

ROTUND GÂTEAUX SMOOTH NORMS WHICH ARE NOT LOCALLY UNIFORMLY ROTUND

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ABSTRACT. We provide, in every infinite-dimensional separable Banach space, an average locally uniformly rotund (and hence rotund) Gâteaux smooth renorming which is not locally uniformly rotund. This solves an open problem posed by A.J. Guirao, V. Montesinos, and V. Zizler in the recent monograph [10].

1. INTRODUCTION

Renorming techniques and relations between smoothness and rotundity play a central role in the study of Banach spaces. Very recently A.J. Guirao, V. Montesinos and V. Zizler published a monograph which collects the most important results, techniques and open problems in renorming theory [10]; we refer also to [6] as a main reference in the field and to [4, 5, 14, 9, 20] for very recent progresses in the topic. Several different notions of rotundity of the unit ball of a Banach space has been introduced and widely studied. The most common, that can be considered already classical, are *strict rotundity* (R), *uniform rotundity* (UR), and *local uniform rotundity* (LUR). It is well-known and easy-to-prove that local uniform rotundity of the unit ball implies that *each point of unit sphere is strongly exposed by any of its supporting functionals* (spaces satisfying this condition are called almost locally uniformly rotund (almost LUR) in the literature [1, 2]), which in turns implies that *each point of unit sphere is a denting point* (spaces satisfying this condition are called average locally uniformly rotund (ALUR) in the literature [19, 22]). Notice that for reflexive spaces the notions of ALUR and almost LUR coincide (see Theorem 2.4 below).

An interesting problem consists in determining in which situation and to what extent the rotundity properties cited above are distinct or not, in particular, when additional smoothness properties are assumed. Despite an extensive literature on the subject, surprisingly enough, the following problem is open (see [10, p.495, Problem 5]).

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Problem 1.1. *Does every infinite-dimensional separable Banach space admit a norm that is rotund and Gâteaux smooth but not LUR?*

Another related problem arises naturally taking into account the fact that each almost LUR Fréchet smooth norm is automatically LUR, the proof of which easily follows by Šmulyan Lemma (see also [1, Corollary 2.11]).

Problem 1.2. *Is every almost LUR Gâteaux smooth space automatically LUR?*

The purpose of our paper is to answer these two questions. The main result we obtained reads as follow (see, Theorem 3.8 below).

Theorem A. *Every infinite-dimensional separable Banach space admits an average locally uniformly rotund (and hence rotund) Gâteaux smooth equivalent norm $|\cdot|$ which is not locally uniformly rotund.*

Theorem A answers in the affirmative the former problem, and, since for reflexive spaces the notions of ALUR and almost LUR coincide, it also solves in the negative the latter problem.

Let us describe the structure of the paper. Section 2 contains some notation, preliminaries, and a brief study of properties ALUR and almost LUR, showing that the two notions do not coincide in general. Section 3 is devoted to the proof of our main result Theorem 3.8, whose main ingredient is a geometrical construction in the separable Hilbert space (that is actually a bit hidden in the proof itself) combined with a suitable use of Markushevich bases. Finally, in the last section we lift our construction to some classes of nonseparable Banach spaces and we present some final remarks. For example, we observe that existence of our renorming $|\cdot|$ can be obtained in each (WCD) space (see Definition 4.2). The class of (WCD) Banach spaces is also known as Vařák, it contains properly the class of weakly compactly generated Banach spaces (in particular it contains each reflexive space) and it shares good renorming properties, indeed each (WCD) has a Gâteaux smooth LUR equivalent norm, we refer to [6, 10, 12, 3, 7] for more information on this class. Finally, we provide some examples of Banach spaces which are not (WCD) that can be renormed by an ALUR Gâteaux smooth norm which is not LUR.

2. NOTATIONS AND PRELIMINARIES

Throughout this paper, all normed and Banach spaces are real and infinite-dimensional. Let X be a normed space, by X^* we denote the dual space of X . By B_X , U_X , and S_X we denote the closed unit ball, the open unit ball, and the unit sphere of X , respectively. Moreover, in situations when more than one norm on X is considered, we denote by $B_{(X, \|\cdot\|)}$, $U_{(X, \|\cdot\|)}$ and $S_{(X, \|\cdot\|)}$ the closed unit ball, the open unit ball, and the closed unit sphere with respect to the norm $\|\cdot\|$, respectively. By $\|\cdot\|^*$ we denote the dual norm of $\|\cdot\|$. Given

a set $A \subset X$ we denote by ∂A the boundary of A . For $x, y \in X$, $[x, y]$ denotes the closed segment in X with endpoints x and y , and $(x, y) = [x, y] \setminus \{x, y\}$ is the corresponding “open” segment; the segment $[x, y)$ is defined similarly. We recall a geometric observation, which turns out to be very useful in the next sections (cf. [17]).

Lemma 2.1. *Let X be a normed space. Let C be a nonempty closed convex subset of X with non-empty interior. If $x \in \text{int } C$ and $y \in \partial C$, then $[x, y) \subset \text{int } C$.*

A *biorthogonal system* in a Banach space X is a system $(e_\gamma; f_\gamma)_{\gamma \in \Gamma} \subset X \times X^*$, such that $f_\alpha(e_\beta) = \delta_{\alpha, \beta}$ ($\alpha, \beta \in \Gamma$). A biorthogonal system is *fundamental* if $\text{span}\{e_\gamma\}_{\gamma \in \Gamma}$ is dense in X ; it is *total* when $\text{span}\{f_\gamma\}_{\gamma \in \Gamma}$ is w^* -dense in X^* . A *Markushevich basis* (M-basis) is a fundamental and total biorthogonal system. We refer to [12] and [21] for good references on M-bases and [11, 15, 16] for some recent progresses in the topic.

Let us recall that the duality map $D_X : S_X \rightarrow 2^{S_{X^*}}$ is the function defined, for each $x \in S_X$, by

$$D_X(x) := \{x^* \in S_{X^*}; x^*(x) = 1\}.$$

Given a subset K of X , a *slice* of K is a set of the form

$$S(K, x^*, \alpha) := \{x^* \in K; x^*(x) > \sup x^*(K) - \alpha\},$$

where $\alpha > 0$ and $x^* \in X^* \setminus \{0\}$ is bounded above on K . For convenience of the reader we recall some standard notions which we list in the following definition.

Definition 2.2. Let $x \in S_X$ and $x^* \in S_{X^*}$. We say that:

- x is an *extreme point* of B_X if it does not lie in any “open” segment contained in B_X ;
- X is *rotund* (R) if each point of S_X is an extreme point of B_X ;
- x is *supported* by x^* if $x^* \in D_X(x)$;
- x is a *denting point* of B_X if for each neighbourhood V of x in the norm topology there exists a slice S of B_X such that $x \in S \subset V$;
- x is *strongly exposed* by x^* if $x^* \in D_X(x)$ and, for each norm neighbourhood V of x , there exists $\alpha > 0$ such that $S(B_X, x^*, \alpha) \subset V$ (equivalently, if $x^* \in D_X(x)$ and $x_n \rightarrow x$ for all sequences $\{x_n\}_n \subset B_X$ such that $\lim_{n \rightarrow \infty} x^*(x_n) = 1$);
- x is a *locally uniformly rotund (LUR) point* of B_X if $x_n \rightarrow x$, whenever $\{x_n\}_n \subset X$ is such that

$$\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0;$$

- x is a *weakly locally uniformly rotund (WLUR) point* of B_X if $x_n \rightarrow x$ weakly, whenever $\{x_n\}_n \subset X$ is such that

$$\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0;$$

- x is a *Gâteaux smooth point* (respectively, *Fréchet smooth point*) of B_X if, the following limit holds

$$\lim_{h \rightarrow 0} \frac{\|x + hy\| + \|x - hy\| - 2}{h} = 0,$$

whenever $y \in S_X$ (respectively, uniformly for $y \in S_X$).

If each point of S_X is LUR, Gâteaux smooth, Fréchet smooth, respectively, we say that X and the corresponding norm $\|\cdot\|$ is LUR, Gâteaux smooth, Fréchet smooth, respectively.

A Banach space X has the *Kadec property* (resp. *Kadec-Klee property*) if the norm and the weak topology coincide on the unit sphere S_X (resp. convergent sequences in (S_X, w) are convergent in $(S_X, \|\cdot\|)$). If ℓ_1 does not embed into X , then Kadec-Klee implies Kadec, see [22].

We say that a Banach space X is *average locally uniformly rotund* (ALUR, in short) if every point of the unit sphere S_X is a denting point, in fact the original definition of ALUR space, given by S. Troyanski in [22], requires some technical preliminaries which are not needed in our paper, see [19] for the equivalence of the two definitions. Clearly, if a norm is LUR, then it is ALUR, on the other hand, the other implication does not hold, see [22, Example 2.2], notice that such an example, provided in $X = \ell_1$, is not Gâteaux differentiable at $x = e_1$. Furthermore, in [22] it is proved that a norm is ALUR if and only if it is rotund and has the Kadec property. Finally, we recall that in [23] it is proved that if X is reflexive, and $\|\cdot\|$ is rotund and has the Kadec property (equivalently the Kadec-Klee property), then $\|\cdot\|^*$ is Fréchet smooth. Since we will make use of these two results, we summarize them in the following theorem.

Theorem 2.3. *Let X be a Banach space. Then:*

- (i) *X is ALUR if and only if X is rotund and has the Kadec property;*
- (ii) *Suppose that X is reflexive. If X is rotund and has the Kadec property, then X^* is Fréchet smooth.*

We conclude this section by comparing the definition of average locally uniformly rotund norm and the one of almost locally uniformly rotund norm. A Banach space X is *almost locally uniformly rotund* (*almost LUR*, in short) if for every $x \in S_X$ and every pair of sequences $\{x_n\}_n \subset S_X$ and $\{x_n^*\}_n \subset S_{X^*}$ such that $\lim_m(\lim_n(x_m^*((x_n + x)/2))) = 1$, $\{x_n\}_n$ converges to x . In [2] it is proved that a Banach space is almost LUR if and only if each point $x \in S_X$

is strongly exposed by $x^* \in S_{X^*}$ whenever x is supported by x^* . This equivalence is used in [1] for proving that an almost LUR, Fréchet smooth norm is actually LUR. Notice that the same result is no longer true if we replace Fréchet by Gâteaux (combine Theorem 3.8 with Theorem 2.4 below). In literature almost LUR norms are abbreviated as ALUR norms. Since this choice might be ambiguous, and we were not able to find references that compare the two notions, we decided to include here some results that clarify the relations between these two definitions.

Theorem 2.4. *Let X be a Banach space, then the following holds true*

- (i) *if X is almost LUR, then X is ALUR.*
- (ii) *if X is reflexive and ALUR, then X is almost LUR.*

Proof. (i) It follows by the fact that if $x \in S_X$ is strongly exposed by $x^* \in X^*$, then x is a denting point of B_X .

(ii) By Theorem 2.3, X is rotund and has the Kadec property. Let $x \in S_X$ and $x^* \in D_X(x)$. By the Kadec property, in order to show that x is strongly exposed by x^* , it is enough to show that, whenever $\{x_n\}_n \subset S_X$ is such that $\lim_n x^*(x_n) = 1$, there exists a subsequence $\{x_{n_k}\}_k$ w -converging to x .

Now, since X is reflexive, there exists a subsequence $\{x_{n_k}\}_k$ which converges weakly to some $y \in B_X$. We obtain $\lim_k x^*(x_{n_k}) = x^*(y) = 1$. Since $\|x^*\|^* = 1$, y lies on S_X . Hence, since X is rotund, y must be equal to x . Thus $\{x_{n_k}\}_k$ converges weakly to x . This concludes the proof. \square

Finally, we provide an equivalent norm in ℓ_1 that is ALUR but not almost LUR. Our example is based on [22, Example 2.2]. In which it is proved that there exists an ALUR norm which is not LUR. We just show that such a norm is not almost LUR.

Example 2.5. Let $X = \ell_1$ endowed with the norm

$$\|x\| = \sum_{n=1}^{\infty} |x_n| + \left(\sum_{n=1}^{\infty} \frac{x_n^2}{n^2} \right)^{1/2}.$$

The norm $\|\cdot\|$ is ALUR, see [22, Example 2.2]. Let us show that $\|\cdot\|$ is not almost LUR. Let $x := \frac{e_1}{2} \in S_X$, where $\{e_n\}_n$ is the standard Schauder basis of ℓ_1 . Let $x^* = (x_n^*)_n \in (\ell_\infty, \|\cdot\|^*)$ defined by

$$x_n^* = \begin{cases} 2 & \text{if } n = 1, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that $\|x^*\|^* = 1$. Indeed, let $y = (y_n)_n \in S_X$, we have

$$|x^*(y)| \leq 2|y_1| + \sum_{n=2}^{\infty} |y_n| = \sum_{n=1}^{\infty} |y_n| + |y_1| \leq \|y\|.$$

Moreover we have $x^*(x) = 1$. Therefore, $\|x^*\|^* = 1$ and $x \in S_X$ is supported by x^* . It remains to show that x is not strongly exposed by x^* . Let $\delta > 0$, for sufficiently large $n \in \mathbb{N}$, both x and $(\frac{n}{1+n})e_n \in S_X$ belongs to the slice $S(B_X, x^*, \delta)$, observing that $\|x - (\frac{n}{1+n})e_n\| > 1/2$, we get that x is not strongly exposed by x^* , hence $\|\cdot\|$ is not almost LUR.

3. MAIN RESULTS

This section is devoted to construct an equivalent norm in any separable infinite-dimensional Banach space, which is ALUR, Gâteaux smooth but not LUR. Before defining such a norm, we need some preliminary constructions and technical assumptions. In the sequel of this section, X denotes a separable Banach space. We shall need the following result that essentially collects some well-known results about renorming of separable Banach spaces and existence of M-bases with additional properties, for the sake of completeness we include a sketch of the proof.

Theorem 3.1. *There exist an equivalent norm $\|\cdot\|$ and an M-basis $(e_n, g_n)_{n \in \mathbb{N}}$ on X such that:*

- (i) $\|\cdot\|$ is LUR and Gâteaux smooth;
 - (ii) for every $x \in X$ we have
- (1) $\|x\|^2 = \|x - g_1(x)e_1\|^2 + [g_1(x)]^2;$
- (iii) $\|e_n\| = 1$, whenever $n \in \mathbb{N}$;
 - (iv) $\|g_1\|^* = \|g_{3n}\|^* = 1$, whenever $n \in \mathbb{N}$.

Proof. Let $\|\cdot\|_1$ be a LUR and Gâteaux smooth equivalent norm on X (see e.g. [8, Theorem 8.2]). By [21, Proposition 8.13], there exists an M-basis $(e_n, g_n)_{n \in \mathbb{N}}$ on X such that:

- $\|e_n\|_1 = 1$, whenever $n \in \mathbb{N}$;
- $\|g_{3n}\|_1^* = 1$, whenever $n \in \mathbb{N}$.

Let us define an equivalent norm $\|\cdot\|$ on X by

$$\|x\|^2 = \|x - g_1(x)e_1\|_1^2 + [g_1(x)]^2, \quad x \in X.$$

It remains to prove (i)-(iv). Since $\|\cdot\|_1$ is Gâteaux smooth then by definition $\|\cdot\|$ is Gâteaux smooth too. Let us prove that $\|\cdot\|$ is LUR: let $\{x_n\}_n \subset X$ and $x \in X$ such that $\lim_n 2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 = 0$, in other words

$$\begin{aligned} \lim_n 2\|x - g_1(x)e_1\|_1^2 + 2[g_1(x)]^2 + 2\|x_n - g_1(x_n)e_1\|_1^2 + 2[g_1(x_n)]^2 \\ - \|x + x_n - g_1(x + x_n)e_1\|_1^2 + [g_1(x + x_n)]^2 = 0. \end{aligned}$$

Since

$$\begin{aligned} 2\|x - g_1(x)e_1\|_1^2 + 2\|x_n - g_1(x_n)e_1\|_1^2 - \|x + x_n - g_1(x + x_n)e_1\|_1^2 \geq 0, \\ 2[g_1(x)]^2 + 2[g_1(x_n)]^2 - [g_1(x + x_n)]^2 \geq 0, \end{aligned}$$

we have

$$\begin{aligned} \lim_n 2\|x - g_1(x)e_1\|_1^2 + 2\|x_n - g_1(x_n)e_1\|_1^2 - \|x + x_n - g_1(x + x_n)e_1\|_1^2 &= 0, \\ \lim_n 2[g_1(x)]^2 + 2[g_1(x_n)]^2 - [g_1(x + x_n)]^2 &= 0, \end{aligned}$$

Therefore, by using the fact that $\|\cdot\|_1$ is LUR, we obtain

$$\lim_n [x_n - g_1(x_n)e_1] = x - g_1(x)e_1, \quad \lim_n g_1(x_n) = g_1(x).$$

Hence $\lim_n \|x - x_n\| = 0$, and (i) follows.

For $x \in X$, we have

$$\begin{aligned} \|x - g_1(x)e_1\|^2 &= \|x - g_1(x)e_1 - g_1(x - g_1(x)e_1)e_1\|_1^2 + [g_1(x - g_1(x)e_1)]^2 \\ &= \|x - g_1(x)e_1\|_1^2, \end{aligned}$$

so (ii) follows. (iii) is trivial. For (iv), let $x \in X$ be such that $\|x\| = 1$. Then we have $1 = \|x\|^2 = \|x - g_1(x)e_1\|_1^2 + [g_1(x)]^2$, from which we get

$$|g_1(x)| \leq \sqrt{1 - \|x - g_1(x)e_1\|_1^2} \leq 1.$$

Combining (iii) with $g_1(e_1) = 1$, we get $\|g_1\|^* = 1$. Finally, let $n \in \mathbb{N}$ and $x \in X$ such that $\|x\| \leq 1$, then, since $\|g_{3n}\|_1^* = 1$, we have

$$\frac{|g_{3n}(x)|}{\|x\|^2} = \frac{|g_{3n}(x - g_1(x)e_1)|}{\|x - g_1(x)e_1\|_1^2 + [g_1(x)]^2} \leq \frac{|g_{3n}(x - g_1(x)e_1)|}{\|x - g_1(x)e_1\|_1^2} \leq 1.$$

Combining (iii) with $g_{3n}(e_{3n}) = 1$ we obtain (iv). \square

In the sequel of this section, let us denote by $\|\cdot\|$ and $(e_n, g_n)_{n \in \mathbb{N}}$ the equivalent norm and the M-basis on X , respectively, given by Theorem 3.1. Let us denote by P_1 the bounded projection onto the one dimensional space generated by e_1 defined by $P_1(x) = g_1(x)e_1$ ($x \in X$), and put $Q_1 := I - P_1$. Then, (1) implies that $Q_1[B_{(X, \|\cdot\|)}] \subset B_{(X, \|\cdot\|)}$.

We consider the linear operator $T: (\ell_2, \|\cdot\|_2) \rightarrow (X, \|\cdot\|)$ defined by

$$T\alpha = \sqrt{2}\alpha_1 e_1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \alpha_n e_n,$$

where $\alpha = (\alpha_n)_n \in \ell_2$. Observing that $\sum_{n=2}^{\infty} \frac{1}{n^2} < 1$, we have $T[R_1 B_{\ell_2}] \subset U_{(X, \|\cdot\|)}$, where $R_1 B_{\ell_2} := \{\alpha = (\alpha_n)_n \in \ell_2: \alpha_1 = 0\}$. We notice that the operator T is well-defined, bounded, linear, one-to-one, and the range $Y := T\ell_2$ contains $\{e_n\}_n$, therefore Y is dense in X . By the injectivity of the operator T , we can consider the subspace Y endowed with the norm $\|\cdot\|_{\theta}$ defined by $\|y\|_{\theta} := \|T^{-1}y\|_2$, for any $y \in Y$. In this way, we obtain that T is an isometric isomorphism between $(\ell_2, \|\cdot\|_2)$ and $(Y, \|\cdot\|_{\theta})$. We set $B := T[B_{\ell_2}]$. In other words, we have that

$$B = \{y \in Y: \|y\|_{\theta} \leq 1\}.$$

Similarly, we define

$$\begin{aligned} U &= \{y \in Y : \|y\|_\theta < 1\}, \\ S &= \{y \in Y : \|y\|_\theta = 1\}. \end{aligned}$$

We claim that the convex subset B , is compact in $(X, \|\cdot\|)$. Indeed, the operator T is bounded, therefore it is w - w -continuous, hence B is closed in $(X, \|\cdot\|)$. In order to prove the claim, it is enough to observe that the operator T is compact.

By the claim, B is a convex compact set in $(X, \|\cdot\|)$ and hence we have that the set

$$D = \text{conv}(B_{(X, \|\cdot\|)} \cup B)$$

is closed in X . Then by our definition and by symmetry, D is the closed unit ball of an equivalent norm $\|\cdot\|$ on X . Unfortunately such a norm is not rotund, therefore we need a further step. Define $f_n = g_n / \|g_n\|^*$ ($n \in \mathbb{N}$), and consider the equivalent norm $|\cdot|$ on X defined by

$$(2) \quad F(x) := |x|^2 = \|x\|^2 + \sum_{n=2}^{\infty} 2^{-n} f_n(x)^2.$$

In order to prove the main theorem of this section we need several technical lemmas that describe the geometry of the norm $|\cdot|$. Next results are stated by using the same notation as in the first part of this section.

Lemma 3.2. *We have $U \cap \partial D = \emptyset$, equivalently $B \cap \partial D \subset S$, equivalently $U \subset \text{int } D$.*

Proof. Suppose on the contrary that there exists $x \in U \cap \partial D$. Combining that $T[R_1 B_{\ell_2}] \subset U_{(X, \|\cdot\|)} \subset \text{int } D$ with the fact that $\|x\|_\theta = \|T^{-1}x\|_2 < 1$, there exists $y_1 \in \mathbb{R}$ such that:

- $y := y_1 e_1 + Q_1 x \in S$ (which is equivalent to $\|y\|_\theta = 1$);
- $x \in [Q_1 x, y]$.

Since $y \in S \subset D$ and $Q_1 x \in \text{int } D$, by Lemma 2.1, we get $x \in [Q_1 x, y] \subset \text{int } D$, that is a contradiction. \square

Lemma 3.3. *Let $x \in \partial D$, then there exist $\lambda \in [0, 1]$, $b \in S_{(X, \|\cdot\|)}$, and $c \in S$ such that $x = \lambda b + (1 - \lambda)c$.*

Proof. Let us observe that D is the union of the following sets:

$$\begin{aligned} &\text{conv}(S_{(X, \|\cdot\|)} \cup S), \quad U_{(X, \|\cdot\|)}, \quad U, \\ &\bigcup_{\mu \in (0,1)} [\mu B_{(X, \|\cdot\|)} + (1 - \mu)U], \quad \bigcup_{\mu \in (0,1)} [\mu U_{(X, \|\cdot\|)} + (1 - \mu)B]. \end{aligned}$$

We observe that $U_{(X, \|\cdot\|)} \subset \text{int } D$, by Lemma 3.2, $U \subset \text{int } D$, and by Lemma 2.1, the sets

$$\bigcup_{\mu \in (0,1)} [\mu B_{(X, \|\cdot\|)} + (1 - \mu)U], \quad \bigcup_{\mu \in (0,1)} [\mu U_{(X, \|\cdot\|)} + (1 - \mu)B].$$

are contained in $\text{int } D$. Hence necessarily $x \in \text{conv}(S_{(X, \|\cdot\|)} \cup S)$, and the thesis holds. \square

Lemma 3.4. *If $x \in \partial D$, then at least one of the following conditions is satisfied:*

- (i) $x + te_1 \notin D$, whenever $t > 0$;
- (ii) $x - te_1 \notin D$, whenever $t > 0$.

In other words, the set ∂D does not contain segments parallel to e_1 .

Proof. Let $x \in \partial D$ and suppose without any loss of generality that $g_1(x) \geq 0$, the case $g_1(x) \leq 0$ is similar. Let us observe that, by (1), we get

$$Q_1(B_{(X, \|\cdot\|)} \setminus Q_1(B_{(X, \|\cdot\|)})) \subset U_{(X, \|\cdot\|)}.$$

Moreover, since $Q_1 B = T[R_1 B_{\ell_2}]$, we have

$$Q_1(B) \subset U_{(X, \|\cdot\|)}.$$

Hence $Q_1(D) \subset B_{(X, \|\cdot\|)}$ and $Q_1(D \setminus Q_1(B_{(X, \|\cdot\|)})) \subset U_{(X, \|\cdot\|)}$. Let us denote $z = Q_1(x) \in B_{(X, \|\cdot\|)}$ and let us consider the following two cases.

Case 1: $\|z\| = 1$. In this case $x = z$ and, since $Q_1(D \setminus Q_1(B_{(X, \|\cdot\|)})) \subset U_{(X, \|\cdot\|)}$, we have that (i) is satisfied.

Case 2: $\|z\| < 1$. First, observe that $z \in \text{int } D$. Suppose on the contrary that (i) does not hold and let $t > 0$ be such that $w := x + te_1 \in D$. Then $x \in [z, w]$ and hence $x \in \text{int } D$, a contradiction.

Let us conclude the proof, by observing that ∂D does not contain segment parallel to e_1 . Indeed, suppose that $[x, x + te_1] \subset \partial D$ for some $t > 0$, and $x \in \partial D$, then the point $x + \frac{t}{2}e_1 \in \partial D$ does not satisfy neither (i) nor (ii). Which is a contradiction. \square

The previous lemmas are necessary for proving that the norm defined in (2) is rotund.

Proposition 3.5. *The norm $|\cdot|$ is rotund.*

Proof. Let us prove that the set of all $x \in X$ such that

$$F(x) = |x|^2 = \|x\|^2 + \sum_{n=2}^{\infty} 2^{-n} f_n(x)^2 \leq 1$$

is strictly convex. Suppose on the contrary that there exist distinct elements $x \in X$ and $y \in X$ such that $F(x) = F(y) = F(\frac{x+y}{2}) = 1$. For $z \in X$, we

define $g(z) = \sqrt{\sum_{n=2}^{\infty} \frac{f_n(z)^2}{2^n}}$. In \mathbb{R}^2 equipped with the euclidean norm, let us consider the norm one vectors $v = (\|x\|, g(x))$, $w = (\|y\|, g(y))$, then we have

$$(3) \quad \left(\frac{\|x\| + \|y\|}{2}\right)^2 + \left(\frac{g(x) + g(y)}{2}\right)^2 \leq 1.$$

On the other hand, by the convexity of $\|\cdot\|$ and $g(\cdot)$ we have

$$(4) \quad 1 = F\left(\frac{x+y}{2}\right) = \left\|\frac{x+y}{2}\right\|^2 + \left(g\left(\frac{x+y}{2}\right)\right)^2 \leq \left(\frac{\|x\| + \|y\|}{2}\right)^2 + \left(\frac{g(x) + g(y)}{2}\right)^2.$$

By combining (3) and (4), we obtain

$$\left(\frac{\|x\| + \|y\|}{2}\right)^2 + \left(\frac{g(x) + g(y)}{2}\right)^2 = 1,$$

and hence, by the strict convexity of the euclidean norm on \mathbb{R}^2 , v and w must coincide. This last fact and (4) easily imply that:

- (i) $\sum_{n=2}^{\infty} 2^{-n} f_n(x)^2 = \sum_{n=2}^{\infty} 2^{-n} f_n(y)^2 = \sum_{n=2}^{\infty} 2^{-n} \left(\frac{f_n(x) + f_n(y)}{2}\right)^2$;
- (ii) $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\|$.

By (i), we have that $f_n(x) = f_n(y)$, whenever $n \geq 2$. Since $x \neq y$, and $(e_n, g_n)_n$ is an M-basis, we have that $f_1(x) \neq f_1(y)$, and hence, by (ii), ∂D contains a nontrivial segment parallel to e_1 , which contradicts Lemma 3.4. Therefore $\|\cdot\|$ is strictly convex. \square

Proposition 3.6. *The norm $\|\cdot\|$ is Gâteaux smooth.*

Proof. By Šmulyan Lemma, it is sufficient to prove that if $x \in S_{(X, \|\cdot\|)} = \partial D$ then there exists a unique $f \in S_{(X^*, \|\cdot\|_*)}$ supporting $B_{(X, \|\cdot\|)} = D$ at x .

Let $x \in \partial D$ and suppose on the contrary that there exist two distinct functionals $h_1, h_2 \in S_{(X^*, \|\cdot\|_*)}$ such that $h_1(x) = h_2(x) = 1$. By Lemma 3.3, there exist $\lambda \in [0, 1]$, $b \in S_{(X, \|\cdot\|)}$, and $c \in S$ such that $x = \lambda b + (1 - \lambda)c$. Let us consider the following two cases.

Case 1: $\lambda > 0$. In this case we proceed as in [18, Theorem 1.5]: observe that if we define $C_\lambda = \lambda B_{(X, \|\cdot\|)} + (1 - \lambda)c$, then $x \in \partial C_\lambda$ and $C_\lambda \subset D$. Hence, h_1 and h_2 are functionals supporting C_λ at x , and we get a contradiction since $\|\cdot\|$ is Gâteaux smooth.

Case 2: $\lambda = 0$. In this case $x = c \in S$ and $h_i(x) = 1 = \sup h_i(B)$ ($i = 1, 2$), moreover $h_1|_Y \neq h_2|_Y$, since Y is a dense subspace of X . Hence, $h_1|_Y$ and $h_2|_Y$ are distinct $\|\cdot\|_\theta$ -continuous functionals (since bounded on B) on Y supporting B at x , moreover $\|h_1|_Y\|_\theta^* = \|h_2|_Y\|_\theta^* = 1$. A contradiction, since B is the closed unit ball of $(Y, \|\cdot\|_\theta)$, which is a Hilbert space (and hence a Gâteaux smooth Banach space). \square

Proposition 3.7. *The norm $\|\cdot\|$ has the Kadec property.*

Proof. Suppose on the contrary that there exists a net $\{x_\xi\}_{\xi \in \Sigma} \subset S_{(X, \|\cdot\|)}$ that is w -convergent to an element $x \in S_{(X, \|\cdot\|)}$ and such that $\|x_\xi - x\|$ is bounded away from 0. By Lemma 3.3, for each $\xi \in \Sigma$, there exist $\lambda_\xi \in [0, 1]$, $b_\xi \in S_{(X, \|\cdot\|)}$, and $c_\xi \in S$ such that $x_\xi = \lambda_\xi b_\xi + (1 - \lambda_\xi)c_\xi$. By compactness of B and $[0, 1]$, we can suppose that $\lambda_\xi \rightarrow \lambda \in [0, 1]$ and that $\{c_\xi\}_{\xi \in \Sigma}$ converges in norm to $c \in B$. Let us consider the following two cases.

Case 1: $\lambda = 0$. In this case, since $\{b_\xi\}_{\xi \in \Sigma}$ is bounded and $\lambda_\xi \rightarrow 0$, we have that the net $\{x_\xi\}_{\xi \in \Sigma}$ converges in norm to $c = x$.

Case 2: $\lambda \in (0, 1]$. In this case, we have that $\{b_\xi\}_{\xi \in \Sigma}$ converges weakly to an element $b \in B_{(X, \|\cdot\|)}$ (indeed, eventually $\lambda_\xi \neq 0$ and hence we can write $b_\xi = \frac{x_\xi - (1 - \lambda_\xi)c_\xi}{\lambda_\xi}$). Since $U_{(X, \|\cdot\|)} \subset \text{int } D$ we necessarily have $b \in S_{(X, \|\cdot\|)}$. Since $\|\cdot\|$ has the Kadec property, we have that $\|b_\xi - b\| \rightarrow 0$ and hence that the net $\{x_\xi\}_{\xi \in \Sigma}$ converges in norm to x .

In any case, we get a contradiction and the proof is concluded. \square

Theorem 3.8. *Let X be a separable Banach space and $|\cdot|$ defined as above. Then the following conditions hold:*

- (i) $(X, |\cdot|)$ is Gâteaux smooth;
- (ii) $(X, |\cdot|)$ is ALUR;
- (iii) if X is reflexive then $(X^*, |\cdot|)$ is Fréchet smooth;
- (iv) $(X, |\cdot|)$ is not LUR.

Proof. (i) It is sufficient to observe that the function $H : X \rightarrow \mathbb{R}$, defined by

$$H(x) = \sum_{n=2}^{\infty} 2^{-n} [f_n(x)]^2, \quad x \in X,$$

is Gâteaux differentiable on X . By Proposition 3.6, the map $F = |\cdot|^2$ is sum of two Gâteaux differentiable functions and hence $|\cdot|$ is Gâteaux smooth.

(ii) By Proposition 3.5 and Theorem 2.3, it is sufficient to prove that $|\cdot|$ has the Kadec property. Let $\{x_\xi\}_{\xi \in \Sigma} \subset S_{(X, |\cdot|)}$ be a net that is w -convergent to an element $x \in S_{(X, |\cdot|)}$. Then we have that, for each $n \in \mathbb{N}$, $\lim_\xi f_n(x_\xi) = f_n(x)$. This easily implies that the function H , defined as in the previous point, satisfies $\lim_\xi H(x_\xi) = H(x)$. Hence, by the definition of $|\cdot|$, we have $\lim_\xi \|x_\xi\| = \|x\|$. By Proposition 3.7 and the fact that $\{x_\xi\}_{\xi \in \Sigma}$ w -converges to x , we have that $\{x_\xi\}_{\xi \in \Sigma}$ $\|\cdot\|$ -converges to x .

(iii) The proof follows by Theorem 2.3, and the previous point.

(iv) Let us prove that $x_0 = \sqrt{2}e_1 \in S_{(X, |\cdot|)}$ is not a LUR point for $B_{(X, |\cdot|)}$. For $n \in \mathbb{N}$, define

$$x_n = \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_{3n}, \quad x_n^* = f_1 + f_{3n} = g_1 + g_{3n},$$

and observe that

- (a) $x_n \in B_{(X,|\cdot|)} \subset D$;
- (b) $\{x_n^*\}_n$ is bounded in X^* ;
- (c) for each $n \in \mathbb{N}$, we have

$$\sup x_n^*(B_{(X,\|\cdot\|)}) = \sqrt{2}, \quad \sup x_n^*(B) \leq \sqrt{2} + \frac{1}{(3n)^2}$$

and hence

$$\sup x_n^*(D) \leq \sqrt{2} + \frac{1}{(3n)^2};$$

- (d) for each $n \in \mathbb{N}$, we have

$$x_n^*(x_n) = x_n^*(x_0) = \sqrt{2} = x_n^*\left(\frac{x_0 + x_n}{2}\right).$$

Now, for $n \in \mathbb{N}$, define $z_n = \frac{x_0 + x_n}{2} + \frac{x_n}{\sqrt{2}(3n)^2}$, and observe that

$$x_n^*(z_n) = \sqrt{2} + \frac{1}{(3n)^2} \geq \sup x_n^*(D).$$

In particular, we have $\|z_n\| \geq 1$ and hence there exists $w_n \in [\frac{x_0 + x_n}{2}, z_n] \cap \partial D$. Hence,

$$\text{dist}_{\|\cdot\|}\left(\frac{x_0 + x_n}{2}, \partial D\right) \leq \left\|\frac{x_0 + x_n}{2} - w_n\right\| \leq \left\|\frac{x_0 + x_n}{2} - z_n\right\| \rightarrow 0,$$

that is, $\left\|\frac{x_0 + x_n}{2}\right\| \rightarrow 1$. Let us also observe that $|x_0| = 1$ and that, for each $n \in \mathbb{N}$, we have

$$|x_n|^2 \leq 1 + 2^{-n-1}, \quad \left|\frac{x_0 + x_n}{2}\right|^2 \geq \left\|\frac{x_0 + x_n}{2}\right\|^2;$$

hence, since $2|x_0|^2 + 2|x_n|^2 - |x_0 + x_n|^2 \geq 0$, we have

$$\lim_n (2|x_0|^2 + 2|x_n|^2 - |x_0 + x_n|^2) = 0.$$

On the other hand, for each $n \in \mathbb{N}$, we have $\|e_1 - e_n\| \geq g_1(e_1 - e_n) = 1$, and hence the sequence $\{x_n\}_n$ does not converge in norm to x_0 , the proof is concluded. \square

We conclude this section with a trivial consequence of the previous result. Recall that a Banach space is said *weakly locally uniformly rotund (WLUR)* if each point of its unit sphere is WLUR (see Definition 2.2).

Corollary 3.9. *Every infinite-dimensional Banach space admits an ALUR Gâteaux smooth equivalent norm which is not WLUR.*

Proof. It follows by observing that the norm defined in Theorem 3.8 satisfies the Kadec-Klee property. \square

4. CONSEQUENCES IN NONSEPARABLE BANACH SPACES

The present section is devoted to lift the construction made in Section 3 to the nonseparable setting. Let us first notice that if a Banach space has an ALUR norm, then it admits an equivalent LUR norm see [22, Proposition 1.4]. Therefore, not every Banach space can be renormed with an ALUR Gâteaux smooth norm which is not LUR. On the other hand, the following result, which is an immediate consequence of Theorem 3.8, provides a sufficient condition for a Banach space to have an equivalent ALUR Gâteaux smooth norm which is not LUR.

Corollary 4.1. *Let $(X, \|\cdot\|)$ be a Banach space and $Y \subset X$ be a separable subspace. Suppose that:*

- *the norm $\|\cdot\|$ is LUR and Gâteaux smooth;*
- *Y is infinite-dimensional and complemented in X .*

Then X admits an equivalent norm which is ALUR, Gâteaux smooth but not LUR.

Proof. By Theorem 3.8 there exists an ALUR, Gâteaux smooth, not LUR norm $|\cdot|_Y$ on Y which is equivalent to the norm $\|\cdot\|$ restricted to Y . Let $P : X \rightarrow X$ be a bounded linear projection onto Y . The norm $|\cdot|$, defined for $x \in X$ by

$$|x|^2 = \|(I - P)x\|^2 + |Px|_Y^2,$$

satisfies the desired properties. \square

It is worth to notice that the hypotheses of Corollary 4.1 are satisfied by a wide class of Banach spaces. To clarify this, let us recall the following definitions (see, e.g., [10, Section 6.1.6]).

Definition 4.2. Let X be a Banach space. Then X is said:

- (i) *weakly compactly generated* (WCG), if there exists a weakly compact subset K of X such that $\overline{\text{span}}(K) = X$;
- (ii) *weakly countably determined* (WCD) if there is a countable collection $\{K_n; n \in \mathbb{N}\}$ of w^* -compact subsets of X^{**} such that for every $x \in X$ and $u \in X^{**} \setminus X$ there is $n_0 \in \mathbb{N}$ for which $x \in K_{n_0}$ and $u \notin K_{n_0}$;
- (iii) *weakly Lindelöf determined* (WLD) if there exist a set Γ and a one-to-one bounded linear operator from X^* into $\ell_\infty^c(\Gamma)$ that is w^* -pointwise-continuous, where $\ell_\infty^c(\Gamma)$ denotes the subspace of $\ell_\infty(\Gamma)$ formed by all elements countably supported;
- (iv) to have the *separable complementation property* (SCP) if every separable subspace of X is contained in a complemented separable subspace of X .

The following implications hold (see [10]):

$$\begin{array}{ccccccc}
& & \langle R^* \rangle & & \langle LUR \rangle & & \\
& & \uparrow & & \uparrow & & \\
(WCG) & \Rightarrow & (WCD) & \Rightarrow & (WLD) & \Rightarrow & (SCP)
\end{array}$$

where $\langle LUR \rangle$ means that the space X admits an equivalent LUR norm, and $\langle R^* \rangle$ means that the space X admits an equivalent norm such that its dual norm is rotund. Notice that none of the above implications can be reversed.

Let us recall that, by Asplund averaging method (see [6, Theorem 4.1]), if X has $\langle R^* \rangle$ and $\langle LUR \rangle$ then it admits an equivalent LUR norm such that its dual norm is rotund. Moreover, if a Banach space X has the (SCP) then, in particular, it contains a complemented infinite-dimensional separable subspace. Hence, it is clear that Corollary 4.1 can be applied to the class (WLD) spaces having $\langle R^* \rangle$. In particular, *each (WCD) space admits an equivalent norm which is ALUR, Gâteaux smooth but not LUR*. Taking into account Theorems 2.3 and 2.4, we clearly obtain the following corollary.

Corollary 4.3. *Let X be a reflexive space. Then X admits an equivalent Gâteaux smooth not LUR norm $|\cdot|$ such that $|\cdot|^*$ is Fréchet smooth.*

The Banach spaces of continuous functions $C([0, \alpha])$, with α uncountable ordinal, constitute another relevant class of nonseparable Banach spaces to which we can apply Corollary 4.1, and that is not contained in the class of (WLD) spaces. Indeed, the containment of an infinite-dimensional separable complemented subspace is trivial and existence of an equivalent norm that is LUR and Gâteaux smooth follows by [13].

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