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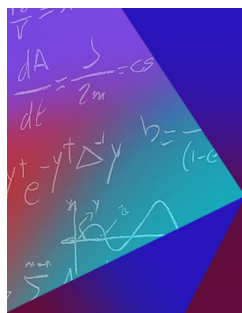
## Large deviations, central limit, and dynamical phase transitions in the atom maser

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# Large deviations, central limit, and dynamical phase transitions in the atom maser

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## ABSTRACT

The theory of quantum jump trajectories provides a new framework for understanding dynamical phase transitions in open systems. A candidate for such transitions is the atom maser, which for certain parameters exhibits strong intermittency in the atom detection counts and has a bistable stationary state. Although previous numerical results suggested that the “free energy” may not be a smooth function, we show that the atom detection counts satisfy a large deviations principle and, therefore, we deal with a phase crossover rather than a genuine phase transition. We argue, however, that the latter occurs in the limit of an infinite pumping rate. As a corollary, we obtain the central limit theorem for the counting process. The proof relies on the analysis of a certain deformed generator whose spectral bound is the limiting cumulant generating function. The latter is shown to be smooth so that a large deviations principle holds by the Gärtner–Ellis theorem. One of the main ingredients is the Krein–Rutman theory, which extends the Perron–Frobenius theorem to a general class of positive compact semigroups.

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## I. INTRODUCTION

The last couple of decades have witnessed a revolution in the experimental realization of quantum systems.<sup>1</sup> Ultracold atomic gases are created and used for the study of complex many body phenomena, such as quantum phase transitions<sup>2</sup> shedding light on open problems in condensed-matter physics.<sup>3</sup>

Real quantum systems are “open” in the sense that they interact with their environment, which leads to an irreversible loss of coherence and to energy dissipation. In many cases, the dynamics can be well described by the Markov approximation in which the environment possesses no memory and interacts weakly with the system. The joint unitary evolution of the system and environment can be described through the input–output formalism<sup>4</sup> using a quantum stochastic calculus.<sup>5</sup> In this framework, the Markov semigroup can be seen as the result of averaging over stochastic quantum trajectories arising from continuous-time measurements performed in the environment. These are described via stochastic Schrödinger (or filtering) equations<sup>6,7</sup> and capture the system’s evolution conditional on the detection record.

In Ref. 8, a new perspective was put forward, which looks at quantum jumps from the viewpoint of non-equilibrium statistical mechanics.<sup>9</sup> Detection trajectories are seen as “configurations” of a stochastic system, and large deviations theory<sup>10,11</sup> is employed to study the dynamical phase transitions arising in this way. Consider for simplicity the case of a counting measurement, which is directly relevant for the model studied in this paper. The interesting scenarios are that of a *phase crossover* in which the counting trajectories show intermittency between long active periods (many counts) and passive ones (few counts) and that of *phase coexistence* where the counting process exhibits a mixture of infinitely long trajectories of either type. In the latter case, the asymptotic cumulant generating function (or “free energy”) of the total counts process  $\Lambda_t$  is singular at the origin, and the total counts do not obey a large deviations principle (LDP). In contrast, in a phase crossover, an LDP may hold, but numerically and practically, there would be a strong resemblance to an actual phase transition.

For finite-dimensional systems, the counting process  $\Lambda_t$  satisfies an LDP when the Markov dynamics is mixing, i.e., irreducible and aperiodic.<sup>12</sup> The proof uses the Gärtner–Ellis theorem according to which it suffices to prove the convergence of the cumulant generating function to a smooth limit. By the Markov property, the former can be expressed in terms of a certain “deformed generator”  $\mathcal{L}_s$ , and the existence of the limit

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_\rho(e^{s\Lambda_t}) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \text{Tr}(\rho e^{t\mathcal{L}_s}(\mathbf{1})) = \lambda(s)$$

follows from the spectral gap property of  $\mathcal{L}_s$ , where  $\lambda(s)$  is the spectral bound of  $\mathcal{L}_s$  and  $\rho$  is the initial state.

In this paper, we investigate the existence of dynamical phase transitions for the *atom maser*, a well-known quantum open system exhibiting interesting properties such as bistability and sub-Poissonian statistics.<sup>13–15</sup> The maser consists of a beam of excited atoms passing through a cavity with which they interact according to the Jaynes–Cummings model. After the interaction, the atoms are measured in the standard basis and the trajectory of measurement outcomes is recorded. For certain values of the interaction strength, the stationary mean photon number changes abruptly (cf. Fig. 1), and the distribution is bistable, having a low and a high energy “phase.” The measurement trajectories alternate between periods of low and high ground state atom counts (cf. Fig. 4), and its limiting moment generating function exhibits characteristic phase separation lines (cf. Fig. 5).

Our main result (Theorem 5) is that the count process satisfies the LDP, and therefore, the atom maser does not have the non-analytic properties characteristic of phase transitions, although it exhibits clear phase crossover(s), which become sharper with the increasing pumping rate. As a corollary, we obtain the central limit theorem for the counting process using a result of Ref. 16. The proof follows the line of Ref. 12, but the novelty here is the treatment of an infinite-dimensional system in continuous time dynamics. We use an  $L^2$ -representation<sup>17,18</sup> of the semigroup generated by  $\mathcal{L}_s$  and show that the corresponding semigroup is compact. We, then, use the Krein–Rutman theory (Ref. 19 and references therein) to establish the uniqueness and strict positivity of the eigenvector of  $\lambda(s)$  and, hence, the existence of the spectral gap. Some steps of the proof rely on a special feature of the maser dynamics, which allows us to restrict the attention to the commutative invariant algebra of diagonal operators. However, the line of the proof is applicable to general infinite-dimensional quantum Markov dynamics.

For recent work on quantum dynamical phase transitions, we refer to Refs. 8 and 20–23. In particular, our investigation was motivated by the numerical results of Ref. 24, indicating a possible non-analytic behavior of  $\lambda(s)$ . In Ref. 12 (see also Ref. 25), a large deviations principle is shown to hold for correlated states on quantum spin chains; large deviations for quantum Markov semigroups are studied in Ref. 26. Metastable behavior in a different atom maser was investigated in Ref. 27. More broadly, there is a large body of large deviations work in quantum systems.<sup>28–33</sup>

In Sec. II, we introduce the background of our problem: the atom maser and its Markov semigroup, the counting processes associated with the jump terms in the Lindblad generator, the static and dynamical phase transitions and the interplay between them, and the general setup of large deviations theory. In addition, the existence and properties of various semigroups are established rigorously. In Sec. III, we formulate the large deviations results and give a point-by-point outline of the proof. The results of a detailed numerical analysis are presented

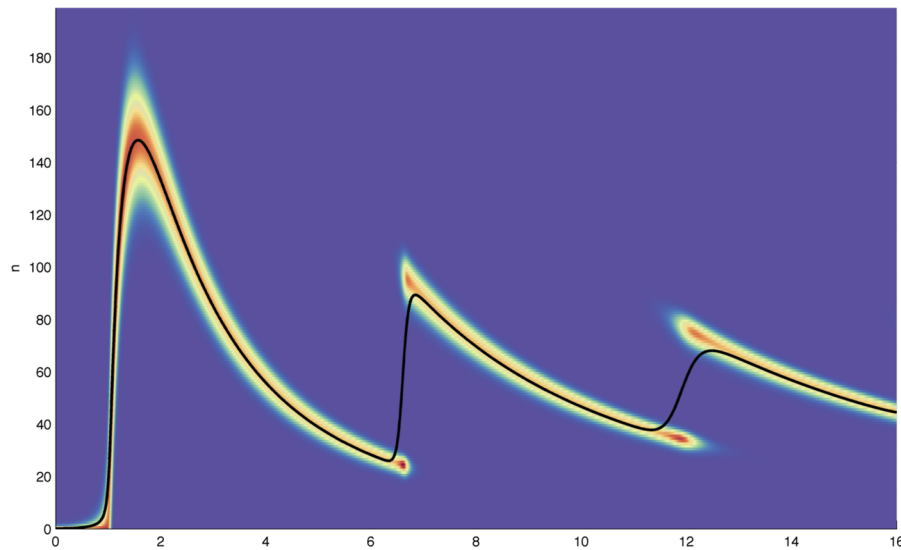


FIG. 1. Mean photon number (black line) and photon number distribution (background) in the stationary state  $\rho_{ss}$  as a function of  $\alpha = \sqrt{N_{ex}}\phi$ .

in Sec. IV, where we argue that “phase transitions” do occur in the limit of a very large pumping rate, at  $\alpha \approx 1$  (second order), at  $\alpha \approx 6.66$ , and further points (first order), where  $\alpha$  is the pumping parameter (see Fig. 1).

## II. BACKGROUND

In this section, we introduce the atom maser dynamics, investigate the counting process associated with the measurement of outgoing atoms, and describe the basic elements of large deviations theory used in this paper. Propositions 2 and 3 establish the mathematical properties of the quantum dynamical semigroups used in this paper.

### A. The atom maser

In the atom maser, two-level atoms pass successively through a cavity and interact resonantly with the electromagnetic field inside the cavity. The two-level atoms are identically and independently prepared in the excited state, and for simplicity, we assume that only a single atom passes through the cavity at any time. In addition, the cavity is also coupled to a thermal bath, which represents the interaction between the (non-ideal) cavity and the environment. The combined effects of the interactions with the atoms and the environment change the state of the cavity, which is described by a *quantum Markov semigroup* in a certain coarse-grained approximation described below (see Refs. 18 and 17 for a mathematical overview and Ref. 34 for the physical derivation of the master equation). In this section, we give an intuitive description of the dynamics starting with a simplified discrete time model, with an emphasis on the statistics of measurements performed on the atoms.

The cavity is described by a one mode continuous variable system with Hilbert space  $\mathfrak{h} = \ell^2(\mathbb{N})$  whose canonical basis vectors  $(|e_n\rangle)_{n \geq 0}$  represent pure states of a fixed number of photons. Therefore, if  $|\psi\rangle \in \mathfrak{h}$  is a pure state, the *photon number distribution* of the cavity is given by  $|\langle e_n, \psi \rangle|^2$ . Mixed states are described by density operators, i.e., trace-class operators  $\rho \in L^1(\mathfrak{h})$ , which are positive and normalized to have unit trace, and the observables are represented by self-adjoint elements of the von Neumann algebra of bounded operators  $\mathcal{B}(\mathfrak{h})$  whose predual is  $L^1(\mathfrak{h})$ .

Recall that the *annihilation operator*  $a$  on  $\mathfrak{h}$  is defined by

$$a|e_n\rangle = \begin{cases} \sqrt{n}|e_{n-1}\rangle & \text{if } n > 0, \\ 0 & \text{if } n = 0, \end{cases}$$

its adjoint is the *creation operator*  $a^*$ , and  $N = a^*a$  is the photon number operator such that  $N|e_n\rangle = n|e_n\rangle$ . For every  $\beta > 0$ , we introduce the notation

$$D(N^\beta) = \left\{ u = \sum_{n=0}^{+\infty} u_n |e_n\rangle : \sum_{n=0}^{+\infty} n^{2\beta} |u_n|^2 < +\infty \right\}$$

for the domain of  $N^\beta$ , and we recall that  $D(a) = D(a^*) = D(N^{\frac{1}{2}})$ . The atom is modeled by a two-dimensional Hilbert space  $\mathbb{C}^2$  with standard orthonormal basis  $\{|0\rangle, |1\rangle\}$  consisting of the “ground” and “excited” states. We denote by  $\sigma_*$  and  $\sigma$  the corresponding raising and lowering operators (i.e.,  $\sigma^*|0\rangle = |1\rangle$ ). The interaction between an atom and the cavity is described by the Jaynes–Cummings Hamiltonian on  $\mathbb{C}^2 \otimes \mathfrak{h}$ ,

$$H_{\text{int}} = -g(\sigma \otimes a^* + \sigma^* \otimes a),$$

where  $g$  is the coupling constant. The free Hamiltonian is given by

$$H_{\text{free}} = \omega \mathbf{1} \otimes a^*a + \omega \sigma^* \sigma \otimes \mathbf{1},$$

where  $\omega$  is the frequency of the resonant mode; however, by passing to the interaction picture, the effect of the free evolution can be ignored. Therefore, if the interaction lasts for a time  $t_0$ , the joint evolution is described by the unitary operator  $U := \exp(-it_0 H_{\text{int}})$  whose action on a product initial state is given by

$$U : |1\rangle \otimes |k\rangle \mapsto \cos(\phi\sqrt{k+1})|1\rangle \otimes |k\rangle + i \sin(\phi\sqrt{k+1})|0\rangle \otimes |k+1\rangle,$$

where  $\phi := t_0 g$  is the *accumulated Rabi angle*. If a measurement is performed on the outgoing atom in the standard basis, then the cavity remains in state  $|k\rangle$  with probability  $\cos^2(\phi\sqrt{k+1})$  or gains an excitation with probability  $\sin^2(\phi\sqrt{k+1})$ . By averaging over the outcomes, we obtain the cavity transfer operator  $\mathcal{T}_* : L^1(\mathfrak{h}) \rightarrow L^1(\mathfrak{h})$ ,

$$\mathcal{T}_*(\rho) = K_1 \rho K_1^* + K_2 \rho K_2^* = \mathcal{K}_{1*}(\rho) + \mathcal{K}_{2*}(\rho), \tag{1}$$

where the Kraus operators  $K_i$  are given by

$$K_1 = a^* \frac{\sin(\phi\sqrt{aa^*})}{\sqrt{aa^*}}, \quad K_2 = \cos(\phi\sqrt{aa^*}),$$

and  $\mathcal{K}_{i^*}$  are the corresponding jump operators on the level of density matrices. Since each atom interacts with the cavity only once, the state of the cavity after  $n$  such interactions is given by  $\rho(n) = \mathcal{T}_*^n(\rho)$ , which can be interpreted as a *discrete time* quantum Markov dynamics. Let us imagine that after the interaction, each atom is measured in the standard basis and found to be either in the excited or the ground state. The master dynamics can be unraveled according to these events as

$$\mathcal{T}_*^n(\rho) = \sum_{\mathbf{i}=(i_1, \dots, i_n)} \mathcal{K}_{i_n^*} \dots \mathcal{K}_{i_1^*}(\rho), \quad (2)$$

where each term of the sum represents the (unnormalized) state of the cavity after a certain sequence  $\mathbf{i} = (i_1, \dots, i_n) \in \{0, 1\}^n$  of measurement outcomes, whose probability is given by

$$\mathbb{P}_\rho(i_1, \dots, i_n) = \text{Tr}(\mathcal{K}_{i_n^*} \dots \mathcal{K}_{i_1^*}(\rho)).$$

If  $\Lambda_n(\mathbf{i}) := \text{No.}\{j : i_j = 0\}$  denotes the number of ground state atoms detected up to time  $n$ , we can use the previous relation to compute its moment generating function,

$$\mathbb{E}_\rho(e^{s\Lambda_n}) = \sum_{k \geq 0} \mathbb{P}_\rho(\Lambda_n = k) e^{sk} = \sum_{\mathbf{i}} e^{s\Lambda_n(\mathbf{i})} \text{Tr}(\mathcal{K}_{i_n^*} \dots \mathcal{K}_{i_1^*}(\rho)) = \text{Tr}(\mathcal{T}_{*s}^n(\rho)), \quad (3)$$

where

$$\mathcal{T}_{*s}(\rho) = e^s \mathcal{K}_{1^*}(\rho) + \mathcal{K}_{2^*}(\rho)$$

is a “deformed” transfer operator, i.e., a completely positive but not trace preserving map on  $L^1(\mathfrak{h})$ . Relation (3) and its continuous time analog (14) will be the key to analyze the large deviations properties of the counting process in terms of spectral properties of operators, such as  $\mathcal{T}_s$  and  $\mathcal{L}_s$ .

To make the model more realistic, we will pass to a continuous time description in which the incoming atoms are Poisson distributed in time with intensity  $N_{\text{ex}}$ , and the cavity is in contact with a thermal bath. If one ignores the details of short-term cavity evolution, the discrete time dynamics can be replaced by the coarse-grained *continuous time* Lindblad (master) equation,<sup>35</sup>

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \mathcal{L}_*(\rho(t)), \\ \mathcal{L}_*(\rho) &= \sum_{i=1}^4 \left( L_i \rho L_i^* - \frac{1}{2} \{L_i^* L_i, \rho\} \right) \\ &= \sum_{i=1}^4 L_i \rho L_i^* + \mathcal{L}_*^{(0)}(\rho) = \sum_{i=1}^4 \mathcal{J}_{i^*}(\rho) + \mathcal{L}_*^{(0)}(\rho), \end{aligned} \quad (4)$$

with jump operators  $L_i$  defined by

$$L_1 = \sqrt{N_{\text{ex}}} a^* \frac{\sin(\phi\sqrt{aa^*})}{\sqrt{aa^*}}, \quad (5)$$

$$L_2 = \sqrt{N_{\text{ex}}} \cos(\phi\sqrt{aa^*}), \quad (6)$$

$$L_3 = \sqrt{\nu + 1} a, \quad (7)$$

$$L_4 = \sqrt{\nu} v a^*. \quad (8)$$

As before, the operators  $L_1$  and  $L_2$  are associated with the detection of an atom in the ground and excited state, respectively. The emission and absorption of photons due to the contact with the bath are represented by operators  $L_3$  and  $L_4$ , respectively. Between jumps, the evolution is described by the semigroup  $e^{t\mathcal{L}_*^{(0)}}(\rho) := e^{tG}(\rho)e^{tG}$  with

$$G := -\frac{1}{2} \sum_{i=1}^4 L_i^* L_i = -\frac{1}{2} (N_{\text{ex}} + \nu + (2\nu + 1)N), \quad D(G) = D(N). \quad (9)$$

Since we deal with an infinite-dimensional space and unbounded jump operators, the above definitions need to be formalized mathematically in order to ensure the existence and uniqueness of the different semigroups (see Proposition 2). As it is customary in the theory of quantum dynamical semigroups with an unbounded generator,<sup>36</sup> so far, the generator  $\mathcal{L}_*$  can be safely defined on the linear manifold generated by the operators  $|u\rangle\langle v|$  for  $u, v \in D(G)$  (this manifold is also a core due to Proposition 2 and Proposition 3.32 of Ref. 36), or equivalently, we can interpret  $\mathcal{L}$  applied to any  $X \in \mathcal{B}(\mathfrak{h})$  as the sesquilinear form on  $D(G) \times D(G)$  given by

$$\langle u, \mathcal{L}(X)v \rangle = \langle Gu, Xv \rangle + \langle u, XGv \rangle + \sum_{i=0}^4 \langle L_i u, X L_i v \rangle \quad \forall u, v \in D(G).$$

*Definition 1* (Ref. 36 and Sec. 3.1.2 in Ref. 37). Let  $\mathcal{B}(\mathfrak{h})$  be the space of bounded operators on  $\mathfrak{h}$  endowed with the  $w^*$ -topology. A quantum dynamical semigroup on  $\mathcal{B}(\mathfrak{h})$  is a family  $\mathcal{S} = (\mathcal{S}(t))_{t \geq 0}$  of bounded operators on  $\mathcal{B}(\mathfrak{h})$  with the following properties:

- (i)  $\mathcal{S}(0) = I$ .
- (ii)  $\mathcal{S}(s+t) = \mathcal{S}(s)\mathcal{S}(t)$  for all  $s, t \geq 0$ .
- (iii)  $\mathcal{S}(t)$  is completely positive for all  $t \geq 0$ .
- (iv)  $\mathcal{S}(t)$  is a  $w^*$ -continuous operator on  $\mathcal{B}(\mathfrak{h})$  for all  $t \geq 0$ .
- (v) For each  $X \in \mathcal{B}(\mathfrak{h})$ , the map  $t \mapsto \mathcal{S}(t)(X)$  is continuous with respect to the  $w^*$ -topology on  $\mathcal{B}(\mathfrak{h})$ .

The dynamical semigroup  $\mathcal{S}(t)$  is called Markov (sub-Markov) if  $\mathcal{S}(t)(\mathbf{1}) = \mathbf{1}$  ( $\mathcal{S}(t)(\mathbf{1}) \leq \mathbf{1}$ ) holds true for every time  $t$ .

We recall that since the maps  $\mathcal{S}(t)$  are positive, the fact that they are  $w^*$ -continuous is equivalent for them to be normal [(Ref. 37, Lemma 2.4.19 and Theorem 2.4.21)]. The  $w^*$ -generator  $\mathcal{Z}$  is the operator defined as

$$\mathcal{Z}(X) := w^* - \lim_{\hbar \downarrow 0} \frac{1}{\hbar} (\mathcal{S}(\hbar)(X) - X)$$

for all  $X \in D(\mathcal{Z}) := \{X \in \mathcal{B}(\mathfrak{h}) : \exists w^* - \lim_{\hbar \downarrow 0} \frac{1}{\hbar} (\mathcal{S}(\hbar)(X) - X)\}$ , which is a  $w^*$ -dense linear space of  $\mathcal{B}(\mathfrak{h})$ . Although no simple expression exists for the operators  $\mathcal{S}(t)$  in terms of the generator  $\mathcal{Z}$ , it is helpful to think of  $\mathcal{S}(t)$  as the exponential of the generator

$$\mathcal{S}(t)(X) = e^{t\mathcal{Z}}(X), \tag{10}$$

especially from the point of view of relating spectral properties of  $\mathcal{Z}$  to those of  $\mathcal{S}(t)$ , e.g., *spectral mapping theorems*. In Proposition 2, we show that the Heisenberg picture Lindbladian  $\mathcal{L}$  is the generator of a quantum Markov semigroup  $(\mathcal{T}(t))_{t \geq 0}$  on  $\mathcal{B}(\mathfrak{h})$ ; we postpone the Proof of Proposition 2 to [Appendix A](#).

*Proposition 2.*

1.  $\mathcal{L}$  generates a unique quantum Markov semigroup  $\mathcal{T}(t)$ , which has the following integral representation: for every  $X \in \mathcal{B}(\mathfrak{h})$ ,

$$\begin{aligned} \mathcal{T}(t)(X) &= e^{t\mathcal{L}^{(0)}}(X) \\ &+ \sum_{k \geq 1} \sum_{i_1, \dots, i_k=1}^4 \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} X(t; t_1, i_1, \dots, t_k, i_k) dt_1 \dots dt_k \end{aligned} \tag{11}$$

and

$$X(t; t_1, i_1, \dots, t_k, i_k) := e^{(t-t_k)\mathcal{L}^{(0)}} \mathcal{J}_{i_k} \dots e^{(t_2-t_1)\mathcal{L}^{(0)}} \mathcal{J}_{i_1} e^{t_1 \mathcal{L}^{(0)}}(X),$$

where the equality is understood in terms of the associated bilinear form  $\langle u, \mathcal{T}(t)(X)v \rangle$  for  $u, v \in \mathfrak{h}$ .

2.  $(\mathcal{T}(t))_{t \geq 0}$  has a unique faithful stationary state

$$\rho_{ss} := \rho_{ss}(0) \sum_{n \geq 0} \prod_{k=1}^n \left( \frac{\nu}{\nu+1} + \frac{N_{\text{ex}}}{\nu+1} \frac{\sin^2(\phi\sqrt{k})}{k} \right) |e_n\rangle\langle e_n| \tag{12}$$

with  $\rho_{ss}(0)$  taken such that  $\text{Tr}(\rho_{ss}) = 1$ .

3.  $(\mathcal{T}(t))_{t \geq 0}$  is ergodic, in the sense that any initial state  $\rho$  converges to the stationary state

$$w - \lim_{t \rightarrow \infty} \mathcal{T}_*(t)(\rho) = \rho_{ss}.$$

The dependence of the stationary mean photon number and photon number distribution on the “pumping parameter”  $\alpha := \sqrt{N_{\text{ex}}}\phi$  is shown in Fig. 1 for  $\nu = 0.15$  and  $N_{\text{ex}} = 150$ . We note two interesting features in Fig. 1: first, there is a sharp change in the mean photon number at  $\alpha \approx 1$  followed by less pronounced jumps near  $\alpha = 6.66$  and  $\alpha = 12$ . The other, related, feature to note is that the photon number distribution has a single peak for most values of  $\alpha$  except in certain regions such as around the critical point  $\alpha \approx 6.66$ , where the stationary state has two local maxima. We will come back to these aspects in Sec. II B and show that they are related to features of the counting trajectories, such as intermittency, which indicates proximity to a dynamical phase transition. The reason for plotting the stationary distribution in terms of  $\alpha$  (with  $N_{\text{ex}}$  fixed) rather than  $\phi$  is because the transitions appear to occur at fixed values of  $\alpha$  and sharpen as  $N_{\text{ex}} \rightarrow \infty$ . This will be further investigated in Sec. IV.

## B. The counting process and the deformed transition operator

To better understand the behavior of the stationary state illustrated in Fig. 1, we unravel the Markov semigroup  $\mathcal{T}_*(t)$  with respect to the four counting processes associated with the jump terms (5–8), each of them corresponding to a counting measurement of the quantum output process. If  $\rho$  is the initial state of the cavity, then  $\rho(t) := \mathcal{T}_*(t)(\rho)$  is the evolved state at time  $t$ , which [in analogy to Eq. (2)] can be seen as an average over all possible counting events in the environment,

$$\begin{aligned} \rho(t) := \mathcal{T}_*(t)(\rho) &= e^{t\mathcal{L}_*^{(0)}}(\rho) \\ &+ \sum_{k \geq 1} \sum_{i_1, \dots, i_k=1}^4 \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} \rho(t; t_1, i_1, \dots, t_k, i_k) dt_1 \dots dt_k, \end{aligned} \tag{13}$$

where the integrand

$$\rho(t; t_1, i_1, \dots, t_k, i_k) := e^{(t-t_k)\mathcal{L}_*^{(0)}} \mathcal{J}_{i_k} \dots e^{(t_2-t_1)\mathcal{L}_*^{(0)}} \mathcal{J}_{i_1} e^{t_1\mathcal{L}_*^{(0)}}(\rho)$$

is the unnormalized state of the cavity, given that detections of type  $i_1, \dots, i_k \in \{1, 2, 3, 4\}$  have occurred at times  $0 \leq t_1 \leq \dots \leq t_k \leq t$ , and no other counting events happened in the meantime. Its trace is interpreted as the probability of observing the given measurement record. Note that Eq. (13) can be obtained by duality from Eq. (11), so this description is mathematically rigorous. Among the four counting processes, we focus on the first one associated with the detection of an atom in the ground state and simultaneous absorption of a photon by the cavity. We denote by  $\Lambda_t$  the total number of such atoms detected up to time  $t$ : for every  $n_1 \in \mathbb{N}$ ,

$$\mathbb{P}_\rho(\Lambda_t = n_1) = \sum_{n_2, n_3, n_4 \geq 0} \sum_{(***)} \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} \text{tr}(\rho(t; t_1, i_1, \dots, t_k, i_k)) dt_1 \dots dt_k,$$

where  $(***)$  stands for

$$\{i_1, \dots, i_{n_1+\dots+n_4} = 1, \dots, 4 : \#\{k : i_k = j\} = n_j \forall j = 1, \dots, 4\}.$$

Similarly to the discrete case, by using the above unraveling and point 3 in Proposition 3, we can show that the moment generating function of  $\Lambda_t$  is given by

$$\mathbb{E}_\rho(e^{s\Lambda_t}) = \text{Tr}(\mathcal{T}_{*s}(t)(\rho)) = \text{Tr}(\rho \mathcal{T}_s(t)(\mathbf{1})), \tag{14}$$

where  $(\mathcal{T}_s(t))_{t \geq 0}$  is the quantum dynamical semigroup on  $\mathcal{B}(\mathfrak{h})$  with the generator

$$\mathcal{L}_s(X) = e^s \mathcal{J}_1(X) + \sum_{i=2}^4 \mathcal{J}_i(X) + \mathcal{L}^{(0)}(X) = (e^s - 1)\mathcal{J}_1(X) + \mathcal{L}(X) \tag{15}$$

and  $(\mathcal{T}_{*s}(t))_{t \geq 0}$  is the predual semigroup on  $L^1(\mathfrak{h})$ . This is formalized in the following proposition whose proof can be found in Appendix A.

*Proposition 3.* For all  $s \in \mathbb{R}$ ,  $\mathcal{L}_s$  generates a semigroup  $\mathcal{T}_s = (\mathcal{T}_s(t))_{t \geq 0}$  such that the following holds:

1.  $\mathcal{T}_s(t)$  is a quantum dynamical semigroup.
2.  $\mathcal{T}_s(t)$  is the unique solution to

$$\langle u, \mathcal{T}_s(t)(X)v \rangle = \langle e^{tG} u, X e^{tG} v \rangle + \sum_{i=1}^4 \int_0^t \langle L_i' e^{rG} u, \mathcal{T}_s(t-r)(X) L_i' e^{rG} v \rangle dr \tag{16}$$

for every  $u, v \in \mathfrak{h}$ , where  $L_1' = e^{s/2}L_1$  and  $L_i' = L_i$  for  $i = 2, 3, 4$ .

3.  $\mathcal{T}_s(t)$  has the integral representation

$$\begin{aligned} \mathcal{T}_s(t)(X) &= e^{t\mathcal{L}^{(0)}}(X) \\ &+ \sum_{k \geq 1} \sum_{i_1, \dots, i_k=1}^4 \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} X_s(t; t_1, i_1, \dots, t_k, i_k) dt_1 \dots dt_k \end{aligned} \tag{17}$$

for every  $t \geq 0$ ,  $X \in B(\mathfrak{h})$ , and

$$X_s(t; t_1, i_1, \dots, t_k, i_k) := e^{(t-t_k)\mathcal{L}^{(0)}} \mathcal{J}'_{i_k} \dots e^{(t_2-t_1)\mathcal{L}^{(0)}} \mathcal{J}'_{i_1} e^{t_1\mathcal{L}^{(0)}}(X),$$

where the equality has to be read for the associated bilinear form  $\langle u, \mathcal{T}_s(t)(X)v \rangle$  for  $u, v \in \mathfrak{h}$  and  $\mathcal{J}'_1 = e^s \mathcal{J}_1$ ,  $\mathcal{J}'_i = \mathcal{J}_i$  for  $i = 2, 3, 4$ .

Equation (14) plays a central role in this paper; we will use it to formulate a large deviations principle for the counting process  $\Lambda_t$  and, in particular, to relate the moment generating function of  $\Lambda_t$  to the spectral properties of  $\mathcal{L}_s$ . Note that  $\mathcal{L}_s$  differs from the Lindblad generator by the factor  $e^s$  multiplying the jump term associated with the detection of a ground state atom. It is still the generator of a completely positive semigroup, but it is no longer identity preserving and, therefore, does not represent a physical evolution except for  $s = 0$ .

Unraveling (13) allows for a classical interpretation of the cavity dynamics. Indeed, the semigroup generated by  $\mathcal{L}$  (and  $\mathcal{L}_s$ ) leaves invariant the commutative subalgebra  $\mathcal{B}_d(\mathfrak{h}) \subset \mathcal{B}(\mathfrak{h})$  generated by the number operator  $N$ , and the restriction of  $(\mathcal{T}(t))_{t \geq 0}$  to the diagonal algebra is the dynamical semigroup of a classical birth-death process on the state space  $\{0, 1, 2, \dots\}$ , with rates

$$\begin{aligned} \lambda_k^2 &:= N_{ex} \sin(\phi\sqrt{k+1})^2 + v(k+1), \quad k \geq 0, \\ \mu_k^2 &:= (v+1)k, \quad k \geq 1. \end{aligned} \tag{18}$$

Figure 2 shows the birth and death rates (minus the common factor  $vk$ ) rescaled by a factor  $N_{ex}$ , in the limit  $N_{ex} \rightarrow \infty$ , as functions of the parameter  $\vartheta := \sqrt{(k+1)/N_{ex}}\alpha = \phi\sqrt{k+1}$ . In this regime, the rates become the functions  $\lambda_\vartheta^2 = \sin(\vartheta)^2$  and  $\mu_\vartheta^2 = \alpha^{-2}\vartheta^2$  of the continuous parameter  $\vartheta$ , and we plot  $\lambda_\vartheta^2$  along with  $\mu_\vartheta^2$  for different values of  $\alpha$ . The intersection points correspond to minima and maxima of the stationary distribution<sup>35</sup> as suggested by the following argument. For  $\alpha < 1$ , the death rate is always larger than the birth rate and the distribution is maximum at the vacuum state. For  $1 < \alpha < 4.6$ , there is a single non-trivial intersection point such that the birth rate is larger to its left and smaller to its right and, therefore, corresponds to the maximum of the stationary distribution. Similarly, when  $4.6 < \alpha < 7.8$ , the rates intersect

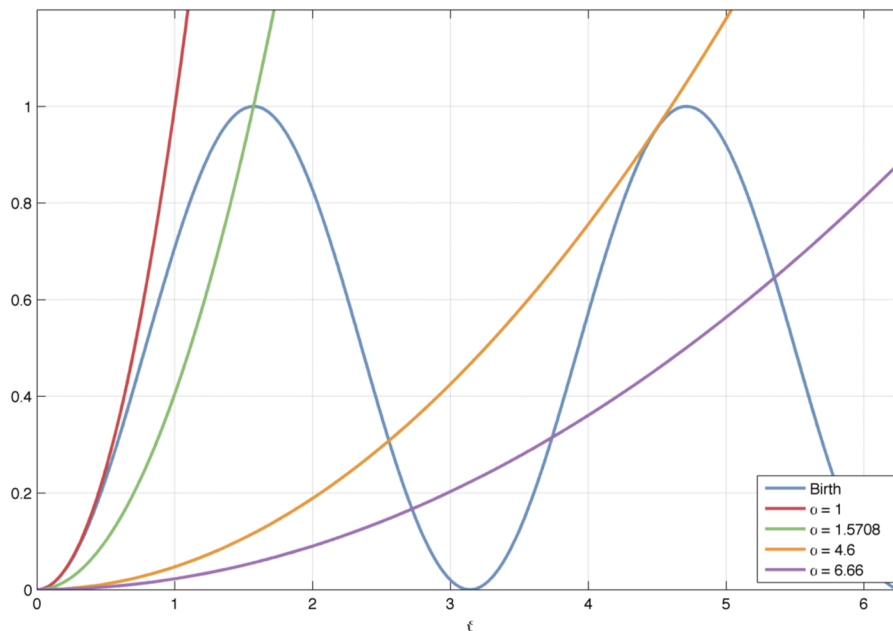


FIG. 2. The birth (blue) and death rates as functions of  $\vartheta$  for different values of  $\alpha$ . The intersection points correspond to minima and maxima of the stationary distribution.

in three points: the first and last are located at local maxima while the middle point is a local minimum, so we deal with a bimodal distribution. However, while this analysis clarifies the emergence of multimodal distributions, it does not explain the sudden jump of the mean photon number at  $\alpha \approx 6.66$  and higher values.

This feature can be intuitively understood by appealing to the effective potential model.<sup>1</sup> If we think of the photon number as a continuous variable and introduce a fictitious potential  $U$  defined by

$$\rho_{ss}(n) = \rho_{ss}(0)e^{-U(n)}, \tag{19}$$

then the photon number distribution appears as the thermal equilibrium distribution of a particle moving in the potential  $U$  (with  $k_B \cdot T = 1$ ); see Fig. 3. When the potential has a single local minimum (for  $0 < \alpha < 4.6$ ), the stationary distribution is unimodal and concentrates around this point. The cavity state fluctuates around the mean, and  $\Lambda_t$  increases steadily with an average rate. When there are two (or more) local minima of different heights, the higher minimum corresponds to a metastable phase from which the system eventually escapes due to thermal fluctuations. The rate of return to the metastable phase is typically much lower due to the larger potential barrier that needs to be climbed. The point  $\alpha \approx 6.66$  where the two local minima are equal plays the role of a “phase transition” and corresponds roughly to the point where the mean photon number changes abruptly. Here, the cavity spends long periods of time around the two local maxima with rare but quick transitions between them. The change from the low energy to the high energy mode is accompanied by a clear change in the slope of the counting process  $\Lambda_t$ .

In the stationary regime, the mean  $\mathbb{E}_{\rho_{ss}}(\Lambda_t)$  grows linearly with time with rate  $\text{Tr}(\rho_{ss}L_1^*L_1)$ . This expression can be obtained by differentiating the moment generating function (14) at  $s = 0$ ,

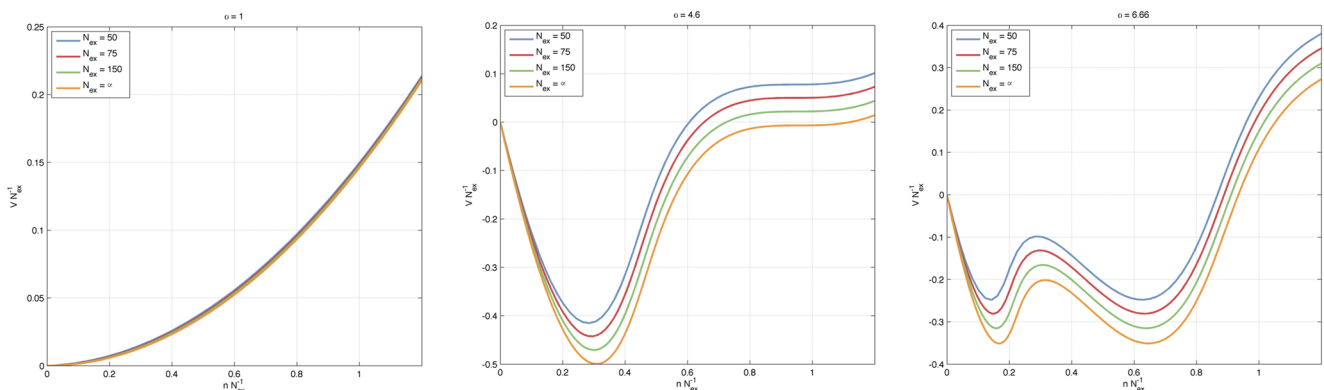
$$\begin{aligned} \frac{\mathbb{E}_{\rho}(\Lambda_t)}{t} &= \frac{1}{t} \frac{d}{ds} \mathbb{E}_{\rho}(e^{s\Lambda_t}) \Big|_{s=0} = \frac{1}{t} \frac{d}{ds} \text{Tr}(\rho \mathcal{T}_s(t)(\mathbf{1})) \Big|_{s=0} \\ &= \frac{1}{t} \int_{u=0}^t du \text{Tr}(\rho \mathcal{T}_s(u) \circ \mathcal{J}_1 \circ \mathcal{T}_s(t-u)(\mathbf{1})) \Big|_{s=0} \\ &= \frac{1}{t} \int_{u=0}^t du \text{Tr}(\mathcal{T}_*(u)(\rho)L_1^*L_1), \end{aligned}$$

where we used the fact that  $\frac{d\mathcal{L}_s}{ds} \Big|_{s=0} = \mathcal{J}_1$ ; cf. (15). The rate is, then, obtained by taking  $t \rightarrow \infty$  and using the fact that  $\rho$  converges to the stationary state.

Using the property of the birth–death process  $\sum_n \rho_{ss}(n)(\lambda_n^2 - \mu_n^2) = 0$ , we can further write the rate as

$$\frac{\mathbb{E}_{\rho_{ss}}(\Lambda_t)}{t} = N_{ex} \sum_n \rho_{ss}(n) \sin^2(\phi \sqrt{n+1}) = \sum_n n \rho_{ss}(n) - \nu. \tag{20}$$

Unlike the “first order transition” occurring at  $\alpha = 6.66$ , a “second order transition” occurs at  $\alpha \approx 1$ . Here, the first derivative of the mean photon number has a jump in the limit of  $N_{ex} \rightarrow \infty$ . This and the scaling of the potential  $U$  with  $N_{ex}$  will be discussed in Sec. IV.



**FIG. 3.** Rescaled potentials  $U(n)/N_{ex}$  as a function of  $n/N_{ex}$  for various finite  $N_{ex}$  converge to a limit potential for  $N_{ex} \rightarrow \infty$ . For  $\alpha < 1$ , the potential is minimum at zero; for  $1 < \alpha < 4.6$ , it has a unique minimum away from  $n = 0$ ; and for  $4.6 < \alpha < 7.8$ , there are two local minima, which become equal at  $\alpha \approx 6.66$ .

Equation (20) shows that the statistics of the trajectories are, therefore, closely related to the dynamics of the cavity and, consequently, to its stationary state. The next step is to think of the time trajectories as “configurations” of the stochastic system draw from ideas in non-equilibrium statistical mechanics and large deviations theory to study their phases and phase transitions.

### C. Large deviations

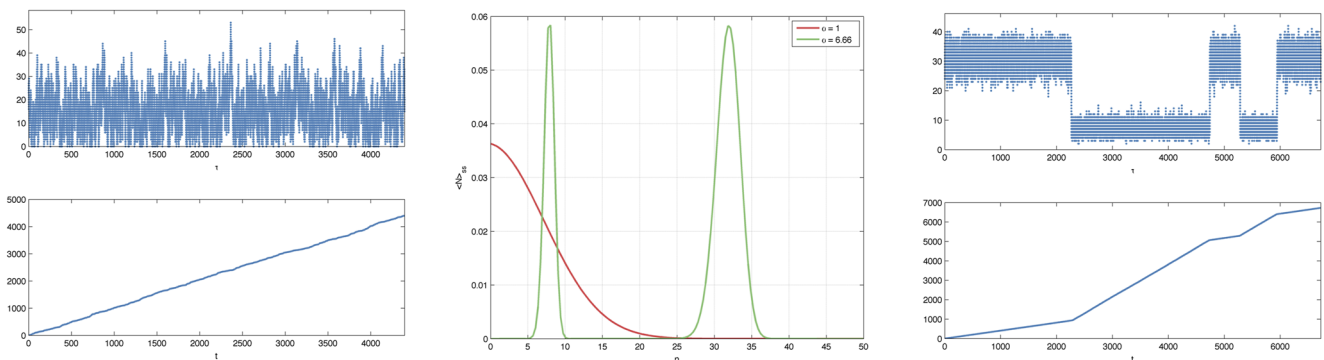
The main result of this paper is the existence of a *large deviations principle* for the counting process  $\Lambda_t$  introduced above. Such results have already been obtained in the context of discrete time quantum Markov chains with finite-dimensional systems,<sup>12</sup> but the novelty here is that we consider a continuous time Markov process with an infinite-dimensional system. The physical motivation lies in the new approach to the study of phase transitions for open systems developed in Refs. 8 and 24. Here, the idea is to identify *dynamical phase transitions* of the open system by analyzing the statistics of jump trajectories in the long time (stationary) regime. The trajectories play an analogous role in the configurations of a statistical mechanics model at equilibrium. In this analogy, the parameter  $s$  of the moment generating function (14) can be seen as a “field” that biases the distribution of trajectories in the direction of active or passive trajectories by effectively changing the probability of a trajectory  $\omega := (i_1, t_1, \dots, i_n, t_n)$  by a factor  $\exp(s\Lambda_t(\omega))$ . When  $\alpha$  is such that the stationary distribution is unimodal, the trajectories’ distribution changes smoothly from passive ones for  $s < 0$  to active ones for  $s > 0$ . However, near  $\alpha \approx 6.66$  (corresponding to the jump in the mean photon number), there is a steep change in the counting rates around  $s = 0$ . The active trajectories are associated with periods when the cavity is in the higher excited phase, while the passive trajectories are connected to the lower phase. Since the cavity makes very rare transitions between the phases, any trajectory—when followed for long but finite periods of time—falls typically into one of the two distinct categories (see Fig. 4). Our goal is to investigate whether this distinction survives the infinite time limit, in which case we would deal with a dynamical phase transition characterized by the non-analyticity of a certain large deviations rate function. We will show that this is not the case, but rather we deal with a *crossover* behavior, that is, the count rate does not jump but has a very steep change around  $s = 0$ , which appears to become a jump only in the limit of an infinite pumping rate  $N_{ex} \rightarrow \infty$  (see Sec. V).

We will now briefly review some basic notions of large deviations needed in this paper. We refer the reader to Ref. 10 for a complete treatment; see Ref. 11 for a comprehensive overview and Ref. 38 for an introduction to large deviations in the context of statistical mechanics. Large deviations is a framework for studying rare events, more precisely events whose probabilities decay exponentially for a sequence of probability distributions. A key result is the *Gärtner–Ellis theorem*, which relates the rate of the exponential decay to the limiting behavior of the moment generating functions associated with the random variables.

Informally, a sequence  $(\mu_n)_{n \in \mathbb{N}}$  of probability distributions on  $\mathbb{R}^d$  endowed with the Borel  $\sigma$ -field  $\mathcal{B}$  satisfies a *large deviations principle* (LDP) if there exists a function  $I : \mathbb{R}^d \rightarrow [0, \infty]$  such that

$$\mu_n(dx) \approx e^{-nI(x)} dx.$$

More rigorously, the function  $I$  is called a *rate function* if it is lower semicontinuous [that is, its level sets  $\{x \in \mathbb{R}^d : I(x) \leq \alpha\}$  are closed]; if, in addition, its level sets are compact, we call it a *good rate function*. The domain of  $I$  is the set of points in  $\mathbb{R}^d$  for which  $I$  is finite. The limiting behavior of the probability measures  $\{\mu_n\}$  is characterized in terms of asymptotic upper and lower bounds on the values that  $\mu_n$  assigns to measurable subsets  $\Gamma \in \mathcal{B}$ . The sequence of probability measures  $\{\mu_n\}$  satisfies a large deviations principle with a rate function  $I$  (or shortly, it satisfies an LDP) if for all  $\Gamma \in \mathcal{B}$ ,



**FIG. 4.** Sample trajectories for the birth–death process describing the cavity state jumping on the ladder of Fock states  $|k\rangle\langle k|$  (top left and right) and total measurement counts  $\Lambda_t$  (bottom left and right) for  $\alpha \approx 1$  (left) and  $\alpha \approx 6.66$  (right) at  $N_{ex} = 50$ . The corresponding stationary state distributions (center) showing large variance at  $\alpha \approx 1$  (red) and bistability at  $\alpha \approx 6.66$  (green).

$$-\inf_{x \in \Gamma^0} I(x) \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log \mu_n(\Gamma) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mu_n(\Gamma) \leq -\inf_{x \in \bar{\Gamma}} I(x). \quad (21)$$

Our goal is to prove an LDP for the counting process  $\Lambda_t$  of the atom maser; we will do this not by showing that  $\Lambda_t$  satisfies the above definition directly but by applying the Gärtner–Ellis theorem, which gives sufficient conditions on the sequence of probability measures in order to satisfy an LDP.

**Theorem 4** (Gärtner–Ellis theorem, Ref. 10, pp. 44–55). Let  $(Z_n)_{n \in \mathbb{N}}$  be a sequence of random variables in  $\mathbb{R}^d$  with laws  $\mu_n$ . Suppose that the (limiting) logarithmic moment generating function

$$\lambda(\mathbf{s}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E} \left[ e^{\langle n\mathbf{s}, Z_n \rangle} \right], \quad \mathbf{s} \in \mathbb{R}^d,$$

exists as an extended real number and is finite in a neighborhood of the origin, and let  $\lambda^*$  denote the Fenchel–Legendre transform of  $\lambda$ , given by

$$\lambda^*(x) = \sup_{\lambda \in \mathbb{R}^d} \{ \langle \mathbf{s}, x \rangle - \lambda(\mathbf{s}) \}.$$

If  $\lambda$  is an essentially smooth, lower semicontinuous function (e.g.,  $\lambda$  is differentiable on  $\mathbb{R}^d$ ), then  $(Z_n)_{n \in \mathbb{N}}$  satisfies a LDP with good rate function  $\lambda^*$ .

The discrete index in the Gärtner–Ellis theorem can be replaced by a continuous one with the obvious modifications in (21). By the Gärtner–Ellis theorem,  $\Lambda_t$  satisfies an LDP if the following limit exists and is a differentiable function:

$$\lambda(s) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_\rho \left( e^{s\Lambda_t} \right) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \text{Tr}(\rho \mathcal{T}_s(t)(\mathbf{1})). \quad (22)$$

We will show that this is, indeed, true and  $\lambda(s)$  is *spectral bound* (i.e., the eigenvalue with the largest real part) of a certain generator  $L_s^{(d)}$ , which is closely related to  $\mathcal{L}_s$ . An essential ingredient is the Krein–Rutman theory, which generalizes the Perron–Frobenius theorem to compact positive semigroups and ensures that  $\lambda(s)$  is real and non-degenerate. In particular, our analysis shows that  $\lambda(s)$  is smooth and its derivatives at  $s = 0$  are the limiting cumulants of  $\Lambda_t$ ,

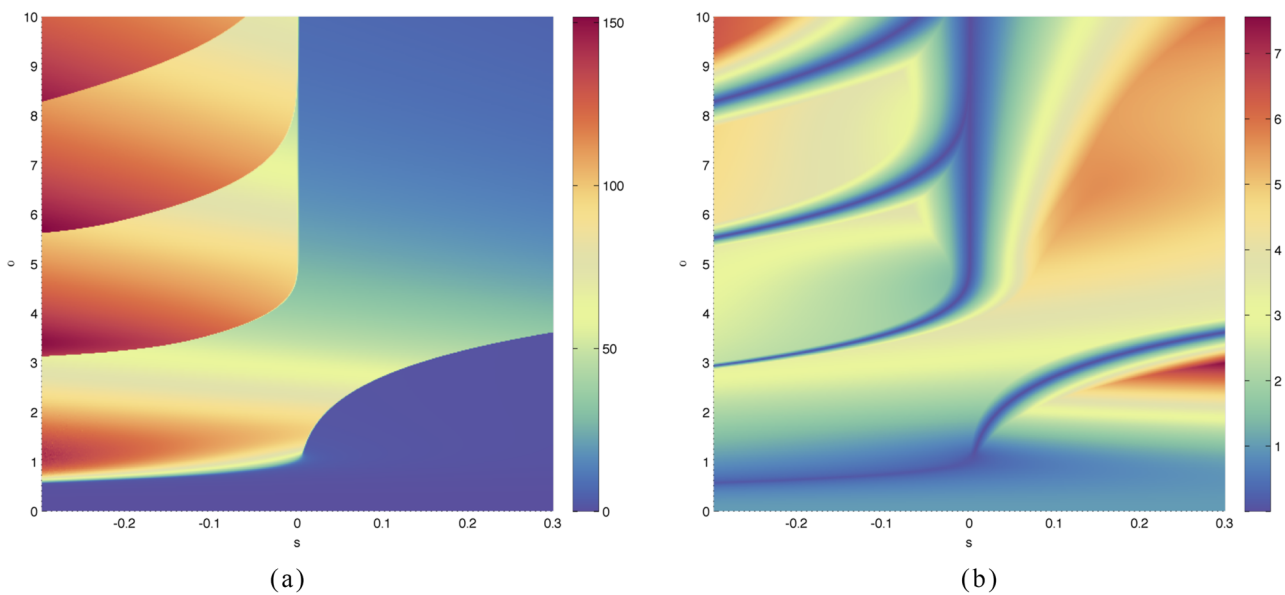


FIG. 5. Derivative  $\lambda'(s)$  (a) and the spectral gap  $g(s)$  of  $L_s$  (b) as functions of  $s$  and  $\alpha = \phi/\sqrt{N_{\text{ex}}}$  (after Fig. 3 in Ref. 24).

$$\lim_{t \rightarrow \infty} \frac{1}{t} C_k(\Lambda_t) = \left. \frac{d^k \lambda(s)}{ds^k} \right|_{s=0}, \quad k \geq 1,$$

with the first two being the mean and the variance. Moreover, the generator  $L_s^{(d)}$  has a non-zero spectral gap; this spectral analysis is illustrated in Fig. 5.

### III. THE MAIN RESULTS

Our main results are the following large deviations and central limit theorems. For reader's convenience, we outline the key steps of the proofs below.

**Theorem 5.** Suppose that the initial state  $\rho$  is a finite rank operator with respect to the Fock basis or, more generally, that  $\sum_{n \geq 0} |(e_n, \rho e_n)|^2 \rho_{ss}(n)^{-1} < +\infty$ . Then, the counting process  $\Lambda_t$  satisfies the large deviations principle with a rate function equal to the Legendre transform of  $\lambda(s)$ , where  $\lambda(s)$  is the limit in (22). The function  $\lambda(s)$  is smooth, and it is equal to the spectral bound of a certain semigroup generator  $L_s^{(d)}$  defined below.

In particular, the atom maser does not exhibit dynamical phase transitions, but rather crossover transitions, which become sharper as  $N_{ex}$  increases. This behavior will be investigated in more detail in Sec. IV.

*Corollary 6.* The counting process  $\Lambda_t$  satisfies the central limit theorem,

$$\frac{1}{\sqrt{t}} (\Lambda_t - t \cdot m) \xrightarrow{\mathcal{D}} N(0, V),$$

where  $\mathcal{D}$  denotes the convergence in distribution and  $m$  and  $V$  are the mean and variance,

$$m = \left. \frac{d\lambda(s)}{ds} \right|_{s=0} = \frac{\mathbb{E} \rho_{ss}(\Lambda_t)}{t}, \quad V = \left. \frac{d^2 \lambda(s)}{ds^2} \right|_{s=0}.$$

*Proof of Theorem 5 and Corollary 6.* For clarity of the exposition, we break the proof into individual steps.

- (i) We introduce  $L^2(\rho_{ss})$  as the completion of  $B(\mathfrak{h})$  endowed with the norm  $\|\cdot\|_2$  induced by the following inner product:

$$\langle Y, X \rangle = \text{Tr}((\rho_{ss}^{1/4} Y \rho_{ss}^{1/4})^* (\rho_{ss}^{1/4} X \rho_{ss}^{1/4})). \quad (23)$$

We recall (see Ref. 18, especially Proposition 2.1) that  $L^2(\rho_{ss})$  is isomorphic as an Hilbert space to the Schatten ideal  $L^2(\text{Tr}) := \{X \in B(\mathfrak{h}) : \text{Tr}(X^* X) < +\infty\}$  via the unique continuous extension of the correspondence

$$i : B(\mathfrak{h}) \rightarrow L^2(\text{Tr}), \\ X \mapsto \rho_{ss}^{1/4} X \rho_{ss}^{1/4}.$$

Hence, with an abuse of notation, we will identify  $Y \in L^2(\rho_{ss})$  with the corresponding operator in  $L^2(\text{Tr})$ . We recall that for every  $X_n, X \in B(\mathfrak{h})$  and  $Y \in L^2(\rho_{ss})$ , the following holds:

1.  $\|X\|_2 \leq \|X\|_\infty$  (Ref. 18, Proposition 2.1) and
2. if  $X_n \xrightarrow{w^*} X$ , then  $\langle Y, X_n \rangle \rightarrow \langle Y, X \rangle$ : indeed,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \langle Y, X_n \rangle &= \lim_{n \rightarrow +\infty} \text{Tr}(Y^* \rho_{ss}^{1/4} X_n \rho_{ss}^{1/4}) = \lim_{n \rightarrow +\infty} \text{Tr}(\rho_{ss}^{1/4} Y^* \rho_{ss}^{1/4} X_n) \\ &= \text{Tr}(\rho_{ss}^{1/4} Y^* \rho_{ss}^{1/4} X) = \langle Y, X \rangle. \end{aligned}$$

*Lemma 7.* For every  $s \in \mathbb{R}$ , the following holds:

1. There exists a unique strongly continuous semigroup  $(T_s(t))_{t \geq 0}$  of bounded linear maps on  $L^2(\rho_{ss})$  such that

$$T_s(t)(X) = \mathcal{T}_s(t)(X), \quad X \in B(\mathfrak{h}).$$

Every  $X \in D(\mathcal{L}_s)$  belongs to the domain of the generator  $L_s$  of  $(T_s(t))_{t \geq 0}$  and

$$L_s(X) = \mathcal{L}_s(X), \quad X \in D(\mathcal{L}_s) = D(\mathcal{L}).$$

As usual, we denote  $(T(t))_{t \geq 0}$  and  $L$  as the semigroup and the corresponding generator in the case  $s = 0$ .

2. The set  $\mathcal{M}(\mathfrak{h})$  of finite rank operators given by finite matrices with respect to the Fock basis forms a core for  $L_s$ .
3.  $L_s = L + \delta_s$ , with  $\delta_s$  being a bounded perturbation.

*Proof of Lemma 7.*

1. First, we show that  $\rho_{ss}$  is sub-invariant for  $\mathcal{T}'$  (as defined in the proof of points 1. and 2. of Proposition 3). Since  $\rho_{ss} \in D(\mathcal{L}'_*)$  and  $\rho_{ss}$  is invariant for  $\mathcal{T}$ , this is equivalent to showing that

$$\mathcal{L}'_*(\rho_{ss}) = (\mathcal{L}'_* - \mathcal{L}_*)(\rho_{ss}) = (e^s - 1) \left( \mathcal{J}_{1*}(\rho_{ss}) - 1_{s>0} N_{ex} \left(1 + \frac{1}{\nu}\right) \rho_{ss} \right) \leq 0.$$

This inequality is trivial for  $s < 0$ , while for  $s > 0$ , we can write the explicit form of  $\rho_{ss}$  and get

$$\begin{aligned} \mathcal{J}_{1*}(\rho_{ss}) &= N_{ex} \sum_{n \geq 0} \sin^2(\phi\sqrt{n+1}) \rho_{ss}(n) |n+1\rangle \langle n+1| \\ &= N_{ex} \sum_{n \geq 0} \sin^2(\phi\sqrt{n+1}) \frac{\rho_{ss}(n)}{\rho_{ss}(n+1)} \rho_{ss}(n+1) |n+1\rangle \langle n+1| \\ &\leq N_{ex} \left(1 + \frac{1}{\nu}\right) \rho_{ss} \end{aligned}$$

since

$$\frac{\rho_{ss}(n)}{\rho_{ss}(n+1)} = \frac{(\nu+1)(n+1)}{\nu(n+1) + N_{ex} \sin^2(\phi\sqrt{n+1})} \leq 1 + \frac{1}{\nu}.$$

Then, we apply Theorem 2.3 of Ref. 18 and we obtain that the semigroup  $\mathcal{T}'$  can be extended to a strongly continuous contraction semigroup on  $L^2(\rho_{ss})$ . In addition, here, the conclusion follows multiplying the semigroup by a suitable exponential factor as for points 1. and 2. of Proposition 3.

2. and 3. The proof of the previous point shows that  $(N_{ex}(1 + \frac{1}{\nu}))^{-1} \mathcal{J}_{1*}(\rho_{ss}) \leq \rho_{ss}$ ; hence, we can apply again Theorem 2.3 of Ref. 18 in order to show that  $(N_{ex}(1 + \frac{1}{\nu}))^{-1} \mathcal{J}_{1*}$  extends to a bounded operator on  $L^2(\rho_{ss})$  and so does  $\mathcal{L}_s - \mathcal{L} = (e^s - 1) \mathcal{J}_1$ ; let us call  $\delta_s$  such extension. Since on  $\mathcal{M}(\mathfrak{h})$  we have  $L_s = L + \delta_s$ , it follows that if we prove that  $\mathcal{M}(\mathfrak{h}) \subset D(\mathcal{L})$  is a core for  $L_s$ , we have that  $D(L_s) = D(L)$  and  $L_s = L + \delta_s$ . Note that both  $L$  and  $\delta_s$  preserve  $\mathcal{M}(\mathfrak{h})$  and that  $\mathcal{M}(\mathfrak{h})$  is dense in  $B(\mathfrak{h})$  in the  $w^*$ -topology and, hence, in  $L^2(\rho_{ss})$  in norm. Hence, we only need to show that  $\mathcal{M}(\mathfrak{h})$  is a set of analytic vectors for  $L_s$  and, then, apply Proposition 14 in Appendix B.

Let us fix  $s \in \mathbb{R}$ . Then, for  $n, m \in \mathbb{N}$ , we have

$$L_s(|e_n\rangle \langle e_m|) = \alpha_{m,n} |e_{n-1}\rangle \langle e_{m-1}| + \beta_{m,n} |e_n\rangle \langle e_m| + \gamma_{m,n} |e_{n+1}\rangle \langle e_{m+1}|, \quad (24)$$

where

$$\begin{aligned} \alpha_{m,n} &= \nu\sqrt{nm} + N_{ex} e^s \sin(\phi\sqrt{n}) \sin(\phi\sqrt{m}), \\ \beta_{m,n} &= -\frac{1}{2}((\nu+1)(m+n) + \nu(m+n+2)) + N_{ex}(\cos(\phi\sqrt{n+1}) \cos(\phi\sqrt{m+1}) - 1), \\ \gamma_{m,n} &= (\nu+1)\sqrt{n+1}\sqrt{m+1}. \end{aligned}$$

Note that  $|\alpha_{m,n}| + |\beta_{m,n}| + |\gamma_{m,n}| \leq \underbrace{3(\nu+1 + 2(e^s \vee 1)N_{ex})}_{=:B} (n+m+1)$ . Hence,

$$\begin{aligned} \|L_s^k(|e_n\rangle\langle e_m|)\|_\infty &\leq B^k(n+m+1) \cdot (n+m+3) \dots (n+m+2k-1) \\ &\leq B^k 2^k (n+m+1)(n+m+2) \dots (n+m+k) \\ &= B^k 2^k k! \binom{n+m+k}{n+m} \leq 2^{n+m} 4^k B^k k!. \end{aligned}$$

If  $\bar{T} = 1/(4B)$ , for any  $t < \bar{T}$ ,  $\sum_{k=0}^{+\infty} \frac{t^k L_s^k(|e_n\rangle\langle e_m|)}{k!}$  converges uniformly on compact intervals in the uniform norm, hence, with respect to  $\|\cdot\|_2$ . By the definition of  $\mathcal{M}(\mathfrak{h})$ , we can deduce the same for every  $X \in \mathcal{M}(\mathfrak{h})$ . Let us call  $X_t := \sum_{k=0}^{+\infty} \frac{t^k L_s^k(X)}{k!}$  and note that since the series of the derivatives converges uniformly on compact intervals, it solves the following abstract Cauchy problem:

$$X_t = X + L_s \int_0^t X_u du, \quad t \leq \bar{T}.$$

Hence, by Proposition 6.4 of Ref. 39,  $X_t = T_s(t)(X)$  for every  $t < \bar{T}$ . □

- (ii) The moment generating function of  $\Lambda_t$  for an initial state  $\rho$  [cf. Eq. (14)] can be expressed in terms of the semigroup acting on  $L^2(\rho_{ss})$  as

$$\mathbb{E}_\rho(e^{s\Lambda_t}) = \text{Tr}(\rho T_s(t)(\mathbf{1})) = \langle \tilde{\rho}, T_s(t)(\mathbf{1}) \rangle,$$

where  $\tilde{\rho} := \rho_{ss}^{-1/2} \rho \rho_{ss}^{-1/2}$  is assumed to belong to  $L^2(\rho_{ss})$  or, equivalently,  $\sum_{n \geq 0} |\langle e_n, \rho e_n \rangle|^2 \rho_{ss}(n)^{-1} < +\infty$  [in this case,  $\tilde{\rho}$  extends to a bounded linear functional on  $L^2(\rho_{ss})$ ]. This holds, for instance, not only if  $\rho$  has a finite number of photons but also if  $\rho = \rho_{ss}$ .

- (iii) As we already mentioned, the commutative von Neumann algebra  $B_d(\mathfrak{h})$  of the operators that are diagonal in the Fock basis plays a fundamental role. We need to introduce some other related linear spaces,

$$\begin{aligned} \mathcal{M}_d(\mathfrak{h}) &:= \mathcal{M}(\mathfrak{h}) \cap B_d(\mathfrak{h}) = \text{span}\{|e_n\rangle\langle e_n| : n \in \mathbb{N}\} \subset D(\mathcal{L}), \\ L_d^2(\rho_{ss}) &:= \overline{\mathcal{M}_d(\mathfrak{h})}^{\|\cdot\|_2}. \end{aligned}$$

Note that  $B_d(\mathfrak{h}) \simeq \ell^\infty(\mathbb{N})$  as von Neumann algebras and  $L_d^2(\rho_{ss}) \simeq \ell^2(\mathbb{N}, \rho_{ss})$  as Banach spaces.

*Proposition 8.* The following statements hold for every  $s \in \mathbb{R}$ .

- The generator  $\mathcal{L}_s$  and the corresponding semigroup  $(T_s(t))_{t \geq 0}$  preserve the algebra  $B_d(\mathfrak{h})$ . Consequently,  $L_s$  and  $T_s$  also preserve the subspace  $L_d^2(\rho_{ss})$ .
- The action of  $\mathcal{L}_s$  on the diagonal is explicitly written as an operator  $\mathcal{L}_s^{(d)}$  acting on  $f = \sum_k f_k |e_k\rangle\langle e_k| \in D(\mathcal{L}_s^{(d)}) := D(\mathcal{L}_s) \cap B_d(\mathfrak{h})$  as  $\mathcal{L}_s^{(d)}(f) = g = \sum_k g_k |e_k\rangle\langle e_k|$  with

$$\begin{aligned} g_k &= k(v+1)(f_k - f_{k-1}) + (e^s - 1)N_{ex} \sin^2(\phi\sqrt{k+1})f_{k+1} \\ &\quad + (v(k+1) + N_{ex} \sin^2(\phi\sqrt{k+1}))(f_{k+1} - f_k). \end{aligned} \tag{25}$$

In particular,  $\mathcal{L}_0^{(d)} = \mathcal{L}^{(d)}$  is the generator of a birth–death process with birth and death rates as in Eq. (18).

*Proof of Proposition 8.*

- $\mathcal{M}_d(\mathfrak{h}) \subset D(\mathcal{L}_s) \cap B_d(\mathfrak{h})$  is  $w^*$ -dense in  $B_d(\mathfrak{h})$ . Hence, it is sufficient to compute explicitly  $\mathcal{L}_s(|e_k\rangle\langle e_k|)$  and observe that it belongs to

$$\text{span}\{|e_j\rangle\langle e_j|, j = k-1, k, k+1\}$$

for all  $k$ . Then,  $\mathcal{L}_s(\mathcal{M}_d(\mathfrak{h})) \subseteq \mathcal{M}_d(\mathfrak{h})$ . The rest follows from the definitions.

- This point is a direct computation by using the definition of the generator  $\mathcal{L}_s$  given by Eqs. (4) and (15). □

We denote by  $L_s^{(d)}$  and  $T_s^{(d)}$  the restrictions of  $L_s$  and  $T_s$  to  $L_d^2(\rho_{ss})$  (as usual, we drop the index  $s$  in the case  $s = 0$ ). Since  $\mathbf{1} \in L_d^2(\rho_{ss})$ , the moment generating function can be expressed as

$$\mathbb{E}_\rho(e^{s\Lambda_t}) = \langle \tilde{\rho}^{(d)}, T_s^{(d)}(t)(\mathbf{1}) \rangle,$$

with  $\tilde{\rho}^{(d)}$  denoting the diagonal of  $\tilde{\rho}$ .

- (iv) The semigroup  $(T_s^{(d)}(t))_{t \geq 0}$  is *immediately compact* [i.e.,  $T_s^{(d)}(t)$  is compact for all  $t > 0$ ]. In order to prove it, we first show it for  $T^{(d)}(t)$  with the classical theory of birth–death processes and then use perturbation theory.

*Lemma 9.* The following statements hold.

1.  $L^{(d)}$  is self-adjoint, and its essential spectrum is empty.
2.  $(T^{(d)}(t))_{t \geq 0}$  is immediately compact.
3.  $(T_s^{(d)}(t))_{t \geq 0}$  is immediately compact for all  $s \in \mathbb{R}$ .

We recall that  $\sigma(L^{(d)})$  is the disjoint union of the discrete and the essential spectrum and the discrete spectrum for a self-adjoint operator is defined as those  $\alpha \in \mathbb{C}$ , which are isolated eigenvalues with finite multiplicity (see Ref. 40, Theorem VII.10).

*Proof of Lemma 9.*

1. The proof of point 2. of Lemma 7 shows that  $\mathcal{M}_d(\mathfrak{h})$  is a dense linear space of analytic vectors for  $L^{(d)}$ ; hence, by the Nelson theorem (Ref. 41, Theorem X.39), in order to establish the self-adjointness of  $L^{(d)}$ , it is enough to check that it is symmetric on  $\mathcal{M}_d(\mathfrak{h})$  [which is a simple computation using Eq. (24)].

In order to show that the essential spectrum of  $L^{(d)}$  is empty, it suffices to check that the following condition on birth and death rates holds (cf. Ref. 42, Theorem 1.2) and we already know that the process is non-explosive and admits a unique invariant state,

$$\lim_{n \rightarrow +\infty} \left( \sum_{i=1}^n \frac{1}{\mu_i^2 \rho_{ss}(i)} \right) \cdot \sum_{j=n+1}^{+\infty} \rho_{ss}(j) = 0.$$

Let us do the following computations:

$$\begin{aligned} \sum_{i=1}^n \frac{1}{\mu_i^2 \rho_{ss}(i)} \cdot \sum_{j=n+1}^{+\infty} \rho_{ss}(j) &= \sum_{i=1}^n \sum_{j=n+1}^{+\infty} \frac{1}{\mu_i^2} \frac{\rho_{ss}(j)}{\rho_{ss}(i)} \\ &= \sum_{i=1}^n \sum_{j=n+1}^{+\infty} \frac{1}{\mu_i^2} \prod_{k=i+1}^j \frac{\nu + N_{ex} \sin^2(\phi\sqrt{k})/k}{\nu + 1}. \end{aligned}$$

Note that for every  $0 < \epsilon < 1$ , there exists  $M \in \mathbb{N}$  such that for every  $k \geq M$ ,  $N_{ex} \sin^2(\phi\sqrt{k}) \leq \epsilon k$ ; therefore, for  $n > M$ , we have

$$\begin{aligned} \sum_{i=1}^n \frac{1}{\mu_i^2 \rho_{ss}(i)} \cdot \sum_{j=n+1}^{+\infty} \rho_{ss}(j) &\leq \underbrace{\sum_{i=1}^M \frac{1}{\mu_i^2 \rho_{ss}(i)} \cdot \sum_{j=n+1}^{+\infty} \rho_{ss}(j)}_{=o(1)} \\ &+ \sum_{i=M+1}^n \frac{1}{\mu_i^2} \cdot \sum_{j=n+1}^{+\infty} \left( \underbrace{\frac{\nu + \epsilon}{\nu + 1}}_{=:q} \right)^{j-i-1} \leq \frac{1}{1-q} \sum_{i=M+1}^n \frac{q^{n-i}}{(\nu + 1)^i} + o(1) \\ &= \frac{1}{1-q} \sum_{i=0}^{n-M-1} \frac{q^i}{(\nu + 1)(n-i)} + o(1). \end{aligned}$$

The result follows from the fact that  $q < 1$  and the dominated convergence theorem.

2.  $L^{(d)}$  is a self-adjoint, unbounded operator with an empty essential spectrum such that  $L^{(d)} \leq 0$ ; therefore, its spectral resolution reads

$$L^{(d)} = -\sum_{n \geq 0} \alpha_n P_n,$$

where  $P_n$ 's are finite-dimensional orthogonal projections and  $\alpha_n$ 's are distinct non-negative real numbers such that  $\lim_{n \rightarrow +\infty} \alpha_n = +\infty$  (they do not accumulate). The semigroup generated by  $L^{(d)}$  can be expressed via the functional calculus as

$$T^{(d)}(t) = \sum_{n \geq 0} e^{-t\alpha_n} P_n, \quad t \geq 0.$$

By its spectral representation, we can conclude that  $T^{(d)}$  is immediately compact.

3. The restriction  $L_s^{(d)}$  is a bounded perturbation of the generator  $L^{(d)}$ ; hence, the semigroup  $(T_s^{(d)}(t))_{t \geq 0}$  is also immediately compact; cf. Ref. 43, Theorem III.1.16.  $\square$

Since  $(T_s^{(d)}(t))$  is an immediately compact semigroup, we have (Ref. 43, Corollary IV.3.12) a spectral mapping theorem of the form

$$e^{t\sigma(L_s^{(d)})} = \sigma(T_s^{(d)}(t)) \setminus \{0\}, \quad t > 0.$$

In particular, the spectral radius of  $T_s^{(d)}(t)$  is given by

$$r_s(t) := r(T_s^{(d)}(t)) = e^{t\lambda(s)},$$

where  $\lambda(s)$  is the spectral bound of  $L_s^{(d)}$ , i.e., the real part of the eigenvalue with the largest real part.

- (v) The semigroup  $(T_s^{(d)}(t))_{t \geq 0}$  is strictly positive, that is,  $T_s^{(d)}(t)(D) > 0$  for all  $D \geq 0$  in  $L_d^2(\rho_{ss})$  and  $t > 0$ .

*Proof.* It is not difficult to see that every  $D \geq 0$  in  $L_d^2(\rho_{ss})$  is of the form  $D = \sum_k D_k |e_k\rangle\langle e_k|$  for some  $D_k \in \ell^2(\mathbb{N}, \rho_{ss})$  and  $D_k \geq 0$  for every  $k$ ; hence, it is enough to show that for every  $k$   $T_s^{(d)}(t)(|e_k\rangle\langle e_k|) > 0$  for every  $t > 0$ . Note that  $(P_{m,n}(t) := e^{t(1-e^t)(1+\frac{1}{v})N_{\alpha,1\{s>0\}}}(e_n, T_s^{(d)}(t)(|e_m\rangle\langle e_m|)e_n))_{n,m \in \mathbb{N}}$  is a standard transition function, and so by the well-known results of continuous time Markov chains, we have that either  $P_{m,n}$  is constant and equal to 0 or it is strictly positive for every  $t > 0$  [Levy's theorem (Ref. 44, Proposition 1.3)]. Since the birth and death rates are all strictly positive,  $P_{m,n}(t) > 0$  for every positive time (for a reference about continuous time Markov chains, see, for instance, Ref. 44).  $\square$

- (vi) Since  $(T_s^{(d)}(t))_{t > 0}$  is compact and strictly positive, Krein–Rutman theory (Ref. 19, Theorem 1.5) implies that the spectral radius of  $T_s^{(d)}(t)$  is an algebraically simple eigenvalue with strictly positive right and left eigenvectors  $r(s)$  and  $l(s)$ .

*Lemma 10.* For every  $s \in \mathbb{R}$ , there exists a strictly positive number  $g(s) > 0$  such that for every  $t > 0$ ,

$$T_s^{(d)}(t) = e^{t\lambda(s)} (|r(s)\rangle\langle l(s)| + R_t^s) \tag{26}$$

and  $\|R_t^s\|_{L_d^2(\rho_{ss}) \rightarrow L_d^2(\rho_{ss})} = O(e^{-g(s)t})$ .

*Proof of Lemma 10.* Because of the compactness of  $T_s^{(d)}(t)$ , we only need to prove that

$$\sigma(T_s^{(d)}(t)) \cap \{z \in \mathbb{C} : |z| = e^{\lambda(s)t}\} = \{e^{\lambda(s)t}\}.$$

Suppose that there exist  $\theta \in [0, 2\pi)$  and  $D \in L_d^2(\rho_{ss})$  such that  $L_s^{(d)}(D) = (\lambda(s) + i\theta)D$ ; then, if we consider  $t = 2\pi/\theta$ ,  $T_s^{(d)}(t)(D) = e^{\lambda(s)t} D$ . Since  $l(s)$  is the unique eigenvector with eigenvalue one, we conclude that  $\theta = 0$ .  $\square$

Using point (iii), this implies that

$$\mathbb{E}_\rho(e^{s\Lambda_t}) = e^{t\lambda(s)} (\langle \tilde{\rho}, r(s) \rangle \langle l(s), \mathbf{1} \rangle + o(1)). \tag{27}$$

Since  $l(s), r(s) > 0$  and  $\tilde{\rho}, \mathbf{1} \geq 0$ , the inner products are non-zero and we obtain the limiting cumulant generating function

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_{\rho_{in}}(e^{s\Lambda_t}) = \lambda(s).$$

- (vii) Using analytic perturbation theory for the generator  $L_s^{(d)}$ , the spectral bound  $\lambda(s)$  can be shown to be a smooth function of  $s$  in a complex neighborhood of the real line. Since for every  $s_0 \in \mathbb{R}$ ,  $\lambda(s_0)$  is an isolated eigenvalue with finite multiplicity, we can apply Proposition 3.25 of Ref. 45, p. 141 to the family of perturbations

$$V_{s-s_0} := L_s^{(d)} = L_{s_0}^{(d)} + \delta_s - \delta_{s_0} = L_{s_0}^{(d)} + e^{s_0} (e^{s-s_0} - 1) J_1^{(d)}$$

and we find that  $\lambda(s)$  is an analytic function of  $s$  and remains isolated in a complex neighborhood of  $s_0$ .

- (viii) Using points (vi) and (vii), we apply the Gärtner–Ellis theorem to conclude that  $\Lambda_t$  satisfies the LD principle with the rate function equal to the Legendre transform of  $\lambda(s)$ .
- (ix) Furthermore, by the result of Ref. 16, it follows that  $\Lambda_t$  satisfies the CLT. In particular, the limiting cumulants of  $\Lambda_t$  can be computed as derivatives of  $\lambda(s)$  at  $s = 0$ ,

$$\lim_{t \rightarrow \infty} \frac{1}{t} C_k(\Lambda_t) = \left. \frac{d^k \lambda(s)}{ds^k} \right|_{s=0}.$$

By  $J_1^{(d)}$ , we denote the operator acting on  $D = \sum_k D_k |e_k\rangle\langle e_k| \in L_d^2(\rho_{ss})$  in the following way:

$$J_1^{(d)}(D) = \sum_{k \geq 1} N_{\text{ex}} \sin^2(\phi \sqrt{k}) D_k |e_{k-1}\rangle\langle e_{k-1}|.$$

*Lemma 11.*

$$\begin{aligned} \left. \frac{d\lambda(s)}{ds} \right|_{s=0} &= \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle, \\ \left. \frac{d^2 \lambda(s)}{ds^2} \right|_{s=0} &= \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle + 2 \langle \mathbf{1}, J_1^{(d)}(D_V) \rangle, \end{aligned}$$

and  $D_V$  can be characterized as the unique solution in  $L_d^2(\rho_{ss})$  of

$$L^{(d)}(D_V) = \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle \mathbf{1} - J_1^{(d)}(\mathbf{1}).$$

*Proof of Lemma 11.* Differentiating first once and then twice

$$L_s^{(d)}(r(s)) = \lambda(s)r(s)$$

and evaluating in  $s = 0$ , we get

$$L^{(d)}(r'(0)) + J_1^{(d)}(\mathbf{1}) = \lambda'(0)\mathbf{1}, \tag{28}$$

$$L^{(d)}(r''(0)) + J_1^{(d)}(\mathbf{1}) + 2J_1^{(d)}(r'(0)) = \lambda''(0)\mathbf{1} + 2\lambda'(0)r'(0), \tag{29}$$

where we used the fact that  $\lambda(0) = 0$  and  $r(0) = \mathbf{1}$ . Note that  $l(0) = \mathbf{1}$  too; hence, taking the scalar product of Eq. (28) against  $\mathbf{1}$ , we get

$$\lambda'(0) = \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle.$$

Substituting the expression we obtained for  $\lambda'(0)$  in Eq. (28), we get that  $r'(0)$  is the unique [remember that  $\ker(L^{(d)})$  has dimension 1] solution of

$$L^{(d)}(r'(0)) = \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle \mathbf{1} - J_1^{(d)}(\mathbf{1}).$$

We can choose  $r(s)$  such that  $\langle \mathbf{1}, r(s) \rangle = 1$ ; hence, substituting the expression we obtained for  $\lambda'(0)$  in Eq. (29) and taking the scalar product against  $\mathbf{1}$ , we get

$$\lambda''(0) = \langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle + 2 \langle \mathbf{1}, J_1^{(d)}(r'(0)) \rangle.$$

Note that  $\lambda'(0)$  is the expected value of  $\frac{\Delta_t}{t}$  if the system starts in the stationary state  $\rho_{ss}$  [Eq. (20)]; indeed, □

$$\langle \mathbf{1}, J_1^{(d)}(\mathbf{1}) \rangle = \text{Tr}(\rho_{ss} L_1^* L_1) = \sum_{n \geq 0} n \rho_{ss}(n) = v.$$

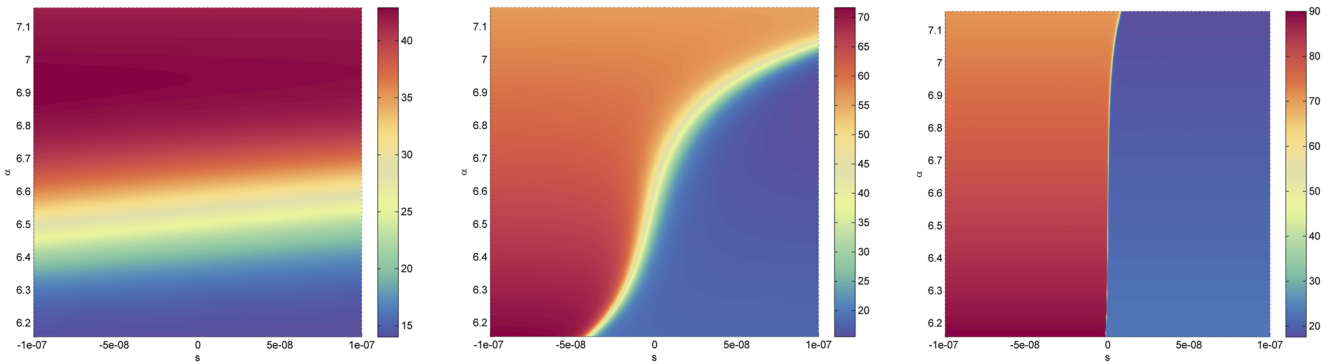


FIG. 6. Phase boundaries at the  $s = 0, \alpha \approx 6.66$  crossover with  $N_{ex} = 75, 100,$  and  $125$ .

#### IV. NUMERICAL ANALYSIS

The existence of a “phase transition” in the atom maser has been discussed in several theoretical physics papers.<sup>13,24,35,46,47</sup> There is a general agreement that if  $N_{ex}$  is sufficiently large (for instance,  $N_{ex} \approx 150$ ), then “for all practical purposes,” we can consider that the mean photon number of the stationary state has a jump at  $\alpha \approx 6.66$  (see Fig. 1), which matches up with a jump between the left and right derivatives of  $\lambda(s)$  at  $s = 0$  in the dynamical scenario (see Fig. 5). However, the question whether we are dealing with a “true” (dynamical) phase transition or rather a steep but smooth crossover was left open and motivated this investigation. Having proved that the latter is the case, we would like to briefly put the result in the context of a numerical analysis.

As the proof suggests, dynamical phase transitions are intimately connected with the closing of the spectral gap of the semigroup generator. Figure 5 shows the close match between the behavior of the first derivative of  $\lambda(s)$  and the spectral gap  $g(s) := \lambda(s) - \text{Re}\lambda_1(s)$ . In particular, at first sight, it would appear that for  $\alpha \geq 4.6$  (the point where the stationary state becomes bistable), the entire  $s = 0$  line is a phase separation line. However, by zooming in a vertical strip of size  $10^{-7}$  in this region (see Fig. 6), we find that the line separating the phases is

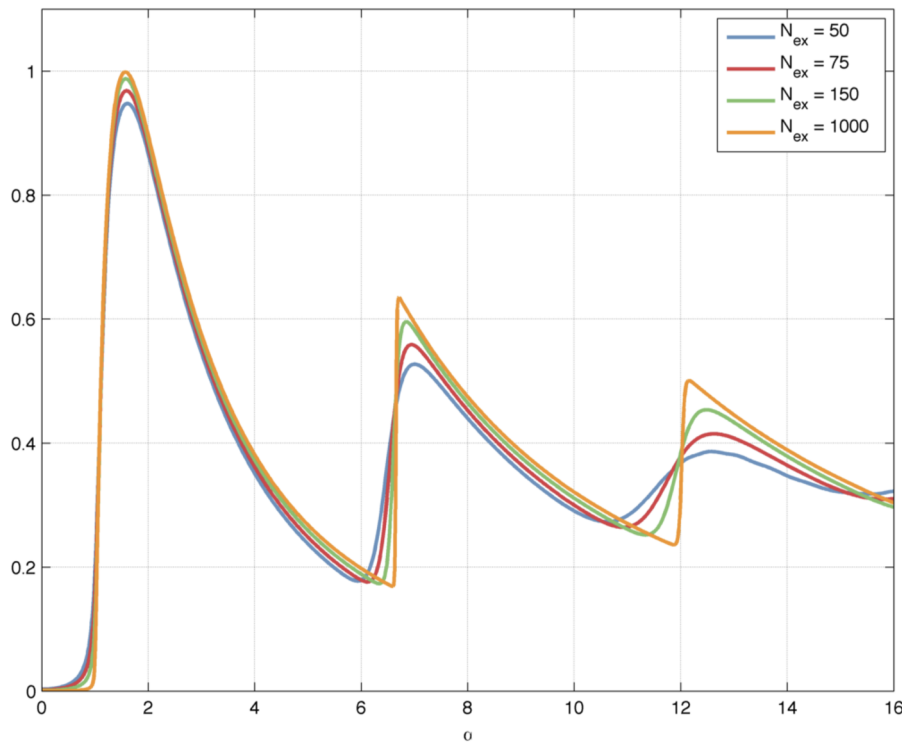


FIG. 7. Rescaled stationary state mean photon numbers,  $\langle \rho_{ss} \rangle / N_{ex}$  for increasing  $N_{ex}$ , showing phase transition becomes sharp as  $N_{ex} \rightarrow \infty$ .

not perfectly vertical but crosses  $s = 0$  at  $\alpha \approx 6.6$ , which corresponds roughly to the transition point for the stationary state. Moreover, on this scale, it is clear that we deal with a steep but smooth transition between phases.

Figure 6 shows that the phase separation lines become sharper with larger  $N_{ex}$ , and a “true” phase transition emerges in the infinite pumping rate limit. A similar conclusion can be drawn by plotting the rescaled stationary mean  $\langle N \rangle / N_{ex}$ ; cf. Fig. 7. This can be intuitively understood by appealing to the effective potential (19). As  $N_{ex}$  increases, the potential barrier becomes larger and two stable phases emerge at the point where the local minima are equal. Indeed, Fig. 3 shows the plot of the rescaled potential  $U/N_{ex}$  as a function of the rescaled variable  $x = n/N_{ex}$ , which approaches the ( $N_{ex}$  independent) limit

$$v(x) = - \int_0^x \log \left( \frac{v}{v+1} + \frac{\sin^2(\alpha\sqrt{y})}{(v+1)y} \right) dy$$

as it can be deduced from formulas (12) and (19). Therefore, in the limit of a large pumping rate, we deal with a particle in a fixed potential  $v(x)$  at inverse temperature  $1/kT = N_{ex}$ . At  $\alpha \approx 1$ , the dependence of the mean on  $N_{ex}$  switches from constant to linear behavior as the minimum of the potential  $v(x)$  moves away from zero. When the two minima are at different heights, the lower one becomes the stable and other one is metastable. Communication between the phases becomes increasingly unlikely, with the probability decreasing exponentially with  $N_{ex}$ . When the two minima are equal, we have two stable states, and the corresponding value of  $\alpha$  is the phase transition point for the mean photon number.

More information about the dynamical phase transitions may be obtained from the rest of the spectrum of the semigroup generator, and for a more in-depth treatment of these numerical aspects, we refer the reader to Ref. 48.

## V. CONCLUSIONS AND OUTLOOK

We have studied the counting process associated with the measurement of the outgoing atoms in the atom maser and shown that this process satisfies the large deviations principle. In particular, this means that the crossover behavior observed in numerical simulations is not associated with the non-analyticity of the limiting log-moment generating function, as one would expect for a genuine phase transition. The rescaled counting process  $\Lambda_t/N_{ex}$  does, however, appear to exhibit such a transition in the limit of infinite rate  $N_{ex}$ , as argued in Sec. IV using the potential model and illustrated in Figs. 3, 6, and 7.

As a corollary, we have showed that the counting process satisfies the central limit theorem, which can be used to develop the statistical estimation theory of local asymptotic normality.<sup>49–51</sup>

The model we have investigated has the property that the stationary state is diagonal in the Fock basis, and all the jump operators leave the set of diagonal states invariant. The proof of the large deviations principle for the non-Markov counting process  $\Lambda_t$  relies on the quantum semigroup’s restriction to the diagonal algebra, which results in a classical birth–death semigroup (when proving the strict positivity and immediate compactness of  $T_s^{(d)}$  and when applying Krein–Rutman theory). We leave for a future investigation the study of the same problem in settings where no classical reduction is possible; an example would be the atom maser where the outgoing atoms are measured in a different basis than the standard one, thus breaking the invariance of the diagonal algebra.

The compactness of the Markov semigroup makes our model tractable as it becomes essentially finite dimensional, as the bath decay dominates the absorption due to the atom interaction. An interesting problem would be to explore more general classes of infinite-dimensional systems (e.g., continuous variables or infinite spin chains) where a similar phenomenon holds. Another issue is the general relation between the “static” transitions, which refer to non-analytic properties of the stationary state, and dynamic transitions, which characterize properties of the measurement process. As shown in Ref. 52, one can construct examples where the stationary state does not change, while the system undergoes a dynamical phase transition.

Finally, it would be interesting to consider a more general large deviations setup, which takes into account the correlations between the detection events rather than looking at the total number of counts.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: PROOFS OF PROPOSITIONS 2 AND 3

*Proof of Proposition 2.*

1. Since  $G$  is negative by (9), it is the generator of a strongly continuous contraction semigroup on  $\mathfrak{h}$ . We have

$$D(G) \subseteq D(L_i) \quad \forall \quad i = 1, \dots, 4$$

and

$$\langle Gu, u \rangle + \langle u, Gu \rangle + \sum_{i=0}^4 \langle L_i u, L_i u \rangle = 0 \quad \forall u \in D(G).$$

Consequently,  $\mathcal{L}$  generates a minimal sub-Markov quantum dynamical semigroup by Ref. 36, Theorem 3.21. In particular,  $\mathcal{L}^{(0)}$  generates the sub-Markov quantum dynamical semigroup that we formally denoted by  $(e^{t\mathcal{L}^{(0)}})_{t \geq 0}$ .

In order to prove conservativity (which implies uniqueness by Ref. 36, Corollary 3.23), we can use Corollary 3.41 of Ref. 36, p. 73 with  $C = \Phi = -2G$  and  $D = D(G)$ ; we just need to check condition (3.41) of Ref. 36, Corollary 3.41, p. 73, which is equivalent to the existence of a constant  $b$  such that for every  $u \in D(G)$ ,

$$-4\langle Gu, Gu \rangle + \sum_{i=0}^4 \langle \sqrt{-2GL_i}u, \sqrt{-2GL_i}u \rangle \leq b \|\sqrt{-2Gu}\|^2. \tag{A1}$$

Let  $S$  be the shift operator, which acts as  $S|e_n\rangle = |e_{n+1}\rangle$  and whose adjoint is  $S^*|e_n\rangle = \delta_{n \geq 1}|e_{n-1}\rangle$ ; note that  $a = S^* \sqrt{N}$  and  $a^* = \sqrt{N}S$ . We recall some relations that are used in the following computation:

$$aa^* = N + 1, \quad a^*a = N,$$

$$Sf(N)|e_n\rangle = f(n)|e_{n+1}\rangle = f(N - 1)S|e_n\rangle$$

for any suitable function  $f$  of the number operator. We have

$$\begin{aligned} -4G^2 - 2\sum_{i=0}^4 L_i^* GL_i &= -(N_{\text{ex}} + \nu + N(2\nu + 1))^2 \\ &\quad + N_{\text{ex}} \sin(\phi\sqrt{N+1})S^*(N_{\text{ex}} + \nu + N(2\nu + 1))S \sin(\phi\sqrt{N+1}) \\ &\quad + N_{\text{ex}} \cos^2(\phi\sqrt{N+1})(N_{\text{ex}} + \nu + N(2\nu + 1)) \\ &\quad + (\nu + 1)a^*(N_{\text{ex}} + \nu + N(2\nu + 1))a \\ &\quad + \nu a(N_{\text{ex}} + \nu + N(2\nu + 1))a^* \\ &= -(N_{\text{ex}} + \nu + N(2\nu + 1))^2 \\ &\quad + N_{\text{ex}} \sin^2(\phi\sqrt{N+1})(N_{\text{ex}} + \nu + (N + 1)(2\nu + 1)) \\ &\quad + N_{\text{ex}} \cos^2(\phi\sqrt{N+1})(N_{\text{ex}} + \nu + N(2\nu + 1)) \\ &\quad + (\nu + 1)N(N_{\text{ex}} + \nu + (N - 1)(2\nu + 1)) \\ &\quad + \nu(N_{\text{ex}} + \nu + (N + 1)(2\nu + 1))(N + 1) \\ &= 0 \cdot N^2 - (2\nu + 1)N + B, \end{aligned}$$

where  $B$  is a bounded operator. Since  $-2G = (N_{\text{ex}} + \nu) + (2\nu + 1)N$ , we obtain (A1).

Equation (11) follows from Ref. 36, Propositions 3.18; by the definition of a minimal quantum dynamical semigroup given by Fagnola and Chebotarev, we have that  $\mathcal{T}$  is approximated in the pointwise  $w^*$ -topology by the following maps: for every  $X \in B(\mathfrak{h})$ ,  $u, \nu \in D(N)$ ,

$$\begin{aligned} \langle u, \mathcal{T}^{(0)}(t)(X)v \rangle &= \langle e^{tG} u, X e^{tG} v \rangle, \\ \langle u, \mathcal{T}^{(n+1)}(t)(X)v \rangle &= \langle e^{tG} u, X e^{tG} v \rangle + \sum_{i=1}^4 \int_0^t \langle L_i e^{sG} u, \mathcal{T}(t-s)^{(n)}(X) L_i e^{sG} v \rangle ds. \end{aligned} \tag{A2}$$

For every  $n \geq 0$ , Eq. (A2) extends uniquely to a bounded bilinear form represented by a bounded operator  $\mathcal{T}^{(n)}(t)(X)$ . In our case, for every  $u \in \mathfrak{h}$  and  $s > 0$ ,  $e^{sG} u \in D(N^\infty) := \bigcap_{n=1}^\infty D(N^n)$  and  $L_i(D(N^\infty)) \subset D(N^\infty)$ ; hence, the expressions in Eq. (A2) make sense for every  $u, v \in \mathfrak{h}$  (the function that is inside the integral is well defined for every  $u, v \in \mathfrak{h}$  unless when  $s = 0$ , which is a set with zero Lebesgue measure). Moreover, we can get by recursion the following explicit expression for  $\langle u, \mathcal{T}^{(n+1)}(t)(X)v \rangle$ ,  $u, v \in \mathfrak{h}$ :

$$\begin{aligned} \langle u, \mathcal{T}^{(n+1)}(t)(X)v \rangle &= \langle e^{tG} u, X e^{tG} v \rangle \\ &+ \sum_{k=1}^n \sum_{i_1, \dots, i_n=1}^4 \int_{0 \leq t_1 \leq \dots \leq t_n \leq t} f(t, u, v, X; t_1, i_1, \dots, t_n, i_n) dt_1 \dots dt_n, \end{aligned} \tag{A3}$$

where

$$\begin{aligned} f(t, u, v, X; t_1, i_1, \dots, t_n, i_n) &= \langle e^{G(t-t_n)} L_{i_n} \dots e^{G(t_2-t_1)} L_{i_1} e^{G t_1} u, X e^{G(t-t_n)} L_{i_n} \dots e^{G(t_2-t_1)} L_{i_1} e^{G t_1} v \rangle. \end{aligned}$$

Since for every  $t \geq 0$ ,  $\mathcal{T}_t^{(n)}(X)$  converges to  $\mathcal{T}_t(X)$  in the  $w^*$ -topology (monotonically for positive  $X$ ), we conclude.

2. and 3. By restricting  $\mathcal{T}_t$  to diagonal operators in the Fock basis, we obtain the semigroup of a birth–death process [see point (iii) in Sec. III for the details]. We can, then, apply standard arguments (Ref. 44, Sec. V D) to see that  $\rho_{ss}$  is a faithful normal invariant state. This implies that the semigroup is positive recurrent<sup>53</sup> and, consequently, the fixed points  $\mathcal{F}(\mathcal{T})$  are contained in the decoherence-free algebra  $\mathcal{N}(\mathcal{T})$ . Moreover, thanks to Ref. 54, Theorem 3.2,  $\mathcal{N}(\mathcal{T})$  is trivial [i.e.,  $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbf{1}$ ]. Then,  $\mathbf{1}$  is the unique harmonic projection and so  $\rho_{ss}$  is the unique invariant state; finally, the semigroup is ergodic by Ref. 53, Theorem 3.3.  $\square$

*Proof of Proposition 3.*

1. and 2. We fix  $s \in \mathbb{R}$ , and we drop the index  $s$  for the rest of the proof. We can proceed similarly as in the Proof of Proposition 2 and use again (Ref. 36, Theorem 3.22) choosing

$$\begin{aligned} L'_1 &= e^{s/2} L_1, \\ L'_k &= L_k \quad \text{for } k = 2, 3, 4, \\ -2G' &= \sum_{k=1}^4 L_k^* L_k + (e^s - 1) 1_{s>0} \left(1 + \frac{1}{\nu}\right) N_{ex} \mathbf{1} \\ &= -2G + (e^s - 1) 1_{s>0} \left(1 + \frac{1}{\nu}\right) N_{ex} \mathbf{1}. \end{aligned}$$

For the purposes of this proof, it would be enough to define  $-2G'$  as  $-2G + (e^s - 1) 1_{s>0} N_{ex} \mathbf{1}$ ; the extra factor  $\left(1 + \frac{1}{\nu}\right)$  is required in order to keep the same notation in the Proof of Lemma 7.

Note that the following holds:

- $G'$  generates a strongly continuous contraction semigroup on  $\mathfrak{h}$ .
- For all  $u \in D(G') = D(N)$ ,

$$\langle G' u, u \rangle + \langle u, G' u \rangle + \sum_{k=1}^4 \langle L'_k u, L'_k u \rangle \leq 0.$$

Then, the operator  $\mathcal{L}'$ , defined as

$$\mathcal{L}'(X) = \{G', X\} + \sum_{k=1}^4 \mathcal{J}'_k(X) = \mathcal{L}^{(s)}(X) + (1 - e^s) 1_{s>0} \left(1 + \frac{1}{\nu}\right) N_{ex} X,$$

generates a (sub-Markovian) quantum dynamical semigroup  $\mathcal{T}'$ , which is the minimal solution to

$$\langle u, \mathcal{T}'(t)(X)v \rangle = \langle e^{tG'} u, X e^{tG'} v \rangle + \sum_{i=1}^4 \int_0^t \langle L'_i e^{rG'} u, \mathcal{T}'_{t-r}(X) L'_i e^{rG'} v \rangle dr \tag{A4}$$

for every  $u, v \in D(G') = D(G)$  and which is approximated by the sequence of maps

$$\begin{aligned} \langle u, \mathcal{T}^{(0)'}(t)(X)v \rangle &= \langle e^{tG'} u, X e^{tG'} v \rangle, \\ \langle u, \mathcal{T}^{(n+1)'}(t)(X)v \rangle &= \langle e^{tG'} u, X e^{tG'} v \rangle + \sum_{i=1}^4 \int_0^t \langle L'_i e^{rG'} u, \mathcal{T}^{(n)'}_{t-r}(X) L'_i e^{rG'} v \rangle dr \end{aligned}$$

for  $u, v \in \mathfrak{h}$ . Note that when  $s \leq 0$ ,  $G = G'$ ; hence, we can take  $\mathcal{T}_s := \mathcal{T}'$ , while when  $s > 0$ , we can take  $\mathcal{T}_s(t) = e^{t(e^s-1)(1+\frac{1}{v})N_{\text{ex}}} \mathcal{T}'(t)$ . Hence, again, the conclusion immediately follows.

Let us prove the uniqueness of the solution to Eq. (16) and that its  $w^*$ -infinitesimal generator  $(\mathcal{L}_s, D(\mathcal{L}_s))$  is equal to  $(\mathcal{L} + (e^s - 1)\mathcal{J}_1, D(\mathcal{L}))$ . Let us consider  $\tilde{\mathcal{T}}$  the semigroup generated by  $(\mathcal{L}_s + \mathcal{C}, D(\mathcal{L}_s))$  (see Theorem 12 in Appendix B), where  $\mathcal{C}$  is the bounded completely positive linear map defined as

$$\mathcal{C}(X) = (1 - e^s)\mathcal{J}_1(X) + \mathbf{1}_{s>0}N_{\text{ex}}(e^s - 1)X, \quad X \in B(\mathfrak{h}). \tag{A5}$$

Because of Corollary 13 in Appendix B,  $\tilde{\mathcal{T}}$  is again a  $w^*$ -continuous quantum dynamical semigroup. Note that Eq. (B1) implies

$$\tilde{\mathcal{T}}(t)(X) = \tilde{\mathcal{T}}_s(t)(X) + \int_0^t \tilde{\mathcal{T}}(t-r)\mathcal{C}\tilde{\mathcal{T}}(r)(X)dr, \tag{A6}$$

and differentiating Eq. (A6) multiplied by  $e^{tN_{\text{ex}}(1-e^s)\mathbf{1}_{\{s>0\}}}$ , we obtain that  $e^{tN_{\text{ex}}(1-e^s)\mathbf{1}_{\{s>0\}}}\tilde{\mathcal{T}}$  solves for

$$\frac{d}{dt} \langle u, e^{tN_{\text{ex}}(1-e^s)\mathbf{1}_{\{s>0\}}}\tilde{\mathcal{T}}(t)(X)v \rangle = \mathcal{L}_*(|v\rangle\langle u|)(e^{tN_{\text{ex}}(1-e^s)\mathbf{1}_{\{s>0\}}}\tilde{\mathcal{T}}(t)(X)) \tag{A7}$$

for  $u, v \in D(G)$ . Since Eq. (A7) admits a unique  $w^*$ -continuous positive solution (see Ref. 36, Corollary 3.23),  $e^{tN_{\text{ex}}(1-e^s)\mathbf{1}_{\{s>0\}}}\tilde{\mathcal{T}}$  must coincide with  $\mathcal{T}$  and so do their  $w^*$ -infinitesimal generators, which are, respectively,  $(\mathcal{L}_s + (1 - e^s)\mathcal{J}_1, D(\mathcal{L}_s))$  and  $(\mathcal{L}, D(\mathcal{L}))$ . Hence, we get that  $(\mathcal{L}_s, D(\mathcal{L}_s))$  is equal to  $(\mathcal{L} + (e^s - 1)\mathcal{J}_1, D(\mathcal{L}))$  and the solution of Eq. (16) is, indeed, unique.

- By the previous arguments, we have, following once again the same line of the Proof of Proposition 2, the integral representation for the semigroup  $\mathcal{T}$ . When  $s > 0$ , we just have to introduce the correcting multiplicative term.  $\square$

## APPENDIX B: ADDITIONAL RESULTS

**Theorem 12** (Theorem 3.1.33, p. 191, Ref. 37). Let  $\mathcal{S}$  be the generator of a  $\sigma$ -continuous semigroup  $(\mathcal{P}(t))_{t \geq 0}$ , with  $\sigma$  equal to either the weak or the weak\* topology. If  $\mathcal{C}$  is a bounded and  $\sigma - \sigma$ -continuous, then  $(\mathcal{S} + \mathcal{C})$  generates a  $\sigma$ -continuous semigroup  $\mathcal{P}^{(\mathcal{S} + \mathcal{C})}$  of bounded operators, and for every  $t \geq 0$ ,  $X \in B(\mathfrak{h})$ ,

$$\begin{aligned} \mathcal{P}^{(\mathcal{S} + \mathcal{C})}(t)(X) &= \mathcal{P}(t)(x) \\ &+ \sum_{k \geq 1} \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} \mathcal{P}(t_1)\mathcal{C}\mathcal{P}(t_2 - t_1)\mathcal{C} \dots \mathcal{P}(t_k - t_{k-1})\mathcal{C}\mathcal{P}(t - t_k)(X)dt_1 \dots dt_k. \end{aligned} \tag{B1}$$

The integrals define a series of bounded operators that converges in norm; the integrals are defined in the norm topology when  $\sigma = w$  and in the weak\* topology when  $\sigma = w^*$ .

*Corollary 13.* In the conditions of the previous theorem, if we additionally suppose that both  $\mathcal{C}$  and the maps  $\mathcal{P}(t)$  are completely positive, then also the perturbed semigroup  $\mathcal{P}^{(\mathcal{S} + \mathcal{C})}$  is completely positive.

*Proof.* Recall that the composition of two completely positive (c.p.) maps is still c.p. Then, all the integrands in the integral form before are c.p. Now, remember another equivalent characterization of c.p.:  $\Phi$  on  $B(\mathfrak{h})$  is c.p. iff for any  $n \in \mathbb{N}$  and for any  $X_1, \dots, X_n$  and  $Y_1, \dots, Y_n$  bounded operators,

$$\sum_{i, j=1}^n X_i^* \Phi(Y_i^* Y_j) X_j \geq 0.$$

Then, it is immediate to see that

$$\sum_{i, j=1}^n X_i^* \mathcal{P}^{(S+C)}(t)(Y_i^* Y_j) X_j = \sum_{i, j=1}^n X_i^* \mathcal{P}(t)(Y_i^* Y_j) X_j + \sum_{k \geq 1} \int_{0 \leq t_1 \leq \dots \leq t_k \leq t} \sum_{j=1}^n X_i^* \mathcal{P}(t_1) \mathcal{C}\mathcal{P}(t_2 - t_1) \dots \mathcal{C}\mathcal{P}(t - t_k)(Y_i^* Y_j) X_j dt_1 \dots dt_k \geq 0.$$

□

**Proposition 14.** Let  $B$  be a Banach space and  $(S(t))_{t \geq 0}$  be a strongly continuous semigroup with generator  $[A, D(A)]$ . If  $\mathcal{M} \subset B$  is such that

1.  $\mathcal{M}$  is dense in  $B$ ,
2.  $\mathcal{M}$  is a set of analytic vectors for  $A$ , that is,  $\mathcal{M} \subset \cap_{n \geq 0} D(A^n)$ , and for every  $X \in \mathcal{M}$ ,

$$\sum_{n=0}^{+\infty} \frac{z^n}{n!} \|A^n(X)\|$$

has a positive radius of convergence, and

3.  $A(\mathcal{M}) \subset \mathcal{M}$ ,

then  $\mathcal{M}$  is a core for  $A$ .

*Proof.* By the definition of core, we need to prove that  $\mathcal{M}$  is dense in  $D(A)$  with respect to the graph norm  $\|X\|_A := \|X\| + \|A(X)\|$ . Fix  $X \in D(A)$ ,  $\epsilon > 0$ . We shall consider successive approximations of  $X$ .

**Step 1.**

$$\lim_{t \rightarrow 0} \left\| X - \frac{1}{t} \int_0^t S(u)(X) du \right\|_A = 0.$$

Note that due to the strong continuity of  $S$ ,

$$\lim_{t \rightarrow 0} \left\| X - \frac{1}{t} \int_0^t S(u)(X) du \right\| = 0$$

is trivial, and note that

$$\lim_{t \rightarrow 0} \left\| A(X) - \frac{1}{t} A \int_0^t S(u)(X) du \right\| = \lim_{t \rightarrow 0} \left\| A(X) - \frac{S(t)(X) - X}{t} \right\| = 0$$

follows from the definition of infinitesimal generator. Therefore, there exists  $\tilde{t} > 0$  such that

$$\left\| X - \frac{1}{\tilde{t}} \int_0^{\tilde{t}} S(u)(X) du \right\|_A \leq \epsilon.$$

**Step 2.** There exists  $Y \in \mathcal{M}$  such that

$$\left\| \frac{1}{\tilde{t}} \int_0^{\tilde{t}} S(u)(X) du - \frac{1}{\tilde{t}} \int_0^{\tilde{t}} S(u)(Y) du \right\|_A < \epsilon.$$

Indeed,  $\mathcal{M}$  is dense in  $B$ ; hence, we can find  $Y \in \mathcal{M}$  such that  $\|X - Y\| < \epsilon \tilde{t}/4$ ; we have

$$\left\| \frac{1}{\tilde{t}} \int_0^{\tilde{t}} S(u)(X) du - \frac{1}{\tilde{t}} \int_0^{\tilde{t}} S(u)(Y) du \right\| \leq \frac{1}{\tilde{t}} \int_0^{\tilde{t}} \|S(u)(X - Y)\| du \leq \epsilon/2$$

and

$$\left\| \frac{1}{\tilde{t}} A \int_0^{\tilde{t}} S(u)(X) du - \frac{1}{\tilde{t}} A \int_0^{\tilde{t}} S(u)(Y) du \right\| = \frac{1}{\tilde{t}} \|S_{\tilde{t}}(X - Y) + (X - Y)\| \leq \epsilon/2,$$

and summing up, we conclude.

**Step 3.** If we assume that  $\bar{t}$  is small enough, there exists  $N > 0$  such that

$$\left\| \frac{1}{\bar{t}} \int_0^{\bar{t}} S(u)(Y) du - \frac{1}{\bar{t}} \int_0^{\bar{t}} \sum_{k=0}^N \frac{u^k A^k(Y)}{k!} du \right\| < \epsilon/2$$

and, since  $A(Y) \in \mathcal{M}$  too,

$$\left\| \frac{1}{\bar{t}} \int_0^{\bar{t}} S(u)(A(Y)) du - \frac{1}{\bar{t}} \int_0^{\bar{t}} \sum_{k=0}^N \frac{u^k A^k(A(Y))}{k!} du \right\| < \epsilon/2.$$

Hence,

$$\left\| \frac{1}{\bar{t}} \int_0^{\bar{t}} S(u)(Y) du - \frac{1}{\bar{t}} \int_0^{\bar{t}} \sum_{k=0}^N \frac{u^k A^k(Y)}{k!} du \right\|_A < \epsilon/2,$$

and note that

$$\frac{1}{\bar{t}} \int_0^{\bar{t}} \sum_{k=0}^N \frac{u^k A^k(Y)}{k!} du = \sum_{k=0}^N \left( \frac{1}{\bar{t}} \int_0^{\bar{t}} \frac{u^k}{k!} du \right) A^k(Y) \in \mathcal{M}.$$

Hence, we have shown that for every  $X \in D(A)$  and for every  $\epsilon > 0$ , there exists an element in  $\mathcal{M}$ , which is far from  $X$  less than  $4\epsilon$  with respect to  $\|\cdot\|_A$ , and we are done.  $\square$

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