



Integer orbits in rectangular lattice billiards

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ABSTRACT

In this paper we consider rectangular billiard tables having vertices with integer coordinates, and side lengths equal to integer multiples of the norms of the side directions. We also assume that all the bouncing points of a billiard ball are constrained to belong to the integer lattice \mathbb{Z}^2 . We address several questions concerning combinatorial and geometric properties of the allowed orbits, that, due to the integer constraint, are called integer orbits. We give a complete classification of integer orbits, and the parameters contributing to their structure are precisely determined. This leads to understand how the orbit fills the lattice billiard before it really propagates. In particular, one can characterize the trajectories that reach a billiard pocket, as well as all the closed orbits, by the simple knowledge of the size of the billiard table, and of the starting moving direction. The characterization bases on the explicit determination of the numerical sequences corresponding to clockwise, and counterclockwise, bouncing.

We also investigate the geometrical structure of an allowed orbit in terms of special sub-patterns, called Z-paths, pointing out the allowed lengths of different Z-paths in a same orbit. This is of independent interest, and is related to the configurations known as switching components, that play a crucial role in discrete tomography, and in problems concerning image reconstruction.

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1. Introduction

In this paper we consider rectangular billiard tables having vertices with integer coordinates, and side lengths equal to integer multiples of the norms of the side directions. We also require that all the bouncing points of a billiard ball belong to the integer lattice \mathbb{Z}^2 . In particular, the billiard ball is allowed to move only along lines having rational slopes, and, when it hits the boundary of the billiard, then it changes its direction according to the mirror law, namely the angle of incidence equals the angle of reflection. Due to the integer constraint, the resulting path is called *integer orbit* of the billiard table. A *singular orbit* is an integer orbit that hits a vertex of the billiard.

Polygons where all angles are rational multiples of π are said to be *rational polygons*, and are extensively studied in the literature concerning billiard tables (see, for instance [1,12,16,18]). However, we are here intrigued about the integer constraint on the billiard ball, and on the points where it hits the boundary of the billiard table, since this impose a severe limitation on the existing orbits, and motivates the study of their geometric, numerical and combinatorial structure. This leads to a number of questions that, differently from the continuous cases, are strongly related to the integer nature of the set of points we are considering. We list below a few of them.

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Question (1) Which orbits are allowed in a lattice billiard table? Which parameters are involved? How these characterize the different orbits?

Question (2) Can the number of bouncing points of an integer orbit be computed by the only knowledge of the starting direction of the billiard ball?

Question (3) How clockwise and counterclockwise bounces of the billiard ball mutually alternate during its traveling? Can the corresponding structural sequences be determined in advance, namely before the orbit propagates?

Question (4) Is there a meaningful geometrical characterization of the zig-zag sub-paths of an integer orbit?

Moving from the above questions, we provide several combinatorial, numerical and geometrical answers concerning the structure of the integer orbits in a lattice billiard table.

On one side, the obtained results can be interpreted as a discrete version of properties of a billiard flow in the unit tangent bundle to a polygonal billiard, where the reflections in the boundary correspond to discontinuities. From this point of view, our proposal could be considered as an additional contribution to the rich literature concerning polygonal billiards (see for example [1,2,6,9,12,17,18,20] and the included bibliographies), where, however, the focus is mainly on the properties of sets of orbits, rather than on the structure of single orbits.

On the other side, our approach represents a complete generalization of the so-called arithmetic billiards (also known under the name of Paper Pool), considered in recreational mathematics, where the angle of incidences is always at $\frac{\pi}{4}$, and questions concerning bouncing points and the shape of the orbits are studied (see, for instance, [19]). Here, differently, we allow any possible angle of incidence related to discrete directions, and we also investigate on different problems, as outlined in the previous questions. Rational constraints in billiards are considered also in some different papers. For example, rational lines are adopted in the so called *plaid model*, which however has different aims, representing a combinatorial model for orbits of outer billiards on kites [21], and it is connected with polytope exchange transformations [15], as well as with renormalization and continued fractions [22].

We also comment briefly on possible applications of our results to the area of discrete tomography, where the orbits can be reinterpreted as special paths called *switching components*. These configurations play a crucial role in discrete tomography, since are responsible of possible ambiguous reconstructions of the image of an unknown object by means of X-rays (see [13,14] for a general overview of discrete tomography). In particular, we give a rigorous proof of a property that has been always considered as true in the *folklore*, even if a bit disputed after the publication of some results in [10], which instead can be now confirmed.

The paper is organized as follows. In Section 2 we provide the main notations, definitions, and preliminary results. In Section 3, Theorem 5, we give a complete characterization of the allowed integer orbits in a given lattice billiard table. Thanks to this result, we describe in Section 4 all the parameters that are involved in the different orbits. By means of Theorems 7, 10, and 15, we show how to distinguish among degenerate, singular, elementary windows and twisted integer orbits, so providing a complete answer to Question (1), as well as a few additional remarks. In Section 5 we consider Question (2) and Question (3), so providing a quantitative description of how admissible orbits are displaced in a lattice billiard table, thanks to a complete characterization of their structural sequences. Theorem 36 summarizes such a characterization. Although the geometry of integer orbits is investigated throughout the whole paper, detailed answers to Question (4) can be obtained by means of Lemma 17 and Corollary 31, while Remark 32 points out a link with *h_v-convex polyominoes*. In Section 6 we resume the obtained results, also commenting on their possible extension in other works.

2. Notations and preliminaries

2.1. Lattice billiard tables

A *rational direction* is a vector $\vec{u} = [u_1, u_2]$, with $u_1, u_2 \in \mathbb{Z}$, $u_1 \geq 0$, and u_1, u_2 coprime if $u_1, u_2 \neq 0$, or $u_2 = 1$ if $u_1 = 0$, or $u_1 = 1$ if $u_2 = 0$. A *lattice line* is a line with rational slope, namely a line having equation $ax + by + c = 0$, with $a, b \in \mathbb{Z}$, $c \in \mathbb{R}$. The horizontal and the vertical directions are denoted by \vec{h}, \vec{v} , respectively, while $L_{\vec{u}}(x)$ is the lattice line in direction \vec{u} through the point $x \in \mathbb{R}^2$.

As a general approach, we define a lattice billiard (table) to be any rectangle having vertices with integer coordinates, sides of rational slopes, and side lengths equal to integer multiples of the norms of the side directions.

For $a, b \in \mathbb{N}$, $a \geq b$, and a direction $\vec{u} = [u_1, u_2]$, $u_1, u_2 \geq 0$, let $\Omega_{a,b}(\vec{u})$ be the rectangular lattice billiard having a vertex at the origin, and sides of lengths $a\|\vec{u}\|$ and $b\|\vec{u}\|$ in directions \vec{u}^\perp and \vec{u} , respectively, where $\|\vec{u}\|$ denotes the norm of \vec{u} , and $\vec{u}^\perp = [1, 0]$ if $\vec{u} = [0, 1]$. In particular, if $\vec{u} = \vec{h}$, then we say that the billiard table is in *canonical position*. The two sides parallel to \vec{u}^\perp contain $a + 1$ lattice points, while the sides parallel to \vec{u} contain $b + 1$ lattice points (vertices are counted on both sides). Note that $\Omega_{a,b}(\vec{u}) = \Omega_{b,a}(\vec{u}^\perp)$. A *corner point* of the billiard is one of the four vertices V_1, V_2, V_3, V_4 of the rectangle, namely,

$$\begin{aligned} V_1 &= (0, 0) \\ V_2 &= (b\|\vec{u}\| \cos \alpha_{\vec{u}}, b\|\vec{u}\| \sin \alpha_{\vec{u}}) \\ V_3 &= (b\|\vec{u}\| \cos \alpha_{\vec{u}} - a\|\vec{u}\| \sin \alpha_{\vec{u}}, b\|\vec{u}\| \sin \alpha_{\vec{u}} + a\|\vec{u}\| \cos \alpha_{\vec{u}}) \\ V_4 &= (a\|\vec{u}\| \sin \alpha_{\vec{u}}, a\|\vec{u}\| \cos \alpha_{\vec{u}}). \end{aligned}$$

We denote by C the set of corner points, and by $\partial\Omega_{a,b}(\vec{u})$ the boundary of the billiard table.

2.2. Switching components

A set of points $A \subset \mathbb{Z}^2$ is *horizontally convex* (resp. *vertically convex*) if, for each $p \in A$, the set $L_{\vec{h}}(p) \cap A$ (resp. $L_{\vec{v}}(p) \cap A$) is connected. A set that is both horizontally and vertically convex is shortly called *hv-convex*. A *polyomino* is a finite set of points in \mathbb{Z}^2 whose elements are 4-connected, namely, any two points of a polyomino can be connected by a path consisting of only horizontal and vertical steps (see [11] for additional details concerning polyominoes and [5] for classes characterizations and related enumerative results). When no confusion with previous notations arises, then (x_1, x_2) , and $[x_1, x_2]$, are also adopted without further explanation to mean, respectively, the open and the closed intervals having x_1, x_2 as endpoints.

A pair $S = (S^0, S^1)$ of sets of points is an *hv-switching* if $S^0 \cap S^1 = \emptyset$, $|S^0| = |S^1|$, and S^0, S^1 have the same number of points on each horizontal, and on each vertical line. Each set S^0 and S^1 is indicated as *hv-switching component*. A discrete set A contains an *hv-switching component* if $S^0 \subseteq A$ and $S^1 \cap A = \emptyset$. In this case, we consider $A = Y \cup S^0$, with Y being a (possibly empty) discrete set; we define the set $A' = Y \cup S^1$ as the *dual* of A , and we say that the *hv-switching* S is *associated* to A and A' . Switching components have a prominent role when studying discrete sets of points having the same number of elements lying on each horizontal and vertical line, one of the main topics in Discrete Tomography [13,14]. In case A, A' are dual *hv-convex polyominoes*, then S is said to be an *hv-convex switching component* (actually, the notion of *hv-convex switching component* can be slightly generalized, see [8, Definition 4]). In particular, an *hv-convex switching component* can be traveled by a sequence of horizontal and vertical steps. In case the clockwise, or the counterclockwise orientation changes during the walk, then the switching component is said to be a *curl* (an example of polyomino-classes characterizations and study according to different curls see [3]). As shown in [7,8], a curl can be uniquely associated to a couple of integers a and b that count the number of points that lie, respectively, on the NE-side, and on the SE-side of an *hv-convex polyomino*. To this, the notation $Curl(a, b)$ is adopted.

2.3. Integer orbits

A billiard ball is a point $(p, q) \in \mathbb{Z}^2$, moving along lattice lines. We also require that all the bouncing points of a billiard ball belong to the integer lattice \mathbb{Z}^2 . When the billiard ball hits the boundary of the billiard, then it changes its direction according to the mirror law, namely the angle of incidence equals the angle of reflection. Due to the integer constraint, the resulting path is called *integer orbit* of the billiard table. A *singular orbit* is an integer orbit that hits a vertex of the billiard. Any orbit of a billiard table $\Omega_{a,b}(\vec{u})$, that is not reduced to a single segment parallel to \vec{u} , hits the side of the billiard table containing \vec{u} in some points $x = (c\|\vec{u}\| \cos \alpha_{\vec{u}}, c\|\vec{u}\| \sin \alpha_{\vec{u}})$, where $c \in [0, b] \cap \mathbb{N}$, and $\alpha_{\vec{u}}$ is the smallest angle between \vec{u} and \vec{h} . In particular, when the billiard table is in canonical position, namely when $\vec{u} = \vec{h}$, then $x = (c, 0)$. Denote by $S_c(\vec{k})$ the orbit of $\Omega_{a,b}(\vec{h})$ passing through x , and having direction $\vec{k} = [k_1, k_2]$.

A point $(x_1, x_2) \in \partial\Omega_{a,b}(\vec{h})$ is a horizontal bouncing point of $S_c(\vec{k})$ if $x_2 = 0$ or $x_2 = a$. Analogously, it is a vertical bouncing point of $S_c(\vec{k})$ if $x_1 = 0$ or $x_1 = b$. Note that, the reflection law implies that, after the reflection in a bouncing point different from a corner point, the new direction of the orbit is of the form $\rho(\vec{k}) = [\pm k_1, \pm k_2]$, where the signs depend on the considered bouncing point. An orbit stops if it reaches a corner point, so identifying corner points with billiard pockets. In this case the orbit is said to be *singular*. An integer orbit is said to be *degenerate* if it reduces to a single segment, namely $\vec{k} \in \{\vec{h}, \vec{v}\}$ (if $\vec{k} = \vec{h}$, the notation $S_c(\vec{h})$ is still adopted, but c denotes the vertical coordinate of the two bouncing points). In case $S_c(\vec{k}) \cap C = \emptyset$, and the orbit is not degenerate, then $S_c(\vec{k})$ closes. Differently, we say that the sub-path of $S_c(\vec{k})$ starting at $(c, 0)$ is a *singular forward orbit*, since it consists of a portion of a singular orbit, the other portion being the *backward singular orbit* $S_c(v(\vec{k}))$, where $v(\vec{k}) = [-k_1, k_2]$. A non-singular and non degenerate orbit has $s \geq 4$ bounces. In case $s = 4$, the orbit is said to be an *elementary window*. We say that a non degenerate non singular integer orbit is *twisted* if it is different from an elementary window (see Fig. 1).

Let $S_c(\vec{k})$ be a non-degenerate integer orbit of a lattice billiard $\Omega_{a,b}(\vec{h})$. Suppose to select any bouncing point x of $S_c(\vec{k})$, and start traveling the orbit by moving from x clockwise (or counterclockwise). A bouncing point p of $S_c(\vec{k})$ is said to be a *counterclockwise point* if, crossing p , implies a counterclockwise change of direction. Differently, p is a *clockwise point*. So, visiting one vertex after the other, we can group them in ordered lists of consecutive clockwise and counterclockwise points.

Once an orientation of an integer orbit has been chosen, let p be a bouncing point where the orientation changes. Starting from p , compute the lengths L_1, L_2, \dots, L_n of the lists of consecutive vertices having, alternatively, the same and the opposite orientation as p . The resulting sequence of integers (L_1, L_2, \dots, L_n) is said to be the *structural sequence* associated to the orbit.

The structural sequence is considered cyclically arranged, namely, its first entry can be selected arbitrarily among L_1, \dots, L_n , and then listing the other entries by preserving the given order. If the sequence (L_1, L_2, \dots, L_n) is the s time repetition of a sequence $\pi = (L_1, \dots, L_{n'})$, then we denote (L_1, L_2, \dots, L_n) by $(\pi)_s$, being $n = n' \cdot s$; if $s = 1$, we omit such index.

Remark 1. In [8, Theorem 26], $Curl(a, b)$ has been shown to be the concatenation of $d = \gcd(a, b)$ equal *hv-convex paths*. By [8, Theorem 26] we also know that all *hv-convex path* of $Curl(a, b)$ have the same sequence of clockwise and counterclockwise alternation of their turning points. Such a sequence $(\pi)_2$ (denoted by π_2 in [8]) is obtained by repeating twice a *generating sequence* π , explicitly computed by means of [8, Algorithm CompSeq].

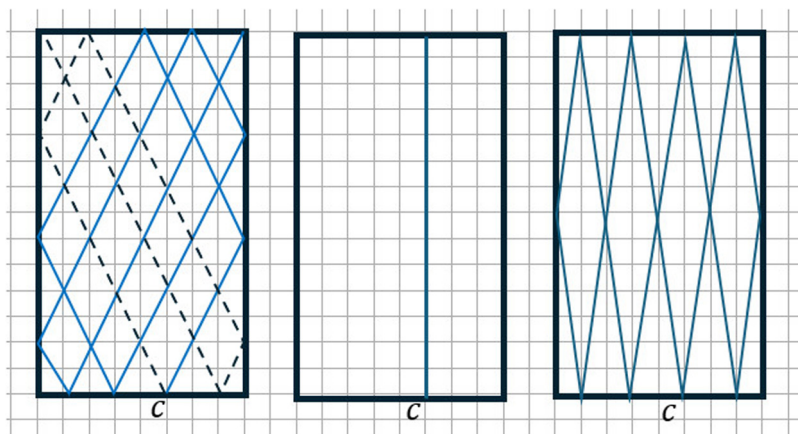


Fig. 1. Different orbits in the lattice billiard table $\Omega_{14,8}(\bar{h})$. Left: Singular orbit $S_c([1, 2])$. The solid line is the singular forward orbit, while the dashed line is the backward singular orbit. Middle; A degenerate orbit. Right: A twisted orbit.

2.4. Computation of the generating sequence

In Section 5, the sequence $(\pi)_2$ is explicitly used. Therefore, for the readers' convenience, we resume how the generating sequence π can be computed:

1. Label the points of $Curl(a, b)$ from 1 to $2(a + b)$, and identify each point with its label $(\text{mod } 2(a + b))$;
2. Define the horizontal operator $H : Curl(a, b) \rightarrow Curl(a, b)$, and the vertical operator $V : Curl(a, b) \rightarrow Curl(a, b)$, as follows:

$$H(x) = 2a + 1 - x \pmod{2a + 2b};$$

$$V(x) = 1 - x \pmod{2a + 2b};$$

3. Assume $h = \lfloor \frac{b}{a-b} \rfloor$, and compute $\alpha = 2h + 1$, $\gamma = 2h + 3$, $N_\alpha = 2(h + 1)a - 2(h + 2)b$, $N_\gamma = 2(h + 1)b - 2ha$. Then, by adopting the simplified continuous notation $(a, b]$, (instead of $(a, b] \cap \mathbb{N}$), define the following intervals of (integer) labels:

$$I_\alpha : \left(\frac{N_\gamma}{2}, \frac{N_\alpha + N_\gamma}{2} \right] \cup \left(b, \frac{N_\alpha}{2} + b \right] \cup \left(a + b + \frac{N_\gamma}{2}, a + b + \frac{N_\alpha + N_\gamma}{2} \right] \cup \left(a + 2b, \frac{N_\alpha}{2} + a + 2b \right]$$

$$I_\gamma : \left[1, \frac{N_\gamma}{2} \right] \cup \left[a + 1 - \frac{N_\gamma}{2}, a \right] \cup \left[a + b + 1, a + b + \frac{N_\gamma}{2} \right] \cup \left[2a + b + 1 - \frac{N_\gamma}{2}, 2a + b \right];$$

4. The generating sequence π , and consequently the sequence $(\pi)_2$, is then computed according to Algorithm *CompSeq* in [8], that we rephrase as Algorithm 1: *GenSeq* below.

Remark 2. In what follows, the term *integer* is sometimes omitted, but when we write *orbit* we always mean integer orbit.

Example 3. For the sake of clarity, we provide an example of how *CompSeq* runs using $a = 27$ and $b = 19$ as input parameters (see also [8, Example 27] for additional details). We have $d = 1$, $h = 2$, $\alpha = 5$, $\gamma = 7$, $N_\alpha = 10$, and $N_\gamma = 6$, and consequently

$$I_\alpha = (3, 8] \cup (19, 24] \cup (49, 54] \cup (65, 70] \text{ and } I_\gamma = [1, 3] \cup [25, 27] \cup [47, 49] \cup [71, 73].$$

The *for* loop provides (see Fig. 2)

$$i = 1: x = 1 \in I_\gamma \Rightarrow \pi(1) = 7. y = (HV)^3(1) = 71, i = 1 \text{ is odd, so } x = V(71) = 1 - 71 = -70 \pmod{92} = 22;$$

$$i = 2: x = 22 \in I_\alpha \Rightarrow \pi(2) = 5. y = (VH)^2(22) = 98 \pmod{92} = 6, i = 2 \text{ is even, so } x = H(6) = 55 - 6 = 49;$$

$$i = 3: x = 49 \in I_\gamma \Rightarrow \pi(3) = 7. y = (HV)^3(49) = 70 + 49 \pmod{92} = 27, i = 3 \text{ is odd, so } x = V(27) = 1 - 27 \pmod{92} = 66;$$

$$i = 4: x = 66 \in I_\alpha \Rightarrow \pi(4) = 5. y = (VH)^2(66) = 76 + 66 \pmod{92} = 50, i = 4 \text{ is even, so } x = H(50) = 55 - 50 = 5;$$

$$i = \dots :$$

The *for* loop runs from 1 to $\frac{N_\alpha + N_\gamma}{2} = 8$, returning the vector $\pi_8 = (7, 5, 7, 5, 7, 5, 7, 5)$. The final output is $\pi = (\pi_8)_2 = (7, 5, 7, 5, 7, 5, 7, 5, 7, 5, 7, 5, 7, 5, 7, 5)$.

For what concerns degenerate orbits, we have the following easy characterization.

Lemma 4. *An orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$ is degenerate if and only if $\vec{k} \in \{-b, a\}, [b, a]$ and $x = (c, 0) \in C$, or $\vec{k} \in \{\vec{h}, \vec{v}\}$ for all $x \in \partial\Omega_{a,b}(\vec{h})$,*

Proof. Suppose that $S_c(\vec{k})$ is a degenerate orbit, namely, a segment where $x \in \partial\Omega_{a,b}(\vec{h})$ is one of the endpoints. If the segment is not horizontal, or vertical, then its endpoints must be two opposite corner points, which implies that $x \in C$ and $\vec{k} \in \{-b, a\}, [b, a]$. Differently, the segment is horizontal, or vertical, and, due to the reflection law, it must reflect on itself. Therefore, the angle of incidence must be equal to $\frac{\pi}{2}$, so $\vec{k} \in \{\vec{h}, \vec{v}\}$.

Conversely, if $x \in C$, and $\vec{k} \in \{-b, a\}, [b, a]$, then $S_c(\vec{k})$ is the segment that joins two opposite corner points, so the orbit is degenerate. In case $\vec{k} \in \{\vec{h}, \vec{v}\}$, then the reflection law implies that, for all $x \in \partial\Omega_{a,b}(\vec{h})$, the orbit reduces to a single horizontal or vertical segment having x as an endpoint, so the orbit is still degenerate. \square

Due to Lemma 4, in what follows we can assume that $S_c(\vec{k})$ is a non-degenerate orbit, in particular that $\vec{k} \notin \{\vec{h}, \vec{v}\}$. This leads to the following characterization of the allowed non-degenerate integer orbits.

Theorem 5. *Let $x = (c_1, 0)$, for some $c_1 \in [0, b] \cap \mathbb{N}$ and $\vec{k} = [k_1, k_2]$, with $k_1, k_2 > 0$. Then, $S_{c_1}(\vec{k})$ is a non-degenerate integer orbit of $\Omega_{a,b}(\vec{h})$ if and only if k_1 divides c_1 and b , and k_2 divides a .*

Proof. Consider any non-degenerate integer orbit $S_{c_1}(\vec{k})$ of $\Omega_{a,b}(\vec{h})$, and suppose that $c_1 \neq b$. Then we can always assume that the orbit is walked moving from $x = (c_1, 0)$ in the direction of the positive x -axis, meaning that $k_1, k_2 > 0$. In case $c_1 = b$, then the same holds after translating along the vector $[b, 0]$ the lattice billiard table, so, up to such a preliminary translation, we can always assume $c_1 \neq b$. Therefore, the first bouncing point after $x = (c_1, 0)$ is of the form (b, q_1) , or of the form (c_2, a) , according as the slope of \vec{k} is lower than, or higher than $\frac{a}{b-c_1}$. In the first case, it is $q_1 = \frac{k_2(b-c_1)}{k_1}$, so k_1 divides $b - c_1$. In the second case, $c_2 = \frac{k_1}{k_2}a + c_1$, so k_2 divides a .

In the first case, let $p_1 = \frac{b-c_1}{k_1}$. After the first reflection on the line $x = b$, we have $n_1 \geq 0$ further reflections on the vertical sides, alternating on the lines $x = 0$ and $x = b$, where n_1 is the greatest integer such that $a - q_1 \geq n_1 b \frac{k_2}{k_1}$. Therefore, three cases arise.

- (a) $n_1 = 0$. In this case, after touching the vertical bouncing point (b, q_1) , and using the reflection law, the orbit reaches the bouncing point (c_2, a) , where $\frac{a-q_1}{b-c_2} = \frac{k_2}{k_1}$, namely $c_2 = b - \frac{k_1}{k_2}a + p_1 k_1$. In particular, k_2 divides a .
- (b) n_1 is even. This implies that the last vertical reflection before reaching the side lying on the line $y = a$ occurs on the line $x = b$, so, the orbit reaches the bouncing point (c_2, a) such that

$$\frac{a - q_1 - n_1 b \frac{k_2}{k_1}}{b - c_2} = \frac{k_2}{k_1},$$

so that $c_2 = (n_1 + 1)b - \frac{k_1}{k_2}a + p_1 k_1$, and k_2 divides a . Also, in this case k_1 divides b , so, since k_1 divides $b - c_1$, then k_1 divides c_1 .

- (c) n_1 is odd. This implies that the last vertical reflection before reaching the side on the line $y = a$ occurs on the line $x = 0$, so, the orbit reaches the bouncing point (c_2, a) such that

$$\frac{a - q_1 - n_1 b \frac{k_2}{k_1}}{c_2} = \frac{k_2}{k_1},$$

so that $c_2 = \frac{k_1}{k_2}a - p_1 k_1 - n_1 b$. Still, k_2 divides a , and k_1 divides b , so k_1 divides c_1 .

In any case, a bouncing point of the form (c_2, a) is obtained.

Now we repeat the above argument by moving from (c_2, a) , in direction $\rho(\vec{k})$, where

- $\rho(\vec{k}) = [k_1, -k_2]$ if $\frac{k_2}{k_1} > \frac{a}{b-c_1}$, or $\frac{k_2}{k_1} \leq \frac{a}{b-c_1}$ and n_1 is odd.
- $\rho(\vec{k}) = [-k_1, -k_2]$ in case $n_1 = 0$ or n_1 is even;

The new bouncing point on the horizontal line $y = 0$ is of the form $(c_3, 0)$, where c_3 is computed differently according as $\rho(\vec{k}) = [k_1, -k_2]$, or $\rho(\vec{k}) = [-k_1, -k_2]$. If $\rho(\vec{k}) = [k_1, -k_2]$, then c_3 is simply obtained as in the above formulas for c_2 , where p_1 is replaced by $p_2 = \frac{b-c_2}{k_1}$, q_1 by $q_2 = p_2 k_2$, and, when $q_2 \leq a$, then n_1 is replaced by the greatest integer n_2 such that $a - q_2 \geq n_2 b \frac{k_2}{k_1}$.

If $\rho(\vec{k}) = [-k_1, -k_2]$, then p_2 must be computed as $p_2 = \frac{c_2}{k_1}$, meaning that k_1 divides c_2 . Note that, in this case, the first possible vertical bouncing point is now on the line $x = 0$. Consequently, c_3 is computed by considering \pmod{b} the same formulas as above, once p_1, q_1, n_1 are replaced by p_2, q_2, n_2 , respectively, and by exchanging the cases corresponding to

the parity of n_1 when reflections on both vertical sides occur. Therefore, we get

$$\left\{ \begin{array}{ll} c_3 = -\frac{k_1}{k_2}a + c_2 & \text{in case } q_2 > a. \\ c_3 = \frac{k_1}{k_2}a - p_2k_1 & \text{in case } q_2 < a \text{ and } n_2 = 0. \\ c_3 = \frac{k_1}{k_2}a - p_2k_1 - n_2b & \text{in case } q_2 < a, \text{ and } n_2 \text{ is even.} \\ c_3 = (n_2 + 1)b - \frac{k_1}{k_2}a + p_2k_1 & \text{in case } q_2 < a, \text{ and } n_2 \text{ is odd.} \end{array} \right. \tag{1}$$

By repeating the argument we get that, after any horizontal bouncing point having c_i as the first coordinate, k_1 divides $b - c_i$, or k_1 divides c_i , according as the first component of the reflected direction is k_1 or $-k_1$. Since x is not a corner point, then the orbit can be traveled by moving from x in the direction having the first component either equal to k_1 , or to $-k_1$, so k_1 must divide both $b - c_i$ and c_i , so it divides also b . In any case k_2 divides a .

Conversely, suppose that k_1 divides c_1 and b , and k_2 divides a , for some choice of a, b, c_1 as stated. With the same argument as above we get that the coordinate c_{i+1} of the horizontal bouncing point found at the $(i + 1)$ -th step depends on the coordinate c_i of the horizontal bouncing point at the i th step as follows

- $c_{i+1} = \frac{\rho_{i1}}{k_2}a + c_i$ if $q_i > a$;
- $c_{i+1} = -\left(\frac{\rho_{i1}}{k_2}a + p_i k_1\right) \pmod{b}$ if $q_i \leq a$,

where $\rho_i = [\rho_{i1}, \rho_{i2}]$ is the moving direction computed after the i th step. Since k_1 divides c_1 and b , then k_1 divides $b - c_1$. Therefore, k_1 divides $b - c_2$ or c_2 , and, by iterating the argument, at each step we get that k_1 divides $b - c_i$ or c_i . But k_1 divides b , so, at each step, k_1 divides both c_i and $b - c_i$. Since k_2 divides a , then any horizontal bouncing point has integer coordinates.

Concerning the vertical bouncing points, the above arguments show that:

- if $\rho_{i1} = k_1$, and i is odd, then we have a first bouncing point of the form $v_0 = (0, q_i)$, while, if i is even, then $v_0 = (0, a - q_i)$. In addition, we get the other vertical bouncing points $v_{2j} = (0, q_i + 2jn_i)$, $v_{2j-1} = (b, (2j - 1)n_i)$, for $1 \leq j \leq n_i$;
- if $\rho_{i1} = -k_1$, and i is odd, then we have a first bouncing point of the form $v_0 = (b, q_i)$, while, if i is even, then $v_0 = (b, a - q_i)$. In addition, we get the other vertical bouncing points $v_{2j} = (b, q_i + 2jn_i)$, $v_{2j-1} = (0, (2j - 1)n_i)$, for $1 \leq j \leq n_i$.

Being q_i, n_i integer numbers, than any vertical bouncing point is a lattice point.

Consequently, all bouncing points have integer coordinates, so $S_{c_1}(k)$ is a non degenerate integer orbit. \square

For a given lattice point $x = (c, 0) \in \partial\Omega_{a,b}(\vec{h})$, a lattice direction $\vec{k} = [k_1, k_2]$ that satisfies all the assumptions of [Theorem 5](#), is said to be *admissible* for the lattice billiard $\Omega_{a,b}(\vec{h})$. We also say that the integer orbit $S_c(\vec{k})$ is admissible for the given lattice billiard. In particular, a singular orbit joins two different corner points, so any lattice billiard contains precisely two singular orbits for any admissible direction k . We define *k-singular path* the union of the two singular orbits in direction \vec{k} . Note that, for any $a, b \in \mathbb{N}$, $a \geq b$, the lattice direction $[1, 1]$ is always admissible, so [Theorem 5](#) includes as a special case the trajectories of any arithmetic billiard (see [\[19\]](#)).

As a matter of fact, [Theorem 5](#) provides an answer to the first part of [Question \(1\)](#).

4. Classification of integer orbits

In the previous section we have investigated when an orbit $S_c(\vec{k})$ is admissible in a given lattice billiard. Now, we wish to find conditions that allow to distinguish singular from twisted orbits. Also, in case the orbit is non singular, we want to understand when it is an elementary window.

Definition 6. An integer orbit $S_c(\vec{k})$ of a lattice billiard $\Omega_{a,b}(\vec{h})$ is said to be *equivalent* to an integer orbit $S_{c'}(\vec{k}')$ of a lattice billiard $\Omega_{a',b'}(\vec{h}')$, if $S_c(\vec{k}), S_{c'}(\vec{k}')$ are described by the same structural sequence.

For example, the orbits in [Fig. 3](#) are both described by the same sequence $\pi = (3, 3)$. Indeed, up to change the starting point, we can travel the orbits by turning 3 consecutive times clockwise, followed by 3 consecutive counterclockwise turns.

Theorem 7. Let $\vec{k} = [k_1, k_2]$ be an admissible direction for the lattice billiard $\Omega_{a,b}(\vec{h})$. Then, for any admissible integer orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$ there exists a lattice billiard $\Omega_{a',b'}(\vec{h}')$, and some integer orbit $S_{c'}([1, 1])$ in $\Omega_{a',b'}(\vec{h}')$, such that $S_{c'}([1, 1])$ is equivalent to $S_c(\vec{k})$.

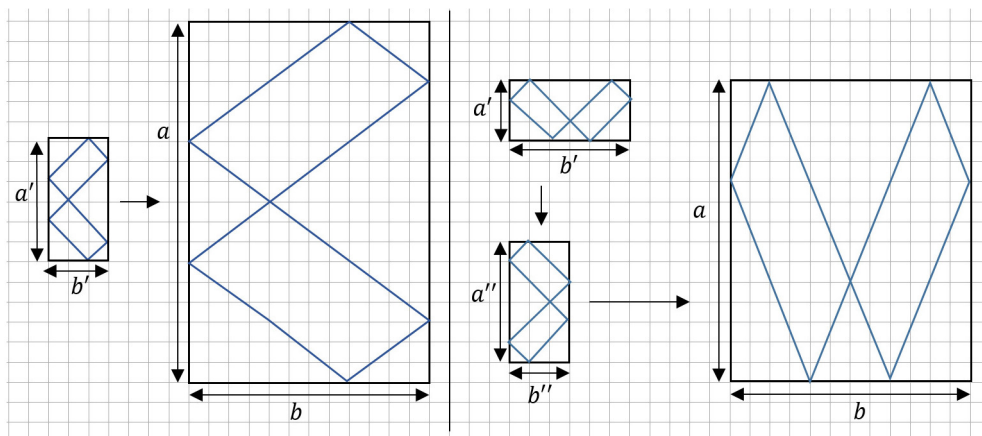


Fig. 3. Left: The orbit $S_8([4, 3])$ in the lattice billiard $\Omega_{18,12}(\vec{h})$. Right: The orbit $S_4([2, 5])$ in the lattice billiard $\Omega_{15,12}(\vec{h})$.

Table 1
Bouncing points in the $\Omega_{a,b}(\vec{h})$ billiard of the non singular orbit $S_c([1, 1])$ when $\frac{a}{d}$ and $\frac{b}{d}$ are odd.

Bottom side on $y = 0$	$x =$	$n2d + (-1)^n c$	$0 \leq n \leq \frac{b-d}{2d}$
Right side on $x = b$	$y =$	$n2d + d - c, \quad n2d + d + c$	$0 \leq n \leq \frac{a-d}{2d}$
Top side on $y = a$	$x =$	$n2d + d - c, \quad n2d + d + c$	$0 \leq n \leq \frac{b-d}{2d}$
Left side on $x = 0$	$y =$	$n2d + (-1)^n c$	$0 \leq n \leq \frac{a-d}{2d}$

Table 2
Bouncing points in the $\Omega_{a,b}(\vec{h})$ billiard of the non singular orbit $S_c([1, 1])$ when $\frac{a}{d}$ is even and $\frac{b}{d}$ is odd.

Bottom side on $y = 0$	$x =$	$n2d + (-1)^n c$	$0 \leq n \leq \frac{b-d}{2d}$
Right side on $x = b$	$y =$	$n2d + d - c, \quad n2d + d + c$	$0 \leq n \leq \frac{a-d}{2d}$
Top side on $y = a$	$x =$	$n2d + (-1)^n c$	$0 \leq n \leq \frac{b-d}{2d}$
Left side on $x = 0$	$y =$	$n2d + (-1)^n c$	$0 \leq n \leq \frac{a-d}{2d}$

Proof. Consider the following affine change of coordinates

$$\sigma \begin{cases} x' = \frac{x}{k_1} \\ y' = \frac{y}{k_2} \end{cases} \tag{2}$$

The transformation σ fixes the horizontal and the vertical directions, and changes the direction $\vec{k} = [k_1, k_2]$ in the direction $[1, 1]$. Also, the lattice points $V_2(b, 0)$ and $V_4(0, a)$ are changed, respectively, in the points $V'_2(\frac{b}{k_1}, 0)$ and $V'_4(0, \frac{a}{k_2})$. Since \vec{k} is admissible for $\Omega_{a,b}(\vec{h})$, then k_1 divides b , and k_2 divides a , so V'_2, V'_4 are still lattice points. In addition, since $c \in [0, b]$, then σ changes c to $c' = \frac{c}{k_1}$. By Theorem 5, k_1 divides c , so $(c', 0)$ is a lattice point. It follows that the orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$ becomes the orbit $S_{c'}([1, 1])$ of the lattice billiard $\Omega_{a',b'}(\vec{h})$, where $a' = \frac{a}{k_2}, b' = \frac{b}{k_1}$.

If $a' \geq b'$, then, by Theorem 5, σ provides a one-to-one correspondence between the bouncing points of $S_c(\vec{k})$ and of $S_{c'}([1, 1])$. In particular, on the horizontal and on the vertical sides of both billiards there is the same number of bouncing points. Since σ preserves the orientations, then the corresponding orbits have the same structural sequences, and consequently the two orbits are equivalent.

If $a' < b'$, then we apply a rotation of $\frac{\pi}{2}$, counterclockwise, which provides the lattice billiard $\Omega_{b',a'}(\vec{v})$, with $S_{c'}([1, 1])$ replaced by the admissible orbit $S_{c''}([1, 1])$, where $c'' = |a' - c'| \pmod{a'}$. Since structural sequences are cyclically arranged, then the orbit $S_{c'}([1, 1])$ of $\Omega_{a',b'}(\vec{h})$, and the orbit $S_{c''}([1, 1])$ of $\Omega_{b',a'}(\vec{v})$ are equivalent, and the statement follows also in this case. \square

Thanks to Theorem 7, in order to investigate the structure of any admissible orbit of any lattice billiard $\Omega_{a,b}(\vec{h})$, it suffices to focus on the orbits having direction $[1, 1]$. In [19], the authors explicitly computed the bouncing points' coordinates of the non singular orbit $S_c([1, 1])$ on a lattice billiard $\Omega_{a,b}(\vec{h})$, according to the parities of $\frac{a}{d}$ and $\frac{b}{d}$, being $d = \text{gcd}(a, b)$ and c varying from 1 and $d - 1$, i.e., being the first bouncing point on the bottom side. The values are listed in Table 1 and in Table 2. Note that both $\frac{a}{d}$ and $\frac{b}{d}$ cannot be even; we can assume w.l.g. that $\frac{b}{d}$ may be even.

Table 3

Computation of the bouncing points of the $\Omega_{18,12}(\vec{h})$ billiard w.r.t. $S_8([4, 3])$. The points are obtained after shifting the problem on the smallest billiard $\Omega_{6,3}(\vec{h})$ w.r.t. $S_2([1, 1])$. The point to point correspondence on each side is highlighted.

	$S_2([1, 1])$ in $\Omega_{6,3}(\vec{h})$		$S_8([4, 3])$ in $\Omega_{18,12}(\vec{h})$	
Bottom side	(2, 0)		(8, 0)	
Right side	(3, 1)	(3, 5)	(12, 3)	(12, 15)
Top side	(2, 6)		(8, 18)	
Left side	(0, 2)	(0, 4)	(0, 6)	(0, 12)

Table 4

Computation of the bouncing points of the integer orbit $S_{(4,0)}([2, 5])$ in the billiard $\Omega_{15,12}(\vec{h})$. The points are obtained after shifting the problem to the integer orbit $S_{(1,0)}([1, 1])$ in the smallest billiard $\Omega_{3,6}(\vec{h})$, then rotating by $\frac{\pi}{2}$ counterclockwise and obtaining the integer orbit $S_{(1,0)}([1, 1])$ in the billiard $\Omega_{6,3}(\vec{v})$.

	$S_1([1, 1])$ in $\Omega_{6,3}(\vec{v})$		$S_4([2, 5])$ in $\Omega_{15,12}(\vec{h})$	
Bottom side	(1, 0)		(4, 0)	(8, 0)
Right side	(3, 2)	(3, 4)	(12, 10)	
Top side	(1, 6)		(2, 15)	(10, 15)
Left side	(0, 1)	(0, 5)	(0, 10)	

So, instead of using the iterative construction exploited in the proof of [Theorem 5](#), the coordinates of the bouncing points of a generic non singular orbit $S_c(\vec{k})$ of a lattice billiard $\Omega_{a,b}(\vec{h})$, can be simply computed by applying [Theorem 7](#), then using [Tables 1](#) and [2](#), and then coming back to the original billiard table by multiplying the two coordinates of each bouncing point by k_1 , and by k_2 , respectively.

Example 8. Let us consider the lattice billiard $\Omega_{18,12}(\vec{h})$, and $c = 8$ be the starting bottom side position of a bouncing point that moves along the lattice direction $\vec{k} = [4, 3]$.

According to [Theorem 7](#), we shift the problem on $\Omega_{a',b'}(\vec{h})$ such that $a' = \frac{a}{k_2} = 6$, $b' = \frac{b}{k_1} = 3$, $c' = \frac{c}{k_1} = 2$ and to its non singular orbit $S_c([1, 1])$.

In particular, $d' = \gcd(a', b') = 3$, $\frac{a'}{d'} = 2$ is even, and $\frac{b'}{d'} = 1$ is odd, so, by computing the bouncing values using [Table 2](#) with a', b', c', d' , we get the coordinates in [Table 3](#), second column.

Now, we move back to the initial billiard by magnifying the a' sides (and the lying bouncing points accordingly) by a factor $k_2 = 3$, and both the b' sides and c' by a factor $k_1 = 4$. The resulting bouncing points are in [Table 3](#), third column. [Fig. 3](#), on the left, depicts the situation.

Example 9. We focus on the lattice billiard be $\Omega_{15,12}(\vec{h})$, $c = 4$ and the non singular orbit $S_c([2, 5])$. Again [Theorem 7](#) allows us to shift the problem to $\Omega_{a',b'}(\vec{h})$ such that $a' = \frac{a}{k_2} = 3$, $b' = \frac{b}{k_1} = 6$, $c' = \frac{c}{k_1} = 2$ and to its non singular orbit $S_{c'}([1, 1])$. Now, $a' < b'$, so, as in the proof of [Theorem 7](#), we consider a counterclockwise $\frac{\pi}{2}$ -rotation of the billiard table $\Omega_{3,6}(\vec{h})$, so obtaining the billiard table $\Omega_{6,3}(\vec{v})$. Now, with $a'' = 6$, $b'' = 3$, and $c'' = |a' - c'| \pmod{a'} = 1$, we compute $d'' = \gcd(a'', b'') = 3$, so that $\frac{a''}{d''} = 2$ is even, and $\frac{b''}{d''} = 1$ is odd. Therefore, [Table 2](#), with a'', b'', c'', d'' can be applied, so obtaining the values in [Table 4](#), second column. [Fig. 3](#), on the right, shows the situation.

Moving to the original billiard, we apply the following transformation: first the billiard table is clockwise rotated in order to get the original position of the sides. Finally, the resulting coordinates are multiplied by $k_1 = 2$, and by $k_2 = 5$, respectively, which provides the bouncing points of the orbit $S_c([2, 5])$ of $\Omega_{15,12}(\vec{h})$ as in [Table 4](#), third column.

Note that, by [Theorem 5](#), the direction $[1, 1]$ is always admissible. The following theorem provides a characterization of singular orbits.

Theorem 10. Let $\vec{k} = [1, 1]$, and let $S_c(\vec{k})$ be a non degenerate orbit of $\Omega_{a,b}(\vec{h})$. Then, $S_c(\vec{k})$ is singular if and only if $\gcd(a, b)$ divides c .

Proof. Since $S_c(\vec{k})$ is not reduced to a single horizontal segment, then bouncing points on both the horizontal sides of the lattice billiard exist.

Suppose that $c \neq b$.

Setting $c = c_1$, and moving from $(c_1, 0)$, then the second horizontal point is of the form (c_2, a) , and it is reached after a last bounce on one of the vertical sides. For some $m \in \mathbb{N}$, it results $a = b - c_1 + mb + c_2$ or $a = b - c_1 + mb + b - c_2$, according as the last vertical bounce is on the line $x = 0$, or $x = b$ (see [Fig. 4](#)).

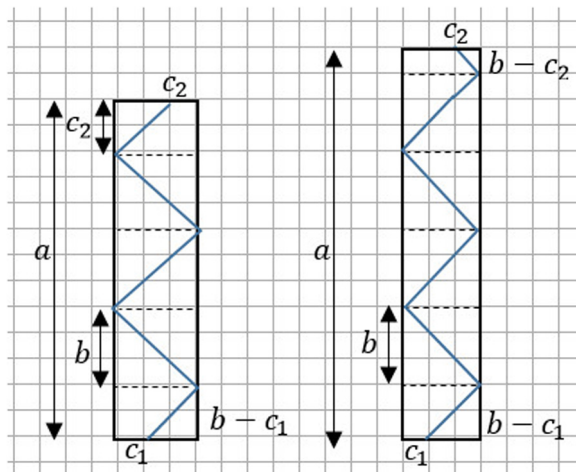


Fig. 4. Consecutive horizontal bouncing points of an orbit in direction $\vec{k} = [1, 1]$, in two different lattice billiards.

Therefore, we have $c_2 = \pm(a + c_1) \pmod{b}$. By iterating the argument, the first coordinate c_i of the i th horizontal bouncing point of $S_c(\vec{k})$ is such that

$$c_i = \pm(a + c_{i-1}) = \pm(a \pm (a + c_{i-2})) = \dots = pa \pm c_1 \pmod{b},$$

for some $p \in \mathbb{Z}$. The orbit $S_c(\vec{k})$ is singular if and only if there exists i such that $c_i = 0 \pmod{b}$, that is

$$pa \pm c_1 = 0 \pmod{b}. \tag{3}$$

Then (3) can be written as

$$pa + tb = \pm c_1, \tag{4}$$

for some $t \in \mathbb{Z}$. It is well known (see, for instance [4]) that a modular congruence $\alpha p + \beta t = \gamma$, with $\alpha, \beta, \gamma \in \mathbb{Z}$ has solution if and only if γ is a multiple of the greatest common divisor between α and β . Therefore, Eq. (4) has solution if and only if $\gcd(a, b)$ divides c_1 . Since the starting horizontal bouncing point of $S_c(\vec{k})$ has been chosen arbitrarily, then, by the same argument we get that $\gcd(a, b)$ must divide the first coordinate of any horizontal bouncing point, and the statement follows.

Now, suppose that $c = b$, then $S_c(\vec{k})$ is singular, and just the backward part is contained in $\Omega_{a,b}(\vec{h})$. Since $S_c(\vec{k})$ is not degenerate, then a bouncing point exists of the form (c_2, a) , with $c_2 \notin \{0, b\}$. Consequently, a bouncing point exists of the form $(c_1, 0)$, with $c_1 \neq b$, so that $S_c(\vec{k}) = S_{c_1}(\vec{k})$. Therefore, the statement follows by the same previous argument. \square

Remark 11. By the proof of Theorem 10 it follows that, in order to show that $\gcd(a, b)$ divides the first coordinate of any horizontal bouncing point, it suffices to check if $\gcd(a, b)$ divides the first component of an arbitrarily selected bouncing horizontal point of the orbit.

Corollary 12. Let $\vec{k} = [k_1, k_2]$ be an admissible direction of $\Omega_{a,b}(\vec{h})$, and let $S_c(\vec{k})$ be a non degenerate orbit of $\Omega_{a,b}(\vec{h})$. Then, $S_c(\vec{k})$ is singular if and only if $\gcd\left(\frac{a}{k_2}, \frac{b}{k_1}\right)$ divides $\frac{c}{k_1}$.

Proof. By Theorem 7, $S_c(\vec{k})$ is singular if and only if $S_{c'}([1, 1])$ is singular in $\Omega_{a',b'}(\vec{h})$, where $a' = \frac{a}{k_2}$, $b' = \frac{b}{k_1}$, $c' = \frac{c}{k_1}$. By Theorem 10, $S_{c'}([1, 1])$ is singular if and only if $\gcd(a', b')$ divides c' , and the statement follows. \square

Remark 13. If $\gcd\left(\frac{a}{k_2}, \frac{b}{k_1}\right) = 1$, then any non degenerate orbit $S_c([k_1, k_2])$ is singular. Also, any orbit of the lattice billiard $\Omega_{a,b}(\vec{h})$ is singular if and only if $\gcd(a, b) = 1$, so, in this case, only the singular path exists.

Example 14. Since the proof of Corollary 12 bases on Theorem 7, then, in some cases, the correspondence between an orbit $S_c([k_1, k_2])$ in $\Omega_{a,b}(\vec{h})$, and an orbit $S_{c'}([1, 1])$ in $\Omega_{a',b'}(\vec{h})$, could be obtained up to a $\frac{\pi}{2}$ counterclockwise rotation (as shown in Fig. 3), in order to have a longer vertical side. For example, if b is even, $[k_1, k_2] = \left[\frac{b}{2\gcd(a,b)}, \frac{a}{\gcd(a,b)}\right]$, and $c = 0$, then the set of bouncing points of $S_c([k_1, k_2])$ is $\{(0, 0), \left(\frac{b}{2}, a\right), (b, 0)\}$. Since $a' = \frac{a}{k_2} = \gcd(a, b)$, and $b' = \frac{b}{k_1} = 2\gcd(a, b) = 2a'$,

then after a $\frac{\pi}{2}$ counterclockwise rotation we get $\left\{ (0, 0), (a', a'), (0, \frac{b'}{2}) \right\}$, which is the set of bouncing points of the orbit $S_{c'}([1, 1])$.

We can now provide a characterization of the elementary windows.

Theorem 15. Let $\vec{k} = [k_1, k_2]$ be an admissible direction of a lattice billiard $\Omega_{a,b}(\vec{h})$. Then, an admissible orbit $S_c(\vec{k})$ is an elementary window if and only if $\frac{a}{b} = \frac{k_2}{k_1}$, and $\gcd(a, b) \neq 1$.

Proof. Assume that $S_c(\vec{k})$ is an elementary window. Then, by Remark 13, $\gcd(a, b) \neq 1$. Up to an affine transformation σ as in (2), we can consider $\vec{k} = [1, 1]$ in the lattice billiard $\Omega_{a',b'}(\vec{h})$, with $a' = \frac{a}{k_2}$, $b' = \frac{b}{k_1}$. Let $S_c([1, 1])$ be an elementary window in $\Omega_{a',b'}(\vec{h})$. If $a' = 1$, or $b' = 1$, then $k_2 = a$, or $k_1 = b$, meaning that $S_c(\vec{k})$ is singular, a contradiction. Therefore, we have $a', b' > 1$. Setting $c = c_1$, with $c_1 \in (0, b') \cap \mathbb{N}$, then the other bouncing points of $S_c([1, 1])$ are $v_1 = (b', b' - c_1)$, $h_2 = (2b' - a' - c_1, a')$, and $v_2 = (0, 2a' - 2b' + c_1)$. Since $S_c([1, 1])$ is an elementary window, then, by continuing in traveling the orbit, the bouncing point after v_2 must coincide with h_1 , so it must be $2a' - 2b' + c_1 = c_1$, and consequently, $a' = b'$, that is $\frac{a}{b} = \frac{k_2}{k_1}$.

Conversely, assume that $\frac{a}{b} = \frac{k_2}{k_1}$, and $\gcd(a, b) \neq 1$. Therefore, $a' = \frac{a}{k_2} = \frac{b}{k_1} = b'$, and, being $\gcd(a, b) \neq 1$, by Remark 13 there exist non singular orbits. Consider a non singular orbit $S_{c_1}([1, 1])$ in $\Omega_{a',b'}(\vec{h})$, $c_1 \in (0, b') \cap \mathbb{N}$. Then $a' - (b' - c_1) = c_1 < b'$, so that the bouncing point after $v_1 = (b', (b' - c_1))$ is the horizontal bouncing point $h_2 = (b' - c_1, a')$, and the bouncing point after h_2 is the vertical bouncing point $v_2 = (0, c_1)$. Consequently, the bouncing point after v_2 coincides with h_1 , and the orbit is an elementary window. \square

The second part of Question (1) can be now answered. By the previous results, the possible structure of an integer orbit $S_c(\vec{k})$, $\vec{k} = [k_1, k_2]$, in a lattice billiard table $\Omega_{a,b}(\vec{h})$, can be summarized as follows

1. $S_c(\vec{k})$ is degenerate if and only if $\vec{k} \in \{\vec{h}, \vec{v}, [a, b], [b, a]\}$.
2. $S_c(\vec{k})$ is admissible if and only if k_1 divides c and $b - c$, and k_2 divides a (see Theorem 5),
3. $S_c(\vec{k})$ is singular if and only if it is admissible, and $\left(\frac{a}{k_2}, \frac{b}{k_1}\right)$ divides $\frac{c}{k_1}$ (see Theorem 10, and Corollary 12),
4. $S_c(\vec{k})$ is an elementary window if and only if it is admissible, $\frac{a}{b} = \frac{k_2}{k_1}$, and $\gcd(a, b) \neq 1$ (see Theorem 15).
5. In any other case an admissible orbit is twisted.

5. Number of bouncing points and structural sequence of a generic integer orbit

In this section we provide a quantitative description of how an admissible orbit $S_c(\vec{k})$ is displaced in a lattice billiard $\Omega_{a,b}(\vec{h})$. In particular, we would like to compute the total number of bouncing points, their coordinates, as well as to determine the structural sequence of $S_c(\vec{k})$, from the only knowledge of \vec{k}, a, b .

As a first step, we introduce the following definition.

Definition 16. The path between two consecutive horizontal bouncing points of an orbit is said to be a *Z-path* of the orbit. Excluding its endpoints, any Z-path consists of $q + 1$ vertical points having alternating orientations, where $q \geq -1$ is said to be the *level* of the Z-path.

So an orbit is a concatenation of Z-paths. Therefore, in order to investigate the geometric structure of an orbit, is important to characterize the possible different lengths of its Z-paths.

Since orbits are closed paths, then for any admissible direction \vec{k} , there exists an integer number $n = n(\vec{k})$ such that any non singular orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$, has the same number n of bouncing vertical points on both vertical sides of $\Omega_{a,b}(\vec{h})$. We wish now to investigate how these bouncing points distribute on each orbit.

Lemma 17. Let $\vec{k} = [k_1, k_2]$ be an admissible direction of $\Omega_{a,b}(\vec{h})$. Then there exists an integer $n(\vec{k})$, depending only on \vec{k} , such that any Z-path of any non singular orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$, has level $q \in \{2n(\vec{k}) - 1, 2n(\vec{k})\}$.

Proof. Let P be a generic Z-path of the orbit $S_c(\vec{k})$, and let n, m be the number of bouncing points of P on the left, respectively the right, vertical side. Assume that $n, m > 0$. Let $v_b(L), v_t(L)$ (possibly $v_b(L) = v_t(L)$) be, respectively, the lowest and the highest bouncing point of P lying between the corner points V_1, V_4 , namely on the left vertical side of $\Omega_{a,b}(\vec{h})$. Analogously, let $v_b(R), v_t(R)$ (possibly $v_b(R) = v_t(R)$) be, respectively, the lowest and the highest bouncing point of P lying between the corner points V_2, V_3 , namely on the right vertical side of $\Omega_{a,b}(\vec{h})$. Denote by L_1, L_2 the distances between $v_b(L)$ and V_1 , and between $v_t(L)$ and V_4 , respectively. Analogously, let R_1, R_2 be the distances between $v_b(R)$ and V_2 , and between $v_t(R)$ and V_3 , respectively. Then we have

$$a = (n - 1)2b \frac{k_2}{k_1} + L_1 + L_2 = (m - 1)2b \frac{k_2}{k_1} + R_1 + R_2,$$

so that

$$|m - n|2b \frac{k_2}{k_1} = |L_1 - R_1 + L_2 - R_2| \leq |L_1 - R_1| + |L_2 - R_2|.$$

Since, independently of the Z -path, $|L_1 - R_1| = |L_2 - R_2| = b \frac{k_2}{k_1}$, then $|n - m| \leq 1$. The level of the Z -path is $q = n + m - 1$, so, if we assume $n(\vec{k}) = \min\{n, m\}$, then $q \in \{2n(\vec{k}) - 1, 2n(\vec{k})\}$, and the statement follows.

In case $n = 0$ (resp. $m = 0$), then, of course, $m \in \{0, 1\}$ (resp. $n \in \{0, 1\}$), and the statement holds for $n(\vec{k}) = 0$. \square

Corollary 18. *The only integer orbits that can be traveled without changing the orientation are all elementary windows or, possibly, singular orbits.*

Proof. Let $S_c(\vec{k})$ be an orbit of a lattice billiard $\Omega_{a,b}(\vec{h})$ that can be traveled without changing the orientation. Then the orbit must be a concatenation of Z -paths such that $n(\vec{k}) = 0$. By using Eq. (2) we can assume $\vec{k} = [1, 1]$ in the lattice billiard $\Omega_{a',b'}(\vec{h})$, with $a' = \frac{a}{k_2}$, $b' = \frac{b}{k_1}$. Since $n([1, 1]) = 0$, then the bouncing points of the orbit alternate between horizontal and vertical ones. Starting from the horizontal bouncing point $h_1 = (c, 0)$, the orbit touches the billiard boundary in bouncing points v_1, h_2, v_2 , and then reaches a bouncing point $h_3 = (c', 0)$. If $c' = c$, then the orbit is an elementary window. Differently $h_3 \neq h_1$, and, without loss of generality, we can assume that $c' > c$. Since the orientation does not change, then by continuing in traveling the orbit, the sequences of the first coordinate of the horizontal bouncing points is increasing on the x -axis, and decreasing on the line $y = a$. Therefore, after a finite number of bounces, a corner is reached, so a singular forward orbit is obtained. In case the opposite orientation is preserved when traveling also the singular backward orbit from $x = (c, 0)$ to the corresponding corner point, then a same orientation is preserved along the whole singular orbit. \square

Remark 19. In general, when traveling a singular orbit the orientation may change.

Since an elementary window consists of just four bouncing points having the same orientation, then the structural sequence of an elementary window is simply $\pi = (4)$, and, by Corollary 18, the elementary windows are the only non-singular integer orbits whose structural sequence reduces to a single entry.

In view of the computation of the number of bouncing points of any other non degenerate integer orbit, and of the characterization of the related structural sequence, we need a few preliminary results.

Lemma 20. *Let $a, b \in \mathbb{N}$, $a > b$, and let \vec{w} be an admissible direction of $\Omega_{a,b}(\vec{h})$. Then, for any admissible orbit $S_c(\vec{w})$ of $\Omega_{a,b}(\vec{h})$, and for any lattice direction $\vec{u} = [u_1, u_2]$, there exists an equivalent orbit $S_{c'}(\vec{h})$ in $\Omega_{a,b}(\vec{u})$*

Proof. Since $\vec{u} = [u_1, u_2]$ is a lattice direction, then also $\vec{u}^\perp = [u_2, -u_1]$ is a lattice direction. Therefore, by a rotation, $\Omega_{a,b}(\vec{h})$ can be mapped to a rectangle having sides in directions \vec{u} and \vec{u}^\perp . By means of a dilatation $\sigma_{\vec{u}}$ of length $\|\vec{u}\| = \sqrt{u_1^2 + u_2^2}$, the lattice billiard $\Omega_{a,b}(\vec{u}) = \sigma_{\vec{u}}(\Omega_{a,b}(\vec{h}))$, is obtained. Since the resulting transformation preserves intersecting lines, then the orbit $S_c(\vec{w})$ in $\Omega_{a,b}(\vec{h})$, passing through $x = (c, 0)$, and the orbit $S_c(\sigma_{\vec{u}}(\vec{w}))$ in $\Omega_{a,b}(\vec{u})$, passing through the point $\sigma_{\vec{u}}(x) = (c\|\vec{u}\| \cos \alpha_{\vec{u}}, c\|\vec{u}\| \sin \alpha_{\vec{u}})$, with α integer, have the same number of bouncing points on corresponding sides of the two billiards. In addition, the clockwise and counterclockwise orientation of the bouncing points is also preserved by $\sigma(\vec{u})$. Therefore, the two orbits are equivalent. \square

Corollary 21. *Let $a, b \in \mathbb{N}$, $a > b$, and let \vec{k} be an admissible direction of $\Omega_{a,b}(\vec{h})$. Then, the structural sequence of any admissible integer orbit $S_c(\vec{k})$ of $\Omega_{a,b}(\vec{h})$, is the same as the structural sequence of an orbit $S_{c'}(\vec{h})$ in $\Omega_{a',b'}([1, -1])$, where $a' = \frac{a}{k_2}$, $b' = \frac{b}{k_1}$.*

Proof. By Theorem 7, the orbit $S_c(\vec{k})$ in $\Omega_{a,b}(\vec{h})$ is equivalent to some orbit $S_{c'}([1, 1])$ in the lattice billiard $\Omega_{a',b'}(\vec{h})$. Assuming $w = [1, 1]$, and $\vec{u} = [1, -1]$, by Lemma 20 there exists an orbit $S_{c'}(\vec{h})$ in $\Omega_{a',b'}([1, -1])$ having the same structural sequence of $S_{c'}(\vec{w})$ in $\Omega_{a',b'}(\vec{h})$. \square

Remark 22. In order to have an easier way of computing the number of bouncing points of an integer orbit, we now adopt the usual strategy of unfolding the billiard $\Omega_{a,b}(\vec{h})$ where $a = qb + s$, $0 \leq s < b$ (see, for instance [2,12]). In particular, thanks to Theorem 7, we can focus on the orbits in direction $\vec{k} = [1, 1]$ (see Fig. 5). By unfolding the billiard, the vertical and the horizontal bouncing points of the orbit $S_c([1, 1])$ (apart from their exact positions), can be represented as the intersections of the line $y = x - c$ with the lines $x = pb$ and $y = ma$, respectively, where $p, m \geq 0$. In particular, we call *vertical intersections* of the orbit the points intersected on the lines $x = pb$, which correspond to vertical bouncing points placed on $x = 0$ for p odd, and on $x = b$ for p even. Analogously, we call *horizontal intersections* of the orbit the points intersected on the lines $y = ma$, which correspond to the bouncing points on the horizontal sides of the original billiard, placed on the lines $y = 0$, or $y = a$, according as m is even or odd, respectively.

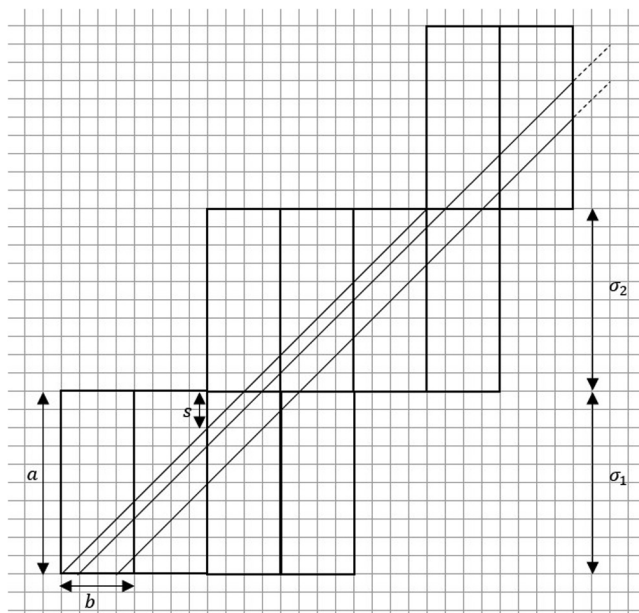


Fig. 5. Representing the orbits as lines in an unfolded billiard. The bottom-left rectangle corresponds to the original billiard table. Here, $a = 10, b = 4$, so $q = 2, s = 2, r = 2$. Three orbits are shown, the upper one being singular.

Lemma 23. Let $a = qb + s, 0 \leq s < b$, and let r be the smallest integer such that $sr = 0 \pmod{b}$. If r is even, then $\frac{rs}{b}$ is an odd integer.

Proof. Since $sr = 0 \pmod{b}$, and r is even, then b is even. So, it holds $b = 2^\alpha \beta$, with $\alpha, \beta \in \mathbb{N}, \beta$ is odd, and by hypothesis, $r = 2^{\alpha'} \beta'$, where β' is odd. Since sr is the minimum multiple of b involving s , we have that $s = 2^{\alpha - \alpha'} \beta''$, with β'' odd. Consequently $\frac{rs}{b}$ is odd. \square

Remark 24. Let $a = qb + s, 0 \leq s < b$, and let r be the least integer such that $sr = 0 \pmod{b}$. It holds that $\gcd(a, b) = \frac{b}{r}$.

Proof. By hypothesis it holds that $ra = 0 \pmod{b}$, so there exists $t > 0$ such that $ra = tb$. By the minimality of r we have $t = \frac{a}{\gcd(a,b)}$, and, consequently, $r = \frac{b}{\gcd(a,b)}$ \square

Theorem 25. In any lattice billiard $\Omega_{a,b}(\vec{h})$ the number of bouncing points of an admissible twisted orbit $S_c([1, 1])$ is twice the number of bouncing points of an admissible singular orbit having the same direction.

Proof. Let $a = qb + s, 0 \leq s < b$, and let r be the least integer such that $sr = 0 \pmod{b}$. Consider a singular integer orbit $S_c([1, 1])$. Since the unfolded billiard is invariant by a translations of the vectors $[b, 0], [0, a]$ or $[b, a]$, then, up to such a translation, we can always assume that the singular orbit starts at $c = 0$, so it is represented by the line $y = x$. The second endpoint of the singular orbit corresponds to the first corner point reached by the line $y = x$ in the unfolded billiard, and consequently, it is of the form (ta, ta) for some integer $t > 0$. A corner point in the unfolded billiard has coordinates $(0 \pmod{b}, ta)$, so it must be $ta = 0 \pmod{b}$. Since $a = bq + s$, then $ts = 0 \pmod{b}$, and consequently $t = r$. Therefore, the bouncing points of the singular orbit correspond (up to sign exchange of the coordinates when needed) to the horizontal and to the vertical intersections of the line $y = x$ in the set of horizontal strips $\sigma_m, 0 \leq m \leq r$.

Now, consider a twisted integer orbit $S_c([1, 1])$, and displace the orbit as the line $y = x - c$ in the unfolded billiard. For $y = ra$ the line reaches the point $(ra + c, ra)$.

If r is odd, this corresponds to a bouncing point placed on the side $y = a$ of the billiard table $\Omega_{a,b}(\vec{h})$, different from the starting point $x = (c, 0)$, meaning that the orbit is not completed at this point. By continuing the line $y = x - c$ across the strips σ_m , for $r < m \leq 2r$, the point $(2ra + c, 2ra)$ is reached. Note that $2ra = 2b(rq + \frac{rs}{b})$, so, by Remark 22, the line $x = 2ra$ corresponds to the side of $\Omega_{a,b}(\vec{h})$ placed on the line $x = 0$. Therefore, the point $(2ra + c, 2ra)$ corresponds to the bouncing point $(c, 0)$ and the orbit $S_c([1, 1])$ closes.

If r is even, then the point $(ra + c, ra)$ corresponds to a bouncing point placed on the side $y = 0$ of the billiard table $\Omega_{a,b}(\vec{h})$. By Lemma 23, $\frac{rs}{b}$ is odd. Therefore, by Remark 22, the line $x = ra$ corresponds to the side of $\Omega_{a,b}(\vec{h})$ placed on

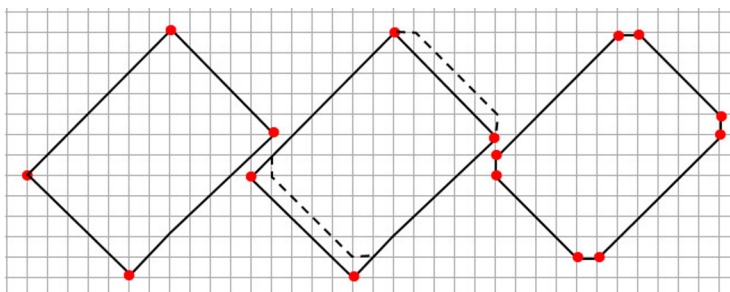


Fig. 6. Changing a rectangular lattice billiard $\Omega_{a,b}([1, -1])$ in a lattice octagon $\Omega_{(a,2,b,2)}([1, -1])$, where $a = 7, b = 5$. Left: The original billiard table. Middle: Reducing the size and doubling the vertices. Right: The resulting lattice octagon.

the line $x = b$. Consequently, the point $(ra + c, ra)$ corresponds to the bouncing point $(b - c, 0)$, meaning that the orbit $S_c([1, 1])$ is not completed at this point. By continuing the line $y = x - c$ across the strips σ_m , for $r < m \leq 2r$, the point $(2ra + c, 2ra)$ is reached. As in the case when r is odd, it corresponds to the bouncing point $(c, 0)$ and the orbit $S_c([1, 1])$ closes.

In any case, the line $y = x - c$ covers $2r$ strips, so the number of bouncing points of the twisted orbit is twice the number of bouncing points of the singular orbit, and the statement follows. \square

Corollary 26. Let $\Omega_{a,b}(\vec{h})$ be a lattice billiard, where $a = qb + s, 0 \leq s < b$. Any admissible singular orbits have $N = \frac{a+b}{\gcd(a,b)}$ bouncing points, while admissible twisted orbits have $2N$ bouncing points,

Proof. Let r be as in Remark 24. By the proof of Theorem 25, it follows that the line $y = x$ has $rq + \frac{rs}{b}$ vertical intersections in the union of the r strips $\sigma_m, 0 \leq m \leq r$. In addition, there exist r horizontal intersections. Since the line $y = x$ corresponds to a singular orbit, then, by Remark 22, the number of bouncing points is $N = rq + r + \frac{rs}{b} = \frac{r(a+b)}{b} = \frac{a+b}{\gcd(a,b)}$, which proves the statement for admissible singular orbits, and, by Theorem 25, also for admissible twisted orbits. \square

Remark 27. If $\gcd(a, b) = 1$, then $r = b$, so $N = a + b$. By Remark 13, since $\gcd(a, b) = 1$, then for any admissible direction \vec{k} there exist only the two corresponding singular orbits. Therefore, in this case, the k -singular path visits all the points of the billiard table, so it can be considered as a kind of ergodic property from the integer point of view.

Theorem 25 and Corollary 26 provide answers to Question (2). We now focus to Question (3), coming back to Corollary 26 after determining the characterization of structural sequences of integer orbits.

Lemma 28. For any $a, b \in \mathbb{N}, a > b$, and any admissible direction $\vec{k} = [k_1, k_2]$, there exist precisely $d' + 1$ different integer orbits $S_c(\vec{k})$ in $\Omega_{a,b}(\vec{h})$, where $d' = \gcd\left(\frac{a}{k_1}, \frac{b}{k_2}\right)$.

Proof. By a cardinality argument, we divide the total number of points of $\partial\Omega_{a',b'}(\vec{h})$, by the number of points of each non singular orbit. Precisely, we get $2(a' + b') \cdot \frac{b'}{2r(a'+b')} = \frac{b'}{r}$. By Remark 24 it holds $\frac{b'}{r} = \gcd(a', b') = d'$. Since one of these orbits is a singular path, it splits into the two singular orbits, reaching the total orbits' number $d' + 1$. \square

Remark 29. The result stated in Lemma 28 can also be obtained by Theorem 7: for any integer orbit $S_c(\vec{k}), 0 \leq c \leq b$ in $\Omega_{a,b}(\vec{h})$, there exists an equivalent integer orbit $S_{c'}([1, 1])$ in $\Omega_{a',b'}(\vec{h})$, where $a' = \frac{a}{k_2}, b' = \frac{b}{k_1}, c' = \frac{c}{k_1}$. It is known that in any arithmetic billiard $\Omega_{a',b'}(\vec{h})$ there exist precisely $d' - 1$ closed integer orbits in direction $[1, 1]$, where $d' = \gcd(a', b')$ (see, for instance, [19]). By adding the two singular orbits, the statement follows.

Lemma 30. For any $a, b, c \in \mathbb{N}, a > b, 0 \leq c \leq b$, the set consisting of all integer orbits $S_c(\vec{h})$ of the billiard table $\Omega_{a,b}([1, -1])$, together with the h -singular path, corresponds one-to-one to the set of $h\nu$ -convex paths of $\text{Curl}(a, b)$.

Proof. Reduce by one, in the $[1, -1]$ direction, the size of the billiard table, then replace the bottom and the upper corner points by a pair of horizontally adjacent lattice points, as well as replace the left and the right corner points by a pair of vertically adjacent lattice points. This changes $\Omega_{a,b}([1, -1])$ in a lattice octagon $\Omega_{(a,2,b,2)}([1, -1])$, having two small vertical sides, and two small horizontal sides, both with two lattice points, two sides in direction $[1, -1]$ of length $(b - 1)\sqrt{2}$, so containing b lattice points, and two sides in direction $[1, 1]$ of length $(a - 1)\sqrt{2}$, so containing a lattice points (see Fig. 6).

Starting from an arbitrary point of $\partial\Omega_{(a,2,b,2)}([1, -1])$, and moving along the boundary, say counterclockwise, we progressively label the lattice points of $\partial\Omega_{(a,2,b,2)}([1, -1])$ from 1 to $2a + 2b$. Let S^0, S^1 be, respectively, the sets of the even and of the odd labeled vertices. Then, $S^0 \cap S^1 = \emptyset, |S^0| = |S^1| = a + b$, and S^0, S^1 have the same number of lattice

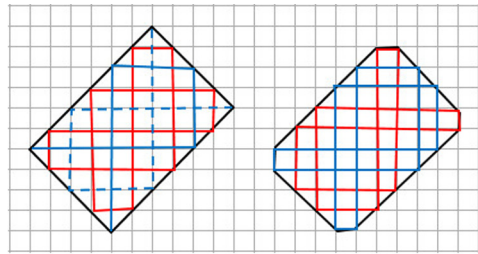


Fig. 7. Correspondence between orbits in $\Omega_{a,b}([1, -1])$ and hv -convex paths in $Curl(a, b)$. Here, $a = 6$, and $b = 4$, so $d = \gcd(a, b) = 2$. The two singular orbits are glued together in a single hv -convex path.

points on each horizontal and on each vertical line. Therefore $S = (S^0, S^1)$ is an hv -convex switching convex component. In particular, since $a \neq b$, then it is an hv -convex curl (see [7,8]), and consequently $S = Curl(a, b)$. Therefore, the set of lattice points of $\partial\Omega_{(a,2,b,2)}([1, -1])$ is in one-to-one correspondence with the set of turning points of $Curl(a, b)$.

Now, by Lemma 28, there exists precisely $d + 1$ integer orbits $S_c(\vec{h})$ in $\Omega_{a,b}([1, -1])$, where $d = \gcd(a, b)$. In particular, $d - 1$ are closed integer orbits, and two are singular orbits (see Fig. 7, left).

By Remark 1, $Curl(a, b)$ consists of d different hv -convex paths of equal lengths (see Fig. 7, right). Note that, due to the corner's doubling, one of these d hv -convex paths corresponds to the h -singular path of $\Omega_{a,b}([1, -1])$, while each one of the remaining $d - 1$ hv -convex paths corresponds to one of the $d - 1$ closed integer orbits, and the statement follows. \square

Corollary 31. Any switching component of an hv -convex polyomino is the joining of Z -paths having only 2 different levels.

Proof. By Lemma 17. and Corollary 21, each Z -path has level $q \in \{2n(\vec{h}) - 1, 2n(\vec{h})\}$ in $\Omega_{a,b}([1, -1])$, so the same holds in the corresponding polyomino \mathcal{P} . By Lemma 30 the statement follows, since any switching component of \mathcal{P} identifies with an orbit $S_c(\vec{h})$ of $\Omega_{a,b}([1, -1])$. \square

Remark 32. Lemma 17 and Corollary 31 provide answers to Question (4). We emphasize that the property of Z -paths having the same level had been already assumed for true in the folklore related to the research concerning hv -convex polyominoes, but, as far as we know, it has been never proved explicitly. The results in [10] raised some doubts on its truth, which, instead, can be now confirmed.

Remark 33. Consider the bouncing points A, B of a twisted orbit that are closest to the origin, and placed, respectively, on the lines $x = 0, y = 0$. Since we always assume $\vec{k} = [k_1, k_2]$ such that $k_1, k_2 > 0$, then it follows that A, B have the same orientation. Let π be the generating sequence of the twisted orbit computed by Algorithm 1: *GenSeq*. Since it is cyclically considered, we can always list the entries such that the first one corresponds to the segment containing the bouncing points A, B . Denote by $\pi^* = (\pi(1), \dots, \pi(\frac{a-b}{d}))$ such a rearrangement of π .

Theorem 34. Let $a, b \in \mathbb{N}$, with $a > b$, and $d = \gcd(a, b) > 1$, and let $\pi^* = (\pi(1), \pi(2), \dots, \pi(\frac{a-b}{d}))$. Then, the structural sequence of the twisted orbits of $\Omega_{a,b}(\vec{h})$ is $(\pi^*)_2$, while the structural sequence of singular orbits is $\pi' = (\pi(1) - 2, \pi(2), \dots, \pi(\frac{a-b}{d}))$.

Proof. By the proof of Lemma 30, it follows that a twisted orbit of $\Omega_{a,b}(\vec{h})$ corresponds to an hv -convex path of $Curl(a, b)$, where the clockwise and counterclockwise alternation of the turning points is preserved. By Remark 1, and Remark 33, the structural sequence of a twisted orbit is $(\pi^*)_2$.

Consider now the singular orbit of $\Omega_{a,b}(\vec{h})$ containing the origin. By Theorem 25, a singular orbit has half of the bouncing points of a twisted orbit. Note that two of the bouncing points of a singular orbit correspond to corner points, and consequently these do not contribute to the construction of the structural sequence. The singular orbit consists of different segments, each one placed between parallel segments belonging to the twisted orbit through the points A, B introduced in Remark 33, so it can be obtained by identifying both A, B with the corner point $O(0, 0)$, and then walking the singular orbit along segments parallel to the segments of the twisted orbit. Therefore, after half of the length of the twisted orbit, we get π' , which consequently reproduces the first part of the structural sequence of the twisted orbit, apart the first entry $\pi(1)$, that must be lowered by 2, corresponding to the identification of A, B with the origin. This implies that $\pi' = (\pi(1) - 2, \pi(2), \dots, \pi(\frac{a-b}{d}))$. Let P be the second corner point belonging to the singular orbit through the origin.

Now, apply a rotation that moves to the origin one of the corner points of the second singular orbit of the same lattice billiard. Then, by the same previous argument, we get that π' is also its structural sequence, and the statement follows. \square

Thanks to this, and using $h = \lfloor \frac{b}{a-b} \rfloor$, $\alpha = 2h + 1$, $\gamma = 2h + 3$, $N_\alpha = 2(h + 1)a - 2(h + 2)b$, and $N_\gamma = 2(h + 1)b - 2ha$ (see Section 2), we get the following result.

Theorem 35. Let $a, b \in \mathbb{N}$, $a > b$, and $d = \gcd(a, b)$. The structural sequence of any admissible twisted orbits $S_x(\vec{h})$ in $\Omega_{a,b}([1, -1])$ consists of $\frac{N_\alpha}{d}$ entries $\alpha = 2h + 1$, and $\frac{N_\gamma}{d}$ entries $\gamma = 2h + 3$. For the structural sequence of admissible singular orbits the number of entries α, γ is $\frac{N_\alpha}{2d} + 1$, and $\frac{N_\gamma}{2d} - 1$, respectively.

Proof. By [8, Theorem 17], the sequence associated to $\text{Curl}(a, b)$ is π_{2d} , and consists of N_α entries equal to α , and N_γ entries equal to γ . Therefore, the number of entries α and γ corresponding to $(\pi^*)_2$ is $\frac{N_\alpha}{d}$, and $\frac{N_\gamma}{d}$, respectively. Note that, as described in Section 2.4, the point having label 1, always belongs to the set I_γ . This means that it always holds $\pi(1) = \gamma$. As described in the proof of Lemma 30 (see also Fig. 6), such a point corresponds to a point of the lattice billiard closest to a corner point. Therefore, by Remark 33, we can always assume that $\pi(1) = \gamma$.

Consider now a singular orbit. By the above argument, and by Theorem 34, the first entry of π' is $\gamma - 2 = \alpha$, and the statement follows. \square

The following result summarizes the description of the structural sequences of any non degenerate integer orbit of any lattice billiard $\Omega_{a,b}(\vec{h})$.

Theorem 36. Let $\vec{k} = [k_1, k_2]$ be an admissible direction of $\Omega_{a,b}(\vec{h})$. Then, as computed by Algorithm GenSeq, the sequence π consists of $\frac{a'-b'}{d'}$ entries, where $a' = \frac{a}{k_2}, b' = \frac{b}{k_1}, d' = \gcd(a', b')$. The structural sequence of an admissible orbit $S_c(\vec{k})$ is $(\pi^*)_2 = (\pi(1), \dots, \pi(\frac{a-b}{d}))_2$ if $S_c(\vec{k})$ is twisted, and $\pi' = (\pi(1) - 2, \dots, \pi(\frac{a-b}{d}))$ if $S_c(\vec{k})$ is singular. The number of bouncing points is $\frac{2(a'+b')}{d'}$ or $\frac{(a'+b')}{d'}$, according as the orbit is twisted or singular, respectively.

Proof. By Corollary 21, $S_c(\vec{k})$, is equivalent to the orbit $S_{c'}(\vec{h})$ in $\Omega_{a',b'}([1, -1])$. By Algorithm GenSeq, applied with entries a', b' , it follows that π consists of $\frac{a'-b'}{d'}$ entries. Therefore, if $S_{c'}(\vec{h})$ is twisted, then its structural sequence is $(\pi^*)_2 = (\pi(1), \dots, \pi(\frac{a-b}{d}))_2$. By Theorem 35, among the $2\frac{a'-b'}{d'}$ entries of $(\pi)_2, \frac{N_\alpha}{d'}$ equal $\alpha = 2h + 1$, and $\frac{N_\gamma}{d'}$, equal $\gamma = 2h + 3$, where $h = \lfloor \frac{b'}{a'-b'} \rfloor$. Therefore, we have a total amount of $\frac{\alpha N_\alpha + \gamma N_\gamma}{d'}$ bouncing points, namely

$$\frac{\alpha N_\alpha + \gamma N_\gamma}{d'} = \frac{(2h + 1)}{d'}(2(h + 1)a - 2(h + 2)b) + \frac{(2h + 3)}{d'}(2(h + 1)b - 2ha) = \frac{2(a' + b')}{d'}$$

bouncing points, and the statement follows for twisted orbits. The two singular orbits in direction \vec{k} correspond to singular orbits of $\Omega_{a',b'}([1, -1])$. By Corollary 21, each one has structural sequence $\pi' = (\pi(1) - 2, \dots, \pi(\frac{a-b}{d}))$ if $S_c(\vec{k})$, and the statement follows also for singular orbits. \square

Example 37. Let us summarize the obtained results with an illustrative example. Assume $\vec{k} = [3, 2]$ $a = 48, b = 45$, and consider the non degenerate admissible orbits $S_c(\vec{k})$ of $\Omega_{48,45}(\vec{h})$, where $0 \leq c \leq 45$ so that $c = 3t$ for some integer $t \geq 0$ (Theorem 5).

Since $\frac{a}{b} \neq \frac{k_2}{k_1}$, then the orbits $S_c(\vec{k})$ are not elementary windows (Theorem 15).

Since $(\frac{a}{k_2}, \frac{b}{k_1}) = (24, 15)$, and $\gcd(24, 15) = 3$ then, by Corollary 12, $S_c(\vec{k})$ is singular if and only if 3 divides $\frac{c}{k_1} = t$, that is $t = 3s$, and consequently $c = 9s$, for some integer $s \geq 0$. Therefore, the two singular orbits are obtained when $c \in \{0, 9, 18, 27, 36, 45\}$, where three different values of c correspond to different horizontal bouncing points of a same orbit.

The other allowed values of c correspond to twisted orbits.

Since $d = \gcd(a, b) = 3$, then there exist $d + 1 = 4$ orbits, two of them are twisted and two are singular (See Fig. 8 for a representation of the orbits).

The structural sequence of any orbit is obtained as stated in Theorem 36, where $a' = 24, b' = 15$, and in particular, being $h = \lfloor \frac{b'}{a'-b'} \rfloor = 1$, The structural sequence of a twisted orbit has $\frac{N_\alpha}{d'} = \frac{2(h+1)a' - 2(h+2)b'}{d'} = 2$ entries equal to $\alpha = 2h + 1 = 3$, and $\frac{N_\gamma}{d'} = \frac{2(h+1)b' - 2ha'}{d'} = 4$ entries equal to $\gamma = 2h + 3 = 5$, where $d' = \gcd(a', b') = 3$. The alternation of these entries can be explicitly computed by means of Algorithm GenSeq, which returns $\pi = (5, 5, 3)$ (see also [8, Example 29]). Note that it consists of $\frac{a'-b'}{d'}$ entries. The structural sequence of the twisted orbits is $(\pi^*)_2 = (5, 5, 3)_2 = (5, 5, 3, 5, 5, 3)$, while $\pi' = (3, 5, 3)$ is the structural sequence of singular orbits. The two singular orbits have $\frac{a'+b'}{d'} = 13$ different bouncing points, while the twisted orbits have 26 different bouncing points. Note that the number N of bouncing points of the singular orbits, and $2N$ for twisted orbits can also be computed by means of Corollary 26, referred to a', b' and the related parameters. Indeed, we have $24 = 1 \cdot 15 + 9$, and $\gcd(24, 15) = 3$, so we get $N = \frac{24+15}{3} = 13$ bouncing points for the singular orbits.

6. Concluding remarks

We have discussed several properties of rational orbits in rectangular lattice billiard tables, where the constraint that all bouncing points must belong to the integer lattice \mathbb{Z}^2 has been assumed. Differently from the continuous case, the

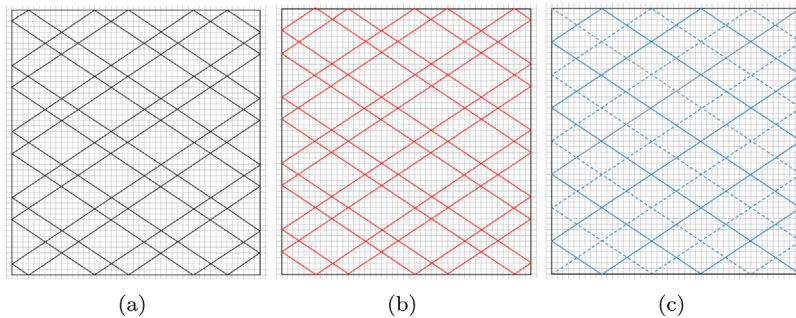


Fig. 8. The orbits $S_c([3, 2])$ in $\Omega_{48,45}(\vec{h})$. (a) The black path provides the twisted orbit corresponding to $c \in \{3, 15, 21, 33, 39\}$, having structural sequence $(5, 5, 3)_2$. (b) The red path provides the twisted orbit corresponding to $c \in \{6, 12, 24, 30, 42\}$, also having structural sequence $(5, 5, 3)_2$. (c) The solid and the dashed blue paths are the two singular orbits (corresponding, respectively, to $c \in \{9, 27, 45\}$, and to $c \in \{0, 18, 36\}$), where $(3, 5, 3)$ is the structural sequence for both of them. These can be interpreted as the single \vec{k} -singular path, $\vec{k} = [3, 2]$, having structural sequence $(3, 5, 3)_2$, up to identify the top-left and the bottom-right corners, and the top-right and the bottom-left corners (see the online version for the different colors). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

main questions have a pure combinatorial nature, and the results strongly depend on the numerical properties of the involved parameters. A complete classification of integer orbits has been provided, and different related problems have been studied, in particular their existence in a given lattice billiard table, their geometric structure, the total number of their bouncing points and the corresponding sequences of clockwise and counterclockwise turns. The main results are summarized in [Theorem 5](#), concerning the characterization of the allowed integer orbits, in [Theorem 7](#) on their mutual equivalence, in the characterization of singular orbits and elementary windows provided by [Theorems 10](#) and [15](#), and in [Theorem 36](#) concerning the geometric and the combinatorial characterization of the structural sequences of integer orbits. As an additional remark, when the orbits have direction $[1, 1]$, then the properties concerning arithmetic billiards (see [\[19\]](#)) can be included as special cases.

These results seem to be promising, and encourage us to deepen and generalize the study of integer orbits, even exploring possible different paths. For example, one could try to extend the results to lattice billiard tables consisting of convex polygons different from a rectangle, or even non convex polygons. Also, as we have already pointed out, connections with polyominoes, as well as with some aspects usually considered in Discrete Tomography exist, so possible additional applications of the obtained results could be further explored.

Data availability

No data was used for the research described in the article.

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