Adaptive Multiple-Surface Sliding Mode Control of Nonholonomic Systems with Matched and Unmatched Uncertainties

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***Abstract*—The problem of stabilizing a class of nonholonomic systems in chained form affected by both matched and unmatched uncertainties is addressed in this paper. The proposed design methodology is based on a discontinuous transformation of the perturbed nonholonomic system to which an adaptive multiple-surface sliding mode technique is applied. The generation of a sliding mode allows to eliminate the effect of matched uncertainties, while a suitable function approximation technique enables to deal with the residual uncertainties, which are unmatched. The control problem is solved by choosing a particular sliding manifold upon which a second order sliding mode is enforced via a continuous control with discontinuous derivative. A positive feature of the present proposal, apart from the fact of being capable of dealing with the presence of both matched and unmatched uncertainties, is that no knowledge of the bounds of the unmatched uncertainty terms is required. Moreover, the fact of producing a continuous control makes the proposed approach particularly appropriate in nonholonomic applications, such as those of mechanical nature.**

***Index Terms*—Higher order sliding mode control, multiple-surface sliding control, adaptive control, function approximation, nonholonomic systems.**

I. Introduction

The problem of controlling and stabilizing nonholonomic dynamic systems has been receiving considerable attention since the nineties (see for instance [1]–[5], among others). This particular class of nonlinear systems is encountered in modeling finite dimensional mechanical systems where non integrable constraints are imposed on the motion, as it happens in many wheeled mobile robots or vehicles.

The main problem in controlling this class of systems is related to the fact that nonholonomic systems do not satisfy Brockett's necessary smooth feedback stabilization condition [6] as shown in [7]. To overcome this problem, several nonlinear approaches have been proposed in the literature. Most of them are based on a discontinuous transformation of the system states and on a back-stepping based design procedure (see, for instance [8]–[14] and the references therein cited).

The control problem is further complicated whenever uncertainties of various nature affect the nonholonomic system due to modeling uncertainties or external disturbances. Such uncertainties can determine a parameter drift, causing an overall degradation of the controlled system performance or even instability. For this reason, the design of control schemes for practical control of real world nonholonomic processes requires the adoption of robust control methodologies. Apart from other types of robust control solutions, sliding mode control schemes have been proposed to address the problem of controlling nonholonomic systems in presence of matched uncertainties [15], [16]. Following the approach developed in [17], the main idea of these approaches is to couple the back-stepping based procedure design with sliding mode control so as to attain a controlled system invariant with respect to the matched uncertainties affecting the nonholonomic dynamics [18]. Yet, even designing a controller capable of suppressing the effect of matched uncertainties, in field implementations the controller has still to face residual uncertainties that can be absolutely deleterious for obtaining the desired closed-loop performance. Possible solutions to this issue have been discussed in [19], where a suitable modelling in presence of skidding and slipping effects is presented for the deployment of various control design techniques. More recently, in [20], the same problem is formulated and addressed via a modified first-order sliding mode controller, while in [21], a robust finite-time stabilization controller is proposed for an arbitrary-order nonholonomic system in chained form.

A. Contribution with respect to the state of the art

In this paper we address the more complicated problem of stabilizing a class of nonholonomic systems affected by both matched and unmatched uncertainties. The presence of unmatched uncertainties is particularly critical for any sliding mode controller. To circumvent the difficulty, in this paper, we propose a controller, which can be classified as an adaptive multiple-surface sliding mode controller, by relying on a suitable function approximation approach.

Function approximation based adaptive multiple-surface sliding mode controllers have been introduced in [22]–[24] to deal with nonlinear systems. In our case, with respect to previous papers, the problem is further complicated by the complexity of the system model necessary to capture the nonholonomic nature of the considered class of processes, where matched and unmatched perturbations are differently categorized also with respect to other works on the same topic as [20] and [21].

The function approximation technique here adopted is used to transform the uncertain terms into a finite combination of orthonormal basis functions in analogy with [25]. Since the coefficients of the approximation series are time-invariant, their update laws are derived relying on a Lyapunov approach with the aim of guaranteeing the closed-loop stability of the overall controlled system. As a novelty with respect to other

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proposals appeared in the literature to deal with nonholonomic uncertain systems (see for instance, [10], [11], [26]–[29]), in this paper it is not assumed that the functions of the system states that constitutes the bounds on uncertain terms are a priori known, while only the knowledge of the bound of some terms which depends in an aggregate way from the uncertainties is required. Then, another novelty in the present proposal is that the control signal is designed so that a second order sliding mode [30]–[32] is enforced. This implies that the discontinuity necessary to produce the sliding mode is confined to the control vector derivative. As a result, while the control vector derivative is a discontinuous signal, the actual control is continuous, which mitigates the problems that may arise when a conventional discontinuous sliding mode control law is applied to a real plant [33], [34]. This can be a fundamental advantage for nonoholonomic systems which often are processes of mechanical nature, and, as such, very sensitive to control induced vibrations.

B. Outline of the paper

This paper is organized as follows. The considered control problem is formulated in Section II, where the control law for the component of index zero of the control signal, and a discontinuous state scaling are introduced. The adaptive multiple-surface sliding mode design procedure is described in Section III. The proposed second order sliding mode control law to complete the design of the control signal is dealt with in Section IV. Section V explores the particular yet relevant case when the initial condition of the first state is zero. The stability of the overall closed-loop control system is proved in Section VI to complete the theoretical discussion. Finally, simulation results and some final comments are reported in Section VII and VIII to conclude the paper.

II. Problem Statement

Consider the following class of maximally nonholonomic [2] systems captured by the perturbed chain model given by

$$\begin{cases} \dot{x}_0 = d_0 u_0 + x_0 f_0(x_0) \\ \dot{x}_i = x_{i+1} u_0 + \delta_i(x_0, \bar{x}_i, u_0), \quad 1 \le i \le n-1 \\ \dot{x}_n = d_n u_1 + \delta_n(x_0, x, u_0) \ , \end{cases} \tag{1}$$

where $[x_0, x^\top]^\top \in \mathbb{R}^{n+1}$ are the system states, with $x_0 \in \mathbb{R}$, $x = [x_1, \dots, x_n]^\top \in \mathbb{R}^n$, and u_0, u_1 are scalar control variables. Moreover, $\bar{x}_i \triangleq [x_1, \dots, x_i]^\top$, $f_0(x_0)$ and $\delta_i(x_0, \bar{x}_i, u_0)$ are unknown functions which represent the possible modeling errors and parametric uncertainties affecting the system, and d_0 and d_n are the unknown control gains. Note that $\delta_i(x_0, \bar{x}_i, u_0)$ can also include uncertain drift terms or parametric uncertainties. As for the uncertain terms $f_0(x_0)$ and $\delta_i(x_0, \bar{x}_i, u_0)$, we assume that a known constant c_0, and unknown smooth nonnegative functions $\phi_j(x_0, \bar{x}_i, u_0)$ exist such that

$$|f_0(x_0)| \le c_0 \tag{2}$$

$$\delta_i(x_0, \bar{x}_i, u_0) \triangleq \sum_{j=1}^{i} x_j \phi_j(x_0, \bar{x}_i, u_0), \quad 0 \le i \le n \ . \tag{3}$$

Assumption (3) implies that the uncertainties $\delta_i(x_0, \bar{x}_i, u_0)$ satisfy a triangularity structure requirement. Note that this assumption is a quite common assumption in the framework of robust and adaptive nonlinear control [8]. As a consequence of (3), the origin is a possible equilibrium point of the considered system (1). It is important to observe that this assumption is significantly less stringent than requiring the knowledge of a function of the state bounding the uncertainty terms as usually done in the classical nonholonomic literature. Moreover, in this paper, we assume that the control gains d_0 and d_n are unknown but bounded as

$$0 < \bar{d}_0 \le d_0 \tag{4}$$

$$0 < d_{n1} \le d_n \le d_{n2} \ , \tag{5}$$

where the known bounds $\bar{d}_0$, d_{n1} and d_{n2} can be retrieved for instance from the physical characteristics of the plant. Taking into account the foregoing problem formulation, the control objective is to design the control laws u_0 and u_1 appearing in (1) such that $[x_0, x^\top]^\top$, as $t \to \infty$, converges to a small vicinity of the equilibrium point, which will be formally defined in the sequel of the paper relying on the concept of Input-to-State Stability [35], and all the other signals in the closed-loop system are bounded. Note that the triangular structure of system (1) allows us to design the control inputs u_0 and u_1 in two separate steps. The control input u_0 is designed so as to globally asymptotically stabilize the x_0-subsystem described by the first equation of (1), while the control input u_1 takes into account the x-subsystem given by the remaining equations in (1).

A. The x_0-subsystem

The case $x_0(t_0) \neq 0$ is now considered. The case when $x_0(t_0) = 0$ deserves a special treatment, and will be dealt with in Section V. When $x_0(t_0) \neq 0$, the following theorem can be proved.

Theorem 1. *Consider the chained form uncertain system* (1)*. Then, for any initial condition $x_0(t_0) \neq 0$, the control law u_0 given by*

$$u_0(x_0) = x_0 g_0 \tag{6}$$

with

$$g_0 = -\frac{c_0 + k_0}{\bar{d}_0} \ , \tag{7}$$

where $k_0 > 0$ is a design parameter, can globally asymptotically regulate the state x_0 to zero, i.e., $\lim_{t\to\infty} x_0(t) = 0$. Moreover, since $x_0(t_0) \neq 0$ is assumed, u_0 ensures that x_0 does not cross zero $\forall\, t \ge t_0$.

Proof. Consider the Lyapunov function candidate

$$V_0 = \frac{1}{2} x_0^2 \ . \tag{8}$$

The first time derivative of (8) is given by

$$\begin{aligned}\dot{V}_0 &= x_0(d_0 g_0 x_0 + x_0 f_0(x_0)) \\ &\leq x_0\left[\frac{d_0}{\bar{d}_0}(-c_0 - k_0)x_0 + c_0 x_0\right] \\ &= -\frac{d_0}{\bar{d}_0}k_0 x_0^2 - \frac{d_0}{\bar{d}_0}c_0 x_0^2 + c_0 x_0^2 \\ &\leq -k_0 x_0^2 \ , \end{aligned} \tag{9}$$

then one can conclude that $x_0 \to 0$ as $t \to \infty$. Applying the control law (6) to system (1), the solution $x_0(t)$ of the closed-loop system is given by

$$x_0(t) = x_0(t_0)\mathrm{e}^{-\int_{t_0}^{t}\left((k_0+c_0)\frac{d_0}{\bar{d}_0} - f_0(\tau)\right)\mathrm{d}\tau} \ . \tag{10}$$

Thus, for any initial instant $t_0 \geq 0$, and any initial condition $x_0(t_0) \neq 0$, u_0 ensures that x_0 does not cross zero $\forall\, t \geq t_0$. □

B. Discontinuous state scaling

As previously proved, the control law (6) can globally asymptotically regulate the state x_0 to zero. However, in doing so, the control u_0 will converge to zero as $t \to \infty$. This causes a serious problem since, in the limiting case, when $u_0 = 0$, the x-subsystem is uncontrollable via the control input u_1. As in [10], [11], to overcome the loss of controllability of the x-subsystem in the limiting case, the following discontinuous state scaling transformation is performed [9], that is

$$z_i \triangleq \frac{x_i}{x_0^{n-i}}, \quad 1 \leq i \leq n \ . \tag{11}$$

The discontinuous state coordinate transformation (11) possesses the property of increasing the resolution around a given point [36] so that x_0 cannot converge to zero before x_i, $i = 1, \ldots, n$. By applying the state transformation (11) to system (1), it yields

$$\begin{aligned}\dot{z}_i &= \frac{\dot{x}_i}{x_0^{n-i}} - (n-i)\frac{\dot{x}_0 x_i}{x_0^{n-i+1}} \\ &= \frac{u_0 x_{i+1} + \delta_i}{x_0^{n-i}} - (n-i)\frac{x_i(d_0 u_0 + x_0 f_0)}{x_0^{n-i+1}} \\ &= g_0 z_{i+1} + \Delta_i(x_0, \bar{z}_i) \end{aligned} \tag{12}$$

where

$$\Delta_i(x_0, \bar{z}_i) = \frac{\delta_i(x_0, \bar{x}_i, u_0)}{x_0^{n-i}} - (n-i)(d_0 g_0 + f_0(x_0))z_i \ . \tag{13}$$

Then, the resulting z-subsystem is given by

$$\begin{cases}\dot{z}_i = g_0 z_{i+1} + \Delta_i(x_0, \bar{z}_i), \quad 1 \leq i \leq n-1 \\ \dot{z}_n = d_n u_1 + \Delta_n(x_0, z)\end{cases} , \tag{14}$$

where $\bar{z}_i \triangleq [z_1, \ldots, z_i]^\top$. In this paper we assume that Δ_i, $i = 1, \ldots, n$ are unknown functions satisfying the Dirichlet conditions [37]. To meet such conditions the functions must be absolutely integrable over a period, with bounded variation in any given bounded interval, and they must have at most a finite number of discontinuities in any given bounded interval, and the discontinuities cannot be infinite. This class of functions is rather broad and it includes all the uncertain terms usually considered in classical sliding mode theory. The z-subsystem formulation in (14) has the advantage of making the matched and unmatched uncertainty terms appear explicitly, thus being instrumental for the design of the proposed control approach. Note that, in the following, the time dependence of all the variables will be omitted for the sake of simplicity.

III. The Adaptive Multiple-Surface Sliding Procedure

Most of the control schemes appeared in the literature capable of stabilizing an uncertain nonholonomic system are based on the back-stepping procedure (see for instance [8], [10], [11], [16], [17] and the references therein). However, the back-stepping design procedure cannot be applied to the z-subsystem (14) due to the time-variant nature of the uncertainties. Moreover, since the bounds of the uncertainty terms $\Delta_i(x_0, \bar{z}_i)$ are unknown, even traditional sliding mode controllers [18] and multiple-surface sliding controllers [24] cannot be designed. To deal with the particularly hard kind of uncertainty considered in this paper, we rely on the function approximation based adaptive multiple-surface sliding control approach proposed in [22]. The function approximation technique is based on the fact that if a piecewise continuous real-valued function $h(t)$ satisfies the Dirichlet conditions, then it can be transformed into the Fourier series within a time interval $[0; T_\mathrm{s}]$ as

$$h(t) = a_0 + \sum_{j=1}^{\infty}\left(a_j \cos(v_j t) + \beta_j \sin(v_j t)\right) \ , \tag{15}$$

where $v_j = \frac{2j\pi}{T_\mathrm{s}}$ are the frequencies of the sinusoidal function. Equation (15) can be rewritten as

$$h(t) = w^\top b(t) + \epsilon(t) \tag{16}$$

where

$$w^\top \triangleq [a_0, a_1, \beta_1, \ldots, a_N, \beta_N] \tag{17a}$$

$$b^\top(t) \triangleq [1, \cos(v_1 t), \sin(v_1 t), \ldots, \cos(v_N t), \sin(v_N t)] \tag{17b}$$

$$\epsilon(t) \triangleq \sum_{j=N+1}^{\infty}\left(a_j \cos(v_j t) + \beta_j \sin(v_j t)\right) \ . \tag{17c}$$

Therefore, if N is chosen large enough, function $h(t)$ can be approximated as

$$\hat{h}(t) = \hat{w}^\top b(t), \tag{18}$$

where $\hat{w}$ is an estimate of w.

In this paper, (18) is used to represent the unmatched uncertainties affecting system (14). Since the vector w of the coefficients of the approximation series is time invariant, the update laws for determining the estimate $\hat{w}$ can be derived relying on a standard Lyapunov approach so as to ensure the closed-loop stability. The control design procedure can be subdivided into several steps:

Step 1: With reference to system (14), the following quantities are defined:

$$s_1 \triangleq z_1 \tag{19}$$

$$s_2 \triangleq z_2 - \alpha_1 \ . \tag{20}$$

By differentiating (19), it yields

$$\dot{s}_1 = g_0 s_2 + g_0 \alpha_1 + \bar{\Delta}_1, \tag{21}$$

where $\bar{\Delta}_1$ is the function approximation of Δ_1 according to the technique introduced in [25], and it can be represented as

$$\bar{\Delta}_1 = w_1^\top b_1 + \epsilon_1 \tag{22}$$

with $w_1 \in \mathbb{R}^{N_1}$ being a weighting vector, $b_1 \in \mathbb{R}^{N_1}$ being a vector of orthonormal basis, $\epsilon_1 \in \mathbb{R}$ representing the approximation error, and N_1 being the number of basis used in the approximation.

Let $\bar{\Delta}_1 = \Delta_1$ and $\hat{\bar{\Delta}}_1 = \hat{w}_1^\top b_1$, with $\hat{w}_1 \in \mathbb{R}^{N_1}$ being a suitable estimate of w_1 such that

$$\dot{\hat{w}}_1 = Q_1 b_1 s_1 \ , \tag{23}$$

where $Q_1 = Q_1^\top \succ 0$. Consider now the Lyapunov function candidate

$$V_1 = \frac{1}{2} s_1^2 + \frac{1}{2} \tilde{w}_1^\top Q_1^{-1} \tilde{w}_1 \ , \tag{24}$$

where $\tilde{w}_1 = w_1 - \hat{w}_1$. By differentiating (24), it yields

$$\begin{aligned} \dot{V}_1 &= s_1 \left(g_0 s_2 + g_0 \alpha_1 + \bar{\Delta}_1 \right) - \tilde{w}_1^\top Q_1^{-1} \dot{\hat{w}}_1 \\ &= s_1 \left(g_0 s_2 + g_0 \alpha_1 + \bar{\Delta}_1 \right) - w_1^\top b_1 s_1 + \hat{w}_1^\top b_1 s_1 \ . \end{aligned} \tag{25}$$

Choosing the virtual control α_1, that is

$$\alpha_1 \triangleq \frac{1}{g_0} \left(-k_1 s_1 - \hat{\bar{\Delta}}_1 \right) \ , \tag{26}$$

where $k_1 > 0$ is a positive parameter design, one has that

$$\dot{V}_1 = g_0 s_1 s_2 - k_1 s_1^2 + \epsilon_1 s_1 \ . \tag{27}$$

Step $i = 2, \dots, n-1$*:* Introduce the quantity

$$s_i \triangleq z_i - \alpha_{i-1} \tag{28}$$

where the virtual control α_{i-1} will be defined later and is such that

$$\dot{\alpha}_{i-1} = \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_k} g_0 z_{k+1} + \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_k} \Delta_k - \frac{\partial \alpha_{i-1}}{\partial \hat{\bar{\Delta}}_{i-1}} \dot{\hat{\bar{\Delta}}}_{i-1} \ . \tag{29}$$

Posing the lumped uncertainty term $\overline{\Delta}_i$ equal to

$$\overline{\Delta}_i = \Delta_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_k} \Delta_k + \frac{\partial \alpha_{i-1}}{\partial \hat{\bar{\Delta}}_{i-1}} \dot{\hat{\bar{\Delta}}}_{i-1} \ , \tag{30}$$

and representing this term as

$$\overline{\Delta}_i = w_i^\top b_i + \epsilon_i \ , \tag{31}$$

from (28), it yields

$$\dot{s}_i = g_0 s_{i+1} + g_0 \alpha_i + \Delta_i - \dot{\alpha}_{i-1} \tag{32}$$

$$= g_0 s_{i+1} + g_0 \alpha_i + \bar{\Delta}_i - \sum_{k=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial z_k} g_0 z_{k+1} \ , \tag{33}$$

where $w_i \in \mathbb{R}^{N_i}$ is a weighting vector, $b_i \in \mathbb{R}^{N_i}$ is a vector of orthonormal basis, and $\epsilon_i \in \mathbb{R}$ is the approximation error, N_i being the number of basis used in the approximation. Consider now the Lyapunov function candidate

$$V_i = V_{i-1} + \frac{1}{2} s_i^2 + \frac{1}{2} \tilde{w}_i^\top Q_i^{-1} \tilde{w}_i \ , \tag{34}$$

where $Q_i = Q_i^\top \succ 0$, yielding

$$\begin{aligned} \dot{V}_i &= \dot{V}_{i-1} + s_i \dot{s}_i - \tilde{w}_i^\top Q_i^{-1} \dot{\hat{w}}_i \\ &= -k_1 s_1 + g_0 s_1 s_2 + \epsilon_1 s_1 \\ &\quad + \sum_{j=2}^{i} s_j \left(g_0 s_{j+1} + g_0 \alpha_j + \bar{\Delta}_j - \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial z_k} g_0 z_{k+1} \right) \\ &\quad - \sum_{j=2}^{i} w_j^\top b_j s_j + \sum_{j=2}^{i} \hat{w}_j^\top b_j s_j \ , \end{aligned} \tag{35}$$

where, analogously to (23), it yields

$$\dot{\hat{w}}_j = Q_j b_j s_j \ . \tag{36}$$

By selecting the virtual control α_j as

$$\alpha_j \triangleq \frac{1}{g_0} \left(-k_j s_j - g_0 s_{j-1} - \hat{\bar{\Delta}}_j + \sum_{k=1}^{j-1} \frac{\partial \alpha_{j-1}}{\partial z_k} g_0 z_{k+1} \right) \ , \tag{37}$$

with $k_j > 0$, substituting in (35), exploiting (31) and posing $\hat{\bar{\Delta}}_j = \hat{w}_j^\top b_j$, one has

$$\begin{aligned} \dot{V}_i &= -\sum_{j=1}^{i} k_j s_j^2 + g_0 s_i s_{i+1} + \epsilon_1 s_1 + \sum_{j=2}^{i} \bar{\Delta}_j - \sum_{j=2}^{i} w_j^\top b_j s_j \\ &= -\sum_{j=1}^{i} k_j s_j^2 + g_0 s_i s_{i+1} + \sum_{j=1}^{i} \epsilon_j s_j \ . \end{aligned} \tag{38}$$

Then, by relying on the concept of Input-to-State-Stability (ISS) [35], the following result can be proved.

Theorem 2. *The dynamic system*

$$\begin{cases} \dot{s}_1 = \dot{z}_1 \\ \dot{s}_i = \dot{z}_i - \dot{\alpha}_{i-1}, \quad 2 \le i \le n-1 \ , \end{cases} \tag{39}$$

where the states s_i*,* $i = 1, \dots, n-1$*, are given by* (19) *and* (28)*,* z_i*,* $i = 1, \dots, n-1$ *are defined in* (14)*,* α_i*,* $i = 1, \dots, n-2$ *as in* (26) *and* (37)*, is ISS with respect to* $\mu \triangleq [0, 0, \dots, g_0 s_n]^\top$ *and* $\epsilon \triangleq [\epsilon_1, \dots, \epsilon_{n-1}]^\top$*, and if both* μ *and* ϵ *go to zero, then* $\lim_{t\to\infty} \|s\| = 0$*, where* $s = [s_1, \dots, s_{n-1}]^\top$*.*

Proof. The Lyapunov function (34) is an ISS Lyapunov function [35]. Indeed, from (39) one has that

$$\begin{aligned} \dot{V}_{n-1} &\le -\underline{k} \|s\|^2 + s^\top \epsilon + s^\top \mu \\ &\le -\underline{k} \|s\|^2 + \|s\| \|\epsilon\| + \|s\| \|\mu\| \ , \end{aligned} \tag{40}$$

where $\underline{k} \triangleq \min_{1 \le j \le n-1} \{k_j\}$. Thus, from (40), it turns out that

$$\forall \, \|s\| \ge \frac{\max\{\|\epsilon\|; \|\mu\|\}}{\sigma \underline{k}} \ , \tag{41}$$

where $\sigma \in (0,1)$, and the following inequality holds

$$\dot{V}_{n-1} \leq -\underline{k}(1-\sigma)\|s\|^2 \ . \tag{42}$$

This implies that there exists a function $\chi(\|s\|, t)$ of class $\mathcal{KL}$ and functions $\rho_\epsilon(\|\epsilon\|)$ and $\rho_\mu(\|\mu\|)$ of class $\mathcal{K}$ (called ISS gain functions) such that, for any initial state $s(t_0)$ one has that

$$\|s(t)\| \leq \chi(\|s(t_0\|, t) + \rho_\epsilon(\|\epsilon\|) + \rho_\mu(\|\mu\|) \ . \tag{43}$$

Hence, if $\|\epsilon\|$ and $\|\mu\|$ are bounded, then $\|s\|$ is bounded [35]. Moreover, if $\epsilon \to 0$ and $\mu \to 0$, for $t \to \infty$, then $s \to 0$. Furthermore, by applying the LaSalle's Invariant Theorem [38], it follows that $\tilde{w}_i$, $1 \leq i \leq n-1$, and, as a consequence, $\hat{w}_i$, $1 \leq i \leq n-1$, are bounded. □

Note that ϵ can be steered to zero by choosing a sufficiently large number of basis functions, while μ can be turned to zero, for instance, by designing a control law u_1 such that s_n is steered to zero in a finite time.

IV. The Control Signal u_1

From Theorem 2 one can observe that, if a sufficiently large number of basis functions are chosen so as to have $\epsilon \approx 0$, it is possible to steer s to zero with a control law u_1 capable of steering s_n to zero. In the present work, a second order sliding mode control law is designed to steer to zero not only s_n but also its first time derivative $\dot{s}_n$, and this is attained in finite time. This implies that a second order sliding mode is generated. Moreover, the design procedure is carried out so that the discontinuity is connected to the control derivative $\dot{u}_1$. As a result, while $\dot{u}_1$ is constructed as a discontinuous signal, guaranteeing the attainment of a second order sliding mode on the sliding manifold $s_n = \dot{s}_n \equiv 0$, the actual control u_1 is continuous, and thus more acceptable, in terms of chattering, in systems of mechanical nature [34].

Step n: From (28), it yields

$$\dot{s}_n = d_n u_1 + \bar{\Delta}_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} g_0 z_{k+1} \ , \tag{44}$$

being the lumped uncertainty term $\overline{\Delta}_n$ equal to

$$\overline{\Delta}_n = \Delta_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} \Delta_k + \frac{\partial \alpha_{n-1}}{\partial \hat{\bar{\Delta}}_{n-1}} \dot{\hat{\bar{\Delta}}}_{n-1} \ . \tag{45}$$

Now, consider the Lyapunov function candidate

$$V_n = \frac{1}{2} s_n^2 + \frac{1}{2} \tilde{w}_n^\top Q_i^{-1} \tilde{w}_n + \frac{1}{2} \gamma_n^{-1} \tilde{d}_n^2 \ , \tag{46}$$

where $Q_n = Q_n^\top \succ 0$, $\gamma_n > 0$, and $\tilde{d}_n = d_n - \hat{d}_n$, with $\hat{d}_n$ being a suitable estimate of d_n. The first time derivative of (46) is

$$\dot{V}_n = s_n \left(d_n u_1 + \bar{\Delta}_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} g_0 z_{k+1} \right) \tag{47}$$

$$- \tilde{w}_n^\top Q_i^{-1} \dot{\hat{w}}_n - \gamma_n^{-1} \tilde{d}_n \dot{\hat{d}}_n \ . \tag{48}$$

Thus, the control signal u_1 can be chosen as

$$u_1 = \bar{u}_1 + \tau_1 \tag{49}$$

with

$$\bar{u}_1 = \frac{1}{\hat{d}_n} \left(-k_n s_n - \hat{\bar{\Delta}}_n + \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} g_0 z_{k+1} \right) \ , \tag{50}$$

where $k_n > 0$ is a design parameter, and τ_1 will be designed later so as to robustly steer s_n to zero in finite time.

By substituting (49) in (46), since $d_n = \tilde{d}_n + \hat{d}_n$, it results

$$\begin{aligned} \dot{V}_n &= s_n(d_n \tau_1 + \epsilon_n) + \tilde{d}_n \bar{u}_1 s_n + \hat{d}_n \bar{u}_1 s_n \\ &\quad + s_n \left(\hat{\bar{\Delta}}_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} g_0 z_{k+1} \right) - \gamma_n^{-1} \tilde{d}_n \dot{\hat{d}}_n \\ &= s_n(d_n \tau_1 + \epsilon_n) - k_n s_n^2 - \gamma_n^{-1} \tilde{d}_n \left(\dot{\hat{d}}_n - \gamma_n \bar{u}_1 s_n \right) \ . \end{aligned} \tag{51}$$

Letting $\pi_n = \gamma_n \bar{u}_1 s_n$, by selecting the initial value for the estimate $d_{n1} \leq \hat{d}_n(t_0) \leq d_{n2}$, and by choosing its update law as follows

$$\dot{\hat{d}}_n = \begin{cases} \pi_n, \text{if } d_{n1} < \hat{d}_n < d_{n2} \\ \pi_n, \text{if } (\hat{d}_n = d_{n1} \cap \pi_n > 0) \cup (\hat{d}_n = d_{n2} \cap \pi_n < 0) \\ 0, \ \ \text{if } (\hat{d}_n = d_{n1} \cap \pi_n \leq 0) \cup (\hat{d}_n = d_{n2} \cap \pi_n \geq 0) \end{cases} \tag{52}$$

it yields

$$\begin{aligned} &\dot{V}_n \leq s_n(d_n \tau_1 + \epsilon_n) - k_n s_n^2 \\ &+ \begin{cases} 0 & \text{if } d_{n1} < \hat{d}_n < d_{n2} \\ 0, & \text{if } (\hat{d}_n = d_{n1} \cap \pi_n > 0) \cup (\hat{d}_n = d_{n2} \cap \pi_n < 0) \\ \gamma_n^{-1} \tilde{d}_n \pi_n, & \text{if } (\hat{d}_n = d_{n1} \cap \pi_n \leq 0) \cup (\hat{d}_n = d_{n2} \cap \pi_n \geq 0). \end{cases} \end{aligned}$$

Since in $\dot{V}_n$ the term which depends on the update law (52) is nonpositive, it follows that

$$\dot{V}_n \leq s_n(d_n \tau_1 + \epsilon_n) - k_n s_n^2 \ . \tag{53}$$

Relying on the concept of ISS, (53) implies that if τ_1 and ϵ_n are bounded, then s_n is also bounded. Moreover, it turns out that $\tilde{w}_n$ and $\tilde{d}_n$, and consequently $\hat{w}_n$ and $\hat{d}_n$ are bounded.

A. The second order sliding mode control

Now the point is to design τ_1 according to the second order sliding mode control technique in order to steer s_n to zero in finite time in presence of uncertainties. To this end, the chosen sliding variable is equal to

$$s_n = z_n - \alpha_{n-1} \ . \tag{54}$$

The first and the second time derivative of (54) are given by

$$\begin{aligned} \dot{s}_n &= \tilde{d}_n \bar{u}_1 + \hat{d}_n \bar{u}_1 + d_n \tau_1 + \bar{\Delta}_n - \sum_{k=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial z_k} g_0 z_{k+1} \\ &= \tilde{d}_n \bar{u}_1 - k_n s_n + \tilde{\bar{\Delta}}_n + d_n \tau_1 \end{aligned} \tag{55}$$

$$\ddot{s}_n = \tilde{d}_n \dot{\bar{u}}_1 - \dot{\hat{d}}_n \bar{u}_1 - k_n \dot{s}_n + \dot{\tilde{\bar{\Delta}}}_n + d_n \dot{\tau}_1 \ , \tag{56}$$

where $\tilde{\bar{\Delta}}_n \triangleq \bar{\Delta}_n - \hat{\bar{\Delta}}_n$, while $\dot{\tau}_1$ can be regarded as an auxiliary control signal. Now, by using the sliding variable and its first time derivative as states of a new dynamical system, i.e., by introducing the auxiliary variables $y_1 = s_n$ and $y_2 = \dot{s}_n$, equations (55) and (56) can be rewritten as

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_1 = \xi + d_n \dot{\tau}_1 \end{cases} . \quad (57)$$

The auxiliary system (57) is a double integrator affected by the uncertainty terms d_n and

$$\xi \triangleq \tilde{d}_n \dot{\bar{u}}_1 - \dot{\hat{d}}_n \bar{u}_1 - k_n \dot{s}_n + \dot{\bar{\Delta}}_n \quad (58)$$

Relying on assumptions (5) and on the previous results, one can observe that the term ξ is uncertain but its components are bounded, i.e.,

$$|\xi| \leq \bar{\xi} \quad (59)$$

where $\bar{\xi} > 0$ is assumed to be a known constant. Note that the quantity y_2 can be viewed as an unmeasurable quantity. Then, the following theorem can be proved.

Theorem 3. *Given the auxiliary system* (57)*, where* ξ*, and* d_n *satisfy* (59) *and* (5)*, respectively, and* y_2 *is not measurable, the auxiliary control signal* $\dot{\tau}_1$ *given by*

$$\dot{\tau}_1(t) \triangleq -U \operatorname{sgn}\left(y_1(t) - \tfrac{1}{2} y_{1\,\max}\right) , \quad (60)$$

where

$$U > \max\left\{ \frac{\bar{\xi}}{d_{n1}}; \frac{4\bar{\xi}}{3d_{n1} - d_{n2}} \right\} , \quad (61)$$

and $y_{1\,\max}$ *is a piece-wise constant function representing the value of the last singular point of* $y_1(t)$ *(i.e., the most recent value* $y_{1\,\max}$ *such that* $\dot{y}_1(t) = 0$*), causes the convergence of the system trajectory to the origin of the auxiliary plane in finite time.*

Proof. The control law (60) can be classified as a suboptimal second order sliding mode control law [30], and by following a theoretical development as that provided in [30] for the general case, it can be proved that the trajectories on the auxiliary plane are connected within limited parabolic arcs which include the origin. As shown in [39], under condition (61), the following relationships hold

$$|y_1| \leq |y_{1\,\max}| \quad (62)$$

$$|y_2| \leq \sqrt{|y_{1\,\max}|} , \quad (63)$$

and the convergence of $y_{1\,\max}$ to zero takes place in finite time. As a consequence, the origin of the plane, i.e., $y_1 = y_2 = 0$, is reached in finite time since y_1 and y_2 are both bounded by $\max\left\{|y_{1\,\max}|; \sqrt{|y_{1\,\max}|}\right\}$. □

V. The case $x_0(t_0) = 0$

As previously mentioned, the case $x_0(t_0) = 0$ is a critical case to cope with separately. Different schemes have been proposed in the literature (see for instance [10], [11], [27] and the references therein) to circumvent the loss of controllability. In this paper, the adaptive switching proposed in [11] is adopted because this approach is rather general and also capable of solving the finite time escape problem for systems with non-Lipschitz nonlinearities. When $x_0(t_0) = 0$, the control signal u_0 is chosen as

$$u_0 = x_0 g_0 + \bar{u}_0 \quad (64)$$

where $\bar{u}_0 \in \mathbb{R}^+$ is a constant. Choosing the Lyapunov function (8), its first time derivative is such that

$$\dot{V}_0 \leq -k_0 x_0^2 + d_0 \bar{u}_0 x_0 , \quad (65)$$

which leads to the boundedness of x_0. Moreover, x_0 does not escape and $x_0(\bar{t}) \neq 0$, $\forall\, \bar{t} > t_0$. Thus, the discontinuous state scaling discussed in Section II-B can be applied. The control law u_0 defined by (64) is applied during time interval $[t_0; \bar{t}]$, and, since $x_0(\bar{t}) \neq 0$, at time $\bar{t}$ we can switch the control inputs u_0 and u_1 to (6) and (49), respectively.

VI. Stability Considerations

In this section, the stability properties of the proposed control scheme are analyzed.

Theorem 4. *Under assumptions* (4) *and* (5)*, the control laws* (6) *and* (49)*, with adaptation laws* (36)*, along with the switching strategy described in Section V, makes the nonholonomic uncertain system* (1) *ISS with respect to the approximation error* $\bar{\epsilon} \triangleq [0, \epsilon_1, \ldots, \epsilon_{n-1}]^\top$*, while keeping the estimated parameters bounded. Moreover, if a sufficiently large number of basis functions are chosen such that* $\epsilon_i \approx 0$*,* $1 \leq i \leq n-1$*, then system* (1) *is globally asymptotically regulated to the origin.*

Proof. To analyze the stability properties of the overall closed-loop system (1) with (6) and (49), consider the Lyapunov function candidate

$$V = V_0 + V_{n-1} = \frac{1}{2} x_0^2 + \sum_{j=1}^{n-1} \frac{1}{2} s_j^2 + \sum_{j=1}^{n-1} \frac{1}{2} \tilde{w}_j^\top Q_j^{-1} \tilde{w}_j . \quad (66)$$

Then, the first time derivative of (66) results in

$$\dot{V} \leq -k_0 x_0^2 - \sum_{j=1}^{n-1} k_j s_j^2 + g_0 s_{n-1} s_n + \sum_{j=1}^{n-1} s_j \epsilon_j . \quad (67)$$

Since s_n is steered to zero in finite time by the control law u_1 as proved by Theorem 4, it yields

$$\dot{V} \leq -k_0 x_0^2 - \sum_{j=1}^{n-1} k_j s_j^2 + \sum_{j=1}^{n-1} s_j \epsilon_j . \quad (68)$$

Now, let $\bar{s} \triangleq [x_0, s_1, \ldots, s_{n-1}]^\top$, then one has that

$$\forall\, \|\bar{s}\| \geq \frac{\|\bar{\epsilon}\|}{\sigma \underline{k}_0} , \quad (69)$$

with $\sigma \in (0, 1)$, and it holds

$$\dot{V} \leq -(1-\sigma) \underline{k}_0 \|\bar{s}\|^2 , \quad (70)$$

where

$$\underline{k}_0 \triangleq \min_{0 \leq j \leq n-1} \{k_j\} . \quad (71)$$

Since (66) is an ISS Lyapunov function, the closed-loop system with state $\bar{s}$ is ISS with respect to ϵ. Moreover, the estimation errors $\tilde{w}_i$, $1 \leq i \leq n-1$, remain bounded. As observed in Section IV, also $\tilde{w}_n$ and $\tilde{d}_n$ are bounded. Since w_i, $1 \leq i \leq n$, and d_n are constant quantities, then, we can conclude that $\hat{w}_i$, $1 \leq i \leq n$, and $\hat{d}_n$ are bounded. Note that, if a sufficient large number of basis functions are chosen so that $\epsilon_i \approx 0$, $1 \le i \le n-1$, then (68) results in

$$\dot{V} \le -k_0 x_0^2 - \sum_{j=1}^{n-1} k_j s_j^2 \,. \tag{72}$$

and $\bar{s} \to 0$ as $t \to \infty$. In this latter case, (28) gives

$$\lim_{t\to\infty} z_i = \alpha_{i-1}, \quad 1 \le i \le n \,, \tag{73}$$

and, from the assumption that $\epsilon_i \approx 0$, it follows that

$$\lim_{t\to\infty} z_i = -\hat{\bar{\Delta}}_{i-1} \approx -\bar{\Delta}_{i-1}, \quad 1 \le i \le n \,. \tag{74}$$

Taking into account (3) and (11), we can conclude that

$$\lim_{t\to\infty} x_i = 0, \quad 1 \le i \le n \,. \tag{75}$$

Thus, the perturbed nonholonomic system (1) is globally asymptotically regulated to the origin. □

VII. Simulation Results

In this section, the proposed control scheme is applied to the parking problem of a wheeled mobile robot of unicycle type affected by parametric uncertainty [40]. The model equations are

$$\begin{cases} \dot{x}_l = d_2 v \cos\theta_l \\ \dot{y}_l = d_2 v \sin\theta_l \\ \dot{\theta}_l = d_0 \omega \end{cases} , \tag{76}$$

where x_l, y_l denote the coordinates of the center of mass on the plane, θ_l denotes the heading angle measured from the $\vec{x}$-axis, v denotes the magnitude of the translational velocity of the center of mass, ω denotes the angular velocity of the robot, and d_0, d_n are unknown positive parameters determined by the radius of the rear wheels and the distance between them.

System (76) can be transformed into the perturbed chain form (1) via the following choice of auxiliary variables

$$\begin{aligned} x_0 &= \theta_l \\ x_1 &= x_l \sin\theta_l - y_l \cos\theta_l \\ x_2 &= x_l \cos\theta_l + y_l \sin\theta_l \\ u_0 &= \omega \\ u_1 &= v \end{aligned} \quad , \tag{77}$$

so that the resulting transformed system is

$$\begin{cases} \dot{x}_0 = d_0 u_0 \\ \dot{x}_1 = x_2 u_0 + \delta_1 \\ \dot{x}_2 = d_2 u_1 + \delta_2 \end{cases} , \tag{78}$$

where $f_0 = 0$, $\delta_1 = (d_0 - 1)x_2 u_0$ and $\delta_2 = -d_0 x_1 u_0$. We assume that the unknown parameters are such that $d_0 \ge \bar{d}_0$ and $d_{21} \le d_2 \le d_{22}$. Letting $t_0 = 0$, the simulation parameters are $[x_0(0), x_1(0), x_2(0)]^\top = [1, 1, 1]^\top$, $k_0 = 1$, $k_1 = 10$, $k_2 = 20$, $U = 10$, $\gamma_2 = 0.1$, $d_0 = d_2 = 2$ (the value of d_0 and d_2 is unknown to the controller), $\bar{d}_0 = 1$, $d_{21} = 1$ and $d_{22} = 5$. All the simulations have been executed using the MATLAB toolbox Simulink, with automatic selection solver, fixed-time step equal to 1×10^{-5} s and simulation window of 3 s. The steady-state value of the estimate of d_2 provided by the estimator (52), with $\hat{d}_2(0) = 3$, is given by $\hat{d}_2 = 2.16$. The time evolution

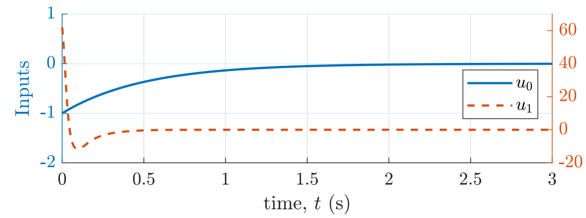

Fig. 1. The time evolution of control signals u_0 and u_1.

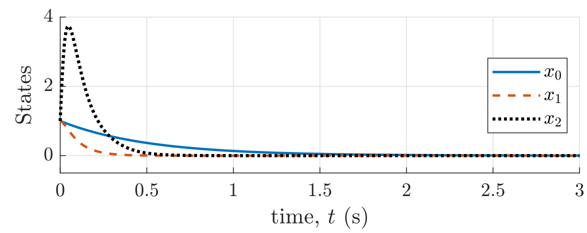

Fig. 2. The time evolution of the transformed system states with initial condition $[x_0(0), x_1(0), x_2(0)]^\top = [1, 1, 1]^\top$.

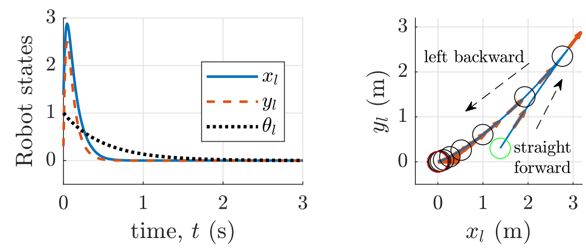

Fig. 3. The time evolution of the robot states (left), and parking motion on the plane (right), with heading angle (orange arrows) and initial condition $[x_l(0), y_l(0), \theta_l(0)]^\top = [1.3818, 0.3012, 1]^\top$ (green circle).

of the control signals u_0 and u_1 is reported in Figure 1. Note that control u_1 is a continuous control signal as previously discussed. From Figure 2 it appears that all the states x_0, x_1, and x_2 converge to zero as expected, while Figure 3 shows the corresponding evolution of the robot states, with initial condition $[x_l(0), y_l(0), \theta_l(0)]^\top = [1.3818, 0.3012, 1]^\top$, and the parking motion on the plane. The time evolution of the sliding variable s_2 is reported in Figure 4. As one can note, the sliding variable s_2 is steered to zero quite rapidly. In Figure 4, also the first time derivative of the sliding variable is shown, which is steered to zero, since a second order sliding mode is enforced. The time evolution of the state of the system (78) starting from the critical initial condition $[x_0(0), x_1(0), x_2(0)]^\top = [0, 1, 1]^\top$ is still satisfactory. The global asymptotic convergence to zero is maintained as can be seen in Figure 5, where $\bar{t} = 0.05$ s, while $\bar{u}_0 = 10$ in the time interval $[0; \bar{t}]$.

VIII. Conclusions

In this paper an adaptive multiple-surface sliding procedure generating second order sliding modes has been proposed for stabilizing a class of nonholonomic systems in chained form affected by matched and unmatched uncertainties. Differently from other proposals appeared in the literature, after the application of the discontinuous state scaling transformation,

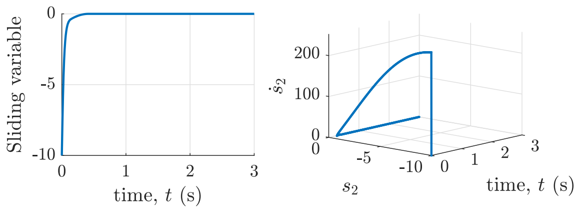

Fig. 4. The sliding variable s_2 (left), and the phase portrait with respect to time of the sliding variable s_2 and its first time derivative $\dot{s}_2$ (right).

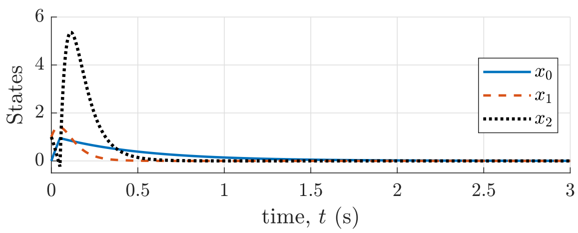

Fig. 5. The time evolution of the system states with initial condition $[x_0(0), x_1(0), x_2(0)]^\top = [0, 1, 1]^\top$.

no knowledge of the bounds of the uncertain terms is required to accomplish the control design. The key idea is to apply a function approximation technique to deal with the unmatched uncertainties, while the matched uncertainties are coped with by the sliding mode controller. By virtue of the second order nature of the generated sliding modes, the overall stabilization problem is solved via a continuous control signal. This fact enables the application of the proposed strategy even to nonholonomic systems, such as the mechanical ones, for which a discontinuous control input may be unacceptable. By applying the proposed control strategy, in spite of the presence of uncertainties, the system states globally asymptotically converge to the origin, while the estimated parameters remain bounded. Simulation results have shown the effectiveness of the proposed control scheme.

Future works will be devoted to the extension of the present proposal to more complex situations, in particular to the case of models characterized by a generic number of inputs.

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