Design of an Easy-to-Implement Fixed-Time Stable Sliding Mode Control

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***Abstract*— This letter introduces a new methodology for the design and tuning of sliding mode controllers with fixed-time stability property for a class of second-order uncertain nonlinear systems. Exploiting the Gauss error function, a novel sliding variable is designed, giving rise to a new control law, whose the main strengths are its ease of implementation and robustness. Indeed, differently from other fixed-time stable techniques in the literature, it only requires the tuning of two design parameters in order to ensure fixed-time convergence, while making the controlled system robust in front of disturbance and uncertainty terms. The properties of the closed-loop systems are theoretically analysed, and the effectiveness of the proposal is shown in simulation on a benchmark example.**

***Index Terms*— Fixed-time stability, sliding mode control, uncertain systems, Gauss error function.**

I. Introduction

SLIDING mode control (SMC) is a powerful adopted approach in different application domains [1], due to its ease of implementation and well-known finite-time convergence property [2], [3]. The latter is indeed significantly convenient in many scenarios, e.g., in the case of electro-mechanical systems that employ robots or positioning of machine tools in manufacturing, even when varying disturbance terms could make the convergence phase of the controlled variable unpredictable. In fact, SMC strategies make the system dynamics insensitive to uncertainties in the sliding mode phase, so that ensuring a bounded convergence time, possibly independent on the initial conditions, is often crucial.

A. Brief overview

Due to the practical impact and the theoretical challenges rising from the finite-time stability properties, this topic has sparked the interest of many researchers in the last decades. Specifically, the majority of finite-time control methodologies rely on homogeneity theory (see, e.g., [4], among many others), with extensions also to the observation problems (e.g., as in [5], [6]). Such a feature has been proved also for high-order sliding mode control algorithms [7], [8], even with state and input constraints [9]–[11], or for specific structure of systems, e.g., [12], to cite a few.

Finite-time convergence is however characterized by a strict dependence on the state initial conditions, which could sometimes prevent a prescribed settling time of the controlled variable towards the desired equilibrium. Therefore, in [13] the new paradigm of fixed-time stability has been originated, and then investigated in many other works in the literature, such as [14]–[19], and also extended to predefined and prescribed-time stability results (see, e.g., [20], [21]). Fixed-time stability indeed guarantees that the settling time is independent of the initial states, thus allowing a predetermined convergence period towards the equilibrium. Nevertheless, in the literature the simplest fixed-time and finite-time sliding variables are non-differentiable, which implies singularity issues [16], [17]. Some solutions to this problem have been recently proposed, e.g., in [18], where variable exponent coefficients in the sliding variable and the controllers are adopted.

B. Novelty with respect to the related literature

Motivated by this open problem, and inspired by [18] and [20], in this letter a novel design procedure to achieve fixed-time stability and robustness in front disturbances is provided for a class of uncertain nonlinear systems.

Differently from [18], the proposal relies on the use of the Gauss error function depending on the system states, so that the settling time is strictly dependent on only the positive control gain, which in turn is subject to the constraint determined by the upper bound of the disturbance terms. Then, considering a class of second-order nonlinear systems, an enhanced fixed-time stable SMC is introduced. More precisely, by exploiting the Gauss error function and the combination of a switching manifold with optimal reaching [20] in a subset of the state space, we prove that the proposed SMC scheme allows fixed-time stabilization of the sliding variable as well as robust fixed-time stabilization of the x-system. Finally, a realistic case-study, relying on a simple inverted pendulum, assesses the effectiveness of the proposal.

It is also worth highlighting that the proposal has a singularity avoidance property, it is time-independent and robust

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against disturbances. Moreover, differently from [16]–[18], the tuning of only two control parameters is required.

C. Outline

The paper is organized as follows. In Section II, some preliminaries on finite- and fixed-time stability are recalled. In Section III, the proposed results on fixed-time stability are discussed, while the novel fixed-time stable SMC is introduced and analysed in Section IV. In Section V, simulation results carried out relying on a single inverted pendulum are illustrated, even in comparison with other methodologies, and some conclusions are finally gathered in Section VI.

Notation

The notation adopted in the paper is mostly standard. Let $\mathbb{R}_{\geq 0}$ be the set of positive real numbers including 0 and $\mathbb{R}_+ := \mathbb{R}_{\geq 0} \backslash \{0\}$. Given a vector $v \in \mathbb{R}^n$, then v' indicates its transpose. A function $\gamma(s) \in \mathcal{PD}$ (positive definite) if $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and $\gamma(0) = 0$, $\gamma(s) > 0$ for all $s > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ (or $\mathcal{K}-$function) if $\gamma \in \mathcal{PD}$ and it is strictly increasing. A scale function $\gamma : \mathbb{R}_{\geq 0} \to [0, 1)$ is of class $\mathcal{K}_1$ if $\gamma \in \mathcal{K}$ and $\lim_{s \to \infty} \gamma(s) = 1$. Then, $\gamma \in \mathcal{DK}_1$ if $\gamma \in \mathcal{K}_1$ and $\gamma \in \mathrm{C}^0$. Finally, let $\mathrm{sign}(s)$ be a function such that $\mathrm{sign}(s) = 1$ if $s > 0$, $\mathrm{sign}(s) \in [-1\,,\,1]$ if $s = 0$, and $\mathrm{sign}(s) = -1$ if $s < 0$.

II. Preliminaries

Before introducing the proposed approach, some definitions about finite- and fixed-time stability are recalled.

Consider the nonlinear system

$$\dot{x}(t) = f(x(t), t), \quad x(0) = x_0, \tag{1}$$

with state vector $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ being a nonlinear (even discontinuous) function with the origin being an equilibrium point. Let $\varphi(t, x)$ be the unique solution of (1) with $\varphi(0, x) = x_0$.

In this letter, we are interested to address the fixed-time stabilization of a class of systems belonging to (1). Therefore, the following definitions will be instrumental in the next sections.

Definition 1 (Global finite-time stability, [2]): The origin is said to be a globally finite-time-stable equilibrium point for system (1) if there exists a function $T : \mathbb{R}^n \backslash \{0\} \to (0, \infty)$, called settling-time function, such that:

(i) For any $x \in \mathbb{R}^n \backslash \{0\}$, $\varphi(t, x)$ is defined on $[0, T(x))$, $\varphi(t, x) \in \mathbb{R}^n \backslash \{0\}$, $\forall\, t \in [0, T(x))$, and $\lim_{t \to T(x)} \|x\| = 0$.

(ii) For any open neighbourhood $\mathcal{X}_\varepsilon$ of 0 there exists $\mathcal{X}_\delta \subset \mathbb{R}^n$ containing the origin such that, $\forall\, x \in \mathcal{X}_\delta \backslash \{0\}$, then $\varphi(t, x) \in \mathcal{X}_\varepsilon$, $\forall\, t \in [0, T(x))$.

Definition 2 (Global fixed-time stability, [13]): The origin is said to be a globally fixed-time-stable equilibrium point for system (1), if it is globally finite-time stable and the settling-time function $T(x)$ is bounded, that is there exists $\bar{T} \in \mathbb{R}_+$ such that, for any $x \in \mathbb{R}^n \backslash \{0\}$, $T(x) \leq \bar{T}$.

III. Fixed-Time Stability Based on Gauss Error Function-Like Coefficient

In this section the proposed fixed-time stability results are shown. For the sake of simplicity, without loss of generality, the scalar case is considered both in the nominal condition (i.e., without disturbance) and in the perturbed one.

More precisely, consider now the class of perturbed systems

$$\dot{x}(t) = a(x(t)) + d(t), \quad x(0) = x_0, \tag{2}$$

where $x \in \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ is a nonlinear (even discontinuous) known drift function, and $d \in \mathbb{R}$ represents the matched parameter uncertainties and/or external disturbances such that the following assumption holds.

$\mathcal{A}_1$: The uncertainty d is such that $|d| < \bar{d}$, where $\bar{d} \in \mathbb{R}_+$ is known.

A. Nominal case

The following theorem proves fixed-time stability for a scalar system belonging to the class of systems (2), when $d = 0$.

Theorem 1: The system

$$\dot{x}(t) = -\bar{U}\,\mathrm{sign}(x(t)), \quad x(0) = x_0, \tag{3}$$

with $x \in \mathbb{R}$, and control gain

$$\bar{U} := U \frac{\sqrt{\pi}}{2} \mathrm{e}^{x^2(t)}, \quad U \in \mathbb{R}_+, \tag{4}$$

is globally fixed-time stable with settling time $T(x_0) \leq \bar{T}$ and

$$\bar{T} = \frac{1}{U}. \tag{5}$$

Proof: To prove fixed-time stability of system (3), according to Definition 2, finite-time stability is a prerequisite. In fact, following the reasoning in [15, §4.2], it can be proved that the origin of system (3) is globally finite-time stable.

Now, we need to prove the existence of a bound $\bar{T}$ for the settling-time function $T(x)$. Consider the following Lyapunov function

$$V(x(t)) := |x(t)|, \tag{6}$$

whose time-derivative is given by

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -U \frac{\sqrt{\pi}}{2} \mathrm{e}^{x^2} \tag{7}$$

By suitably swapping the numerator and denominator of the left and right sides in the previous expression, and left and right multiplying by the denominator of the left-side term, one has

$$\mathrm{d}t = -\frac{1}{U} \frac{2}{\sqrt{\pi}} \mathrm{e}^{-x^2} \mathrm{d}V \tag{8}$$

Noticing that $x^2 = V^2$ and

$$\frac{2}{\sqrt{\pi}} \mathrm{e}^{-V^2} = \frac{\mathrm{d}\,\mathrm{erf}(V)}{\mathrm{d}V},$$

where $\mathrm{erf}(V) := \frac{2}{\sqrt{\pi}} \int_0^V \mathrm{e}^{-\tau^2} \mathrm{d}\tau \in \mathcal{DK}_1$ is the Gauss error function, by integrating (8) from 0 to $T(x)$, one obtains

$$T(x) = \frac{1}{U} \left(\mathrm{erf}(V(x_0)) - \mathrm{erf}(V(x(T(x)))) \right) \tag{9}$$

Therefore, from (9), since $V(x(T(x))) = V(0) = 0$, it yields

$$T(x_0) = \frac{1}{U}\,\mathrm{erf}(V(x_0)) \leq \frac{1}{U}, \quad \forall\, x_0 \in \mathbb{R}\backslash\{0\}, \tag{10}$$

which concludes the proof. ∎

B. Perturbed case

The previous result is now extended to the perturbed case as follows.

Theorem 2: The system

$$\dot{x}(t) = -\bar{U}\,\mathrm{sign}(x(t)) + d(t), \quad x(0) = x_0, \tag{11}$$

with $x \in \mathbb{R}$, $d \in \mathbb{R}$ such that $\boldsymbol{\mathcal{A}_1}$ holds, and control gain

$$\bar{U} := U\frac{\sqrt{\pi}}{2}\mathrm{e}^{x^2(t)}, \quad U > \tfrac{2}{\sqrt{\pi}}\bar{d}, \tag{12}$$

is globally fixed-time stable with settling time $T(x_0) \leq \bar{T}$ and

$$\bar{T} = \frac{1}{U - \frac{2}{\sqrt{\pi}}\bar{d}}. \tag{13}$$

Proof: Following the same reasoning in the proof of Theorem 1, given $U > \frac{2}{\sqrt{\pi}}\bar{d}$, according to [15, §4.2], the system is globally finite-time stable.

Now, consider again the Lyapunov function (6) over (11), whose derivative is

$$\begin{aligned}\dot{V} &= -U\frac{\sqrt{\pi}}{2}\mathrm{e}^{V^2} + d\,\mathrm{sign}(x)\\ &\leq -\left(U - \tfrac{2}{\sqrt{\pi}}\bar{d}\right)\frac{\sqrt{\pi}}{2}\mathrm{e}^{V^2}.\end{aligned} \tag{14}$$

Therefore, analogously to the proof of Theorem 1, one has

$$T(x_0) = \frac{1}{U - \frac{2}{\sqrt{\pi}}\bar{d}}\,\mathrm{erf}(V(x_0)) \leq \frac{1}{U - \frac{2}{\sqrt{\pi}}\bar{d}}, \tag{15}$$

$\forall\, x_0 \in \mathbb{R}\backslash\{0\}$, which concludes the proof. ∎

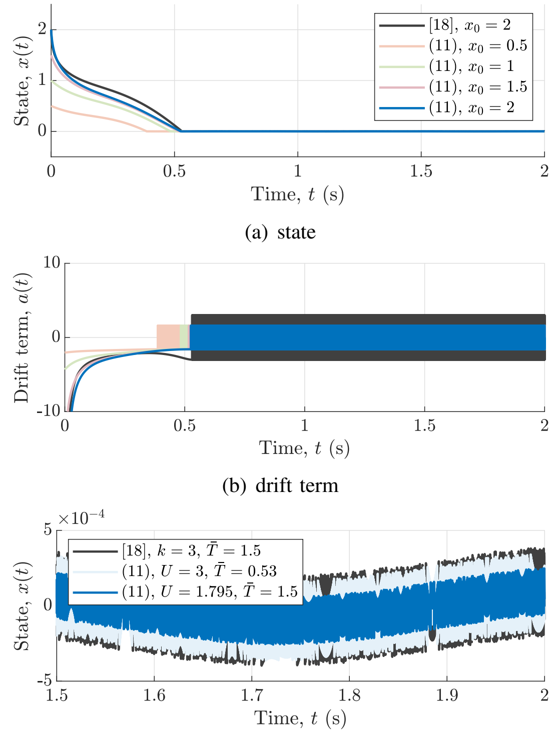
C. Illustrative example

In order to further highlight the enhanced fixed-time stability results previously presented, let us consider the following example

$$\dot{x}(t) = a(x(t)) + \sin(6t), \quad x(0) = x_0, \tag{16}$$

where $a(x(t)) = -\bar{U}\,\mathrm{sign}(x(t))$, and $\bar{U}$ is chosen as in (12), with $U > \frac{2}{\sqrt{\pi}}\bar{d}$ and $\bar{d} = 1$. The control objective is to regulate the state x to zero with settling time bounded by $\bar{T} = 1.5\,\mathrm{s}$. Therefore, according to Theorem 2, the gain $\bar{U}$ is selected such that $U = 1.795$. Fig. 1 shows the results for four different initial conditions $x_0 \in \{0.5, 1, 1.5, 2\}$, and it can be noticed that, independently on x_0, the state trajectory is regulated to zero within $\bar{T} = 1.5\,\mathrm{s}$. Furthermore, the time evolution of the drift term $a(t)$ allows one to better appreciate the convergence time, when a sliding motion is enforced.

Moreover, the performance of our proposal when $x_0 = 2$ is fairly compared with that of the methodology developed in [18, §II.B] in the case of variable exponent coefficient with parameters $\lambda = 2$, $\mu = 0.1$, and $k = 3$, to achieve $T(x_0) \leq 1.5\,\mathrm{s}$. Such a comparison is also reported in Figs. 1(a), 1(b) for both the state and the drift term (black lines).



(a) state

(b) drift term

(c) zoom in on the state

Fig. 1. Time evolution of state $\boldsymbol{x(t)}$ (top) and drift term $\boldsymbol{a(t)}$ (middle) for $\boldsymbol{x_0 \in \{0.5, 1, 1.5, 2\}}$, both in comparison with respect to the strategy in [18], and chattering comparison (bottom) with respect to the control law in [18] under the same control gains $\boldsymbol{k = U = 3}$ and the same bound of the settling time, $\boldsymbol{\bar{T} =}\ 1.5\,\mathrm{s}$.

Finally, the method in [18, §II.B] is compared in terms of chattering phenomenon. A zoom in of the state in the interval $[1.5, 2]\,\mathrm{s}$ is reported in Fig. 1(c), when the same value of $\bar{T}$ is achieved by the two compared strategies with different gains, and also when the gains k and U assume the same value. As it can be observed, the choice of the coefficient as in our proposal allows to achieve the desired convergence time within the expected upper-bound, i.e., $T(x_0) \leq 1.5\,\mathrm{s}$, as in the case of the strategy in [18], but with a better chattering reduction due to the need of a smaller gain.

IV. Sliding Mode Control Design

In this section, the fixed-time stability results previously presented are employed to design a sliding mode control approach relying on the class of second-order systems, which is widely adopted to model many kind of plants, such as the electromechanical ones. Consider now the following system

$$\begin{cases}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= h(x(t)) + g(x(t))u(t) + \delta(t)\end{cases}, \quad x(0) = x_0 \tag{17}$$

where $x \in \mathbb{R}^2$ is the state, $u \in \mathbb{R}$ is the input, $h : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuous smooth functions, such that $h(0) = 0$, and, without loss of generality, $g(x) \geq \underline{g} > 0, \forall\, x \in \mathbb{R}^2$ (the specular case $g(x) \leq \underline{g} < 0$ is analogous). Moreover, $\delta \in \mathbb{R}$ is a disturbance such that $\boldsymbol{\mathcal{A}_1}$ holds with $d = \delta$.

Let us introduce the sliding variable $s(x(t)) : \mathbb{R}^2 \to \mathbb{R}$ defined by exploiting the results in Theorem 2, that is

$$s(x(t)) := \begin{cases} x_2(t) + U_1 \frac{\sqrt{\pi}}{2} \mathrm{e}^{x_1^2(t)} \operatorname{sign}(x_1(t)), & x \notin \mathcal{N}_\varepsilon \\ \frac{x_2(t)|x_2(t)|}{2U_\mathrm{r}} + x_1(t), & x \in \mathcal{N}_\varepsilon \end{cases} \tag{18}$$

with $s_0 \in \mathbb{R}\backslash\{0\}$, $U_1 \in \mathbb{R}_+$, $U_\mathrm{r} := \underline{g}U_2 - \bar{\delta} > 0$, $U_2 \in \mathbb{R}_+$, and the set $\mathcal{N}_\varepsilon := \{x \in \mathbb{R}^2 \mid |x_1| \leq \varepsilon\}$, with $\varepsilon \in \mathbb{R}_+$ given as the solution of the nonlinear system (18) by posing $s = 0$. Therefore, the following result holds.

Theorem 3: Given system (17), controlled by

$$u(t) := \begin{cases} -\frac{1}{g(x(t))} \Big(h(x(t)) + U_1 \sqrt{\pi} |x_1(t)| x_2(t) \mathrm{e}^{x_1^2(t)} \\ \quad + U_2 \frac{\sqrt{\pi}}{2} \mathrm{e}^{s^2(x(t))} \operatorname{sign}(s(x(t))) \Big), & x \notin \mathcal{N}_\varepsilon \\ -\frac{1}{g(x(t))} \left(h(x(t)) + U_2 \operatorname{sign}(s(x(t))) \right), & x \in \mathcal{N}_\varepsilon \end{cases} \tag{19}$$

with sliding variable (18), if $\boldsymbol{\mathcal{A}_1}$ holds, $U_1 \in \mathbb{R}_+$ and $U_2 > \frac{2}{\sqrt{\pi}}\bar{\delta}$, then the origin of the closed-loop system is fixed-time stable with settling time $T(x_0) \leq \bar{T}$ and

$$\bar{T} = \frac{\sqrt{\pi}(U_1 + U_2) - 2\bar{\delta}}{U_1(\sqrt{\pi}U_2 - 2\bar{\delta})} + \frac{U_1}{U_\mathrm{r}} \frac{\sqrt{\pi}}{2} \mathrm{e}^{\varepsilon^2}. \tag{20}$$

Proof: The proof directly follows from Theorems 1 and 2. Let us consider $x \notin \mathcal{N}_\varepsilon$, by choosing a Lyapunov function $V(s) = |s|$ with time-derivative $\dot{V}(s) = \dot{s}\operatorname{sign}(s)$, and computing the first-time derivative of the sliding variable for any x in $\mathbb{R}^2\backslash\mathcal{N}_\varepsilon$, i.e.,

$$\dot{s}(x(t)) := h(x(t)) + g(x(t))u(t) + \delta(t) \\ + U_1 \sqrt{\pi} |x_1(t)| x_2(t) \mathrm{e}^{x_1^2(t)}, \tag{21}$$

substituting the control input (19), one obtains

$$\dot{V}(s) = -U_2 \frac{\sqrt{\pi}}{2} \mathrm{e}^{s^2} + \delta \operatorname{sign}(s). \tag{22}$$

Hence, by virtue of Theorem 2, with $U_2 > \frac{2}{\sqrt{\pi}}\bar{\delta}$, the s-system is globally fixed-time stable, that is the sliding manifold $s = 0$ is reached with settling time

$$T_1(s_0) \leq \frac{1}{U_2 - \frac{2}{\sqrt{\pi}}\bar{\delta}}, \forall s_0 \in \mathbb{R}\backslash\{0\}.$$

Now, let compute the equivalent dynamics in sliding mode $s = 0$, that is, from (18), one has $x_2 = -U_1 \frac{\sqrt{\pi}}{2} \mathrm{e}^{x_1^2} \operatorname{sign}(x_1)$, and in turn the reduced-order system equivalent to (17) becomes

$$\dot{x}_1(t) = -U_1 \frac{\sqrt{\pi}}{2} \mathrm{e}^{x_1^2(t)} \operatorname{sign}(x_1(t)), \quad \forall t \geq T_1(s_0). \tag{23}$$

By virtue of Theorem 1, the latter is fixed-time stable, and the set $\mathcal{N}_\varepsilon$ is reached with settling time

$$T_2(x_{1_0}) < \frac{1}{U_1}, \forall x_0 \notin \mathcal{N}_\varepsilon.$$

Consider now $x \in \mathcal{N}_\varepsilon$, so that the switching manifold becomes the one with optimal reaching [20]. Since the sliding variable is defined as a continuous function, and by virtue of the choice of U_r, it is possible to prove that, in the worst realization of the uncertain term, starting on the manifold with initial condition given by $x = [\pm\varepsilon, \mp U_1 \frac{\sqrt{\pi}}{2} \mathrm{e}^{(\pm\varepsilon)^2}]'$, the origin is reached in a predefined time equal to

$$T_\varepsilon = \frac{U_1}{U_\mathrm{r}} \frac{\sqrt{\pi}}{2} \mathrm{e}^{\varepsilon^2}. \tag{24}$$

Therefore, the origin of system (17) is fixed-time stable with settling time

$$T(x_0) = T_1(s_0) + T_2(x_{1_0}) + T_\varepsilon \\ \leq \frac{1}{U_2 - \frac{2}{\sqrt{\pi}}\bar{\delta}} + \frac{1}{U_1} + \frac{U_1}{U_\mathrm{r}} \frac{\sqrt{\pi}}{2} \mathrm{e}^{\varepsilon^2}, \tag{25}$$

and $\bar{T}$ as in (20), which concludes the proof. ■

Remark 4.1 (Singularities): Although the designed sliding variable (18) is not differentiable for $x_1 = \pm\varepsilon$, however, differently from [16]–[18], the SMC input (19) does not present singularity for $x_1 = 0$ by virtue of the singularity avoidance property of the proposed sliding variable. ▽

Remark 4.2 (Chattering alleviation): It is worth noticing that the controller (19) consists of two components: one is continuous and capable of compensating the nominal dynamics of the system, whereas the other one is the discontinuous component aimed at rejecting the disturbance term. In fact, the latter is designed as a discontinuous law with time-varying gain [22], such that it becomes equal to $U_2 > \frac{2}{\sqrt{\pi}}\bar{\delta}$ whenever in sliding mode, which is sufficiently high to dominate the disturbance δ, while reducing the control effort with beneficial effects in terms of chattering alleviation. ▽

Remark 4.3 (Ease of implementation): Note that the proposed SMC has the advantage to require the simpler tuning of only the two parameters U_1 and U_2, with respect to other methods as those in [16]–[18], where 14 and 6 parameters are needed, respectively. ▽

V. CASE-STUDY

The proposed approach is now assessed in a more realistic case-study relying on a simple inverted pendulum (SIP) (see [16]), to solve a tracking control problem.

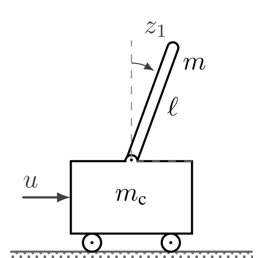


Fig. 2. The considered simple inverted pendulum.

A. Settings

Consider Fig. 2, where the simple inverted pendulum state vector is $z = [z_1 \,,\, z_2]'$, with z_1 being the swing angle and z_2 being the swing speed, u is the force applied to the cart and δ is a disturbance affecting the SIP. Having in mind to track a reference trajectory $r(t)$, the error model of the SIP is captured by equations (26), where $x_1 = z_1 - r$, $x_2 = z_2 - \dot{r}$,

$$
\begin{cases}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \underbrace{\dfrac{\bar{g}\sin(x_1+r) - m\ell(x_2+\dot{r})^2\cos(x_1+r)\sin(x_1+r)/(m_c+m)}{\ell[4/3 - m\cos^2(x_1+r)/(m_c+m)]}}_{h(x)} + \underbrace{\dfrac{\cos(x_1+r)/(m_c+m)}{\ell[4/3 - m\cos^2(x_1+r)/(m_c+m)]}}_{g(x)} u - \ddot{r} + \delta
\end{cases}
\tag{26}
$$

$\underline{g} = 0.5$, the parameters are those reported in Table I, and the dependence of all the variables on time is omitted for the sake of simplicity. The bounded perturbation is given by

TABLE I
SIP PARAMETERS

$\bar{g}$	gravitational acceleration	$9.8\,\mathrm{m\,s^{-2}}$
m_c	mass of the cart	$1\,\mathrm{kg}$
m	mass of the pendulum	$0.1\,\mathrm{kg}$
ℓ	length to the pendulum centre of mass	$0.5\,\mathrm{m}$

$\delta = \sin(10z_1) + \cos(z_2)$, such that $\bar{\delta} = 2$, while the reference trajectory is $r = \sin(0.5\pi t)$. The initial conditions are $z_0 = [1\,,\,0.5]'$.

As for the proposed SMC, practical tuning rules are provided by the conditions in Theorem 3, so that the control parameters are chosen as $U_1 = 1$ and $U_2 = 9.0264$, such that $U_r = 2.5132$, determining a value $\varepsilon = 0.165$, and an upper-bound of the settling time equal to $\bar{T} = 1.5101\,\mathrm{s}$, according to (20). All the simulations have been executed using MATLAB, with fixed-time step equal to $1 \times 10^{-4}\,\mathrm{s}$ in order to avoid performance limitations or lack of robustness.

B. Results

The outcome of the simulations of the SIP motion under the suggested fixed-time controller are shown in Fig. 3. It can be observed that the swing angle z_1 of the SIP tracks the time-varying reference r with a settling time of $1\,\mathrm{s}$. Analogously, the swing speed z_2 follows its reference signal $\dot{r}$, as well as the sliding variable is steered to zero within $T_1(s_0) \le 0.15\,\mathrm{s}$ with settling time equal to $T_1(s_0) = 0.1\,\mathrm{s}$. As for the control input u, Fig. 3 shows also its time evolution, characterized by the superposition of a continuous component and a discontinuous one with time-varying gain.

Comparison and discussion: In order to further assess the proposed fixed-time stable SMC in (19), in the following its performance is compared with those achieved by using the SMC approaches in [18] and [16]. In order to compare the strategies as fairly as possible in terms of chattering phenomenon, to the best of our possibilities, the parameters of the compared methods (summarized in Table II), are selected as in [18, §IV.B] and [16, §4] but with the same control gains, i.e., $k = \gamma = U_2 = 9.0264$.

According to [18, §IV.B], the 6 control parameters are selected as $\beta = 1$, $\lambda_1 = 2$, $\mu_1 = 0.1$, $\lambda_2 = 4$, $\mu_2 = 1$, $k = 9.0264$, thus leading to a settling time within $3.2393\,\mathrm{s}$. As for the approach in [16, §4], the 14 parameters are chosen as $\alpha_1 = \beta_1 = 2.35$, $\alpha_2 = \beta_2 = 1.5$, $m_1 = 9$, $n_1 = 5$, $p_1 = 7$, $q_1 = 9$, $m_2 = 5$, $n_2 = 3$, $p_2 = 5$, $q_2 = 9$, $\gamma = 9.0264$, $\tau = 0.1$, such that the settling time is bounded by $6.625\,\mathrm{s}$. For the

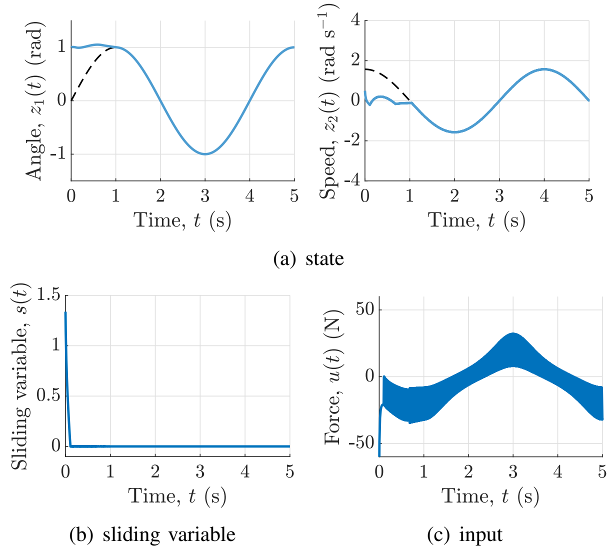


Fig. 3. Time evolution of the swing angle $z_1(t)$ and of the swing speed $z_2(t)$ with respect to the reference signals (top), of the sliding variable $s(t)$, and of the input $u(t)$ (bottom), when the proposal is applied.

TABLE II
CONTROL PARAMETERS TO TUNE.

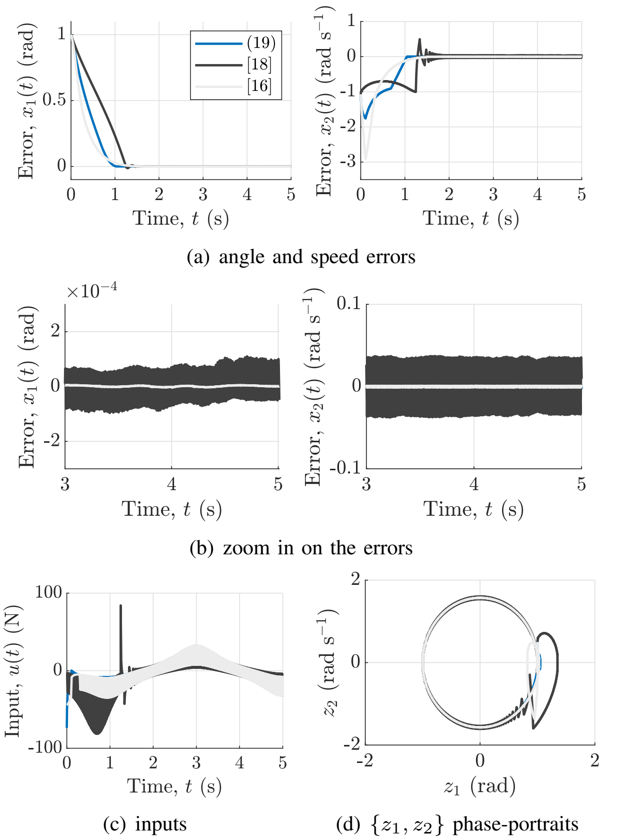
(19)	$U_1,\ U_2$
[18]	$\beta,\ \lambda_1,\ \mu_1,\ \lambda_2,\ \mu_2,\ k$
[16]	$\alpha_1,\ \beta_1,\ \alpha_2,\ \beta_2,\ m_1,\ n_1,\ p_1,\ q_1,\ m_2,\ n_2,\ p_2,\ q_2,\ \gamma,\ \tau$

comparison, some performance indexes are finally introduced, that is the root mean square (RMS) value of the error x_{RMS}, of the sliding variable s_{RMS}, both in steady state for $t \in [3, 5]\,\mathrm{s}$, and of the input u_{RMS}, apart from the settling time $\bar{T}$, and the different number of control parameters to tune, namely n_{par}.

TABLE III
PERFORMANCE METRICS.

	x_{RMS}	s_{RMS}	u_{RMS} (N)	$\bar{T}$ (s)	n_{par}
(19)	7.79×10^{-4}	1.71×10^{-7}	17.14	1.51	2
[18]	0.02	0.99	26.77	3.24	6
[16]	5.24×10^{-4}	2.67×10^{-5}	17.05	6.625	14

The outcome of the simulations is reported in Table III and illustrated in Fig. 4. Specifically, the latter shows the time evolution of the errors x_1 and x_2 of the swing angle z_1 and of the swing speed z_2 with respect to their reference signals r and $\dot{r}$, respectively. In all the cases the error is steered to zero, with a bigger oscillation in the reaching phase when the approach in [16] is used. In the same figure, in order to evaluate the chattering phenomenon, a comparison in steady state in the interval $[3,\ 5]\,\mathrm{s}$ is reported. The proposed strategy



(a) angle and speed errors

(b) zoom in on the errors

(c) inputs

(d) $\{z_1, z_2\}$ phase-portraits

Fig. 4. Time evolution of the angle error $\boldsymbol{x_1(t)}$, speed error $\boldsymbol{x_2(t)}$ (top), comparison in terms of chattering with respect to the strategies in [18] and [16] (middle), inputs $\boldsymbol{u(t)}$, and phase portraits $\boldsymbol{\{z_1, z_2\}}$ (bottom).

and the one in [16] are comparable in terms RMS error and RMS value of the input, resulting also better than [18], while the proposal outperforms all the others in terms of RMS value of the sliding variable. It is worth noticing that the control law in [16] presents some singularity issues and implementation problems when the initial velocity is equal to zero (see [16, Remarks 3–4] for further details). Also the control inputs are illustrated in Fig. 4. Apart from some initial overshoot in the case of [18], all the input signals are practically feasible, and the effect of the discontinuous component of the control law can be observed overlapped to a continuous component leading to a smoother time-varying signal in all the cases. The phase portraits $\{z_1, z_2\}$ of the three closed-loop systems are also reported, from which it is visible that the proposed controller guarantees smoother trajectories than those with the methods in [18] and [16]. Overall, the proposal has the merit to require only 2 control parameters, instead of the 6 or 14 parameters in the other methods, whose tuning could be not obvious.

VI. Conclusions

In this letter we have proposed an alternative SMC strategy for a class of uncertain nonlinear systems, which exploits the so-called Gauss error function in the control design, in order to achieve robust fixed-time stability of the x-system. Such a property, together with its robustness in front of bounded disturbance terms, has been theoretically analysed. Finally, numerical simulations have confirmed the theoretical results.

Future works are devoted to the extension of the proposed approach to more complex settings, for instance in presence of state constraints, unmatched uncertainties, high-frequency reference signals, measurement noises, or partial knowledge of the systems dynamics.

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