

A Mathematical Framework for the Evaluation of System Expected Utility Not Satisfied Under Periodic Demand

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In this paper, we consider a general system whose reliability can be characterized with respect to a periodic time-dependent utility function related to the system performance in time. When an anomaly occurs in the system operation, a loss of utility is incurred that depends on the instance of the anomaly's occurrence and its duration. Under exponential anomalies' inter-arrival times and general distributions of maintenance time duration, we analyze the long-term average utility loss and we show that the expected utility loss can be written in a simple form. This allows us to evaluate the expected utility loss of the system in a relatively simple way, which is quite useful for the dimensioning of the system at the design stage. To validate our results, we consider as a use case scenario a cellular network consisting of 660 base stations. Using data provided by the network operator, we validate the periodic nature of users' traffic and the exponential distribution of the anomalies inter-arrival times, thus allowing us to leverage our results and provide reliability scores to the aforementioned network.

Keywords: Cellular networks, network reliability, expected utility not satisfied, periodic utility function, semi-Markov.

1. Introduction

The evaluation of reliability in large-scale systems, such as electric power grids and cellular networks, is necessary for the planning, designing and operation of these systems. The objective of such an evaluation is to derive measures of the service provision capability by these systems, considering the various hazardous events and anomalous conditions that may occur and impair the functioning of the various interconnected components of the system Al-Shaalan (2020); Benidris et al. (2015). Events such as component failures, system outages, and eventually maintenance actions are all to be accounted for in the evaluation of the system-wide reliability and the assessment of the efficacy of proposed reliability improvements to the system. Among these measures, we cite the loss of load probability (LOLP), expected frequency of load curtailment (EFLC), expected duration of load curtailment (EDLC), expected duration of curtailment (EDC), and expected demand not satisfied (EDNS), and many others Al-Shaalan (2020); Benidris et al. (2015); Bellani et al. (2020); Medjoudj et al. (2017).

Computing these measures is challenging due to the complexity of the systems and randomness of the failure and outage events, as well as the maintenance duration. In this paper, we focus on quantitatively characterizing the expected utility not satisfied as a reliability measure of a system. In other words, we analyze systems whose reliability can be characterized with respect to a time-dependent utility function $U(t)$ related to the system performance in time. The utility function $U(t)$ can represent, for example, the electricity demand of customers, data traffic in cellular networks, or other quantities alike. The goal, then, becomes to derive the expected utility loss $\bar{\mathcal{L}}$ of the system considering the failures that may occur at its components. A challenging aspect of such a derivation originates from the dependence of $U(t)$ on time in relation to external factors which influence it. For example, if we consider a cellular network, an anomaly occurring at 2 AM when user activity is low would lead to a lower utility loss than an anomaly taking place at peak users' activity hour.

Given this dependence on time, the analysis of $\bar{\mathcal{L}}$ requires careful attention.

Commonly, the theoretical evaluation of $\bar{\mathcal{L}}$ necessitates the formulation of the stochastic differential equations governing its evolution. Then, tools such as stochastic hybrid systems and Dynkin's formula are leveraged to analyze $\bar{\mathcal{L}}$ Hespanha (2004); Yates and Kaul (2019); Maatouk et al. (2020). However, obtaining a closed-form expression of $\bar{\mathcal{L}}$ is heavily contingent on the complexity of the differential equations involved. Generally, only approximations can be obtained by such analytical frameworks Fan et al. (2018); Maatouk et al. (2021). Another approach to characterize $\bar{\mathcal{L}}$ consists of running Monte Carlo simulations of the system Bouissou et al. (2014); Zio (2013). However, Monte Carlo simulations can be computationally costly, especially when a large number of system components interact with one another. Additionally, the absence of closed-form expressions reduces the interpretability of $\bar{\mathcal{L}}$ and hinders the optimization process of the system's parameters in the design stage. The goal of our paper is to address these challenges and provide a theoretical framework to obtain an expression of $\bar{\mathcal{L}}$ under a periodicity assumption on $U(t)$. To that end, the following are the key contributions of this paper:

- We first start our stochastic analysis by formulating the expected utility not satisfied $\bar{\mathcal{L}}$ as function of various elements such as the inter-arrival times of anomalies and their repair times. Then, we show by Fourier analysis of the anomalies' inter-arrival times distribution that a key stochastic process of interest converges to a uniform distribution. This convergence is, then, leveraged to provide limiting distributions of several quantities that dictate the system's performance.
- Afterward, the above results and the periodicity of the utility function are combined to show that the expected utility not satisfied can be written in a simple intuitive form. Additionally, links between the resulting expression and stan-

- Lastly, we consider a large-scale cellular network consisting of 660 cells and serving over 20,000 users. Using data provided by the network operator, we showcase the validity of the periodicity assumption on user traffic and the exponential distribution of anomalies inter-arrival times. We, then, leverage our theoretical results to characterize the expected data traffic not satisfied and the expected users affected by the anomalies in the network.

The rest of the paper is organized as follows. Section 2 introduces the system model adopted in the paper. In Section 3, we present the mathematical analysis of our system, and we provide our main theoretical findings. Then, a use case scenario consisting of a large-scale cellular network is considered, and our theoretical findings are then corroborated. Lastly, Section 5 concludes the paper.

2. System model

Without loss of generality, we consider a system consisting of one component and operating in its useful-life phase, where anomalies occur with a Poisson rate λ . In other words, the inter-arrival time X_j between anomalies $j - 1$ and j is exponentially distributed with rate λ

$$\Pr(X_j < t) = 1 - e^{-\lambda t}. \tag{1}$$

After an anomaly takes place, the system operator triggers a maintenance procedure. We let Y_j denote the maintenance time of anomaly j . We suppose that Y_j is independent of X_j and has an arbitrary distribution characterized by the following cumulative distribution function

$$\Pr(Y_j < t) = \begin{cases} F_Y(t), & \text{for } t \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We consider that when an anomaly takes place at time t_0 and the issue is addressed at time t_1 , a utility loss $\int_{t_0}^{t_1} U(t)dt$ is incurred. Therefore, by letting $W(t)$ be a binary random variable that is equal to 1 when the system is suffering from an

anomaly, we can define the expected utility not satisfied as follows

$$\bar{\mathcal{L}} = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T U(t)W(t)dt. \tag{3}$$

In practice, T being large translates to the system being operated long enough before the expected utility loss assessment. An illustration of the evolution of the utility loss can be found in Fig. 1.

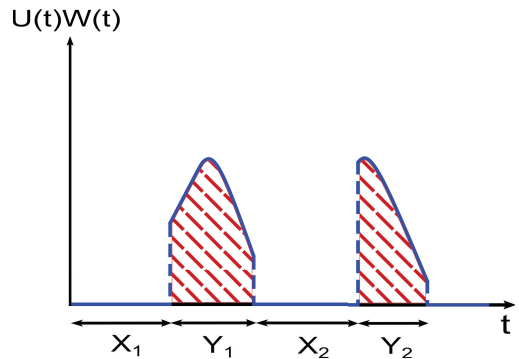


Fig. 1. Illustration of the loss function evolution.

Note that the function $U(t)$ can represent a large variety of system quantities depending on the system’s operator priorities. For example:

- $U(t)$ can denote the customer demand (e.g., electricity, communication traffic) served by the system at time t . In this scenario, the expected loss score coincides with the notion of Expected Demand Not Satisfied (EDNS) Medjoudj et al. (2017).
- $U(t)$ can represent the number of users served by the system at time t . Thus, the expected utility loss in this case represents the expected number of users affected by the anomalies.

Additionally, one can also define $U(t)$ as a combination of multiple system quantities. Given the various quantities $U(t)$ can represent, the generality of our framework and its flexibility can be therefore showcased.

In the next section, we will provide a mathematical framework to characterize $\bar{\mathcal{L}}$. We show that when $U(t)$ is a periodic function of period p , $\bar{\mathcal{L}}$

ends up having a relatively simple form, thus alleviating the difficulties reported in the introduction with respect to the quantitative characterization of the reliability of complex systems. It is worth noting that the periodicity of the system utility has been observed in various applications due to the nature of human behavior with respect to service demand. For example, in cellular networks, it is shown that user traffic exhibits a periodical pattern on the scale of a week, where Sunday's traffic is less than weekday's traffic Xu et al. (2017). Similar trends have been found in the data^a gathered from a cellular network operator, as seen in Fig. 2. Specifically, both the LTE (Long Term Evolution) traffic demand and the number of connected users exhibit a periodic behavior similar to what has been reported in Xu et al. (2017). It is worth noting that such trends are not exclusive to cellular networks. For instance, this periodic behavior has been witnessed in electricity demands in power grid networks Yukseltan et al. (2017).

3. Mathematical Analysis

To proceed with our mathematical analysis, we first decompose the time horizon T reported in eq. (3) into multiple stages. Specifically, we let $D_n = \sum_{j=1}^n (X_j + Y_j)$ and we rewrite the expected utility loss of the system as

$$\bar{\mathcal{L}} = \lim_{n \rightarrow +\infty} \frac{\int_0^{D_n} U(t)W(t)dt}{D_n}. \quad (4)$$

Next, by multiplying by $\frac{1}{n}$ both the numerator and denominator, we end up with

$$\bar{\mathcal{L}} = \lim_{n \rightarrow +\infty} \frac{\frac{1}{n} \sum_{j=1}^n \int_{D_{j-1}}^{D_j} U(t)W(t)dt}{\frac{1}{n} \sum_{j=1}^n (X_j + Y_j)}. \quad (5)$$

Afterward, we note that $W(t)$ is equal to 0 by definition in every interval $[D_j, D_j + X_j]$. Therefore, we can rewrite the expected loss as

$$\bar{\mathcal{L}} = \lim_{n \rightarrow +\infty} \frac{\frac{1}{n} \sum_{j=1}^n \int_{D_{j-1}}^{D_j} U(t)dt}{\frac{1}{n} \sum_{j=1}^n (X_j + Y_j)}. \quad (6)$$

As one can see, the challenging part about evaluating the expected loss originates from the numer-

^aAll the data used in our paper have been scaled when necessary for confidentiality reasons.

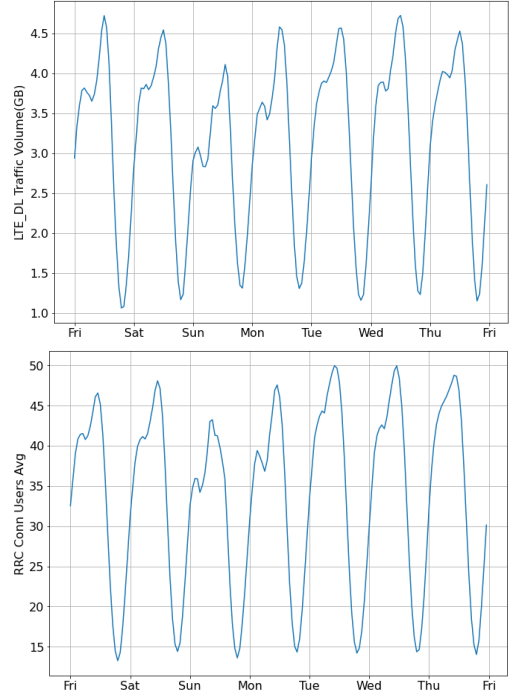


Fig. 2. Illustration of the LTE traffic demand and the number of users connected to the base station.

ator. To deal with this, we leverage the periodicity of the function $U(t)$.

Lemma 3.1. *If $U(t)$ is a periodic function of period p , then the expected loss $\bar{\mathcal{L}}$ can be rewritten as*

$$\bar{\mathcal{L}} = \lim_{n \rightarrow +\infty} \frac{\frac{1}{n} \sum_{j=1}^n \int_{D_j^{[p]} - Y_j}^{D_j^{[p]}} U(t)dt}{\frac{1}{n} \sum_{j=1}^n (X_j + Y_j)}, \quad (7)$$

where $D_j^{[p]} = D_j \bmod p$ is the remainder of the Euclidean division of D_j by p .

Proof. The proof consists of leveraging the definition of the modulo function and the periodicity of the utility function $U(t)$. For conciseness, the details of the proof are omitted in the paper, but will be presented at the conference. \square

As we can see, the next fundamental step in our analysis consists of finding the distribution of $D_j^{[p]}$. To do so, we first rewrite $D_j^{[p]}$ as follows

$$D_j^{[p]} = (X_j^{[p]} + Y_j^{[p]}) \bmod p, \quad (8)$$

where

$$\begin{aligned}
 X_j^{[p]} &= \left(\sum_{k=1}^j X_k\right) \bmod p, \\
 Y_j^{[p]} &= \left(\sum_{k=1}^j Y_k\right) \bmod p.
 \end{aligned}
 \tag{9}$$

Next, we investigate the distribution of $X_j^{[p]}$ more closely.

Theorem 3.1. *The random variable*

$$X^{[p]} = \lim_{j \rightarrow +\infty} X_j^{[p]}
 \tag{10}$$

is uniformly distributed on $[0, p]$.

Proof. Our proof revolves around a Fourier analysis of the distribution of $X^{[p]}$. In essence, we first analyze the behavior of the Fourier coefficients of the random variables making up $X^{[p]}$. Then, by leveraging theorems such as the dominated convergence theorem Bartle (2014) and Benford’s law for the product of random variables Miller and Nigrini (2008), we can derive the desired results. For conciseness, the details of the proof are omitted in the paper, but will be presented at the conference. □

In the theorem above, we have shown that $X_j^{[p]}$ converges to a uniform distribution when j gets large. However, to characterize the distribution of $D_j^{[p]}$, we need to take into account the distribution of $Y_j^{[p]}$, which can be quite general as we impose no restriction on $F_Y(t)$. To that end, we provide in the following a theorem that alleviates this difficulty.

Theorem 3.2. *Let A be a uniformly distributed RV on $[0, p]$ and B a RV of arbitrary distribution defined on $[0, p]$, independent of A . Then, the sum $A + B \bmod p$ is uniformly distributed on $[0, p]$.*

Proof. Given the independence between A and B , the proof revolves around the notion of probability distributions’ convolution. Then, by leveraging the particularity of the uniform distribution along with the definition of the modulo function, we can derive the desired results. For conciseness,

the details of the proof are omitted in the paper, but will be presented at the conference. □

Given the above, we can conclude that the RV

$$D^{[p]} = (X^{[p]} + Y^{[p]}) \bmod p
 \tag{11}$$

is uniformly distributed on $[0, p]$ where

$$\begin{aligned}
 D^{[p]} &= \lim_{j \rightarrow +\infty} D_j^{[p]}, \\
 Y^{[p]} &= \lim_{j \rightarrow +\infty} Y_j^{[p]}.
 \end{aligned}
 \tag{12}$$

In other words, as j gets larger, $D_j^{[p]}$ approaches the uniform distribution. With this in mind, we provide below the main results of our mathematical analysis.

Theorem 3.3. *If $U(t)$ is a periodic function of period p , then the expected loss $\bar{\mathcal{L}}$ can be rewritten as*

$$\bar{\mathcal{L}} = \frac{\mathbb{E}[Y]\bar{U}}{\mathbb{E}[X] + \mathbb{E}[Y]},
 \tag{13}$$

where $\bar{U} = \frac{1}{p} \int_0^p U(t)dt$ is the average utility in a period p .

Proof. To prove our theorem, we first show the ergodicity of the stochastic processes found in the expression of $\bar{\mathcal{L}}$ reported in eq. (7). Next, we leverage the results of Theorems 3.1 and 3.2, along with the law of large numbers to simplify further the expression. Afterward, by making use of the periodicity of $U(t)$, we can obtain the desired results. Again, for conciseness, the details of the proof are omitted in the paper, but will be presented at the conference. □

As can be seen above, the expression of $\bar{\mathcal{L}}$ turns out to have a relatively simple formulation thanks to the periodicity of the function $U(t)$. Additionally, the expression has an intuitive meaning and relationships with well-established metrics. In fact, let us define the system’s availability as follows

$$\text{Availability} = \frac{\text{MTBF}}{\text{MTBF} + \text{MTTR}},
 \tag{14}$$

where $\text{MTBF} = \mathbb{E}[X] = \frac{1}{\lambda}$ and $\text{MTTR} = \mathbb{E}[Y]$ denote the Mean Time Between Failure and Mean

Time To Repair respectively Medjoudj et al. (2017). By examining the expression of $\bar{\mathcal{L}}$, we can deduce that

$$\bar{\mathcal{L}} = \underbrace{(1 - \text{Availability})}_{(A)} \times \underbrace{\bar{U}}_{(B)}. \quad (15)$$

The term (A) can be seen as the probability that the system will be suffering from an anomaly. On the other hand, the term (B) is the average utility that the system delivers in a period p . Consequently, one can see how the periodicity of the utility $U(t)$, the exponential nature of anomalies inter-arrival times and the mathematical analysis provided in the paper lead to the intuitive form of $\bar{\mathcal{L}}$. In the next section, we will consider a particular application of interest and showcase the usefulness of the analytical results derived.

4. Use case: Cellular Networks

To showcase the usefulness of our analysis, we consider a large LTE cellular network consisting of 660 cellular base stations and serving approximately 22k users. As previously stated and seen in Fig. 2, the users' traffic data in this network exhibits a periodical pattern on the scale of a week, where Sunday's traffic is less than week-day's traffic. Next, our goal is to verify whether or not the inter-arrival times of anomalies are exponentially distributed. To do so, we leverage the trouble tickets data provided by the operator of the network. Specifically, when an anomaly takes place at any base station in the network, a trouble ticket is issued by the operator. This ticket contains details about the anomaly (e.g., anomaly ID), its location, and its occurrence and resolution times. As seen in Fig 3, the distribution of the anomalies inter-arrival times in the network is very close to an exponential distribution of rate $\lambda = 12.6$ anomalies/hour, in this case. Supposing that the base stations are all identical, and given the splitting property of Poisson processes Bertsekas and Tsitsiklis (2008), we can conclude that the anomalies rate for each base station is $\lambda = 0.019$ anomalies/hour. With the above results in mind, along with the periodic nature of users' traffic, we can conclude that the results of Theorem 3.3 can be leveraged to find the expected

utility not satisfied in the network.

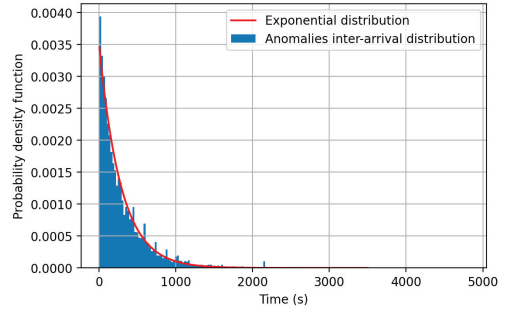


Fig. 3. Illustration of the anomalies inter-arrival times distribution.

On another note, we report in Fig. 4 an extract of the anomalies maintenance time distribution. As one can see, this distribution is far from straightforward. Specifically, we can see that a part of the anomalies are resolved almost instantaneously by the network itself. On the other hand, other anomalies require either remote or on-site interventions that take longer time (hours, days, and sometimes weeks). Modeling such a distribution is a challenging task, differently from the anomalies inter-arrival times. However, we recall that the results reported in Section 3 hold for any general maintenance time distribution. In fact, all we need to characterize the expected utility not satisfied is the average maintenance duration, which puts into perspective the generality of our results and their usefulness. To that end, using the trouble tickets data, we can conclude that the average maintenance time is 2 hours and 8 minutes.

Given all the above, and knowing that the average traffic per hour for each base station is $\bar{U} = 3.1$ GBs/hour, the operator can, then, conclude that the expected data traffic not satisfied in the entire network is

$$\bar{\mathcal{L}}_{\text{data}} = 660 \times \frac{2.13}{\frac{1}{0.019} + 2.13} \times 3.1 \simeq 80 \text{ GBs/hour} \quad (16)$$

Similarly, knowing that the average connected users to each base station is $\bar{N} = 33.07$, we can conclude that the expected number of users affected by anomalies at each hour is $\bar{\mathcal{L}}_{\text{users}} =$

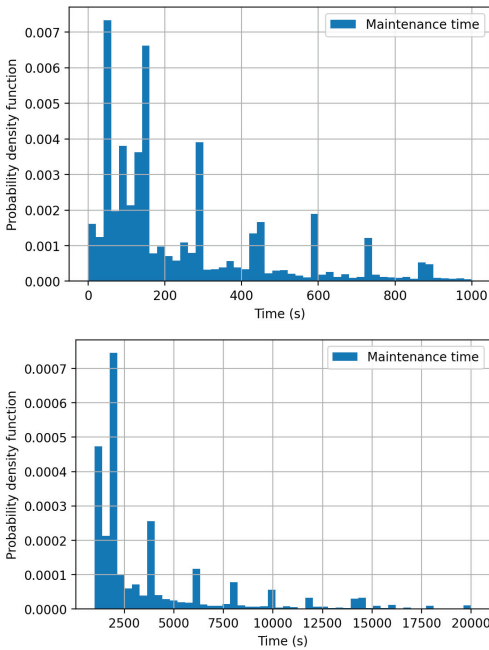


Fig. 4. Extracts of the maintenance time distribution.

854 users/hour. All in all, we can conclude that the network loses on average around $\frac{2.13}{\frac{1}{0.019} + 2.13} \simeq 3.8\%$ of its traffic due to the various anomalies that may occur. This reliability score can be used by the operator to assess the network performance, with respect to its objective. Further planning and network upgrades can, then, take place if necessary, and a reevaluation of the score can be done to conclude the efficacy of the proposed upgrades.

5. Conclusions and Future Work

In this paper, we have studied a general system whose reliability can be formulated with respect to a periodic time-dependent utility function related to the system's performance in time. Under exponential anomalies' inter-arrival times and general distributions of maintenance time duration, we have leveraged the periodicity of the utility function to derive the expected utility loss due to the system's anomalies. In these settings, we have shown that the expected utility loss has a simple form. A cellular network use case scenario was, then, considered where the usefulness of the presented analysis was highlighted. Future research

directions include the extension of the analysis to a more general family of distributions, the investigation of partial utility loss, and a deeper examination of the probability distributions convergence.

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