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Pinning control of linear systems on hypergraphs

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ABSTRACT

When steering the dynamics of network systems, the control design needs to cope with constraints on actuation and sensing, which often imply that the same control input is injected to each node in a given subset, and this input signal is a function of the state of this node subset. This common situation cannot be modeled in terms of standard pairwise interactions on digraphs, and we propose to use directed hypergraphs as the mathematical object suitable to describe this kind of directed, multibody interactions. We apply this framework to the pinning control problem in networks of coupled linear systems, and derive necessary and sufficient conditions for convergence onto the desired trajectory set by the pinner. Furthermore, we provide a dedicated control algorithm to identify the interconnections that are critical for network control.

1. Introduction

In the last decades, controlling the dynamics of coupled dynamical systems has been the subject of a vast and interdisciplinary research effort [9,15,26]. A fundamental result obtained by the research community has been the discovery of the essential mechanisms to induce the emergence of collective behaviors such as synchronization or consensus [4,17,19,27]. The theoretical discoveries have been used in a wide range of applications, such as formation control and flocking [2,18,20,25], epidemics [10,22,28], power grids [11], and dynamics of human mobility [6]. A common feature in most of these diverse applications is that it may not be feasible to inject the control input into each unit.

The so-called *pinning control* strategy has been introduced to cope with this scenario: an additional unit, the pinner, is added to the system, and injects a control signal only to a (possibly small) subset of the system units, called *pinned nodes* [27]. Conditions for pinning controllability towards a point in the state space or onto a synchronous trajectory have been provided in the literature, see e.g. [4,17,19]. An implicit assumption in the literature on pinning control is that it is possible to measure the state of each of the pinned nodes and individually inject a different control signal which is proportional to the error between their state and that of the pinner. Such a control action can be suitably modeled through a directed link from the pinner to the pinned nodes.

In real-world applications, the implementation of traditional pinning control may be jeopardized by limitations on sensing, which may prevent the measurement of the state of a specific node. Rather, it might only be possible to measure a function of the state of a node subset, and thus the feedback input can only be designed by using such an aggregated information. A paradigmatic example could be the feedback control of microbial consortia, where only aggregated measurements of the fluorescence of cell groups can be performed, and the limited resolution of the actuation constrains to inject the same control signal to more than one cell [23,24]. To study these kinds of interactions, directed links are not suitable anymore, leading us to resort to directed hyperlinks [5,12]. Indeed, the hyperedge heads define a set of nodes whose state is affected by a common input function of the aggregated measurements of their state and of that of a second set of nodes defined by the hyperedge tails.

In this paper, leveraging the formalism introduced in [5], we study pinning control of linear dynamical systems in the presence of higher-order, directed interactions. The focus on networks of linear systems allowed us to derive a necessary and sufficient condition for convergence of the node dynamics onto the desired trajectory defined by the pinner. Using an analogy between directed hypergraphs and a class of signed graphs, we show that a crucial difference with the case of pinning control of standard digraphs is that the addition of hyperedges may jeopardize the convergence

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onto the pinner's trajectory. Therefore, we propose an algorithm that guides the control designer in the selection of the hyperedges. Two numerical examples are used to demonstrate the approach, the first dealing with synchronization of harmonic oscillators, and the second studying opinion consensus on hypergraphs.

The outline of the paper is as follows. In Section 2, we introduce the notation and provide the required background on directed hypergraphs and signed graphs. Then, in Section 3 we introduce the pinning control problem for linear systems that are coupled through directed hypergraphs and, in Section 4, we derive a necessary and sufficient conditions for convergence of the network dynamics onto the pinner's trajectory, together with a dedicated algorithm for identifying the key interconnections, critical for control. Conclusions and possible future work are finally discussed in Section 6.

2. Mathematical preliminaries

Given a positive integer n, I_n denotes the identity matrix in $\mathbb{R}^n$, with $\mathfrak{e}_i$ being its i-th column, and $\mathbb{1}_n$ the vector of all ones in $\mathbb{R}^n$. Given a matrix $M \in \mathbb{R}^{n\times n}$, M^T denotes its transpose, $M_{\mathrm{sym}} = (M + M^\mathrm{T})/2$ its symmetric part, and $M > 0$ ($M \geq 0$) means that it is positive (semi-)definite. Further, we sort the eigenvalues $\lambda_1(M), \ldots, \lambda_n(M)$ of matrix M so that their real part is in ascending order, that is, $\Re(\lambda_1(M)) \leq \ldots \leq \Re(\lambda_n(M))$.

Given two matrices $M_1 \in \mathbb{R}^{a\times b}$ and $M_2 \in \mathbb{R}^{c\times d}$, we denote we denote $(M_1 \otimes M_2) \in \mathbb{R}^{ac\times bd}$ their Kronecker product [14]. When M_1 and M_2 have the same number of columns ($b = d$), $[M_1; M_2] \in \mathbb{R}^{(a+c)\times b}$ denotes their vertical concatenation, whereas, when they have the same number of rows ($a = c$), $[M_1, M_2] \in \mathbb{R}^{a\times(b+d)}$ denotes their horizontal concatenation.

Next, we provide the definitions on signed graphs and directed hypergraphs that will be used in the rest of the manuscript.

2.1. Weighted signed graphs [1]

A weighted signed graph $\mathscr{S}$ is the triple $\{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$, where $\mathcal{V}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges, and $\mathcal{W}: \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ is the function that associates zero to every pair $(i, j) \in \mathcal{V}\times\mathcal{V}$ that is not in $\mathcal{E}$, and a non-zero weight to every pair $(i, j) \in \mathcal{E}$. Note that, different from the case of digraphs, the weights can also be negative. Extending the notation and definitions used in network control of unsigned digraphs [16], the ji-th entry of the adjacency matrix A associated to $\mathscr{S}$ is the weight $\mathcal{W}(i, j)$ associated to the pair (i, j), and the Laplacian matrix for $\mathscr{S}$ is $L = D - A$, where $D = \mathrm{diag}\{d_1^{\mathrm{in}}, \ldots, d_{|\mathcal{V}|}^{\mathrm{in}}\}$, with $d_i^{\mathrm{in}} = \sum_{j=1}^{|\mathcal{V}|} \mathcal{W}(j, i)$ being the weighted in-degree of node i. Note that when $|\mathcal{W}(i, j)| = 1$ for all $(i, j) \in \mathcal{E}$, then $\mathscr{S}$ is unweighted, and d_i^{in} is the (unweighted) in-degree, and that all definitions fall back to those for standard digraphs if $\mathscr{S}$ has all positive weights. By definition, L is zero row-sum, which implies that $0 \in \mathrm{spec}(L)$, and that $\mathbb{1}_N$ is its associated (right) eigenvector.

Given two signed graphs $\mathscr{S}_1 = \{\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1\}$ and $\mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2\}$, their superimposition is the graph $\mathscr{S} = \mathscr{S}_1 \oplus \mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$ where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, and $\mathcal{W}(i, j) = \mathcal{W}_1(i, j) + \mathcal{W}_2(i, j)$.

2.2. Directed hypergraphs [12]

A directed hypergraph $\mathscr{H}$ is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_N\}$ is the set of nodes, and $\mathcal{E} = \{\varepsilon_1, \ldots, \varepsilon_M\}$ is the set of directed hyperedges. The i-th directed hyperedge $\varepsilon_i \in \mathscr{H}$ is an ordered pair $(\mathcal{T}(\varepsilon_i), \mathcal{H}(\varepsilon_i))$ of (possibly empty) disjoint subsets of the hypergraph nodes, where the ordered subsets $\mathcal{T}(\varepsilon_i)$ and $\mathcal{H}(\varepsilon_i)$ of $\mathcal{V}$, are the set of tails and heads of ε_i, respectively. Note that an edge can be viewed as a particular instance of hyperedge with one tail and one head. We define the cardinality $|\varepsilon_i|$ of a directed hyperedge ε_i as the number $|\mathcal{T}(\varepsilon_i)| + |\mathcal{H}(\varepsilon_i)|$ of nodes composing it. Given $\varepsilon \in \mathcal{E}$, the functions $\mathfrak{t}(\varepsilon, i)$ and $\mathfrak{h}(\varepsilon, j)$ associate to the i-th tail and j-th head of ε the corresponding labels in $\mathcal{V}$, respectively.

Given $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathcal{V}$, we denote $\mathcal{E}^{\mathcal{V}_1,\mathcal{V}_2} = \{\varepsilon \in \mathcal{E} : \mathcal{V}_1 \subseteq \mathcal{T}(\varepsilon) \wedge \mathcal{V}_2 \subseteq \mathcal{H}(\varepsilon)\}$, that is, the set of hyperedges with (at least) one tail in $\mathcal{V}_1$ and one head in $\mathcal{V}_2$. With a slight abuse of notation, when $\mathcal{V}_1$ or $\mathcal{V}_2$ is a singleton, we will refer to it by its only element. For instance, if $\mathcal{V}_1 = \{v_j\}$ we write $\mathcal{E}^{j,\mathcal{V}_2}$ instead of $\mathcal{E}^{\{v_j\},\mathcal{V}_2}$. Finally, given a node v_j, we introduce $\mathcal{E}^{\cdot,j} = \{\varepsilon_i \in \mathcal{E} : v_j \in \mathcal{H}(\varepsilon_i)\}$ as the subset of hyperedges having v_j as a head, and $\mathcal{E}^{j,\cdot} = \{\varepsilon_i \in \mathcal{E} : v_j \in \mathcal{T}(\varepsilon_i)\}$ as the set of hyperedges having v_j as a tail. Note that, when $\mathscr{H}$ is a graph, the cardinality of the sets $\mathcal{E}^{\cdot,j}$ and $\mathcal{E}^{j,\cdot}$ coincide with the in-degree and out-degree of j, respectively.

3. Pinning control of linear systems on hypergraphs

We consider an ensemble of N linear dynamical systems coupled through a directed hypergraph $\mathscr{H}_c = \{\mathcal{V}_c, \mathcal{E}_c\}$, where the sets $\mathcal{V}_c = \{v_1, \ldots, v_N\}$ and $\mathcal{E}_c = \{\varepsilon_1, \ldots, \varepsilon_M\}$ are the set of nodes and hyperedges, respectively. Given a node $v_i \in \mathcal{V}_c$, its state will be a vector x_i in $\mathbb{R}^n$, whereas, given a hyperedge $\varepsilon \in \mathcal{E}_c$ and denoting $x_{\mathfrak{t}(\varepsilon,i)}$ and $x_{\mathfrak{h}(\varepsilon,j)}$ the state of its i-th tail and its j-th head, respectively, we associate to ε a tail state matrix $x_\varepsilon^\tau = [x_{\mathfrak{t}(\varepsilon,1)}, \ldots, x_{\mathfrak{t}(\varepsilon,|\mathcal{T}(\varepsilon)|)}]$ and a head state matrix $x_\varepsilon^h = [x_{\mathfrak{h}(\varepsilon,1)}, \ldots, x_{\mathfrak{h}(\varepsilon,|\mathcal{H}(\varepsilon)|)}]$. We can describe the node dynamics as

$$\dot{x}_i = Fx_i + \sum_{\varepsilon\in\mathcal{E}_c^{\cdot,i}} \sigma_\varepsilon\left(x_\varepsilon^\tau \alpha_\varepsilon - x_\varepsilon^h \beta_\varepsilon\right) + u_i, \tag{1}$$

where $F \in \mathbb{R}^{n\times n}$ is the dynamic matrix describing the individual dynamics, σ_ε is the coupling gain associated to ε; $\alpha_\varepsilon = [(\alpha_\varepsilon)_{\mathfrak{t}(\varepsilon,1)}, \ldots, (\alpha_\varepsilon)_{\mathfrak{t}(\varepsilon,|\mathcal{T}(\varepsilon)|)}]^\mathrm{T}$ and $\beta_\varepsilon = [(\beta_\varepsilon)_{\mathfrak{h}(\varepsilon,1)}, \ldots, (\beta_\varepsilon)_{\mathfrak{h}(\varepsilon,|\mathcal{H}(\varepsilon)|)}]^\mathrm{T}$ are the (ordered) vectors stacking the non-negative weights associated to the tails and heads of ε, respectively, defined such that $\alpha_\varepsilon^\mathrm{T}\mathbb{1}_{|\mathcal{T}(\varepsilon)|} = \beta_\varepsilon^\mathrm{T}\mathbb{1}_{|\mathcal{H}(\varepsilon)|} = 1$.

According to coupling protocol (1), called hyperdiffusive in [5], a head of a hyperedge ε computes the difference between a convex combination of the states of the tails and a convex combination of the states of the heads of ε. Note that i) such a protocol is synchronization noninvasive [13] since the summation in (1) would be naught when the nodes are synchronized, and ii) when $\mathscr{H}_c$ is a graph (no hyperedges of cardinality greater than two exist) the protocol reduces to the standard diffusive protocol defined for digraphs [3].

We design the controller u_i so that each of the nodes in (1) is steered towards the trajectory of the pinner, which is modeled as a further node in the network, denoted v_s, and is characterized by the same dynamics as the controlled nodes. Denoting the state of the pinner as $x_s \in \mathbb{R}^n$, we can then write its dynamics as

$$\begin{aligned} \dot{x}_s &= Fx_s, \\ x_s(0) &= x_s^0, \end{aligned} \tag{2}$$

where x_s^0 is the initial condition of the pinner. Now, we can introduce an enlarged hypergraph $\mathscr{H} = \{\mathcal{V}, \mathcal{E}\}$ which also includes the pinner v_s and the directed hyperedges connecting it to the nodes in $\mathcal{V}_c$. Namely, this implies that $\mathcal{V} = \mathcal{V} \cup \{v_s\}$, whereas $\mathcal{E} = \mathcal{E}_c \cup \mathcal{E}^{s,\cdot}$, with $\mathcal{E}^{s,\cdot}$ being the set of hyperedges having the pinner as a tail. Denoting $\mathcal{E}^{s,i}$ the set of hyperedges having the pinner v_s as a tail and v_i as a head, the definition of set of pinned nodes can be extended to hypergraphs as $\mathcal{P} = \{i \in \mathcal{V} : \mathcal{E}^{s,i} \neq \emptyset\}$. As in digraphs, the set $\mathcal{P}$ identifies the nodes that are directly influenced by the pinner. In what follows, we consider the following hyperdiffusive control protocol [5]:

$$u_i = \begin{cases} \sum_{\varepsilon\in\mathcal{E}^{s,i}} k_\varepsilon (x_\varepsilon^\tau \alpha_\varepsilon - x_\varepsilon^h \beta_\varepsilon), & i \in \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

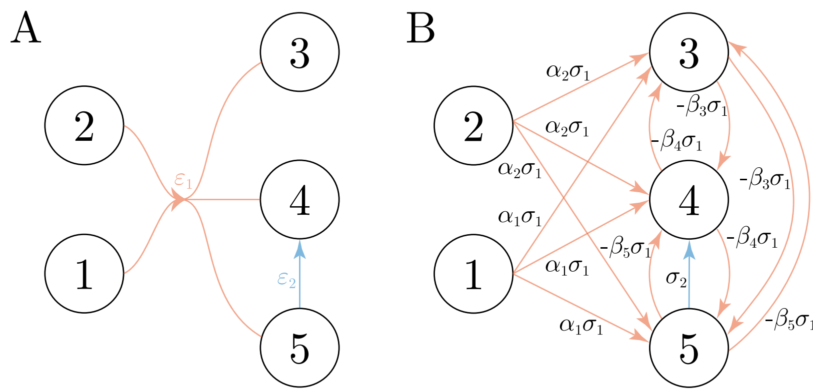

Fig. 1. Panel A depicts a hypergraph with two hyperedges ε_1 and ε_2, whereas its equivalent signed graph is reported in panel B, where σ_1 and σ_2 are the coupling gains associated to ε_1 and ε_2, respectively, and where $\alpha_{\varepsilon_1} = [\alpha_1, \alpha_2]^{\mathrm{T}}$, and $\beta_{\varepsilon_1} = [\beta_3, \beta_4, \beta_5]^{\mathrm{T}}$.

where k_ε is the control gain associated to the hyperedge ε. Since in (3) $\varepsilon \in \mathcal{E}^{s,i}$, we have that the state x_s of the pinner v_s is one of the columns of the state matrix $x_\varepsilon^{\mathcal{T}}$.

Next, we define the network pinning error as $e = [e_1; \ldots; e_N]$, where $e_i = x_i - x_s$ is the pinning error of node i. The scope of the control law (3) is to steer the dynamics of the nodes in $\mathcal{V}_c$ onto the trajectory of the pinner, that is,

$$\lim_{t\to+\infty} \|e(t)\| = 0, \tag{4}$$

through a suitable choice of the sets $\mathcal{P}$ and $\mathcal{E}^{s,i}$, $i \in \mathcal{P}$, and of the control gains k_ε.

4. Convergence analysis

Before illustrating our main result, we leverage a useful equivalence between hypergraphs and signed graphs first derived in [5] to rewrite the dynamics (1)-(3) over a suitably defined network on a signed graph $\mathscr{S}$. Introducing the state $z = [z_1; \ldots; z_{N+1}]$, so that $z_i = x_i$ for $i = 1, \ldots, N$, and $z_{N+1} = x_s$, we can now give the following proposition:

Proposition 1. *The dynamics (1)-(3) on the hypergraph $\mathscr{H}$ can be equivalently written as*

$$\dot z_i = (I_N \otimes F) z_i + \sum_{j=1}^{N+1} a_{ij}(z_j - z_i), \tag{5}$$

where a_{ij} is the ij-th entry of the adjacency matrix A of the signed graph $\mathscr{S} = \{\mathcal{V}, \mathcal{E}_s, \mathcal{W}\}$, defined as

$$a_{ij} = \sum_{\varepsilon\in\mathcal{E}_c^{j,i}} (\alpha_\varepsilon)_j \sigma_\varepsilon - \sum_{\varepsilon\in\mathcal{E}_c^{\cdot,\{i,j\}}} (\beta_\varepsilon)_j \sigma_\varepsilon + \mathcal{I}_{\mathcal{P}}(i)\left(\sum_{\varepsilon\in\mathcal{E}^{\{s,j\},i}} (\alpha_\varepsilon)_j k_\varepsilon - \sum_{\varepsilon\in\mathcal{E}^{s,\{i,j\}}} (\beta_\varepsilon)_j k_\varepsilon \right), \tag{6}$$

with the indicator function $\mathcal{I}_{\mathcal{P}}(i) = 1$ if $i \in \mathcal{P}$, and zero otherwise.

Proof. From [5, Proposition 1], the thesis follows. □

This equivalence is graphically illustrated in Fig. 1.

Next, introducing the Laplacian matrix L associated to $\mathscr{S}$, we can rewrite the overall system dynamics in compact matrix form as

$$\dot z = (I_N \otimes F) z - (L \otimes I_n) z. \tag{7}$$

When pinning control is achieved, network (7) synchronizes, whereby $z_1(t) = \ldots = z_{N+1}(t)$.

Note that, as the pinner's dynamics are independent of those of the controlled nodes, the Laplacian matrix L can be cast in block-triangular form as follows

$$L = \begin{bmatrix} M & \xi \\ 0_N^{\mathrm{T}} & 0 \end{bmatrix}, \tag{8}$$

where M is the reduced Laplacian that can be obtained by removing the last row and column. Recalling that the pinning error $e_i = x_i - x_s$ can be equivalently written as $e_i = z_i - z_{N+1}$, its dynamics can be described by

$$\dot e = (I_N \otimes F) e - (M \otimes I_n) e. \tag{9}$$

For ease of illustration, we focus on the case in which matrix M is diagonalizable, but the convergence analysis that we are going to perform can be trivially extended by considering that M can always be put in Jordan form. Sorting the eigenvalues of M and F in ascending order by their real part, so that $\Re(\lambda_1(M)) \le \ldots \le \Re(\lambda_N(M))$ and $\Re(\lambda_1(F)) \le \ldots \le \Re(\lambda_n(F))$, we are now ready to give the following theorem:

Theorem 1. *Network (1)-(3) is asymptotically controlled to the pinner's trajectory if and only if*

$$\Re(\lambda_n(F)) - \Re(\lambda_1(M)) < 0,$$

Proof. We denote by $T \in \mathbb{R}^{N\times N}$ the matrix that diagonalizes M, that is, such that $T^{-1} M T = \Lambda$, with Λ being the diagonal matrix whose i-th diagonal element is the i-th eigenvalue of M. Next, we define the transformed state ζ as

$$\zeta = (T^{-1} \otimes I_n) e,$$

whose dynamics are given by

$$\dot\zeta = (I_N \otimes F)\zeta - (\Lambda \otimes I_n)\zeta.$$

Decomposing the vector $\zeta \in \mathbb{R}^{nN}$ as $[\zeta_1; \ldots; \zeta_N]$, where $\zeta_i \in \mathbb{R}^n$ for all i, we obtain a set of N decoupled n-dimensional equations, that is,

$$\dot\zeta_i = (F - \lambda_i(M) I_n)\zeta_i.$$

The thesis then follows. □

The result proposed in Theorem 1 can be used to highlight the main differences between the standard pinning control over digraphs and its hyper-diffusive counterpart. We illustrate these differences through the following two paradigmatic examples.

First, in pinning control over standard, positively weighted, digraphs, the communication protocol, which tunes the interactions

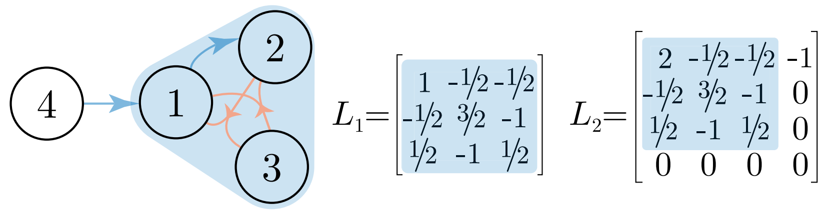

Fig. 2. Sample hypergraph $\mathscr{H}$, where node 4 is the pinner, and the Laplacian matrix L_1 and L_2 associated to the unpinned and pinned hypergraph, respectively.

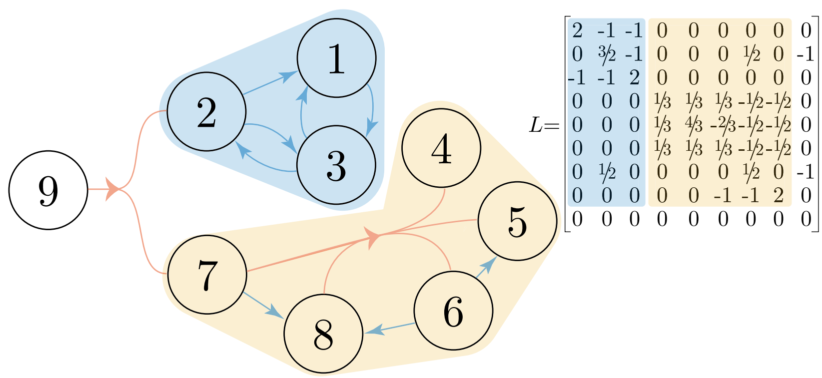

Fig. 3. Sample hypergraph $\mathscr{H}$, where node 9 is the pinner, and its associated Laplacian matrix L.

proportionally to the differences between the nodes, cannot destabilize an otherwise stable system. This implies that, if we are controlling a network of physical systems, a trial-and-error strategy for the choice of the pinned nodes would be viable. Unfortunately, this result does not hold on hypergraphs, as illustrated below.

Example 1. Let us consider the 4-node hypergraph $\mathscr{H}$ illustrated in the left panel of Fig. 2, where node 4 is the pinner, and suppose that the each node is a first-order integrator, i.e. $F = 0$. The dynamics of the unpinned network are then described by the Laplacian L_1 of the signed graphs associated to the hypergraph $\mathscr{H}_c$ connecting the controlled nodes. The eigenvalues of L_1 are 0, 0.8, and 2.2, thus implying that consensus would be achieved in the absence of the pinner. If we attempt to change the consensus value to the state of the pinner (node 4 in Fig. 2) by pinning node 1, the eigenvalues of the Laplacian L_2 of the signed graphs associated to enlarged hypergraph $\mathscr{H}$ would be -0.06, 0, 1.68, and 2.38. Therefore, the pinning action has destabilized an otherwise stable system, something that cannot occur on digraphs.

A second standard result on pinning control on digraphs is that, if the coupling between any two nodes is sufficiently high, the existence of a path from the pinner to any other network node suffices to control the entire network onto the pinner trajectory, whereby pinning a node in every strongly connected component (SCC) of the graph would be sufficient [19]. This classic result does not trivially extend to hypergraphs. Indeed, the existence of a hyperpath [12] from the pinner to all other nodes does not ensure that $\Re(\lambda_2(L)) > 0$, as illustrated in the next example.

Example 2. Let us consider the hypergraph $\mathscr{H}$ depicted in Fig. 3, where node 9 is the pinner, and each node is a one-dimensional integrator, i.e., $F = 0$. The hypergraph has two disjoint strongly connected components and a hyperedge with one tail, corresponding to the pinner, and two heads, one in each strongly connected component. Denoting by L the Laplacian associated to $\mathscr{H}$, it can be noted that matrix M has a 0 eigenvalue with eigenvector v, whose i-th element is equal to any scalar γ if i belongs to the first SCC, and to $-\gamma$ if it belongs to the second. This implies that 0 has multiplicity 2 as an eigenvalue of L, whereby $\Re(\lambda_2) = 0$, and therefore that $\lambda_1(M) \leq 0$. Theorem 1 then implies that the nodes in the controlled network will not asymptotically converge to the pinner's state. We emphasize that this result can be generalized to any hypergraph where two root strongly connected components are pinned with a single hyperedge, and whose heads are partly in one root and partly in the other.

These simple counterexamples support the need for a sufficient condition that allows us to algorithmically verify if we can successfully pinning control a hypergraph by only increasing the gains of the hyperedges. Toward this goal, we start by recasting Eqs. (1) and (3) as

$$\dot{x}_i = Fx_i + \sigma \sum_{\varepsilon \in \mathcal{E}_c^{:,i}} \varsigma_\varepsilon \left(\alpha_\varepsilon^{\mathrm{T}} x_\varepsilon^\tau - \beta^{\mathrm{T}} x_\varepsilon^h \right) + u_i, \tag{10}$$

$$u_i = \begin{cases} \sigma \sum_{\varepsilon \in \mathcal{E}^{s,i}} c_\varepsilon \left(\alpha_\varepsilon^{\mathrm{T}} x_\varepsilon^\tau - \beta_\varepsilon^{\mathrm{T}} x_\varepsilon^h \right), & i \in \mathcal{P}, \\ 0, & \text{otherwise}, \end{cases} \tag{11}$$

where $\sigma > 0$ is the overall coupling strength between interacting nodes, and $\varsigma_\varepsilon = \sigma_\varepsilon / \sigma$ and $c_\varepsilon = k_\varepsilon / \sigma$. Furthermore, we denote $\mathscr{S}^+ = \{\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+\}$ the subgraph of $\mathscr{S}$ whose edge set only contains the positive edges of $\mathscr{S}$, that is, such that $\mathcal{E}^+ = \{(i,j) : a_{ij} > 0\}$, and $\mathcal{W}^+(i,j) = \mathcal{W}(i,j)$ if $(i,j) \in \mathcal{E}^+$, whereas $\mathcal{W}(i,j) = 0$ otherwise. Let us define the edge set $\mathcal{E}^- = \{(i,j) \in \mathcal{E}, j > i : a_{ij} < 0\}$, and let us denote by η its cardinality. We label the edges of $\mathcal{E}^-$ as $(i_1, j_1), \dots, (i_\eta, j_\eta)$, and compute iteratively the binary variable b as follows:

Step 0. Set $\mathscr{S}_{\text{curr}} = \mathscr{S}^+$, and denote as M_{curr} the reduced Laplacian matrix M of $\mathscr{S}_{\text{curr}}$. If 0 is an eigenvalue of M_{curr}, set $b = 0$ and terminate the algorithm. Otherwise, set $b = 1$ and $\mathscr{S}_{\text{curr}} = \mathscr{S}^+$, and proceed to the following step.

Step k. Part 1. Set $\mathscr{S}_1 = \mathscr{S}_{\text{curr}}$ and $\mathscr{S}_2 = \{\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2\}$, where $\mathcal{E}_2 = \{(i_k, j_k) \cup (j_k, i_k)\}$, $\mathcal{W}_2(i_k, j_k) = -\delta\sigma_{ij}$ and $\mathcal{W}_2(j_k, i_k) = -\delta\sigma_{ji}$, with $\sigma_{ij} = -a_{i_k, j_k}$ and $\sigma_{ji} = -a_{j_k, i_k}$ if $a_{j_k, i_k} < 0$, whereas $\sigma_{ji} = 0$ otherwise.

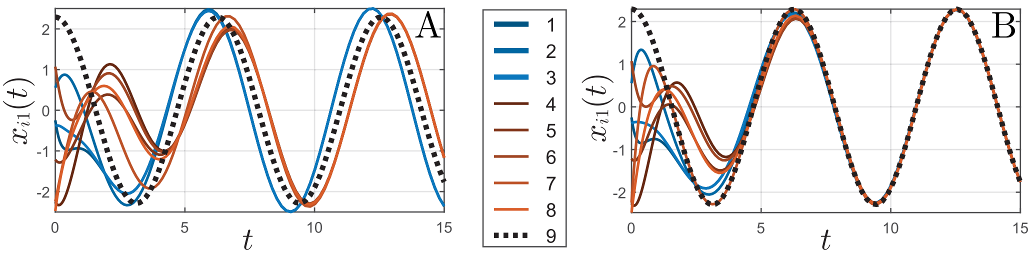

Fig. 4. Dynamics of the first component of the states of all nodes for a controlled network of 8 harminc oscillators when the hypergraph is that in Fig. 3 (panel A) and when edges (4,6) and (2,7) are added (panel B).

Step k. Part 2. Update $\mathscr{S}_{\text{curr}}$ to $\mathscr{S}_1 \oplus \mathscr{S}_2$, and compute

$$\delta^\star = \min_{\delta>0,\omega\geq 0} \delta \quad \text{subject to } \Re(\lambda_1(M_{\text{curr}})) = 0 \tag{12}$$

Step k. Part 3. If $\delta^\star \leq 1$, set $b = 0$ and terminate the algorithm. If $\delta^\star > 1$ and $k = \eta$ terminate the algorithm with $b = 1$. If $\delta^\star > 1$ and $k < \eta$, keep $b = 1$ and go to step $k + 1$.

Remark 1. The continuity of the eigenvalues allows computing $\delta^\star$ as the normalized weight that would induce an eigenvalue, or a pair of complex eigenvalues, to cross the imaginary axes. This implies that, when the algorithm outputs $b = 1$, then $\lambda_1(M) > 0$, thus implying the following corollary.

Corollary 1. *If $b = 1$, then there exist a finite σ in (10)-(11) such that network (1)-(3) is asymptotically controlled to the pinner's trajectory.*

Proof. The thesis follows as a consequence of Remark 1 and the application of Theorem 1. □

Remark 2. Step 0 of the above algorithm is equivalent to checking that $\mathscr{S}^+$ contains a directed spanning tree rooted in s, which is a necessary and sufficient condition for having $\lambda_1(M_{\text{curr}}) > 0$. The next steps of the algorithms are used to check that the addition of negative edges still preserves the property that all the eigenvalues of M_{curr} have positive real part.

Note that Corollary 1 can then be used to design the hyperedges connecting the pinner to the rest of the network, so as to ensure that i) there is a directed hyperpath from the pinner to the rest of the network, and ii) that the negative edges are not detrimental for pinning control.

Remark 3. Should the hypergraph $\mathscr{H}$ be undirected, then the dynamics (1) could be equivalently studied in terms of a standard, undirected graphs with positive weights [21]. Indeed, we could replace each hyperedge ε by an all-to-all subgraph connecting the nodes in $\varepsilon \in \mathcal{E}$, with associated positive weights $\sigma_\varepsilon/|\varepsilon|$. However, the hypergraph $\mathscr{H}$ is intrinsically directed because of the presence of the pinner and therefore, even in the case of a linear coupling protocol as in (1), cannot be reduced to a standard digraph with positive weights.

5. Applications

5.1. Pinning control of harmonic oscillators

In the general Eq. (3), we consider as individual dynamics the harmonic oscillator with unitary frequency. Namely, we set

$$F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Initially, the hypergraph through which the nodes are coupled is the one in Fig. 3, where the pinner is node 9, $\sigma_\varepsilon = 1$ for all ε, and the hyperdiffusive protocol is homogeneous, that is, $\alpha_\varepsilon = \mathbb{1}_{|\mathcal{T}(\varepsilon)|}/|\mathcal{T}(\varepsilon)|$ and $\beta_\varepsilon = 1_{|\mathcal{H}(\varepsilon)|}/|\mathcal{H}(\varepsilon)|$ [5]. Fig. 4 A shows that pinning control is not achieved. Consistently, the algorithm in Section 4 yields $b = 0$. Interestingly, the algorithm stops when either the negative edge (4,6) or (2,7) are added. This suggests to add two corresponding directed positive edges in $\mathscr{H}$. Running the algorithm again, we finally obtain $b = 1$, which, according to Corollary 1, implies that the network is asymptotically controlled on the pinner trajectory, as illustrated in Fig. 4B.

5.2. Opinion consensus on hypergraphs

We consider a controlled network of $N = 16$ individuals interconnected through hypergraph $\mathscr{H}$ (Fig. 5A). The nodes are grouped in 3 (fully connected) strongly connected components (that we call communities), depicted in blue, yellow and red, respectively. Standard edges connect a blue node to two yellow nodes, and a yellow node to a red node. The pinner (in black) injects control signals into three red nodes through standard edges, and can add two hyperedges ε_1 and ε_2 (in red), with weights $k_{\varepsilon_1} = k_1$ and $k_{\varepsilon_2} = k_2$; the set of heads of ε_1 coincide with the yellow nodes, whereas that of ε_2 coincide with the blue nodes.

We consider first-order integrators ($F = 0$) as in the seminal DeGroot opinion dynamics model [7]. The goal of the pinner is to exploit its connections to steer all the individuals towards its own opinion. We perform 5 simulations where the hyperdiffusive protocol is homogeneous, $\sigma_\varepsilon = 1$ for all $\varepsilon \in \mathcal{E}_c$, $x_i(0) = i$ for all $i \in \mathcal{V}$, and we vary the pinning gains (k_1, k_2) as shown in Fig. 5B-F.

Fig. 5 B shows that, if the pinner does not use either ε_1 or ε_2 and sets σ_ε and k_ε equal to 1 for all ε, no nodes converge onto its opinion, while the states of the nodes of the yellow community converge onto the opinion of those of the blue community. Fig. 5C shows that leveraging ε_1 and setting $k_{\varepsilon_1} = 2$, the pinner can prevent the achievement of consensus between the yellow and blue communities. If the pinner increases the gain k_{ε_1} to 4 (panel D), the state of the nodes in the red community and that of a subset of the yellow community will converge to the pinner's opinion. As shown in Fig. 5E, if the pinner further increases the control gain to $k_{\varepsilon_1} = 8$, then again no nodes will converge to the pinner's opinion, with half of the yellow community having an opinion that is more extreme than that of the pinner. Finally, Fig. 5F shows that when the pinner uses the hyper-edge ε_2 with gain $k_{\varepsilon_1} = 10$, the state of the nodes in all the communities will converge onto the pinner's opinion, as the unique root strongly connected component of the controlled hypergraphs encompasses pinned nodes.

In agreement with the numerical results, applying the algorithm in Section 4 to the augmented hypergraph parametrized as in the simulations shown in Fig. 5B-E yields $b = 0$ at step 0. On the other hand, applied to the parametrization corresponding to Fig. 5F, the algorithm in Section 4 runs for a total of 10 steps and yields $b = 1$, in line with the numerical observation that consensus onto the state of the pinner is attained.

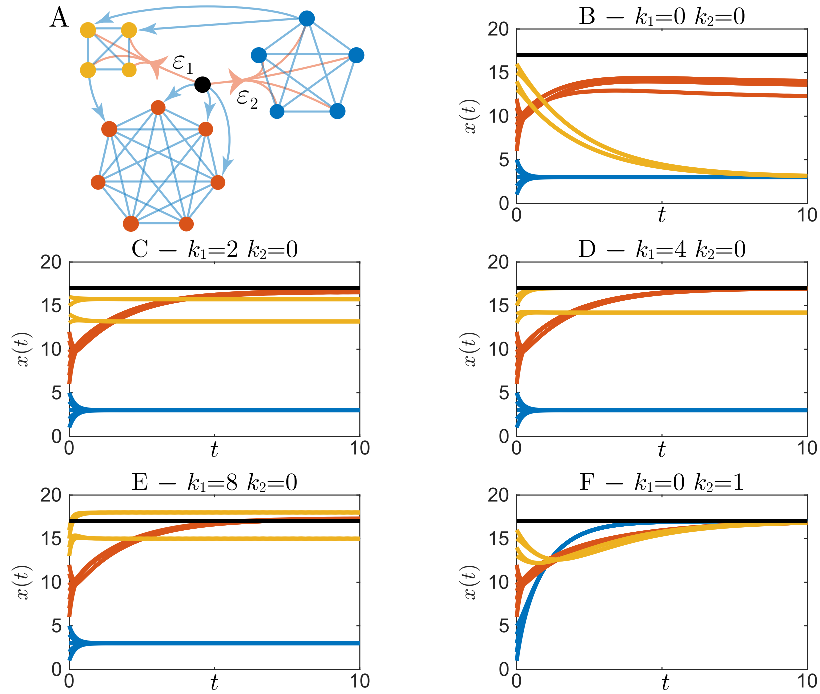

Fig. 5. Pinning control for consensus on hypegraphs. Panel A depicts the hypergraph topology with blue, yellow and red nodes identifying the 3 SCCs and with the pinner being the black node. Panels B-E show that, if $\sigma_{\varepsilon} = 1$ and the pinner sets $k_{\varepsilon_2} = 0$, consensus is not achieved on the pinner's state, regardless of the value of k_{ε_1}. On the other hand, panel F shows that when $k_{\varepsilon_2} = 1$, consensus is achieved onto the pinner's opinion even if $k_{\varepsilon_1} = 0$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

6. Conclusions

In this paper, we have further expanded the results of [5] to provide a necessary and sufficient condition for pinning control of linear dynamical systems in the presence of higher-order, directed interactions. The analogy between directed hypergraphs and a class of signed graphs allowed us to stress that, different from pinning control on digraphs, the addition of hyperedges is not always beneficial for control, and may even make the dynamics of the controlled network divergent. In addition, we also gave an algorithm that can be used to identify the critical hyperedges that may prevent convergence onto the pinner's trajectory. We demonstrated the approach on two numerical examples on synchronization of harmonic oscillators, and opinion consensus on hypergraphs. Interestingly, we observed that dynamics over hypergraphs exhibit a richer behavior than that observed on digraphs, whereby partial pinning control [8] may be reached even without pinning any root strongly connected component, thereby calling for further research on this control problem.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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