with $p \in \partial H$, a spherical cap such that $\alpha_\lambda(E') = \frac{|E' \Delta B^\lambda(|B^\lambda|, p)|}{|B^\lambda|}$. From the first inequality in (4.15), recalling that $|E| = |B^\lambda|$, we then get

$$\begin{aligned} \alpha_\lambda(E) &\leq \frac{|E \Delta B^\lambda(|B^\lambda|, p/\sigma)|}{|B^\lambda|} \\ &\leq \frac{|E \Delta \tilde{E}|}{|B^\lambda|} + \frac{|\tilde{E} \Delta B^\lambda(|B^\lambda|/\sigma^n, p/\sigma)|}{|B^\lambda|} + \frac{|B^\lambda(|B^\lambda|/\sigma^n, p/\sigma) \Delta B^\lambda(|B^\lambda|, p/\sigma)|}{|B^\lambda|} \\ &= \frac{|E \setminus \tilde{E}|}{|B^\lambda|} + \frac{\alpha_\lambda(E')}{\sigma^n} + \frac{|B^\lambda(|B^\lambda|) \setminus B^\lambda(|B^\lambda|/\sigma^n)|}{|B^\lambda|} \\ &\stackrel{(4.15)}{\leq} C_1 D_\lambda(E) + \alpha_\lambda(E') + C_1(\sigma - 1) \\ &\leq \alpha_\lambda(E') + C_1 D_\lambda(E). \end{aligned}$$

Thus the set E' satisfies (4.5) and points in E' have first coordinate contained in an interval of length bounded by l' .

Starting from E' , we can repeat the same construction finitely many times with respect to the axes x_2, \dots, x_{n-1} , thus getting a new set, still denoted by E' , satisfying (4.5).

It remains to adapt the construction with respect to the coordinate axis x_n . In this case we eventually aim at truncating the set E' in some controlled slab of the form $\{0 < x_n < \bar{\tau}_2\}$. Define

$$\bar{v}(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (x', t) \in E'\}), \quad \bar{E}_t^- := \{x \in E' : x_n < t\}, \quad \bar{g}(t) := \frac{|\bar{E}_t^-|}{|B^\lambda|},$$

for $t > 0$. It is readily checked that, arguing as above, one estimates

$$\bar{v}(t) \geq \frac{1}{2} P_\lambda(B^\lambda) (\psi(\bar{g}(t)) - D_\lambda(E)), \tag{4.16}$$

which is analogous to (4.9), for any t such that $\bar{g}(t) \in (0, 1)$. Similarly as before, we define $0 < \bar{t}_1 < \bar{t}_2$ such that $\bar{g}(\bar{t}_1) = 1 - \bar{g}(\bar{t}_2)$ and $\psi(\bar{g}(\bar{t}_1)) = \psi(\bar{g}(\bar{t}_2)) = 2D_\lambda(E')$. Therefore, using (4.16) and the concavity of ψ , arguing as before one estimates

$$\bar{v}(t) \geq \frac{n|B^\lambda|}{4} \psi(\bar{g}(t)) \quad \forall t \in [\bar{t}_1, \bar{t}_2], \tag{4.17}$$

which is analogous to (4.13).

Let

$$A := \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} - 1 \right) > 0.$$

We claim that there exists $\bar{\varepsilon} = \bar{\varepsilon}(n, \lambda) > 0$ such that if $D_\lambda(E) < \bar{\varepsilon}$ then

$$\bar{v}(t) \geq \frac{A}{2} \quad \text{for a.e. } t \in (0, \bar{t}_1). \tag{4.18}$$

Indeed, since $D_\lambda(E') \leq C_1 D_\lambda(E)$ and $\psi(\bar{g}(\bar{t}_1)) = 2D_\lambda(E')$, then for any $\omega > 0$ there is $\bar{\varepsilon} = \bar{\varepsilon}(n, \lambda) > 0$ such that $\bar{g}(\bar{t}_1) < \omega$ whenever $D_\lambda(E) < \bar{\varepsilon}$. Applying Lemma 4.1 with $F = E'$, for almost every $t \in (0, \bar{t}_1)$ we find

$$\begin{aligned} \bar{v}(t) &\geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \bar{g}(t))^{\frac{n-1}{n}} - 1 - D_\lambda(E') \right) \\ &\geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \bar{g}(\bar{t}_1))^{\frac{n-1}{n}} - 1 - D_\lambda(E') \right) \\ &\geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|} \right)^{\frac{1}{n}} (1 - \omega)^{\frac{n-1}{n}} - 1 - C_1 \bar{\varepsilon} \right) \geq \frac{A}{2}, \end{aligned} \tag{4.19}$$

provided $\bar{\varepsilon}$ is small enough.

Therefore, assuming without loss of generality that $D_\lambda(E) < \bar{\varepsilon}$, since $\bar{g}'(t) = \bar{v}(t)/|B^\lambda|$ we estimate

$$\begin{aligned} \bar{t}_2 &\stackrel{(4.17)}{\leq} \bar{t}_1 + \frac{4}{n} \int_{\bar{t}_1}^{\bar{t}_2} \frac{\bar{g}'(t)}{\psi(\bar{g}(t))} dt \stackrel{(4.18)}{\leq} \frac{2}{A} \int_0^{\bar{t}_1} \bar{v}(t) dt + \frac{4}{n} \int_0^1 \frac{1}{\psi} \\ &\leq \frac{2|B^\lambda|}{A} + \frac{4}{n} \int_0^1 \frac{1}{\psi} =: \alpha'(n, \lambda). \end{aligned} \tag{4.20}$$

The rest of the construction follows analogously as above by defining

$$\bar{\tau}_2 := \min \left\{ t \geq \bar{t}_2 : \bar{g}(t) < 1, \bar{v}(t) \leq \frac{n|B^\lambda|D_\lambda(E')}{2} \right\},$$

estimating $\bar{\tau}_2 - \bar{t}_2 \leq \beta'(n, \lambda)$, hence finally taking the set

$$\left(\frac{|B^\lambda|}{|E' \cap \{x_n < \bar{\tau}_2\}|} \right)^{\frac{1}{n}} (E' \cap \{x_n < \bar{\tau}_2\}). \tag{4.21}$$

Up to translation along ∂H , the set defined in (4.21) yields the final one satisfying the claim of the lemma. \square

Corollary 4.3 (Non-quantitative stability) *For any $\bar{\varepsilon} > 0$ there exists $\bar{\delta} = \bar{\delta}(n, \lambda, \bar{\varepsilon}) > 0$ such that if $E \subset \mathbb{R}^n \setminus H$ is a Borel set such that $D_\lambda(E) \leq \bar{\delta}$, then $\alpha_\lambda(E) \leq \bar{\varepsilon}$.*

Proof By scale-invariance of the asymmetry and of the deficit, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. We argue by contradiction. Suppose there exist a number $\bar{\varepsilon} > 0$ and a sequence of sets $\{E_i\}_i$, with $E_i \subset \mathbb{R}^n \setminus H$ and $|E_i| = |B^\lambda|$, such that $D_\lambda(E_i) < \frac{1}{i}$ and $\alpha_\lambda(E_i) > \bar{\varepsilon}$ for all $i \in \mathbb{N}$. Let us consider the sequence of sets $\{E'_i\}_i$, with $E'_i \subset Q_i \cap (\mathbb{R}^n \setminus H)$ and $|E'_i| = |B^\lambda|$, given by Lemma 4.2. Moreover the Lemma assures that $\alpha_\lambda(E'_i) > \bar{\varepsilon}/2$ for large i , and $D_\lambda(E'_i) \rightarrow 0$. Since each set E'_i is contained in the same Q_i , by Corollary 3.9

we can assume, up to a subsequence, that $E'_i \xrightarrow{L^1} E'$ for some set E' of finite perimeter with $|E'| = |B^\lambda|$. By the lower semicontinuity of the perimeters we get $P_\lambda(E') \leq P_\lambda(B^\lambda)$, hence $E' = B^\lambda(|B^\lambda|, x)$ for some $x \in \partial H \cap Q_i$ by uniqueness of minimizers. The convergence of E'_i to E' implies that $|E'_i \Delta E'| \rightarrow 0$, against the assumption $\alpha_\lambda(E'_i) > \frac{\bar{\varepsilon}}{2}$. \square

Corollary 4.4 *There exist $A_\lambda, T_\lambda, \eta > 0$ depending on n, λ such that for any set of finite perimeter $E \subset \mathbb{R}^n \setminus H$ with $||E| - |B^\lambda|| \leq \eta$ and $D_\lambda(E) \leq \eta$ there holds*

$$\mathcal{H}^{n-1}(E \cap \{x_n = t\}) \geq A_\lambda,$$

for almost every $t \in (0, T_\lambda)$.

Proof Let us prove the inequality assuming $|E| = |B^\lambda|$ first. Fix $T'_\lambda > 0$ such that

$$\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}} \left(1 - \frac{|B^\lambda(|B^\lambda|, 0) \cap \{x_n < T'_\lambda\}|}{|B^\lambda|}\right)^{\frac{n-1}{n}} \geq 1 + a,$$

for some $a > 0$. By Corollary 4.3, for any $\omega > 0$ there is $\eta > 0$ such that if $D_\lambda(E) < \eta$ then $|E \cap \{x_n < T'_\lambda\}| \leq |B^\lambda(|B^\lambda|, 0) \cap \{x_n < T'_\lambda\}| + \omega$. For such a set E , Lemma 4.1 implies that for almost every $t \in (0, T'_\lambda)$ there holds

$$\begin{aligned} \mathcal{H}^{n-1}(E \cap \{x_n = t\}) &\geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}} \left(1 - \frac{|E \cap \{x_n < t\}|}{|B^\lambda|}\right)^{\frac{n-1}{n}} - 1 - D_\lambda(E) \right) \\ &\geq \frac{1}{2} P_\lambda(B^\lambda) \left(\left(\frac{\omega_n}{|B^\lambda|}\right)^{\frac{1}{n}} \left(1 - \frac{|B^\lambda(|B^\lambda|, 0) \cap \{x_n < T'_\lambda\}| + \omega}{|B^\lambda|}\right)^{\frac{n-1}{n}} - 1 - \eta \right). \end{aligned}$$

Hence for sufficiently small $\eta > 0$ the right hand side in the previous estimate is bounded below by some constant $A'_\lambda(n, \lambda) > 0$.

For a generic set E such that $||E| - |B^\lambda|| \leq \eta$ and $D_\lambda(E) \leq \eta$, the set $E' = \left(|B^\lambda|^{\frac{1}{n}}/|E|^{\frac{1}{n}}\right) E$ has measure equal to $|B^\lambda|$ and deficit $D_\lambda(E') \leq \eta$. Up to decreasing $\eta > 0$, applying the first part of the proof to E' , the desired estimate holds on E for $T_\lambda = T'_\lambda/2$ and $A_\lambda = A'_\lambda/2$. \square

4.2 Reduction to $(n - 1)$ -symmetric sets

In this section we prove that, in order to prove Theorem 1.1, it is sufficient to further reduce to show (1.6) among $(n - 1)$ -symmetric sets, i.e., sets which are symmetric with respect to reflection across $n - 1$ orthogonal hyperplanes, each one orthogonal to $\{x_n = 0\}$. The results are analogous to [46, Section 6].

Lemma 4.5 *Let $E \subset \mathbb{R}^n \setminus H$ be a Borel set with finite measure, symmetric with respect to $k \in \{1, \dots, n - 1\}$ orthogonal half-hyperplanes $H_j = \{x \in \mathbb{R}^n \setminus H : \langle x, v_j \rangle = 0\}$ for*

$1 \leq j \leq k$, where $|v_j| = 1$ and $\langle v_j, e_n \rangle = 0$ for any $1 \leq j \leq k$. Then

$$\min_{x \in \partial H} |E \Delta B^\lambda(|E|, x)| \leq \min_{y \in \partial H \cap \bigcap_{j=1}^k H_j} |E \Delta B^\lambda(|E|, y)| \leq 3 \min_{x \in \partial H} |E \Delta B^\lambda(|E|, x)|. \tag{4.22}$$

Proof We can suppose for simplicity that $\forall j \in \{1, \dots, k\}$ we have $v_j = e_j$. If $x^0 = (x_1^0, \dots, x_{n-1}^0, 0)$ is such that $\alpha_\lambda(E)$ is achieved by $B^\lambda(|E|, x^0)$, then $\alpha_\lambda(E)$ is achieved also by $B^\lambda(|E|, \bar{x}^0)$, where $\bar{x}^0 = (-x_1^0, \dots, -x_k^0, x_{k+1}^0, \dots, x_{n-1}^0, 0)$. Since $|B^\lambda(|E|, x^0) \Delta B^\lambda(|E|)| \leq |B^\lambda(|E|, x^0) \Delta B^\lambda(|E|, \bar{x}^0)|$ we have

$$\begin{aligned} |E \Delta B^\lambda(|E|)| &\leq |E \Delta B^\lambda(|E|, x^0)| + |B^\lambda(|E|, x^0) \Delta B^\lambda(|E|)| \\ &\leq |E \Delta B^\lambda(|E|, x^0)| + |B^\lambda(|E|, x^0) \Delta B^\lambda(|E|, \bar{x}^0)| \\ &\leq |E \Delta B^\lambda(|E|, x^0)| + |B^\lambda(|E|, x^0) \Delta E| + |E \Delta B^\lambda(|E|, \bar{x}^0)| \\ &= 3|E \Delta B^\lambda(|E|, x^0)|. \end{aligned}$$

□

Given a Borel set $E \subset \mathbb{R}^n \setminus H$ with finite measure and a unit vector v with $\langle v, e_n \rangle = 0$, we denote by $H_v^+ = \{x \in \mathbb{R}^n : \langle x, v \rangle > t\}$ an open half-space orthogonal to v where $t \in \mathbb{R}$ is chosen in such a way that

$$|E \cap H_v^+| = \frac{|E|}{2}.$$

We also denote by $r_v : \mathbb{R}^n \setminus H \rightarrow \mathbb{R}^n \setminus H$ the reflection with respect to $H_v := \partial H_v^+$, and by $H_v^- := r_v(H_v^+)$ the open half-space complementary to H_v^+ . Finally we write $E_v^\pm := E \cap H_v^\pm$.

Observe that

$$D_\lambda(E_v^\pm \cup r_v(E_v^\pm)) \leq 2D_\lambda(E). \tag{4.23}$$

Indeed

$$\begin{aligned} P_\lambda(E_v^\pm \cup r_v(E_v^\pm)) - P_\lambda(B^\lambda(|E|)) &\leq 2P_\lambda(E) - P_\lambda(E_v^\mp \cup r_v(E_v^\mp)) - P_\lambda(B^\lambda(|E|)) \\ &= 2(P_\lambda(E) - P_\lambda(B^\lambda(|E|))) + P_\lambda(B^\lambda(|E|)) \\ &\quad - P_\lambda(E_v^\mp \cup r_v(E_v^\mp)) \\ &\leq 2(P_\lambda(E) - P_\lambda(B^\lambda(|E|))), \end{aligned}$$

where in the last inequality we used the isoperimetric inequality of Theorem 3.3.

Lemma 4.6 *There exist $\bar{C}_2, \bar{\delta}_2 > 0$ depending on n, λ such that, if $E \subset \mathbb{R}^n \setminus H$ is a Borel set with finite measure such that $D_\lambda(E) \leq \bar{\delta}_2$, and if v_1 and v_2 are two orthogonal vectors, with $\langle v_i, e_n \rangle = 0$, such that H_{v_1} and H_{v_2} divide E in four parts of equal measure, then there exist $i \in \{1, 2\}$ and $s \in \{+, -\}$ such that, setting $E' = E_{v_i}^s \cup r_{v_i}(E_{v_i}^s)$, there holds*

$$\alpha_\lambda(E) \leq \bar{C}_2 \alpha(E'). \tag{4.24}$$

Proof By scale-invariance of Fraenkel asymmetry, it is sufficient to prove the claim assuming also $|E| = |B^\lambda|$. If $i \in \{1, 2\}$ and $s \in \{+, -\}$, let $E_{v_i}^s$ denote the sets obtained by reflecting $E_{v_i}^s$ along H_{v_i} and let $B_i^{\lambda, s} = B^\lambda(|B^\lambda|, x_i^s)$ be four spherical caps such that

$$|E_{v_i}^s \Delta B_i^{\lambda, s}| = \min_{x \in H_{v_i} \cap \partial H} |E_{v_i}^s \Delta B^\lambda(|B^\lambda|, x)|.$$

For $i = 1, 2$, by the triangular inequality we have

$$\begin{aligned}
 \min_{x \in \partial H} |E \Delta B^\lambda(|B^\lambda|, x)| &\leq |E \Delta B_i^{\lambda,+}| \\
 &= |(E \Delta B_i^{\lambda,+}) \cap H_{v_i}^+| + |(E \Delta B_i^{\lambda,+}) \cap H_{v_i}^-| \\
 &\leq |(E \Delta B_i^{\lambda,+}) \cap H_{v_i}^+| + |(E \Delta B_i^{\lambda,-}) \cap H_{v_i}^-| + |(B_i^{\lambda,+} \Delta B_i^{\lambda,-}) \cap H_{v_i}^-| \\
 &= \frac{1}{2} |E_{v_i}^+ \Delta B_i^{\lambda,+}| + \frac{1}{2} |E_{v_i}^- \Delta B_i^{\lambda,-}| + \frac{1}{2} |B_i^{\lambda,+} \Delta B_i^{\lambda,-}|. \tag{4.25}
 \end{aligned}$$

Once we show that if $D_\lambda(E)$ is sufficiently small then there exists $c_{n,\lambda} > 0$ such that at least one the following

$$\begin{aligned}
 |B_1^{\lambda,+} \Delta B_1^{\lambda,-}| &\leq 2c_{n,\lambda} \left(|E_{v_1}^+ \Delta B_1^{\lambda,+}| + |E_{v_1}^- \Delta B_1^{\lambda,-}| \right) \\
 |B_2^{\lambda,+} \Delta B_2^{\lambda,-}| &\leq 2c_{n,\lambda} \left(|E_{v_2}^+ \Delta B_2^{\lambda,+}| + |E_{v_2}^- \Delta B_2^{\lambda,-}| \right)
 \end{aligned} \tag{4.26}$$

holds, then we soon conclude the proof. Indeed, assume for example that the first inequality in (4.26) holds. Then, from (4.22) and (4.25) with $i = 1$, we get

$$\begin{aligned}
 \min_{x \in \partial H} |E \Delta B^\lambda(|B^\lambda|, x)| &\stackrel{(4.25)}{\leq} (c_{n,\lambda} + 1/2) \left(|E_{v_1}^+ \Delta B_1^{\lambda,+}| + |E_{v_1}^- \Delta B_1^{\lambda,-}| \right) \\
 &\stackrel{(4.22)}{\leq} 3(c_{n,\lambda} + 1/2) \left(\min_{x \in \partial H} |E_{v_1}^+ \Delta B^\lambda(|B^\lambda|, x)| + \min_{x \in \partial H} |E_{v_1}^- \Delta B^\lambda(|B^\lambda|, x)| \right),
 \end{aligned}$$

thus proving (4.24) with $\bar{C}_2 = 6(c_{n,\lambda} + 1/2)$ and E' equal to $E_{v_1}^+$ or $E_{v_1}^-$.

Observe that, given $\tilde{\epsilon} > 0$, Corollary 4.3, (4.22) and (4.23) imply that there exists $\bar{\delta}_2(n, \lambda) > 0$ such that if $D_\lambda(E) < \bar{\delta}_2$ then

$$\max \left\{ \alpha_\lambda(E), \frac{|E_{v_i}^\pm \Delta B_i^{\lambda,\pm}|}{|B^\lambda|} : i = 1, 2 \right\} < \tilde{\epsilon}. \tag{4.27}$$

Thanks to (4.27), we can show that the caps $B_i^{\lambda,\pm}$ get closer and closer to the optimal ones for E , as $\bar{\delta}_2$ decreases.

Indeed, let us assume by contradiction that there exists $\eta > 0$ such that for every $j \in \mathbb{N}$ there exist E_j , with $|E_j| = |B^\lambda|$, $D_\lambda(E_j) < \frac{1}{j}$, with $B_j^\lambda := B^\lambda(|B^\lambda|, x_j)$ realizing the asymmetry of E_j , but for $i \in \{1, 2\}$ and $s \in \{+, -\}$, if $B_{i,j}^{\lambda,s} := B^\lambda(|B^\lambda|, x_{i,j}^s)$ is such that

$$|E_{j,v_i^j}^{s'} \Delta B_{i,j}^{\lambda,s}| = \min_{x \in H_{v_i^j} \cap \partial H} |E_{j,v_i^j}^{s'} \Delta B^\lambda(|B^\lambda|, x)|,$$

where $E_{j,v_i^j}^{s'}$ is given by reflections of truncations of E_j along orthogonal subspaces $H_{v_i^j}$, $H_{v_i^j}$, then $|x_{i,j}^s - x_j| > \eta$ for some $i \in \{1, 2\}$, $s \in \{+, -\}$ and any j . Without loss of generality we can assume that that $i = 1$ and $s = +$.

Let us translate every set in the above contradiction assumption by $-x_j$. Without relabeling the objects involved, up to subsequences, we have that $E_j \rightarrow B_0^\lambda := B^\lambda(|B^\lambda|, 0)$ in L^1 . We can show that $E_{j,v_1^j}^+ \rightarrow B_0^\lambda$ as well. Indeed, up to a rotation we can further assume that

$$H_{v_1^j} = \left\{ (a_j, 0, \dots, 0) + e_1^\perp \right\}, \quad v_1^j = e_1, \quad \forall j,$$

with $a_j \in \mathbb{R}$. By the definition of $H_{v_1^j}$, we have

$$\frac{|B^\lambda|}{2} = \frac{|E_j|}{2} = |E_j \cap \{x_1 > a_j\}|.$$

Then $\{a_j\}$ is bounded and, up to subsequence, converges to $a_\infty = 0$ because, if $a_\infty \neq 0$, then

$$E_j \cap \{x_1 > a_j\} \rightarrow B_0^\lambda \cap \{x_1 > a_\infty\}$$

with $|B_0^\lambda \cap \{x_1 > a_\infty\}| \neq \frac{|B^\lambda|}{2}$. In particular $E_{j,v_1}^{'+} \rightarrow B_0^\lambda$. Finally by (4.27)

$$\tilde{\varepsilon}|B^\lambda| \geq \lim_j \left| E_{j,v_1}^{'+} \Delta B_{1,j}^{\lambda,+} \right| \geq \min_{x \in \partial H, |x| \geq \eta} \{ |B^\lambda(|B^\lambda|, x) \Delta B_0^\lambda \} =: C(\eta) > 0.$$

But for j sufficiently large, since $D_\lambda(E_j) \rightarrow 0$, by (4.27) we can choose $\tilde{\varepsilon}$ such that $\tilde{\varepsilon}|B^\lambda| < C(\eta)/2$, getting a contradiction.

We observe that for $\tilde{\varepsilon} > 0$ sufficiently small, that is for $\bar{\delta}_2 > 0$ sufficiently small, there exists $c_{n,\lambda} > 0$ such that for all possible choices of $s, t \in \{+, -\}$ there holds

$$|(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{v_1}^s \cap H_{v_2}^t)| > \frac{|B_1^{\lambda,s} \Delta B_2^{\lambda,t}|}{c_{n,\lambda}}. \tag{4.28}$$

We only sketch the argument for (4.28). Letting $Q := (H_{v_1}^s \cap H_{v_2}^t)$ and $B_1(h) := B^\lambda(|B^\lambda|, h x_1^s)$, $B_2(h) := B^\lambda(|B^\lambda|, h x_2^t)$ for $h \in [0, 1]$, one can compute

$$\begin{aligned} \frac{d}{dh} |(B_1(h) \Delta B_2(h)) \cap Q| &= \int_{\partial B_1(h) \cap B_2(h) \cap Q} \langle \nu^{B_1(h)}, x_2^t - x_1^s \rangle d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial B_2(h) \cap B_1(h) \cap Q} \langle \nu^{B_2(h)}, x_1^s - x_2^t \rangle d\mathcal{H}^{n-1} \\ &= \left(\int_{\partial B_1(h) \cap B_2(h) \cap Q} \left\langle \nu^{B_1(h)}, \frac{x_2^t - x_1^s}{|x_1^s - x_2^t|} \right\rangle \right. \\ &\quad \left. + \int_{\partial B_2(h) \cap B_1(h) \cap Q} \left\langle \nu^{B_2(h)}, \frac{x_1^s - x_2^t}{|x_1^s - x_2^t|} \right\rangle \right) |x_1^s - x_2^t| \\ &= \int_{\partial(B_1(h) \cap B_2(h)) \cap Q} \langle \nu^{B_1(h) \cap B_2(h)}, \nu_{12}^{s,t} \rangle d\mathcal{H}^{n-1} |x_1^s - x_2^t| \\ &\geq c|x_1^s - x_2^t|, \end{aligned}$$

where $\nu_{12}^{s,t}$ is obviously defined, provided $\tilde{\varepsilon}$ is small enough, for some $c = c(n, \lambda) > 0$ that will change from line to line. The last estimate follows since $\langle \nu^{B_1(h) \cap B_2(h)}, \nu_{12}^{s,t} \rangle \geq 0$ pointwise and, for $\tilde{\varepsilon}$ small, centers x_1^s, x_2^t are so close that $\langle \nu^{B_1(h) \cap B_2(h)}, \nu_{12}^{s,t} \rangle$ can be estimated from below by a positive constant on a set of \mathcal{H}^{n-1} -measure uniformly bounded from below away from zero. On the other hand one can estimate

$$|(B_1^{\lambda,s} \Delta B_2^{\lambda,t})| \leq c|x_1^s - x_2^t|.$$

Hence

$$\begin{aligned} |(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{v_1}^s \cap H_{v_2}^t)| &= \int_0^1 \frac{d}{dh} |(B_1(h) \Delta B_2(h)) \cap Q| dh \geq c|x_1^s - x_2^t| \\ &\geq c|(B_1^{\lambda,s} \Delta B_2^{\lambda,t})|, \end{aligned}$$

and (4.28) follows.

Letting

$$S_1 = (B_1^{\lambda,+} \cap H_{v_1}^+) \cup (B_1^{\lambda,-} \cap H_{v_1}^-), \quad S_2 = (B_2^{\lambda,+} \cap H_{v_2}^+) \cup (B_2^{\lambda,-} \cap H_{v_2}^-),$$

we deduce

$$|S_1 \Delta S_2| \geq |(S_1 \Delta S_2) \cap (H_{v_1}^s \cap H_{v_2}^t)| = |(B_1^{\lambda,s} \Delta B_2^{\lambda,t}) \cap (H_{v_1}^s \cap H_{v_2}^t)| > \frac{|B_1^{\lambda,s} \Delta B_2^{\lambda,t}|}{c_{n,\lambda}}.$$

In particular we have

$$\begin{aligned} |B_1^{\lambda,+} \Delta B_1^{\lambda,-}| &\leq |B_1^{\lambda,+} \Delta B_2^{\lambda,+}| + |B_2^{\lambda,+} \Delta B_1^{\lambda,-}| < 2c_{n,\lambda} |S_1 \Delta S_2|, \\ |B_2^{\lambda,+} \Delta B_2^{\lambda,-}| &\leq |B_2^{\lambda,+} \Delta B_1^{\lambda,+}| + |B_1^{\lambda,+} \Delta B_2^{\lambda,-}| < 2c_{n,\lambda} |S_1 \Delta S_2|. \end{aligned} \tag{4.29}$$

If by contradiction (4.26) were false, then

$$\begin{aligned} |E_{v_1}^{\prime,+} \Delta B_1^{\lambda,+}| + |E_{v_1}^{\prime,-} \Delta B_1^{\lambda,-}| &< \frac{|B_1^{\lambda,+} \Delta B_1^{\lambda,-}|}{2c_{n,\lambda}} \quad \text{and} \\ |E_{v_2}^{\prime,+} \Delta B_2^{\lambda,+}| + |E_{v_2}^{\prime,-} \Delta B_2^{\lambda,-}| &< \frac{|B_2^{\lambda,+} \Delta B_2^{\lambda,-}|}{2c_{n,\lambda}}. \end{aligned} \tag{4.30}$$

Hence

$$\begin{aligned} |S_1 \Delta S_2| &\leq |S_1 \Delta E| + |E \Delta S_2| = \frac{1}{2} \sum_{i=1}^2 \left(|E_{v_i}^{\prime,+} \Delta B_i^{\lambda,+}| + |E_{v_i}^{\prime,-} \Delta B_i^{\lambda,-}| \right) \\ &\stackrel{(4.30)}{<} \frac{1}{4c_{n,\lambda}} \sum_{i=1}^2 |B_i^{\lambda,+} \Delta B_i^{\lambda,-}| \stackrel{(4.29)}{\leq} |S_1 \Delta S_2|, \end{aligned}$$

getting a contradiction. □

Lemma 4.7 (Reduction to $(n - 1)$ -symmetric sets) *There exist $C_2, \delta_2 > 0$ depending on n, λ such that, if E is a Borel set with $E \subset \mathbb{R}^n \setminus H$, $E \subset Q_l$, $|E| = |B^\lambda|$ and $D_\lambda(E) \leq \delta_2$, there exists a Borel set $F \subset \mathbb{R}^n \setminus H$, $F \subset Q_{2l}$, $|F| = |B^\lambda|$, symmetric with respect to $n - 1$ orthogonal half-hyperplanes (each orthogonal to ∂H) and such that*

$$\alpha_\lambda(E) \leq C_2 \alpha_\lambda(F), \quad D_\lambda(F) \leq 2^{n-1} D_\lambda(E).$$

Proof Let us define $\delta_2 := \bar{\delta}_2 2^{-(n-2)}$, where $\bar{\delta}_2$ is the constant appearing in Lemma 4.6. We can apply Lemma 4.6 $n - 2$ times to different pairs of orthogonal vectors in $\{e_1, \dots, e_{n-2}\}$ normal to corresponding pairs of affine hyperplanes splitting the measure of E in two halves. Therefore, also recalling (4.23), we find an $(n - 2)$ -symmetric set E' such that $|E'| = |B^\lambda|$ and

$$\alpha_\lambda(E) \leq \bar{C}_2^{n-2} \alpha_\lambda(E'), \quad D_\lambda(E') \leq 2^{n-2} D_\lambda(E).$$

To perform the last symmetrization, let us consider a half-hyperplane H_{n-1} orthogonal to e_{n-1} and dividing E' into two parts of equal measure. For simplicity let us assume that $H_{n-1} = \{x_{n-1} = 0\} \setminus H$. We denote by E'^+ (resp. E'^-) the set obtained by the union of $E' \cap \{x_{n-1} > 0\}$ (resp. $E' \cap \{x_{n-1} < 0\}$) with its reflection along H_{n-1} . By (4.23) we have

$$D_\lambda(E'^{\pm}) \leq 2 D_\lambda(E') \leq 2^{n-1} D_\lambda(E).$$

Regarding the asymmetry of E'^{\pm} note that since E' is symmetric with respect to the first $n - 2$ coordinate hyperplanes, E'^+ and E'^- are $(n - 1)$ -symmetric. By Lemma 4.5 we get

$$\begin{aligned} |B^\lambda|\alpha_\lambda(E') &\leq |E' \Delta B^\lambda(|B^\lambda|)| \\ &= |(E' \Delta B^\lambda(|B^\lambda|)) \cap \{x_{n-1} > 0\}| + |(E' \Delta B^\lambda(|B^\lambda|)) \cap \{x_{n-1} < 0\}| \\ &= \frac{1}{2} \left(|E'^+ \Delta B^\lambda(|B^\lambda|)| + |E'^- \Delta B^\lambda(|B^\lambda|)| \right) \\ &\leq \frac{3|B^\lambda|}{2} \left(\alpha_\lambda(E'^+) + \alpha_\lambda(E'^-) \right). \end{aligned}$$

Therefore at least one of the sets E'^+ , E'^- has asymmetry greater than $\frac{1}{3}\alpha_\lambda(E')$ and, denoting by F this set, we have

$$\begin{aligned} D_\lambda(F) &\leq 2D_\lambda(E') \leq 2^{n-1}D_\lambda(E) \\ \alpha_\lambda(E) &\leq \bar{C}_2^{n-2}\alpha_\lambda(E') \leq 3\bar{C}_2^{n-2}\alpha_\lambda(F). \end{aligned}$$

Finally, the inclusion $F \subset Q_{2l}$ follows since F was obtained by performing reflections of $E \subset Q_l$ along affine hyperplanes of the form $\{x_j = a_j\}$ for $j = 1, \dots, n - 1$ with $a_j \in (-l, l)$. □

4.3 Reduction to Schwarz-symmetric sets

In this section we observe that, in order to prove Theorem 1.1, it is sufficient to further reduce to show (1.6) just among Schwarz-symmetric sets. The proof is analogous to [26, Proposition 4.9].

Lemma 4.8 (Reduction to Schwarz-symmetric sets) *There exists $C_3 = C_3(n, \lambda) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| = |B^\lambda|$. Suppose that E is symmetric with respect to the coordinate hyperplanes $\{x_1 = 0\}, \dots, \{x_{n-1} = 0\}$ and that*

$$D_\lambda(E) < 1, \quad E \subset Q_{2l},$$

where $l = l(n, \lambda)$ is as in Lemma 4.2. Then

$$|E \Delta E^*| \leq C_3\sqrt{D_\lambda(E)} \quad \text{and} \quad D_\lambda(E^*) \leq D_\lambda(E), \tag{4.31}$$

where E^* denotes the Schwarz symmetrization of E with respect to the n -th axis.

Proof The second inequality in (4.31) follows from the fact $|E^*| = |E|$ and $P_\lambda(E^*) \leq P_\lambda(E)$. Exploiting Lemma 2.4, we may assume

$$\mathcal{H}^{n-1}(\{x \in \partial^*E \setminus H : \nu^E(x) = \pm e_n\}) = 0, \tag{4.32}$$

and thus that $\nu_E \in W^{1,1}(\mathbb{R})$. Indeed, if E_i is given by Lemma 2.4 and the claim is proved for E_i , by the contractivity of Schwartz rearrangement ([47, Exercise 19.14]), for every $\tilde{\varepsilon} > 0$ and with i sufficiently large, we get

$$|E \Delta E^*| \leq |E \Delta E_i| + |E_i \Delta E_i^*| + |E_i^* \Delta E^*| \leq 2|E \Delta E_i| + |E_i \Delta E_i^*| \leq 2\tilde{\varepsilon} + C_3\sqrt{D_\lambda(E_i)},$$

and the last term tends to $C_3\sqrt{D_\lambda(E)}$ as $i \rightarrow \infty$.

For \mathcal{H}^1 -a.e. $t \in (0, \infty)$ denote

$$v_E(t) := \mathcal{H}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (x', t) \in E\}),$$

$$p_E(t) := \mathcal{H}^{n-2}(\partial^* \{x' \in \mathbb{R}^{n-1} : (x', t) \in E\}),$$

and employ analogous notation for E^* . Since $|\partial^* E \cap \partial H| = |\partial^* E^* \cap \partial H|$, we get

$$P_\lambda(E) - P_\lambda(B^\lambda) \geq P_\lambda(E) - P_\lambda(E^*) = P(E, \mathbb{R}^n \setminus H) - P(E^*, \mathbb{R}^n \setminus H).$$

It is therefore possible to reproduce the computation in [26, Proposition 4.9] verbatim up to [26, Eq. (4.29)] to estimate the right hand side in the last equation from below. We include the computation for the convenience of the reader. By [47, Theorem 19.11, (19.30)] one has

$$\begin{aligned} P_\lambda(E) - P_\lambda(B^\lambda) &\geq P(E, \mathbb{R}^n \setminus H) - P(E^*, \mathbb{R}^n \setminus H) \geq \int_0^\infty \left(\sqrt{p_E^2 + v_E^2} - \sqrt{p_{E^*}^2 + v_{E^*}^2} \right) dt \\ &= \int_0^\infty \frac{p_E^2 - p_{E^*}^2}{\sqrt{p_E^2 + v_E^2} + \sqrt{p_{E^*}^2 + v_{E^*}^2}} dt \\ &\geq \left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2} dt \right)^2 \frac{1}{\int_0^\infty \sqrt{p_E^2 + v_E^2} + \sqrt{p_{E^*}^2 + v_{E^*}^2} dt} \\ &\geq \left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2} dt \right)^2 \frac{1}{P(E, \mathbb{R}^n \setminus H) + P(E^*, \mathbb{R}^n \setminus H)} \\ &\geq c(\lambda) \left(\int_0^\infty \sqrt{p_E^2 - p_{E^*}^2} dt \right)^2 \frac{1}{P_\lambda(E) + P_\lambda(E^*)}, \end{aligned}$$

where in the last inequality we used Corollary 2.5. Since $D_\lambda(E) < 1$ and $p_E \geq p_{E^*}$, we have $P_\lambda(E^*) \leq P_\lambda(E) \leq 2P_\lambda(B^\lambda)$ and

$$\begin{aligned} \sqrt{D_\lambda(E)} &\geq c \int_0^\infty \sqrt{p_E^2 - p_{E^*}^2} dt = c \int_0^\infty \sqrt{p_E + p_{E^*}} \sqrt{p_{E^*}} \sqrt{\frac{p_E - p_{E^*}}{p_{E^*}}} dt \\ &\geq \sqrt{2}c \int_0^\infty p_{E^*} \sqrt{\frac{p_E - p_{E^*}}{p_{E^*}}} dt, \end{aligned} \tag{4.33}$$

for some constant $c = c(n, \lambda) > 0$ changing from line to line. Note that $(E^*)_t$ is a $(n - 1)$ -dimensional ball with the same \mathcal{H}^{n-1} measure of E_t . Then the quantity

$$\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}$$

is the classical isoperimetric deficit in \mathbb{R}^{n-1} of E_t with respect to the standard perimeter. By standard the quantitative isoperimetric inequality in \mathbb{R}^{n-1} [29], the fact that E_t is $n - 1$ symmetric with $(E^*)_t$ centered at the center of symmetry of E_t and Lemma 4.5, we have

$$\frac{\mathcal{H}^{n-1}(E_t \Delta E_t^*)}{\mathcal{H}^{n-1}((E^*)_t)} \leq c(n) \sqrt{\frac{p_E(t) - p_{E^*}(t)}{p_{E^*}(t)}}.$$

By (4.33) and the inclusion $E \subset Q_{2l}$ we conclude

$$\sqrt{D_\lambda(E)} \geq c \int_0^\infty \frac{p_{E^*}(t)}{\mathcal{H}^{n-1}((E^*)_t)} \mathcal{H}^{n-1}(E_t \Delta E_t^*) dt \geq \frac{c}{l} \int_0^\infty \mathcal{H}^{n-1}(E_t \Delta E_t^*) dt = \frac{c}{l} |E \Delta E^*|.$$

□

Putting together Lemmas 4.2, 4.7 and 4.8, one immediately gets the following corollary.

Corollary 4.9 *There exist $\delta_4, \tilde{l} > 0$ depending on n, λ such that for every $0 < \delta \leq \delta_4$, if the quantitative isoperimetric inequality (1.6) holds (with some constant depending on n, λ) for every Schwarz-symmetric set F , with $|F| = |B^\lambda|$, contained in a cube $Q_{\tilde{l}}$ and with $D_\lambda(F) < \delta$, then (1.6) holds for any measurable set of finite measure (up to changing the multiplicative constant, depending on n, λ only).*

5 First quantitative isoperimetric inequality

5.1 Coupling

We start by recalling the general definition and properties of the restricted envelopes introduced in [17].

Definition 5.1 [17, Definition 3.1] Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{E}) \cap C^2(E)$. The K -envelope of u is the function $\bar{u}^K : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$\bar{u}^K(x) := \sup\{a + \langle \xi, x \rangle : \xi \in K, a + \langle \xi, y \rangle \leq u(y) \ \forall y \in \overline{E}\}.$$

The desired coupling corresponding to a competitor E will be essentially the K -envelope of the solution to the elliptic problem (3.3), for K equal to the closure of the optimal bubble B^λ .

Definition 5.2 [17, Definition 3.9] Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set and $u \in C^0(\overline{E}) \cap C^2(E)$. Given $x \in \mathbb{R}^n$ and $\xi \in K$, we define

$$S_\xi := \arg \min_{x \in E} \{u(x) - \langle \xi, x \rangle\}$$

and

$$H(x, \xi, K) := \left\{ \begin{array}{l} \sum_{i=1}^m \lambda_i \nabla^2 u(s_i) : \begin{array}{l} 1 \leq m \leq n + 1 \\ \lambda_i \geq 0, \sum \lambda_i = 1 \\ s_i \in S_\xi \cap E \\ x - \sum \lambda_i s_i \in N(\xi, K) \end{array} \end{array} \right\},$$

where $N(\xi, K)$ is the normal cone of K at ξ , defined as

$$N(\xi, K) := \{v \in \mathbb{R}^n : \langle v, \xi' - \xi \rangle \leq 0 \text{ for all } \xi' \in K\}.$$

We will need the following

Proposition 5.3 [17, Proposition 3.10] *Let $K \subset \mathbb{R}^n$ be a compact convex set, $E \subset \mathbb{R}^n$ be a bounded open set, and $u \in C^0(\overline{E}) \cap C^2(E)$. Assume that for any $\xi \in K$ it holds $S_\xi \subset E$, then $\bar{u}^K : \mathbb{R}^n \rightarrow \mathbb{R}$ is a $C^{1,1}$ convex function such that*

$$\nabla \bar{u}^K(\mathbb{R}^n) = \nabla \bar{u}^K(E) = K.$$

Moreover, for any $x \in \mathbb{R}^n$ and any $H_x \in H(x, \nabla \bar{u}^K(x), K) \neq \emptyset$, it holds $\nabla^2 \bar{u}^K(x) \leq H_x$.

We will need to associate a coupling only to Lipschitz-regular connected competitors, having C^1 relative boundary in $\mathbb{R}^n \setminus H$. This is established in the next result, whose proof is analogous to the one in [17].

Proposition 5.4 *There exists $\hat{C} = \hat{C}(n, \lambda) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a connected bounded open set. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \geq 0\}$ is a hypersurface of class C^1 with boundary.*

Then there exists a 1-Lipschitz convex function $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}$ of class $C^{1,1}$ such that $\Delta \Psi \leq \frac{P_\lambda(E)}{|E|}$ and such that $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(E) = B^\lambda$ up to negligible sets. Moreover, if $|E| = |B^\lambda|$ and $D_\lambda(E) \leq 1$, then

$$\int_E |\nabla^2 \Psi - \text{id}| \, dx \leq \hat{C} \sqrt{D_\lambda(E)} \tag{5.1}$$

$$\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} (1 - |\nabla \Psi|) \, d\mathcal{H}^{n-1} \leq \hat{C} D_\lambda(E). \tag{5.2}$$

Proof Let us assume first that $\partial E \setminus \partial H$ is a smooth hypersurface with smooth boundary intersecting ∂H orthogonally. Let $u : E \rightarrow \mathbb{R}$ be a solution of (3.3) and $(K_i)_{i \in \mathbb{N}}$ a sequence of compact convex sets such that $K_i \subset \subset \overset{\circ}{K}_{i+1}$ and $\cup_{i \in \mathbb{N}} K_i = B^\lambda$. We showed in the proof of Theorem 3.3 that for any $\xi \in B^\lambda$ the minimum of $u(x) - \langle \xi, x \rangle$ cannot be achieved on the boundary of E . Moreover, at any point $x \in E$ such that $\nabla^2 u(x) \geq 0$, it holds

$$0 \leq \nabla^2 u(x) \leq \Delta u(x) \text{id} = \frac{P_\lambda(E)}{|E|} \text{id}.$$

Therefore, recalling Remark 3.2, by Proposition 5.3 we get that \bar{u}^{K_i} is a sequence of 1-Lipschitz functions, being suprema of 1-Lipschitz functions, that is uniformly bounded in $C^{1,1}$ on compact sets; hence up to subsequence it converges to a limit function Ψ in $C^1_{\text{loc}}(\mathbb{R}^n)$. Since $\nabla \bar{u}^{K_i}(\mathbb{R}^n) = \nabla \bar{u}^{K_i}(E) = K_i$, then $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(\bar{E}) = \overline{B^\lambda}$, Ψ is a convex function of class $C^{1,1}$, and writing the inequality $\Delta \bar{u}^{K_i} \leq \frac{P_\lambda(E)}{|E|}$ in the sense of distributions, one checks that it readily passes to the limit as $i \rightarrow +\infty$ for the function Ψ .

To prove that $|\nabla \Psi(E) \Delta B^\lambda| = 0$, let $Z \subset E$ be a compact set and notice that

$$\begin{aligned} |\nabla \bar{u}^{K_i}(E \setminus Z)| &\leq c(n, |E|, P_\lambda(E)) |E \setminus Z|, \\ |\nabla \bar{u}^{K_i}(Z)| &= |\nabla \bar{u}^{K_i}(E \setminus (E \setminus Z))| \geq |\nabla \bar{u}^{K_i}(E)| - |\nabla \bar{u}^{K_i}(E \setminus Z)| = |K_i| - c |E \setminus Z|. \end{aligned}$$

Passing to the limit we find

$$|\nabla \Psi(Z)| \geq \left| \limsup_i \nabla \bar{u}^{K_i}(Z) \right| \geq \limsup_i |\nabla \bar{u}^{K_i}(Z)| \geq |B^\lambda| - c |E \setminus Z|,$$

hence letting $Z \nearrow E$, we get that $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(E) = B^\lambda$ up to negligible sets.

Suppose now that E is a generic connected set as in the assumptions. If $n \geq 3$ we can apply the above argument to a sequence of sets E_i approximating E given by Lemma 2.4, suitably modified connecting possibly disconnected components with thin tubes vanishing in the limit. If $n = 2$, then $\partial E \setminus \partial H$ is a union of C^1 curves, which thus can be approximated by smooth ones touching ∂H orthogonally preserving the connectedness of the set. Applying the first part of the proof on the approximating sequence E_i we get a corresponding sequence of functions Ψ_i uniformly bounded in $C^{1,1}$ on compact sets, hence converging in C^1_{loc} up to subsequence to a convex function Ψ of class $C^{1,1}$ with $\Delta \Psi \leq P_\lambda(E)/|E|$. Also, since $\nabla \Psi_i(\mathbb{R}^n) = \overline{B^\lambda}$, then $\nabla \Psi(\mathbb{R}^n) \subset \overline{B^\lambda}$ by C^1_{loc} -convergence. Moreover, since E has Lipschitz boundary, there exists a sequence of compact sets $Z_j \subset E$ such that $Z_j \subset E_i$ for any $i \geq i_j$ and such that $Z_j \nearrow E$. Hence one can repeat the above argument with Ψ_i, E, Z_j in place of \bar{u}^{K_i}, E, Z , respectively, to deduce that

$$|\nabla \Psi_i(Z_j)| \geq |B^\lambda| - c(n, |E_i|, P_\lambda(E_i)) |E_i \setminus Z_j| \geq |B^\lambda| - c(n, |E|, P_\lambda(E)) |E_i \setminus Z_j|.$$

Letting $i \rightarrow \infty$ first, and then $j \rightarrow \infty$, we get that $\nabla \Psi(\mathbb{R}^n) = \nabla \Psi(E) = B^\lambda$ up to negligible sets.

We now prove (5.1) and (5.2). The symbol \hat{C} shall denote a positive constant depending on n, λ changing from line to line. By the area formula, the arithmetic–geometric mean inequality and the properties of Ψ we get

$$\begin{aligned} |B^\lambda| &= |\nabla \Psi(E)| \leq \int_E \det(\nabla^2 \Psi) \, d\mathcal{H}^n \\ &\leq \int_E \left(\frac{\Delta \Psi}{n}\right)^n \, dx \leq \int_E \left(\frac{P_\lambda(E)}{n|E|}\right)^n \, dx = \int_E \left(\frac{P_\lambda(E)}{n|B^\lambda|}\right)^n \, dx \\ &= \left(\frac{P_\lambda(E)}{P_\lambda(B^\lambda)}\right)^n |B^\lambda| = (1 + D_\lambda(E))^n |B^\lambda|. \end{aligned} \tag{5.3}$$

Hence

$$\int_E \left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2 \Psi) \right) \, dx \leq |B^\lambda|(1 + D_\lambda(E))^n - |B^\lambda| \leq \hat{C} D_\lambda(E). \tag{5.4}$$

By [17, Lemma A.1] applied with $m = n, \lambda_1 = \dots = \lambda_n = 1, (x_1, \dots, x_n)$ equal to the eigenvalues of $\nabla^2 \Psi$ and $c = \frac{P_\lambda(E)}{n|E|} = \frac{P_\lambda(E)}{P_\lambda(B^\lambda)} \geq 1$, we obtain

$$\begin{aligned} |\nabla^2 \Psi - \text{id}|^2 &\leq 2|\nabla^2 \Psi - c \, \text{id}|^2 + 2|(c - 1)\text{id}|^2 \\ &\leq \hat{C} \left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2 \Psi) \right) + 2n \left(\frac{P_\lambda(E) - n|E|}{n|E|} \right)^2 \\ &\leq \hat{C} \left(\left(\frac{P_\lambda(E)}{n|E|}\right)^n - \det(\nabla^2 \Psi) + D_\lambda(E)^2 \right). \end{aligned} \tag{5.5}$$

Therefore by (5.4) and (5.5) we get

$$\int_E |\nabla^2 \Psi - \text{id}|^2 \, dx \leq \hat{C} D_\lambda(E),$$

which implies (5.1).

Arguing as in (5.3), by the divergence theorem and using that $\langle \nabla \Psi, -e_n \rangle \leq -\lambda$ since $\nabla \Psi(\mathbb{R}^n) \subset B^\lambda$, we get

$$\begin{aligned} |B^\lambda| &\leq \int_E \left(\frac{\Delta \Psi}{n}\right)^n \, dx \leq \frac{P_\lambda(E)^{n-1}}{n^n |E|^{n-1}} \int_E \Delta \Psi \, dx = \frac{P_\lambda(E)^{n-1}}{n^n |E|^{n-1}} \int_{\partial^* E} \langle \nabla \Psi, \nu^E \rangle \, d\mathcal{H}^{n-1} \\ &\leq \frac{P_\lambda(E)^{n-1}}{n^n |E|^{n-1}} \left(\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} \langle \nabla \Psi, \nu^E \rangle \, d\mathcal{H}^{n-1} + \int_{\partial^* E \cap \partial H} \langle \nabla \Psi, -e_n \rangle \, d\mathcal{H}^{n-1} \right) \\ &\leq \frac{P_\lambda(E)^{n-1}}{n^n |E|^{n-1}} \left(\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} \langle \nabla \Psi, \nu^E \rangle - 1 \, d\mathcal{H}^{n-1} + P_\lambda(E) \right) \\ &\leq \frac{P_\lambda(E)^n}{n^n |E|^{n-1}} - \frac{P_\lambda(E)^{n-1}}{n^n |E|^{n-1}} \int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} (1 - |\nabla \Psi|) \, d\mathcal{H}^{n-1}. \end{aligned}$$

Rearranging terms, since $D_\lambda(E) \leq 1$ and $|E| = |B^\lambda|$, we obtain

$$\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} (1 - |\nabla \Psi|) \, d\mathcal{H}^{n-1} \leq \frac{P_\lambda(E)^n - P_\lambda(B^\lambda)^n}{P_\lambda(E)^{n-1}} \leq \hat{C} D_\lambda(E).$$

□

We now want to translate the quantitative estimates obtained on the coupling in Proposition 5.4 into quantitative estimates on the asymmetry of a competitor. We will need some technical results first.

The next lemma is analogous to [17, Lemma 6.2], but with a varying range of parameters that here must depend on λ .

Lemma 5.5 *Let $E \subset [0, \infty)$ be a 1-dimensional set of locally finite perimeter with $|E| < \infty$ and set*

$$r_\lambda := \min \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\}, \quad R_\lambda := \max \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\}.$$

There exists $\hat{c} = \hat{c}(n, \lambda) > 0$ such that for any $\frac{7}{8}r_\lambda \leq l \leq \frac{9}{8}R_\lambda$ there holds

$$\int_{E \Delta [0, l]} t^{n-1} dt \leq \hat{c} \left(\int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt + \int_{\partial^* E} t^{n-1} |l - t| d\mathcal{H}^0 \right). \tag{5.6}$$

Proof It holds $E \Delta [0, l] = ((l, \infty) \cap E) \cup ([0, l] \setminus E)$. We claim that

$$\begin{aligned} & \max \left\{ \int_{[l, \infty) \cap E} t^{n-1} dt, \int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt \right\} \\ & \leq \hat{c}(n, \lambda) \left(\int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt + \int_{\partial^* E} t^{n-1} |l - t| d\mathcal{H}^0(t) \right). \end{aligned} \tag{5.7}$$

We will estimate the two terms on the left-hand side separately.

Without loss of generality, suppose $[l, \infty) \cap E \neq \emptyset$. Since $|E| < \infty$, then $\partial^* E \cap [l, \infty)$ is nonempty and we can assume it has finite supremum \bar{t} (otherwise the right hand side in (5.7) equals $+\infty$). In particular the right-hand side in (5.7) is finite. It holds

$$\int_{[l, \infty) \cap E} t^{n-1} dt \leq \int_l^{\bar{t}} t^{n-1} dt \leq \bar{t}^{n-1} |\bar{t} - l| \leq \int_{\partial^* E} t^{n-1} |l - t| d\mathcal{H}^0(t),$$

and the first term in the left-hand side of (5.7) is bounded as wished.

Let us now consider $\int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt$. Its value is a priori bounded by $(\frac{9}{8}R_\lambda)^n$. If $\partial^* E \cap [\frac{r_\lambda}{4}, \frac{3}{4}r_\lambda] \neq \emptyset$ and τ is one of its elements, then

$$\begin{aligned} \int_{\partial^* E} t^{n-1} |l - t| d\mathcal{H}^0(t) & \geq \tau^n |l - \tau| \geq \left(\frac{r_\lambda}{4}\right)^n \frac{1}{8} r_\lambda \geq \hat{c}(n, \lambda) \left(\frac{9}{8}R_\lambda\right)^{n-1} \left| \frac{9}{8}R_\lambda - \frac{r_\lambda}{2} \right| \\ & \geq \hat{c}(n, \lambda) \int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt. \end{aligned}$$

So from now on we can assume that $\partial^* E \cap [\frac{r_\lambda}{4}, \frac{3}{4}r_\lambda] = \emptyset$. If $[\frac{r_\lambda}{4}, \frac{3}{4}r_\lambda] \setminus E \neq \emptyset$, then $E \cap [\frac{r_\lambda}{4}, \frac{3}{4}r_\lambda] = \emptyset$ and

$$\begin{aligned} \int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt + \int_{\partial^* E} t^{n-1} |l - t| d\mathcal{H}^0(t) & \geq \int_{[\frac{r_\lambda}{4}, \frac{r_\lambda}{2}]} t^{n-1} dt = \hat{c}(n, \lambda) \\ & \geq \hat{c}(n, \lambda) \left(\frac{9}{8}R_\lambda\right)^{n-1} \left| \frac{9}{8}R_\lambda - \frac{r_\lambda}{2} \right| \\ & \geq \hat{c}(n, \lambda) \int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt. \end{aligned}$$

So we can further assume $[\frac{r_\lambda}{4}, \frac{3}{4}r_\lambda] \subset E$. If $\partial^* E \cap [\frac{r_\lambda}{4}, l] = \emptyset$, then $[\frac{r_\lambda}{2}, l] \subset E$ and there is nothing to prove. Finally, if $\partial^* E \cap [\frac{r_\lambda}{4}, l] \neq \emptyset$, let us denote by \underline{t} the infimum of $\partial^* E \cap [\frac{r_\lambda}{4}, l]$. Then

$$\begin{aligned} \int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt &\leq \int_{\underline{t}}^l t^{n-1} dt \leq l^{n-1}|l - \underline{t}| = \underline{t}^{n-1}|l - \underline{t}| \left(\frac{l}{\underline{t}}\right)^{n-1} \leq \hat{c}(n, \lambda) \underline{t}^{n-1}|l - \underline{t}| \\ &\leq \hat{c}(n, \lambda) \int_{\partial^* E} t^{n-1}|l - t| d\mathcal{H}^0(t). \end{aligned}$$

This concludes the proof of the claim (5.7). Hence

$$\begin{aligned} \int_{E \Delta [0, l]} t^{n-1} dt &\leq \max \left\{ \int_{(l, \infty) \cap E} t^{n-1} dt, \int_{[\frac{r_\lambda}{2}, l] \setminus E} t^{n-1} dt \right\} + \int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt \\ &\stackrel{(5.7)}{\leq} \hat{c}(n, \lambda) \left(\int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt + \int_{\partial^* E} t^{n-1}|l - t| d\mathcal{H}^0(t) \right) + \int_{[0, \frac{r_\lambda}{2}] \setminus E} t^{n-1} dt, \end{aligned}$$

and the proof follows. □

We shall also need the following standard technical result stating that a Vol’pert property holds for the intersections of a set of finite perimeter with rays from the origin, cf. [52]. The proof follows, for example, by adapting the proof of [25, Theorem 3.21] working in polar coordinates rather than in Cartesian coordinates.

Lemma 5.6 *Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with $|E| < +\infty$. If $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$, we define*

$$E_\vartheta := \{t \geq 0 : t\vartheta \in E\}.$$

Then, for \mathcal{H}^{n-1} -almost every $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$, E_ϑ is a 1-dimensional set of locally finite perimeter such that

$$\partial^* E_\vartheta \cap \{t > 0\} = \{t > 0 : t\vartheta \in \partial^* E\}.$$

Moreover, if $\eta \in L^1(\partial^ E)$ is nonnegative, we have*

$$\int_{\partial^* E \setminus H} \eta d\mathcal{H}^{n-1} \geq \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{\partial^* E_\vartheta} t^{n-1} \eta(t\vartheta) d\mathcal{H}^0(t) \right) d\mathcal{H}^{n-1}(\vartheta). \tag{5.8}$$

Combining Lemma 5.5 with Lemma 5.6 we get the following result that estimates the symmetric difference of a competitor with a bubble that is just close to a standard bubble $B^\lambda(v, x)$. The result is analogous to [17, Proposition 6.1].

Lemma 5.7 *There exist $\varepsilon, \tilde{c} > 0$ depending on n, λ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a bounded set of finite perimeter. If $\left| E \cap B_{\frac{r_\lambda}{2}}(0) \right| \geq \frac{1}{2} \left| B_{\frac{r_\lambda}{2}}(0) \setminus H \right|$, then*

$$|E \Delta (B_1(x_0) \setminus H)| \leq \tilde{c} \int_{\partial^* E \setminus H} ||x - x_0| - 1| d\mathcal{H}^{n-1}(x),$$

for any $x_0 \in \mathbb{R}^n$ such that $|x_0 - (0, \dots, 0, -\lambda)| < \varepsilon$.

Proof If ε is sufficiently small, depending only on n, λ , then for any $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$ the set $\{t > 0 : |t\vartheta - x_0| < 1\}$ is an open segment $(0, t(\vartheta))$, with $t(\vartheta)$ close to the number

$$T_\vartheta \in \left[\min \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\}, \max \left\{ \sqrt{1 - \lambda^2}, 1 - \lambda \right\} \right] =: [r_\lambda, R_\lambda]$$

such that $|T_\vartheta \vartheta + \lambda e_n| = 1$. In particular, if ε is sufficiently small, then $\frac{9}{8}R\lambda \geq t(\vartheta) \geq \frac{7}{8}r\lambda$ for any $\vartheta \in \mathbb{S}^{n-1} \cap (\mathbb{R}^n \setminus H)$.

As before, for any $\vartheta \in \mathbb{S}^{n-1} \setminus H$ let

$$E_\vartheta := \{t \geq 0 : t\vartheta \in E\}.$$

By coarea formula we get

$$|E \Delta B_1(x_0) \cap (\mathbb{R}^n \setminus H)| = \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{E_\vartheta \Delta [0, t(\vartheta)]} t^{n-1} dt \right) d\mathcal{H}^{n-1}(\vartheta). \tag{5.9}$$

By Lemma 5.5 we obtain

$$\begin{aligned} &|E \Delta B_1(x_0) \setminus H| \\ &\leq c(n, \lambda) \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{[0, \frac{r_\lambda}{2}] \setminus E_\vartheta} t^{n-1} dt + \int_{\partial^* E_\vartheta} t^{n-1} |t(\vartheta) - t| d\mathcal{H}^0 \right) d\mathcal{H}^{n-1}(\vartheta). \end{aligned} \tag{5.10}$$

For every $t > 0$, we claim that

$$||t\vartheta - x_0| - 1| \geq c(n, \lambda)|t - t(\vartheta)|. \tag{5.11}$$

Note that there exists $\delta = \delta(n, \lambda, \varepsilon) \in (0, r_\lambda/8)$ such that for $t \in [t(\vartheta) - \delta, t(\vartheta) + \delta]$ there holds

$$\left| \frac{d}{dt} |t\vartheta - x_0| \right| = |\langle (t\vartheta - x_0)/|t\vartheta - x_0|, \vartheta \rangle| \geq c(n, \lambda, \varepsilon) > 0.$$

Then, for $t \in [t(\vartheta) - \delta, t(\vartheta) + \delta]$,

$$|t(\vartheta) - t| \leq c(n, \lambda)|t\vartheta - x_0| - 1|,$$

and in this case the claim follows. Regarding the remaining cases, note that

$$\frac{||t\vartheta - x_0| - 1|}{|t(\vartheta) - t|} \rightarrow 1 \text{ as } |t| \rightarrow +\infty$$

and the claim follows for $t \geq R = R(n, \lambda, \varepsilon) > 0$ big enough. Finally, if $0 < t < R$ and $t \notin [t(\vartheta) - \delta, t(\vartheta) + \delta]$, then

$$\begin{aligned} &||t\vartheta - x_0| - 1| \geq c(n, \lambda, \delta) > 0 \\ &|t(\vartheta) - t| \leq c(n, \lambda, R) \end{aligned}$$

hence the claim follows as well.

Therefore

$$\begin{aligned} &\int_{\mathbb{S}^{n-1} \setminus H} \int_{\partial^* E_\vartheta} t^{n-1} |t - t(\vartheta)| d\mathcal{H}^0(t) d\mathcal{H}^{n-1}(\vartheta) \\ &\stackrel{(5.11)}{\leq} c(n, \lambda) \int_{\mathbb{S}^{n-1} \setminus H} \int_{\partial^* E_\vartheta} t^{n-1} ||t\vartheta - x_0| - 1| d\mathcal{H}^0(t) d\mathcal{H}^{n-1}(\vartheta). \end{aligned} \tag{5.12}$$

By Lemma 5.6 we deduce

$$\begin{aligned} &\int_{\partial^* E \setminus H} ||x - x_0| - 1| d\mathcal{H}^{n-1}(x) \\ &\geq \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{\partial^* E_\vartheta} t^{n-1} ||t\vartheta - x_0| - 1| d\mathcal{H}^0(t) \right) d\mathcal{H}^{n-1}(\vartheta). \end{aligned} \tag{5.13}$$

Since $||x - x_0| - 1| \geq c(n, \lambda) > 0$ in $B_{r_\lambda/2}(0)$, by coarea formula and relative isoperimetric inequality we get

$$\begin{aligned} \int_{\mathbb{S}^{n-1} \setminus H} \left(\int_{[0, r_\lambda/2] \setminus E_\vartheta} t^{n-1} dt \right) d\mathcal{H}^{n-1}(\vartheta) &= \left| \left(B_{r_\lambda/2}(0) \setminus H \right) \setminus E \right| \\ &= \left| \left(B_{r_\lambda/2}(0) \setminus H \right) \setminus E \right|^{\frac{n-1}{n}} \left| \left(B_{r_\lambda/2}(0) \setminus H \right) \setminus E \right|^{\frac{1}{n}} \\ &\leq c(n, \lambda) \int_{\partial^* E \cap \left(B_{r_\lambda/2}(0) \setminus H \right)} d\mathcal{H}^{n-1} \\ &\leq c(n, \lambda) \int_{\partial^* E \setminus H} ||x - x_0| - 1| d\mathcal{H}^{n-1}(x). \end{aligned} \tag{5.14}$$

(5.10), (5.12), (5.13) and (5.14), the proof follows. □

We can finally show that if a suitably regular Schwarz-symmetric set satisfies a trace inequality, then the quantitative estimates in Proposition 5.4 imply a quantitative isoperimetric inequality.

Proposition 5.8 *There exists $\delta_5 = \delta_5(n, \lambda) > 0$ such that for any $c_T > 0$ there exists $\gamma = \gamma(n, \lambda, c_T) > 0$ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be a bounded connected open set with $|E| = |B^\lambda|$. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \geq 0\}$ is a hypersurface of class C^1 with boundary. Assume that E is Schwarz-symmetric with respect to the n -th axis and that there exists a constant c_T such that for every function $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ there is a constant $c \in \mathbb{R}$ such that the following holds*

$$\int_E d|Df|(x) \geq c_T \int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} \text{tr}_E(|f - c|) d\mathcal{H}^{n-1}(x). \tag{5.15}$$

If $D_\lambda(E) < \delta_5$, then

$$\alpha_\lambda^2(E) \leq \gamma D_\lambda(E).$$

Proof Let Ψ be given by Proposition 5.4. By (5.15) and (5.1) we get

$$\int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} |(\nabla \Psi - x) + x_0| d\mathcal{H}^{n-1}(x) \leq c(n, \lambda, c_T) \sqrt{D_\lambda(E)},$$

where $x_0 = (x_0^1, \dots, x_0^n)$ is the vector whose i -th component is the constant c of (5.15) corresponding to the i -th component of $\nabla \Psi - x$. Therefore

$$\begin{aligned} \int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} ||x - x_0| - 1| d\mathcal{H}^{n-1}(x) &\leq \int_{\partial^* E \cap (\mathbb{R}^n \setminus H)} |\nabla \Psi - (x - x_0)| + |1 - |\nabla \Psi|| d\mathcal{H}^{n-1}(x) \\ &\stackrel{(5.2)}{\leq} c(n, \lambda, c_T) \sqrt{D_\lambda(E)}. \end{aligned} \tag{5.16}$$

We observe that if $\varepsilon = \varepsilon(n, \lambda)$ is given by Lemma 5.7, then for δ_5 small enough depending on ε , we ensure that $|x_0 - (0, \dots, 0, -\lambda)| < \varepsilon$. Indeed, if for every $i \in \mathbb{N}$ there were E_i satisfying the hypotheses of Proposition 5.8 such that $D_\lambda(E) < \frac{1}{i}$ with corresponding $x_{0,i}$ verifying $|x_{0,i} - (0, \dots, 0, -\lambda)| \geq \varepsilon$, passing to limit in (5.16) we would get a contradiction with the fact that E_i converges to $B^\lambda(|B^\lambda|)$.

Hence we can apply Lemma 5.7. Since E is Schwarz-symmetric, we get

$$\begin{aligned} |E \Delta B_1(0, \dots, 0, x_0^n) \cap (\mathbb{R}^n \setminus H)| &\leq |E \Delta B_1(x_0) \cap (\mathbb{R}^n \setminus H)| \\ &\leq c(n, \lambda, c_T) \sqrt{D_\lambda(E)}. \end{aligned} \tag{5.17}$$

Arguing as above, up to taking a smaller δ_5 , we can assume that $|x_0^n + \lambda| \leq |x_0 - (0, \dots, 0, -\lambda)|$ is so small that

$$\left| \frac{d}{dt} |B_1(0, \dots, 0, t) \setminus H| \right| \geq \frac{1}{2} \omega_{n-1} (1 - \lambda^2)^{\frac{n-1}{2}},$$

for any $t \in [-|x_0^n + \lambda|, |x_0^n + \lambda|]$. Hence

$$\begin{aligned} c(n, \lambda, c_T) \sqrt{D_\lambda(E)} &\stackrel{(5.17)}{\geq} \left| |B_1(0, \dots, 0, x_0^n) \setminus H| - |E| \right| \\ &= \left| |B_1(0, \dots, 0, x_0^n) \setminus H| - |B_1(0, \dots, 0, -\lambda) \setminus H| \right| \\ &\geq \frac{1}{2} \omega_{n-1} (1 - \lambda^2)^{\frac{n-1}{2}} |x_0^n + \lambda|, \end{aligned}$$

which implies

$$|x_0^n + \lambda| \leq c(n, \lambda, c_T) \sqrt{D_\lambda(E)}, \tag{5.18}$$

for a suitable constant. Therefore

$$\begin{aligned} \left| (B_1(0, \dots, 0, x_0^n) \setminus H) \Delta (B_1(0, \dots, 0, -\lambda) \setminus H) \right| &\leq c(n, \lambda) |x_0^n + \lambda| \\ &\stackrel{(5.18)}{\leq} c(n, \lambda, c_T) \sqrt{D_\lambda(E)}, \end{aligned} \tag{5.19}$$

where in the first inequality we used that $t \mapsto |(B_1(0, \dots, 0, t) \setminus H) \Delta (B_1(0, \dots, 0, -\lambda) \setminus H)|$ is Lipschitz for some Lipschitz constant $c(n, \lambda) > 0$.

Finally

$$\begin{aligned} |E \Delta B_1(0, \dots, 0, x_0^n) \cap (\mathbb{R}^n \setminus H)| &\geq |B^\lambda(|B^\lambda|) \Delta E| - |(B_1(0, \dots, 0, x_0^n) \setminus H) \Delta (B_1(0, \dots, 0, -\lambda) \setminus H)| \\ &\stackrel{(5.19)}{\geq} \alpha_\lambda(E) - c(n, \lambda, c_T) \sqrt{D_\lambda(E)}. \end{aligned}$$

□

In the next lemma we observe that optimal bubbles do satisfy trace inequalities.

Lemma 5.9 *There exists $\bar{c} = \bar{c}(n, \lambda) > 0$ such that for every function $f \in BV(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ there is a constant $c \in \mathbb{R}$ such that the following holds*

$$\int_{B^\lambda(|B^\lambda|)} d|Df|(x) \geq \bar{c} \int_{\partial B^\lambda(|B^\lambda|) \setminus H} \text{tr}_{B^\lambda(|B^\lambda|)}(|f - c|) d\mathcal{H}^{n-1}(x). \tag{5.20}$$

Proof The proof follows combining the classical Poincaré inequality [2, Theorem 3.44] with the boundary trace theorem [2, Theorem 3.87]. □

We now introduce a notion of C^1 -distance from $B^\lambda(|B^\lambda|)$ for sets in the half-space $\mathbb{R}^n \setminus H$, and we deduce that Schwarz-symmetric sets sufficiently close in C^1 to $B^\lambda(|B^\lambda|)$ enjoy a quantitative isoperimetric inequality.

Definition 5.10 Let $\varphi_\lambda : \partial B_1 \setminus H \rightarrow \mathbb{R}$ be such that $\partial B^\lambda(|B^\lambda|) \setminus H = \{\varphi_\lambda(x)x : x \in \partial B_1 \setminus H\}$.

Let $E \subset \mathbb{R}^n \setminus H$ be a bounded open set. Suppose that E has Lipschitz boundary and that $\partial E \cap \{x_n \geq 0\}$ is a hypersurface of class C^1 with boundary. Assume that E is Schwarz-symmetric. Suppose that there exists a C^1 functions

$$\varphi : \partial B_1 \setminus H \rightarrow \mathbb{R}$$

whose graph parametrizes the boundary of E in $\mathbb{R}^n \setminus H$, that is

$$\partial E \setminus H = \{\varphi(x)x : x \in \partial B_1 \setminus H\}.$$

We define the C^1 distance of E to $B^\lambda(|B^\lambda|)$ by $d_{C^1}(E, B^\lambda(|B^\lambda|)) := \|\varphi - \varphi_\lambda\|_{C^1(\partial B_1 \cap \mathbb{R}^n \setminus H)}$. A sequence of sets E_j as above is said to converge to $B^\lambda(|B^\lambda|)$ in C^1 if $d_{C^1}(E_j, B^\lambda(|B^\lambda|)) \rightarrow 0$ as $j \rightarrow +\infty$.

Corollary 5.11 *There exist $\hat{\varepsilon}, \hat{\gamma} > 0$ depending only on n, λ such that the following holds. Let $E \subset \mathbb{R}^n \setminus H$ be as in Definition 5.10. If $d_{C^1}(E, B^\lambda(|B^\lambda|)) \leq \hat{\varepsilon}$, then*

$$\alpha_\lambda^2(E) \leq \hat{\gamma}(n, \lambda)D_\lambda(E).$$

Proof Let φ, φ_λ be as in Definition 5.10. If $\chi : [0, +\infty) \rightarrow [0, 1]$ is a smooth cut-off function such that $\chi(t) = 0$ for $t < \frac{1}{4} \min\{\sqrt{1-\lambda^2}, 1-\lambda\}$ and such that $\chi(t) = 1$ for $t > \frac{1}{2} \min\{\sqrt{1-\lambda^2}, 1-\lambda\}$, we define the diffeomorphism

$$\psi : \mathbb{R}^n \setminus H \rightarrow \mathbb{R}^n \setminus H \quad \psi(x) = \left(1 - \chi(|x|) + \chi(|x|) \frac{\varphi\left(\frac{x}{|x|}\right)}{\varphi_\lambda\left(\frac{x}{|x|}\right)} \right) x.$$

Note that

$$\|\psi - \text{id}\|_{C^1} \leq c\hat{\varepsilon}, \quad \psi(\partial B^\lambda(|B^\lambda|) \setminus H) = \partial E \setminus H,$$

for some $c = c(n, \lambda, \chi)$, if $d_{C^1}(E, B^\lambda(|B^\lambda|)) \leq \hat{\varepsilon} < 1$.

Let $g \in \text{Lip}_c(\mathbb{R}^n)$ and define $f := g \circ \psi$. If c is the constant in (5.20) corresponding to f , then by area formula and (5.20) we get

$$\begin{aligned} \int_{\partial^* E \cap H} |g - c| d\mathcal{H}^{n-1} &\leq C(n, \lambda) \int_{\partial B^\lambda \setminus H} |f - c| d\mathcal{H}^{n-1} \leq C(n, \lambda) \int_{B^\lambda} |\nabla f| dx \\ &\leq C(n, \lambda) \int_E |\nabla g| dx. \end{aligned}$$

Therefore, if $\hat{\varepsilon}$ is small enough, we can apply Proposition 5.8 with c_T therein depending on n, λ only, and we get

$$\alpha_\lambda(E) \leq \hat{\gamma}(n, \lambda)D_\lambda(E).$$

□

5.2 Proof of the first quantitative isoperimetric inequality

We are ready to prove the main quantitative isoperimetric inequality. Let us recall the following immediate result, completely analogous to [26, Lemma 5.3].

Lemma 5.12 *The standard bubble $B^\lambda(|B^\lambda|)$ is the unique solution, up to translations along ∂H , of*

$$\min \{ P_\lambda(F) + \Lambda \left| |F| - |B^\lambda| \right| : F \subset \mathbb{R}^n \setminus H \},$$

for any $\Lambda > n$.

Proof of Theorem 1.1 Let $\hat{\gamma}(n, \lambda)$ be the constant given by Corollary 5.11 and let δ_4, \tilde{l} be given by Corollary 4.9. By Corollary 4.9 it is sufficient to prove that there exists $\delta \in (0, \delta_4)$ such that, if E is a Schwarz-symmetric set contained in $Q_{\tilde{l}}$ such that $|E| = |B^\lambda|$ and $D_\lambda(E) < \delta$, then $\alpha_\lambda(E) \leq 2\hat{\gamma}(n, \lambda)\sqrt{D_\lambda(E)}$.

We argue by contradiction. Let $\{E_j\}_j$ be a sequence of Schwarz-symmetric sets contained in $Q_{\tilde{l}}$ such that $|E_j| = |B^\lambda|$, with $P_\lambda(E_j) \rightarrow P_\lambda(B^\lambda)$ and

$$\alpha_\lambda(E_j) > 2\hat{\gamma}(n, \lambda)\sqrt{D_\lambda(E_j)}. \tag{5.21}$$

For every j we consider a minimizer F_j of the problem

$$\min\{P_\lambda(F) + |\alpha_\lambda(F) - \alpha_\lambda(E_j)| + \Lambda||F| - |E_j|| : F \text{ Schwarz-symmetric contained in } Q_{\tilde{l}}\}, \tag{5.22}$$

chosen large. Up to subsequence, F_j converges in L^1 to a minimizer of $F \mapsto P_\lambda(F) + \alpha_\lambda(F) + \Lambda||F| - |B^\lambda||$, hence, taking $\Lambda > n$, we have that F_j converges to $B^\lambda(|B^\lambda|)$ by Lemma 5.12. Also, by comparison with E_j , we have that $P_\lambda(F_j) \rightarrow P_\lambda(B^\lambda)$.

We prove that F_j is a local (Λ_1, r_0) -minimizer in $\mathbb{R}^n \setminus H$, for some $\Lambda_1, r_0 > 0$ and j large. Let us consider a ball $B_r(x) \subset\subset \mathbb{R}^n \setminus H$, with $r < \min\{r_0, d(x, \partial H)\}$, and a set G such that $F_j \Delta G \subset\subset B_r(x)$. Denoting by $(\cdot)^*$ the Schwarz symmetrization with respect to the n -th axis and by $Z := G \cap Q_{\tilde{l}}$, we have

$$\begin{aligned} P(F_j, \mathbb{R}^n \setminus H) &\leq P(Z^*, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(E_j)| - |\alpha_\lambda(F_j) - \alpha_\lambda(E_j)| \\ &\quad + \Lambda[||Z| - |E_j|| - ||F_j| - |E_j||] \\ &\leq P(Z, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(F_j)| + \Lambda|Z \Delta F_j| \\ &\leq P(G, \mathbb{R}^n \setminus H) + |\alpha_\lambda(Z^*) - \alpha_\lambda(F_j)| + \Lambda|G \Delta F_j|. \end{aligned}$$

For r_0 small enough and j sufficiently large we have that $|G| \geq |Z| \geq c(n, \lambda) > 0$. Assume for instance that $\alpha_\lambda(Z^*) \geq \alpha_\lambda(F_j)$ (the opposite case being symmetric), then

$$\begin{aligned} \alpha_\lambda(Z^*) - \alpha_\lambda(F_j) &\leq |Z|^{-1} (|Z^* \Delta F_j| + |F_j \Delta B^\lambda(|F_j)|) + ||Z| - |F_j|| \\ &\quad - |F_j|^{-1} |F_j \Delta B^\lambda(|F_j)| \\ &\leq c(n, \lambda)|Z \Delta F_j| + c(n, \lambda) (|F_j| - |Z|) + |Z \Delta F_j| \\ &\leq c(n, \lambda)|G \Delta F_j|. \end{aligned}$$

Arguing analogously in case $\alpha_\lambda(Z^*) < \alpha_\lambda(F_j)$, we deduce

$$P(F_j, \mathbb{R}^n \setminus H) \leq P(G, \mathbb{R}^n \setminus H) + \Lambda_1|G \Delta F_j|,$$

for some $\Lambda_1 = (\Lambda, n, \lambda)$.

By Theorem A.3 we know that $\partial^* F_j \cap \{x_n > 0\}$ is a $C^{1, \frac{1}{2}}$ manifold and $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ has Hausdorff dimension $\leq n - 8$. Since F_j is Schwarz-symmetric, if there exists a point $(r\vartheta, t) \in \partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ for some $r, t > 0, \vartheta \in \mathbb{S}^{n-2}$, then $(r\vartheta', t) \in \partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ for any $\vartheta' \in \mathbb{S}^{n-2}$. Hence $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j \subset \{te_n : t > 0\}$. However by Theorem A.3 for every $\varepsilon > 0$ the set $\partial F_j \cap \{x_n \geq \varepsilon\}$ converges to $\partial B^\lambda(|B^\lambda|) \cap \{x_n \geq \varepsilon\}$ in $C^{1, \alpha}$ for any $0 < \alpha < \frac{1}{2}$. Also, for j large we can apply Corollary 4.4 which implies that $\mathcal{H}^{n-1}(F_j \cap \{x_n = t\}) \geq A_\lambda$ for a.e. $t \in (0, T_\lambda)$, for some $A_\lambda, T_\lambda > 0$ depending on n, λ . Then points te_n for $t \in (0, T_\lambda/2)$ are points of density 1 for F_j , hence they belong to the interior of F_j . Therefore, for j large enough, $\partial F_j \cap \{x_n > 0\} \setminus \partial^* F_j$ must be empty and $\partial F_j \cap \{x_n > 0\}$ is an axially symmetric hypersurface of class $C^{1, \frac{1}{2}}$.

By the minimality of the F_j , (5.21) and Lemma 5.12 we observe that

$$\begin{aligned}
 P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \left| \alpha_\lambda(F_j) - \alpha_\lambda(E_j) \right| &\leq P_\lambda(E_j) \\
 &\leq P_\lambda(B^\lambda(|B^\lambda|)) + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n, \lambda)} \alpha_\lambda^2(E_j) \\
 &\leq P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n, \lambda)} \alpha_\lambda^2(E_j).
 \end{aligned}
 \tag{5.23}$$

Therefore, we have that

$$\left| \alpha_\lambda(F_j) - \alpha_\lambda(F_j) - \alpha_\lambda(E_j) \right| \leq \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n, \lambda)} \alpha_\lambda^2(E_j).$$

Since $\alpha_\lambda(E_j) \rightarrow 0$ we get that

$$\frac{\alpha_\lambda(F_j)}{\alpha_\lambda(E_j)} \rightarrow 1.$$

Let $\{\hat{\lambda}_j\} \subset (0, \infty)$ such that, setting $\tilde{F}_j := \hat{\lambda}_j F_j$, then $|\tilde{F}_j| = |B^\lambda|$. Clearly $\hat{\lambda}_j \rightarrow 1$ since $|F_j| \rightarrow |B^\lambda|$. Since $P_\lambda(F_j) \rightarrow P_\lambda(B^\lambda(|B^\lambda|))$ and $\Lambda > n$, for j sufficiently large we have $P_\lambda(F_j) < \Lambda|F_j|$ and

$$\begin{aligned}
 \left| P_\lambda(\tilde{F}_j) - P_\lambda(F_j) \right| &= P_\lambda(F_j) \left| \hat{\lambda}_j^{n-1} - 1 \right| \leq P_\lambda(F_j) \left| \hat{\lambda}_j^n - 1 \right| \leq \Lambda \left| \hat{\lambda}_j^n - 1 \right| |F_j| \\
 &= \Lambda \left| |\tilde{F}_j| - |F_j| \right|.
 \end{aligned}$$

Hence, by definition of $\hat{\lambda}_j$ and by (5.23) we get

$$\begin{aligned}
 P_\lambda(\tilde{F}_j) &\leq P_\lambda(F_j) + \Lambda \left| |\tilde{F}_j| - |F_j| \right| = P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| \\
 &\stackrel{(5.23)}{\leq} P_\lambda(B^\lambda(|B^\lambda|)) + \frac{P_\lambda(B^\lambda(|B^\lambda|))}{4\hat{\gamma}^2(n, \lambda)} \alpha_\lambda^2(E_j).
 \end{aligned}
 \tag{5.24}$$

Since $\alpha_\lambda(F_j)/\alpha_\lambda(E_j) \rightarrow 1$ as $j \rightarrow \infty$ we have $\alpha_\lambda(E_j)^2 < 2\alpha_\lambda(\tilde{F}_j)^2$ for j sufficiently large. Hence from (5.24) we finally obtain

$$\alpha_\lambda(\tilde{F}_j) > \sqrt{2}\hat{\gamma}(n, \lambda)\sqrt{D_\lambda(\tilde{F}_j)}.
 \tag{5.25}$$

For $t > 0$ let

$$\varphi_{\tilde{F}_j}^-(t) := \begin{cases} \min_{x \in \partial \tilde{F}_j \cap \{x_n = t\}} \{|x - te_n|\} & \text{if } \partial \tilde{F}_j \cap \{x_n = t\} \neq \emptyset, \\ 0 & \text{if } \partial \tilde{F}_j \cap \{x_n = t\} = \emptyset. \end{cases}$$

be the function measuring the distance of $\partial \tilde{F}_j \cap \{x_n = t\}$ from the n -th axis, set to zero in case $\partial \tilde{F}_j \cap \{x_n = t\} = \emptyset$. For j large we can apply Corollary 4.4 again to deduce that there exists $T_\lambda, A_\lambda > 0$ such that $\mathcal{H}^{n-1}(\tilde{F}_j \cap \{x_n = t\}) \geq A_\lambda$ for almost every $t \in (0, T_\lambda)$. Since \tilde{F}_j is Schwarz-symmetric and its relative boundary in $\{x_n > 0\}$ is C^1 regular, then we can write that $\varphi_{\tilde{F}_j}^-(t) \geq A'_\lambda > 0$ for j large and for any $t \in (0, T_\lambda)$.

Recalling that F_j is a local (Λ_1, r_0) -minimizer, by Lemma A.2 its boundary has generalized mean curvature bounded by Λ_1 for any j . Since $\tilde{F}_j = \hat{\lambda}_j F_j$ with $\hat{\lambda}_j \rightarrow 1$, then $\partial \tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ is a hypersurface of class $C^{1, \frac{1}{2}}$ with generalized mean curvature $H_{\partial \tilde{F}_j}$ bounded

by $2\Lambda_1$ for any j . Observe that if we locally parametrize $\partial\tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ with the graph of a function Φ_j , then Φ_j weakly solves the mean curvature equation

$$\operatorname{div} \left(\frac{\nabla\Phi_j}{\sqrt{1 + |\nabla\Phi_j|^2}} \right) = \langle H_{\partial\tilde{F}_j}, N_{\Phi_j} \rangle,$$

where N_{Φ_j} is the unit normal corresponding to Φ_j and $H_{\partial\tilde{F}_j}$ is evaluated along the graph of Φ_j . Since $H_{\partial\tilde{F}_j}$ is bounded, we get that Φ_j is of class $W^{2,p}$ for every $p < \infty$ (see [31]).

Fix $p_0 \in \partial\tilde{F}_j \cap \{x_1 > 0, 0 < x_n \leq T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$. Since $\partial\tilde{F}_j \cap (\mathbb{R}^n \setminus H)$ is $C^{1,\frac{1}{2}}$, there exists a curve $\gamma_j = (\alpha_j, 0, \dots, 0, \beta_j) : (a, b) \rightarrow \operatorname{span}\{e_1, e_n\} \setminus H$ such that the map $\mathbb{S}^{n-2} \times (a, b) \ni (\vartheta, t) \mapsto (\alpha_j(t)\vartheta, \beta_j(t))$ parametrizes $\partial\tilde{F}_j$ in a neighborhood of p_0 . We claim that $\alpha_j, \beta_j \in W^{2,p}$, up to reparametrization.

Indeed, we can also parametrize $\partial\tilde{F}_j$ in a neighborhood U of p_0 as the graph of a function Φ_j with domain contained in some affine hyperplane of the form $p_0 + V$, and without loss of generality we can assume that either $V = \{x_1 = 0\}$ or $V = \{x_n = 0\}$. If $n = 2$, then the claimed regularity immediately follows from the regularity of Φ_j . Then assume $n \geq 3$, and suppose for example that $V = \{x_1 = 0\}$. The image of the curve γ_j in U can be parametrized as the graph of a function $t \mapsto (f(t), 0, \dots, 0, t)$. Writing as $(x', x_n) \in p_0 + V$ the variable for Φ_j , the fact that the distance from the n -th axis is constant on the intersection of $\partial\tilde{F}_j$ with any horizontal hyperplane yields the identity

$$(\Phi_j(x', x_n) + \operatorname{dist}_{x_n}(p_0))^2 + |x'|^2 = f(x_n)^2,$$

where $\operatorname{dist}_{x_n}(p_0)$ denotes distance of p_0 from the n -th axis. Since Φ_j is of class C^1 and $W^{2,p}$ and $\operatorname{dist}_{x_n}(p_0) > 0$ because $\varphi_{\tilde{F}_j}^- \geq A'_\lambda > 0$, inverting the above identity we find that f is of class $W^{2,p}$, hence so is γ_j , up to reparametrization. In case $V = \{x_n = 0\}$, the observation follows analogously relating Φ_j with a parametrization for γ_j .

We further observe that, for $\alpha_j, \beta_j : (a, b) \rightarrow (0, \infty)$ as above, since α_j, β_j are of class $W^{2,p}_{\text{loc}}$, up to reparametrization by arclength we can apply Lemma A.4 to get that

$$H_{\partial\tilde{F}_j}|_{(\alpha_j(t)\vartheta, \beta_j(t))} = \left(\langle k_{\gamma_j}, v \rangle - (n-2) \frac{\beta'_j}{\alpha_j} \right) (-\beta'_j \vartheta, \alpha'_j),$$

in the notation of Lemma A.4. Recalling that $\varphi_{\tilde{F}_j}^- \geq A'_\lambda > 0$ on $(0, T_\lambda)$, we have that $|\alpha_j| \geq A'_\lambda$ and thus

$$|k_{\gamma_j}| \leq 2\Lambda_1 + \frac{n-2}{A'_\lambda}. \tag{5.26}$$

Observe that the upper bound in (5.26) is independent of j and of the initially chosen point p_0 .

Fix now $q_0 \in \partial\tilde{F}_j \cap \{x_1 > 0, x_n = T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$, let $\gamma_j^0 : [0, l_0] \rightarrow \operatorname{span}\{e_1, e_n\}$ be part of a curve defined as before, parametrized by arclength, such that $\langle \gamma_j^0(t), e_n \rangle \leq T_\lambda$ for any t . If $\lim_{t \rightarrow l_0^-} \gamma_j^0(t) \notin \partial H$, the curve can be extended to a longer one, parametrized by arclength, by joining γ_j^0 with a curve defined as before for the choice $p_0 = \lim_{t \rightarrow l_0^-} \gamma_j^0(t)$. Hence we can consider $\sigma_j : [0, L_j] \rightarrow \operatorname{span}\{e_1, e_n\}$ the maximal extension of γ_j^0 parametrized by arclength that parametrizes $\tilde{F}_j \cap \{x_1 > 0, 0 < x_n \leq T_\lambda\} \cap \operatorname{span}\{e_1, e_n\}$.

Since the perimeter $P(\tilde{F}_j, \mathbb{R}^n \setminus H)$ is uniformly bounded, then $\sup_j L_j < +\infty$. Obviously $\lim_{t \rightarrow L_j} \sigma_j(t) \in \partial H$, for otherwise the curve could be further extended. By construction, the uniform bound in (5.26) holds pointwise for the curvature of σ_j . Therefore σ_j can be extended to a curve $\gamma_j : [0, L_j] \rightarrow \text{span}\{e_1, e_n\}$ such that

$$\|\gamma_j\|_{C^{1,1}([0, L_j])} \leq C, \tag{5.27}$$

with C independent of j , depending only on n, λ, \tilde{l} and the upper bound on the curvature given by (5.26).

Up to a subsequence, since $\tilde{F}_j \rightarrow B^\lambda(|B^\lambda|)$ and we already know that for every $\varepsilon > 0$ the set $\partial \tilde{F}_j \cap \{x_n \geq \varepsilon\}$ converges to $\partial B^\lambda(|B^\lambda|) \cap \{x_n \geq \varepsilon\}$ in $C^{1,\alpha}$ for any $0 < \alpha < \frac{1}{2}$, the bound (5.27) implies that \tilde{F}_j converges in C^1 sense to $B^\lambda(|B^\lambda|)$ in the sense of Definition 5.10. Hence, by Corollary 5.11, for j sufficiently large there holds

$$\alpha_\lambda^2(\tilde{F}_j) \leq \hat{\gamma}(n, \lambda) D_\lambda(\tilde{F}_j),$$

in contradiction with (5.25). □

6 Second quantitative isoperimetric inequality

We will need the following technical lemma, proving that if the energy $P_\lambda(E_i)$ a sequence of sets E_i converges to the energy of the limit, then the sequence strictly converges in the sense of BV functions. The proof essentially follows by analyzing the equality case in the Reshetnyak lower semicontinuity theorem, see [2, Theorem 2.38].

Lemma 6.1 *Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of sets of finite perimeter in $\mathbb{R}^n \setminus H$ such that $E_i \rightarrow E$ in L^1 , for some set of finite perimeter E with $|E| < +\infty$. If $P_\lambda(E_i) \rightarrow P_\lambda(E)$, then*

$$\lim_i P(E_i, \mathbb{R}^n \setminus H) = P(E, \mathbb{R}^n \setminus H), \quad \lim_i \mathcal{H}^{n-1}(\partial^* E_i \cap \partial H) = \mathcal{H}^{n-1}(\partial^* E \cap \partial H).$$

Proof Let $f(v) := |v| - \lambda \langle e_n, v \rangle$, for any $v \in \mathbb{R}^n$, and let $\nu_i := |D\chi_{E_i}| \otimes \delta_{\nu_{E_i}}$ be a measure on $\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}$. Since $|\nu_i|(\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}) \leq P(E_i) \leq 2P_\lambda(E_i)/(1 - \lambda)$ by Corollary 2.5, up to subsequence, ν_i weakly* converges to a finite measure ν . Up to subsequence, also $|D\chi_{E_i}|$ weakly* converges to a finite measure μ on $\mathbb{R}^n \setminus H$. Denoting by $\pi : \mathbb{R}^n \setminus H \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n \setminus H$ the natural projection, we have $\pi_\# \nu_i = |D\chi_{E_i}| \rightarrow \mu = \pi_\# \nu$. Moreover, $\mu \geq |D\chi_E|$ by lower semicontinuity. By the disintegration theorem [2, Theorem 2.28], we can write $\nu = \mu \otimes \nu_x$, for a μ -measurable map $\mathbb{R}^n \setminus H \ni x \mapsto \nu_x$, where ν_x is a probability measure on \mathbb{S}^{n-1} . Analogously to [2, Eq. (2.30)], we observe that

$$\int_{\mathbb{S}^{n-1}} v \, d\nu_x(v) = \nu^E(x) \frac{|D\chi_E|}{\mu}(x), \tag{6.1}$$

at μ -a.e. $x \in \mathbb{R}^n \setminus H$. Indeed, for any continuous function g with $\text{spt}(g) \subset\subset \mathbb{R}^n \setminus H$ we find

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus H} g(x) \int_{\mathbb{S}^{n-1}} v \, dv_x(v) \, d\mu(x) = \int_{\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}} g(x) v \, dv(x, v) \\ &= \lim_i \int_{\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}} g(x) v \, dv_i(x, v) \\ &= - \lim_i \int_{\mathbb{R}^n \setminus H} g(x) \, dD\chi_{E_i}(x) = - \int_{\mathbb{R}^n \setminus H} g(x) \, dD\chi_E(x) \\ &= \int_{\mathbb{R}^n \setminus H} g(x) v^E(x) \frac{|D\chi_E|}{\mu}(x) \, d\mu(x). \end{aligned}$$

Since f is nonnegative, convex and continuous, by Remark 2.2 we find

$$\begin{aligned} \lim_i P_\lambda(E_i) &= \lim_i \int_{\mathbb{R}^n \setminus H} f(v^{E_i}) \, d|D\chi_{E_i}| = \lim_i \int_{\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}} f(v) \, dv_i(x, v) \\ &\geq \int_{\mathbb{R}^n \setminus H \times \mathbb{S}^{n-1}} f(v) \, dv(x, v) \\ &= \int_{\mathbb{R}^n \setminus H} \int_{\mathbb{S}^{n-1}} f(v) \, dv_x(v) \, d\mu(x) \geq \int_{\mathbb{R}^n \setminus H} f \left(\int_{\mathbb{S}^{n-1}} v \, dv_x(v) \right) \, d\mu(x) \tag{6.2} \\ &\stackrel{(6.1)}{=} \int_{\mathbb{R}^n \setminus H} f \left(v^E(x) \frac{|D\chi_E|}{\mu}(x) \right) \, d\mu(x) = \int_{\mathbb{R}^n \setminus H} f(v^E(x)) \, d|D\chi_E|(x) \\ &= P_\lambda(E), \end{aligned}$$

where in the second inequality we applied Jensen inequality, and where the last equality follows since f is positively 1-homogeneous. Since $\lim_i P_\lambda(E_i) = P_\lambda(E)$ by assumption and since f is not affine, equality in Jensen inequality implies that the identity map $\mathbb{S}^{n-1} \ni v \mapsto v$ is constant v_x -a.e., for μ -a.e. $x \in \mathbb{R}^n \setminus H$. This means that $v_x = \delta_{v_x}$ for some $v_x \in \mathbb{S}^{n-1}$ for μ -a.e. $x \in \mathbb{R}^n \setminus H$. Hence (6.1) implies

$$v_x = v^E(x) \frac{|D\chi_E|}{\mu}(x),$$

at μ -a.e. $x \in \mathbb{R}^n \setminus H$, and since $|v_x| = |v^E(x)| = 1$, then $|D\chi_E|/\mu(x) = 1$ at μ -a.e. $x \in \mathbb{R}^n \setminus H$, and $v_x = v^E(x)$ μ -almost everywhere. Inserting in (6.2) we deduce

$$\int_{\mathbb{R}^n \setminus H} f(v^E(x)) \, d\mu(x) = \int_{\mathbb{R}^n \setminus H} f \left(\int_{\mathbb{S}^{n-1}} v \, dv_x(v) \right) \, d\mu(x) = \int_{\mathbb{R}^n \setminus H} f(v^E(x)) \, d|D\chi_E|(x).$$

Since $f(v^E(x)) > 0$ and $\mu \geq |D\chi_E|$, we deduce that $\mu = |D\chi_E|$, and then $|D\chi_{E_i}|$ weakly* converges to $|D\chi_E|$.

We can now fix an increasing sequence of Lipschitz bounded open sets $\Omega_j \subset\subset \mathbb{R}^n \setminus H$ such that $\cup_j \Omega_j = \mathbb{R}^n \setminus H$ and $P(E_i, \partial\Omega_j) = P(E, \partial\Omega_j) = 0$ for every i, j . Hence $\lim_i P(E_i, \Omega_j) = P(E, \Omega_j)$ for any j . Moreover

$$P(E_i, \mathbb{R}^n \setminus (H \cup \Omega_j)) \leq \frac{1}{1 - |\lambda|} \left(P_\lambda(E_i) - \int_{\Omega_j} f(v^{E_i}) \, d|D\chi_{E_i}| \right),$$

for any i, j . Applying Reshetnyak continuity theorem [2, Theorem 2.39] on Ω_j we get

$$\begin{aligned} \limsup_i P(E_i, \mathbb{R}^n \setminus (H \cup \Omega_j)) &\leq \frac{1}{1 - |\lambda|} \int_{\mathbb{R}^n \setminus (H \cup \Omega_j)} f(v^E) \, d|D\chi_E| \\ &\leq \frac{1 + |\lambda|}{1 - |\lambda|} P(E, \mathbb{R}^n \setminus (H \cup \Omega_j)), \end{aligned}$$

for any j . Therefore

$$\limsup_i P(E_i, \mathbb{R}^n \setminus H) \leq P(E, \Omega_j) + \frac{1 + |\lambda|}{1 - |\lambda|} P(E, \mathbb{R}^n \setminus (H \cup \Omega_j)),$$

for any j . Letting $j \rightarrow \infty$, the proof follows. □

We will also exploit the concept of (K, r_0) -quasiminimal set.

Definition 6.2 Let $E \subset \mathbb{R}^n \setminus H$ be a set of finite perimeter with finite measure, and let $K \geq 1, r_0 > 0$. We say that E is a (K, r_0) -quasiminimal set (relatively in $\mathbb{R}^n \setminus H$) if

$$P(E, \mathbb{R}^n \setminus H) \leq KP(F, \mathbb{R}^n \setminus H),$$

for any $F \subset \mathbb{R}^n \setminus H$ such that $E \Delta F \subset \subset B_r(x)$, for some ball $B_r(x) \subset \mathbb{R}^n$ with $r \leq r_0$ and $x \in \{x_n \geq 0\}$.

Quasiminimal sets have well-known topological regularity properties following from uniform density estimates at boundary points. We recall these facts in the following statement. The proof follows, for example, by repeatedly applying [41, Theorem 4.2] with $X = \{x_n \geq 0\}$ in domains $\Omega = X \cap B_{r_0}(x)$ for $x \in X$, in the notation of [41, Theorem 4.2]. Observe that in [41], the perimeter functional coincides with the relative perimeter in $\mathbb{R}^n \setminus H$, hence the definition of quasiminimal set in [41, Definition 3.1] coincides with our Definition 6.2. Alternatively, the proof follows by adapting the proof of [47, Theorem 21.11] working with (K, r_0) -quasiminimal sets instead of (Λ, r_0) -minimizers.

Theorem 6.3 Let $E \subset \mathbb{R}^n \setminus H$ be a (K, r_0) -quasiminimal set, for some $K \geq 1, r_0 > 0$. Then there exist $m = m(n, K, r_0) \in (0, 1)$ and $r'_0 = r'_0(n, K, r_0) \in (0, r_0]$ such that

$$m \leq \frac{|E \cap B_r(x)|}{|B_r(x) \setminus H|} \leq 1 - m \quad \forall x \in \overline{\partial E \setminus H}, \quad \forall r \in (0, r'_0].$$

In particular the set $E^{(1)}$ of points of density 1 for E is an open representative for E .

We will identify a (K, r_0) -quasiminimal set with its open representative $E^{(1)}$. In order to prove Theorem 1.2 we need two preparatory lemmas.

Lemma 6.4 For any $K \geq 1, r_0 > 0$ there exist $\delta_6, C_5, C_6 > 0$ depending on n, λ, K, r_0 such that the following holds. If $E \subset \mathbb{R}^n \setminus H$ is a bounded (K, r_0) -quasiminimal set with $|E| = |B^\lambda|$ and $D_\lambda(E) \leq \delta_6$, then

$$d_{\mathcal{H}} \left(\overline{\partial E \setminus H}, \overline{\partial B^\lambda(|B^\lambda|, x) \setminus H} \right) \leq C_5 \alpha_\lambda(E)^{\frac{1}{n}}, \tag{6.3}$$

where $B^\lambda(|B^\lambda|, x)$ is a bubble realizing the asymmetry of E . Moreover

$$\beta_\lambda(E) \leq C_6 D_\lambda(E)^{\frac{1}{2n}}. \tag{6.4}$$

Proof Up to translation, we can assume that $x = 0$. Also, letting m, r'_0 be given by Theorem 6.3, up to decreasing r_0 we can assume that $r_0 = r'_0$. Let $p \in \overline{\partial E \setminus H}$ be such that

$$d_0 := \text{dist} \left(p, \overline{\partial B^\lambda(|B^\lambda|) \setminus H} \right) = \max \left\{ \text{dist} \left(y, \overline{\partial B^\lambda(|B^\lambda|) \setminus H} \right) : y \in \overline{\partial E \setminus H} \right\}.$$

Hence $B_{d_0}(p) \cap \overline{\partial B^\lambda(|B^\lambda|) \setminus H} = \emptyset$. Then either $B_{d_0}(p) \setminus H \subset B^\lambda(|B^\lambda|)$ or $B_{d_0}(p) \setminus H \subset \mathbb{R}^n \setminus (H \cup B^\lambda(|B^\lambda|))$. In the first case Theorem 6.3 implies

$$m|B_r(x) \setminus H| \leq |B_r(x) \setminus (H \cup E)| \leq |B^\lambda(|B^\lambda|) \setminus E| = \frac{1}{2} \alpha_\lambda(E) \quad \forall r \in (0, \min\{d_0, r_0\}),$$

while in the second case Theorem 6.3 implies

$$m|B_r(x) \setminus H| \leq |B_r(x) \cap E| \leq |E \setminus B^\lambda(|B^\lambda|)| = \frac{1}{2}\alpha_\lambda(E) \quad \forall r \in (0, \min\{d_0, r_0\}).$$

Since $|B_R(x) \setminus H| \geq Cr^n$, then $\min\{d_0, r_0\}^n \leq C\alpha_\lambda(E)$, for $C = C(n, \lambda, K, r_0)$. So by Corollary 4.3, choosing δ_6 small enough we have that $\alpha_\lambda(E)$ is so small that $\min\{d_0, r_0\} = d_0$ and then

$$d_0^n \leq C\alpha_\lambda(E).$$

Since density estimates as those in Theorem 6.3 hold for $B^\lambda(|B^\lambda|)$, repeating the above argument exchanging the roles of E and $B^\lambda(|B^\lambda|)$, (6.3) follows.

From (6.3), we deduce that

$$\overline{\partial E \setminus H} \subset \left\{ y \in \{x_n \geq 0\} : \text{dist} \left(y, \overline{\partial B^\lambda(|B^\lambda|) \setminus H} \right) \leq C_5\alpha_\lambda(E)^{\frac{1}{n}} \right\}.$$

Hence

$$\begin{aligned} &\mathcal{H}^{n-1}(\partial^* E \Delta \partial B^\lambda(|B^\lambda|) \cap \partial H) \\ &\leq \mathcal{H}^{n-1} \left(\left\{ (x', 0) \in \mathbb{R}^n : (1 - \lambda^2)^{\frac{1}{2}} - C_5\alpha_\lambda(E)^{\frac{1}{n}} \leq |x'| \leq (1 - \lambda^2)^{\frac{1}{2}} + C_5\alpha_\lambda(E)^{\frac{1}{n}} \right\} \right) \\ &\leq C\alpha_\lambda(E)^{\frac{1}{n}} \leq CD_\lambda(E)^{\frac{1}{2n}}, \end{aligned}$$

for some $C = C(n, \lambda, K, r_0)$, where we used Theorem 1.1 in the last inequality. Hence (6.4) follows. □

Lemma 6.5 *There exists $\delta_7, C_7 > 0$ depending on n, λ such that for any measurable set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $D_\lambda(E) \leq \delta_7$ there holds*

$$\beta_\lambda(E) \leq C_7 D_\lambda(E)^{\frac{1}{2n}}. \tag{6.5}$$

Proof Fix $\Lambda > n$. Let $Q \subset \mathbb{R}^n$ be a large cube whose interior contains the closure of $B^\lambda(|B^\lambda|)$, and let $F \subset Q \setminus H$ be such that $|B^\lambda|/2 \leq |F| \leq 2|B^\lambda|$. Let $G \subset \mathbb{R}^n \setminus H$ be such that $G \Delta F \subset \subset B_{r_0}(x)$, for $x \in \{x_n \geq 0\}$ and $r_0 \in (0, 1)$ to be chosen small. Let $Z := G \cap Q$. Observe that

$$\begin{aligned} ||Z| - |F|| &\leq |Z \Delta F| \leq |G \Delta F|^{\frac{1}{n}} |G \Delta F|^{\frac{n-1}{n}} \leq \omega_n^{\frac{1}{n}} r \left(|G|^{\frac{n-1}{n}} + |F|^{\frac{n-1}{n}} \right) \\ &\leq \overline{C}(n)r_0 \left(P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H) \right), \end{aligned} \tag{6.6}$$

where in the last inequality we used the relative isoperimetric inequality in a half-space (see [14] for the sharp inequality). Let $y, z \in \partial H$ be such that

$$\beta_\lambda(F) = \frac{\mathcal{H}^{n-1}(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H)}, \quad \beta_\lambda(Z) = \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial B^\lambda(|Z|, z) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, z) \cap \partial H)},$$

Observe that y, z exist since $F, Z \subset Q$. Suppose, for instance, that $\beta_\lambda(Z) \geq \beta_\lambda(F)$. Then

$$\begin{aligned} \beta_\lambda(Z) - \beta_\lambda(F) &\leq \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial B^\lambda(|Z|, y) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)} - \frac{\mathcal{H}^{n-1}(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H)} \\ &\leq \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial^* F \cap \partial H) + \mathcal{H}^{n-1}(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H) + \mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \Delta \partial B^\lambda(|Z|, y) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)} \\ &\quad - \frac{\mathcal{H}^{n-1}(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H)} \tag{6.7} \\ &= \frac{\mathcal{H}^{n-1}(\partial^* Z \Delta \partial^* F \cap \partial H)}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)} + \frac{|\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H) - \mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)|}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)} \\ &\quad + \mathcal{H}^{n-1}(\partial^* F \Delta \partial B^\lambda(|F|, y) \cap \partial H) \left(\frac{1}{\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)} - \frac{1}{\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H)} \right) \end{aligned}$$

By a trace inequality [2, Theorem 3.87] we estimate

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^* Z \Delta \partial^* F \cap \partial H) &\leq C(n) (|Z \Delta F| + P(Z, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)) \\ &\leq C(n) (|G \Delta F| + P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)) \\ &\leq C(n) (P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)), \end{aligned}$$

where the last inequality follows as in (6.6), and C denotes a constant depending on suitable parameters that changes from line to line. For r_0 small, depending only on n, λ , we can ensure that

$$\mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H) \geq C(n, \lambda) > 0.$$

Finally

$$\begin{aligned} |\mathcal{H}^{n-1}(\partial B^\lambda(|F|, y) \cap \partial H) - \mathcal{H}^{n-1}(\partial B^\lambda(|Z|, y) \cap \partial H)| &\leq L ||Z| - |F|| \\ &\stackrel{(6.6)}{\leq} C(n, \lambda) (P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)), \end{aligned}$$

for a suitable Lipschitz constant $L = L(n, \lambda)$. Therefore (6.7) becomes

$$\beta_\lambda(Z) - \beta_\lambda(F) \leq C(n, \lambda) (P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)).$$

In case $\beta_\lambda(Z) < \beta_\lambda(F)$, the very same argument leads to an analogous estimate. Hence

$$|\beta_\lambda(Z) - \beta_\lambda(F)| \leq \tilde{C}(n, \lambda) (P(G, \mathbb{R}^n \setminus H) + P(F, \mathbb{R}^n \setminus H)). \tag{6.8}$$

Up to taking a smaller r_0 , we fix $r_0 = r_0(n, \lambda, \Lambda) \in (0, 1)$ and $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) \in (0, 1)$ such that $1 - |\lambda| - \varepsilon_0 \tilde{C}(n, \lambda) - \Lambda \bar{C}(n) r_0 > 0$, and we define

$$K := \frac{1 + |\lambda| + \varepsilon_0 \tilde{C}(n, \lambda) + \Lambda \bar{C}(n) r_0}{1 - |\lambda| - \varepsilon_0 \tilde{C}(n, \lambda) - \Lambda \bar{C}(n) r_0} > 1.$$

Let δ_6, C_6 be given by Lemma 6.4 corresponding to the parameters $K, r_0/2$. We want to prove that if δ_7 is sufficiently small, then

$$\beta_\lambda(E) \leq 2C_6 D_\lambda(E)^{\frac{1}{2n}}.$$

We argue by contradiction assuming that there exist sets $E_j \subset \mathbb{R}^n \setminus H$ with $|E_j| = |B^\lambda|$ and $D_\lambda(E_j) \leq 1/j$ such that

$$\beta_\lambda(E_j) > 2C_6 D_\lambda(E_j)^{\frac{1}{2n}}, \tag{6.9}$$

for any j . Up to translation, $E_j \rightarrow B^\lambda(|B^\lambda|, 0)$ and $P_\lambda(E_j) \rightarrow P_\lambda(B^\lambda)$. Since the trace operator is continuous with respect to strict convergence of BV functions, see [2, Theorem 3.88], by Lemma 6.1 we deduce that $\beta_\lambda(E_j) \rightarrow 0$.

Let F_j be a minimizer of the problem

$$\min \{ P_\lambda(E) + \varepsilon_0 |\beta_\lambda(E) - \beta_\lambda(E_j)| + \Lambda \left| |E| - |E_j| \right| : E \subset Q \}. \tag{6.10}$$

By Corollary 3.9, up to subsequence F_j converges to a limit set F in L^1 . If by contradiction $\beta_j(F_j) \not\rightarrow 0$, by Lemma 5.12 for large j we would have that

$$P_\lambda(B^\lambda(|B^\lambda|)) + \varepsilon_0 \beta_\lambda(E_j) < P_\lambda(F_j) + \varepsilon_0 |\beta_\lambda(F_j) - \beta_\lambda(E_j)| + \Lambda \left| |F_j| - |E_j| \right|,$$

contradicting the minimality of F_j . Hence $\beta_\lambda(F_j) \rightarrow 0$. It follows that, up to translation, F_j converges to $B^\lambda(|B^\lambda|)$ in L^1 . Comparing with E_j , we also see that $P_\lambda(F_j) \rightarrow P_\lambda(B^\lambda)$.

We want to show that F_j is (K, r_0) -quasiminimal for j large. Indeed, $|B^\lambda|/2 \leq |F_j| \leq 2|B^\lambda|$ for j large. Hence we can apply (6.6) and (6.8) with $F = F_j$. Letting $G \subset \mathbb{R}^n \setminus H$ such that $G \Delta F \subset \subset B_{r_0}(x)$, for $x \in \{x_n \geq 0\}$, denoting $Z := G \cap Q$, by minimality of F_j for (6.10) we find

$$\begin{aligned} (1 - |\lambda|)P(F_j, \mathbb{R}^n \setminus H) &\leq (1 + |\lambda|)P(Z, \mathbb{R}^n \setminus H) + \varepsilon_0 |\beta_\lambda(Z) - \beta_\lambda(F_j)| + \Lambda \left| |Z| - |F_j| \right| \\ &\leq (1 + |\lambda|)P(G, \mathbb{R}^n \setminus H) + (\varepsilon_0 \tilde{C}(n, \lambda) + \Lambda \bar{C}(n)r_0) (P(G, \mathbb{R}^n \setminus H) + P(F_j, \mathbb{R}^n \setminus H)), \end{aligned}$$

proving that F_j is (K, r_0) -quasiminimal.

By minimality of F_j , we have

$$\begin{aligned} P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \varepsilon_0 |\beta_\lambda(F_j) - \beta_\lambda(E_j)| &\leq P_\lambda(E_j) \\ &\leq P_\lambda(B^\lambda) + \frac{P_\lambda(B^\lambda)}{(2C_6)^{2n}} \beta_\lambda^{2n}(E_j) \leq P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| + \frac{P_\lambda(B^\lambda)}{(2C_6)^{2n}} \beta_\lambda^{2n}(E_j) \end{aligned} \tag{6.11}$$

Therefore

$$|\beta_\lambda(F_j) - \beta_\lambda(E_j)| \leq \frac{P_\lambda(B^\lambda)}{\varepsilon_0 (2C_6)^{2n}} \beta_\lambda^{2n}(E_j),$$

and then

$$\frac{\beta_\lambda(F_j)}{\beta_\lambda(E_j)} \rightarrow 1.$$

Next we select $\{\hat{\lambda}_j\} \subset (0, \infty)$ such that, setting $\tilde{F}_j := \hat{\lambda}_j F_j$, then $|\tilde{F}_j| = |B^\lambda|$. Clearly $\hat{\lambda}_j \rightarrow 1$ since $|F_j| \rightarrow |B^\lambda|$. Since $P_\lambda(F_j) \rightarrow P_\lambda(B^\lambda)$ and $\Lambda > n$, for j sufficiently large we have $P_\lambda(F_j) < \Lambda |F_j|$ and

$$\begin{aligned} \left| P_\lambda(\tilde{F}_j) - P_\lambda(F_j) \right| &= P_\lambda(F_j) \left| \hat{\lambda}_j^{n-1} - 1 \right| \leq P_\lambda(F_j) \left| \hat{\lambda}_j^n - 1 \right| \\ &\leq \Lambda \left| \hat{\lambda}_j^n - 1 \right| |F_j| = \Lambda \left| |\tilde{F}_j| - |F_j| \right|. \end{aligned}$$

Hence, by definition of $\hat{\lambda}_j$ and by (6.11) we get

$$\begin{aligned} P_\lambda(\tilde{F}_j) &\leq P_\lambda(F_j) + \Lambda \left| |\tilde{F}_j| - |F_j| \right| = P_\lambda(F_j) + \Lambda \left| |F_j| - |B^\lambda| \right| \\ &\stackrel{(6.11)}{\leq} P_\lambda(B^\lambda) + \frac{P_\lambda(B^\lambda)}{(2C_6)^{2n}} \beta_\lambda^{2n}(E_j). \end{aligned} \tag{6.12}$$

Since $\beta_\lambda(F_j)/\beta_\lambda(E_j) \rightarrow 1$ as $j \rightarrow \infty$ and β_λ is scale invariant, we have $\beta_\lambda(E_j)^{2n} < 2\beta_\lambda(\tilde{F}_j)^{2n}$ for j sufficiently large. Hence from (6.12) we obtain

$$\beta_\lambda(\tilde{F}_j)^{2n} \geq 2^{2n-1} C_6^{2n} D_\lambda(\tilde{F}_j),$$

that is $\beta_\lambda(\tilde{F}_j) \geq 2^{1-\frac{1}{2n}} C_6 D_\lambda(\tilde{F}_j)^{\frac{1}{2n}}$. On the other hand, for j large, \tilde{F}_j is $(K, \hat{\lambda}_j r_0)$ -quasiminimal. As $\hat{\lambda}_j \rightarrow 1$, then \tilde{F}_j is $(K, r_0/2)$ -quasiminimal for j large. Moreover $D_\lambda(\tilde{F}_j) \rightarrow 0$. By the choice of C_6 above, Lemma 6.4 implies that

$$\beta_\lambda(\tilde{F}_j) \leq C_6 D_\lambda(\tilde{F}_j)^{\frac{1}{2n}},$$

giving a contradiction. □

Proof of Theorem 1.2 By Lemma 6.5 it follows that for any $A > 0$ there exists $C_A > 0$ such that for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \leq A$ there holds

$$\beta_\lambda(E) \leq C_A D_\lambda(E)^{\frac{1}{2n}}. \tag{6.13}$$

Indeed, if $D_\lambda(E) \leq \delta_7$, for δ_7 as in Lemma 6.5, then (6.13) follows with $C_A = C_7$. Otherwise we just have

$$\begin{aligned} \beta_\lambda(E) &\leq C(n, \lambda) (\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \mathcal{H}^{n-1}(\partial^* B^\lambda(|B^\lambda|, 0) \cap \partial H)) \\ &\leq C(n, \lambda, A) \frac{\delta_7^{\frac{1}{2n}}}{\delta_7^{\frac{1}{2n}}} \leq C(n, \lambda, A) D_\lambda^{\frac{1}{2n}}. \end{aligned}$$

Next we observe that, letting C_λ such that $P_\lambda(B^\lambda) \leq C_\lambda \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$, then for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ and $\mathcal{H}^{n-1}(\partial^* E \cap \partial H) \geq \frac{2C_\lambda}{1-\lambda} \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$ there holds

$$\beta_\lambda(E) \leq C_8 D_\lambda(E), \tag{6.14}$$

for a constant $C_8 = C_8(n, \lambda) > 0$. Indeed

$$\begin{aligned} P_\lambda(E) - P_\lambda(B^\lambda) &\geq (1 - \lambda) \mathcal{H}^{n-1}(\partial^* E \cap \partial H) - C_\lambda \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H) \\ &\geq \frac{1 - \lambda}{2} \mathcal{H}^{n-1}(\partial^* E \cap \partial H), \end{aligned}$$

and

$$\begin{aligned} \beta_\lambda(E) &\leq C(n, \lambda) (\mathcal{H}^{n-1}(\partial^* E \cap \partial H) + \mathcal{H}^{n-1}(\partial^* B^\lambda(|B^\lambda|, 0) \cap \partial H)) \\ &\leq C(n, \lambda, C_\lambda) \mathcal{H}^{n-1}(\partial^* E \cap \partial H). \end{aligned}$$

Setting now $A := \frac{2C_\lambda}{1-\lambda} \mathcal{H}^{n-1}(\partial B^\lambda(|B^\lambda|) \cap \partial H)$ in (6.13), taking into account (6.14) we conclude that for any set $E \subset \mathbb{R}^n \setminus H$ with $|E| = |B^\lambda|$ there holds

$$\beta_\lambda(E) \leq \max\{C_A, C_8\} \max \left\{ D_\lambda(E), D_\lambda(E)^{\frac{1}{2n}} \right\}.$$

□

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Appendix A. Auxiliary results

A.1. Regularity of (Λ, r_0) -minimizers

We recall definitions and basic properties of local (Λ, r_0) -minimizers of the perimeter. A detailed account on the theory of (Λ, r_0) -minimizers can be found in [47].

Definition A.1 Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a set of finite perimeter. We say that E is a local (Λ, r_0) -minimizer of the perimeter in Ω , with $\Lambda, r_0 > 0$, if

$$P(E, B_r(x)) \leq P(F, B_r(x)) + \Lambda |E \Delta F|,$$

whenever $E \Delta F \subset \subset B_r(x) \subset \subset \Omega$ and $r \leq r_0$.

It is well-known that local (Λ, r_0) -minimizers have bounded mean curvature in a generalized sense. We could not find an explicit reference in the literature, hence we provide a proof in the following result.

Lemma A.2 Let $\Omega \subset \mathbb{R}^n$ be an open set and let $E \subset \mathbb{R}^n$ be a local (Λ, r_0) -minimizer of the perimeter in Ω . Then there exists $H \in L^\infty(P(E, \cdot), \mathbb{R}^n)$ such that $\|H\|_{L^\infty} \leq \Lambda$ and

$$\int_{\partial^* E} \operatorname{div}_T X = - \int_{\partial^* E} \langle X, H \rangle \quad \forall X \in C_c^1(\Omega, \mathbb{R}^n),$$

where $\operatorname{div}_T X$ is the tangential divergence of X along the $(\mathcal{H}^{n-1}$ -a.e. defined) tangent space of $\partial^* E$. We shall refer to H as to the (generalized) mean curvature of E .

Proof Let $X \in C^1_c(B_r(x))$, with $B_r(x) \subset\subset \Omega$ and $r \leq r_0$. Let $\{g_t\}_{t < |\eta]}$ be the flow of the vector field X , and define $F_t = g_t(E)$. Then $E \Delta F_t \subset\subset B_r(x)$. Let $f_t = g_t^{-1} = g_{-t}$. If $u \in C^1(\Omega)$, for $t > 0$ we have

$$\begin{aligned} \int_{B_r(x)} |u(f_t(y)) - u(y)| \, dy &= \int_{B_r(x)} \left| \int_0^t \partial_s [u(f_s(y))] \, ds \right| \, dy \\ &\leq \int_{B_r(x)} \int_0^t |\nabla u(f_s(y))| |X(f_s(y))| \, ds \, dy \\ &= \int_0^t \int_{B_r(x)} |\nabla u(f_s(y))| |X(f_s(y))| \frac{Jf_s(y)}{Jf_s(y)} \, dy \, ds \tag{A.1} \\ &\leq (1 + o(1)) \int_0^t \int_{B_r(x)} |\nabla u(f_s(y))| |X(f_s(y))| Jf_s(y) \, dy \, ds \\ &= (1 + o(1)) \int_0^t \int_{B_r(x)} |\nabla u(z)| |X(z)| \, dz \, ds \\ &= (1 + o(1))t \int_{B_r(x)} |\nabla u(z)| |X(z)| \, dz, \end{aligned}$$

where $o(1) \rightarrow 0$ as $t \rightarrow 0^+$.

Setting $u = u_\varepsilon = \chi_E * \rho_\varepsilon$, then $|\nabla u_\varepsilon| \mathcal{L}^n \rightarrow P(E, \cdot)$ as $\varepsilon \rightarrow 0$ by [47, Proposition 12.20].

Also

$$\int_{B_r(x)} |u_\varepsilon(f_t(y)) - \chi_{f_{-t}(E)}(y)| \frac{Jf_t}{Jf_t} \, dy \leq 2 \int_{B_r(x)} |u_\varepsilon(z) - \chi_E(z)| \, dz.$$

Hence setting $u = u_\varepsilon$ in (A.1) and letting $\varepsilon \rightarrow 0$ implies

$$|E \Delta F_t| = \int_{B_r(x)} |\chi_{F_t} - \chi_E| \leq (1 + o(1))t \int_{B_r(x)} |X| \, dP(E, \cdot).$$

By (Λ, r_0) -minimality we deduce

$$P(E, B_r(x)) - P(F_t, B_r(x)) \leq (1 + o(1))\Lambda t \int_{B_r(x)} |X| \, dP(E, \cdot).$$

Dividing by $t > 0$ and letting $t \rightarrow 0^+$ we get

$$- \int_{\partial^* E} \operatorname{div}_T X \, dP(E, \cdot) \leq \Lambda \int_{\partial^* E} |X| \, dP(E, \cdot).$$

Up to changing X with $-X$ we obtain

$$\left| \int_{\partial^* E} \operatorname{div}_T X \, dP(E, \cdot) \right| \leq \Lambda \int_{\partial^* E} |X| \, dP(E, \cdot),$$

that implies the existence of the generalized mean curvature $H \in L^\infty(P(E, \cdot) \llcorner B_r(x))$ for E in $B_r(x)$ with $\|H\|_{L^\infty} \leq \Lambda$. Since $B_r(x)$ was arbitrary in Ω , by a partition of unity argument the claim follows. \square

Let us further recall the following fundamental regularity properties of local (Λ, r_0) -minimizers.

Theorem A.3 ([51], [47, Theorem 26.3, Theorem 26.6]) *Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $E \subset \Omega$ be a local (Λ, r_0) -minimizer in Ω . Then the set $E^{(1)}$ of points of density 1 for E is*

an open representative for E . Moreover, representing E with $E^{(1)}$, we have that $\partial^* E \cap \Omega$ is a $C^{1, \frac{1}{2}}$ manifold and $\mathcal{H}^d(\partial E \cap \Omega \setminus \partial^* E) = 0$ for any $d > n - 8$.

Let $E_i \subset \Omega$ be a sequence of local (Λ, r_0) -minimizers in Ω that converges to E in $L^1(\Omega)$. If $\partial E \cap B_r(x)$ is of class C^2 , for some $B_r(x) \subset \Omega$, then $\partial E_i \cap B_{r/2}(x)$ is of class $C^{1, \frac{1}{2}}$ for large i and converges to $\partial E \cap B_{r/2}(x)$ in $C^{1, \alpha}$ for any $\alpha \in (0, 1/2)$.

Thanks to Theorem A.3, we will always identify a local (Λ, r_0) -minimizer E with the open set $E^{(1)}$.

A.2. Axially symmetric hypersurfaces

We recall a formula for the mean curvature of axially symmetric hypersurfaces of class $W^{2,p}$.

Lemma A.4 *Let $a < b$. Let $\alpha, \beta : (a, b) \rightarrow (0, \infty)$ be $W^{2,p}$ functions, with $p \in (1, \infty]$, parametrizing the curve $\gamma : (a, b) \rightarrow \text{span}\{e_1, e_n\} \subset \mathbb{R}^n$ given by $\gamma(t) = (\alpha(t), 0, \dots, 0, \beta(t))$, and assume that $|\gamma'(t)| = 1$ and that $\inf_{(a,b)} \alpha > 0$. Let S be the axially symmetric hypersurface around the n -th axis parametrized by*

$$\begin{aligned} \varphi_S : \mathbb{S}^{n-2} \times (a, b) &\rightarrow \mathbb{R}^n \\ \varphi_S(\vartheta, t) &= (\alpha(t)\vartheta, \beta(t)). \end{aligned}$$

Then the vector

$$H = \left(\langle k_\gamma, \nu \rangle - (n - 2) \frac{\beta'}{\alpha} \right) (-\beta' \vartheta, \alpha'),$$

for every ϑ and a.e. t , where k_γ is the curvature of γ and $\nu(t) = (-\beta', 0, \dots, 0, \alpha')$, is the (generalized) mean curvature of S . More precisely

$$\int_S \text{div}_T X = - \int_S \langle X, H \rangle, \tag{A.2}$$

for any $X \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ such that $\text{spt} X \cap \partial S = \emptyset$, where $\text{div}_T X$ is the tangential divergence of X along S .

Proof If α, β are smooth, the claimed formula follows by a direct computation. The statement then follows by approximating α, β in $W^{2,p}$. The approximating hypersurfaces S_i converge to S in C^1 and the measures $H_i \mathcal{H}^{n-1} \llcorner S_i$ converge to $H \mathcal{H}^{n-1} \llcorner S$ in duality with compactly supported continuous fields. Hence (A.2) passes to the limit. \square

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