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Weak and strong solutions to the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system

Andrea Giorgini, Roger Temam *

Department of Mathematics & Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405, USA

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A B S T R A C T

We study the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system in a bounded smooth domain in $\mathbb{R}^d$, $d = 2, 3$. This model arises from the Diffuse Interface theory of binary mixtures accounting for density variation, capillarity effects at the interface and partial mixing. We consider the case of initial density away from zero and concentration-depending viscosity with free energy potential equal to either the Landau potential or the Flory-Huggins logarithmic potential. In this setting, we prove the existence of global weak solutions in two and three dimensions, and the existence of strong solutions with bounded and strictly positive density. The strong solutions are local in time in three dimensions and global in time in two dimensions.

R É S U M É

Dans cet article nous étudions le système de Navier-Stokes-Cahn-Hilliard, correspondant à un fluide non homogène et incompressible, posé dans un domaine borné de $\mathbb{R}^d$, $d = 2$ ou 3. Ce modèle correspond à l'étude d'un mélange binaire de fluides avec une interface diffuse et tient compte des variations de la densité, des effets de capillarité à l'interface et d'un mélange partiel des deux fluides. Nous supposons que la densité initiale est bornée inférieurement par un nombre strictement positif et que la viscosité dépend de la concentration. Nous supposons enfin que le potentiel d'énergie libre est soit le potentiel de Landau, soit le potentiel logarithmique de Flory-Huggins. Dans ce contexte nous prouvons l'existence globale de solutions faibles en dimension deux et trois, et l'existence de solutions fortes avec une densité bornée et strictement positive. Les solutions fortes sont locales en temps en dimension trois et globales en temps en dimension deux.

* Corresponding author.
E-mail addresses: agiorgin@iu.edu (A. Giorgini), temam@indiana.edu (R. Temam).

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1. Introduction

The subject of this article is the analysis of a Diffuse Interface model for isothermal nonhomogeneous incompressible viscous binary fluid mixtures. The model takes into account the capillarity effect and the partial mixing of the fluids in the interfacial region. In a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and given a time $T > 0$, for $(x,t) \in \Omega \times (0,T)$, $\rho = \rho(x,t)$ is the density of the mixture, $\boldsymbol{u} = \boldsymbol{u}(x,t)$ is the (mass-averaged) velocity of the mixture, $p = p(x,t)$ is the pressure of the mixture, $\phi = \phi(x,t)$ is the difference of fluids concentrations, $\mu = \mu(x,t)$ is the chemical potential. The dynamics of the state variables is described by the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system

$$\begin{cases} \partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0 \\ \rho \partial_t \boldsymbol{u} + \rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div}(\nu(\phi)\mathbb{D}\boldsymbol{u}) + \nabla p = -\operatorname{div}(\nabla \phi \otimes \nabla \phi) \\ \operatorname{div} \boldsymbol{u} = 0 \\ \rho \partial_t \phi + \rho \boldsymbol{u} \cdot \nabla \phi = \Delta \mu \\ \rho \mu = -\Delta \phi + \rho \Psi'(\phi) \end{cases} \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

subject to the boundary and initial conditions

$$\begin{cases} \boldsymbol{u} = 0, \quad \partial_{\mathbf{n}} \mu = \partial_{\mathbf{n}} \phi = 0 & \text{on } \partial\Omega \times (0,T), \\ \rho(\cdot,0) = \rho_0, \ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0, \ \phi(\cdot,0) = \phi_0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

Throughout this work we will assume that the viscosity $\nu = \nu(s) \in W^{1,\infty}(\mathbb{R})$ is such that $0 < \nu_* \leq \nu(s) \leq \nu^*$ for all $s \in \mathbb{R}$. For the potential $\Psi(s)$ (also called homogeneous free energy density), we will consider the physically relevant Flory-Huggins logarithmic potential

$$\Psi(s) = \frac{\theta}{2}\Big[(1+s)\ln(1+s) + (1-s)\ln(1-s)\Big] - \frac{\theta_0}{2}s^2 \quad \forall\, s \in [-1,1], \tag{1.3}$$

where $0 < \theta < \theta_0$, and the Landau potential[1]

$$\Psi_0(s) = \frac{1}{4}(s^2-1)^2 \quad \forall\, s \in \mathbb{R}. \tag{1.4}$$

In the last thirty years, the application of the Diffuse Interface (or Phase-Field) theory has became a fundamental method in fluid mechanics to model and simulate large deformations and topological transitions in two-phase flows. Originally developed for phase transition/separation phenomena, the Diffuse Interface methodology is based on the description of the interface separating the two fluids as a narrow region with finite thickness across which continuous fields can change smoothly their values. The evolution of the macroscopic state variables is derived through the combination of the continuum theory of mixtures, classical thermodynamics and statistical mechanics. From the mathematical viewpoint, the main advantage of the Diffuse Interface approach is the Eulerian representation of the interface recovered as the level set of the concentration (order) parameter. This is in contrast to the Lagrangian representation of free boundary problems. This aspect has been particularly significant for the computational analysis since the system is numerically solved on a fixed domain rather than on a domain with moving boundary. Among many others, two well-known applications of Diffuse Interface models have been the approximation of Sharp Interface problems through smearing of the interface and the description of the multiscale nature of complex

[1] We recall that the Landau potential is an approximation of the Flory-Huggins potential (up to an additive constant) in the limit $\frac{\theta}{\theta_0} \to 1$. We refer the reader to [38] for the physical derivation of the potentials.

fluids (e.g. polymers, liquid crystals, emulsions), which are characterized by a molecular interaction at the microscopic scale affecting the macroscopic dynamics. We refer the reader to the review articles [9,27,31] and the references therein for a more detailed description of the versatile features of this framework.

A paradigm model of the Diffuse Interface theory for two-phase flows is the Navier-Stokes-Cahn-Hilliard (NSCH) system, also called Model H after the seminal work [34] on dynamic critical phenomena. Over the past years there have been important developments concerning the mathematical modeling and analysis of NSCH systems for binary mixtures. Proceeding along the historical course, let us give an overview of the main achievements here:

- *Homogeneous incompressible mixtures*: The original Model H describes incompressible binary mixtures whose total density is constant. It corresponds to (1.1) with density $\rho = 1$. The system was derived in [32] within the framework of continuum mechanics and in [42] through an energetic variational approach (see also [27] for a recent review). The mathematical analysis of the Model H have been addressed in several papers. In the case of the Flory-Huggins potential (1.3), the existence, uniqueness and regularity of weak and strong solutions have been established in [1] and [30]. The existence of global weak solutions in the case of degenerate mobility (cf. Section 7) was earlier proven in [12]. In the case of the Landau potential (1.4) and its polynomial generalizations, the analysis has been carried out in [12,16,24,30]. We also mention the existence of global weak solutions achieved in [25] for boundary conditions which account for moving contact lines.

- *Compressible mixtures*: A first generalization of the Model H was derived in the remarkable work [44] for compressible binary mixtures. This compressible NSCH system was initially studied in [6] taking the Helmholtz free energy of the form $\int_\Omega \frac{1}{2}|\nabla\phi|^2 + \rho F(\rho,\phi)\,\mathrm{d}x$ where F assumes a specific form (see [6, Eqns. (1.11) − (1.14)] for more details). The authors of [6] proved the existence of global weak solutions in three dimensions. This result has been recently extended in [17] to the case of dynamic boundary conditions. Secondly, the compressible NSCH system with (different) Helmholtz free energy of the form $\int_\Omega \frac{1}{2}\rho|\nabla\phi|^2 + \rho\overline{\Psi}(\rho,\phi)\,\mathrm{d}x$, where $\overline{\Psi} \in \mathcal{C}^5(\mathbb{R}^2)$ is a general function, has been investigated in [36], where the authors proved existence and uniqueness of local strong solutions in the L^p setting. Further generalizations of the compressible NSCH model proposed in [44] have been developed in [23] and [33] through a framework based on entropy production.

- *Concentration-depending density/quasi-incompressible mixtures*: A second generalization of the Model H is concerned with mixtures of incompressible fluids having different constant densities. As observed in [44], the incompressible nature of the fluids involves a degeneracy in the Legendre transformation between the Helmholtz and Gibbs free energy, which leads to a constitutive equation that connects the density with the concentration $\rho = \rho(\phi)$.[2] In this direction, a model was derived in [44] in terms of the mass-averaged velocity and the difference of fluid (mass) concentrations. In this case, the mixture becomes slightly compressible because of the non-zero divergence of the mass-averaged velocity due to interfacial diffusion, and both the density and the pressure appear in the equation for the chemical potential. This quasi-incompressible NSCH model was studied in [2] and [3] in two and three dimensions. In the former, the author proves the existence of global weak solutions assuming the Helmholtz free energy of the form $\int_\Omega a(\phi)\frac{|\nabla\phi|^q}{q} + \Psi(\phi)\,\mathrm{d}x$, where $q > d$, a and Ψ are suitable functions on $\mathbb{R}$. In the latter, the author shows the existence and uniqueness of local strong solutions in the case of the Helmholtz free energy $\int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x$, where Ψ is a general function in $\mathcal{C}^3(\mathbb{R})$. It is worth mentioning that a new version of the quasi-incompessible NSCH model has been derived in [45]. Later on, several works have been devoted to the derivation and analysis of NSCH systems with non-matching densities whose variables are the volume-averaged velocity and the difference

[2] Examples of this relation are $\frac{1}{\rho(\phi)} = \frac{\phi}{\rho_1} + \frac{1-\phi}{\rho_2}$ (simple mixture) and $\rho(\phi) = \rho_1\frac{1+\phi}{2} + \rho_2\frac{1-\phi}{2}$, where ρ_1 and ρ_2 are the constant densities of the two fluids.

of fluid volume fractions. It is noteworthy that, contrary to the systems in [44] and [45] based on the mass-averaged velocity, the volume-averaged velocity leads to an incompressibility relation for the total flow. Two models were initially derived in [14] and [22]. The first one has been studied in [13], where the author proves the existence and uniqueness of local strong solutions and the existence of weak solutions under a smallness conditions of the relative difference of the two fluids densities. Both models however do not satisfy an energy balance law. To overcome this issue, a new thermodynamically consistent NSCH model was derived in [7]. This system was subsequently studied in [4] and [5] where the existence of global weak solutions has been proven in the case of logarithmic potential and positive mobility and of logarithmic potential and degenerate mobility, respectively, in two and three dimensions. The existence result was recently extended to the case with dynamic boundary conditions in [26].

• *Nonhomogeneous incompressible mixtures*: Another generalization of the Model H lies at the intersection between the aforementioned classes. This relies on the assumptions that the density of the mixture is an independent variable of the system and the mass-averaged velocity of the mixture is divergence-free. This leads to the system (1.1)-(1.2), whose derivation is based on the conservation of mass and linear momentum and the second law of thermodynamic in the form of dissipation inequality as in [6,33,32]. To the best of our knowledge, the system (1.1)-(1.2) has only been recently studied in [52], where is proven the local existence of classical solutions in the case of constant viscosity and Landau potential.

In this paper we present a mathematical analysis concerning weak and strong solutions for the nonhomogeneous incompressible NSCH system (1.1)-(1.2). The analysis is inspired by the classical techniques for the nonhomogeneous incompressible Navier-Stokes equations, i.e. eqns. $(1.1)_1$-$(1.1)_3$ with a driving force in the right-hand side of $(1.1)_2$, devised in [10,11,37,40,41,46], and the more recent methods for the Model H introduced in [4,29,30]. Our specific purpose is twofold. First, we aim to prove the existence of global weak solutions in the general setting with only bounded but positive density, physically consistent non-constant viscosity parameter, and Landau or Flory-Huggins potential. We note that to date the existence of (global) solutions for NSCH systems with thermodynamically relevant Flory-Huggins potential has only shown for the classical Model H (cf. [2,30]) and the density-depending NSCH model proposed in [7] (cf. [4,5]). In both cases, however, the density does not enter in the Cahn-Hilliard equation, and a main issue concerning the bound of the L^1-norm of $\Psi'(\phi)$ (which implies a control of μ in $H^1(\Omega)$) is overcome by exploiting the monotonicity of the logarithmic part $F'(s)$ of $\Psi'(s)$. Our first contribution consists in extending this technique to the non-constant density case in $(1.1)_5$. Having at our disposal the energy balance (2.4), our argument is based on three elements: a preliminary estimate of $\Delta\phi$, a suitable test function $\frac{1}{\rho}\big(\rho\phi - \overline{\rho_0\phi_0}\big)$ (where $\overline{u} = \frac{1}{|\Omega|}\int_\Omega u \,\mathrm{d}x$), and a monotonicity estimate (cf. (5.26))

$$\int_\Omega |F'(\phi)| \,\mathrm{d}x \le C_0 \int_\Omega F'(\phi)\big(\rho\phi - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x + C_1, \tag{1.5}$$

for some constants C_0, C_1 depending on the lower and upper bound of the density. Notice that $\overline{\rho(t)\phi(t)}$ is an invariant of the flow. In particular, the estimate (1.5) holds under the assumption $-1 < \frac{\overline{\rho_0\phi_0}}{\rho_*} < 1$, where $\rho_* = \operatorname{ess\,inf}_\Omega \rho$, which replaces the condition $-1 < \overline{\phi_0} < 1$ of the constant density case (cf. [4,30]). Next, the second object is showing the existence of suitable strong solutions by maintaining the most general assumptions on the density. Despite the presence of the density in several terms of the Cahn-Hilliard equation and its fourth-order structure, it turns out that a higher-order differential inequality (cf. (6.17)) can be achieved without any differential operator acting on the density function. This provides higher regularity estimates for the solution provided that the following compatibility condition on the initial data is imposed

$$\rho_0 \in L^\infty(\Omega),\ \phi_0 \in H^2(\Omega) \quad \text{such that} \quad \partial_{\mathbf{n}}\phi_0 = 0 \text{ on } \partial\Omega, \quad \mu_0 = \frac{-\Delta\phi_0}{\rho_0} + \Psi_0'(\phi_0) \in H^1(\Omega).$$

We notice that a suitable approximation of the initial data is necessary to guarantee the validity of these estimates for the approximated regular solutions. As a result, in three dimensions we derive the existence of local strong solutions. In the two dimensional case, thanks to some refined estimates and a logarithmic type Gronwall argument, we obtain the global existence of strong solutions. We point out that such solutions have bounded and strictly positive density ρ, as opposed to the $H^3(\Omega)$ regularity required in [52].

Plan of the paper. In Section 2 we derive the energy balance and the conservation of mass. In Section 3 we report the notation and the mathematical tools used in the analysis. Sections 4 and 5 are devoted to the existence of global weak solution in the case of Landau potential and Flory-Huggins potential, respectively. In Section 6 we prove the existence of strong solutions in the Landau potential case. Future directions are collected in Section 7. In Appendixes A and B we show the existence of approximated regular solutions and a suitable approximation of the initial data.

2. Energy balance and conservation of mass

In this section we report the total energy balance and the conservation of mass associated with the system (1.1)-(1.2), which will play a crucial role in our analysis, assuming that the solution of (1.1)-(1.2) is sufficiently regular.

Total energy balance. Multiplying $(1.1)_2$ by $\boldsymbol{u}$ and integrating over Ω, we have

$$\int_\Omega \rho\partial_t\Big(\frac{|\boldsymbol{u}|^2}{2}\Big)\,\mathrm{d}x + \int_\Omega \rho\boldsymbol{u}\cdot\nabla\Big(\frac{|\boldsymbol{u}|^2}{2}\Big)\,\mathrm{d}x + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x = \int_\Omega -\operatorname{div}(\nabla\phi\otimes\nabla\phi)\cdot\boldsymbol{u}\,\mathrm{d}x.$$

Observing that

$$\int_\Omega \rho\partial_t\Big(\frac{1}{2}|\boldsymbol{u}|^2\Big)\,\mathrm{d}x + \int_\Omega \rho\boldsymbol{u}\cdot\nabla\Big(\frac{1}{2}|\boldsymbol{u}|^2\Big)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2\,\mathrm{d}x - \int_\Omega \big(\partial_t\rho + \operatorname{div}(\rho\boldsymbol{u})\big)\frac{|\boldsymbol{u}|^2}{2}\,\mathrm{d}x,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x = \int_\Omega -\operatorname{div}(\nabla\phi\otimes\nabla\phi)\cdot\boldsymbol{u}\,\mathrm{d}x.$$

Moreover, in light of the relations

$$\begin{aligned} -\operatorname{div}(\nabla\phi\otimes\nabla\phi) &= -\Delta\phi\nabla\phi - \nabla\Big(\frac{1}{2}|\nabla\phi|^2\Big) \\ &= \rho\mu\nabla\phi - \rho\Psi'(\phi)\nabla\phi - \nabla\Big(\frac{1}{2}|\nabla\phi|^2\Big) \\ &= \rho\mu\nabla\phi - \rho\nabla\Psi(\phi) - \nabla\Big(\frac{1}{2}|\nabla\phi|^2\Big), \end{aligned} \tag{2.1}$$

we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\boldsymbol{u}|^2\,\mathrm{d}x + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2\,\mathrm{d}x = \int_\Omega \rho\mu\nabla\phi\cdot\boldsymbol{u}\,\mathrm{d}x - \int_\Omega \rho\boldsymbol{u}\cdot\nabla\Psi(\phi)\,\mathrm{d}x. \tag{2.2}$$

Next, multiplying $(1.1)_4$ by μ and integrating over Ω, we find

$$\int_\Omega \rho\partial_t\phi\mu\,\mathrm{d}x + \int_\Omega \rho\boldsymbol{u}\cdot\nabla\phi\mu\,\mathrm{d}x + \int_\Omega |\nabla\mu|^2\,\mathrm{d}x = 0.$$

By exploiting $(1.1)_4$ and (1.2), we have

$$\int_\Omega \rho \partial_t \phi \mu \, \mathrm{d}x = -\int_\Omega \partial_t \phi \Delta \phi \, \mathrm{d}x + \int_\Omega \rho \Psi'(\phi) \partial_t \phi \, \mathrm{d}x$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla \phi|^2 \, \mathrm{d}x + \int_\Omega \rho \partial_t \Psi(\phi) \, \mathrm{d}x$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla \phi|^2 + \rho \Psi(\phi) \, \mathrm{d}x - \int_\Omega \partial_t \rho \Psi(\phi) \, \mathrm{d}x.$$

Thus, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla \phi|^2 + \rho \Psi(\phi) \, \mathrm{d}x + \int_\Omega \rho \boldsymbol{u} \cdot \nabla \phi \mu \, \mathrm{d}x + \int_\Omega |\nabla \mu|^2 \, \mathrm{d}x = \int_\Omega \partial_t \rho \Psi(\phi) \, \mathrm{d}x. \tag{2.3}$$

By summing (2.2) and (2.3), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla \phi|^2 + \rho \Psi(\phi) \, \mathrm{d}x + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2 \, \mathrm{d}x + \int_\Omega |\nabla \mu|^2 \, \mathrm{d}x$$
$$= \int_\Omega \partial_t \rho \Psi(\phi) \, \mathrm{d}x - \int_\Omega \rho \boldsymbol{u} \cdot \nabla \Psi(\phi) \, \mathrm{d}x.$$

After integrating by parts, and owing to $(1.1)_1$, we observe that

$$\int_\Omega \partial_t \rho \Psi(\phi) \, \mathrm{d}x - \int_\Omega \rho \boldsymbol{u} \cdot \nabla \Psi(\phi) \, \mathrm{d}x = \int_\Omega \big(\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u})\big) \Psi(\phi) \, \mathrm{d}x = 0.$$

Therefore, we infer that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla \phi|^2 + \rho \Psi(\phi) \, \mathrm{d}x + \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2 \, \mathrm{d}x + \int_\Omega |\nabla \mu|^2 \, \mathrm{d}x = 0.$$

Integrating the above equation in time, we obtain for all $t \in [0, T]$

$$E(\rho(t), \boldsymbol{u}(t), \phi(t)) + \int_0^t \int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2 + |\nabla \mu|^2 \, \mathrm{d}x \, \mathrm{d}\tau = E(\rho_0, \boldsymbol{u}_0, \phi_0), \tag{2.4}$$

having set

$$E(\rho, \boldsymbol{u}, \phi) = \int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla \phi|^2 + \rho \Psi(\phi) \, \mathrm{d}x.$$

Conservation of mass. First, integrating $(1.1)_1$ over Ω, and using the boundary condition of $\boldsymbol{u}$, we have for all $t \in [0, T]$

$$\int_\Omega \rho(t) \, \mathrm{d}x = \int_\Omega \rho_0 \, \mathrm{d}x. \tag{2.5}$$

Next, integrating $(1.1)_4$ over Ω and using the boundary condition of μ, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\phi\,\mathrm{d}x - \int_\Omega \big(\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u})\big)\phi\,dx = 0.$$

Thus, by $(1.1)_1$, we obtain for all $t\in[0,T]$

$$\int_\Omega \rho(t)\phi(t)\,\mathrm{d}x = \int_\Omega \rho_0\phi_0\,\mathrm{d}x. \tag{2.6}$$

3. Functional setting

Let X be a (real) Banach or Hilbert space with norm $\|\cdot\|_X$. We denote by X' the dual space of X and by $\langle\cdot,\cdot\rangle$, the duality product between X and X'. Let Ω be a bounded domain in $\mathbb{R}^d$, where $d=2$ or $d=3$, with boundary $\partial\Omega$. We denote by $W^{k,p}(\Omega)$, $k\in\mathbb{N}$, the Sobolev space of functions in $L^p(\Omega)$ with distributional derivatives of order less than or equal to k in $L^p(\Omega)$ and by $\|\cdot\|_{W^{k,p}(\Omega)}$ its norm. For $k\in\mathbb{N}$, the Hilbert space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ with norm $\|\cdot\|_{H^k(\Omega)}$. We denote by $H^1_0(\Omega)$ the closure of $\mathcal{C}^\infty_0(\Omega)$ in $H^1(\Omega)$ and by $H^{-1}(\Omega)$ its dual space. The inner product and norm in $L^2(\Omega)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_{L^2(\Omega)}$, respectively. We denote by $\overline{u}$ the average of u over Ω, that is $\overline{u}=|\Omega|^{-1}\langle u,1\rangle$, for all $u\in(H^1(\Omega))'$. By the generalized Poincaré inequality (see [48, Chapter II, Section 1.4]), we recall that

$$u\to\Big(\|\nabla u\|^2_{L^2(\Omega)}+\Big|\frac{1}{|\Omega|}\int_\Omega u\,\mathrm{d}x\Big|^2\Big)^{\frac12}\quad\text{and}\quad u\to\Big(\|\nabla u\|^2_{L^2(\Omega)}+\Big|\int_\Omega \eta u\,\mathrm{d}x\Big|^2\Big)^{\frac12}, \tag{3.1}$$

where $\eta\in L^\infty(\Omega)$ is such that $0<\eta_*\le\eta(x)\le\eta^*$ for almost every $x\in\Omega$, are norms on $H^1(\Omega)$ equivalent to $\|u\|_{H^1(\Omega)}$. In particular, there exists a positive constant $C=C(\Omega,\eta_*,\eta^*)$ such that

$$\|u\|_{H^1(\Omega)}\le C\Big(\|\nabla u\|^2_{L^2(\Omega)}+\Big|\int_\Omega \eta u\,\mathrm{d}x\Big|^2\Big)^{\frac12},\quad \forall\, u\in H^1(\Omega). \tag{3.2}$$

We recall the following Gagliardo-Nirenberg and Agmon inequalities

$$\|u\|_{L^4(\Omega)}\le C\|u\|^{\frac12}_{L^2(\Omega)}\|u\|^{\frac12}_{H^1(\Omega)},\qquad \forall\, u\in H^1(\Omega),\quad \text{if } d=2, \tag{3.3}$$

$$\|u\|_{L^3(\Omega)}\le C\|u\|^{\frac12}_{L^2(\Omega)}\|u\|^{\frac12}_{H^1(\Omega)},\qquad \forall\, u\in H^1(\Omega),\quad \text{if } d=3, \tag{3.4}$$

$$\|u\|_{L^\infty(\Omega)}\le C\|u\|^{\frac12}_{L^2(\Omega)}\|u\|^{\frac12}_{H^2(\Omega)},\qquad \forall\, u\in H^2(\Omega),\quad \text{if } d=2, \tag{3.5}$$

$$\|u\|_{L^\infty(\Omega)}\le C\|u\|^{\frac12}_{H^1(\Omega)}\|u\|^{\frac12}_{H^2(\Omega)},\qquad \forall\, u\in H^2(\Omega),\quad \text{if } d=3. \tag{3.6}$$

We also report the logarithmic estimate of the product (see [30, Proposition C.1])

$$\|uv\|_{L^2(\Omega)}\le C\|u\|_{H^1(\Omega)}\|v\|_{L^2(\Omega)}\ln^{\frac12}\Big(\mathrm{e}\frac{\|v\|_{H^1(\Omega)}}{\|v\|_{L^2(\Omega)}}\Big),\quad \forall\, u,v\in H^1(\Omega),\quad \text{if } d=2. \tag{3.7}$$

We now introduce the Hilbert spaces of solenoidal vector-valued functions. We denote by $\mathcal{C}^\infty_{0,\sigma}(\Omega)$ the space of divergence free vector fields in $\mathcal{C}^\infty_0(\Omega)$. We define $\mathbf{H}_\sigma$ and $\mathbf{V}_\sigma$ as the closure of $\mathcal{C}^\infty_{0,\sigma}(\Omega)$ with respect to the $L^2(\Omega)$ and $H^1_0(\Omega)$ norms, respectively. We also use $(\cdot,\cdot)$ and $\|\cdot\|_{L^2(\Omega)}$ for the inner product and norm in $\mathbf{H}_\sigma$. The space $\mathbf{V}_\sigma$ is endowed with the inner product and norm $(\boldsymbol{u},\boldsymbol{v})_{\mathbf{V}_\sigma}=(\nabla\boldsymbol{u},\nabla\boldsymbol{v})$ and $\|\boldsymbol{u}\|_{\mathbf{V}_\sigma}=\|\nabla\boldsymbol{u}\|_{L^2(\Omega)}$, respectively. We denote by $\mathbf{V}_\sigma'$ its dual space. We recall that the Korn's inequality entails

$$\|\nabla \boldsymbol{u}\|_{L^2(\Omega)} \leq \sqrt{2}\|\mathbb{D}\boldsymbol{u}\|_{L^2(\Omega)} \leq \sqrt{2}\|\nabla \boldsymbol{u}\|_{L^2(\Omega)}, \quad \forall \boldsymbol{u} \in \mathbf{V}_\sigma, \tag{3.8}$$

where $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t)$. In turn, the above inequality gives that $\boldsymbol{u} \to \|\mathbb{D}\boldsymbol{u}\|_{L^2(\Omega)}$ is a norm on $\mathbf{V}_\sigma$ equivalent to the initial norm. We consider the Hilbert space $\mathbf{W}_\sigma = \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ with inner product and norm $(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{W}_\sigma} = (\mathbf{A}\boldsymbol{u}, \mathbf{A}\boldsymbol{v})$ and $\|\boldsymbol{u}\|_{\mathbf{W}_\sigma} = \|\mathbf{A}\boldsymbol{u}\|$, where $\mathbf{A}$ is the Stokes operator. We recall that there exists $C > 0$ such that

$$\|\boldsymbol{u}\|_{H^2(\Omega)} \leq C\|\boldsymbol{u}\|_{\mathbf{W}_\sigma}, \quad \forall \boldsymbol{u} \in \mathbf{W}_\sigma. \tag{3.9}$$

Given $1 \leq p \leq \infty$ and an interval $I \subseteq [0, \infty)$, the set $L^p(I; X)$ consists of all Bochner measurable p-integrable functions defined on I with values in X. We define the space $W^{1,p}(0, T; X)$ to consist of all functions $f \in L^p(0, T; X)$ with the vector-valued distributional derivative $\partial_t f$ in $L^p(0, T; X)$. In particular, we set $H^1(0, T; X) = W^{1,2}(0, T; X)$. The set of continuous functions $f : I \to X$ is denoted by $\mathcal{C}(I; X)$. The space $\mathcal{C}(I; X_w)$ consists of all functions $f \in L^\infty(I; X)$ such that the map $t \in I \mapsto \langle \phi, f(t)\rangle$ is continuous, for all $\phi \in X'$. The Besov spaces denoted by $B^s_{p,\infty}(I; X)$, with $s \in (0, 1)$, consist of the sets of functions $f \in L^p(I; X)$ with finite norm

$$\|f\|_{B^s_{p,\infty}(I;X)} = \|f\|_{L^p(I;X)} + \sup_{0<h\leq 1} h^{-s}\|\Delta_h f\|_{L^p(I_h;X)},$$

where $\Delta_h f(t) = f(t + h) - f(t)$ and $I_h = \{t \in I : t + h \in I\}$. We recall that $B^s_{\infty,\infty}(I; X) = \mathcal{C}^s(I; X)$. We report the following compactness and embedding results (see [15,46,47,49])

Lemma 3.1. *Let X, Y, Z be three Banach spaces such that $X \subset Y \subset Z$. Assume that $X \overset{c}{\hookrightarrow} Y$ and $Y \hookrightarrow Z$. Then, for all $1 \leq q \leq \infty$ and $0 < \sigma < 1$, we have*

$$L^q(0, T; X) \cap B^\sigma_{q,\infty}(0, T; Z) \overset{c}{\hookrightarrow} L^q(0, T; Y), \qquad 1 \leq q < \infty,$$
$$\mathcal{C}([0, T]; X) \cap B^\sigma_{q,\infty}(0, T; Z) \overset{c}{\hookrightarrow} \mathcal{C}([0, T]; Y), \qquad q = \infty.$$

Lemma 3.2. *Let X, Y be two Banach spaces such that $X \hookrightarrow Y$ and $Y' \hookrightarrow X'$ densely. Then, $L^\infty(0, T; X) \cap \mathcal{C}([0, T]; Y_w) \hookrightarrow \mathcal{C}([0, T]; X_w)$.*

4. Existence of global weak solutions: the Landau potential case

In this section we prove the first result concerning the existence of global weak solutions to system (1.1)-(1.2) with Landau potential (1.4).

Theorem 4.1. *Let Ω be a bounded domain of class $\mathcal{C}^3$ in $\mathbb{R}^d$, $d = 2, 3$, and let T be a positive time. Assume that $\rho_0 \in L^\infty(\Omega)$ such that $0 < \rho_* \leq \rho_0(x) \leq \rho^*$ a.e. in Ω, $\boldsymbol{u}_0 \in L^2(\Omega)$ and $\phi_0 \in H^1(\Omega)$. Then, there exists a weak solution $(\rho, \boldsymbol{u}, \phi, \mu)$ to system (1.1)-(1.2) in the following sense:*

(i) *The solution $(\rho, \boldsymbol{u}, \phi, \mu)$ satisfies*

$$\begin{aligned}
&\rho \in \mathcal{C}([0, T]; L^r(\Omega)) \cap L^\infty(\Omega \times (0, T)) \cap W^{1,\infty}(0, T; H^{-1}(\Omega)),\\
&\boldsymbol{u} \in \mathcal{C}([0, T]; (\mathbf{H}_\sigma)_w) \cap L^2(0, T; \mathbf{V}_\sigma) \cap B^{\frac{1}{4}}_{2,\infty}(0, T; \mathbf{H}_\sigma),\\
&\phi \in \mathcal{C}([0, T]; (H^1(\Omega))_w) \cap L^4(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)) \cap B^{\frac{1}{4}}_{\infty,\infty}(0, T; L^2(\Omega)),\\
&\mu \in L^4(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)),
\end{aligned} \tag{4.1}$$

for any $r \in [1,\infty)$*, and* $q = 6$ *if* $d = 3$ *and for any* $q \in [2,\infty)$ *if* $d = 2$*. In addition,*

$$\rho_* \leq \rho(x,t) \leq \rho^* \quad a.e.\ in\ \Omega \times (0,T). \tag{4.2}$$

(ii) *The system* (1.1) *is satisfied as follows:*

$$\int_0^T \int_\Omega \rho \partial_t \psi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \rho \boldsymbol{u} \cdot \nabla \psi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{4.3}$$

for all $\psi \in \mathcal{C}_c^\infty(\Omega \times (0,T))$*;*

$$\begin{aligned} &-\int_0^T \int_\Omega \rho \boldsymbol{u} \cdot \partial_t \boldsymbol{w} \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \rho \boldsymbol{u} \otimes \boldsymbol{u} : \nabla \boldsymbol{w} \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega \nu(\phi) \mathbb{D}\boldsymbol{u} : \nabla \boldsymbol{w} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_0 \boldsymbol{u}_0 \cdot \boldsymbol{w}(0)\,\mathrm{d}x + \int_0^T \int_\Omega \nabla\phi \otimes \nabla \phi : \nabla \boldsymbol{w} \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.4}$$

for all $\boldsymbol{w} \in \mathcal{C}_c^1([0,T); \mathbf{V}_\sigma)$*;*

$$\begin{aligned} -\int_0^T \int_\Omega \rho \phi \partial_t w \,\mathrm{d}x\, dt - \int_0^T \int_\Omega \rho \boldsymbol{u} \phi \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \nabla \mu \cdot \nabla w \,\mathrm{d}x\,\mathrm{d}t \\ = \int_\Omega \rho_0 \phi_0 w(0)\,\mathrm{d}x \end{aligned} \tag{4.5}$$

for all $w \in \mathcal{C}_c^1([0,T); H^1(\Omega))$*;*

$$\rho\mu = -\Delta \phi + \rho \Psi_0'(\phi) \quad a.e.\ in\ \Omega \times (0,T). \tag{4.6}$$

Furthermore, $\partial_{\mathbf{n}} \phi = 0$ *almost everywhere on* $\partial\Omega \times (0,T)$*.*

(iii) *The initial data is assumed in the following sense:* $\rho(t) \to \rho(0) = \rho_0$ *in* $L^r(\Omega)$*, for all* $r \in [1,\infty)$*,* $\boldsymbol{u}(t) \to \boldsymbol{u}(0) = \boldsymbol{u}_0$ *in* $L^2(\Omega)$*,* $\phi(t) \to \phi(0) = \phi_0$ *in* $H^1(\Omega)$*, as* $t \to 0^+$*.*

(iv) *The conservation of total mass and of mass of fluids difference holds as follows*

$$\int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \rho_0 \,\mathrm{d}x, \quad \int_\Omega \rho(t)\phi(t)\,\mathrm{d}x = \int_\Omega \rho_0\phi_0\,\mathrm{d}x \quad \forall\, t \in [0,T]. \tag{4.7}$$

(v) *The energy inequality*

$$E_0(\rho(t), \boldsymbol{u}(t), \phi(t)) + \int_0^t \int_\Omega \nu(\phi) |\mathbb{D}\boldsymbol{u}|^2 + |\nabla \mu|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq E_0(\rho_0, \boldsymbol{u}_0, \phi_0) \tag{4.8}$$

holds for all $t \in [0,T]$*, where*

$$E_0(\rho, \boldsymbol{u}, \phi) = \int_\Omega \frac{1}{2}\rho |\boldsymbol{u}|^2 + \frac{1}{2}|\nabla\phi|^2 + \rho \Psi_0(\phi)\,\mathrm{d}x.$$

Remark 4.2. The proof of Theorem 4.1 presented here below can be easily adapted with minor changes to the following cases:

(i) Density-depending viscosity $\nu = \nu(\rho)$. We refer to [33,41] for physical motivations;
(ii) General potential $\Psi_0 \in C^2(\mathbb{R})$ satisfying the growth conditions

$$-C_1 \leq \Psi_0(s), \quad -C_2 \leq \Psi_0''(s) \leq C_3|s|^2 + C_4 \quad \forall\, s \in \mathbb{R},$$

where C_1, C_2, C_3, C_4 are positive constants.

Remark 4.3. The pressure p is recovered in a classical way. We refer the reader to [41,46,49].

Proof of Theorem 4.1. The proof consists of several steps.

The approximate problem. Given $\rho_0 \in L^\infty(\Omega)$ such that $\rho_* \leq \rho_0(x) \leq \rho^*$ for almost every $x \in \Omega$, we find by classical mollification a sequence $\{\rho_{0m}\}_{m=1}^\infty$ such that

$$\rho_{0m} \in \mathcal{C}^\infty(\overline{\Omega}), \ \forall\, m \in \mathbb{N}, \quad \rho_* \leq \rho_{0m}(x) \leq \rho^* \quad \forall\, x \in \overline{\Omega}, \ \forall\, m \in \mathbb{N},$$

and

$$\rho_{0m} \to \rho_0 \quad \text{strongly in } L^r(\Omega), \ \forall\, r \in [1, \infty), \quad \rho_{0m} \rightharpoonup \rho_0 \quad \text{weak-star in } L^\infty(\Omega). \tag{4.9}$$

We consider the family of eigenfunctions $\{w_j\}_{j=1}^\infty$ and eigenvalues $\{\lambda_j\}_{j=1}^\infty$ of the Laplace operator $A = -\Delta + I$ with homogeneous Neumann boundary condition and the family of eigenfunctions $\{\boldsymbol{w}_j\}_{j=1}^\infty$ and eigenvalues $\{\lambda_j^S\}_{j=1}^\infty$ of the Stokes operator $\mathbf{A}$. For any integer $m \geq 1$, we define the finite-dimensional subspaces of $H^1(\Omega)$ and $\mathbf{V}_\sigma$, respectively, by $V_m = \text{span}\{w_1, ..., w_m\}$ and $\mathbf{V}_m = \text{span}\{\boldsymbol{w}_1, ..., \boldsymbol{w}_m\}$. We denote by Π_m and $\mathbb{P}_m$ the orthogonal projections on V_m and $\mathbf{V}_m$ with respect to the inner product in $L^2(\Omega)$ and in $\mathbf{H}_\sigma$, respectively.

For any $m \in \mathbb{N}$, we consider the triplet $(\rho_{0m}, \boldsymbol{u}_{0m}, \phi_{0m})$, where $\boldsymbol{u}_{0m} = \mathbb{P}_m \boldsymbol{u}_0$ and $\phi_{0m} = \Pi_m \phi_0$ satisfy

$$\boldsymbol{u}_{0m} \to \boldsymbol{u}_0 \quad \text{strongly in } L^2(\Omega), \quad \phi_{0m} \to \phi_0 \quad \text{strongly in } H^1(\Omega). \tag{4.10}$$

We determine the approximate solutions $(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$ that solve system (1.1)-(1.2) as follows:

$$\rho_m \in \mathcal{C}^1(\overline{Q_T}), \quad \boldsymbol{u}_m \in \mathcal{C}^1([0,T]; \mathbf{V}_m), \quad \phi_m \in \mathcal{C}^1([0,T]; V_m), \quad \mu_m \in \mathcal{C}([0,T]; V_m), \tag{4.11}$$

where $Q_T = \Omega \times (0,T)$, such that

$$\partial_t \rho_m + \boldsymbol{u}_m \cdot \nabla \rho_m = 0 \quad \text{in } \Omega \times (0,T), \tag{4.12}$$

and

$$\begin{aligned}(\rho_m \partial_t \boldsymbol{u}_m, \boldsymbol{w}) + (\rho_m(\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}) + (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla \boldsymbol{w}) \\ = (\rho_m \mu_m \nabla \phi_m, \boldsymbol{w}) - (\rho_m \nabla \Psi_0(\phi_m), \boldsymbol{w}),\end{aligned} \tag{4.13}$$

$$(\rho_m \partial_t \phi_m, w) + (\rho_m \boldsymbol{u}_m \cdot \nabla \phi_m, w) + (\nabla \mu_m, \nabla w) = 0, \tag{4.14}$$

$$(\rho_m \mu_m, w) = (\nabla \phi_m, \nabla w) + (\rho_m \Psi_0'(\phi_m), w), \tag{4.15}$$

for all $\boldsymbol{w} \in \mathbf{V}_m$ and $w \in V_m$, and for all $t \in [0,T]$. In particular, the density is defined by

$$\rho_m(x,t) = \rho_{0m}(\mathbf{X}_m(0,t,x)), \tag{4.16}$$

where

$$\mathbf{X}_m(s,t,x) = x + \int_t^s \boldsymbol{u}_m(\mathbf{X}_m(\tau,t,x),\tau)\,\mathrm{d}\tau \quad \forall\, s,t \in [0,T]. \tag{4.17}$$

Notice that the right-hand side in (4.13) has been rewritten according to (2.1). Furthermore, the approximate solution $(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$ satisfies the boundary and initial conditions

$$\begin{cases} \boldsymbol{u}_m = 0, \quad \partial_{\mathbf{n}}\mu_m = \partial_{\mathbf{n}}\phi_m = 0 & \text{on } \partial\Omega \times (0,T), \\ \rho_m(\cdot,0) = \rho_{0m},\ \boldsymbol{u}_m(\cdot,0) = \boldsymbol{u}_{0m},\ \phi(\cdot,0) = \phi_{0m} & \text{in } \Omega. \end{cases} \tag{4.18}$$

The existence of the approximated solutions $(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$ satisfying (4.11)-(4.18) is proven in Appendix A.

Estimates for the approximate solutions. First, we recall that ρ_m satisfies the estimates

$$\rho_* \le \rho_m(x,t) \le \rho^* \quad \forall\,(x,t) \in \overline{Q_T}. \tag{4.19}$$

By arguing as in Appendix A (cf. (A.10), see also Section 2), we take $\boldsymbol{w} = \boldsymbol{u}_m$ in (4.13), $w = \mu_m$ in (4.14) and $w = \partial_t\phi_m$ in (4.15). We find for all $t \in [0,T]$

$$E_0(\rho_m(t), \boldsymbol{u}_m(t), \phi_m(t)) + \int_0^t \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2 + |\nabla\mu_m|^2 \,\mathrm{d}x\,\mathrm{d}\tau = E_0(\rho_{0m}, \boldsymbol{u}_{0m}, \phi_{0m}). \tag{4.20}$$

In light of (4.9) and (4.10), we deduce that

$$E_0(\rho_{0m}, \boldsymbol{u}_{0m}, \phi_{0m}) \to E_0(\rho_0, \boldsymbol{u}_0, \phi_0),$$

as $m \to \infty$. Then, observing that $\Psi_0(s) \ge 0$ for all $s \in \mathbb{R}$, we have for m sufficiently large

$$\|\boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2 + \|\nabla\phi_m(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega \nu(\phi_m(\tau))|\mathbb{D}\boldsymbol{u}_m(\tau)|^2 + |\nabla\mu_m(\tau)|^2 \,\mathrm{d}x\,\mathrm{d}\tau \\ \le \frac{1}{\min\{\frac{\rho_*}{2}, \frac{1}{2}\}}\Big(1 + E_0(\rho_0, \boldsymbol{u}_0, \phi_0)\Big).$$

Then, there exists a constant $C = C(\rho_*, \rho^*, \nu_*)$ such that

$$\|\boldsymbol{u}_m\|_{L^\infty(0,T;\mathbf{H}_\sigma)} \le C(1 + E_0^{\frac{1}{2}}), \tag{4.21}$$

$$\|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{V}_\sigma)} \le C(1 + E_0^{\frac{1}{2}}), \tag{4.22}$$

$$\|\nabla\phi_m\|_{L^\infty(0,T;L^2(\Omega))} \le C(1 + E_0^{\frac{1}{2}}), \tag{4.23}$$

$$\|\nabla\mu_m\|_{L^2(0,T;L^2(\Omega))} \le C(1 + E_0^{\frac{1}{2}}). \tag{4.24}$$

Here we use the notation $E_0 = E_0(\rho_0, \boldsymbol{u}_0, \phi_0)$. In the sequel, we will specify the dependency of the constants C in brackets, but we will omit the parameters ρ_*, ρ^*, ν_*, ν^*. Since $w_1 = 1$, we infer from (4.12) and (4.14) that

$$\Big|\int_\Omega \rho_m(t)\phi_m(t)\,\mathrm{d}x\Big| = \Big|\int_\Omega \rho_{0m}\phi_{0m}\,\mathrm{d}x\Big| \le 1+|\Omega||\overline{\rho_0\phi_0}|.$$

In light of (3.1), together with (4.23), the above inequality implies that

$$\|\phi_m\|_{L^\infty(0,T;H^1(\Omega))} \le C(\overline{\rho_0\phi_0},E_0). \tag{4.25}$$

Taking also $w=\mu_m$ in (4.15), and using (4.19), we obtain

$$\begin{aligned}\rho_*\|\mu_m\|^2_{L^2(\Omega)} &\le \|\nabla\phi_m\|_{L^2(\Omega)}\|\nabla\mu_m\|_{L^2(\Omega)} + \rho^*\|\Psi_0'(\phi_m)\|_{L^2(\Omega)}\|\mu_m\|_{L^2(\Omega)}\\ &\le \|\nabla\phi_m\|_{L^2(\Omega)}\|\nabla\mu_m\|_{L^2(\Omega)} + C(1+\|\phi_m\|^3_{L^6(\Omega)})\|\mu_m\|_{L^2(\Omega)}.\end{aligned}$$

In light of (4.25), we have

$$\|\mu_m\|^2_{L^2(\Omega)} \le C(\overline{\rho_0\phi_0},E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)}). \tag{4.26}$$

Thus, we deduce from (4.24) and (4.25) that

$$\|\mu_m\|_{L^4(0,T;L^2(\Omega))} \le C(\overline{\rho_0\phi_0},E_0,T), \quad \|\mu_m\|_{L^2(0,T;H^1(\Omega))} \le C(\overline{\rho_0\phi_0},E_0,T). \tag{4.27}$$

We take $w=-\Delta\phi_m$ in (4.15). Using integration by parts, and (4.19), we find

$$\begin{aligned}\|\Delta\phi_m\|^2_{L^2(\Omega)} &= -\int_\Omega \rho_m\mu_m\Delta\phi_m\,\mathrm{d}x + \int_\Omega \rho_m\Psi_0'(\phi_m)\Delta\phi_m\,\mathrm{d}x\\ &\le \rho^*\|\mu_m\|_{L^2(\Omega)}\|\Delta\phi_m\|_{L^2(\Omega)} + \rho^*\|\Psi_0'(\phi_m)\|_{L^2(\Omega)}\|\Delta\phi_m\|_{L^2(\Omega)}\\ &\le C\big(1+\|\mu_m\|_{L^2(\Omega)}+\|\phi_m\|^3_{L^6(\Omega)}\big)\|\Delta\phi_m\|_{L^2(\Omega)}.\end{aligned} \tag{4.28}$$

Thanks to (4.25) and (4.27), the above estimate yields

$$\|\phi_m\|_{L^4(0,T;H^2(\Omega))} \le C(\overline{\rho_0\phi_0},E_0,T). \tag{4.29}$$

Let us consider $v\in H^1_0(\Omega)$ such that $\|v\|_{H^1_0(\Omega)}\le 1$. We multiply the transport equation by v and integrate over Ω. Since the boundary term vanishes, we find

$$\langle \partial_t\rho_m, v\rangle = \int_\Omega \rho_m\boldsymbol{u}_m\nabla v\,\mathrm{d}x.$$

Thus, using (4.19) and (4.21), we deduce that

$$\|\partial_t\rho_m\|_{L^\infty(0,T;H^{-1}(\Omega))} \le C(E_0). \tag{4.30}$$

Translation estimates. Since we are not able to estimate time derivatives of $\boldsymbol{u}_m$ and ϕ_m, and hence to use the Aubin-Lions compactness theorem, we will estimate translation differences of $\boldsymbol{u}_m$ and ϕ_m, namely $\boldsymbol{u}_m(t+h)-\boldsymbol{u}_m(t)$ and $\phi_m(t+h)-\phi_m(t)$. This type of estimates has been used in [10,11,40,46,50].[3] Now,

[3] A similar estimate has been used in the context of stochastic processes, first proven in [8].

recalling that ρ_m solves $\partial_t \rho_m + \mathrm{div}(\rho_m \boldsymbol{u}_m) = 0$, we multiply the transport equation by $\boldsymbol{u}_m \cdot \boldsymbol{w}$, where $\boldsymbol{w} \in \mathbf{V}_m$. Integrating over Ω, we find

$$(\partial_t \rho_m \boldsymbol{u}_m, \boldsymbol{w}) + (\mathrm{div}(\rho_m \boldsymbol{u}_m)\boldsymbol{u}_m, \boldsymbol{w}) = 0. \tag{4.31}$$

By summing (4.13) and (4.31), we obtain

$$\begin{aligned} &\int_\Omega \partial_t(\rho_m \boldsymbol{u}_m) \cdot \boldsymbol{w} \, \mathrm{d}x + \int_\Omega \mathrm{div}(\rho_m \boldsymbol{u}_m \otimes \boldsymbol{u}_m) \cdot \boldsymbol{w} \, \mathrm{d}x + \int_\Omega \nu(\phi_m) \mathbb{D}\boldsymbol{u}_m : \nabla \boldsymbol{w} \, \mathrm{d}x \\ &\qquad = \int_\Omega \rho_m \mu_m \nabla \phi_m \cdot \boldsymbol{w} \, \mathrm{d}x - \int_\Omega \rho_m \nabla \Psi_0(\phi_m) \, \mathrm{d}x. \end{aligned} \tag{4.32}$$

Integrating (4.32) on the time interval $(t, t+h)$, where $t \in [0, T-h]$, we have

$$\begin{aligned} &(\rho_m(t+h)\boldsymbol{u}_m(t+h) - \rho_m(t)\boldsymbol{u}_m(t), \boldsymbol{w}) \\ &= -\int_t^{t+h} \int_\Omega \mathrm{div}(\rho_m(\tau)\boldsymbol{u}_m(\tau) \otimes \boldsymbol{u}_m(\tau)) \cdot \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}\tau - \int_t^{t+h} \int_\Omega \nu(\phi_m(\tau)) \mathbb{D}\boldsymbol{u}_m(\tau) : \nabla \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}\tau \\ &\quad + \int_t^{t+h} \int_\Omega \rho_m(\tau)\mu_m(\tau) \nabla \phi_m(\tau) \cdot \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}\tau - \int_t^{t+h} \int_\Omega \rho_m(\tau) \nabla \Psi_0(\phi_m(\tau)) \cdot \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}\tau. \end{aligned}$$

Taking $\boldsymbol{w} = \boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)$, we obtain

$$\begin{aligned} &\int_\Omega \rho_m(t+h)|\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)|^2 \, \mathrm{d}x \\ &= \underbrace{\int_\Omega -(\rho_m(t+h) - \rho_m(t))\boldsymbol{u}_m(t) \cdot (\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)) \, \mathrm{d}x}_{I_1(t)} \\ &\quad + \underbrace{\int_t^{t+h} \int_\Omega -\mathrm{div}(\rho_m(\tau)\boldsymbol{u}_m(\tau) \otimes \boldsymbol{u}_m(\tau)) \cdot \big(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\big) \, \mathrm{d}x \, \mathrm{d}\tau}_{I_2(t)} \\ &\quad + \underbrace{\int_t^{t+h} \int_\Omega -\nu(\phi_m(\tau)) \mathbb{D}\boldsymbol{u}_m(\tau) : \nabla(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)) \, \mathrm{d}x \, \mathrm{d}\tau}_{I_3(t)} \\ &\quad + \underbrace{\int_t^{t+h} \int_\Omega \rho_m(\tau)\mu_m(\tau) \nabla \phi_m(\tau) \cdot \big(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\big) \, \mathrm{d}x \, \mathrm{d}\tau}_{I_4(t)} \\ &\quad + \underbrace{\int_t^{t+h} \int_\Omega -\rho_m(\tau) \Psi_0'(\phi_m(\tau)) \nabla \phi_m(\tau) \cdot \big(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\big) \, \mathrm{d}x \, \mathrm{d}\tau}_{I_5(t)}. \end{aligned} \tag{4.33}$$

In order to estimate $I_1(t)$, by using the transport equation and integration by parts, we rewrite $I_1(t)$ as follows

$$
\begin{aligned}
I_1(t) &= \int_\Omega \int_t^{t+h} \operatorname{div}(\rho_m(\tau)\boldsymbol{u}_m(\tau))\,\mathrm{d}\tau \left(\boldsymbol{u}_m(t)\cdot(\boldsymbol{u}_m(t+h)-\boldsymbol{u}_m(t))\right)\mathrm{d}x \\
&= \int_\Omega \int_t^{t+h} \rho_m(\tau)\boldsymbol{u}_m(\tau)\,\mathrm{d}\tau \cdot \nabla\Big(\boldsymbol{u}_m(t)\cdot(\boldsymbol{u}_m(t+h)-\boldsymbol{u}_m(t))\Big)\,\mathrm{d}x.
\end{aligned}
$$

By exploiting (3.4), (4.19), (4.21), (4.22), and the Sobolev embedding, we deduce that

$$
\begin{aligned}
|I_1(t)| &\le \Big\| \int_t^{t+h} \rho_m(\tau)\boldsymbol{u}_m(\tau)\,\mathrm{d}\tau\Big\|_{L^3(\Omega)} \Big(\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\big(\|\boldsymbol{u}_m(t+h)\|_{L^6(\Omega)}+\|\boldsymbol{u}_m(t)\|_{L^6(\Omega)}\big) \\
&\quad + \big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\big)\|\boldsymbol{u}_m(t)\|_{L^6(\Omega)}\Big) \\
&\le C\rho^* \int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^3(\Omega)}\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^2+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2\Big) \\
&\le C \int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac12}\|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac12}\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^2+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2\Big) \\
&\le C h^{\frac34} \|\boldsymbol{u}_m\|_{L^\infty(t,t+h;\mathbf{H}_\sigma)}^{\frac12}\|\nabla \boldsymbol{u}_m\|_{L^2(t,t+h;L^2(\Omega))}^{\frac12} \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^2+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2\Big) \\
&\le C(E_0) h^{\frac34} \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^2+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2\Big).
\end{aligned}
$$

Therefore, thanks to (4.22), we have

$$
\int_0^{T-h} |I_1(t)|\,\mathrm{d}t \le C(E_0) h^{\frac34}, \tag{4.34}
$$

where the positive constant C is independent of m and h. Using integration by parts, the estimates (4.19) and (4.21), and an interpolation argument, we have

$$
\begin{aligned}
|I_2(t)| &= \Big| \int_t^{t+h}\int_\Omega \rho_m(\tau)\boldsymbol{u}_m(\tau)\otimes \boldsymbol{u}_m(\tau) : \nabla(\boldsymbol{u}_m(t+h)-\boldsymbol{u}_m(t))\,\mathrm{d}x\,\mathrm{d}\tau\Big| \\
&\le \rho^* \int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^4(\Omega)}^2\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\Big) \\
&\le C \int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac12}\|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac32}\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\Big) \\
&\le C(E_0) \int_t^{t+h} \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac32}\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}+\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\Big).
\end{aligned}
$$

Integrating $|I_2(t)|$ on the time interval $(0, T-h)$, and using (4.22) and Fubini's theorem, we find

$$
\begin{aligned}
&\int_0^{T-h} |I_2(t)|\,\mathrm{d}t \\
&\leq C(E_0) \int_0^h \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} \int_0^\tau \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad + C(E_0) \int_h^{T-h} \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} \int_{\tau-h}^{\tau} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\quad + C(E_0) \int_{T-h}^{T} \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} \int_{\tau-h}^{T-h} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}\,\mathrm{d}t\,\mathrm{d}\tau \\
&\leq C(E_0) \int_0^h \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} \tau^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}\,\mathrm{d}\tau \\
&\quad + C(E_0) \int_h^{T-h} \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} h^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}\,\mathrm{d}\tau \\
&\quad + C(E_0) \int_{T-h}^{T} \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}} (T-\tau)^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}\,\mathrm{d}\tau \\
&\leq C(E_0) h^{\frac{1}{2}} \int_0^T \|\nabla \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^{\frac{3}{2}}\,\mathrm{d}\tau.
\end{aligned}
$$

Thus, by exploiting (4.22), it yields

$$
\int_0^{T-h} |I_2(t)|\,\mathrm{d}t \leq C(E_0, T) h^{\frac{1}{2}}. \tag{4.35}
$$

Since $\nu(s) \leq \nu^*$ for all $s \in \mathbb{R}$, we find

$$
\begin{aligned}
|I_3(t)| &\leq \nu^* \int_t^{t+h} \|\mathbb{D}\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}\,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)} \Big) \\
&\leq \nu^* h^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2(t,t+h;L^2(\Omega))} \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)} \Big) \\
&\leq C(E_0) h^{\frac{1}{2}} \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)} \Big).
\end{aligned}
$$

This implies that

$$
\int_0^{T-h} |I_3(t)|\,\mathrm{d}t \leq C(E_0, T) h^{\frac{1}{2}}. \tag{4.36}
$$

Owing to (3.4), (4.21), and (4.25), we have

$$
\begin{aligned}
|I_4(t)| &\le \rho^* \int_t^{t+h} \|\mu_m(\tau)\|_{L^6(\Omega)} \|\nabla \phi_m(\tau)\|_{L^2(\Omega)} \,\mathrm{d}\tau \left(\|\boldsymbol{u}_m(t+h)\|_{L^3(\Omega)} + \|\boldsymbol{u}_m(t)\|_{L^3(\Omega)} \right) \\
&\le C(E_0) \int_t^{t+h} \|\mu_m(\tau)\|_{H^1(\Omega)} \,\mathrm{d}\tau \Big(\|\boldsymbol{u}_m(t+h)\|_{L^2(\Omega)} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac12} \\
&\quad + \|\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12} \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac12} \Big) \\
&\le C(E_0) \int_t^{t+h} \|\mu_m(\tau)\|_{H^1(\Omega)} \,\mathrm{d}\tau \Big(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac12} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac12} \Big).
\end{aligned}
$$

We integrate $|I_4(t)|$ on the time interval $(0, T-h)$, and we use the estimate (4.22). We obtain

$$
\begin{aligned}
&\int_0^{T-h} |I_4(t)| \,\mathrm{d}t \\
&\le C(E_0) \int_0^{h} \|\mu_m(\tau)\|_{H^1(\Omega)} \int_0^{\tau} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac12} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac12} \,\mathrm{d}t \,\mathrm{d}\tau \\
&\quad + C(E_0) \int_h^{T-h} \|\mu_m(\tau)\|_{H^1(\Omega)} \int_{\tau-h}^{\tau} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac12} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac12} \,\mathrm{d}t \,\mathrm{d}\tau \\
&\quad + C(E_0) \int_{T-h}^{T} \|\mu_m(\tau)\|_{H^1(\Omega)} \int_{\tau-h}^{T-h} \|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac12} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac12} \,\mathrm{d}t \,\mathrm{d}\tau \\
&\le C(E_0) \int_0^{h} \|\mu_m(\tau)\|_{H^1(\Omega)} \tau^{\frac34} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}^{\frac12} \,\mathrm{d}\tau \\
&\quad + C(E_0) \int_h^{T-h} \|\mu_m(\tau)\|_{H^1(\Omega)} h^{\frac34} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}^{\frac12} \,\mathrm{d}\tau \\
&\quad + C(E_0) \int_{T-h}^{T} \|\mu_m(\tau)\|_{H^1(\Omega)} (T-\tau)^{\frac34} \|\nabla \boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega))}^{\frac12} \,\mathrm{d}\tau \\
&\le C(E_0) h^{\frac34} \int_0^{T} \|\mu_m(\tau)\|_{H^1(\Omega)} \,\mathrm{d}\tau.
\end{aligned}
$$

In turn, owing to (4.27), we infer that

$$
\int_0^{T-h} |I_4(t)| \,\mathrm{d}t \le C(\overline{\rho_0 \phi_0}, E_0, T) h^{\frac34}. \tag{4.37}
$$

By using (3.4), (4.21), and (4.25), we find

$$
\begin{aligned}
|I_5(t)| &\le \rho^* \int_t^{t+h} \|\Psi_0'(\phi_m(\tau))\|_{L^2(\Omega)} \|\nabla \phi_m(\tau)\|_{L^6(\Omega)} \,\mathrm{d}\tau \, \|\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\|_{L^3(\Omega)} \\
&\le C(E_0) \int_t^{t+h} \|\phi(\tau)\|_{H^2(\Omega)} \,\mathrm{d}\tau \left(\|\nabla \boldsymbol{u}_m(t+h)\|_{L^2(\Omega)}^{\frac{1}{2}} + \|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^{\frac{1}{2}}\right).
\end{aligned}
$$

Arguing as above for $I_4(t)$, we find

$$
\int_0^{T-h} |I_5(t)| \,\mathrm{d}t \le C(E_0) h^{\frac{3}{4}} \int_0^T \|\phi_m(\tau)\|_{H^2(\Omega)} \,\mathrm{d}\tau.
$$

Thus, in light of (4.29), we deduce that

$$
\int_0^{T-h} |I_5(t)| \,\mathrm{d}t \le C(\overline{\rho_0 \phi_0}, E_0, T) h^{\frac{3}{4}}. \tag{4.38}
$$

By exploiting (4.19), integrating (4.33) on the time interval $(0, T-h)$, and using (4.34)-(4.38), we arrive at

$$
\int_0^{T-h} \|\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2 \,\mathrm{d}t \le C(\overline{\rho_0 \phi_0}, E_0, T) h^{\frac{1}{2}}, \tag{4.39}
$$

where C is independent of m and h. We now multiply the transport equation by $\phi_m w$, where $w \in V_m$ and integrate over Ω. This gives

$$
(\partial_t \rho_m \phi_m, w) + (\operatorname{div}(\rho_m \boldsymbol{u}_m) \phi_m, w) = 0. \tag{4.40}
$$

By summing (4.14) and (4.40), we obtain

$$
\int_\Omega \partial_t(\rho_m \phi_m) w \,\mathrm{d}x + \int_\Omega \operatorname{div}(\rho_m \boldsymbol{u}_m \phi_m) w \,\mathrm{d}x + \int_\Omega \nabla \mu_m \cdot \nabla w \,\mathrm{d}x = 0. \tag{4.41}
$$

Integrating (4.41) on the time interval $(t, t+h)$, where $t \in [0, T-h]$, we are led to

$$
\begin{aligned}
&(\rho_m(t+h)\phi_m(t+h) - \rho_m(t)\phi_m(t), w) \\
&= - \int_t^{t+h} \int_\Omega \operatorname{div}(\rho_m(\tau) \boldsymbol{u}_m(\tau) \phi_m(\tau)) w \,\mathrm{d}x \,\mathrm{d}\tau - \int_t^{t+h} \int_\Omega \nabla \mu_m(\tau) \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}\tau \\
&= \int_t^{t+h} \int_\Omega \rho_m(\tau) \boldsymbol{u}_m(\tau) \phi_m(\tau) \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}\tau - \int_t^{t+h} \int_\Omega \nabla \mu_m(\tau) \cdot \nabla w \,\mathrm{d}x \,\mathrm{d}\tau.
\end{aligned}
$$

We take $w = \phi_m(t+h) - \phi_m(t)$, and we rewrite the above equation as follows

$$\begin{aligned}
&\int_\Omega \rho_m(t+h)|\phi_m(t+h)-\phi_m(t)|^2 \,\mathrm{d}x \\
&= \underbrace{\int_\Omega (\rho_m(t+h)-\rho_m(t))\phi_m(t)(\phi_m(t+h)-\phi_m(t))\,\mathrm{d}x}_{J_1(t)} \\
&\quad + \underbrace{\int_t^{t+h}\int_\Omega \rho_m(\tau)\boldsymbol{u}_m(\tau)\phi_m(\tau)\cdot\nabla(\phi_m(t+h)-\phi_m(t))\,\mathrm{d}x\,\mathrm{d}\tau}_{J_2(t)} \\
&\quad + \underbrace{\int_t^{t+h}\int_\Omega -\nabla\mu_m(\tau)\cdot(\nabla\phi_m(t+h)-\phi_m(t))\,\mathrm{d}x\,\mathrm{d}\tau}_{J_3(t)}.
\end{aligned} \tag{4.42}$$

By following the argument used above for $I_1(t)$, we use the transport equation to rewrite $J_1(t)$ as follows

$$\begin{aligned}
J_1(t) &= \int_\Omega \int_t^{t+h} \operatorname{div}(\rho_m(\tau)\boldsymbol{u}_m(\tau))\,\mathrm{d}\tau\,\phi_m(t)(\phi_m(t+h)-\phi_m(t))\,\mathrm{d}x \\
&= -\int_\Omega \int_t^{t+h} \rho_m(\tau)\boldsymbol{u}_m(\tau)\,\mathrm{d}\tau\cdot\nabla\big(\phi_m(t)(\phi_m(t+h)-\phi_m(t))\big)\,\mathrm{d}x.
\end{aligned}$$

By using (4.22) and (4.25), we have

$$\begin{aligned}
|J_1(t)| &\le \Big\| \int_t^{t+h} \rho_m(\tau)\boldsymbol{u}_m(\tau)\,\mathrm{d}\tau\Big\|_{L^3(\Omega)} \Big(\|\nabla\phi_m(t)\|_{L^2(\Omega)}\|\phi_m(t+h)-\phi_m(t)\|_{L^6(\Omega)} \\
&\quad + \|\phi_m(t)\|_{L^6(\Omega)}\|\nabla(\phi_m(t+h)-\phi_m(t))\|_{L^2(\Omega)}\Big) \\
&\le C(\overline{\rho_0\phi_0}, E_0)\int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^3(\Omega)}\,\mathrm{d}\tau \\
&\le C(\overline{\rho_0\phi_0}, E_0)h^{\frac12}\|\nabla\boldsymbol{u}_m\|_{L^2(t,t+h;L^2(\Omega))} \le C(\overline{\rho_0\phi_0}, E_0)h^{\frac12},
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
|J_2(t)| &\le \rho^* \int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^3(\Omega)}\|\phi_m(\tau)\|_{L^6(\Omega)}\,\mathrm{d}\tau\,\|\nabla(\phi_m(t+h)-\phi_m(t))\|_{L^2(\Omega)} \\
&\le C(\overline{\rho_0\phi_0}, E_0)\int_t^{t+h} \|\boldsymbol{u}_m(\tau)\|_{L^3(\Omega)}\,\mathrm{d}\tau \le C(\overline{\rho_0\phi_0}, E_0)h^{\frac12},
\end{aligned} \tag{4.44}$$

$$\begin{aligned}|J_3(t)| &\leq \int_t^{t+h} \|\nabla \mu_m(\tau)\|_{L^2(\Omega)}\, \mathrm{d}\tau\, \|\nabla(\phi_m(t+h)-\phi_m(t))\|_{L^2(\Omega)} \\ &\leq C(\overline{\rho_0\phi_0}, E_0) \int_t^{t+h} \|\nabla \mu_m(\tau)\|_{L^2(\Omega)}\, \mathrm{d}\tau \leq C(\overline{\rho_0\phi_0}, E_0) h^{\frac{1}{2}}.\end{aligned} \tag{4.45}$$

Therefore, collecting (4.43), (4.44) and (4.45) together, we deduce that[4]

$$\sup_{0\leq t\leq T-h} \|\phi_m(t+h)-\phi_m(t)\|_{L^2(\Omega)} \leq C(\overline{\rho_0\phi_0}, E_0) h^{\frac{1}{4}}, \tag{4.46}$$

where C is a positive constant independent of m and h. We conclude from (4.39) and (4.46) that

$$\|\boldsymbol{u}_m\|_{B^{\frac{1}{4}}_{2,\infty}(0,T;\mathbf{H}_\sigma)} \leq C(\overline{\rho_0\phi_0}, E_0, T), \quad \|\phi_m\|_{B^{\frac{1}{4}}_{\infty,\infty}(0,T;L^2(\Omega))} \leq C(\overline{\rho_0\phi_0}, E_0), \tag{4.47}$$

where C is a positive constant independent of m and h.

Passage to the limit. Thanks to (4.19), (4.21), (4.22), (4.25), (4.27) and (4.29), we infer that (up to a subsequence)

$$\begin{aligned}
&\rho_m \rightharpoonup \rho && \text{weak-star in } L^\infty(\Omega\times(0,T)),\\
&\partial_t\rho_m \rightharpoonup \partial_t\rho && \text{weak-star in } L^\infty(0,T;H^{-1}(\Omega)),\\
&\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} && \text{weak-star in } L^\infty(0,T;\mathbf{H}_\sigma),\\
&\boldsymbol{u}_m \rightharpoonup \boldsymbol{u} && \text{weakly in } L^2(0,T;\mathbf{V}_\sigma),\\
&\phi_m \rightharpoonup \phi && \text{weak-star in } L^\infty(0,T;H^1(\Omega)),\\
&\phi_m \rightharpoonup \phi && \text{weakly in } L^4(0,T;H^2(\Omega)),\\
&\mu_m \rightharpoonup \mu && \text{weakly in } L^4(0,T;L^2(\Omega)),\\
&\mu_m \rightharpoonup \mu && \text{weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned} \tag{4.48}$$

In light of (4.22), (4.29), and (4.47), we deduce from Lemma 3.1 that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;L^q(\Omega)), \forall\, q\in[2,6), \tag{4.49}$$

$$\phi_m \to \phi \quad \text{strongly in } L^4(0,T;W^{1,q}(\Omega)), \forall\, q\in[2,6), \tag{4.50}$$

$$\phi_m \to \phi \quad \text{strongly in } \mathcal{C}([0,T];L^q(\Omega)), \forall\, q\in[2,6). \tag{4.51}$$

The strong convergence of the density is obtained from [41, Lemma 2.4] which entails that

$$\rho_m \to \rho \quad \text{strongly in } \mathcal{C}([0,T];L^r(\Omega)), \forall\, r\in[1,\infty). \tag{4.52}$$

Thanks to the above estimates and convergence results, we infer that

[4] It is also possible to show by arguing as for the velocity $\boldsymbol{u}_m$ that ϕ_m is uniformly bounded in $B^{\frac{1}{2}}_{2,\infty}(0,T;L^2(\Omega))$.

$$
\begin{aligned}
&\rho_m \boldsymbol{u}_m \to \rho \boldsymbol{u} && \text{strongly in } L^2(0,T;L^q(\Omega)), \forall\, q\in[2,6),\\
&\rho_m \phi_m \to \rho\phi && \text{strongly in } \mathcal{C}([0,T];L^q(\Omega)), \forall\, q\in[2,6),\\
&\rho_m \boldsymbol{u}_m\otimes \boldsymbol{u}_m \rightharpoonup \rho \boldsymbol{u}\otimes \boldsymbol{u} && \text{weakly in } L^{\frac43}(0,T;L^2(\Omega)),\\
&\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m \rightharpoonup \nu(\phi)\mathbb{D}\boldsymbol{u} && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
&\rho_m\mu_m \rightharpoonup \rho\mu && \text{weakly in } L^2(0,T;L^6(\Omega)),\\
&\rho_m\mu_m\nabla\phi_m \rightharpoonup \rho\mu\nabla\phi && \text{weakly in } L^{\frac43}(0,T;L^2(\Omega)),\\
&\rho_m\Psi_0'(\phi_m) \rightharpoonup \rho\Psi_0'(\phi) && \text{strongly in } \mathcal{C}([0,T];L^{\frac32}(\Omega)),\\
&\rho_m\Psi_0'(\phi_m)\nabla\phi_m \rightharpoonup \rho\Psi_0'(\phi)\nabla\phi && \text{weakly in } L^2(0,T;L^{\frac32}(\Omega)),\\
&\rho_m\boldsymbol{u}_m\phi_m \to \rho\boldsymbol{u}\phi && \text{strongly in } L^2(0,T;L^2(\Omega)).
\end{aligned}
\tag{4.53}
$$

We are now in position to pass to the limit in the weak formulations. First, thanks to [41, Lemma 2.4], we know that

$$
-\int_0^T\int_\Omega \rho\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \rho\boldsymbol{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t=0, \tag{4.54}
$$

for all $\psi\in\mathcal{C}_c^1([0,T),H_0^1(\Omega))$. Next, let us fix $l\in\mathbb{N}$, and take $m\geq l$. By using the transport equation (4.12), we rewrite (4.13) and (4.14) as follows

$$
\begin{aligned}
(\partial_t(\rho_m\boldsymbol{u}_m),\boldsymbol{w})+(\mathrm{div}(\rho_m\boldsymbol{u}_m\otimes\boldsymbol{u}_m),\boldsymbol{w})+(\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m,\nabla\boldsymbol{w})&\\
=(\rho_m\mu_m\nabla\phi_m,\boldsymbol{w})-(\rho_m\Psi_0'(\phi_m)\nabla\phi_m,\boldsymbol{w})&,
\end{aligned}
\tag{4.55}
$$

for all $\boldsymbol{w}\in\mathbf{V}_l$, and

$$
(\partial_t(\rho_m\phi_m),w)+(\mathrm{div}(\rho_m\boldsymbol{u}_m\phi_m),w)+(\nabla\mu_m,\nabla w)=0, \tag{4.56}
$$

for all $w\in V_l$. Taking $\psi,\chi\in\mathcal{C}_c^1([0,T))$, we multiply (4.55) and (4.56) by ψ and χ, respectively, and integrate on the time interval $(0,T)$. We recall that $\rho_{0m}\to\rho_0$ in $L^r(\Omega)$ for all $r\in[2,\infty)$, $\boldsymbol{u}_{0m}\to\boldsymbol{u}_0$ in $\mathbf{H}_\sigma$, and $\phi_{0m}\to\phi$ in $H^1(\Omega)$. By exploiting the convergence results (4.48) and (4.53), we can pass to the limit as $m\to\infty$ getting

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho\boldsymbol{u}\cdot\boldsymbol{w}\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t&\\
+\int_0^T\int_\Omega \nu(\phi)\mathbb{D}\boldsymbol{u}:\nabla\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t=\psi(0)\int_\Omega \rho_0\boldsymbol{u}_0\cdot\boldsymbol{w}\,\mathrm{d}x&\\
+\int_0^T\int_\Omega \rho\mu\nabla\phi\cdot\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \rho\Psi_0'(\phi)\nabla\phi\cdot\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t&,
\end{aligned}
\tag{4.57}
$$

for all $\boldsymbol{w}\in\mathbf{V}_l$, and

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho\phi w \partial_t\chi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho\boldsymbol{u}\phi\cdot\nabla w\chi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\mu\cdot\nabla w\chi\,\mathrm{d}x\,\mathrm{d}t \\
= \chi(0)\int_\Omega \rho_0\phi_0 w\,\mathrm{d}x,
\end{aligned}
\tag{4.58}
$$

for all $w \in V_l$. Next, passing to the limit as $l \to \infty$, and using a classical density argument, we deduce that

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho\boldsymbol{u}\cdot\partial_t\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u} : \nabla\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t \\
+ \int_0^T\int_\Omega \nu(\phi_m)\mathbb{D}\boldsymbol{u} : \nabla\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \rho_0\boldsymbol{u}_0\cdot\boldsymbol{w}(0)\,\mathrm{d}x \\
+ \int_0^T\int_\Omega \rho\mu\nabla\phi\cdot\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho\Psi_0'(\phi)\nabla\phi\cdot\boldsymbol{w}\psi\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{4.59}
$$

for all $\boldsymbol{w} \in \mathcal{C}_c^1([0,T), \mathbf{V}_\sigma)$, and

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho\phi\partial_t w\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \rho\boldsymbol{u}\phi\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \nabla\mu\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t \\
= \int_\Omega \rho_0\phi_0 w(0)\,\mathrm{d}x,
\end{aligned}
\tag{4.60}
$$

for all $w \in \mathcal{C}_c^1([0,T), H^1(\Omega))$. We then take a test function $\varphi \in \mathcal{C}_c((0,T))$, multiply (4.15) by φ, and integrate over $[0,T]$. By using (4.48) and (4.53), we pass to the limit as $m \to \infty$, and then as $l \to \infty$. We easily infer from a density argument that

$$
\rho\mu = -\Delta\phi + \rho\Psi_0'(\phi) \tag{4.61}
$$

almost everywhere in $\Omega \times (0,T)$. Finally, in light of (4.61) and the relation (2.1), we are in position to rewrite the right-hand side of (4.59) as in (4.4).

Time continuity. Thanks to (4.51) and (4.52), we have that $\rho \in \mathcal{C}([0,T]; L^r(\Omega))$ for any $r \in [1,\infty)$, and $\phi \in \mathcal{C}([0,T]; L^q(\Omega))$ for any $q \in [2,6)$. In addition, since $\phi \in L^\infty(0,T; H^1(\Omega))$, we infer from Lemma 3.2 that $\phi \in \mathcal{C}([0,T]; (H^1(\Omega))_w)$. Next, we rewrite (4.55) as follows

$$
\begin{aligned}
\langle \partial_t \mathbb{P}_m(\rho_m\boldsymbol{u}_m), \boldsymbol{w}\rangle = (\rho_m\boldsymbol{u}_m\otimes\boldsymbol{u}_m, \nabla\mathbb{P}_m\boldsymbol{w}) - (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla\mathbb{P}_m\boldsymbol{w}) \\
+ (\rho_m\mu_m\nabla\phi_m, \mathbb{P}_m\boldsymbol{w}) - (\rho_m\Psi_0'(\phi_m)\nabla\phi_m, \mathbb{P}_m\boldsymbol{w}),
\end{aligned}
\tag{4.62}
$$

for all $\boldsymbol{w} \in \mathbf{V}_\sigma$. Here we have used that $\mathbb{P}_m$ commutes with the time derivative. By using (3.4), (4.21), (4.25), we find

$$
\begin{aligned}
\left|\langle \partial_t \mathbb{P}_m(\rho_m\boldsymbol{u}_m), \boldsymbol{w}\rangle\right| &\le C\big(\rho^*\|\boldsymbol{u}_m\|_{L^4(\Omega)}^2 + \nu^*\|\mathbb{D}\boldsymbol{u}_m\|_{L^2(\Omega)} + \rho^*\|\mu_m\|_{L^3(\Omega)}\|\nabla\phi_m\|_{L^2(\Omega)} \\
&\quad + \rho^*\|\Psi_0'(\phi_m)\|_{L^2(\Omega)}\|\nabla\phi_m\|_{L^3(\Omega)}\big)\|\nabla\mathbb{P}_m\boldsymbol{w}\|_{L^2(\Omega)} \\
&\le C\underbrace{\left(\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac{3}{2}} + \|\mathbb{D}\boldsymbol{u}_m\|_{L^2(\Omega)} + \|\mu_m\|_{L^3(\Omega)} + \|\nabla\phi_m\|_{L^3(\Omega)}\right)}_{K}\|\nabla\boldsymbol{w}\|_{L^2(\Omega)},
\end{aligned}
$$

where the constant C is independent of m. In light of (4.22), (4.27), and (4.29), K is bounded in $L^{\frac{4}{3}}(0,T)$. Thus, we infer that $\partial_t \mathbb{P}_m(\rho_m \boldsymbol{u}_m)$ is bounded in $L^{\frac{4}{3}}(0,T;\mathbf{V}_\sigma')$. Recalling that $\rho_m \boldsymbol{u}_m$ is bounded in $L^\infty(0,T;L^2(\Omega))$, we deduce by the Aubin-Lions lemma that (up to a subsequence) $\mathbb{P}_m(\rho_m \boldsymbol{u}_m) \to g_1$ in $\mathcal{C}([0,T];\mathbf{V}_\sigma')$. Since $\rho_m \boldsymbol{u}_m \to \rho \boldsymbol{u}$ strongly in $L^2(0,T;L^2(\Omega))$ (cf. (4.53)), we have that $\mathbb{P}_m(\rho_m \boldsymbol{u}_m) \to \mathbb{P}(\rho \boldsymbol{u})$ in $L^2(0,T;\mathbf{H}_\sigma)$. So, we deduce that $g_1 = \mathbb{P}(\rho \boldsymbol{u})$. In light of $\mathbb{P}(\rho \boldsymbol{u}) \in L^\infty(0,T;\mathbf{H}_\sigma)$, Lemma 3.2 entails that

$$\mathbb{P}(\rho \boldsymbol{u}) \in \mathcal{C}([0,T];(\mathbf{H}_\sigma)_w).$$

This is equivalent to $\mathbb{P}(\rho \boldsymbol{u}) \in \mathcal{C}([0,T];(L^2(\Omega))_w)$. Next, we claim that $\rho \boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$. Indeed, since $\rho \boldsymbol{u} \in L^2(0,T;L^2(\Omega))$, by the properties of the Leray projection $\mathbb{P}$, we have that $\rho \boldsymbol{u}(t) = \mathbb{P}(\rho \boldsymbol{u})(t) + \nabla q(t)$ almost everywhere in $[0,T]$, for some $q \in L^2(0,T;H^1(\Omega))$. Dividing by ρ and using that $\operatorname{div} \boldsymbol{u} = 0$, we find that

$$\int_\Omega \frac{1}{\rho} \nabla q \cdot \nabla w \, \mathrm{d}x = -\int_\Omega \frac{1}{\rho} \mathbb{P}(\rho \boldsymbol{u}) \cdot \nabla w, \quad \forall\, w \in H^1(\Omega), \tag{4.63}$$

for almost every $t \in [0,T]$. Following [4, Section 5.2] (cf. also [41, Chapter 2]), we deduce from $\mathbb{P}(\rho \boldsymbol{u}) \in \mathcal{C}([0,T];(L^2(\Omega))_w)$ and the unique solvability of the problem (4.63) that $\nabla q \in \mathcal{C}([0,T];(L^2(\Omega))_w)$. Thus, by a redefinition of $\boldsymbol{u}$ on a set of measure zero, we find that $\rho \boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$. Since $\rho \in \mathcal{C}([0,T];L^r(\Omega))$ for all $r \in [2,\infty)$ and $\rho_* \leq \rho(x,t) \leq \rho^*$ almost everywhere in $\Omega \times (0,T)$, we eventually infer that $\boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$ and $\sqrt{\rho} \boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$.

Conservation of mass and energy inequality. We notice that the approximated solutions satisfy

$$\int_\Omega \rho_m(t) \, \mathrm{d}x = \int_\Omega \rho_{0m} \, \mathrm{d}x, \quad \int_\Omega \rho_m(t) \phi_m(t) \, \mathrm{d}x = \int_\Omega \rho_{0m} \phi_{0m} \, \mathrm{d}x \quad \forall\, t \in [0,T]. \tag{4.64}$$

The first equality follows from (4.12), whereas the second equality relies on the fact that $w_1 \equiv 1$, where w_1 is the first eigenfunction of A. Since $\rho_m \to \rho$ in $\mathcal{C}([0,T];L^1(\Omega))$ and $\rho_{0m} \to \rho_0$ in $L^1(\Omega)$, passing to the limit as $m \to \infty$, we find the first equality of (4.7). In a similar manner, since $\rho_m \phi_m \to \rho \phi$ in $\mathcal{C}([0,T];L^1(\Omega))$ and $\rho_{0m} \phi_{0m} \to \rho_0 \phi_0$ in $L^1(\Omega)$, we obtain the second equality of (4.7). In order to prove the energy inequality (4.8), we multiply (4.20) by $\psi(t)$, where $\psi \in \mathcal{C}_c^\infty(0,T)$ such that $\psi(t) \geq 0$. After integrating in time, we can pass to the limit by exploiting (4.48), (4.52), and (4.53). We find

$$\begin{aligned}
&\int_0^T \Big[\frac{1}{2} \| \sqrt{\rho(t)} \boldsymbol{u}(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla \phi(t) \|_{L^2(\Omega)}^2 + \int_\Omega \rho(t) \Psi_0(\phi(t)) \, \mathrm{d}x \Big] \psi(t) \, \mathrm{d}t \\
&\quad + \int_0^T \int_0^t \| \sqrt{\nu(\phi(\tau))} \mathbb{D} \boldsymbol{u}(\tau) \|_{L^2(\Omega)}^2 \, \mathrm{d}\tau \, \psi(t) \, \mathrm{d}t + \int_0^T \int_0^t \| \nabla \mu(\tau) \|_{L^2(\Omega)}^2 \, \mathrm{d}\tau \, \psi(t) \, \mathrm{d}t \\
&\quad \leq \int_0^T \int_\Omega \frac{1}{2} \rho_0 |\boldsymbol{u}_0|^2 + \frac{1}{2} |\nabla \phi_0|^2 + \rho_0 \Psi_0(\phi_0) \, \mathrm{d}x \, \psi(t) \, \mathrm{d}t,
\end{aligned}$$

which, in turn, entails (4.8).

The initial data. The convergences (4.9), (4.10), (4.51), (4.52) imply that $\phi(t) \to \phi(0) = \phi_0$ in $L^q(\Omega)$ for all $q \in [2,6)$, and $\rho(t) \to \rho(0) = \rho_0$ in $L^r(\Omega)$ for all $r \in [2,\infty)$ as $t \to 0^+$. Since $\mathbb{P}_m(\rho_m \boldsymbol{u}_m) \to \mathbb{P}(\rho \boldsymbol{u})$ in

$\mathcal{C}([0,T];V'_\sigma)$, by using (4.9) and (4.10), we find as $m \to \infty$

$$\int_\Omega \mathbb{P}(\rho_0 \boldsymbol{u}_0) \cdot \boldsymbol{w} \, \mathrm{d}x \leftarrow \int_\Omega \mathbb{P}_m(\rho_{0m} \boldsymbol{u}_{0m}) \cdot \boldsymbol{w} \, \mathrm{d}x = \int_\Omega \mathbb{P}_m(\rho_m(0) \boldsymbol{u}_m(0)) \cdot \boldsymbol{w} \, \mathrm{d}x \to \int_\Omega \mathbb{P}(\rho(0) \boldsymbol{u}(0)) \cdot \boldsymbol{w} \, \mathrm{d}x,$$

for all $\boldsymbol{w} \in \mathbf{V}_\sigma$. By using a density argument, the properties of $\mathbb{P}$, and $\rho(0) = \rho_0$, we obtain

$$\int_\Omega \rho_0 \boldsymbol{u}(0) \cdot \boldsymbol{w} \, \mathrm{d}x = \int_\Omega \rho_0 \boldsymbol{u}_0 \cdot \boldsymbol{w} \, \mathrm{d}x \quad \forall \, \boldsymbol{w} \in \mathbf{H}_\sigma,$$

which, in turn, entails that $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Next, recalling that $\sqrt{\rho}\boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$ and $\phi \in \mathcal{C}([0,T];(H^1(\Omega))_w)$, we have

$$\int_\Omega \frac{\rho_0}{2} |\boldsymbol{u}_0|^2 \, \mathrm{d}x \leq \liminf_{t \to 0^+} \int_\Omega \frac{\rho(t)}{2} |u(t)|^2 \, \mathrm{d}x, \quad \int_\Omega \frac{1}{2} |\nabla \phi_0|^2 \, \mathrm{d}x \leq \liminf_{t \to 0^+} \int_\Omega \frac{1}{2} |\nabla \phi(t)|^2 \, \mathrm{d}x. \tag{4.65}$$

In light of (4.51), (4.52) and of $\rho(0) = \rho_0$, $\phi(0) = \phi_0$, we infer that

$$\int_\Omega \rho(t) \Psi_0(\phi(t)) \, \mathrm{d}x \to \int_\Omega \rho_0 \Psi_0(\phi_0) \, \mathrm{d}x \quad \text{as } t \to 0^+.$$

Therefore, taking the upper limit in (4.8) as $t \to 0^+$, we have

$$\limsup_{t \to 0^+} \int_\Omega \frac{\rho(t)}{2} |\boldsymbol{u}(t)|^2 \, \mathrm{d}x + \int_\Omega \frac{1}{2} |\nabla \phi(t)|^2 \, \mathrm{d}x \leq \int_\Omega \frac{\rho_0}{2} |\boldsymbol{u}_0|^2 + \int_\Omega \frac{1}{2} |\nabla \phi_0|^2 \, \mathrm{d}x. \tag{4.66}$$

We infer from (4.65) and (4.66) that

$$\lim_{t \to 0^+} \int_\Omega \frac{\rho(t)}{2} |\boldsymbol{u}(t)|^2 \, \mathrm{d}x + \int_\Omega \frac{1}{2} |\nabla \phi(t)|^2 \, \mathrm{d}x = \int_\Omega \frac{\rho_0}{2} |\boldsymbol{u}_0|^2 + \int_\Omega \frac{1}{2} |\nabla \phi_0|^2 \, \mathrm{d}x. \tag{4.67}$$

By combining (4.67) with $\sqrt{\rho}\boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$ and $\phi \in \mathcal{C}([0,T];(H^1(\Omega))_w)$, we finally deduce that

$$\lim_{t \to 0^+} \int_\Omega |\sqrt{\rho(t)}\boldsymbol{u}(t) - \sqrt{\rho_0}\boldsymbol{u}_0|^2 + \int_\Omega |\nabla \phi(t) - \nabla \phi_0|^2 \, \mathrm{d}x = 0.$$

Further properties of weak solutions. The pointwise relation (4.6) allows us to prove the regularity of ϕ in $L^2(0,T;W^{2,q}(\Omega))$ as in (4.1). If $d = 2$, $\phi \in L^\infty(0,T;H^1(\Omega))$ implies that $\Psi'_0(\phi) \in L^\infty(0,T;L^q(\Omega))$ for any $q \in [2,\infty)$. So, $\rho\mu - \rho\Psi'(\phi)$ belongs to $L^2(0,T;L^q(\Omega))$ for any $q \in [2,\infty)$. Thus, the conclusion follows from the regularity theory of the Laplace equation with homogeneous Neumann boundary condition. If $d = 3$, multiplying (4.61) by $|\phi|^q\phi$ with $q = 14$, integrating over Ω, and using integration by parts, we find

$$\begin{aligned}\int_\Omega q|\phi|^q |\nabla \phi|^2 \, \mathrm{d}x + \int_\Omega \rho |\phi|^{q+4} \, \mathrm{d}x &= \int_\Omega \rho(\mu + \phi)|\phi|^q \phi \, \mathrm{d}x \\ &\leq \rho^* \big(\|\mu\|_{L^6(\Omega)} + \|\phi\|_{L^6(\Omega)} \big) \Big(\int_\Omega |\phi|^{\frac{6(q+1)}{5}} \, \mathrm{d}x \Big)^{\frac{5}{6}}.\end{aligned}$$

Therefore, by the choice of q, we deduce that

$$\rho_* \|\phi\|^3_{L^{q+4}(\Omega)} \le \rho^* \big(\|\mu\|_{L^6(\Omega)} + \|\phi\|_{L^6(\Omega)} \big).$$

This gives $\phi \in L^6(0,T;L^{18}(\Omega))$, which implies that $\Psi_0'(\phi) \in L^2(0,T;L^6(\Omega))$. By exploiting again the elliptic regularity of the Laplace equation, we conclude that $-\Delta\phi \in L^2(0,T;L^6(\Omega))$, and so $\phi \in L^2(0,T;W^{2,6}(\Omega))$. □

Remark 4.4. For the purpose of the existence of global weak solutions to (1.1)-(1.2) with Flory-Huggins logarithmic potential (see Section 5), we pass to the limit in the time translation estimates. First, we notice that, by combining (4.20) and (4.42) with (4.43), (4.44) and (4.45), we have

$$\sup_{0\le t\le T-h} \|\phi_m(t+h) - \phi_m(t)\|^2_{L^2(\Omega)} \le C(\overline{\rho_0 \phi_0}) h^{\frac12} (1 + E_0(\rho_0, \boldsymbol{u}_0, \phi_0)^{\frac32}).$$

Therefore, passing to the limit as $m \to \infty$, and using (4.51), we find

$$\sup_{0\le t\le T-h} \|\phi(t+h) - \phi(t)\|^2_{L^2(\Omega)} \le C(\overline{\rho_0 \phi_0}) h^{\frac12} (1 + E_0(\rho_0, \boldsymbol{u}_0, \phi_0)^{\frac32}). \tag{4.68}$$

For the velocity $\boldsymbol{u}_m$ a different argument is needed because the terms I_4 and I_5 involve the norms $\|\mu_m\|_{H^1}$ and $\|\Psi_0'(\phi_m)\|_{L^2(\Omega)}$, whose estimates depend explicitly on the form of the potential Ψ_0. To overcome this issue, we observe that

$$\begin{aligned}
&\rho_* \int_0^{T-h} \|\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\|^2_{L^2(\Omega)} \,\mathrm{d}t \\
&\le C h^{\frac12} (1 + E_0(\rho_0, \boldsymbol{u}_0, \phi_0)^{\frac32}) \\
&\quad + \Big| \int_0^{T-h} \int_\Omega \underbrace{\Big(\int_t^{t+h} \rho_m(\tau)\mu_m(\tau)\nabla\phi_m(\tau) \,\mathrm{d}\tau \Big)}_{R_m(x,t)} \cdot \big(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\big) \,\mathrm{d}x \,\mathrm{d}t \Big| \\
&\quad + \Big| \int_0^{T-h} \int_\Omega \underbrace{\Big(\int_t^{t+h} \rho_m(\tau)\Psi_0'(\phi_m(\tau))\nabla\phi_m(\tau) \,\mathrm{d}\tau \Big)}_{S_m(x,t)} \cdot \big(\boldsymbol{u}_m(t+h) - \boldsymbol{u}_m(t)\big) \,\mathrm{d}x \,\mathrm{d}t \Big|.
\end{aligned} \tag{4.69}$$

Thanks to (4.19), (4.25), (4.27), (4.29), and (4.53), we deduce that, as $m \to \infty$

$$\begin{aligned}
R_m &\to \int_t^{t+h} \rho(\tau)\mu(\tau)\nabla\phi(\tau) \,\mathrm{d}\tau \qquad && \text{strongly in } L^2(0,T-h;L^{\frac32}(\Omega)), \\
S_m &\to \int_t^{t+h} \rho(\tau)\Psi_0'(\phi(\tau))\nabla\phi(\tau) \,\mathrm{d}\tau \qquad && \text{strongly in } L^2(0,T-h;L^{\frac32}(\Omega)).
\end{aligned}$$

Since $\boldsymbol{u}_m(\cdot+h) - \boldsymbol{u}_m(\cdot) \to \boldsymbol{u}(\cdot+h) - \boldsymbol{u}(\cdot)$ strongly in $L^2(0,T-h;L^3(\Omega))$ by (4.49), passing to the limit in (4.69), we arrive at

$$\begin{aligned}
&\|\boldsymbol{u}(t+h)-\boldsymbol{u}(t)\|^2_{L^2(0,T-h;L^2(\Omega)}\\
&\leq Ch^{\frac12}(1+E_0(\rho_0,\boldsymbol{u}_0,\phi_0)^{\frac32})\\
&\quad + C\Big|\int_0^{T-h}\int_\Omega\Big(\int_t^{t+h}\rho(\tau)\mu(\tau)\nabla\phi(\tau)\,\mathrm{d}\tau\Big)\cdot\big(\boldsymbol{u}(t+h)-\boldsymbol{u}(t)\big)\,\mathrm{d}x\,\mathrm{d}t\Big|\\
&\quad + C\Big|\int_0^{T-h}\int_\Omega\Big(\int_t^{t+h}\rho(\tau)\Psi_0'(\phi(\tau))\nabla\phi(\tau)\,\mathrm{d}\tau\Big)\cdot\big(\boldsymbol{u}(t+h)-\boldsymbol{u}(t)\big)\,\mathrm{d}x\,\mathrm{d}t\Big|.
\end{aligned} \tag{4.70}$$

We emphasize that the constants C in (4.68) and (4.70) may depend on ρ_*, ρ^*, ν_*, ν^*, and $\overline{\rho_0\phi_0}$, but they are independent of the specific form of the potential Ψ_0.

5. Existence of global weak solutions: the Flory-Huggins potential case

In this section we show the existence of global weak solutions to system (1.1)-(1.2) in the case of the thermodynamically relevant Flory-Huggins potential (1.3).

Theorem 5.1. *Let Ω be a bounded domain of class $\mathcal{C}^3$ in $\mathbb{R}^d$, $d=2,3$, and let T be a positive time. Assume that $\rho_0\in L^\infty(\Omega)$, $\boldsymbol{u}_0\in L^2(\Omega)$, and $\phi_0\in H^1(\Omega)\cap L^\infty(\Omega)$ are such that*

$$0<\rho_*\leq\rho(x)\leq\rho^* \quad a.e.\ in\ \Omega,\quad \|\phi_0\|_{L^\infty(\Omega)}\leq 1,\quad -1<\frac{\overline{\rho_0\phi_0}}{\rho_*}<1. \tag{5.1}$$

Then, there exists a solution $(\rho,\boldsymbol{u},\phi,\mu)$ to the system (1.1)-(1.2) in the following sense:

(i) *The solution $(\rho,\boldsymbol{u},\phi,\mu)$ satisfies*

$$\begin{aligned}
&\rho\in\mathcal{C}([0,T];L^r(\Omega))\cap L^\infty(\Omega\times(0,T))\cap W^{1,\infty}(0,T;H^{-1}(\Omega)),\\
&\boldsymbol{u}\in\mathcal{C}([0,T];(\mathbf{H}_\sigma)_w)\cap L^2(0,T;\mathbf{V}_\sigma)\cap B^{\frac14}_{2,\infty}(0,T;\mathbf{H}_\sigma),\\
&\phi\in\mathcal{C}([0,T];(H^1(\Omega))_w)\cap L^\infty(\Omega\times(0,T))\cap L^4(0,T;H^2(\Omega))\\
&\phi\in L^2(0,T;W^{2,q}(\Omega))\cap B^{\frac14}_{\infty,\infty}(0,T;L^2(\Omega)),\\
&\mu\in L^2(0,T;H^1(\Omega)),
\end{aligned} \tag{5.2}$$

for any $r\in[1,\infty)$, and $q=6$ if $d=3$ and for any $q\in[2,\infty)$ if $d=2$. In addition,

$$\rho_*\leq\rho(x,t)\leq\rho^*,\quad -1\leq\phi(x,t)\leq 1\quad a.e.\ in\ \Omega\times(0,T). \tag{5.3}$$

(ii) *The system (1.1) is satisfied as follows:*

$$\int_0^T\int_\Omega\rho\partial_t\psi\,\mathrm{d}x\,\mathrm{d}t\int_0^T\int_\Omega\rho\boldsymbol{u}\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t=0 \tag{5.4}$$

for all $\psi\in\mathcal{C}_c^\infty(\Omega\times(0,T))$;

$$
\begin{aligned}
&-\int_0^T\int_\Omega \rho\boldsymbol{u}\cdot\partial_t\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \rho\boldsymbol{u}\otimes\boldsymbol{u}:\nabla\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T\int_\Omega \nu(\phi)\mathbb{D}\boldsymbol{u}:\nabla\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega \rho_0\boldsymbol{u}_0\cdot\boldsymbol{w}(0)\,\mathrm{d}x+\int_0^T\int_\Omega \nabla\phi\otimes\nabla\phi:\nabla\boldsymbol{w}\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{5.5}
$$

for all $\boldsymbol{w}\in\mathcal{C}^1_c([0,T);\mathbf{V}_\sigma)$*;*

$$
\begin{aligned}
-\int_0^T\int_\Omega \rho\phi\partial_t w\,\mathrm{d}x\,\mathrm{d}t-\int_0^T\int_\Omega \rho\boldsymbol{u}\phi\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega \nabla\mu\cdot\nabla w\,\mathrm{d}x\,\mathrm{d}t&\\
=\int_\Omega \rho_0\phi_0 w(0)\,\mathrm{d}x&
\end{aligned}
\tag{5.6}
$$

for all $w\in\mathcal{C}^1_c([0,T);H^1(\Omega))$*;*

$$
\rho\mu=-\Delta\phi+\rho\Psi'(\phi)\quad a.e.\ in\ \Omega\times(0,T). \tag{5.7}
$$

Furthermore, $\partial_{\mathbf{n}}\phi=0$ *almost everywhere on* $\partial\Omega\times(0,T)$*.*

(iii) *The initial data is assumed in the following sense:* $\rho(t)\to\rho(0)=\rho_0$ *in* $L^r(\Omega)$*, for all* $r\in[1,\infty)$*,* $\boldsymbol{u}(t)\to\boldsymbol{u}(0)=\boldsymbol{u}_0$ *in* $L^2(\Omega)$*,* $\phi(t)\to\phi(0)=\phi_0$ *in* $H^1(\Omega)$*, as* $t\to0^+$*.*

(iv) *The conservation of total mass and of mass of fluids difference holds as follows*

$$
\int_\Omega \rho(t)\,\mathrm{d}x=\int_\Omega \rho_0\,\mathrm{d}x,\quad \int_\Omega \rho(t)\phi(t)\,\mathrm{d}x=\int_\Omega \rho_0\phi_0\,\mathrm{d}x\quad\forall\,t\in[0,T]. \tag{5.8}
$$

(v) *The energy inequality*

$$
E(\rho(t),\boldsymbol{u}(t),\phi(t))+\int_0^t\int_\Omega \nu(\phi)|\mathbb{D}\boldsymbol{u}|^2+|\nabla\mu|^2\,\mathrm{d}x\,\mathrm{d}\tau\le E(\rho_0,\boldsymbol{u}_0,\phi_0), \tag{5.9}
$$

holds for all $t\in[0,T]$ *where*

$$
E(\rho,\boldsymbol{u},\phi)=\int_\Omega \frac12\rho|\boldsymbol{u}|^2+\frac12|\nabla\phi|^2+\rho\Psi(\phi)\,\mathrm{d}x.
$$

Remark 5.2. In comparison with the analysis of the homogeneous Model H with Flory-Huggins potential requiring $-1<\overline{\phi_0}<1$, the last condition $-1<\frac{\overline{\rho_0\phi_0}}{\rho_*}<1$ in (5.1) seems to be a natural generalization to the nonhomogeneous case. In fact, in the case of constant density these two assumptions coincide.

Proof of Theorem 5.1. The proof is divided in several parts.

Approximation of the logarithmic potential. We introduce a family of regular potentials Ψ_ε which approximate the logarithmic potential $\Psi(s)=F(s)-\frac{\theta_0}{2}s^2$. For any $\varepsilon\in(0,\frac12)$, we set

$$
\Psi_\varepsilon(s)=F_\varepsilon(s)-\frac{\theta_0}{2}s^2\quad\forall\,s\in\mathbb{R}, \tag{5.10}
$$

where F_ε is defined by

$$F_\varepsilon(s) = \begin{cases} \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(1-\varepsilon)\,[s-(1-\varepsilon)]^j & \forall\, s \geq 1-\varepsilon, \\ F(s) & \forall\, s \in [-1+\varepsilon, 1-\varepsilon], \\ \sum_{j=0}^{2} \frac{1}{j!} F^{(j)}(-1+\varepsilon)\,[s-(-1+\varepsilon)]^j & \forall\, s \leq -1+\varepsilon. \end{cases} \tag{5.11}$$

By definition, the approximating function Ψ_ε belongs to $\mathcal{C}^2(\mathbb{R})$ and it satisfies the properties

$$-\frac{\theta_0}{2} \leq \Psi_\varepsilon(s), \quad -\alpha \leq \Psi_\varepsilon''(s) \leq L_\varepsilon \quad \forall\, s \in \mathbb{R}, \tag{5.12}$$

where $\alpha = \theta - \theta_0$ is a positive constant independent of ε, and $L_\varepsilon = F''(1-\varepsilon)$ is a positive constant depending on ε. Furthermore, we have that

$$\Psi_\varepsilon(s) \leq \Psi(s) \quad \forall\, s \in [-1,1], \quad |\Psi_\varepsilon'(s)| \leq |\Psi'(s)| \quad \forall\, s \in (-1,1). \tag{5.13}$$

Approximate solutions. For any $\varepsilon \in (0, \frac{1}{2})$, we consider the approximate problem

$$\begin{cases} \partial_t \rho_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla \rho_\varepsilon = 0 \\ \rho_\varepsilon \partial_t \boldsymbol{u}_\varepsilon + \rho_\varepsilon(\boldsymbol{u}_\varepsilon \cdot \nabla)\boldsymbol{u}_\varepsilon - \operatorname{div}(\nu(\phi_\varepsilon)\mathbb{D}\boldsymbol{u}_\varepsilon) + \nabla p_\varepsilon = -\operatorname{div}(\nabla \phi_\varepsilon \otimes \nabla \phi_\varepsilon) \\ \operatorname{div} \boldsymbol{u}_\varepsilon = 0 \\ \rho_\varepsilon \partial_t \phi_\varepsilon + \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla \phi_\varepsilon = \Delta \mu_\varepsilon \\ \rho_\varepsilon \mu_\varepsilon = -\Delta \phi_\varepsilon + \rho_\varepsilon \Psi_\varepsilon'(\phi_\varepsilon) \end{cases} \tag{5.14}$$

which is supplemented with the boundary conditions (1.2). Thanks to Theorem 4.1 (cf. also Remark 4.2), for any $\varepsilon \in (0, \frac{1}{2})$, there exists a global weak solution $(\rho_\varepsilon, \boldsymbol{u}_\varepsilon, \phi_\varepsilon, \mu_\varepsilon)$ satisfying (4.1)-(4.8).

Estimates for the approximate solutions. First of all, we have the density bounds

$$\rho_* \leq \rho_\varepsilon(x,t) \leq \rho^* \quad \text{a.e. in } \Omega \times (0,T). \tag{5.15}$$

The energy inequality (4.8) entails that

$$E_\varepsilon(\rho_\varepsilon(t), \boldsymbol{u}_\varepsilon(t), \phi_\varepsilon(t)) + \int_0^t \int_\Omega \nu(\phi_\varepsilon)|\mathbb{D}\boldsymbol{u}_\varepsilon|^2 + |\nabla \mu_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq E_\varepsilon(\rho_0, \boldsymbol{u}_0, \phi_0), \tag{5.16}$$

for almost every $t \in (0,T)$, where

$$E_\varepsilon(\rho, \boldsymbol{u}, \phi) = \int_\Omega \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\nabla \phi|^2 + \rho \Psi_\varepsilon(\phi)\,\mathrm{d}x.$$

Since $\|\phi_0\|_{L^\infty(\Omega)} \leq 1$, by the monotonicity property (5.13) we infer that

$$E_\varepsilon(\rho_0, \boldsymbol{u}_0, \phi_0) \leq E(\rho_0, \boldsymbol{u}_0, \phi_0). \tag{5.17}$$

Then, we deduce from (5.12) and (5.16) that

$$\|\boldsymbol{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{H}_\sigma)} \le C(1+E^{\frac12}), \tag{5.18}$$

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(0,T;\mathbf{V}_\sigma)} \le C(1+E^{\frac12}), \tag{5.19}$$

$$\|\nabla\phi_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C(1+E^{\frac12}), \tag{5.20}$$

$$\|\nabla\mu_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \le C(1+E^{\frac12}). \tag{5.21}$$

Here $E=E(\rho_0,\boldsymbol{u}_0,\phi_0)$ and the generic constant C may depend on the parameters ρ_*, ρ^*, ν_*,ν^*, but not on ε. In light of (4.7), we have

$$\int_\Omega \rho_\varepsilon(t)\phi_\varepsilon(t)\,\mathrm{d}x = \int_\Omega \rho_0\phi_0\,\mathrm{d}x \quad \forall\, t\in[0,T]. \tag{5.22}$$

In turn, by (3.2), (5.1), and (5.20), we obtain

$$\|\phi_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C(1+E^{\frac12}). \tag{5.23}$$

Recalling that $(5.14)_5$ holds almost everywhere, we multiply this equation by $\frac{-\Delta\phi_\varepsilon}{\rho_\varepsilon}$ and integrate over Ω. After integrating by parts and using the boundary condition for ϕ_ε, we find

$$\int_\Omega \frac{|\Delta\phi_\varepsilon|^2}{\rho_\varepsilon}\,\mathrm{d}x + \int_\Omega \Psi_\varepsilon''(\phi_\varepsilon)|\nabla\phi_\varepsilon|^2\,\mathrm{d}x = \int_\Omega \nabla\mu_\varepsilon\cdot\nabla\phi_\varepsilon\,\mathrm{d}x.$$

By (5.12), (5.15), and (5.23), we have

$$\frac{1}{\rho^*}\int_\Omega |\Delta\phi_\varepsilon|^2\,\mathrm{d}x \le \alpha\|\nabla\phi_\varepsilon\|_{L^2(\Omega)}^2 + \|\nabla\mu_\varepsilon\|_{L^2(\Omega)}\|\nabla\phi_\varepsilon\|_{L^2(\Omega)} \le C(E)(1+\|\nabla\mu_\varepsilon\|_{L^2(\Omega)}).$$

Then, combining the above estimate with (5.21), (5.23), and using the elliptic regularity of the Neumann problem, we find

$$\|\phi_\varepsilon\|_{L^4(0,T;H^2(\Omega))} \le C(E,T). \tag{5.24}$$

Next, we multiply $(5.14)_5$ by $\frac{1}{\rho_\varepsilon}\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}\big)$ and integrate over Ω. We have

$$\begin{aligned}\int_\Omega F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}\big)\,\mathrm{d}x &= \int_\Omega \frac{-\Delta\phi_\varepsilon}{\rho_\varepsilon}\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}\big)\,\mathrm{d}x + \int_\Omega \mu_\varepsilon\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}\big)\,\mathrm{d}x\\ &\quad + \theta_0\int_\Omega \phi_\varepsilon\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}\big)\,\mathrm{d}x.\end{aligned}$$

We notice that $\overline{\rho_\varepsilon\phi_\varepsilon - \overline{\rho_\varepsilon\phi_\varepsilon}} = \overline{\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}} = 0$. By exploiting the Poincaré inequality, and (5.15), (5.22), (5.23), and (5.24), we have

$$\begin{aligned}\int_\Omega F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x &\le \frac{C(E)}{\rho^*}\|\Delta\phi_\varepsilon\|_{L^2(\Omega)} + \Big|\int_\Omega (\mu_\varepsilon - \overline{\mu_\varepsilon})\rho_\varepsilon\phi_\varepsilon\,\mathrm{d}x\Big| + C(E)\\ &\le C(E)\big(1+\|\Delta\phi_\varepsilon\|_{L^2(\Omega)} + \|\nabla\mu_\varepsilon\|_{L^2(\Omega)}\big).\end{aligned} \tag{5.25}$$

Now we show that

$$\int_\Omega |F'_\varepsilon(\phi_\varepsilon)|\,\mathrm{d}x \le K_0 \int_\Omega F'_\varepsilon(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x + K_1, \tag{5.26}$$

where K_0 and K_1 are positive constants that depend on ρ_*, ρ^*, $\overline{\rho_0\phi_0}$, the function F and the initial energy E, but are independent of ε. We consider the case $\overline{\rho_0\phi_0} \ge 0$. The case $\overline{\rho_0\phi_0} < 0$ can be handled in the same way. By assumption, we have

$$-1 < 0 \le \frac{\overline{\rho_\varepsilon\phi_\varepsilon}}{\rho_\varepsilon} \le \frac{\overline{\rho_0\phi_0}}{\rho_*} < 1 \quad \text{a.e. in } \Omega\times(0,T). \tag{5.27}$$

We set

$$\delta_0 = \frac{1}{2}\Big(1 - \frac{\overline{\rho_0\phi_0}}{\rho_*}\Big), \qquad \delta_1 = \delta_0 + \max\Big\{\frac{1}{2}, \frac{\overline{\rho_0\phi_0}}{\rho_*}\Big\}.$$

Notice that $\frac{1}{2} < \delta_1 < 1$. For $t\in(0,T)$ we define the sets

$$\Omega_0(t) = \Big\{x\in\Omega : \phi_\varepsilon(x,t) \le -\frac{1}{2}\Big\}, \quad \Omega_1(t) = \Big\{x\in\Omega : -\frac{1}{2} \le \phi_\varepsilon(x,t) \le \delta_1\Big\},$$

and

$$\Omega_2(t) = \Big\{x\in\Omega : \phi_\varepsilon(x,t) \ge \delta_1\Big\}.$$

Since $\overline{\rho_0\phi_0} \ge 0$, we have the following estimate

$$\phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_*} \le \phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_\varepsilon} \le \phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho^*} \quad \text{a.e. in } \Omega\times(0,T).$$

Observing that

$$\phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_\varepsilon} \ge \phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_*} \ge \delta_0 \quad \text{in } \Omega_2(t), \tag{5.28}$$

we find

$$\begin{aligned}\int_{\Omega_2(t)} |F'_\varepsilon(\phi_\varepsilon)|\,\mathrm{d}x &= \int_{\Omega_2(t)} F'_\varepsilon(\phi_\varepsilon)\,\mathrm{d}x \\ &\le \frac{1}{\delta_0}\int_{\Omega_2(t)} F'_\varepsilon(\phi_\varepsilon)\Big(\phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_\varepsilon}\Big)\,\mathrm{d}x \le \frac{1}{\delta_0\rho_*}\int_{\Omega_2(t)} F'_\varepsilon(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x.\end{aligned}$$

Since $\phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_\varepsilon} \le -\frac{1}{2}$ in $\Omega_0(t)$, we find

$$\begin{aligned}\int_{\Omega_0(t)} |F'_\varepsilon(\phi_\varepsilon)|\,\mathrm{d}x &= \int_{\Omega_0(t)} -F'_\varepsilon(\phi_\varepsilon)\,\mathrm{d}x \\ &\le 2\int_{\Omega_0(t)} F'_\varepsilon(\phi_\varepsilon)\Big(\phi_\varepsilon - \frac{\overline{\rho_0\phi_0}}{\rho_\varepsilon}\Big)\,\mathrm{d}x \\ &\le \frac{2}{\rho_*}\int_{\Omega_0(t)} F'_\varepsilon(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x \le \frac{1}{\delta_0\rho_*}\int_{\Omega_0(t)} F'_\varepsilon(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\,\mathrm{d}x.\end{aligned}$$

By using (5.13), we notice that

$$
\begin{aligned}
&\int_{\Omega_1(t)} |F_\varepsilon'(\phi_\varepsilon)| \,\mathrm{d}x \\
&\le \frac{1}{\delta_0\rho_*} \int_{\Omega_1(t)} F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big) \,\mathrm{d}x + \int_{\Omega_1(t)} |F_\varepsilon'(\phi_\varepsilon)| \,\mathrm{d}x \\
&\quad + \frac{1}{\delta_0\rho_*} \int_{\Omega_1(t)} \Big| F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big)\Big| \,\mathrm{d}x \\
&\le \frac{1}{\delta_0\rho_*} \int_{\Omega_1(t)} F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big) \,\mathrm{d}x + \frac{F_\varepsilon'(\delta_1)}{\delta_0\rho_*}\Big(\rho^*|\Omega_1(t)| + \rho^*\|\phi_\varepsilon\|_{L^1(\Omega)} + \overline{\rho_0\phi_0}|\Omega_1(t)|\Big) \\
&\le \frac{1}{\delta_0\rho_*} \int_{\Omega_1(t)} F_\varepsilon'(\phi_\varepsilon)\big(\rho_\varepsilon\phi_\varepsilon - \overline{\rho_0\phi_0}\big) \,\mathrm{d}x + C(\rho_*, \rho^*, |\Omega|, \overline{\rho_0\phi_0}, E, F'(\delta_1)).
\end{aligned}
$$

Combining the above inequalities, we deduce that (5.26) holds with the choices $K_0 = \frac{1}{\delta_0\rho_*}$ and $K_1 = C(\rho_*, \rho^*, \overline{\rho_0\phi_0}, F'(\delta_1), E, |\Omega|)$. Collecting (5.25) and (5.26) together, we infer that

$$
\|F_\varepsilon'(\phi_\varepsilon)\|_{L^1(\Omega)} \le C(E)(1 + \|\Delta\phi_\varepsilon\|_{L^2(\Omega)} + \|\nabla\mu_\varepsilon\|_{L^2(\Omega)}), \tag{5.29}
$$

which, in turn, implies that

$$
\|F_\varepsilon'(\phi_\varepsilon)\|_{L^2(0,T;L^1(\Omega))} \le C(E,T). \tag{5.30}
$$

By using (5.15), we also have

$$
\|\rho_\varepsilon F_\varepsilon'(\phi_\varepsilon)\|_{L^2(0,T;L^1(\Omega))} \le C(E,T). \tag{5.31}
$$

Since $\int_\Omega \rho_\varepsilon\mu_\varepsilon \,\mathrm{d}x = \int_\Omega \rho_\varepsilon \Psi_\varepsilon'(\phi_\varepsilon) \,\mathrm{d}x$, by using (5.23) and (5.31), we conclude that

$$
\Big\| \int_\Omega \rho_\varepsilon\mu_\varepsilon \,\mathrm{d}x \Big\|_{L^2(0,T)} \le C(E,T).
$$

In light of (3.2) and (5.21), we finally arrive at

$$
\|\mu_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C(E,T). \tag{5.32}
$$

Besides, by using (5.15), (5.24), and (5.32), we obtain

$$
\|F_\varepsilon'(\phi_\varepsilon)\|_{L^2(0,T;L^2(\Omega))} \le C(E,T). \tag{5.33}
$$

Now we deduce the time translation estimates for $\boldsymbol{u}_m$ and ϕ_m. Thanks to Remark 4.4, there exists a positive constant C independent of ε such that

$$\begin{aligned}
&\|\boldsymbol{u}_\varepsilon(t+h)-\boldsymbol{u}_\varepsilon(t)\|^2_{L^2(0,T-h;L^2(\Omega))}\\
&\le Ch^{\frac12}\big(1+E_\varepsilon(\rho_0,\boldsymbol{u}_0,\phi_0)^{\frac32}\big)\\
&\quad+\underbrace{C\Big|\int_0^{T-h}\int_\Omega\Big(\int_t^{t+h}\rho_\varepsilon(\tau)\mu_\varepsilon(\tau)\nabla\phi_\varepsilon(\tau)\,\mathrm{d}\tau\Big)\cdot\big(\boldsymbol{u}_\varepsilon(t+h)-\boldsymbol{u}_\varepsilon(t)\big)\,\mathrm{d}x\,\mathrm{d}t\Big|}_{T_1}\\
&\quad+\underbrace{C\Big|\int_0^{T-h}\int_\Omega\Big(\int_t^{t+h}\rho_\varepsilon(\tau)\Psi_\varepsilon'(\phi_\varepsilon(\tau))\nabla\phi_\varepsilon(\tau)\,\mathrm{d}\tau\Big)\cdot\big(\boldsymbol{u}_\varepsilon(t+h)-\boldsymbol{u}_\varepsilon(t)\big)\,\mathrm{d}x\,\mathrm{d}t\Big|}_{T_2}.
\end{aligned}\tag{5.34}$$

and

$$\sup_{0\le t\le T-h}\|\phi_\varepsilon(t+h)-\phi_\varepsilon(t)\|^2_{L^2(\Omega)}\le Ch^{\frac12}(1+E_\varepsilon(\rho_0,\boldsymbol{u}_0,\phi_0)^{\frac32}).\tag{5.35}$$

Arguing as in the proof of Theorem 4.1 (cf. (4.37) and (4.38)), and using (5.19), (5.23), (5.24), (5.32), (5.33), we are lead to

$$T_1+T_2\le C(E)h^{\frac34}\big(\sqrt{T}\|\mu_\varepsilon\|_{L^2(0,T;H^1(\Omega))}+\|\Psi_\varepsilon'(\phi_\varepsilon)\|_{L^2(0,T;L^2(\Omega))}\|\phi_\varepsilon\|_{L^2(0,T;H^2(\Omega))}\big)\le C(E)h^{\frac34}.$$

Then, recalling (5.17), we have

$$\|\boldsymbol{u}_\varepsilon\|_{B^{\frac14}_{2,\infty}(0,T;\mathbf{H}_\sigma)}\le C(E,T),\quad \|\phi_\varepsilon\|_{B^{\frac14}_{\infty,\infty}(0,T;L^2(\Omega))}\le C(E,T),\tag{5.36}$$

where C is a positive constant independent of ε. Lastly, we recall that any approximate weak solution $(\rho_\varepsilon,\boldsymbol{u}_\varepsilon,\phi_\varepsilon,\mu_\varepsilon)$ satisfies (4.4), which can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_\varepsilon\boldsymbol{u}_\varepsilon,\boldsymbol{w})=(\rho_\varepsilon\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon,\nabla\boldsymbol{w})-(\nu(\phi_\varepsilon)\mathbb{D}\boldsymbol{u}_\varepsilon,\nabla\boldsymbol{w})+(\nabla\phi_\varepsilon\otimes\nabla\phi_\varepsilon,\nabla\boldsymbol{w}),\tag{5.37}$$

for all $\boldsymbol{w}\in\mathbf{V}_\sigma$, in the sense of distributions on $(0,T)$. Since $\rho_\varepsilon\boldsymbol{u}_\varepsilon\otimes\boldsymbol{u}_\varepsilon$, $\nu(\phi_\varepsilon)\mathbb{D}\boldsymbol{u}_\varepsilon$, and $\nabla\phi_\varepsilon\otimes\nabla\phi_\varepsilon$ are bounded in $L^{\frac43}(0,T;L^2(\Omega))$ independently of ε, we infer from the definition of weak time derivative that

$$\|\partial_t\mathbb{P}(\rho_\varepsilon\boldsymbol{u}_\varepsilon)\|_{L^{\frac43}(0,T;\mathbf{V}_\sigma')}\le C.\tag{5.38}$$

Passage to the limit. As a consequence of the estimates (5.15), (5.18), (5.19), (5.23), (5.24), and (5.32), we deduce that (up to a subsequence)

$$\begin{aligned}
&\rho_\varepsilon\rightharpoonup\rho &&\text{weak-star in } L^\infty(\Omega\times(0,T)),\\
&\boldsymbol{u}_\varepsilon\rightharpoonup\boldsymbol{u} &&\text{weak-star in } L^\infty(0,T;\mathbf{H}_\sigma),\\
&\boldsymbol{u}_\varepsilon\rightharpoonup\boldsymbol{u} &&\text{weakly in } L^2(0,T;\mathbf{V}_\sigma),\\
&\phi_\varepsilon\rightharpoonup\phi &&\text{weak-star in } L^\infty(0,T;H^1(\Omega)),\\
&\phi_\varepsilon\rightharpoonup\phi &&\text{weakly in } L^4(0,T;H^2(\Omega)),\\
&\mu_\varepsilon\rightharpoonup\mu &&\text{weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned}\tag{5.39}$$

Thanks to the time translation estimates (5.36), we infer from Lemma 3.1 that

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{strongly in } L^2(0,T;L^q(\Omega)), \forall\, q \in [2,6), \tag{5.40}$$

$$\phi_\varepsilon \to \phi \quad \text{strongly in } L^4(0,T;W^{1,q}(\Omega)), \forall\, q \in [2,6), \tag{5.41}$$

$$\phi_\varepsilon \to \phi \quad \text{strongly in } \mathcal{C}([0,T];L^q(\Omega)), \forall\, q \in [2,6). \tag{5.42}$$

As above the strong convergence of the density follows from [41, Lemma 2.4], which implies that

$$\rho_\varepsilon \to \rho \quad \text{strongly in } \mathcal{C}([0,T];L^r(\Omega)), \forall\, r \in [1,\infty). \tag{5.43}$$

Thanks to the above estimates and convergence results, we infer that

$$\begin{aligned}
&\rho_\varepsilon \boldsymbol{u}_\varepsilon \to \rho \boldsymbol{u} && \text{strongly in } L^2(0,T;L^q(\Omega)), \forall\, q \in [2,6),\\
&\rho_\varepsilon \phi_\varepsilon \to \rho\phi && \text{strongly in } \mathcal{C}([0,T];L^q(\Omega)), \forall\, q \in [2,6),\\
&\rho_\varepsilon \boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon \rightharpoonup \rho \boldsymbol{u} \otimes \boldsymbol{u} && \text{weakly in } L^{\frac{4}{3}}(0,T;L^2(\Omega)),\\
&\nu(\phi_\varepsilon)\mathbb{D}\boldsymbol{u}_\varepsilon \rightharpoonup \nu(\phi)\mathbb{D}\boldsymbol{u} && \text{weakly in } L^2(0,T;L^2(\Omega)),\\
&\nabla\phi_\varepsilon \otimes \nabla\phi_\varepsilon \to \nabla\phi \otimes \nabla\phi && \text{strongly in } L^2(0,T;L^2(\Omega)),\\
&\rho_\varepsilon \mu_\varepsilon \rightharpoonup \rho\mu && \text{weakly in } L^2(0,T;L^6(\Omega)),\\
&\rho_\varepsilon \boldsymbol{u}_\varepsilon \phi_\varepsilon \to \rho\boldsymbol{u}\phi && \text{strongly in } L^2(0,T;L^2(\Omega)).
\end{aligned} \tag{5.44}$$

Moreover, we deduce the pointwise convergence (again up to the extraction of a subsequence)

$$\phi_\varepsilon \to \phi \quad \text{a.e. in } \Omega \times (0,T). \tag{5.45}$$

We now claim that the limit function ϕ fulfills

$$\phi \in L^\infty(\Omega \times (0,T)) \quad \text{and} \quad |\phi(x,t)| < 1 \quad \text{a.e. in } \Omega \times (0,T). \tag{5.46}$$

By the definition, $F'_\varepsilon(s)$ is monotone increasing for $s \in \mathbb{R}$. Then, for any $\gamma \in (0, \frac{1}{2})$ we introduce the sets

$$\begin{aligned}
M_\gamma^\varepsilon &= \{(x,t) \in \Omega \times (0,T) :\ |\phi_\varepsilon(x,t)| > 1-\gamma\}, \quad \varepsilon \in (0,\gamma],\\
M_\gamma &= \{(x,t) \in \Omega \times (0,T) :\ |\phi(x,t)| > 1-\gamma\}.
\end{aligned}$$

From the pointwise convergence of ϕ_ε and Fatou's Lemma, we infer that for any fixed γ,

$$\text{meas}(M_\gamma) \le \liminf_{\varepsilon \to 0^+} \text{meas}(M_\gamma^\varepsilon).$$

On the other hand, by using (5.30), we have

$$\min\{F'(1-\gamma), -F'(-1+\gamma)\}\, \text{meas}(M_\gamma^\varepsilon) \le \|F'_\varepsilon(\phi_\varepsilon)\|_{L^1(\Omega\times(0,T))} \le C,$$

where the constant C is independent of γ and ε. Thus, we obtain

$$\text{meas}(M_\gamma) \le \frac{C}{\min\{F'(1-\gamma), -F'(-1+\gamma)\}}.$$

Passing to the limit as $\gamma \to 0^+$, we deduce that $\mathrm{meas}(\{(x,t) \in \Omega \times (0,T) : |\phi(x,t)| \geq 1\}) = 0$, which yields the conclusion (5.46). As a consequence, since F'_ε converge uniformly to F' on every compact set in $(-1,1)$, we obtain

$$\Psi'_\varepsilon(\phi_\varepsilon) \to \Psi'(\phi) \quad \text{a.e. } (x,t) \in \Omega \times (0,T),$$

as $\varepsilon \to 0^+$. Hence, up to a subsequence, it holds

$$\Psi'_\varepsilon(\phi_\varepsilon) \rightharpoonup \Psi'(\phi) \quad \text{weakly in } L^2(0,T;L^2(\Omega)).$$

In light of (5.15) and (5.43), we conclude that

$$\rho_\varepsilon \Psi'_\varepsilon(\phi_\varepsilon) \rightharpoonup \rho \Psi'(\phi) \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{5.47}$$

The convergence results established in this section allow us to pass to the limit in the weak formulation implying that $(\rho, \boldsymbol{u}, \phi, \mu)$ satisfy (5.4)-(5.7). The conservation of mass (5.8) and the energy inequality (5.9) follows from the above convergences (5.39)-(5.44) and the weak lower-semicontinuity of the norm.

Time continuity and initial data. We observe that $\rho \in \mathcal{C}([0,T];L^r(\Omega))$ for all $r \in [1,\infty)$ and $\rho(0) = \rho_0$ by [41, Lemma 2.4]. It follows from (5.42) and $\phi_\varepsilon(0) = \phi_0$ for all $\varepsilon \in (0, \frac{1}{2})$ that $\phi(0) = \phi_0$. By using Lemma 3.2, we also infer that $\phi \in \mathcal{C}([0,T];(H^1(\Omega))_w)$. In order to deduce the time continuity of the velocity, we first notice that (5.15), (5.18) and (5.38) imply $\mathbb{P}(\rho_\varepsilon \boldsymbol{u}_\varepsilon) \to \mathbb{P}(\rho \boldsymbol{u})$ in $\mathcal{C}([0,T];\mathbf{V}'_\sigma)$ by Aubin-Lions lemma. Since $\rho \boldsymbol{u} \in L^\infty(0,T;L^2(\Omega))$, we obtain from Lemma 3.2 that $\mathbb{P}(\rho \boldsymbol{u}) \in \mathcal{C}([0,T];(\mathbf{H}_\sigma)_w)$. This entails that $\mathbb{P}(\rho \boldsymbol{u}) \in \mathcal{C}([0,T];(L^2(\Omega))_w)$ and $\boldsymbol{u}(0) = \boldsymbol{u}_0$. Now, arguing as in the proof of Theorem 4.1 (cf. Time Continuity), we find that $\rho \boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$, and in turn $\boldsymbol{u} \in \mathcal{C}([0,T];(L^2(\Omega))_w)$. Finally, to prove the strong convergence to the initial data as in the proof Theorem 4.1 (cf. The Initial Data), we are only left to show that

$$\int_\Omega \rho(t) F(\phi(t))\,\mathrm{d}x \to \int_\Omega \rho_0 F(\phi_0)\,\mathrm{d}x \quad \text{as } t \to 0^+,$$

where F is the convex part of the logarithmic potential. Taking a sequence $t_n \to 0^+$, it follows from (5.42) and (5.43) that $\phi(t_n) \to \phi_0$ and $\rho(t_n) \to \rho_0$ almost everywhere in Ω. Since F is continuous on $[-1,1]$, we obtain that $\rho(t_n) F(\phi(t_n)) \to \rho_0 F(\phi_0)$ almost everywhere in Ω. Since ρ and $F(\phi)$ are bounded, by the Lebesgue dominated theorem we reach the desired conclusion.

Further properties of weak solutions. We claim that $F'(\phi) \in L^2(0,T;L^q(\Omega))$, where $q = 6$ if $d = 3$ and any $q \in [2,\infty)$ if $d = 2$. To show this, inspired by [28] (see also [1]), we define for all $k > 1$

$$\phi_k = h_k \circ \phi, \quad h_k(s) = \begin{cases} 1 - \frac{1}{k} & s > 1 - \frac{1}{k}, \\ s & -1 + \frac{1}{k} \leq s \leq 1 - \frac{1}{k}, \\ -1 + \frac{1}{k} & s < -1 + \frac{1}{k}. \end{cases}$$

It follows that $\phi_k \in \mathcal{C}([0,T];H^1(\Omega))$ and the chain rule holds $\nabla \phi_k = \nabla \phi \chi_{[-1+\frac{1}{k}, 1-\frac{1}{k}]}(\phi)$. Accordingly, for any $q \geq 2$, $|F'(\phi_k)|^{q-2} F'(\phi_k)$ is well defined, it belongs to $\mathcal{C}([0,T];H^1(\Omega))$ and satisfies

$$\nabla \Big(|F'(\phi_k)|^{q-2} F'(\phi_k) \Big) = (q-1) |F'(\phi_k)|^{q-2} F''(\phi_k) \nabla \phi_k.$$

Now, multiplying (5.7) by $|F'(\phi_k)|^{q-2}F'(\phi_k)$ and integrating over Ω, we find

$$(q-1)\int_\Omega |F'(\phi_k)|^{q-2}F''(\phi_k)\nabla\phi\cdot\nabla\phi_k \, dx + \int_\Omega \rho|F'(\phi_k)|^{q-2}F'(\phi_k)F'(\phi)\, dx = \int_\Omega \big(\rho\mu+\theta_0\rho\phi\big)|F'(\phi_k)|^{q-2}F'(\phi_k)\, dx.$$

Since F is strictly convex, the first term on the left-hand side is non-negative. We also have that $F'(\phi_k)^2 \leq F'(\phi)F'(\phi_k)$ almost everywhere. Thus, by Young's inequality, we obtain

$$\frac{\rho_*}{2}\|F'(\phi_k)\|^q_{L^q(\Omega)} \leq C\|\rho\mu+\theta_0\rho\phi\|^q_{L^q(\Omega)}.$$

Since $\rho\mu+\theta_0\rho\phi \in L^2(0,T;L^q(\Omega))$, where $q=6$ if $d=3$ and any $2\leq q<\infty$ if $d=2$, we deduce that

$$\|F'(\phi_k)\|_{L^2(0,T;L^q(\Omega))} \leq C,$$

where C is independent of k. By the Fatou lemma, we are lead to $F'(\phi)\in L^2(0,T;L^q(\Omega))$. Finally, since $-\Delta\phi = \rho\mu - \rho\Psi'(\phi) \in L^2(0,T;L^q(\Omega))$, by the elliptic regularity of the Laplace equation with Neumann boundary condition, we infer that $\phi\in L^2(0,T;W^{2,q}(\Omega))$ for $q=6$ if $d=3$ and any $2\leq q<\infty$ if $d=2$. □

6. Existence of strong solutions: the Landau potential case

This section is devoted to the existence of strong solutions to system (1.1)-(1.2) with Landau potential (1.4). These solutions are global-in-time in two dimensions, and local-in-time in three dimensions.

Theorem 6.1. *Let Ω be a bounded domain of class $\mathcal{C}^3$ in $\mathbb{R}^d$, $d=2,3$. Assume that $\rho_0\in L^\infty(\Omega)$, $\boldsymbol{u}_0\in \mathbf{V}_\sigma(\Omega)$ and $\phi_0\in H^2(\Omega)$ are given such that $0\leq\rho_*\leq\rho_0\leq\rho^*$, $\partial_{\boldsymbol{n}}\phi_0=0$ on $\partial\Omega$, and $\mu_0=-\frac{\Delta\phi_0}{\rho_0}+\Psi_0'(\phi_0)\in H^1(\Omega)$. Then, we have the following:*

(i) *If $d=2$, for any $T>0$ there exists a strong solution $(\rho,\boldsymbol{u},p,\phi,\mu)$ to the system (1.1)-(1.2) satisfying*

$$\begin{aligned}
&\rho\in\mathcal{C}([0,T];L^r(\Omega))\cap L^\infty(\Omega\times(0,T))\cap L^\infty(0,T;H^{-1}(\Omega)),\\
&\boldsymbol{u}\in\mathcal{C}([0,T];\mathbf{V}_\sigma)\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;\mathbf{H}_\sigma),\\
&p\in L^2(0,T;H^1(\Omega)),\\
&\phi\in\mathcal{C}([0,T];(W^{2,q}(\Omega))_w)\cap H^1(0,T;H^1(\Omega)),\\
&\mu\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;W^{2,q}(\Omega)),
\end{aligned}$$

for any $r\in[2,\infty)$ and any $q\in[2,\infty)$.

(ii) *If $d=3$, there exist $T_0>0$, depending on the norms of the initial data, and a strong solution $(\rho,\boldsymbol{u},p,\phi,\mu)$ to the system (1.1)-(1.2) satisfying*

$$\begin{aligned}
&\rho\in\mathcal{C}([0,T_0];L^r(\Omega))\cap L^\infty(\Omega\times(0,T_0))\cap L^\infty(0,T_0;H^{-1}(\Omega)),\\
&\boldsymbol{u}\in\mathcal{C}([0,T_0];\mathbf{V}_\sigma)\cap L^2(0,T_0;H^2(\Omega))\cap H^1(0,T_0;\mathbf{H}_\sigma),\\
&p\in L^2(0,T_0;H^1(\Omega)),\\
&\phi\in\mathcal{C}([0,T_0];(W^{2,6}(\Omega))_w)\cap H^1(0,T_0;H^1(\Omega)),\\
&\mu\in L^\infty(0,T_0;H^1(\Omega))\cap L^2(0,T_0;W^{2,6}(\Omega)),
\end{aligned}$$

for any $r\in[2,\infty)$.

Besides, (1.1)$_1$ *holds in* $\mathcal{D}'(Q_{T_0})$ *if* $d = 3$ *and in* $\mathcal{D}'(Q_T)$ *if* $d = 2$*, and* (1.1)$_2$-(1.1)$_5$ *hold almost everywhere in* Q_{T_0} *if* $d = 3$ *and in* Q_T *if* $d = 2$*. The boundary conditions* $\partial_{\mathbf{n}}\phi = 0$ *and* $\partial_{\mathbf{n}}\mu = 0$ *hold almost everywhere on* $\partial\Omega \times (0, T_0)$ *if* $d = 3$*, and on* $\partial\Omega \times (0, T)$ *if* $d = 2$*, and the initial data are assumed* $\boldsymbol{u}(\cdot, 0) = \boldsymbol{u}_0$ *and* $\phi(\cdot, 0) = \phi_0$*. Furthermore,* (2.4)*,* (2.5) *and* (2.6) *hold for all* $t \in [0, T_0]$ *if* $d = 3$ *and for all* $t \in [0, T]$ *if* $d = 2$*.*

Proof. The proof is divided in several steps.

Smoothing of the initial data. Thanks to Lemma B.1 in Appendix B, for $k \in \mathbb{N}$ there exists a sequence $(\rho_0^k, \phi_0^k, \mu_0^k)$ such that

(i) The approximated densities $\rho_0^k \in \mathcal{C}^\infty(\overline{\Omega})$ satisfy $\rho_* \le \rho_0^k(x) \le \rho^*$, $\forall\, x \in \overline{\Omega}$, and as $k \to \infty$

$$\rho_0^k \to \rho_0 \quad \text{strongly in } L^r(\Omega),\ \forall\, r \in [1, \infty), \qquad \rho_0^k \rightharpoonup \rho_0 \quad \text{weak-star in } L^\infty(\Omega).$$

(ii) The approximated concentrations $\phi_0^k \in H^5(\Omega)$ are such that

$$\partial_{\mathbf{n}}\phi_0^k = 0 \quad \text{on } \partial\Omega, \quad \phi_0^k \to \phi_0 \quad \text{strongly in } H^2(\Omega), \tag{6.1}$$

and

$$\left\| \frac{-\Delta\phi_0^k}{\rho_0^k} \right\|_{H^1(\Omega)} \le C\|\mu_0\|_{H^1(\Omega)} + C\|\phi_0\|_{H^2(\Omega)}^3. \tag{6.2}$$

(iii) The approximated chemical potentials $\mu_0^k \in H^3(\Omega)$ defined by $\mu_0^k = -\frac{\Delta\phi_0^k}{\rho_0^k} + \Psi_0'(\phi_0^k)$ satisfy

$$\mu_0^k \to \mu_0 \quad \text{strongly in } H^1(\Omega).$$

Approximated regular solutions. For any $k \in \mathbb{N}$ we consider the approximate solutions constructed in the proof of Theorem 4.1 through the semi-Galerkin discretization. More precisely, there exist

$$\rho_m \in \mathcal{C}^1(\overline{Q_T}), \quad \boldsymbol{u}_m \in \mathcal{C}^1([0, T]; \mathbf{V}_m), \quad \phi_m \in \mathcal{C}^1([0, T]; V_m), \quad \mu_m \in \mathcal{C}([0, T]; V_m),$$

with $Q_T = \Omega \times (0, T)$. The approximated densities $\rho_m(x, t) = \rho_0^k(\mathbf{X}_m(0, t, x))$, where the characteristic $\mathbf{X}_m(s, t, x)$ is given by (4.17), solve

$$\partial_t\rho_m + \operatorname{div}(\rho_m\boldsymbol{u}_m) = 0 \quad \forall\, (x, t) \in Q_T. \tag{6.3}$$

The approximated solutions $(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$ satisfy

$$(\rho_m\partial_t\boldsymbol{u}_m, \boldsymbol{w}) + (\rho_m(\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}) + (\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m, \nabla\boldsymbol{w}) = (\rho_m\mu_m\nabla\phi_m, \boldsymbol{w}) - (\rho_m\nabla\Psi_0(\phi_m), \boldsymbol{w}), \tag{6.4}$$

$$(\rho_m\partial_t\phi_m, w) + (\rho_m\boldsymbol{u}_m \cdot \nabla\phi_m, w) + (\nabla\mu_m, \nabla w) = 0, \tag{6.5}$$

$$(\rho_m\mu_m, w) = (-\Delta\phi_m, w) + (\rho_m\Psi_0'(\phi_m), w), \tag{6.6}$$

for all $\boldsymbol{w} \in \mathbf{V}_m$ and $w \in V_m$, and for all $t \in [0, T]$. The initial conditions are $\rho(\cdot, 0) = \rho_0^k$,[5] $\boldsymbol{u}_m(\cdot, 0) = \mathbb{P}_m\boldsymbol{u}_0$, $\phi_m(\cdot, 0) = \Pi_m\phi_0^k$. Moreover, recalling that $\boldsymbol{u}_m(\cdot, 0) \to \boldsymbol{u}_0$ in $\mathbf{V}_\sigma$ and $\phi_m(\cdot, 0) \to \phi_0^k$ in $H^3(\Omega)$, it follows from (4.19), (4.20), (4.26), and (4.28) that

[5] We observe that, since $\rho_0^k \in \mathcal{C}^\infty(\overline{\Omega})$, we do not perform any regularization for ρ_0^k in the semi-Galerkin approximation, as opposed to the beginning of the proof of Theorem 4.1 whose arguments can be easily adapted to this case.

$$\rho_* \le \rho_m(x,t) \le \rho^* \quad \forall\,(x,t) \in \overline{Q_T}, \tag{6.7}$$

$$\|\boldsymbol{u}_m\|_{L^\infty(0,T;\mathbf{H}_\sigma)} \le C(E_0), \quad \|\boldsymbol{u}_m\|_{L^2(0,T;\mathbf{V}_\sigma)} \le C(E_0), \tag{6.8}$$

and

$$\|\phi_m\|_{L^\infty(0,T;H^1(\Omega))} \le C(E_0),\ \|\phi_m\|_{L^2(0,T;H^2(\Omega))} \le C(E_0),\ \|\mu_m\|_{L^2(0,T;H^1(\Omega))} \le C(E_0), \tag{6.9}$$

where C is independent of m and k, and $E_0 = E_0(\rho_0, \boldsymbol{u}_0, \phi_0)$.

Higher-order energy inequalities. First, taking $\boldsymbol{w} = \partial_t \boldsymbol{u}_m$ in (6.4), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\nu(\phi_m)}{2}|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x + \int_\Omega \rho_m|\partial_t\boldsymbol{u}_m|^2\,\mathrm{d}x + \int_\Omega \rho_m(\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x \\
&= \int_\Omega \rho_m\mu_m\nabla\phi_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x - \int_\Omega \rho_m\nabla\Psi_0(\phi_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x + \int_\Omega \nu'(\phi_m)\partial_t\phi_m|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x.
\end{aligned} \tag{6.10}$$

We take $w = \partial_t\mu_m$[6] in (6.5) and we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu_m\|^2_{L^2(\Omega)} + (\rho_m\partial_t\phi_m, \partial_t\mu_m) + (\rho_m\boldsymbol{u}_m\cdot\nabla\phi_m, \partial_t\mu_m) = 0. \tag{6.11}$$

Since $\partial_t\phi_m(t) \in V_m$, by using (6.3) and (6.6), we find

$$\begin{aligned}
(\rho_m\partial_t\phi_m, \partial_t\mu_m) &= \int_\Omega \partial_t(\rho_m\mu_m)\partial_t\phi_m\,\mathrm{d}x - \int_\Omega \mu_m\partial_t\rho_m\partial_t\phi_m\,\mathrm{d}x \\
&= \int_\Omega -\Delta\partial_t\phi_m\partial_t\phi_m\,\mathrm{d}x + \int_\Omega \partial_t(\rho_m\Psi_0'(\phi_m))\partial_t\phi_m\,\mathrm{d}x \\
&\quad + \int_\Omega \mathrm{div}(\rho_m\boldsymbol{u}_m)\mu_m\partial_t\phi_m\,\mathrm{d}x \\
&= \int_\Omega |\nabla\partial_t\phi_m|^2\,\mathrm{d}x + \int_\Omega \partial_t\rho_m\Psi_0'(\phi_m)\partial_t\phi_m\,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m\Psi_0''(\phi_m)|\partial_t\phi_m|^2\,\mathrm{d}x - \int_\Omega \rho_m\boldsymbol{u}_m\cdot\nabla(\mu_m\partial_t\phi_m)\,\mathrm{d}x \\
&= \int_\Omega |\nabla\partial_t\phi_m|^2\,\mathrm{d}x - \int_\Omega \mathrm{div}(\rho_m\boldsymbol{u}_m)\Psi_0'(\phi_m)\partial_t\phi_m\,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m\Psi_0''(\phi_m)|\partial_t\phi_m|^2\,\mathrm{d}x - \int_\Omega \rho_m\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\mu_m\,\mathrm{d}x \\
&\quad - \int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x \\
&= \int_\Omega |\nabla\partial_t\phi_m|^2\,\mathrm{d}x + \int_\Omega \rho_m\Psi_0''(\phi_m)\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m\Psi_0'(\phi_m)\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x + \int_\Omega \rho_m\Psi_0''(\phi_m)|\partial_t\phi_m|^2\,\mathrm{d}x \\
&\quad - \int_\Omega \rho_m\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\mu_m\,\mathrm{d}x - \int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x.
\end{aligned} \tag{6.12}$$

[6] This is a rigorous operation because $\mu_m \in \mathcal{C}^1([0,T], V_m)$. Indeed, recalling the ODEs system (A.9), we have that $\mathbf{C}^m(t) = (\mathbf{M}_1^m(t))^{-1}[-\mathbf{L}_4^m\mathbf{B}^m(t) + F_2^m(t,\mathbf{B}^m)]$. Since $(\mathbf{M}_1^m)^{-1} \in \mathcal{C}^1([0,T])$ and $\mathbf{B}^m \in \mathcal{C}^1[0,T]$, it easily follows that $\mathbf{C}^m \in \mathcal{C}^1[0,T]$.

Besides, using once again (6.3), we have

$$
\begin{aligned}
&(\rho_m \boldsymbol{u}_m \cdot \nabla \phi_m, \partial_t \mu_m) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left\{ \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \right\} - \int_\Omega \partial_t \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \\
&\qquad - \int_\Omega \rho_m \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x - \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left\{ \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \right\} + \int_\Omega \mathrm{div}(\rho_m \boldsymbol{u}_m) \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \\
&\qquad - \int_\Omega \rho_m \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x - \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}t}\left\{ \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \right\} - \int_\Omega \rho_m (\boldsymbol{u}_m \cdot \nabla \mu_m)(\boldsymbol{u}_m \cdot \nabla \phi_m) \,\mathrm{d}x \\
&\qquad - \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla (\boldsymbol{u}_m \cdot \nabla \phi_m) \,\mathrm{d}x - \int_\Omega \rho_m \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \\
&\qquad - \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x.
\end{aligned}
\tag{6.13}
$$

By substituting (6.12) and (6.13) in (6.11), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{ \frac{1}{2}\|\nabla \mu_m\|_{L^2(\Omega)}^2 + \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \right\} + \int_\Omega |\nabla \partial_t \phi_m|^2 \,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m \Psi_0''(\phi_m) |\partial_t \phi_m|^2 \,\mathrm{d}x = - \int_\Omega \rho_m \Psi_0''(\phi_m) \partial_t \phi_m \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x \\
&\quad - \int_\Omega \rho_m \Psi_0'(\phi_m) \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x + \int_\Omega \rho_m \partial_t \phi_m \boldsymbol{u}_m \cdot \nabla \mu_m \,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m (\boldsymbol{u}_m \cdot \nabla \mu_m)(\boldsymbol{u}_m \cdot \nabla \phi_m) \,\mathrm{d}x + \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla (\boldsymbol{u}_m \cdot \nabla \phi_m) \,\mathrm{d}x \\
&\quad + \int_\Omega \rho_m \mu_m \partial_t \boldsymbol{u}_m \cdot \nabla \phi_m \,\mathrm{d}x + 2 \int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x.
\end{aligned}
\tag{6.14}
$$

Taking $w = \partial_t \phi_m$ in (6.5), we find

$$
\int_\Omega \rho_m |\partial_t \phi_m|^2 \,\mathrm{d}x = - \int_\Omega \rho_m \boldsymbol{u}_m \cdot \nabla \phi_m \partial_t \phi_m \,\mathrm{d}x - \int_\Omega \nabla \mu_m \cdot \nabla \partial_t \phi_m \,\mathrm{d}x. \tag{6.15}
$$

Since $\Psi_0''(s) = 3s^2 - 1 \geq -1$, multiplying (6.15) by a factor 2 and adding it to (6.14), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{2}\|\nabla\mu_m\|_{L^2(\Omega)}^2+\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x\right\}+\alpha_0\|\partial_t\phi_m\|_{H^1(\Omega)}^2\\
&\le -\int_\Omega \rho_m\Psi_0''(\phi_m)\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x-\int_\Omega \rho_m\Psi_0'(\phi_m)\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x\\
&\quad+\int_\Omega \rho_m\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\mu_m\,\mathrm{d}x+\int_\Omega \rho_m(\boldsymbol{u}_m\cdot\nabla\mu_m)(\boldsymbol{u}_m\cdot\nabla\phi_m)\,\mathrm{d}x\\
&\quad+\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla(\boldsymbol{u}_m\cdot\nabla\phi_m)\,\mathrm{d}x+\int_\Omega \rho_m\mu_m\partial_t\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x\\
&\quad+2\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x-2\int_\Omega \rho_m\boldsymbol{u}_m\cdot\nabla\phi_m\partial_t\phi_m\,\mathrm{d}x-2\int_\Omega \nabla\mu_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x,
\end{aligned}
\tag{6.16}
$$

where $\alpha_0=\min\{1,\rho_*\}>0$. By summing (6.10) and (6.14), we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_\Omega \frac{\nu(\phi_m)}{2}|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x+\frac{1}{2}\|\nabla\mu_m\|_{L^2(\Omega)}^2+\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x\right\}\\
&\quad+\rho_*\int_\Omega |\partial_t\boldsymbol{u}_m|^2\,\mathrm{d}x+\alpha_0\|\partial_t\phi_m\|_{H^1(\Omega)}^2\\
&\le \underbrace{\int_\Omega -\rho_m(\boldsymbol{u}_m\cdot\nabla)\boldsymbol{u}_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x}_{I_1}+\underbrace{2\int_\Omega \rho_m\mu_m\nabla\phi_m\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x}_{I_2}\\
&\quad+\underbrace{\int_\Omega -\rho_m\nabla\Psi_0(\phi_m)\cdot\partial_t\boldsymbol{u}_m\,\mathrm{d}x}_{I_3}+\underbrace{\int_\Omega \nu'(\phi_m)\partial_t\phi_m|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x}_{I_4}\\
&\quad+\underbrace{\int_\Omega -\rho_m\Psi_0''(\phi_m)\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x}_{I_5}+\underbrace{\int_\Omega -\rho_m\Psi_0'(\phi_m)\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x}_{I_6}\\
&\quad+\underbrace{\int_\Omega \rho_m\partial_t\phi_m\boldsymbol{u}_m\cdot\nabla\mu_m\,\mathrm{d}x}_{I_7}+\underbrace{\int_\Omega \rho_m(\boldsymbol{u}_m\cdot\nabla\mu_m)(\boldsymbol{u}_m\cdot\nabla\phi_m)\,\mathrm{d}x}_{I_8}\\
&\quad+\underbrace{\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla(\boldsymbol{u}_m\cdot\nabla\phi_m)\,\mathrm{d}x}_{I_9}+\underbrace{2\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x}_{I_{10}}\\
&\quad+\underbrace{2\int_\Omega -\rho_m\boldsymbol{u}_m\cdot\nabla\phi_m\partial_t\phi_m\,\mathrm{d}x}_{I_{11}}+\underbrace{2\int_\Omega -\nabla\mu_m\cdot\nabla\partial_t\phi_m\,\mathrm{d}x}_{I_{12}}.
\end{aligned}
\tag{6.17}
$$

Before proceeding to estimate the right-hand side of (6.17), we define

$$
H(\rho_m,\boldsymbol{u}_m,\phi_m,\mu_m)=\int_\Omega \frac{\nu(\phi_m)}{2}|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x+\frac{1}{2}\|\nabla\mu_m\|_{L^2(\Omega)}^2+\int_\Omega \rho_m\mu_m\boldsymbol{u}_m\cdot\nabla\phi_m\,\mathrm{d}x.
\tag{6.18}
$$

For the sake of brevity, we will denote $H_m = H(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$. Thanks to (3.4), (4.26), and (6.9), we observe that

$$\begin{aligned}\int_\Omega \rho_m \mu_m \boldsymbol{u}_m \cdot \nabla \phi_m \, \mathrm{d}x &\leq \rho^* \|\mu_m\|_{L^6(\Omega)} \|\boldsymbol{u}_m\|_{L^3(\Omega)} \|\nabla \phi_m\|_{L^2(\Omega)} \\ &\leq C(E_0)(1 + \|\nabla \mu_m\|_{L^2(\Omega)}) \|\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac{1}{2}} \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac{1}{2}} \\ &\leq \int_\Omega \frac{\nu(\phi_m)}{4} |\mathbb{D}\boldsymbol{u}_m|^2 \, \mathrm{d}x + \frac{1}{4} \|\nabla \mu_m\|_{L^2(\Omega)}^2 + C(E_0),\end{aligned}$$

where the constant C depends on ρ^*, ν^*, but it is independent of m. In turn, this implies that

$$H_m \geq \int_\Omega \frac{\nu(\phi_m)}{4} |\mathbb{D}\boldsymbol{u}_m|^2 \, \mathrm{d}x + \frac{1}{4} \|\nabla \mu_m\|_{L^2(\Omega)}^2 - C(E_0). \tag{6.19}$$

Similarly, we have

$$H_m \leq C(E_0) \int_\Omega \frac{\nu(\phi_m)}{4} |\mathbb{D}\boldsymbol{u}_m|^2 \, \mathrm{d}x + C(E_0) \|\nabla \mu_m\|_{L^2(\Omega)}^2. \tag{6.20}$$

Three-dimensional case ($d = 3$). We estimate successively all terms in the right-hand side of (6.17). To this aim, we recall the estimates (cf. (4.26) and (4.28))

$$\|\mu_m\|_{H^1(\Omega)} \leq C(E_0)(1 + \|\nabla \mu_m\|_{L^2(\Omega)}), \quad \|\phi_m\|_{H^2(\Omega)} \leq C(E_0)(1 + \|\nabla \mu_m\|_{L^2(\Omega)}^{\frac{1}{2}}), \tag{6.21}$$

where C is independent of m. Let ϖ be a positive (small) constant which will be determined later. By exploiting (3.4), (3.6), (6.9) and (6.21), we obtain

$$\begin{aligned}|I_1| &\leq \rho^* \|\boldsymbol{u}_m\|_{L^6(\Omega)} \|\nabla \boldsymbol{u}_m\|_{L^3(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac{3}{2}} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq \frac{\rho_*}{6} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{\varpi}{4} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^6,\end{aligned} \tag{6.22}$$

$$\begin{aligned}|I_2| &\leq 2\rho^* \|\mu_m\|_{L^6(\Omega)} \|\nabla \phi_m\|_{L^3(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq C \|\mu_m\|_{H^1(\Omega)} \|\nabla \phi_m\|_{L^2(\Omega)}^{\frac{1}{2}} \|\phi_m\|_{H^2(\Omega)}^{\frac{1}{2}} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq \frac{\rho_*}{6} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0) \|\phi_m\|_{H^2(\Omega)} (1 + \|\nabla \mu_m\|_{L^2(\Omega)}^2) \\ &\leq \frac{\rho_*}{6} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(1 + \|\nabla \mu_m\|_{L^2(\Omega)}^{\frac{5}{2}}),\end{aligned} \tag{6.23}$$

$$\begin{aligned}|I_3| &\le \rho^*\|\Psi_0'(\phi_m)\|_{L^\infty(\Omega)}\|\nabla\phi_m\|_{L^2(\Omega)}\|\partial_t \boldsymbol{u}\|_{L^2(\Omega)}\\ &\le \frac{\rho_*}{6}\|\partial_t\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(\|\phi_m\|_{L^\infty(\Omega)}^2 + \|\phi_m\|_{L^\infty(\Omega)}^6)\\ &\le \frac{\rho_*}{6}\|\partial_t\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(\|\phi_m\|_{H^1(\Omega)}\|\phi_m\|_{H^2(\Omega)} + \|\phi_m\|_{H^1(\Omega)}^3\|\phi_m\|_{H^2(\Omega)}^3)\\ &\le \frac{\rho_*}{6}\|\partial_t\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(\|\phi_m\|_{H^2(\Omega)} + \|\phi_m\|_{H^2(\Omega)}^3)\\ &\le \frac{\rho_*}{6}\|\partial_t\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)}^{\frac32}),\end{aligned} \tag{6.24}$$

$$\begin{aligned}|I_4| &\le C\|\partial_t\phi_m\|_{L^6(\Omega)}\|\nabla\boldsymbol{u}_m\|_{L^3(\Omega)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + C\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^3\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + \frac{\varpi}{4}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^6,\end{aligned} \tag{6.25}$$

$$\begin{aligned}|I_5| &\le \rho^*\|\Psi_0''(\phi_m)\|_{L^3(\Omega)}\|\partial_t\phi_m\|_{L^6(\Omega)}\|\boldsymbol{u}_m\|_{L^\infty(\Omega)}\|\nabla\phi_m\|_{L^2(\Omega)}\\ &\le C(E_0)\|\partial_t\phi_m\|_{H^1(\Omega)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + \frac{\varpi}{4}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.26}$$

$$\begin{aligned}|I_6| &\le \rho^*\|\Psi_0'(\phi_m)\|_{L^2(\Omega)}\|\boldsymbol{u}_m\|_{L^\infty(\Omega)}\|\partial_t\phi_m\|_{H^1(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + C(E_0)\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + \frac{\varpi}{4}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.27}$$

$$\begin{aligned}|I_7| &\le \rho^*\|\partial_t\phi_m\|_{L^6(\Omega)}\|\boldsymbol{u}_m\|_{L^3(\Omega)}\|\nabla\mu_m\|_{L^2(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + C\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2\|\nabla\mu_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.28}$$

$$\begin{aligned}|I_8| &\le \rho^*\|\boldsymbol{u}_m\|_{L^6(\Omega)}^2\|\nabla\mu_m\|_{L^2(\Omega)}\|\nabla\phi_m\|_{L^6(\Omega)}\\ &\le C\|\nabla\mu_m\|_{L^2(\Omega)}\|\phi_m\|_{H^2(\Omega)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2\\ &\le C(E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)}^{\frac32})\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.29}$$

$$\begin{aligned}|I_9| &\le \rho^*\|\mu_m\|_{L^6(\Omega)}\|\boldsymbol{u}_m\|_{L^6(\Omega)}(\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}\|\nabla\phi_m\|_{L^6(\Omega)} + \|\boldsymbol{u}_m\|_{L^6(\Omega)}\|\phi_m\|_{H^2(\Omega)})\\ &\le C(E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)})\|\phi_m\|_{H^2(\Omega)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2\\ &\le C(E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)}^{\frac32})\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.30}$$

$$\begin{aligned}|I_{10}| &\le 2\rho^*\|\mu_m\|_{L^6(\Omega)}\|\boldsymbol{u}_m\|_{L^3(\Omega)}\|\nabla\partial_t\phi_m\|_{L^2(\Omega)}\\ &\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|_{H^1(\Omega)}^2 + C(E_0)(1+\|\nabla\mu_m\|_{L^2(\Omega)}^2)\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.31}$$

$$\begin{aligned}|I_{11}| &\leq 2\rho^* \|\boldsymbol{u}_m\|_{L^3(\Omega)} \|\nabla \phi_m\|_{L^2(\Omega)} \|\partial_t \phi_m\|_{L^6(\Omega)} \\ &\leq \frac{\alpha_0}{16} \|\partial_t \phi_m\|_{H^1(\Omega)}^2 + C(E_0) \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2,\end{aligned} \tag{6.32}$$

$$|I_{12}| \leq \frac{\alpha_0}{16} \|\partial_t \phi_m\|_{H^1(\Omega)}^2 + C \|\nabla \mu_m\|_{L^2(\Omega)}^2. \tag{6.33}$$

Collecting (6.22)-(6.33) all together, we find the following differential inequality

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} H_m + \frac{\rho_*}{2} \int_\Omega |\partial_t \boldsymbol{u}_m|^2 \,\mathrm{d}x + \frac{\alpha_0}{2} \|\partial_t \phi_m\|_{H^1(\Omega)}^2 \\ &\leq \varpi \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^6 \\ &\quad + C(E_0)(1 + \|\nabla \mu_m\|_{L^2(\Omega)}^2)(\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \|\nabla \mu_m\|_{L^2(\Omega)}^2).\end{aligned} \tag{6.34}$$

Here the constant C depends on ϖ. Next, taking $\boldsymbol{w} = \mathbf{A}\boldsymbol{u}_m$ ($\mathbf{A}$ is the Stokes operator) in (6.4), we obtain

$$\begin{aligned}\int_\Omega -\operatorname{div}(\nu(\phi_m)\mathbb{D}\boldsymbol{u}_m) \cdot \mathbf{A}\boldsymbol{u}_m \,\mathrm{d}x &= -(\rho_m \partial_t \boldsymbol{u}_m, \mathbf{A}\boldsymbol{u}_m) - (\rho_m (\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \mathbf{A}\boldsymbol{u}_m) \\ &\quad + (\rho_m \mu_m \nabla \phi_m, \mathbf{A}\boldsymbol{u}_m) - (\rho_m \nabla \Psi_0(\phi_m), \mathbf{A}\boldsymbol{u}_m).\end{aligned} \tag{6.35}$$

By arguing as in [30], there exists $\pi_m \in L^2(0,T;H^1(\Omega))$ such that $-\Delta \boldsymbol{u}_m + \nabla \pi_m = \mathbf{A}\boldsymbol{u}_m$ almost everywhere in $\Omega \times (0,T)$ and such that

$$\|\pi_m\|_{L^2(\Omega)} \leq C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12}, \quad \|\pi_m\|_{H^1(\Omega)} \leq C \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}, \tag{6.36}$$

where C is independent of m. Then, we rewrite (6.35) as follows

$$\begin{aligned}\int_\Omega \frac{\nu(\phi_m)}{2} |\mathbf{A}\boldsymbol{u}_m|^2 \,\mathrm{d}x &= \underbrace{(-\rho_m \partial_t \boldsymbol{u}_m, \mathbf{A}\boldsymbol{u}_m)}_{K_1} + \underbrace{(-\rho_m (\boldsymbol{u}_m \cdot \nabla)\boldsymbol{u}_m, \mathbf{A}\boldsymbol{u}_m)}_{K_2} \\ &\quad + \underbrace{(\rho_m \mu_m \nabla \phi_m, \mathbf{A}\boldsymbol{u}_m)}_{K_3} + \underbrace{(-\rho_m \nabla \Psi(\phi_m), \mathbf{A}\boldsymbol{u}_m)}_{K_4} \\ &\quad + \underbrace{(\nu'(\phi_m)\mathbb{D}\boldsymbol{u}_m \nabla \phi_m, \mathbf{A}\boldsymbol{u}_m)}_{K_5} + \underbrace{(-\frac12 \nu'(\phi_m)\pi_m \nabla \phi_m, \mathbf{A}\boldsymbol{u}_m)}_{K_6}.\end{aligned} \tag{6.37}$$

Using (3.6), (6.9) and (6.36), we estimate $K_1, \ldots, K_6$ as follows

$$|K_1| \leq \rho^* \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \leq \frac{\nu_*}{24} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{6(\rho^*)^2}{\nu_*} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2, \tag{6.38}$$

$$\begin{aligned}|K_2| &\leq \rho^* \|\boldsymbol{u}_m\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac32} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac32} \leq \frac{\nu_*}{24} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^6,\end{aligned} \tag{6.39}$$

$$\begin{aligned}|K_3| &\leq \rho^* \|\mu_m\|_{L^6(\Omega)} \|\nabla \phi_m\|_{L^3(\Omega)} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\ &\leq \frac{\nu_*}{24} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C \|\mu_m\|_{H^1(\Omega)}^2 \|\phi_m\|_{H^2(\Omega)}^2,\end{aligned} \tag{6.40}$$

$$
\begin{aligned}
|K_4| &\le \rho^* \|\Psi_0'(\phi_m)\|_{L^\infty(\Omega)} \|\nabla \phi_m\|_{L^2(\Omega)} \|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C(E_0)(1+\|\phi_m\|_{L^\infty(\Omega)}^3)\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C(E_0)(1+\|\phi_m\|_{H^1(\Omega)}^{\frac32}\|\phi_m\|_{H^2(\Omega)}^{\frac32})\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)(1+\|\phi_m\|_{H^2(\Omega)}^3),
\end{aligned}
\tag{6.41}
$$

$$
\begin{aligned}
|K_5| &\le C\|\nabla \boldsymbol{u}_m\|_{L^3(\Omega)}\|\nabla\phi_m\|_{L^6(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac32}\|\phi_m\|_{H^2(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2\|\phi_m\|_{H^2(\Omega)}^4,
\end{aligned}
\tag{6.42}
$$

$$
\begin{aligned}
|K_6| &\le C\|\pi_m\|_{L^3(\Omega)}\|\nabla\phi_m\|_{L^6(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C\|\pi_m\|_{L^2(\Omega)}^{\frac12}\|\pi_m\|_{H^1(\Omega)}^{\frac12}\|\phi_m\|_{H^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac14}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac74}\|\phi_m\|_{H^2(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2\|\phi_m\|_{H^2(\Omega)}^8.
\end{aligned}
\tag{6.43}
$$

Combining the above estimates (6.38)-(6.43) with (6.37), and using (6.21), we arrive at

$$
\begin{aligned}
\frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 \le \frac{6(\rho^*)^2}{\nu_*}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^6 + C(1+\|\nabla\mu_m\|_{L^2(\Omega)}^3) \\
+ C(1+\|\nabla\mu_m\|_{L^2(\Omega)}^4)\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{6.44}
$$

We now set

$$
\delta = \frac{\rho_*\nu_*}{24(\rho^*)^2}, \quad \varpi = \frac{\rho_*\nu_*^2}{192(\rho^*)^2}, \tag{6.45}
$$

and

$$
F_m = F(\boldsymbol{u}_m, \phi_m) = \varpi\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{\rho_*}{4}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{\alpha_0}{2}\|\partial_t\phi_m\|_{H^1(\Omega)}^2.
$$

Multiplying (6.44) by δ, adding it to (6.34), and using (6.19), we eventually obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}H_m + F_m \le C(E_0)(C(E_0)+H_m)^3, \tag{6.46}
$$

where the positive constant $C(E_0)$ is independent of the parameters m and k (cf. initial data).

Two-dimensional case ($d = 2$). In comparison with the three-dimensional case, the only difference consists in the estimates of the terms I_1, I_4, and K_2, K_5, K_6. By exploiting (3.3) and (3.7), and using (6.9), we find

$$
\begin{aligned}
|I_1| &\le \rho^*\|\boldsymbol{u}_m\|_{L^4(\Omega)}\|\nabla \boldsymbol{u}_m\|_{L^4(\Omega)}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C(E_0)\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^{\frac12}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le \frac{\rho_*}{6}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \varpi\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C(E_0)\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^4,
\end{aligned}
\tag{6.47}
$$

$$
\begin{aligned}
|I_4| &\le C\|\partial_t\phi_m\|_{H^1(\Omega)}\|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)} \ln^{\frac12}\Big(C\frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}\Big)\\
&\le \frac{\alpha_0}{16}\|\partial_t\phi_m\|^2_{H^1(\Omega)} + C\|\nabla \boldsymbol{u}_m\|^4_{L^2(\Omega)} \ln\Big(C\frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}\Big).
\end{aligned} \tag{6.48}
$$

Collecting (6.23)-(6.24), (6.26)-(6.33), and (6.47)-(6.48) all together, and using (6.19), we find

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_m + \frac{\rho_*}{2}\int_\Omega |\partial_t \boldsymbol{u}_m|^2\,\mathrm{d}x + \frac{\alpha_0}{2}\|\partial_t\phi_m\|^2_{H^1(\Omega)}\\
&\le \varpi\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} + C\|\nabla \boldsymbol{u}_m\|^4_{L^2(\Omega)} \ln\Big(C\frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}\Big)\\
&\quad + C(E_0)(\|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)} + \|\nabla \mu_m\|^2_{L^2(\Omega)})H_m.
\end{aligned} \tag{6.49}
$$

Next we estimate K_2, K_5 and K_6 of (6.37). By using (3.3), (6.8) and (6.9), we have

$$
\begin{aligned}
|K_2| &\le \rho^*\|\boldsymbol{u}_m\|_{L^4(\Omega)}\|\nabla \boldsymbol{u}_m\|_{L^4(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\
&\le C(E_0)\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|^{\frac32}_{L^2(\Omega)} \le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} + C(E_0)\|\nabla \boldsymbol{u}_m\|^4_{L^2(\Omega)},
\end{aligned} \tag{6.50}
$$

$$
\begin{aligned}
|K_5| &\le C\|\nabla \boldsymbol{u}_m\|_{L^4(\Omega)}\|\nabla\phi_m\|_{L^4(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\
&\le C(E_0)\|\nabla \boldsymbol{u}_m\|^{\frac12}_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|^{\frac32}_{L^2(\Omega)}\|\nabla\phi_m\|^{\frac12}_{L^2(\Omega)}\|\phi_m\|^{\frac12}_{H^2(\Omega)}\\
&\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} + C\|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)}\|\phi_m\|^2_{H^2(\Omega)},
\end{aligned} \tag{6.51}
$$

$$
\begin{aligned}
|K_6| &\le C\|\pi_m\|_{L^4(\Omega)}\|\nabla\phi_m\|_{L^4(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\
&\le C\|\pi_m\|^{\frac12}_{L^2(\Omega)}\|\pi_m\|^{\frac12}_{H^1(\Omega)}\|\nabla\phi_m\|^{\frac12}_{L^2(\Omega)}\|\phi_m\|^{\frac12}_{H^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}\\
&\le C(E_0)\|\nabla \boldsymbol{u}_m\|^{\frac14}_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|^{\frac74}_{L^2(\Omega)}\|\phi_m\|^{\frac12}_{H^2(\Omega)}\\
&\le \frac{\nu_*}{24}\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} + C(E_0)\|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)}\|\phi_m\|^4_{H^2(\Omega)}.
\end{aligned} \tag{6.52}
$$

Therefore, using (6.21) once again, we are lead to

$$
\begin{aligned}
\frac{\nu_*}{4}\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} &\le \frac{6(\rho^*)^2}{\nu_*}\|\partial_t \boldsymbol{u}_m\|^2_{L^2(\Omega)} + C(E_0)\|\nabla \boldsymbol{u}_m\|^4_{L^2(\Omega)}\\
&\quad + C(E_0)(1+\|\nabla\mu_m\|^2_{L^2(\Omega)})(\|\nabla\mu_m\|^2_{L^2(\Omega)} + \|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)}).
\end{aligned} \tag{6.53}
$$

Choosing δ and ϖ as in (6.45), multiplying (6.53) by δ and adding to (6.49), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}H_m + \varpi\|\mathbf{A}\boldsymbol{u}_m\|^2_{L^2(\Omega)} + \frac{\rho_*}{4}\int_\Omega |\partial_t \boldsymbol{u}_m|^2\,\mathrm{d}x + \frac{\alpha_0}{2}\|\partial_t\phi_m\|^2_{H^1(\Omega)}\\
&\le C(E_0)(\|\nabla \boldsymbol{u}_m\|^2_{L^2(\Omega)} + \|\nabla\mu_m\|^2_{L^2(\Omega)})H_m + C\|\nabla \boldsymbol{u}_m\|^4_{L^2(\Omega)} \ln\Big(C\frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}\Big).
\end{aligned} \tag{6.54}
$$

By using the basic inequality

$$
x\ln(Cy) \le y + x\ln\big(Cx\big) \quad \forall\, x, y > 0,
$$

with $x = \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2$, $y = \frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}$, we have

$$\begin{aligned}
&C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^4 \ln\Big(C\frac{\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}}{\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}}\Big)\\
&\le C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)} + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^4 \ln\big(C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2\big)\\
&\le \frac{\varpi}{2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^4 \ln\big(C\big(1+\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2\big)\big).
\end{aligned}$$

Thus, setting

$$G_m = G(\boldsymbol{u}_m, \phi_m) = \frac{\varpi}{2}\|\mathbf{A}\boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{\rho_*}{4}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \frac{\alpha_0}{2}\|\partial_t \phi_m\|_{H^1(\Omega)}^2,$$

and exploiting (6.19), we eventually arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}H_m + G_m \le C(E_0)(\|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)}^2 + \|\nabla \mu_m\|_{L^2(\Omega)}^2)(C(E_0)+H_m)\ln\big(C(E_0)+H_m\big). \tag{6.55}$$

Control of the initial data. The next aim is to bound $H_m(0)$ by a constant which is independent of m and k. The crucial term is $\nabla \mu_m(0)$. To do so, evaluating (6.6) at $t=0$, we have

$$(\rho_0^k \mu_m(0), w) = (-\Delta \phi_m(0), w) + (\rho_0^k \Psi_0'(\phi_m(0)), w) \quad \forall\, w \in V_m. \tag{6.56}$$

Taking $w = \mu_m(0)$ in (6.56), we find

$$\begin{aligned}
\|\mu_m(0)\|_{L^2(\Omega)} &\le \frac{1}{\rho_*}\|\Delta \phi_m(0)\|_{L^2(\Omega)} + \frac{\rho^*}{\rho_*}\|\Psi_0'(\phi_m(0))\|_{L^2(\Omega)}\\
&\le C\|\phi_m(0)\|_{H^1(\Omega)}^3 + C\|\phi_m(0)\|_{H^2(\Omega)}\\
&\le C\|\phi_0^k\|_{H^1(\Omega)}^3 + C\|\phi_0^k\|_{H^2(\Omega)} \le C(\|\phi_0\|_{H^2(\Omega)}),
\end{aligned} \tag{6.57}$$

where C is independent of m and k. Next, we take $w = -\Delta \mu_m(0)$ in (6.56)

$$\begin{aligned}
&(\rho_0^k \nabla \mu_m(0), \nabla \mu_m(0) + (\mu_m(0)\nabla \rho_0^k, \nabla \mu_m(0))\\
&\qquad = (-\nabla \Delta \phi_m(0), \nabla \mu_m(0)) + (\nabla(\rho_0^k \Psi_0'(\phi_m(0))), \nabla \mu_m(0)).
\end{aligned} \tag{6.58}$$

Recalling that $\phi_m(0) \to \phi_0^k$ in $H^3(\Omega)$, and using the properties of ρ_0^k and (6.57), we infer that

$$\begin{aligned}
\rho_*\|\nabla \mu_m(0)\|_{L^2(\Omega)} &\le \|\mu_m(0)\|_{L^2(\Omega)}\|\nabla \rho_0^k\|_{L^\infty(\Omega)} + \|\phi_m(0)\|_{H^3(\Omega)}\\
&\quad + \|\nabla \rho_0^k\|_{L^\infty(\Omega)}\|\Psi_0'(\phi_m(0))\|_{L^2(\Omega)} + \rho^*\|\Psi_0''(\phi_m(0))\nabla \phi_m(0)\|_{L^2(\Omega)}\\
&\le C(\|\phi_0\|_{H^2(\Omega)})\|\nabla \rho_0^k\|_{L^\infty(\Omega)} + \|\phi_0^k\|_{H^3(\Omega)} + \|\nabla \rho_0^k\|_{L^\infty(\Omega)}\|\phi_m(0)\|_{H^1(\Omega)}^3\\
&\quad + C\|\phi_m(0)\|_{H^1(\Omega)}^2\|\phi_m(0)\|_{H^2(\Omega)}\\
&\le C(\|\phi_0\|_{H^2(\Omega)})\|\nabla \rho_0^k\|_{L^\infty(\Omega)}(1+\|\phi_0^k\|_{H^1(\Omega)}^3) + \|\phi_0^k\|_{H^3(\Omega)} + \|\phi_m(0)\|_{H^2(\Omega)}^3\\
&\le C(\|\phi_0\|_{H^2(\Omega)}, \|\nabla \rho_0^k\|_{L^\infty(\Omega)}, \|\phi_0^k\|_{H^3(\Omega)}).
\end{aligned}$$

Here the constant C depends on k through the norms of ρ_0^k and ϕ_0^k. This implies that, for any $k \in \mathbb{N}$, $\mu_m(0)$ is bounded in $H^1(\Omega)$. We infer that (up to a subsequence), as $m \to \infty$,

$$\mu_m \to \mu^* \quad \text{strongly in } L^2(\Omega), \quad \nabla \mu_m \rightharpoonup \nabla \mu^* \quad \text{weakly in } L^2(\Omega). \tag{6.59}$$

We claim that $\mu^* = \mu_0^k$. Indeed, taking $v \in L^2(\Omega)$ and recalling that $\Pi_m v \to v$ strongly in $L^2(\Omega)$, we deduce from (6.56) that

$$(\rho_0^k \mu^*, v) \leftarrow (\rho_0^k \mu_m(0), \Pi_m v) = (-\Delta\phi_m(0) + \rho_0^k \Psi_0'(\phi_m(0)), \Pi_m v) \to (-\Delta\phi_0^k + \rho_0^k \Psi_0'(\phi_0^k), v).$$

This entails that

$$\mu^* = \frac{-\Delta\phi_0^k}{\rho_0^k} + \Psi_0'(\phi_0^k) = \mu_0^k.$$

Now, combining (6.58) with (6.59) with $\mu^* = \mu_0^k$, and recalling that $\phi_m(0) \to \phi_0^k$ strongly in $H^3(\Omega)$, we obtain

$$\begin{aligned}
&\lim_{m\to\infty} (\rho_0^k \nabla\mu_m(0), \nabla\mu_m(0)) \\
&= \lim_{m\to\infty} -(\mu_m(0)\nabla\rho_0^k, \nabla\mu_m(0)) - (\nabla\Delta\phi_m(0), \nabla\mu_m(0)) + (\nabla(\rho_0^k \Psi_0'(\phi_m(0))), \nabla\mu_m(0)) \\
&= -(\mu_0^k \nabla\rho_0^k, \nabla\mu_0^k) - (\nabla\Delta\phi_0^k, \nabla\mu_0^k) + (\nabla(\rho_0^k \Psi_0'(\phi_0^k)), \nabla\mu_0^k) \\
&= (\rho_0^k \nabla\mu_0^k, \nabla\mu_0^k).
\end{aligned} \tag{6.60}$$

Thanks to (6.2) and (6.60), it immediately follows that

$$\lim_{m\to\infty} \|\nabla\mu_m(0)\|_{L^2(\Omega)}^2 \le \frac{\rho^*}{\rho_*} \|\nabla\mu_0^k\|_{L^2(\Omega)}^2 \le C\|\mu_0\|_{H^1(\Omega)}^2 + C\|\phi_0\|_{H^2(\Omega)}^6.$$

In turn, this implies that, for any $k \in \mathbb{N}$, there exists m_k such that

$$\|\nabla\mu_m(0)\|_{L^2(\Omega)}^2 \le 1 + C\|\mu_0\|_{H^1(\Omega)}^2 + C\|\phi_0\|_{H^2(\Omega)}^6 \quad \forall\, m \ge m_k. \tag{6.61}$$

Uniform estimates and passage to the limit. First, we consider the case $d = 3$. Thanks to (6.46), we have that, whenever $\widetilde{T}$ satisfies $1 - 2C(E_0)(C(E_0) + H_m(0))^2 \widetilde{T} > 0$,

$$H_m(t) \le \frac{C(E_0) + H_m(0)}{\sqrt{1 - 2C(E_0)(C(E_0) + H_m(0))^2 t}} \quad \forall\, t \in [0, \widetilde{T}], \quad \text{if } d = 3.$$

Since (6.20) and (6.61) entail that, for $m \ge m_k$,

$$H_m(0) \le 1 + C\|\boldsymbol{u}_0\|_{\mathbf{V}_\sigma}^2 + C\|\mu_0\|_{H^1(\Omega)}^2 + C\|\phi_0\|_{H^2(\Omega)}^6,$$

we infer that there exist $T_0 > 0$ and $C_1 > 0$, depending only on the norms $\|\boldsymbol{u}_0\|_{\mathbf{V}_\sigma}$, $\|\mu_0\|_{H^1(\Omega)}$, and $\|\phi_0\|_{H^2(\Omega)}$, as well as ρ_* and ρ^*, such that

$$\sup_{t\in[0,T_0]} \left(\|\nabla\boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2 + \|\nabla\mu_m(t)\|_{L^2(\Omega)}^2 \right) \le C_1 \quad \text{if } d = 3, \tag{6.62}$$

where T_0 and C_1 are independent of m and k. On the other hand, if $d = 2$, integrating (6.55) we deduce that

$$H_m(t) \le (C(E_0) + H_m(0))^{\mathrm{e}^{C(E_0)\int_0^T \|\nabla\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu_m(\tau)\|_{L^2(\Omega)}^2 \, \mathrm{d}\tau}} \quad \forall\, t \in [0, T], \quad \text{if } d = 2.$$

Then, recalling (6.8) and (6.9), for any $T>0$ there exists $C_2=C_2(T)$, also depending on the norms $\|\boldsymbol{u}_0\|_{\mathbf{V}_\sigma}$, $\|\mu_0\|_{H^1(\Omega)}$, and $\|\phi_0\|_{H^2(\Omega)}$, ρ_* and ρ^*, such that

$$\sup_{t\in[0,T]}\Big(\|\nabla \boldsymbol{u}_m(t)\|_{L^2(\Omega)}^2+\|\nabla \mu_m(t)\|_{L^2(\Omega)}^2\Big)\leq C_2 \quad \text{if } d=2. \tag{6.63}$$

We will now use the notations $\overline{C}_1$ and $\overline{C}_2$ to denote generic constants depending on C_1 and C_2, but independent of m and k. We also infer from (6.46) that

$$\int_0^{T_0}\|\mathbf{A}\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^2+\|\partial_t \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^2+\|\partial_t \phi_m(\tau)\|_{H^1(\Omega)}^2\,\mathrm{d}\tau\leq \overline{C}_1 \quad \text{if } d=3, \tag{6.64}$$

and from (6.55) that

$$\int_0^{T}\|\mathbf{A}\boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^2+\|\partial_t \boldsymbol{u}_m(\tau)\|_{L^2(\Omega)}^2+\|\partial_t \phi_m(\tau)\|_{H^1(\Omega)}^2\,\mathrm{d}\tau\leq \overline{C}_2 \quad \text{if } d=2. \tag{6.65}$$

By exploiting (6.21), we have

$$\sup_{t\in[0,T_0]}\|\phi_m(t)\|_{H^2(\Omega)}\leq \overline{C}_1 \quad \text{if } d=3, \qquad \sup_{t\in[0,T]}\|\phi_m(t)\|_{H^2(\Omega)}\leq \overline{C}_2 \quad \text{if } d=2. \tag{6.66}$$

Finally, we infer from (6.5) and the above estimates that

$$\int_0^{T_0}\|\mu_m(\tau)\|_{H^2(\Omega)}^2\,\mathrm{d}\tau\leq \overline{C}_1 \quad \text{if } d=3, \qquad \int_0^{T}\|\mu_m(\tau)\|_{H^2(\Omega)}^2\,\mathrm{d}\tau\leq \overline{C}_2 \quad \text{if } d=2. \tag{6.67}$$

Now we are in position to pass to the limit as k is fixed and $m\to\infty$, and then as $k\to\infty$. More precisely, thanks to [41, Lemma 2.4] and the above estimates (6.62)-(6.67), we deduce that (up to a subsequence)

$$\begin{aligned}
&\rho_m \rightharpoonup \rho_k && \text{weak-star in } L^\infty(\Omega\times(0,T_0)),\\
&\rho_m \to \rho_k && \text{strongly in } \mathcal{C}([0,T_0];L^r(\Omega)), \forall\, r\in[1,\infty),\\
&\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}_k && \text{weak-star in } L^\infty(0,T_0;\mathbf{V}_\sigma),\\
&\boldsymbol{u}_m \rightharpoonup \boldsymbol{u}_k && \text{weakly in } L^2(0,T_0;H^2)\cap H^1(0,T_0;\mathbf{H}_\sigma), \quad \text{if } d=3,\\
&\phi_m \rightharpoonup \phi_k && \text{weak-star in } L^\infty(0,T_0;H^2(\Omega)),\\
&\mu_m \rightharpoonup \mu_k && \text{weak-star in } L^\infty(0,T_0;H^1(\Omega)),\\
&\mu_m \rightharpoonup \mu_k && \text{weakly in } L^2(0,T_0;H^2(\Omega)).
\end{aligned} \tag{6.68}$$

We notice that, if $d=2$, (6.68) holds by replacing $[0,T_0]$ with $[0,T]$. The strong convergences of $\boldsymbol{u}_m$ and ϕ_m are recovered through the Aubin-Lions lemma, which imply the convergence of the nonlinear terms. Thus, in a standard manner, we can pass to the limit as $m\to\infty$ in (6.4)-(6.6). Next, we observe that, by the weak lower semicontinuity of the norm, (6.64)-(6.67) hold for $(\rho_k,\boldsymbol{u}_k,\phi_k,\mu_k)$. This allows us to pass further to the limit as $k\to\infty$, and to obtain a limit solution $(\rho,\boldsymbol{u},p,\phi,\mu)$[7] satisfying (1.1)-(1.2) as stated in Theorem 6.1.

[7] The pressure p is recovered in standard way. We refer the reader to [46] and [49].

Further properties of the solution. We conclude by observing that $\Delta\mu = \rho\partial_t\phi + \rho\boldsymbol{u}\cdot\nabla\phi \in L^2(0,T_0;L^6(\Omega))$ if $d=3$, and $\Delta\mu \in L^2(0,T;L^q(\Omega))$ for all $q\in[2,\infty)$ if $d=2$. Thus, by elliptic regularity, we infer that $\mu \in L^2(0,T_0;W^{2,6}(\Omega))$ if $d=3$ and $\mu\in L^2(0,T;W^{2,q}(\Omega))$ for all $q\in[2,\infty)$ if $d=2$. Next, writing $(1.1)_5$ as $-\Delta\phi = \rho\mu - \rho\Psi_0'(\phi)$, it is immediate that $\Delta\phi\in L^\infty(0,T_0;L^6(\Omega))$ if $d=3$, implying that $\phi\in L^\infty(0,T_0;W^{2,6}(\Omega))$. A similar argument entails that $\phi\in L^\infty(0,T;W^{2,q}(\Omega))$ for all $q\in[2,\infty)$ if $d=2$. □

7. Conclusions

In this work we studied the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system in a bounded smooth domain in $\mathbb{R}^d$, $d=2,3$, with no-slip boundary condition for the velocity and homogeneous Neumann boundary conditions for the concentration and the chemical potential. First, we showed the existence of global weak solutions if the free energy density is the Landau potential or the Flory-Huggins potential. Then, we proved the existence of global strong solutions in two dimensions and of local strong solutions in three dimensions (with initial velocity in $\mathbf{V}_\sigma$) in the case of the Landau potential. In our analysis, the density is bounded from above and bounded away from zero, and the viscosity depends on the concentration. We conclude this paper by mentioning few open questions which will be the object of future investigations:

• **Well-posedness of strong solutions.** An interesting issue is to prove the uniqueness of the strong solutions whose existence is proved in Theorem 6.1 (cf. [20]). In connection to this question, it would be interesting to show the existence and uniqueness of strong solutions to (1.1)-(1.2) having *regular* density and velocity as shown in [19] and [37]. Furthermore, a natural question is whether strong solutions as in Theorem 6.1 exist in the case of the physically relevant Flory-Huggins potential (cf. [1] and [30] for the homogeneous case).

• **Vacuum case.** A possible extension of this paper is studying the case of initial density which is bounded from above but may vanish in Ω. We mention that some important results in this direction have been proven in [18,20,39,46] and the references therein for the nonhomogeneous Navier-Stokes equations. However, the assumption of the model according to which the sum of concentrations is equal to one may fail in this case, and the physical model is no longer valid as such.

• **Mobility and viscosity.** Further interesting questions concern the analysis of system (1.1) with different choices of the mobility and viscosity coefficients. It is well-known that the mobility $m(\phi)$ in the Cahn-Hilliard equation is either constant or degenerate. In the latter case, $-\Delta\mu$ in (1.1) is replaced by $-\operatorname{div}(m(\phi)\nabla\mu)$ with $m(\phi)=1-\phi^2$. We refer the reader to [12] for the existence of global weak solutions for the model H and to [5] for the concentration-depending density model proposed in [7]. Another relevant case is the density-depending viscosity coefficient. This amounts to replacing $\nu(\phi)$ in (1.1) with $\nu(\rho)$. As mentioned in Remark 4.2, the existence of global weak solutions to (1.1)-(1.2) can be adapted to this case. On the other hand, it remains an open issue whether strong solutions can be proven in this case since ρ is generally less regular than ϕ. We refer the reader to [21,35,51].

• **Boundary conditions.** A lot of interests have been devoted in the past few years to Navier-Stokes-Cahn-Hilliard systems (as mentioned in the Introduction) with Navier boundary condition for the velocity and dynamic boundary conditions for the concentration. We mention in this direction the works [17,25,26,43] and the references therein.

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Appendix A. Existence of approximate solutions

We prove the existence of the approximate solution $(\rho_m, \boldsymbol{u}_m, \phi_m, \mu_m)$ satisfying (4.11)-(4.18). To this aim, we perform a fixed point argument. For any $m \in \mathbb{N}$, we fix

$$\widetilde{\boldsymbol{u}}_m \in \mathcal{C}([0,T]; \mathbf{V}_m), \quad \widetilde{\phi}_m \in \mathcal{C}([0,T]; V_m). \tag{A.1}$$

Thanks to [37, Lemma 1.3], there exists a unique $\rho_m \in C^1(\overline{Q_T})$ defined by the formula

$$\rho_m(x,t) = \rho_{0m}(\widetilde{\mathbf{X}}_m(0,t,x)), \tag{A.2}$$

where

$$\widetilde{\mathbf{X}}_m(s,t,x) = x + \int_t^s \widetilde{\boldsymbol{u}}_m(\widetilde{\mathbf{X}}_m(\tau,t,x),\tau)\,\mathrm{d}\tau \quad \forall\, s,t \in [0,T]. \tag{A.3}$$

The solution ρ_m satisfies the following estimates

$$\rho_* \le \rho_m(x,t) \le \rho^* \quad \forall\, (x,t) \in \overline{Q_T}, \tag{A.4}$$

and

$$\max_{0 \le t \le T} \|\nabla \rho_m(t)\|_{L^\infty(\Omega)} \le C \|\nabla \rho_{0m}\|_{L^\infty(\Omega)} \mathrm{e}^{\int_0^T \|\widetilde{\boldsymbol{u}}_m(\tau)\|_{W^{1,\infty}(\Omega)}\,\mathrm{d}\tau}, \tag{A.5}$$

where the positive constant C is independent of m. Next, we look for the triple

$$\boldsymbol{u}_m(x,t) = \sum_{j=1}^m a_j^m(t)\boldsymbol{w}_j(x), \quad \phi_m(x,t) = \sum_{j=1}^m b_j^m(t) w_j(x), \quad \mu_m(x,t) = \sum_{j=1}^m c_j^m(t) w_j(x)$$

which solves

$$\begin{aligned}(\rho_m \partial_t \boldsymbol{u}_m, \boldsymbol{w}_l) + (\rho_m (\widetilde{\boldsymbol{u}}_m \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}_l) + (\nu(\widetilde{\phi}_m)\mathbb{D}\boldsymbol{u}_m, \nabla \boldsymbol{w}_l)& \\ = (\rho_m \mu_m \nabla \widetilde{\phi}_m, \boldsymbol{w}_l) - (\rho_m \nabla \Psi(\phi_m), \boldsymbol{w}_l)&,\end{aligned} \tag{A.6}$$

$$(\rho_m \partial_t \phi_m, w_l) + (\rho_m \boldsymbol{u}_m \cdot \nabla \widetilde{\phi}_m, w_l) + (\nabla \mu_m, \nabla w_l) = 0, \tag{A.7}$$

$$(\rho_m \mu_m, w_l) = (\nabla \phi_m, \nabla w_l) + (\rho_m \Psi_0'(\phi_m), w_l), \tag{A.8}$$

for all $l = 1, \dots, m$. The system (A.6)-(A.8) corresponds to a system of differential equations. We define the vectors

$$\mathbf{A}^m(t) = (a_1^m(t), \dots, a_m^m(t)), \quad \mathbf{B}^m(t) = (b_1^m(t), \dots, b_m^m(t)), \quad \mathbf{C}^m(t) = (c_1^m(t), \dots, c_m^m(t)).$$

The system (A.6)-(A.8) is equivalent to the system

$$\begin{cases} \mathbf{M}_1^m(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^m = \mathbf{L}_1^m(t)\mathbf{A}^m + \mathbf{L}_2^m(t)\mathbf{C}^m + \mathbf{F}_1^m(t,\mathbf{B}^m) \\ \mathbf{M}_2^m(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}^m = \mathbf{L}_3^m(t)\mathbf{A}^m + \mathbf{L}_4^m(t)\mathbf{C}^m \\ \mathbf{M}_2^m(t)\,\mathbf{C}^m = -\mathbf{L}_4^m\mathbf{B}^m + \mathbf{F}_2^m(t,\mathbf{B}^m), \end{cases} \tag{A.9}$$

where the matrices read

$$(\mathbf{M}_1^m(t))_{l,j} = \int_\Omega \rho_m(x,t)\boldsymbol{w}_j(x)\cdot\boldsymbol{w}_l(x)\,\mathrm{d}x, \quad (\mathbf{M}_2^m(t))_{l,j} = \int_\Omega \rho_m(x,t)w_j(x)w_l(x)\,\mathrm{d}x,$$

$$(\mathbf{L}_1^m(t))_{l,j} = -\int_\Omega \rho_m(x,t)(\widetilde{\boldsymbol{u}}_m(x,t)\cdot\nabla)\boldsymbol{w}_j(x)\cdot\boldsymbol{w}_l(x) - \nu(\widetilde{\phi}_m(x,t))\mathbb{D}\boldsymbol{w}_j(x):\nabla\boldsymbol{w}_l(x)\,\mathrm{d}x,$$

$$(\mathbf{L}_2^m(t))_{l,j} = \int_\Omega \rho_m(x,t)w_j(x)\nabla\widetilde{\phi}_m(x,t)\cdot\boldsymbol{w}_l(x)\,\mathrm{d}x, \quad (\mathbf{L}_3^m(t))_{l,j} = (\mathbf{L}_2^m(t))_{j,l},$$

$$(\mathbf{L}_4^m)_{l,j} = \int_\Omega \nabla w_j\cdot\nabla w_l\,\mathrm{d}x = \mathrm{diag}(\lambda_1,\dots,\lambda_m),$$

for $l,j=1,\dots,m$, where $\lambda_1,\dots,\lambda_m$ are the eigenvalues of $-\Delta+I$ with homogeneous Neumann boundary conditions. The nonlinear vector terms are

$$(\mathbf{F}_1^m(t,\mathbf{B}^m))_l = \int_\Omega \rho_m(x,t)\nabla\Psi_0(\phi_m(x,t))\cdot\boldsymbol{w}_l(x,t)\,\mathrm{d}x,$$

$$(\mathbf{F}_2^m(t,\mathbf{B}^m))_l = \int_\Omega \rho_m(x,t)\Psi_0'(\phi_m(x,t))w_l(x,t)\,\mathrm{d}x,$$

and the initial conditions are given by

$$\mathbf{A}^m(0) = ((\boldsymbol{u}_{0m},\boldsymbol{w}_1),\dots,(\boldsymbol{u}_{0m},\boldsymbol{w}_m)), \quad \mathbf{B}^m(0) = ((\phi_{0m},w_1),\dots,(\phi_{0m},w_m)).$$

In light of the regularity properties of $\widetilde{\boldsymbol{u}}_m$, $\widetilde{\phi}_m$ and ρ_m (cf. (A.1)-(A.4)), the matrices $\mathbf{M}_1^m$ and $\mathbf{M}_2^m$ are continuous on $[0,T]$ and invertible[8] for all $t\in[0,T]$, $\mathbf{L}_1^m$, $\mathbf{L}_2^m$, $\mathbf{L}_4^m$ are continuous on $[0,T]$, and $\mathbf{F}_1^m$ and $\mathbf{F}_2^m$ are continuous on $[0,T]\times\mathbb{R}^m$ and locally Lipschitz in $\mathbb{R}^m$ uniformly in t. The classical Cauchy-Lipschitz theorem entails the existence and uniqueness of a local solution $(\mathbf{A}^m,\mathbf{B}^m)\in\mathcal{C}^1([0,T_0),\mathbb{R}^m\times\mathbb{R}^m)$, $\mathbf{C}^m\in\mathcal{C}([0,T_0);\mathbb{R}^m)$. Next, we show that the solution exists on $[0,T]$. Multiplying (A.6) by a_l^m and summing over l, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho_m|\boldsymbol{u}_m|^2\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2\,\mathrm{d}x = \int_\Omega \rho_m\mu_m\nabla\widetilde{\phi}_m\cdot\boldsymbol{u}_m\,\mathrm{d}x - \int_\Omega \rho_m\nabla\Psi_0(\phi_m)\cdot\boldsymbol{u}_m\,\mathrm{d}x.$$

Here we have used that ρ_m is the solution to the transport equation with velocity $\widetilde{\boldsymbol{u}}_m$. Multiplying (A.7) by c_l^m and (A.8) by $\frac{\mathrm{d}}{\mathrm{d}t}b_l^m$, respectively, and summing over l, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}|\nabla\phi_m|^2 + \rho_m\Psi_0(\phi_m)\,\mathrm{d}x + \int_\Omega \rho_m\boldsymbol{u}_m\cdot\nabla\widetilde{\phi}_m\mu_m\,\mathrm{d}x + \int_\Omega |\nabla\mu_m|^2\,\mathrm{d}x = \int_\Omega \partial_t\rho_m\Psi_0(\phi_m)\,\mathrm{d}x.$$

[8] For any fixed $t\in[0,T]$, and a vector $\mathbf{x}\in\mathbb{R}^m\backslash\{\mathbf{0}\}$, we consider $(\mathbf{M}_1^m(t)\mathbf{x},\mathbf{x}) = \sum_{l=1}^m\sum_{j=1}^m\int_\Omega \rho_m(t)\boldsymbol{w}_l x_l\cdot\boldsymbol{w}_j x_j\,\mathrm{d}x = \int_\Omega \rho_m(t)|\xi|^2\,\mathrm{d}x \geq \underline{\rho_0}\|\xi\|_{L^2(\Omega)}^2$, where $\xi=\sum_{l=1}^m \boldsymbol{w}_l x_l$. If $\|\xi\|_{L^2(\Omega)}^2=0$, we infer that $\xi=0$ a.e. in Ω, and so there exists $\overline{l}$ such that $\boldsymbol{w}_{\overline{l}} = \frac{1}{x_{\overline{l}}}(-\sum_{l=1,l\neq\overline{l}}^m \boldsymbol{w}_l x_l)$. This contradicts the properties of the eigenfunctions $\boldsymbol{w}_l$ of the Stokes operator. Therefore, $\|\xi\|_{L^2(\Omega)}^2>0$, and $\mathbf{M}_1^m(t)$ is definite positive. The same argument applies for $\mathbf{M}_2^m(t)$. Furthermore, the regularity of ρ_m ensures that $\mathbf{M}_1^m$ and $\mathbf{M}_2^m$ belong to $\mathcal{C}^1([0,T])$. This, in turn, entails that $\det\mathbf{M}_1^m$ and $\det\mathbf{M}_2^m$ belong to $\mathcal{C}^1([0,T])$ and are strictly positive functions on $[0,T]$. Thus, we conclude that the inverse matrices $(\mathbf{M}_1^m)^{-1}$ and $(\mathbf{M}_2^m)^{-1}$ belong to $\mathcal{C}^1([0,T])$.

By summing the two above relations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho_m|\boldsymbol{u}_m|^2+\frac{1}{2}|\nabla\phi_m|^2+\rho_m\Psi_0(\phi_m)\,\mathrm{d}x+\int_\Omega \nu(\phi_m)|\mathbb{D}\boldsymbol{u}_m|^2+|\nabla\mu_m|^2\,\mathrm{d}x=0. \tag{A.10}$$

Integrating in time over $[0,t]$ where $0<t<T_0$, we deduce that

$$\int_\Omega \frac{1}{2}\rho_m(t)|\boldsymbol{u}_m(t)|^2+\frac{1}{2}|\nabla\phi_m(t)|^2+\rho_m(t)\Psi_0(\phi_m(t))\,\mathrm{d}x+\int_0^t\int_\Omega \nu(\phi_m(\tau))|\mathbb{D}\boldsymbol{u}_m(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega |\nabla\mu_m(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau=\int_\Omega \frac{1}{2}\rho_{0m}|\boldsymbol{u}_{0m}|^2+\frac{1}{2}|\nabla\phi_{0m}|^2+\rho_{0m}\Psi_0(\phi_{0m})\,\mathrm{d}x.$$

By using (A.4), the properties of the projector operator Π_m and $\mathbb{P}_m$, the Young's inequality, and the Sobolev embedding, we find

$$\begin{aligned}&\int_\Omega \frac{1}{2}\rho_{0m}|\boldsymbol{u}_{0m}|^2+\frac{1}{2}|\nabla\phi_{0m}|^2+\rho_{0m}\Psi(\phi_{0m})\,\mathrm{d}x\\&\qquad\le\frac{\rho^*}{2}\|\boldsymbol{u}_{0m}\|^2_{L^2(\Omega)}+\frac{1}{2}\|\phi_{0m}\|^2_{H^1(\Omega)}+C\rho^*\big(1+\|\phi_{0m}\|^4_{L^4(\Omega)}\big)\\&\qquad\le\frac{\rho^*}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}+\frac{1}{2}\|\phi_0\|^2_{H^1(\Omega)}+C\rho^*\big(1+\|\phi_0\|^4_{H^1(\Omega)}\big),\end{aligned} \tag{A.11}$$

where the constant C is independent of m. Since $\Psi_0(s)\ge\frac{1}{8}s^4-\frac{1}{4}$, by exploiting (A.4), we obtain

$$\begin{aligned}&\frac{\rho_*}{2}\|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega)}+\frac{1}{2}\|\nabla\phi_m(t)\|^2_{L^2(\Omega)}+\frac{\rho_*}{8}\|\phi_m(t)\|^4_{L^4(\Omega)}\\&\qquad\le\frac{\rho^*}{2}\|\boldsymbol{u}_0\|^2_{L^2(\Omega)}+\frac{1}{2}\|\phi_0\|^2_{H^1(\Omega)}+C\rho^*\big(1+\|\phi_0\|^4_{H^1(\Omega)}\big)+\frac{\rho_*}{4}.\end{aligned}$$

Therefore, there exists a positive constant $C_0=C_0(\rho_*,\rho^*,\boldsymbol{u}_0,\phi_0)$ such that

$$\sup_{0\le t<T_0}\|\boldsymbol{u}_m(t)\|_{L^2(\Omega)}+\sup_{0\le t<T_0}\|\phi_m(t)\|_{H^1(\Omega)}\le C_0.$$

By a classical argument on ODEs systems, we conclude that $T_0=T$. Moreover, we infer that

$$\max_{0\le t\le T}\|\boldsymbol{u}_m(t)\|_{L^2(\Omega)}+\max_{0\le t\le T}\|\phi_m(t)\|_{H^1(\Omega)}\le C_0. \tag{A.12}$$

For any $m\in\mathbb{N}$, we introduce the set

$$\mathcal{S}_m=\{(\boldsymbol{v},\psi)\in\mathcal{C}([0,T];\mathbf{V}_m\times V_m):\max_{0\le t\le T}\|\boldsymbol{v}_m(t)\|_{L^2(\Omega)}+\max_{0\le t\le T}\|\psi_m(t)\|_{H^1(\Omega)}\le C_0\},$$

and we consider the map $\Lambda:\mathcal{S}_m\to\mathcal{S}_m$ defined by

$$\Lambda(\widetilde{\boldsymbol{u}}_m,\widetilde{\phi}_m)=(\boldsymbol{u}_m,\phi_m).$$

Next, multiplying (A.6) by $\frac{\mathrm{d}}{\mathrm{d}t}a_l^m$ and summing over l, we find

$$\begin{aligned}
&\rho_* \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}^2 \\
&\le -\int_\Omega \rho_m (\widetilde{\boldsymbol{u}}_m \cdot \nabla)\boldsymbol{u}_m \cdot \partial_t \boldsymbol{u}_m \,\mathrm{d}x - \int_\Omega \nu(\widetilde{\phi}_m)\mathbb{D}\boldsymbol{u}_m : \nabla \partial_t \boldsymbol{u}_m \,\mathrm{d}x \\
&\quad - \int_\Omega \rho_m \mu_m \nabla \widetilde{\phi}_m \partial_t \boldsymbol{u}_m \,\mathrm{d}x - \int_\Omega \rho_m \Psi_0'(\phi_m)\nabla\phi_m \partial_t \boldsymbol{u}_m \,\mathrm{d}x \\
&\le \rho^* \|\widetilde{\boldsymbol{u}}_m\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} + \nu^* \|\mathbb{D}\boldsymbol{u}_m\|_{L^2(\Omega)} \|\nabla\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\quad + \rho^* \|\mu_m\|_{L^2(\Omega)} \|\widetilde{\phi}_m\|_{W^{1,6}(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^3(\Omega)} + \rho^* \|\Psi_0'(\phi_m)\|_{L^3(\Omega)} \|\phi_m\|_{W^{1,6}(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\le C\rho^* \|\widetilde{\boldsymbol{u}}_m\|_{H^2(\Omega)} \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)} \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} + \nu^* \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)} \|\nabla\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} \\
&\quad + C\rho^* \|\mu_m\|_{L^2(\Omega)} \|\widetilde{\phi}_m\|_{H^2(\Omega)} \|\nabla\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)} + \rho^* C(1 + \|\phi_m\|_{H^2(\Omega)}^4) \|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)}.
\end{aligned}$$

We now observe that

$$\|\widetilde{\boldsymbol{u}}_m\|_{H^2(\Omega)} \le \lambda_m^S \|\widetilde{\boldsymbol{u}}_m\|_{L^2(\Omega)}, \quad \|\boldsymbol{u}_m\|_{\mathbf{V}_\sigma} \le \sqrt{\lambda_m^S}\|\boldsymbol{u}_m\|_{L^2(\Omega)}, \quad \|\partial_t \boldsymbol{u}_m\|_{\mathbf{V}_\sigma} \le \sqrt{\lambda_m^S}\|\partial_t \boldsymbol{u}_m\|_{L^2(\Omega)},$$

where the λ_m^S are the eigenvalues of the Stokes operator $\mathbf{A}$, and

$$\|\widetilde{\phi}_m\|_{H^2(\Omega)} \le \sqrt{\lambda_m}\|\widetilde{\phi}_m\|_{H^1(\Omega)}, \quad \|\phi_m\|_{H^2(\Omega)} \le \sqrt{\lambda_m}\|\phi_m\|_{H^1(\Omega)}.$$

Also, we have

$$\rho_* \|\mu_m\|_{L^2(\Omega)} \le \|\phi_m\|_{H^2(\Omega)} + \rho^* \|\Psi'(\phi)\|_{L^2(\Omega)} \le \lambda_m \|\phi_m\|_{L^2(\Omega)} + C\rho^* \|\phi_m\|_{H^1(\Omega)}^3.$$

Thus, there exists a constant $C_1 = C_1(m, \rho_*, \rho^*, \nu^*, C_0)$ such that

$$\max_{0\le t\le T} \|\partial_t \boldsymbol{u}_m(t)\|_{L^2(\Omega)} \le C_1. \tag{A.13}$$

In a similar way, multiplying (A.7) by $\frac{\mathrm{d}}{\mathrm{d}t}b_l^m$ and summing over l, we find

$$\begin{aligned}
\rho_* \|\partial_t \phi_m\|_{L^2(\Omega)} &\le -\int_\Omega \rho_m \boldsymbol{u}_m \cdot \nabla\widetilde{\phi}_m \partial_t \phi_m \,\mathrm{d}x - \int_\Omega \nabla\mu_m \cdot \nabla\partial_t\phi_m \,\mathrm{d}x \\
&\le \rho^* \|\boldsymbol{u}_m\|_{L^3(\Omega)} \|\nabla\widetilde{\phi}_m\|_{L^6(\Omega)} \|\partial_t\phi_m\|_{L^2(\Omega)} + \|\nabla\mu_m\|_{L^2(\Omega)} \|\nabla\partial_t\phi_m\|_{L^2(\Omega)} \\
&\le C\rho^* \|\nabla \boldsymbol{u}_m\|_{L^2(\Omega)} \|\widetilde{\phi}_m\|_{H^2(\Omega)} \|\partial_t\phi_m\|_{L^2(\Omega)} + \lambda_m \|\mu_m\|_{L^2(\Omega)} \|\partial_t\phi_m\|_{L^2(\Omega)} \\
&\le \left(C\rho^* \sqrt{\lambda_m^S}\sqrt{\lambda_m} \|\boldsymbol{u}_m\|_{L^2(\Omega)} \|\widetilde{\phi}_m\|_{H^1(\Omega)} + \lambda_m \|\mu_m\|_{L^2(\Omega)} \right) \|\partial_t\phi_m\|_{L^2(\Omega)}.
\end{aligned}$$

So, there exists $C_2 = C_2(m, \rho_*, \rho^*, C_0)$ such that

$$\max_{0\le t\le T} \|\partial_t \phi_m(t)\|_{L^2(\Omega)} \le C_2. \tag{A.14}$$

Let us set $\widetilde{C}_0 = C_1 + C_2$. We consider the subset $\widetilde{\mathcal{S}}_m$ of $\mathcal{S}_m$ defined as

$$\begin{aligned}
\widetilde{\mathcal{S}}_m = \Big\{ (\boldsymbol{v}, \psi) \in \mathcal{C}^1([0,T]; \mathbf{V}_m \times V_m) : \max_{0\le t\le T} \|\boldsymbol{v}_m(t)\|_{L^2(\Omega)} + \max_{0\le t\le T} \|\psi_m(t)\|_{H^1(\Omega)} \le C_0, \\
\max_{0\le t\le T} \|\partial_t \boldsymbol{v}_m(t)\|_{L^2(\Omega)} + \max_{0\le t\le T} \|\partial_t \psi_m(t)\|_{H^1(\Omega)} \le \widetilde{C}_0 \Big\}.
\end{aligned}$$

Thanks to (A.13) and (A.14), we notice that

$$\Lambda : \mathcal{S}_m \to \widetilde{\mathcal{S}}_m.$$

By the Ascoli-Arzelà theorem, we deduce that the set $\widetilde{\mathcal{S}}_m$ is compact in $\mathcal{S}_m$.

We are left to prove the continuity of the map Λ in order to apply the Schauder fixed point theorem. To this aim, for any $m \in \mathbb{N}$, let us consider a sequence of $\{(\boldsymbol{v}^n, \psi^n)\}_{n=1}^{\infty}$ such that $(\boldsymbol{v}^n, \psi^n) \to (\boldsymbol{v}, \psi)$ in $\mathcal{C}([0,T]; \mathbf{V}_m \times V_m)$ as $n \to \infty$. For any $n \in \mathbb{N}$, we find $\rho_m^n \in C^1(\overline{Q_T})$ and $\rho_m \in C^1(\overline{Q_T})$ given by the formula (A.2), where $\widetilde{\mathbf{X}}_m$ satisfies (A.3) with $\widetilde{\boldsymbol{u}}_m$ replaced by $\boldsymbol{v}^n$ and $\boldsymbol{v}$, respectively. We notice that $\rho_m^n(\cdot, 0) = \rho_m(\cdot, 0) = \rho_{0m}$. We now write $\delta\rho_m = \rho_m^n - \rho_m$, which solves

$$\partial_t \delta\rho_m + \boldsymbol{v}^n \cdot \nabla \delta\rho_m + (\boldsymbol{v}^n - \boldsymbol{v}) \cdot \nabla \rho_m = 0.$$

Multiplying the above equation by $|\delta\rho_m|^{r-2}\delta\rho_m$ for $r \geq 2$, integrating over Ω, using (A.5) with $\widetilde{\boldsymbol{u}}_m = \boldsymbol{v}$ and $\delta\rho_m(\cdot, 0) = 0$, we infer that

$$\max_{0 \leq t \leq T} \|\delta\rho_m(t)\|_{L^r(\Omega)} \leq C \int_0^T \|\boldsymbol{v}^n(\tau) - \boldsymbol{v}(\tau)\|_{L^r(\Omega)} \, d\tau,$$

where C is independent of n. This entails that $\delta\rho_m \to 0$ strongly in $\mathcal{C}([0,T]; L^r(\Omega))$ for any $r \geq 2$. Thus, $\rho_m^n \to \rho_m$ strongly in $\mathcal{C}([0,T]; L^r(\Omega))$. Next, given $(\rho_m^n, \boldsymbol{v}^n, \psi^n)$ and $(\rho_m, \boldsymbol{v}, \psi)$, there exist $(\boldsymbol{u}_m^n, \phi_m^n)$ and $(\boldsymbol{u}_m, \phi_m)$ in $\mathcal{C}^1([0,T]; \mathbf{V}_m \times V_m)$ such that

$$\Lambda(\boldsymbol{v}^n, \psi^n) = (\boldsymbol{u}_m^n, \phi_m^n), \quad \Lambda(\boldsymbol{v}, \psi) = (\boldsymbol{u}_m, \phi_m).$$

In particular, $(\boldsymbol{u}_m^n, \phi_m^n)$ and $(\boldsymbol{u}_m, \phi_m)$ satisfy (A.6)-(A.8) with $(\widetilde{\boldsymbol{u}}_m, \widetilde{\phi}_m)$ replaced by $(\boldsymbol{v}^n, \psi^n)$ and $(\boldsymbol{v}, \psi)$, respectively. Arguing as above, we perform the energy estimates for $(\boldsymbol{u}_m^n, \phi_m^n)$ that give us

$$\max_{0 \leq t \leq T} \|\boldsymbol{u}_m^n(t)\|_{L^2(\Omega)} + \max_{0 \leq t \leq T} \|\phi_m^n(t)\|_{H^1(\Omega)} \leq \widetilde{K}_0, \tag{A.15}$$

where the positive constant $\widetilde{K}_0$ depends on m, ρ_{0m}, $\boldsymbol{u}_{0m}$, and ϕ_{0m}, but it is independent of n. In the same way, we also infer the following bound on the time derivatives

$$\max_{0 \leq t \leq T} \|\partial_t \boldsymbol{u}_m^n(t)\|_{L^2(\Omega)} + \max_{0 \leq t \leq T} \|\partial_t \phi_m^n(t)\|_{H^1(\Omega)} \leq \widetilde{K}_1, \tag{A.16}$$

where the positive constant $\widetilde{K}_1$ depends on m, ρ_{0m}, $\boldsymbol{u}_{0m}$, and ϕ_{0m}, but it is independent of n. Thanks to (A.15) and (A.16), the Ascoli-Arzelà theorem entails that there exist a subsequence $(\boldsymbol{u}_m^{n_j}, \phi_m^{n_j})$ and $(\mathbf{U}, \Phi) \in \mathcal{C}([0,T]; \mathbf{V}_m \times V_m)$ such that

$$\boldsymbol{u}_m^{n_j} \to \mathbf{U} \quad \text{strongly in } \mathcal{C}([0,T]; \mathbf{V}_m), \quad \phi_m^{n_j} \to \Phi \quad \text{strongly in } \mathcal{C}([0,T]; V_m).$$

In light of (A.16), we also have

$$\partial_t \boldsymbol{u}_m^{n_j} \to \partial_t \mathbf{U} \quad \text{weak-star in } L^\infty([0,T]; \mathbf{V}_m), \quad \partial_t \phi_m^{n_j} \to \partial_t \Phi \quad \text{weak-star in } L^\infty([0,T]; V_m).$$

Moreover, we also deduce from the above estimates that

$$\mu_m^{n_j} \to \Upsilon \quad \text{weakly in } L^2(0,T; V_m).$$

Thus, we pass to the limit $n_j \to \infty$ in the system (A.6)-(A.8) in $\mathcal{D}'(0,T)$ with $(\widetilde{\boldsymbol{u}}_m, \widetilde{\phi}_m)$ replaced by $(\boldsymbol{v}^n, \psi^n)$. Recalling that $(\boldsymbol{v}^n, \psi^n) \to (\boldsymbol{v}, \psi)$ in $\mathcal{C}([0,T]; \mathbf{V}_m \times V_m)$ and $\rho_m^n \to \rho_m$ strongly in $\mathcal{C}([0,T]; L^r(\Omega))$ for any $r \geq 2$, we find that

$$(\rho_m \partial_t \mathbf{U}, \boldsymbol{w}_l) + (\rho_m (\boldsymbol{v} \cdot \nabla)\mathbf{U}, \boldsymbol{w}_l) + (\nu(\psi)\mathbb{D}\mathbf{U}, \nabla \boldsymbol{w}_l) = (\rho_m \Upsilon \nabla \psi, \boldsymbol{w}_l) - (\rho_m \nabla \Psi_0(\Phi), \boldsymbol{w}_l), \tag{A.17}$$

$$(\rho_m \partial_t \Phi, w_l) + (\rho_m \mathbf{U} \cdot \nabla \psi, w_l) + (\nabla \Upsilon, \nabla w_l) = 0, \tag{A.18}$$

$$(\rho_m \Upsilon, w_l) = (\nabla \Phi, \nabla w_l) + (\rho_m \Psi_0'(\Phi), w_l), \tag{A.19}$$

for all $l = 1, \dots, m$, for all $t \in [0,T]$. On the other hand, (u_m, ϕ_m) solves

$$(\rho_m \partial_t \boldsymbol{u}_m, \boldsymbol{w}_l) + (\rho_m (\boldsymbol{v} \cdot \nabla)\boldsymbol{u}_m, \boldsymbol{w}_l) + (\nu(\psi)\mathbb{D}\boldsymbol{u}_m, \nabla \boldsymbol{w}_l) = (\rho_m \mu_m \nabla \psi, \boldsymbol{w}_l) - (\rho_m \nabla \Psi_0(\phi_m), \boldsymbol{w}_l), \tag{A.20}$$

$$(\rho_m \partial_t \phi_m, w_l) + (\rho_m \boldsymbol{u}_m \cdot \nabla \psi, w_l) + (\nabla \mu_m, \nabla w_l) = 0, \tag{A.21}$$

$$(\rho_m \mu_m, w_l) = (\nabla \phi_m, \nabla w_l) + (\rho_m \Psi_0'(\phi_m), w_l), \tag{A.22}$$

for all $l = 1, \dots, m$, for all $t \in [0,T]$. By uniqueness, it follows that $(\mathbf{U}, \Phi, \Upsilon) = (\boldsymbol{u}_m, \phi_m, \mu_m)$. This argument entails that every subsequence of $\Lambda(\boldsymbol{v}^n, \psi^n)$ possesses a subsequence whose limit coincide with $\Lambda(\boldsymbol{v}, \psi)$. Therefore, we conclude that the whole sequence converges, namely

$$\Lambda(\boldsymbol{v}^n, \psi^n) \to \Lambda(\boldsymbol{v}, \psi) \quad \text{strongly in } \mathcal{C}([0,T]; \mathbf{V}_m \times V_m).$$

Summing up, by setting $X_m = \mathcal{C}([0,T]; \mathbf{V}_m \times V_m)$, we have shown that the map $\Lambda : \mathcal{S}_m \subset X_m \to \mathcal{S}_m$ is a continuous map such that $\overline{\Lambda(\mathcal{S}_m)} = \widetilde{\mathcal{S}}_m$ is compact in X_m. By the Schauder fixed point theorem, we conclude that, for any $m \in \mathbb{N}$, there exists $(\boldsymbol{u}_m, \phi_m) \in X_m$ such that $\Lambda(\boldsymbol{u}_m, \phi_m) = (\boldsymbol{u}_m, \phi_m)$. More precisely, we have found $\rho_m \in \mathcal{C}^1(\overline{Q_T})$, $\boldsymbol{u}_m \in \mathcal{C}^1([0,T]; \mathbf{V}_m)$, $\phi_m \in \mathcal{C}^1([0,T]; V_m)$, and $\mu_m \in \mathcal{C}([0,T]; V_m)$, which solve (4.16)-(4.15).

Appendix B. Suitable approximation of initial data

Lemma B.1. *Let Ω be a bounded domain of class $\mathcal{C}^3$ in $\mathbb{R}^d$, $d = 2, 3$. Let $\rho_0 \in L^\infty(\Omega)$ and $\phi_0 \in H^2(\Omega)$ be such that $0 \leq \rho_* \leq \rho_0 \leq \rho^*$, $\partial_{\mathbf{n}} \phi_0 = 0$ on $\partial\Omega$, and $\mu_0 = -\frac{\Delta \phi_0}{\rho_0} + \Psi_0'(\phi_0) \in H^1(\Omega)$. Then, there exists a sequence $(\rho_0^k, \phi_0^k, \mu_0^k)$, $k \in \mathbb{N}$, such that*

(i) *The approximated densities $\rho_0^k \in \mathcal{C}^\infty(\overline{\Omega})$ and $\rho_* \leq \rho_0^k(x) \leq \rho^*$, $\forall\, x \in \overline{\Omega}$ and $k \in \mathbb{N}$. As $k \to \infty$,*

$$\rho_0^k \to \rho_0 \quad \textit{strongly in } L^r(\Omega),\ \forall\, r \in [1, \infty), \quad \rho_0^k \rightharpoonup \rho_0 \quad \textit{weak-star in } L^\infty(\Omega).$$

(ii) *The approximated concentrations $\phi_0^k \in H^5(\Omega)$ are such that*

$$\partial_{\mathbf{n}} \phi_0^k = 0 \quad \textit{on } \partial\Omega, \quad \phi_0^k \to \phi_0 \quad \textit{strongly in } H^2(\Omega),$$

and there exists a constant C (independent of k) such that

$$\Big\| \frac{-\Delta \phi_0^k}{\rho_0^k} \Big\|_{H^1(\Omega)} \leq C \|\mu_0\|_{H^1(\Omega)} + C \|\phi_0\|_{H^2(\Omega)}^3.$$

(iii) *The approximated chemical potentials* $\mu_0^k \in H^3(\Omega)$ *defined by* $\mu_0^k = -\frac{\Delta\phi_0^k}{\rho_0^k} + \Psi_0'(\phi_0^k)$ *satisfy*

$$\mu_0^k \to \mu_0 \quad \textit{strongly in } H^1(\Omega).$$

Proof. First, by a classical approximation argument (convolution with mollifiers), we find a sequence ρ_0^k satisfying the statement of Lemma B.1. Consider

$$\widetilde{\mu} = -\frac{\Delta\phi_0}{\rho_0} + \phi_0.$$

It is clear that $\widetilde{\mu} \in H^1(\Omega)$. Then, we define $\widetilde{\mu}_0^k = \Pi_k \widetilde{\mu} \in V_k$. We observe that $\widetilde{\mu}_0^k \to \widetilde{\mu}$ in $H^1(\Omega)$. Next, we introduce ϕ_0^k as the solution to Neumann problem

$$\begin{cases} -\Delta\phi_0^k + \rho_0^k\phi_0^k = \rho_0^k\widetilde{\mu}_0^k & \text{in } \Omega \\ \partial_{\mathbf{n}}\phi_0^k = 0 & \text{on } \partial\Omega. \end{cases} \tag{B.1}$$

Since $\rho_0^k\widetilde{\mu}_0^k \in H^3(\Omega)$[9] and $\rho_0^k \geq \rho_*$, there exists a unique solution $\phi_0^k \in H^5(\Omega)$ to problem (B.1). Moreover, we have the following estimates

$$\|\phi_0^k\|_{H^1(\Omega)} \leq C\|\widetilde{\mu}_0^k\|_{L^2(\Omega)} \leq C\|\widetilde{\mu}\|_{L^2(\Omega)}$$

and

$$\|\phi_0^k\|_{H^2(\Omega)} \leq C\|\widetilde{\mu}_0^k\|_{L^2(\Omega)} \leq C\|\widetilde{\mu}\|_{L^2(\Omega)},$$

where the constant C only depends on Ω, ρ_* and ρ^*, but is independent of k. In light of (B.1), we also have

$$\Big\|\nabla\Big(\frac{\Delta\phi_0^k}{\rho_0^k}\Big)\Big\|_{L^2(\Omega)} = \|\nabla(\widetilde{\mu}_0^k - \phi_0^k)\|_{L^2(\Omega)} \leq C\|\widetilde{\mu}\|_{H^1(\Omega)}.$$

We observe that $\rho_0^k\widetilde{\mu}_0^k \to \rho_0\widetilde{\mu}$ in $H^1(\Omega)$. Denote $v_k = \phi_0^k - \phi_0$. By definition of $\widetilde{\mu}$ and ϕ_0^k, we have

$$-\Delta v_k + \rho_0^k v_k = f_k + g_k,$$

where $f_k = \rho_0^k\widetilde{\mu}_0^k - \rho_0\widetilde{\mu}$, and $g_k = (\rho_0 - \rho_0^k)\phi_0$. Multiplying the above equality by v_k and integrating over Ω, we obtain

$$\|\nabla v_k\|_{L^2(\Omega)}^2 + \rho_*\|v_k\|_{L^2(\Omega)}^2 \leq \|f_k\|_{L^2(\Omega)}\|v_k\|_{L^2(\Omega)} + \|\rho_0^k - \rho\|_{L^2(\Omega)}\|\phi_0\|_{L^\infty(\Omega)}\|v_k\|_{L^2(\Omega)}.$$

Thanks to the convergence properties of f_k and $\rho_0^k - \rho_0$, the above inequality implies that $v_k \to 0$ in $H^1(\Omega)$ as $k \to \infty$. This entails that $\phi_0^k \to \phi_0$ in $H^1(\Omega)$. By elliptic regularity, we easily deduce that $\phi_0^k \to \phi_0$ in $H^2(\Omega)$. Finally, we define μ_0^k as follows

$$\mu_0^k = -\frac{\Delta\phi_0^k}{\rho_0^k} + \Psi_0'(\phi_0^k).$$

[9] Here we are using that $V_k \subset H^3(\Omega)$ which follows from the assumptions on the domain and the elliptic theory of the Laplace equation.

By (B.1), we have

$$\mu_0^k = \widetilde{\mu}_0^k - \phi_0^k + \Psi_0'(\phi_0^k).$$

Therefore, it is apparent from the above convergences that $\mu_0^k \to \mu_0$ in $H^1(\Omega)$. □

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