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Global well-posedness and convergence to equilibrium for the Abels-Garcke-Grün model with nonlocal free energy ☆

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A B S T R A C T

We investigate the nonlocal version of the Abels-Garcke-Grün (AGG) system, which describes the motion of a mixture of two viscous incompressible fluids. This consists of the incompressible Navier-Stokes-Cahn-Hilliard system characterized by concentration-dependent density and viscosity, and an additional flux term due to interface diffusion. In particular, the Cahn-Hilliard dynamics of the concentration (phase-field) is governed by the aggregation/diffusion competition of the nonlocal Helmholtz free energy with singular (logarithmic) potential and constant mobility. We first prove the existence of global *strong* solutions in general two-dimensional bounded domains and their uniqueness when the initial datum is strictly separated from the pure phases. The key points are a novel well-posedness result of strong solutions to the nonlocal convective Cahn-Hilliard equation with singular potential and constant mobility under minimal integral assumption on the incompressible velocity field, and a new two-dimensional interpolation estimate for the $L^4(\Omega)$ control of the pressure in the stationary Stokes problem. Secondly, we show that any weak solution, whose existence was already known, is globally defined, enjoys the propagation of regularity and converges towards an equilibrium (i.e., a stationary solution) as $t \to \infty$. Furthermore, we demonstrate the uniqueness of strong solutions and their continuous dependence with respect to general (not necessarily separated) initial data in the case of matched densities and unmatched viscosities (i.e., the nonlocal model H with variable viscosity, singular potential and constant mobility). Finally, we provide a stability estimate between the strong solutions to the nonlocal AGG model and the nonlocal Model H in terms of the difference of densities.

R É S U M É

Nous étudions la version non locale du système d'Abels-Garcke-Grün (AGG), qui décrit l'évolution d'un mélange de deux fluides incompressibles. Il est constitué du système de Navier-Stokes-Cahn-Hilliard incompressible caractérisé par des densités et viscosités dépendant de la concentration, et d'un terme de flux additionnel dû à la diffusion d'interface. En particulier, la dynamique de Cahn-

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Hilliard de la concentration (champ de phase) est gouvernée par la compétition aggrégation/diffusion de l'énergie libre de Helmholtz non locale avec potentiel singulier (logarithmique) et mobilité constante. Nous démontrons d'abord l'existence globale de solutions *fortes* dans un domaine borné de dimension deux, ainsi que leur unicité lorsque la donnée initiale est strictement séparée des phases pures. Les points clefs sont un nouveau résultat sur le caractère bien-posé de solutions fortes de l'équation de Cahn-Hilliard convective avec potentiel singulier et mobilité constante sous une hypothèse intégrale minimale sur le champ de vitesse incompressible et une nouvelle estimation d'interpolation en dimension deux pour le contrôle $L^4(\Omega)$ de la pression dans le problème de Stokes stationnaire. Dans un deuxième temps, nous montrons que toute solution faible, dont l'existence est déjà connue, est définie globalement, propage la régularité et converge vers un équilibre (i.e., une solution stationnaire) lorsque $t \to \infty$. De plus, nous montrons l'unicité des solutions fortes et leur dépendance continue par rapport à des données initiales générales (pas nécessairement séparées) dans le cas de densités égales et viscosités différentes (i.e., the modèle H non local avec viscosité variable, potentiel singulier et mobilité constante). Finalement, nous donnons une estimation de stabilité entre les solutions fortes du modèle AGG non local et le modèle H non local en terme de la différence des densités.

1. Introduction and main results

In the Diffuse Interface theory, the motion of a mixture of two incompressible viscous Newtonian fluids and the evolution of the interface separating the bulk phases have been originally modeled by the so-called Model H (see, e.g., [7,41,43]). This leads to the following Navier-Stokes-Cahn-Hilliard system

$$\begin{cases} \rho\partial_t \mathbf{u} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \operatorname{div}(\nu(\phi)D\mathbf{u}) + \nabla\Pi = -\operatorname{div}(\nabla\phi\otimes\nabla\phi), \\ \operatorname{div}\mathbf{u} = 0, \\ \partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}(m(\phi)\nabla\mu), \\ \mu = -\Delta\phi + \Psi'(\phi), \end{cases} \tag{1.1}$$

in $\Omega \times (0,\infty)$. Here, Ω is a bounded domain in $\mathbb{R}^d$, $d = 2,3$, $\mathbf{u}$ represents the (volume averaged) velocity, $D\mathbf{u} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ is the symmetric strain tensor, Π denotes the pressure and ϕ is the order parameter (i.e., the relative concentration difference of the two mixture components). Also $\nu(\cdot) > 0$ is the viscosity of the mixture, ρ is the *constant* mixture density, $m(\cdot) \geq 0$ is the mobility function, and Ψ is the Flory-Huggins double-well potential defined by

$$\Psi(s) = \frac{\alpha}{2}\left((1+s)\ln(1+s) + (1-s)\ln(1-s)\right) - \frac{\alpha_0}{2}s^2 = F(s) - \frac{\alpha_0}{2}s^2, \quad \forall\, s \in [-1,1], \tag{1.2}$$

where the two positive parameters α, α_0 satisfy the relations $0 < \alpha < \alpha_0$. This potential, in particular, ensures the existence of *physical solutions*, that is, solutions such that $\phi \in [-1,1]$. One of the fundamental modeling assumptions of (1.1) is that the densities of both components match and thereby the density of the mixture ρ is constant. This restricts the applicability of the model to those fluid mixtures having a non-negligible difference between the two densities $\rho_1, \rho_2 > 0$. To overcome it, the so-called Abels-Garcke-Grün (AGG) system has been introduced in the seminal work [9] as a thermodynamically consistent generalization of the Model H, allowing to treat fluids with unmatched densities. The AGG model reads as follows

$$\begin{cases} \partial_t(\rho(\phi)\mathbf{u}) + \operatorname{div}\,(\mathbf{u}\otimes(\rho(\phi)\mathbf{u}+\mathbf{J})) - \operatorname{div}\,(\nu(\phi)D\mathbf{u}) + \nabla\Pi = -\operatorname{div}\,(\nabla\phi\otimes\nabla\phi)\,, \\ \operatorname{div}\mathbf{u} = 0, \\ \partial_t\phi + \mathbf{u}\cdot\nabla\phi = \operatorname{div}\,(m(\phi)\nabla\mu), \\ \mu = -\Delta\phi + \Psi'(\phi), \end{cases} \tag{1.3}$$

in $\Omega\times(0,\infty)$, where

$$\rho(\phi) = \rho_1\frac{1+\phi}{2} + \rho_2\frac{1-\phi}{2}, \quad \mathbf{J} = -\frac{\rho_1-\rho_2}{2}m(\phi)\nabla\mu. \tag{1.4}$$

System (1.3) is usually supplemented with the boundary and initial conditions

$$\begin{cases} \mathbf{u}=0, \quad \partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}\mu = 0, & \text{on } \partial\Omega\times(0,\infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \phi|_{t=0}=\phi_0, & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

In both Model H and AGG model, the fluid mixture is driven by the capillary forces $-\operatorname{div}(\nabla\phi\otimes\nabla\phi)$, accounting for the surface tension effect, together with a partial diffusive mixing. The latter is assumed in the interfacial region and it is modeled by $\operatorname{div}(m(\phi)\nabla\mu)$. The specificity of the AGG model compared to the Model H lies in the presence of the flux term $\mathbf{J}$. In contrast to the one-phase flow, the (average) density $\rho(\phi)$ in (1.3) does not satisfy the continuity equation with respect to the flux associated with the velocity $\mathbf{u}$. Instead, the density satisfies the continuity equation with a flux given by the sum of the transport term $\rho(\phi)\mathbf{u}$ and the term $\mathbf{J}$, which is due to the diffusion of the concentration in the unmatched densities case[1] (see also [37] and the references therein). Notice that we recover (1.1) when $\rho_1=\rho_2$ in (1.3)-(1.4).

Concerning the mathematical analysis of the AGG model (1.3)-(1.5), the existence of global weak solutions were proven in [5] and [6] in the case of strictly positive and degenerate mobility $m(\cdot)$, respectively. The existence of global weak solutions has been extended in [4] to viscous non-Newtonian binary fluids (with constant mobility) and in [31] to the case of dynamic boundary conditions (with strictly positive mobility). The convergence of a fully discrete numerical scheme to weak solutions was shown in [42]. More advanced issues related to the well-posedness, regularity and longtime behavior have obtained a renewed interest in the last years. In [12], the local well-posedness of strong solutions has been proven in three dimensions for polynomial-like potentials Ψ_{pol} and strictly positive mobility. We point out that the solution in [12] may not satisfy $|\phi(x,t)|\le 1$ in the space-time domain. In [37], the well-posedness of (local-in-time) strong solutions in two dimensional bounded domains has been obtained for the logarithmic potential (1.2) and constant mobility. In this case, ϕ takes its values in the physical range $[-1,1]$. If the boundary conditions are periodic then the strong solutions are globally defined in time (see [37]). The case of bounded three-dimensional domains has been investigated in [38], where the well-posedness of local strong solutions is shown. More recently, the propagation of regularity in time of any weak solutions in three dimensions and its stabilization towards an equilibrium state as $t\to\infty$ have been achieved in [8] in the case of constant mobility (see also [2,39] for the matched density case). In the latter, the authors also discussed the global well-posedness in bounded two-dimensional domains and the deep quench limit. We conclude this part by mentioning that the realm of Diffuse Interface (phase field) models for fluid mixtures has been widely deepened in the past decades. Several models have been proposed to describe binary mixtures with non-constant density relying on different assumptions. We refer the interested reader to the models derived, e.g., in [16,36,45,51,56,59] and the analysis carried out in [1,3,15,40,47].

[1] Indeed, $\rho=\rho(\phi)$ satisfies the continuity equation $\partial_t\rho + \operatorname{div}\,(\rho\mathbf{u}+\mathbf{J})=0$.

The evolution of the phase-field variable in both Model H and AGG model is modeled by the local Cahn-Hilliard equation driven by divergence-free drift. The chemical potential μ in (1.3) is defined as the first variation of the Ginzburg-Landau free energy

$$\mathcal{E}_{\mathrm{loc}}(\phi) = \int_\Omega \frac{1}{2}|\nabla\phi|^2 + \Psi(\phi)\,\mathrm{d}x. \tag{1.6}$$

The free energy $\mathcal{E}_{\mathrm{loc}}(\phi)$ only focuses on short range interactions between particles. Indeed, the gradient square term accounts for the fact that the local interaction energy is spatially dependent and varies across the interfacial surface due to spatial inhomogeneities in the concentration. Going back to the general approach of statistical mechanics, the mutual short and long-range interactions between particles are described through convolution integrals weighted by interactions kernels. Based on this ancient approach (see [55]), Giacomin and Lebowitz ([33–35]) observed that a physically more rigorous derivation leads to a nonlocal dynamics, which is the nonlocal Cahn-Hilliard equation. In particular, this equation is rigorously justified as a macroscopic limit of microscopic phase segregation models with particle-conserving dynamics. In this case, the gradient term is replaced by a nonlocal spatial interaction integral, namely

$$\mathcal{E}_{\mathrm{nloc}}(\phi) = \int_\Omega F(\phi(x))\,\mathrm{d}x - \frac{1}{2}\int_\Omega\int_\Omega K(x-y)\phi(x)\phi(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{1.7}$$

where K is a sufficiently smooth symmetric interaction kernel. As shown in [34] (see also [29] and the references therein), the energy $\mathcal{E}_{\mathrm{loc}}$ can be seen as an approximation of $\mathcal{E}_{\mathrm{nloc}}$, as long as we suitably redefine Ψ as $\widetilde{\Psi}(x,s) = F(s) - (K*1)(x)s^2/2$. The physical relevance of nonlocal interactions was already pointed out in the pioneering paper [55] (see also [46, 4.2] and references therein) and studied for different kind of evolution equations, mainly Cahn-Hilliard and phase field systems, see, e.g., [14,18,27,28,30,48–50]. In this context, the nonlocal AGG model we want to analyze reads as

$$\begin{cases} \partial_t(\rho(\phi)\mathbf{u}) + \mathrm{div}\,(\mathbf{u}\otimes(\rho(\phi)\mathbf{u}+\mathbf{J})) - \mathrm{div}\,(\nu(\phi)D\mathbf{u}) + \nabla\Pi = \mu\nabla\phi,\\ \mathrm{div}\,\mathbf{u} = 0,\\ \partial_t\phi + \mathbf{u}\cdot\nabla\phi = \mathrm{div}\,(m(\phi)\nabla\mu),\\ \mu = F'(\phi) - K*\phi, \end{cases} \tag{1.8}$$

in $\Omega\times(0,\infty)$, where F, ρ and $\mathbf{J}$ are given in (1.2) and (1.4). System (1.8) is endowed with the boundary and initial conditions

$$\begin{cases} \mathbf{u} = 0, \quad \partial_{\mathbf{n}}\mu = 0 & \text{on } \partial\Omega\times(0,\infty),\\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \phi|_{t=0} = \phi_0 & \text{in } \Omega. \end{cases} \tag{1.9}$$

Regarding the mathematical analysis of the nonlocal AGG system, there are much fewer contributions than in the local case. The nonlocal AGG system (1.8)-(1.9) has only been analyzed so far in [22] and in [23]. In the former, the existence of weak solutions is shown in the case of singular (logarithmic) potential and strictly positive mobility in two and three dimensional bounded domains. In the latter, the existence of weak solutions is proven in the case of singular potential and degenerate mobility. Another nonlocal variant, focused on fractional diffusion, has been investigated in [10], where the authors proved the existence of weak solutions in two and three dimensional bounded domains in the case of a singular nonlocal free energy and strictly positive mobility. More precisely, the "diffusive" term $|\nabla\phi|^2$ in E_{loc} is replaced by a singular nonlocal operator which controls the $H^{\frac{\alpha}{2}}(\Omega)$ norm of the concentration ϕ for $\alpha\in(0,2)$ (as a consequence, $\Delta\phi$ in $(1.3)_4$ is replaced by a regional fractional Laplacian). More recently, the authors in [11] have shown that such weak solutions converge to those of (1.3) in the setting introduced in [19]-[20]. In addition, in [23, Theorem 3.5], the author also proved that, under suitable assumptions on the initial datum, the kernel K, the potential F and the degenerate mobility, the concentration function ϕ enjoys the regularity $L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ in two dimensions (cf. Theorem 1.3 for weak solutions reported below). In particular, it is worth pointing out that the assumptions (**A4**) and (**A1b**) in [23], i.e. $m(\cdot)F''(\cdot)\in C^1([-1,1])$ such that $m(\cdot)F''(\cdot)$ is strictly positive, allows to rewrite $(1.8)_{3,4}$ as

$$\partial_t\phi+\mathbf{u}\cdot\nabla\phi=\operatorname{div}\left(m(\phi)F''(\phi)\nabla\phi-\nabla K*\phi\right),\quad \text{in } \Omega\times(0,\infty), \tag{1.10}$$

where the main diffusion operator is a second-order divergence form operator with positive and bounded coefficients. On the other hand, if m is constant (or even non-degenerate) at the endpoints ±1, the product $mF''(\cdot)$ no longer enjoys the typical cancellation effect seen with phase segregation phenomena when $m(\pm1)=0$ (see [25]). In particular, the (variable) diffusion coefficient $mF''(\cdot)$ in (1.10) is highly singular and unbounded since $mF''(s)=\alpha m(1-s^2)^{-1}$, owing to (1.2). To the best of our knowledge, there are yet no results concerning the global well-posedness for the nonlocal AGG model (1.8)-(1.9) in dimension two in the case of singular potentials and constant mobility. As a matter of fact, the same open questions remain still unresolved even for the nonlocal Model H with unmatched viscosities and logarithmic-like potential, which corresponds to (1.8)-(1.9) with $\rho_1=\rho_2$. In fact, beyond the global existence of weak solution for the nonlocal Model H established in [26], the uniqueness of weak solutions, their propagation of regularity and their longtime behavior in two dimensions has been discussed in [24] and [28] in the case of constant viscosity only.

The aim of the present paper is to present the first well-posedness result concerning the nonlocal AGG model (with unmatched densities and viscosities) in presence of singular-like potentials and constant mobility. In fact, following [29], we consider a general class of entropy potentials, commonly employed for complex binary particle systems experiencing long range interactions. This class generalizes the classical logarithmic density function (1.2). Recall that the latter is uniquely generated by Boltzmann-Gibbs statistics of macroscopic mixing of the fluid constituents. To this end, let us first state the main assumptions, which will be adopted throughout our analysis:

(H_1) Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ of class C^2.
(H_2) The interaction kernel $K\in W^{1,1}(\mathbb{R}^2)$ is such that $K(x)=K(-x)$ for all $x\in\mathbb{R}^2$.
(H_3) $F\in C([-1,1])\cap C^2(-1,1)$ fulfills

$$\lim_{s\to-1}F'(s)=-\infty,\quad \lim_{s\to1}F'(s)=+\infty,\quad F''(s)\geq\alpha,\quad \forall\, s\in(-1,1).$$

We extend $F(s)=+\infty$ for any $s\notin[-1,1]$. Without loss of generality, $F(0)=0$ and $F'(0)=0$. In particular, this entails that $F(s)\geq0$ for any $s\in[-1,1]$.

(H_4) We assume[2] that F'' is monotone non-decreasing on $[1-\varepsilon_0,1)$, for some $\varepsilon_0>0$ and there exist $p\in[2,\infty)$ and a continuous function $h:(0,1)\to\mathbb{R}_+$, $h(\delta)=o(\delta^{4/p})$, as $\delta\to0^+$, such that

$$\begin{cases} F''(s)\leq C\mathrm{e}^{C|F'(s)|^\beta}, & \text{for all } s\in(-1,1),\\ F''(1-2\delta)\,h(\delta)\geq1, & \text{for all } \delta\leq\dfrac{\varepsilon_0}{2}, \end{cases} \tag{1.11}$$

for some $C>0$ and $\beta\in[1,2)$.

(H_5) The mobility is constant and equal to unity, i.e., $m\equiv1$.

[2] Without loss of generality we also assume that F'' is symmetric on $(-1,1)$, see [29]. Other statistical entropy density functionals (that can be more singular at ±1 than the logarithmic density (1.2)) from information theory are included in this study.

(H_6) The density values ρ_1 and ρ_2 are positive. The viscosity $\nu \in W^{1,\infty}(\mathbb{R})$ satisfies

$$0 < \nu_* \le \nu(s) \le \nu^*, \quad \forall\, s \in \mathbb{R},$$

for some positive values ν_*, ν^*.

Remark 1.1. The logarithmic convex function F in (1.2) fulfills (H_3) and (H_4) (cf. [29]). A common form for the viscosity is the following

$$\nu(s) = \nu_1 \frac{1+s}{2} + \nu_2 \frac{1-s}{2}, \quad s \in [-1,1],$$

which can be easily extended on the whole $\mathbb{R}$ in such way to comply with $((H_6))$. Many other examples of entropy densities satisfying (H_3) and (H_4) (including the Tsallis entropy) can be found in [29, Sections 6.2, 6.3].

Remark 1.2. Among radially symmetric kernels K that satisfy (H_2) are Newtonian, Bessel and Riesz like potentials. For instance, consider for $x \in \mathbb{R}^2 \setminus \{0\}$, the Bessel potential

$$b_s(|x|) = \frac{e^{-|x|}}{(2\pi)\, 2^{s/2} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{3-s}{2}\right)} \int_0^\infty e^{-|x|t} \left(t + \frac{t^2}{2}\right)^{\frac{1-s}{2}} \mathrm{d}t,$$

where Γ is the Gamma function and $s > 0$. Note that on $\mathbb{R}^2$, $(I - \Delta)^{-s/2} v = b_s * v$. In particular, b_s behaves as the Riesz potential, asymptotically as $|x| \to 0^+$, since

$$b_s(|x|) = \frac{\Gamma(2-s)}{2^s \pi^{s/2}} \frac{1}{|x|^{2-s}} (1 + o(1)), \quad \text{if } \ 0 < s < 2.$$

Logarithmically behaving kernels are also included in this analysis, as

$$b_2(|x|) = -\frac{1}{2\pi} \log|x| (1 + o(1)), \quad \text{as } \ |x| \to 0^+.$$

Other smooth kernels (i.e., either Gaussian or multimodal probability density functions) are allowed. We also refer the interested readers to [13].

Before proceeding with the statements of the main results, we report the only available result for the nonlocal AGG system (1.8)-(1.9), which concerns the existence of weak solutions on any fixed time interval $(0, T)$ proven in [22]. For the sake of completeness, we report it in the original form with the non-degenerate mobility. We refer the reader to Section 2 for functional space notation. For instance, $(\cdot, \cdot)$ will denote the inner product in $L^2(\Omega; \mathbb{R}^2)$ and in $L^2(\Omega; \mathbb{R}^{2\times 2})$.

Theorem 1.3. *Let (H_1)-(H_3) and (H_6) hold and let $m \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$ such that $0 < m_* \le m(s) \le m^*$ for all $s \in \mathbb{R}$ for some m_* and m^*. Assume that $\mathbf{u}_0 \in L^2_\sigma(\Omega)$ and $\phi_0 \in L^\infty(\Omega)$ with $F(\phi_0) \in L^1(\Omega)$ and $|\overline{\phi_0}| < 1$. Then, for any $T > 0$, there exists a weak solution to* (1.8)-(1.9) *in $(0, T)$ such that* *(i) The pair* $(\mathbf{u}, \phi)$ *satisfies the properties*

$$\begin{cases} \mathbf{u} \in C_{\mathrm{w}}([0,T]; L^2_\sigma(\Omega)) \cap L^2(0,T; H^1_{0,\sigma}(\Omega)), \\ \phi \in L^\infty(\Omega \times (0,T)) \cap L^2(0,T; H^1(\Omega)) \text{ with } |\phi| < 1 \text{ a.e. in } \Omega \times (0,T), \\ \partial_t \left(\rho(\phi)\mathbf{u}\right) \in L^{\frac{4}{3}}(0,T; V^2_{0,\sigma}(\Omega)'), \quad \partial_t \phi \in L^2(0,T; H^1(\Omega)'), \\ \mu = F'(\phi) - K * \phi \in L^2(0,T; H^1(\Omega)). \end{cases} \tag{1.12}$$

(ii) The solution $(\mathbf{u}, \phi)$ *fulfills the system in weak sense:*

$$\langle \partial_t(\rho(\phi)\mathbf{u}), \mathbf{w}\rangle_{V^2_{0,\sigma}(\Omega)} - (\rho(\phi)\mathbf{u} \otimes \mathbf{u}, D\mathbf{w}) + (\nu(\phi)D\mathbf{u}, D\mathbf{w}) - (\mathbf{u}, (\mathbf{J} \cdot \nabla)\, \mathbf{w}) = -\left(\phi \nabla \mu, \mathbf{w}\right), \tag{1.13}$$

$$\langle \partial_t \phi, v\rangle_{H^1(\Omega)} - (\phi\, \mathbf{u}, \nabla v) + (m(\phi)\nabla \mu, \nabla v) = 0, \tag{1.14}$$

for any $\mathbf{w} \in V^2_{0,\sigma}(\Omega)$, $v \in H^1(\Omega)$ *and almost everywhere in* $(0,T)$.

(iii) The initial conditions $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ *and* $\phi(\cdot, 0) = \phi_0$ *hold in* Ω.

(iv) The energy inequality

$$E(\mathbf{u}(t), \phi(t)) + \int_s^t \left\| \sqrt{\nu(\phi(\tau))} D\mathbf{u}(\tau) \right\|^2_{L^2(\Omega)} + \left\| \sqrt{m(\phi)} \nabla \mu(\tau) \right\|^2_{L^2(\Omega)} \mathrm{d}\tau \le E(\mathbf{u}(s), \phi(s)) \tag{1.15}$$

holds for all $t \in [s,T)$ *and almost all* $s \in [0,T)$ *(including* $s = 0$*), where the total energy is defined as*

$$E(\mathbf{u}, \phi) := \frac{1}{2} \int_\Omega \rho(\phi)|\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega F(\phi) \,\mathrm{d}x - \frac{1}{2} \int_\Omega (K * \phi)\phi \,\mathrm{d}x. \tag{1.16}$$

Remark 1.4. The result of [22, Theorem 1] actually holds for a slightly different model than (1.8). Indeed, the nonlocal Helmholtz free energy considered in [22] is

$$E^\star_{\mathrm{nloc}}(\phi) = \frac{1}{4} \int_\Omega \int_\Omega K(x-y)\left(\phi(x) - \phi(y)\right)^2 \mathrm{d}x\, \mathrm{d}y + \int_\Omega F(\phi) - \frac{\alpha_0}{2}\phi^2 \,\mathrm{d}x.$$

As a consequence, the chemical potential is $\mu = a\phi - K * \phi + F'(\phi) - \alpha_0 \phi$. On the other, as explained in [28] the two problems are strictly related and [22, Theorem 1] can be extended also to the problem in our analysis. More precisely, the two models are equivalent as long as we suppose $\alpha_0(x) = a(x)$, where $a(x) = K * 1$, and the differences in the analysis provided in [22] are related only to lower order terms.

Our first main result concerns the global existence and uniqueness of strong solutions to (1.8)-(1.9).

Theorem 1.5. *Let the assumptions* (H_1)*-*(H_6) *hold. Assume that* $\mathbf{u}_0 \in H^1_{0,\sigma}(\Omega)$, $\phi_0 \in H^1(\Omega)$, *with* $|\overline{\phi_0}| < 1$, $F'(\phi_0) \in L^2(\Omega)$ *and* $F''(\phi_0)\nabla \phi_0 \in L^2(\Omega; \mathbb{R}^2)$. *Then, there exists a global strong solution* $(\mathbf{u}, \Pi, \phi) : \Omega \times [0, \infty) \to \mathbb{R}^2 \times \mathbb{R} \times [-1, 1]$ *to* (1.8)*-*(1.9) *such that:*

(i) The solution $(\mathbf{u}, \Pi, \phi)$ *satisfies the properties*

$$\begin{cases} \mathbf{u} \in BC([0,\infty); H^1_{0,\sigma}(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); V^2_{0,\sigma}(\Omega)) \cap H^1_{\mathrm{uloc}}([0,\infty); L^2_\sigma(\Omega)), \\ \Pi \in L^2_{\mathrm{uloc}}([0,\infty); H^1_{(0)}(\Omega)), \\ \phi \in BC_{\mathrm{w}}([0,\infty); H^1(\Omega)) \cap L^q_{\mathrm{uloc}}([0,\infty); W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \phi \in L^\infty(0,\infty; L^\infty(\Omega)) : \quad |\phi(x,t)| < 1 \text{ for a.a. } x \in \Omega, \forall\, t \in [0,\infty), \\ \partial_t \phi \in L^\infty(0,\infty; H^1(\Omega)') \cap L^2(0,\infty; L^2(\Omega)), \quad F'(\phi) \in L^\infty(0,\infty; H^1(\Omega)), \\ \mu \in BC_{\mathrm{w}}([0,\infty); H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); H^2(\Omega)) \cap H^1_{\mathrm{uloc}}([0,\infty); H^1(\Omega)'). \end{cases} \tag{1.17}$$

(ii) $(\mathbf{u}, \Pi, \phi)$ *fulfills the system* (1.8) *almost everywhere in* $\Omega \times (0,\infty)$ *and the boundary condition* $\partial_{\mathbf{n}} \mu = 0$ *almost everywhere on* $\partial\Omega \times (0,\infty)$*.*

(iii) $(\mathbf{u}, \Pi, \phi)$ *is such that* $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ *and* $\phi(\cdot, 0) = \phi_0$ *in* Ω*.*

(iv) For any $\tau > 0$*, there exists* $\delta = \delta(\tau) \in (0,1)$ *(depending on the norm of the initial datum) such that*

$$\sup_{t \in [\tau,\infty)} \|\phi(t)\|_{L^\infty(\Omega)} \le 1 - \delta. \tag{1.18}$$

Moreover, if we additionally assume that $\|\phi_0\|_{L^\infty(\Omega)} \le 1 - \delta_0$*, for some* $\delta_0 \in (0,1)$*, then there exists* $\delta^\star > 0$ *such that*

$$\sup_{t \in [0,\infty)} \|\phi(t)\|_{L^\infty(\Omega)} \le 1 - \delta^\star. \tag{1.19}$$

As a consequence, $\partial_t \mu \in L^2_{\mathrm{uloc}}([0,\infty); L^2(\Omega))$*, and* $(\mathbf{u}, \phi)$ *is unique and depends continuously on the initial data in* $L^2_\sigma(\Omega) \times L^2(\Omega)$ *on* $[0,T]$*, for any* $T > 0$*. More precisely, if* $(\mathbf{u}_j, \Pi_j, \phi_j)$ *is the strong solutions to* (1.8)*-*(1.9) *originating from the initial datum* $(\mathbf{u}_0^j, \phi_0^j)$*,* $j = 1, 2$*, then*

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{L^2(\Omega)} + \|(\phi_1(t) - \phi_2(t)\|^2_{L^2(\Omega)} \le C \left(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|^2_{L^2(\Omega)} + \|\phi_0^1 - \phi_0^2\|^2_{L^2(\Omega)} \right) \mathrm{e}^{\int_0^T H(\tau)\,\mathrm{d}\tau}, \tag{1.20}$$

for any $t \in [0,T]$*, where* $C > 0$ *is a constant depending on the norms of both the initial data and*

$$H(t) = C \left(1 + \|\partial_t \mathbf{u}_2(t)\|^2_{L^2(\Omega)} + \|\nabla \mathbf{u}_2(t)\|^4_{L^4(\Omega)} + \|\mathbf{u}_2(t)\|^2_{H^2(\Omega)} + \|\nabla \phi_2(t)\|^4_{L^4(\Omega)} \right). \tag{1.21}$$

Remark 1.6 *(Unique continuation property).* Let us consider two strong solutions according Theorem 1.5 which depart from the same initial data with ϕ_0 not necessarily strictly separated. If there exists $\widetilde{\tau} > 0$ such that the two solutions coincide at $t = \widetilde{\tau}$, they coincide over the entire interval $[\widetilde{\tau}, \infty)$. In fact, since both solutions are strictly separated in $[\widetilde{\tau}, \infty)$ by (1.18), the claim follows from (1.20).

The strategy of our proof relies on two new tools. First, we show a novel well-posedness result of strong solutions to the nonlocal Cahn-Hilliard system driven by an incompressible velocity field:

$$\begin{aligned} &\partial_t \phi + \mathbf{u} \cdot \nabla \phi = \Delta \mu, \qquad \mu = F'(\phi) - K * \phi, \qquad \text{in } \Omega \times (0,\infty), \\ &\partial_{\mathbf{n}} \mu = 0, \qquad \text{on } \partial\Omega \times (0,\infty), \qquad \phi(0) = \phi_0, \qquad \text{in } \Omega. \end{aligned} \tag{1.22}$$

We recall that the regularity achieved in [28, Lemma 6.1] has been obtained by assuming that

$$\mathbf{u} \in L^2(0,T; L^\infty_\sigma(\Omega)) \cap H^1(0,T; L^2_\sigma(\Omega)).$$

To the best of our knowledge, no other results in the case of singular potential and constant mobility are available. In Theorem 4.1, we prove the existence and uniqueness of the strong solutions (1.22) under the solely assumption that $\mathbf{u} \in L^4(0,T;L^4_\sigma(\Omega))$, which holds when $\mathbf{u}$ just belongs to the Leray-Hopf class. In doing so, we exploit the specific form of the chemical potential μ to rewrite the apparently unmanageable term $\int_\Omega \mathbf{u}\cdot\nabla\phi\,\partial_t\mu\,\mathrm{d}x$ (cf. (4.35)). The second tool is a new interpolation estimate for the pressure of the Stokes operator given in Lemma 3.1 which improves [39, Lemma B.2]. Once these two preliminary results are proven, the strong couplings in (1.8) are handled through a suitable approximation scheme to obtain global-in-time higher-order Sobolev/energy estimates. Finally, we mention that it is unlikely to study the three-dimensional case in the functional framework considered in Theorem 1.5. Indeed, assuming that μ and $\mathbf{u}$ satisfy the properties (1.17), in three-dimensions the nonlinear term $(\nabla\mu\cdot\nabla)\mathbf{u}$ does not even belong to $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$.

Our second main result regards the global behavior and propagation of regularity of any weak solution. Weak uniqueness or weak-strong uniqueness results are not available in this context, therefore we need to exploit a different proof to obtain that *any* weak solution enjoys an instantaneous propagation of regularity and converges to an equilibrium, i.e., to a stationary state as time goes to $+\infty$.

Theorem 1.7. *Let the assumptions (H_1)-(H_6) hold. Assume that $\mathbf{u}_0 \in L^2_\sigma(\Omega)$ and $\phi_0 \in L^\infty(\Omega)$ with $F(\phi_0) \in L^1(\Omega)$ and $|\overline{\phi_0}| < 1$. For any $T>0$, we consider a weak solution $(\mathbf{u},\phi)$ to (1.8)-(1.9) defined on $\Omega\times[0,T)$ given by Theorem 1.3. Then, $(\mathbf{u},\phi)$ is uniquely extended to $\Omega\times[0,\infty)$ and, for any $\tau>0$, $(\mathbf{u},\phi)$ is a strong solution on $[\tau,+\infty)$. More precisely, for any $\tau>0$, $(\mathbf{u},\phi)$ satisfies*

$$\begin{cases} \mathbf{u} \in BC([\tau,\infty);H^1_{0,\sigma}(\Omega)) \cap L^2_{\mathrm{uloc}}([\tau,\infty);V^2_{0,\sigma}(\Omega)) \cap H^1_{\mathrm{uloc}}([\tau,\infty);L^2_\sigma(\Omega)),\\ \Pi \in L^2_{\mathrm{uloc}}([\tau,\infty);H^1_{(0)}(\Omega)),\\ \phi \in L^\infty(\tau,\infty;H^1(\Omega)) \cap L^q_{\mathrm{uloc}}([\tau,\infty);W^{1,p}(\Omega)), \quad q=\dfrac{2p}{p-2}, \quad \forall\, p\in(2,\infty),\\ \phi \in L^\infty(\tau,\infty;L^\infty(\Omega)): \quad |\phi(x,t)|<1 \quad \textit{for a.a. } x\in\Omega,\ \forall\, t\in[\tau,\infty),\\ \partial_t\phi \in L^\infty(\tau,\infty;H^1(\Omega)') \cap L^2(\tau,\infty;L^2(\Omega)), \quad F'(\phi)\in L^\infty(\tau,\infty;H^1(\Omega)),\\ \mu \in BC_{\mathrm{w}}([\tau,\infty);H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([\tau,\infty);H^2(\Omega)) \cap H^1_{\mathrm{uloc}}([\tau,\infty);H^1(\Omega)'), \end{cases} \tag{1.23}$$

and the energy identity

$$E(\mathbf{u}(t),\phi(t)) + \int_\tau^t \left\|\sqrt{\nu(\phi(s))}D\mathbf{u}(s)\right\|^2_{L^2(\Omega)} + \|\nabla\mu(s)\|^2_{L^2(\Omega))}\,\mathrm{d}s = E(\mathbf{u}(\tau),\phi(\tau)) \tag{1.24}$$

holds for every $0<\tau\le t<\infty$. Moreover, there exists a constant $\delta=\delta(\tau)\in(0,1)$, also depending on $\overline{\phi_0}$, such that

$$\sup_{t\in[\tau,+\infty)} \|\phi(t)\|_{L^\infty(\Omega)} \le 1-\delta.$$

If, in addition, F is real analytic in $(-1,1)$, then $(\mathbf{u}(t),\phi(t)) \to (\mathbf{0},\phi_\infty)$ in $L^2_\sigma(\Omega)\times L^\infty(\Omega)$ as $t\to+\infty$, where $\phi_\infty \in L^\infty(\Omega)\cap H^1(\Omega)$ is a solution to the stationary nonlocal Cahn-Hilliard equation

$$\begin{cases} F'(\phi_\infty) - K*\phi_\infty = \mu_\infty, \quad \textit{in } \Omega,\\ \dfrac{1}{|\Omega|}\displaystyle\int_\Omega \phi_\infty(x)\mathrm{d}x = \overline{\phi_0}, \quad \mu_\infty\in\mathbb{R}. \end{cases} \tag{1.25}$$

Remark 1.8. The convergence to a single equilibrium stated in Theorem 1.7 also holds for the global strong solutions constructed in Theorem 1.5.

Let us now consider the matched densities case, i.e. $\rho = \rho_1 = \rho_2$. Both Theorems 1.5 and 1.7 remain true for the nonlocal Navier-Stokes-Cahn-Hilliard (NSCH) system with unmatched viscosities, singular potential and constant mobility (also called nonlocal Model H, see [24,28]). Thanks to the aforementioned novel interpolation result (see Lemma 3.1 below), we prove the following continuous dependence estimate for strong solutions to the nonlocal NSCH. In particular, this guarantees the uniqueness of strong solutions (not necessarily "separated" as in Theorem 1.5) to the nonlocal NSCH system. This result has been an open question since [24] where only the constant viscosity case was considered.

Theorem 1.9. *Let the assumptions (H_1)-(H_6) hold. Suppose that $(\mathbf{u}_1, \Pi_1, \phi_1)$ and $(\mathbf{u}_2, \Pi_2, \phi_2)$ are two strong solutions given by Theorem 1.5 with constant density $\rho = \rho_1 = \rho_2 > 0$ corresponding to the initial data $(\mathbf{u}_0^1, \phi_0^1)$ and $(\mathbf{u}_0^2, \phi_0^2)$, respectively. Then, there holds*

$$\begin{aligned}
&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{H_\sigma^{-1}(\Omega)} + \|\phi_1(t) - \phi_2(t)\|^2_{H^1(\Omega)'} \\
&\quad \le C \left(\|\mathbf{u}_0^1 - \mathbf{u}_0^2\|^2_{H_\sigma^{-1}(\Omega)} + \|\phi_0^1 - \phi_0^2\|^2_{H^1(\Omega)'} \right) \mathrm{e}^{S(t)} + R(t) \left| \overline{\phi}_0^1 - \overline{\phi}_0^2 \right| \mathrm{e}^{S(t)} + Ct \left| \overline{\phi}_0^1 - \overline{\phi}_0^2 \right|^2 \mathrm{e}^{S(t)},
\end{aligned} \tag{1.26}$$

for all $t > 0$, where

$$S(t) = C \int_0^t \left(1 + \|\mathbf{u}_1(s)\|^4_{L^4(\Omega)} + \|\mathbf{u}_2(s)\|^4_{L^4(\Omega)} + \|D\mathbf{u}_2(s)\|^4_{L^4(\Omega)} + \|\nabla \phi_1(s)\|^4_{L^4(\Omega)} \right) \mathrm{d}s$$

and

$$R(t) = C \int_0^t \left(\|F'(\phi_1(s))\|_{L^1(\Omega)} + \|F'(\phi_2(s))\|_{L^1(\Omega)} \right) \mathrm{d}s,$$

as well as some positive constant C depending on the norm of the initial data.

Finally, we estimate the difference between the strong solutions to the nonlocal AGG model and the nonlocal Model H originating from the same initial datum, in terms of the density values. In particular, the following result gives a rigorous justification of the nonlocal Model H as the *constant density approximation* of the nonlocal AGG model:

Theorem 1.10. *Let the assumptions (H_1)-(H_6) hold. Given an initial datum $(\mathbf{u}_0, \phi_0)$ as in Theorem 1.5, we consider a strong solution $(\mathbf{u}, \Pi, \phi)$ to the AGG model and the strong solution (u_H, Π_H, ϕ_H) to the AGG model with constant density $\overline{\rho} > 0$ (nonlocal Model H). Then, for any given $T > 0$, there exists a constant $C > 0$, that depends on the norm of the initial datum, the time T and the parameters of the systems, such that*

$$\sup_{t \in [0,T]} \|\mathbf{u}(t) - \mathbf{u}_H(t)\|_{H_\sigma^{-1}(\Omega)} + \sup_{t \in [0,T]} \|\phi(t) - \phi_H(t)\|_{H^1(\Omega)'} \le C \left(\left| \frac{\rho_1 - \rho_2}{2} \right| + \left| \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right| \right). \tag{1.27}$$

Remark 1.11. Assuming that $\rho_1 = \overline{\rho}$ and $\rho_2 = \overline{\rho} + \varepsilon$, for some (small) $\varepsilon > 0$, estimate (1.27) reads

$$\sup_{t \in [0,T]} \|\mathbf{u}(t) - \mathbf{u}_H(t)\|_{H_\sigma^{-1}(\Omega)} + \sup_{t \in [0,T]} \|\phi(t) - \phi_H(t)\|_{H^1(\Omega)'} \le C\varepsilon.$$

2. Mathematical setting

Let Ω be a bounded domain of class C^2 in $\mathbb{R}^2$. The Sobolev spaces of functions $u : \Omega \to \mathbb{R}$ and of vector fields $\mathbf{u} : \Omega \to \mathbb{R}^d$ $(d \in \mathbb{N})$ are denoted by $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega; \mathbb{R}^d)$, respectively, where $k \in \mathbb{N}$ and $1 \le p \le \infty$. For simplicity of notation, we will denote their norms by $\|\cdot\|_{W^{k,p}(\Omega)}$ in both cases. If $p = 2$, the Hilbert spaces $W^{k,2}(\Omega)$ and $W^{k,2}(\Omega; \mathbb{R}^d)$ are denoted by $H^k(\Omega)$ and $H^k(\Omega; \mathbb{R}^d)$, respectively, with norm $\|\cdot\|_{H^k(\Omega)}$. We will adopt the notation $(\cdot,\cdot)$ for the inner product in $L^2(\Omega)$ and in $L^2(\Omega; \mathbb{R}^d)$. The dual spaces of $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega; \mathbb{R}^d)$ (as well as $H^k(\Omega)$ and $H^k(\Omega; \mathbb{R}^d)$) are denoted by $W^{k,p}(\Omega)'$ and $W^{k,p}(\Omega; \mathbb{R}^d)'$, respectively, and the duality product by $\langle\cdot,\cdot\rangle_{W^{k,p}(\Omega)}$ and $\langle\cdot,\cdot\rangle_{W^{k,p}(\Omega;\mathbb{R}^d)}$. In addition, we define the linear subspaces

$$L^2_{(0)}(\Omega) = \left\{ u \in L^2(\Omega) : \overline{u} = \frac{(u,1)}{|\Omega|} = 0 \right\}, \quad H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega)$$

and

$$H^{-1}_{(0)}(\Omega) = \left\{ u \in H^1(\Omega)' : \overline{u} = |\Omega|^{-1} \langle u, 1\rangle = 0 \right\},$$

endowed with the norms of $L^2(\Omega)$, $H^1(\Omega)$ and $H^1(\Omega)'$, respectively. By the Poincaré-Wirtinger inequality, it follows that $\left(\|\nabla u\|^2_{L^2(\Omega)} + |\overline{u}|^2\right)^{\frac{1}{2}}$ is a norm in $H^1(\Omega)$, that is equivalent to $\|u\|_{H^1(\Omega)}$. The Laplace operator $A_0 : H^1_{(0)}(\Omega) \to H^{-1}_{(0)}(\Omega)$ defined by $\langle A_0 u, v\rangle_{H^1_{(0)}(\Omega)} = (\nabla u, \nabla v)$, for any $v \in H^1_{(0)}(\Omega)$, is a bijective map between $H^1_{(0)}(\Omega)$ and $H^{-1}_{(0)}(\Omega)$. We denote its inverse by $\mathcal{N} = A_0^{-1} : H^{-1}_{(0)}(\Omega) \to H^1_{(0)}(\Omega)$, namely for any $u \in H^{-1}_{(0)}(\Omega)$, $\mathcal{N}u$ is the unique function in $H^1_{(0)}(\Omega)$ such that $(\nabla \mathcal{N}u, \nabla v) = \langle u, v\rangle_{H^1_{(0)}(\Omega)}$ for any $v \in H^1_{(0)}(\Omega)$. As a consequence, for any $u \in H^{-1}_{(0)}(\Omega)$, we set $\|u\|_* := \|\nabla \mathcal{N}u\|_{L^2(\Omega)}$, which is a norm in $H^{-1}_{(0)}(\Omega)$, that is equivalent to the canonical dual norm. In turn, $\left(\|u - \overline{u}\|^2_* + |\overline{u}|^2\right)^{\frac{1}{2}}$ is a norm $H^1(\Omega)'$, that is equivalent to $\|u\|_{H^1(\Omega)'}$. Moreover, by the regularity theory for the Laplace operator with homogeneous Neumann boundary conditions, there is a constant $C > 0$ such that

$$\|\nabla \mathcal{N}u\|_{H^1(\Omega)} \le C\|u\|_{L^2(\Omega)}, \quad \forall\, u \in L^2_{(0)}(\Omega). \tag{2.1}$$

Lastly, we report the following Gagliardo-Nirenberg and Agmon inequalities

$$\|u\|_{L^4(\Omega)} \le C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}, \qquad \forall\, u \in H^1(\Omega), \tag{2.2}$$

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^2(\Omega)}, \qquad \forall\, u \in H^2(\Omega), \tag{2.3}$$

$$\|u\|_{W^{1,4}(\Omega)} \le C\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|u\|^{\frac{1}{2}}_{H^2(\Omega)}, \qquad \forall\, u \in H^2(\Omega), \tag{2.4}$$

for some suitable constants $C > 0$ depending only on Ω.

Next, we introduce the solenoidal function spaces

$$L^p_\sigma(\Omega) = \{\mathbf{u} \in L^p(\Omega; \mathbb{R}^2) : \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad \forall\, p \in (1, \infty),$$
$$W^{1,p}_{0,\sigma}(\Omega) = \{\mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^2) : \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega\}, \quad \forall\, p \in (1, \infty).$$

We recall that $L^p_\sigma(\Omega)$ and $W^{1,p}_{0,\sigma}(\Omega)$ corresponds to the completion of $C^\infty_{0,\sigma}(\Omega; \mathbb{R}^2)$, namely the space of divergence-free vector fields in $C^\infty_0(\Omega; \mathbb{R}^2)$, in the norm of $L^p(\Omega; \mathbb{R}^2)$ and $W^{1,p}(\Omega; \mathbb{R}^2)$, respectively. For simplicity of notation, we will also use $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{W^{1,p}(\Omega)}$ to denote the norms in $L^p_\sigma(\Omega)$ and $W^{1,p}_{0,\sigma}(\Omega)$, respectively. The space $H^1_{0,\sigma}(\Omega) = W^{1,2}_{0,\sigma}(\Omega)$ is endowed with the inner product and norm $(\mathbf{u}, \mathbf{v})_{H^1_{0,\sigma}(\Omega)} = (\nabla \mathbf{u}, \nabla \mathbf{v})$ and $\|\mathbf{u}\|_{H^1_{0,\sigma}(\Omega)} = \|\nabla \mathbf{u}\|_{L^2(\Omega)}$, respectively. By the Korn inequality,

$$\|\nabla \mathbf{u}\|_{L^2(\Omega)} \le \sqrt{2}\|D\mathbf{u}\|_{L^2(\Omega)} \le \sqrt{2}\|\nabla \mathbf{u}\|_{L^2(\Omega)}, \quad \forall\, \mathbf{u} \in H^1_{0,\sigma}(\Omega). \tag{2.5}$$

The Stokes operator is $\mathbf{A} = -\mathbb{P}\Delta$, with domain $D(\mathbf{A}) = V^2_{0,\sigma}(\Omega)$, where $V^2_{0,\sigma}(\Omega) := H^2(\Omega;\mathbb{R}^2) \cap H^1_{0,\sigma}(\Omega)$ and $\mathbb{P}$ is the Leray orthogonal projector from $L^2(\Omega)$ onto $L^2_\sigma(\Omega)$. We denote by $\mathbf{A}^{-1}$ the inverse map of the Stokes operator. In particular, $\mathbf{A}^{-1}$ is an isomorphism between $H^{-1}_\sigma(\Omega) = H^1_{0,\sigma}(\Omega)'$ and $H^1_{0,\sigma}(\Omega)$ such that, for any $\mathbf{u} \in H^{-1}_\sigma(\Omega)$, $\mathbf{A}^{-1}\mathbf{u}$ is the unique function in $H^1_{0,\sigma}(\Omega)$ such that $(\nabla \mathbf{A}^{-1}\mathbf{u}, \nabla \mathbf{v}) = \langle \mathbf{u}, \mathbf{v}\rangle_{H^1_{0,\sigma}(\Omega)}$ for any $\mathbf{v} \in H^1_{0,\sigma}(\Omega)$. Then, it follows that $\|\mathbf{u}\|_\sharp := \|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^2(\Omega)}$ is an equivalent norm on $H^{-1}_\sigma(\Omega)$. By the regularity theory of the Stokes operator, there exists a constant C such that

$$\|\mathbf{u}\|_{H^2(\Omega)} \le C\|\mathbf{A}\mathbf{u}\|_{L^2(\Omega)}, \quad \forall\, \mathbf{u} \in V^2_{0,\sigma}(\Omega). \tag{2.6}$$

Then, we define $\|\mathbf{u}\|_{V^2_{0,\sigma}(\Omega)} = \|\mathbf{A}\mathbf{u}\|_{L^2(\Omega)}$, which is a norm in $V^2_{0,\sigma}(\Omega)$, that is equivalent to $\|\mathbf{u}\|_{H^2(\Omega)}$.

Let X be a real Banach space and consider an interval $I \subseteq [0,\infty)$. The Banach space $BC(I;X)$ consists of all bounded and continuous $f : I \to X$ equipped with the supremum norm. The subspace $BUC(I;X)$ denotes the set of all bounded and uniformly continuous functions $f : I \to X$. We denote by $BC_{\mathrm{w}}(I;X)$ the topological vector space of all bounded and weakly continuous functions $f : I \to X$. If I is a compact interval, then we simply use the notation $C(I;X)$ or $C_{\mathrm{w}}(I;X)$. The set $C^\infty_0(I;X)$ denotes the vector space of all smooth functions $f : I \to X$ whose support is compactly embedded in I. Given $p \in [1,\infty]$, the Lebesgue space $L^p(I;X)$ denotes the set of all strongly measurable $f : I \to X$ that are p-integrable/essentially bounded. In particular, $L^p_{\mathrm{uloc}}([0,\infty);X)$ is the set of all strongly measurable $f : [0,\infty) \to X$ such that

$$\|f\|_{L^p_{\mathrm{uloc}}([0,\infty);X)} = \sup_{t\ge 0} \|f\|_{L^q(t,t+1;X)} < \infty.$$

The Bochner spaces $W^{1,p}(I;X)$ consists of all $f \in L^p(I;X)$ with $\partial_t f \in L^p(I;X)$. If $p = 2$, then $H^1(I;X) = W^{1,2}(I;X)$. In a similar way, we also define $H^1_{\mathrm{uloc}}([0,\infty);X)$.

In the following sections, we will denote by C a generic positive constant, which may even vary within the same line, possibly depending on Ω as well as on the parameters of the system.

3. Interpolation estimate for the L^4-norm of the pressure in the Stokes problem

The regularity theory of the Stokes operator ensures that, for any $\mathbf{f} \in L^2_\sigma(\Omega) \subset L^2(\Omega;\mathbb{R}^2)$, there exist $\mathbf{u} = \mathbf{A}^{-1}\mathbf{f} \in V^2_{0,\sigma}(\Omega)$ and $P \in H^1_{(0)}(\Omega)$ such that

$$-\Delta \mathbf{u} + \nabla P = \mathbf{f}, \quad \text{a.e. in } \Omega. \tag{3.1}$$

We refer the reader to [32] and the references therein for the comprehensive theory. The L^p-norms of the pressure P are usually controlled by the norms of negative Sobolev spaces of the external force $\mathbf{f}$ (see, for instance, [32, Lemma IV.2.1]). Notwithstanding these results are sharp from the viewpoint of the regularity theory of steady Stokes flows, an estimate of the L^p-norms of P in terms of the norms in $H^{-1}_\sigma(\Omega) = (H^1_{0,\sigma}(\Omega))'$ and in $L^2_\sigma(\Omega)$ of $\mathbf{f}$ is more effective for some purposes in the context of evolutionary Navier-Stokes flows. A first interpolation result for the L^2-norm of the pressure P was established in [39, Lemma B.2]. We present a novel interpolation result for the L^4-norm of the pressure P, which improves the one in [39, Lemma B.2]. This is essential to perform some crucial estimates in the sequel in order to deal with the low regularity guaranteed by the nonlocal setting.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ of class C^2. There exists C such that*

$$\|P\|_{L^4(\Omega)} \le C\|\nabla \mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad \forall \mathbf{f} \in L^2_\sigma(\Omega), \tag{3.2}$$

where P is the pressure in (3.1).

Proof. We know from [32, Lemma IV.2.1]) that there exists C such that

$$\|P\|_{L^4(\Omega)} \le C\|\mathbf{f}\|_{W^{-1,4}(\Omega)}, \tag{3.3}$$

where $W^{-1,4}(\Omega;\mathbb{R}^2) = (W_0^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2))'$ and $W_0^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)$ is the completion of $C_0^\infty(\Omega;\mathbb{R}^2)$ with respect to the norm of $W^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)$. In order to estimate the right-hand side in (3.3), let us consider $\mathbf{v} \in W_0^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)$. By using the integration by parts and the properties of $\mathbb{P}$, we have

$$(\mathbf{f},\mathbf{v}) = (\mathbb{P}(-\Delta)\mathbf{A}^{-1}\mathbf{f},\mathbf{v}) = ((-\Delta)\mathbf{A}^{-1}\mathbf{f},\mathbb{P}\mathbf{v}) = (\nabla\mathbf{A}^{-1}\mathbf{f},\nabla\mathbb{P}\mathbf{v}) - \int_{\partial\Omega} \left(\nabla\mathbf{A}^{-1}\mathbf{f}\,\mathbf{n}\right)\cdot\mathbb{P}\mathbf{v}\,\mathrm{d}\sigma. \tag{3.4}$$

Since $\mathbf{f} \in L^2_\sigma(\Omega)$, we observe from (2.6) that $\|\mathbf{A}^{-1}\mathbf{f}\|_{H^2(\Omega)} \le C\|\mathbf{f}\|_{L^2(\Omega)}$. By using this fact, together with (2.2) and the continuity of the projection operator $\mathbb{P}$ from $W_0^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)$ onto $W^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega)$, we easily infer that

$$\begin{aligned}(\nabla\mathbf{A}^{-1}\mathbf{f},\nabla\mathbb{P}\mathbf{v}) &\le \|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^4(\Omega)}\|\mathbb{P}\mathbf{v}\|_{W^{1,\frac{4}{3}}(\Omega)}\\ &\le C\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{v}\|_{W_0^{1,\frac{4}{3}}(\Omega)}.\end{aligned}$$

Next, by the interpolation trace estimate, there exists C such that

$$\|f\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\Omega)}^{\frac{1}{2}}\|f\|_{H^1(\Omega)}^{\frac{1}{2}}, \quad \forall f \in H^1(\Omega). \tag{3.5}$$

Furthermore, we also report the following trace estimate (see [32, Thm II.4.1], with $n = 2$, $m = 1$, $q = 4/3$ and $r = 2$): there exists C such that

$$\|f\|_{L^2(\partial\Omega)} \le C\|f\|_{W^{1,\frac{4}{3}}(\Omega)}, \quad \forall f \in W^{1,\frac{4}{3}}(\Omega). \tag{3.6}$$

Thus, exploiting (3.5) and (3.6), together with (2.6), we obtain

$$\begin{aligned}\left|\int_{\partial\Omega}\left(\nabla\mathbf{A}^{-1}\mathbf{f}\,\mathbf{n}\right)\cdot\mathbb{P}\mathbf{v}\,\mathrm{d}\sigma\right| &\le C\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^2(\partial\Omega)}\|\mathbb{P}\mathbf{v}\|_{L^2(\partial\Omega)}\\ &\le C\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{H^1(\Omega)}^{\frac{1}{2}}\|\mathbb{P}\mathbf{v}\|_{W^{1,\frac{4}{3}}(\Omega)}\\ &\le C\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{v}\|_{W_0^{1,\frac{4}{3}}(\Omega)}.\end{aligned}$$

Therefore, we deduce that

$$\|\mathbf{f}\|_{W^{-1,4}(\Omega)} = \sup_{0\neq\mathbf{v}\in W_0^{1,\frac{4}{3}}(\Omega;\mathbb{R}^2)} \frac{|(\mathbf{f},\mathbf{v})|}{\|\mathbf{v}\|_{W_0^{1,\frac{4}{3}}(\Omega)}} \le C\|\nabla\mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}, \tag{3.7}$$

which entails the desired conclusion in light of (3.3). □

Remark 3.2. A similar result can be obtained in the three-dimensional case. Replacing the L^4-norm with the L^3-norm of P and exploiting the corresponding Gagliardo-Nirenberg and trace estimates in dimension three, one can repeat word by word the arguments of the above proof to obtain

$$\|P\|_{L^3(\Omega)} \le C\|\nabla \mathbf{A}^{-1}\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{f}\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad \forall\, \mathbf{f} \in L^2_\sigma(\Omega).$$

4. The nonlocal Cahn-Hilliard equation with divergence-free drift

Let $\mathbf{u}$ be a divergence-free vector field. We consider the initial-boundary value problem for the nonlocal Cahn-Hilliard equation with divergence-free drift

$$\begin{cases} \partial_t \phi + \mathbf{u}\cdot\nabla\phi = \Delta\mu, \quad \mu = F'(\phi) - K*\phi, & \text{in } \Omega\times(0,T),\\ \partial_{\mathbf{n}}\mu = 0, & \text{on } \partial\Omega\times(0,T),\\ \phi(\cdot,0) = \phi_0, & \text{in } \Omega. \end{cases} \tag{4.1}$$

We present herein novel well-posedness and regularity results under minimal assumptions on the velocity field for the system (4.1) (cf. with the analysis in [28]). Beyond its own interest *per se*, these statements will play an essential role in the proof of Theorem 1.5.

Theorem 4.1. *Let the assumptions* (H_1)-(H_5) *hold and let* $T > 0$. *Assume that* $\mathbf{u} \in L^4(0,T; L^4_\sigma(\Omega))$ *and* $\phi_0 \in L^\infty(\Omega)$ *with* $F(\phi_0) \in L^1(\Omega)$ *and* $|\overline{\phi_0}| < 1$. *Then, there exists a unique weak solution to* (4.1) *such that*

$$\begin{cases} \phi \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^1(\Omega)'),\\ \phi \in L^\infty(\Omega\times(0,T)): \quad |\phi| < 1 \text{ a.e. in } \Omega\times(0,T),\\ \mu \in L^2(0,T; H^1(\Omega)), \quad F'(\phi) \in L^2(0,T; H^1(\Omega)), \end{cases} \tag{4.2}$$

which satisfies

$$\begin{gathered} \langle \partial_t\phi, v\rangle_{H^1(\Omega)} - (\phi\,\mathbf{u}, \nabla v) + (\nabla\mu, \nabla v) = 0, \quad \forall\, v \in H^1(\Omega), \text{ a.e. in } (0,T),\\ \mu = F'(\phi) - K*\phi, \quad \text{a.e. in } \Omega\times(0,T), \end{gathered} \tag{4.3}$$

and $\phi(\cdot,0) = \phi_0$ *almost everywhere in* Ω. *The weak solution fulfills the energy identity*

$$\mathcal{E}_{\text{nloc}}(\phi(t)) + \int_0^t \|\nabla\mu(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau + \int_0^t\int_\Omega \phi\,\mathbf{u}\cdot\nabla\mu\,\mathrm{d}x\,\mathrm{d}\tau = \mathcal{E}_{\text{nloc}}(\phi_0), \quad \forall\, t \in [0,T]. \tag{4.4}$$

In addition, given two weak solutions ϕ^1 *and* ϕ^2 *corresponding to the initial data* ϕ_0^1 *and* ϕ_0^2, *respectively, we have*

$$\begin{aligned} &\left\|\phi^1 - \phi^2\right\|_{C([0,T];H^1(\Omega)')}\\ &\quad \le \left(\left\|\phi_0^1 - \phi_0^2\right\|_{H^1(\Omega)'} + \left|\overline{\phi_0^1} - \overline{\phi_0^2}\right|^{\frac{1}{2}} \|\Lambda\|_{L^1(0,T)}^{\frac{1}{2}} + CT^{\frac{1}{2}}\left|\overline{\phi_0^1} - \overline{\phi_0^2}\right|\right) \exp\left(C\left(T + \|\mathbf{u}\|_{L^4(0,T;L^4(\Omega))}^4\right)\right), \end{aligned} \tag{4.5}$$

for all $t \in [0,T]$, *where* $\Lambda = 2\left\|F'(\phi^1)\right\|_{L^1(\Omega)} + 2\left\|F'\left(\phi^2\right)\right\|_{L^1(\Omega)}$ *and* C *only depends on* α, K *and* Ω.

Furthermore, the following regularity results hold:

(i) If additionally $\phi_0 \in H^1(\Omega)$ such that $F'(\phi_0) \in L^2(\Omega)$ and $F''(\phi_0)\nabla\phi_0 \in L^2(\Omega;\mathbb{R}^2)$, then

$$\begin{cases} \phi \in L^\infty(0,T;L^\infty(\Omega)): \quad |\phi(x,t)| < 1 \quad \text{for a.a. } x \in \Omega, \, \forall\, t \in [0,T], \\ \phi \in L^\infty(0,T;H^1(\Omega)) \cap L^q(0,T;W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \partial_t \phi \in L^4(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega)), \\ \mu \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)'), \\ F'(\phi) \in L^\infty(0,T;H^1(\Omega)), \quad F''(\phi) \in L^\infty(0,T;L^p(\Omega)), \quad \forall\, p \in [2,\infty). \end{cases} \tag{4.6}$$

In particular, we have the estimates

$$\begin{aligned} &\|\nabla\mu\|_{L^\infty(0,T;L^2(\Omega))} \\ &\le \left(\|F''(\phi_0)\nabla\phi_0 - \nabla K * \phi_0\|_{L^2(\Omega)} + C \left(\int_0^T \|\mathbf{u}(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu(\tau)\|_{L^2(\Omega)}^2 \, \mathrm{d}\tau \right)^{\frac{1}{2}} \right) \\ &\quad \times \exp\left(C \int_0^T \|\mathbf{u}(\tau)\|_{L^4(\Omega)}^4 \, \mathrm{d}\tau \right) =: \Xi_1, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\int_0^T \|\partial_t\phi(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu(\tau)\|_{H^1(\Omega)}^2 \, \mathrm{d}\tau \\ &\quad \le C \left(\|F''(\phi_0)\nabla\phi_0 - \nabla K * \phi_0\|_{L^2(\Omega)}^2 + C \int_0^T \|\mathbf{u}(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu(\tau)\|_{L^2(\Omega)}^2 \, \mathrm{d}\tau \right) \\ &\qquad \times \left(1 + \int_0^T \|\mathbf{u}(\tau)\|_{L^4(\Omega)}^4 \, \mathrm{d}\tau \right) \exp\left(C \int_0^t \|\mathbf{u}(\tau)\|_{L^4(\Omega)}^4 \, \mathrm{d}\tau \right) =: \Xi_2, \end{aligned} \tag{4.8}$$

and the bounds

$$\begin{cases} \|\mu\|_{L^\infty(0,T;H^1(\Omega))} + \|\phi\|_{L^\infty(0,T;H^1(\Omega))} + \|F'(\phi)\|_{L^\infty(0,T;H^1(\Omega))} \le \mathcal{Q}\left(\overline{\phi_0}, \Xi_1, \alpha, \|K\|_{W^{1,1}(\mathbb{R}^2)}, \Omega\right), \\ \|F''(\phi)\|_{L^\infty(0,T;L^p(\Omega))} \le \mathcal{Q}\left(p, \overline{\phi_0}, \Xi_1, \alpha, \|K\|_{W^{1,1}(\mathbb{R}^2)}, \Omega\right), \quad \forall\, p \in [2,\infty), \\ \|\phi\|_{L^q(0,T;L^p(\Omega))} \le \mathcal{Q}\left(\overline{\phi_0}, \Xi_1, \Xi_2, \alpha, \|K\|_{W^{1,1}(\mathbb{R}^2)}, \Omega, T\right), \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \|\mu\|_{L^2(0,T;H^2(\Omega))} + \|\partial_t\mu\|_{L^2(0,T;H^1(\Omega)')} \le \mathcal{Q}\left(\overline{\phi_0}, \Xi_1, \Xi_2, \alpha, \|K\|_{W^{1,1}(\mathbb{R}^2)}, \Omega, T\right), \end{cases} \tag{4.9}$$

where C only depends on α, K and Ω and $\mathcal{Q}$ is a generic increasing and continuous function of its arguments. Moreover, if $\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\Omega))$, we also have $\partial_t\phi \in L^\infty(0,T;H^1(\Omega)')$.

(ii) Let the assumptions of (i) hold. Suppose also $\|\phi_0\|_{L^\infty(\Omega)} \le 1-\delta_0$, for some $\delta_0 \in (0,1)$. Then, there exists $\delta > 0$ such that

$$\sup_{t\in[0,T]} \|\phi(t)\|_{L^\infty(\Omega)} \le 1-\delta. \tag{4.10}$$

As a consequence, $\partial_t\mu \in L^2(0,T;L^2(\Omega))$ and $\mu \in C([0,T];H^1(\Omega))$.

Remark 4.2. The existence of (at least) one weak solution to (4.1) satisfying (4.2), (4.3) as well as (4.4) holds under the milder regularity $\mathbf{u} \in L^2(0,T;L^2_\sigma(\Omega))$. We refer the reader to the proof of Theorem 4.1 (see below).

Remark 4.3. In the case (i) above, the assumption $\phi_0 \in H^1(\Omega)$ can be relaxed by only requiring that $\nabla\phi_0 \in L^1_{\mathrm{loc}}(\Omega)$. Indeed, it implies that $\nabla\phi_0$ is measurable. Then, $\|\nabla\phi_0\|_{L^2(\Omega)} < \infty$ follows from $F''(\phi_0)\nabla\phi_0 \in L^2(\Omega;\mathbb{R}^2)$ due to the strict convexity of F. Besides, our set of assumptions in the case (i) entails that $F'(\phi_0) \in H^1(\Omega)$ (similarly to [21, Theorem 4.1]). In fact, we first observe that the chain rule $\nabla F'(\phi_0) = F''(\phi_0)\nabla\phi_0$ holds almost everywhere in Ω. More precisely, by exploiting the approximation ϕ_0^k of the initial datum provided in the proof of Theorem 4.1, it can be shown (cf. also [44, Lemma 3.2]) that

$$\int_\Omega F'(\phi_0)\,\partial_i\varphi \mathrm{d}x = \lim_{k\to\infty}\int_\Omega F'(\phi_0^k)\,\partial_i\varphi\,\mathrm{d}x = -\lim_{k\to\infty}\int_\Omega F''(\phi_0^k)\partial_i\phi_0^k\,\varphi\,\mathrm{d}x = -\lim_{k\to\infty}\int_\Omega F''(\phi_0)\partial_i\phi_0\varphi\,\mathrm{d}x$$

for any $i = 1,2$ and $\varphi \in C_0^\infty(\Omega)$. Then, owing to this, we immediately infer that $F'(\phi_0) \in H^1(\Omega)$. On the other hand, by the previous reasoning together with the Fatou lemma, it is possible to show that $\phi_0 \in H^1(\Omega)$ with $F'(\phi_0) \in H^1(\Omega)$ guarantees that $F''(\phi_0)\nabla\phi_0 \in L^2(\Omega;\mathbb{R}^2)$.

Remark 4.4. In the case (i), the separation property holds for positive times. More precisely, for any $0 < \tau \le T$ there exists $\delta = \delta(\tau) \in (0,1)$ such that it holds

$$\sup_{t\in[\tau,T]} \|\phi(t)\|_{L^\infty(\Omega)} \le 1-\delta. \tag{4.11}$$

Although we only have $\phi_0 \in H^1(\Omega)$, thereby ϕ_0 might not be strictly separated. Nevertheless, since $\mu \in L^2(0,T;H^2(\Omega))$ and $K*\phi \in L^\infty(\Omega\times(0,T))$, we observe that $F'(\phi) \in L^2(0,T;L^\infty(\Omega))$. In turn, this implies for any $\tau > 0$ there exists $\tau^* \in (0,\tau)$ such that $F'(\phi(\tau^\star)) \in L^\infty(\Omega)$, which gives $\|\phi_0\|_{L^\infty(\Omega)} < 1$. Thus, (4.11) also follows from (ii).

Remark 4.5. In light of the assumptions in Theorem 4.1, while in the case (i), the assumption $\|\phi_0\|_{L^\infty(\Omega)} < 1$ in *(ii)* is ensured if (additionally) $\phi_0 \in W^{1,p}(\Omega)$ for some $p > 2$. In fact, by Remark 4.3, $F'(\phi_0) \in H^1(\Omega)$. Then, the Trudinger-Moser inequality and the assumption (H_4) (exactly as in [28, Theorem 5.2]) ensure that $F''(\phi_0) \in L^r(\Omega)$ for every $r \in [2,\infty)$. Thus, by the chain rule in Remark 4.3, we conclude

$$\|\nabla F'(\phi_0)\|_{L^s(\Omega)} \le \|F''(\phi_0)\|_{L^{\frac{sp}{p-s}}(\Omega)} \|\nabla\phi_0\|_{L^p(\Omega)} < \infty, \quad \text{where } s \in (2,p).$$

Since $s > 2$, $F'(\phi_0) \in W^{1,s}(\Omega) \hookrightarrow L^\infty(\Omega)$ implies that the initial datum ϕ_0 is strictly separated from ± 1.

Remark 4.6. The existence of a weak solution to (4.1) and the first part of the regularity result (cf. (i) above) of Theorem 4.1 can be readily extended in three dimensions by requiring that $\mathbf{u} \in L^4(0,T;L^6_\sigma(\Omega))$.

Proof of Theorem 4.1. The proof is divided into several steps.

Uniqueness and continuous dependence estimate. Let us consider two weak solutions ϕ^1 and ϕ^2 satisfying (4.2) and (4.3), and originating from two initial data ϕ_0^1 and ϕ_0^2 (where possibly $\overline{\phi_0^1} \ne \overline{\phi_0^2}$). Setting $\phi = \phi^1 - \phi^2$ and $\mu = F'(\phi^1) - F'(\phi^2) - K*\phi$, it is easy to realize that

$$\langle \partial_t\phi, v\rangle_{H^1(\Omega)} - (\phi\,\mathbf{u}, \nabla v) + (\nabla\mu, \nabla v) = 0, \quad \forall\, v \in H^1(\Omega), \text{ a.e. in } (0,T). \tag{4.12}$$

Taking $v = \mathcal{N}(\phi - \overline{\phi})$, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi-\overline{\phi}\right\|_*^2+\left(\mathbf{u}\cdot\nabla\phi,\mathcal{N}\left(\phi-\overline{\phi}\right)\right)+\left(\mu,\phi-\overline{\phi}\right)=0.$$

Exploiting (H_2) and (H_3), together with Young inequality, we have

$$\begin{aligned}\left(\mu,\phi-\overline{\phi}\right)&\geq\alpha\|\phi\|_{L^2(\Omega)}^2-\left(F'\left(\phi^1\right)-F'\left(\phi^2\right),\overline{\phi}\right)-\left(K*\phi,\phi-\overline{\phi}\right)\\&=\alpha\|\phi\|_{L^2(\Omega)}^2-\left(F'\left(\phi^1\right)-F'\left(\phi^2\right),\overline{\phi}\right)-\left(\nabla K*\phi,\nabla\mathcal{N}\left(\phi-\overline{\phi}\right)\right)\\&\geq\alpha\|\phi\|_{L^2(\Omega)}^2-\left(F'\left(\phi^1\right)-F'\left(\phi^2\right),\overline{\phi}\right)-\|K\|_{W^{1,1}(\mathbb{R}^2)}\|\phi\|_{L^2(\Omega)}\left\|\phi-\overline{\phi}\right\|_*\\&\geq\frac{3\alpha}{4}\|\phi\|_{L^2(\Omega)}^2-C\|\phi-\overline{\phi}\|_*^2-\left|\overline{\phi^1}-\overline{\phi^2}\right|\left(\|F'(\phi^1)\|_{L^1(\Omega)}+\|F'(\phi^2)\|_{L^1(\Omega)}\right).\end{aligned}\tag{4.13}$$

Concerning the convective term, by (2.1) and (2.2), we obtain

$$\begin{aligned}\left|\left(\mathbf{u}\cdot\phi,\nabla\mathcal{N}\left(\phi-\overline{\phi}\right)\right)\right|&\leq C\|\mathbf{u}\|_{L^4(\Omega)}\|\phi\|_{L^2(\Omega)}\left\|\nabla\mathcal{N}\left(\phi-\overline{\phi}\right)\right\|_{L^2(\Omega)}^{\frac{1}{2}}\left\|\phi-\overline{\phi}\right\|_{L^2(\Omega)}^{\frac{1}{2}}\\&\leq\frac{\alpha}{8}\|\phi\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^4(\Omega)}^2\left(\|\phi\|+C\left|\overline{\phi}\right|\right)\left\|\phi-\overline{\phi}\right\|_*\\&\leq\frac{\alpha}{4}\|\phi\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^4(\Omega)}^4\left\|\phi-\overline{\phi}\right\|_*^2+C\left|\overline{\phi}\right|^2.\end{aligned}\tag{4.14}$$

Then, recalling the conservation of mass, i.e. $\overline{\phi^i}(t)=\overline{\phi_0^i}$ for all $t\in[0,T]$ and $i=1,2$, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{H^1(\Omega)'}^2+\alpha\|\phi\|_{L^2(\Omega)}^2\leq C\left(1+\|\mathbf{u}\|_{L^4(\Omega)}^4\right)\|\phi\|_{H^1(\Omega)'}^2+\Lambda\left|\overline{\phi}(0)\right|+C\left|\overline{\phi}(0)\right|^2,$$

where $\Lambda=2\left\|F'\left(\phi^1\right)\right\|_{L^1(\Omega)}+2\left\|F'\left(\phi^2\right)\right\|_{L^1(\Omega)}$. Therefore, an application of Gronwall's Lemma implies (4.5), which, in particular, entails the uniqueness of the weak solutions.

Definition of the regularized problem. Let us consider a sequence $\{\mathbf{u}^k\}_{k\in\mathbb{N}}\subset C_0^\infty((0,T);C_{0,\sigma}^\infty(\Omega;\mathbb{R}^2))$ such that $\mathbf{u}^k\to\mathbf{u}$ in $L^4(0,T;L_\sigma^4)$. We assume first that $\phi_0\in H^1(\Omega)\cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)}\leq 1$ and $|\overline{\phi_0}|<1$. For any $k\in\mathbb{N}$, we introduce the Lipschitz function $h_k:\mathbb{R}\to\mathbb{R},\ k\in\mathbb{N}$ such that

$$h_k(s)=\begin{cases}-1+\dfrac{1}{k}, & s<-1+\dfrac{1}{k},\\ s, & s\in\left[-1+\dfrac{1}{k},1-\dfrac{1}{k}\right],\\ 1-\dfrac{1}{k}, & s>1-\dfrac{1}{k}.\end{cases}$$

We define $\phi_0^k:=h_k(\phi_0)$. It follows from the Stampacchia superposition principle [52] that $\phi_0^k\in H^1(\Omega)\cap L^\infty(\Omega)$ such that $\nabla\phi_0^k=\nabla\phi_0\cdot\chi_{[-1+\frac{1}{k},1-\frac{1}{k}]}(\phi_0)$ almost everywhere in Ω, where $\chi_A(\cdot)$ is the indicator function of the set A. By definition, we have

$$\left|\phi_0^k\right|\leq|\phi_0|,\quad\left|\nabla\phi_0^k\right|\leq|\nabla\phi_0|,\quad\text{a.e. in }\Omega.\tag{4.15}$$

As a consequence, we infer that $\phi_0^k\to\phi_0$ in $H^1(\Omega)$ as $k\to\infty$. Observe also that $\left|\overline{\phi_0^k}\right|\to\left|\overline{\phi_0}\right|$. Then, there exist $\varpi>0$ and $\overline{k}>0$ such that

$$\left|\overline{\phi_0^k}\right|\leq 1-\varpi,\quad\forall\,k>\overline{k}.\tag{4.16}$$

Thanks to Theorem A.1, there exists a sequence of functions $\{\phi^k,\mu^k\}_{k\in\mathbb{N}}$ satisfying

$$
\begin{aligned}
&\phi^k \in L^\infty(0,T;H^1(\Omega)\cap L^\infty(\Omega)): \quad \sup_{t\in[0,T]} \|\phi^k(t)\|_{L^\infty(\Omega)} \le 1-\delta_k,\\
&\phi^k \in L^q(0,T;W^{1,p}(\Omega)), \quad q=\frac{2p}{p-2}, \quad \forall\, p\in(2,\infty),\\
&\partial_t\phi^k \in L^\infty(0,T;H^1(\Omega)')\cap L^2(0,T;L^2(\Omega)),\\
&\mu^k \in C([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega)),
\end{aligned}
\tag{4.17}
$$

where $\delta_k \in (0,1)$ depends on k. The solutions satisfy

$$
\partial_t\phi^k + \mathbf{u}^k\cdot\nabla\phi^k = \Delta\mu^k,\ \mu^k = F'(\phi^k) - K*\phi^k, \qquad \text{a.e. in } \Omega\times(0,T). \tag{4.18}
$$

In addition, $\partial_{\mathbf{n}}\mu^k = 0$ almost everywhere on $\partial\Omega\times(0,T)$ and $\phi^k(\cdot,0)=\phi_0^k$ almost everywhere in Ω.

Energy estimates. Integrating $(4.18)_1$ over $\Omega\times(0,t)$ for any $t\in(0,T]$, we obtain the conservation of mass

$$
\overline{\phi^k}(t) = \overline{\phi_0^k}, \quad \forall\, t\in[0,T]. \tag{4.19}
$$

We multiply $(4.18)_1$ by μ^k and integrate over Ω. By exploiting the convexity of F, the regularity (4.17) and [18, Proposition 4.2], we obtain

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{\mathrm{nloc}}(\phi^k) + \int_\Omega \mathbf{u}^k\cdot\nabla\phi^k\,\mu^k\,\mathrm{d}x + \int_\Omega |\nabla\mu^k|^2\,\mathrm{d}x = 0. \tag{4.20}
$$

Since $\operatorname{div}\mathbf{u} = 0$ in $\Omega\times(0,T)$ and $\mathbf{u}\cdot\mathbf{n}=0$ on $\partial\Omega\times(0,T)$, by using the uniform bound $\|\phi^k(t)\|_{L^\infty(\Omega)}\le 1$, we find

$$
\left|\int_\Omega \mathbf{u}^k\cdot\nabla\phi^k\,\mu^k\,\mathrm{d}x\right| = \left|\int_\Omega \mathbf{u}^k\cdot\nabla\mu^k\,\phi^k\,\mathrm{d}x\right| \le \frac{1}{2}\int_\Omega |\nabla\mu^k|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\mathbf{u}^k|^2\,\mathrm{d}x.
$$

Then, we easily infer from (4.20) that

$$
\mathcal{E}_{\mathrm{nloc}}(\phi^k(t)) + \frac{1}{2}\int_0^t \|\nabla\mu^k(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau \le \mathcal{E}_{\mathrm{nloc}}(\phi_0^k) + \frac{1}{2}\int_0^t \|\mathbf{u}^k(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau, \quad \forall\, t\in[0,T]. \tag{4.21}
$$

We observe that $0\le F(s)\le C$, for $s\in[-1,1]$. By the Young inequality, we also have that $|(K*u,u)| \le \|K\|_{L^1(\mathbb{R}^2)}\|u\|_{L^1(\Omega)}\|u\|_{L^\infty(\Omega)}$, for any $u\in L^\infty(\Omega)$. Therefore, we simply deduce that

$$
\int_0^T \|\nabla\mu^k(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau \le C + \int_0^T \|\mathbf{u}^k(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau, \tag{4.22}
$$

where C depends on Ω, F and K, but is independent of k. Along the proof, we will adopt the same agreement for all the other constants C appearing in the following subsections. Next, we compute the gradient of $(4.18)_2$. In light of the regularity (4.17), we notice that $F'(\phi^k(t))\in H^1(\Omega)$ almost everywhere in $(0,T)$ and, in particular, $\nabla F'(\phi) = F''(\phi^k)\nabla\phi$ almost everywhere in $\Omega\times(0,T)$ by the Stampacchia superposition principle [52]. Then, by the convexity of F, we have

$$
\int_\Omega \left|F''(\phi^k)\nabla\phi^k\right|^2\,\mathrm{d}x \le \|\nabla\mu^k\|_{L^2(\Omega)} + \|\nabla K*\phi^k\|_{L^2(\Omega)}. \tag{4.23}
$$

Thus, we infer from (H_2) and the uniform L^∞ bound of ϕ^k that

$$\|\nabla \phi^k\|_{L^2(\Omega)} \le C\left(1 + \|\nabla \mu^k\|_{L^2(\Omega)}\right). \tag{4.24}$$

Thanks to (4.22), the above inequality entails that

$$\int_0^T \|\nabla \phi^k(\tau)\|^2_{L^2(\Omega)}\, \mathrm{d}\tau \le C(1+T) + C\int_0^T \|\mathbf{u}^k(\tau)\|_{L^2(\Omega)}\, \mathrm{d}\tau. \tag{4.25}$$

In addition, by duality in (4.18), we easily infer that

$$\|\partial_t \phi^k\|_{H^1(\Omega)'} \le \|\mathbf{u}^k\|_{L^2(\Omega)} + \|\nabla \mu^k\|_{L^2(\Omega)}, \tag{4.26}$$

which implies that

$$\int_0^T \|\partial_t \phi^k(\tau)\|^2_{H^1(\Omega)'}\, \mathrm{d}\tau \le C + \int_0^T \|\mathbf{u}^k(\tau)\|^2_{L^2(\Omega)}\, \mathrm{d}\tau. \tag{4.27}$$

Existence of weak solutions. Let us consider $k \ge \overline{k}$ such that (4.16) holds. As such, we have from (4.19) that $|\overline{\phi^k}(t)| \le 1-\xi$ for all $t \in [0,T]$ uniformly in k. Then, recalling that $\mathbf{u}^k \to \mathbf{u}$ in $L^4(0,T; L^4_\sigma(\Omega)) \hookrightarrow L^2(0,T; L^2_\sigma(\Omega))$, we infer from (4.22), (4.25) and (4.27) that

$$\|\phi^k\|_{L^\infty(\Omega\times(0,T))} \le 1, \quad \|\phi^k\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t \phi^k\|_{L^2(0,T;H^1(\Omega)')} + \|\nabla \mu^k\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{4.28}$$

In order to recover a uniform estimate of the full H^1 norm of μ^k, we multiply $(4.18)_2$ by $\phi^k - \overline{\phi^k}$ and integrate over Ω. By the generalized Poincaré inequality, the assumption (H_2) and the uniform L^∞ bound of ϕ^k, we find

$$\left|\int_\Omega F'(\phi^k)\big(\phi^k - \overline{\phi^k}\big)\, \mathrm{d}x\right| \le C\left(1 + \|\nabla \mu^k\|_{L^2(\Omega)}\right).$$

We report that there exist two positive constants C_F^1 and C_F^2 such that (see, e.g. [53])

$$\|F'(\phi^k)\|_{L^1(\Omega)} \le C_F^1 \left|\int_\Omega F'(\phi^k)\big(\phi^k - \overline{\phi^k}\big)\, \mathrm{d}x\right| + C_F^2, \tag{4.29}$$

where C_F^1 and C_F^2 only depends on F, Ω and ϖ. Then, we conclude that

$$\|\mu^k\|_{L^1(\Omega)} + \|F'(\phi^k)\|_{L^1(\Omega)} \le C\left(1 + \|\nabla \mu^k\|_{L^2(\Omega)}\right). \tag{4.30}$$

Thus, we deduce from (4.30), the Poincaré-Wirtinger inequality and the definition of μ^k that

$$\|\mu^k\|_{L^2(0,T;H^1(\Omega))} + \|F'(\phi^k)\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{4.31}$$

Therefore, we infer from the Banach-Alaoglu theorem and the Aubin-Lions theorem that

$$\begin{aligned}
&\phi^k \rightharpoonup \phi \quad \text{weakly}^\star \text{ in } L^\infty(\Omega\times(0,T)), \quad \phi^k \rightharpoonup \phi \quad \text{weakly in } L^2(0,T;H^1(\Omega)),\\
&\phi^k \to \phi \quad \text{strongly in } L^2(0,T;L^2(\Omega)), \quad \partial_t\phi^k \rightharpoonup \partial_t\phi \quad \text{weakly in } L^2(0,T;H^1(\Omega)'),\\
&\mu^k \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \quad F'(\phi^k) \rightharpoonup \xi \quad \text{weakly in } L^2(0,T;H^1(\Omega)).
\end{aligned} \tag{4.32}$$

Clearly, the limit function ϕ satisfies $|\phi(x,t)| \le 1$ almost everywhere in $\Omega\times(0,T)$. Thanks to the strong convergence in (4.32), we infer that $\phi^k \to \phi$ almost everywhere in $\Omega\times(0,T)$. Then, $F'(\phi^k) \to \widetilde{F'}(\phi)$ almost everywhere in $\Omega\times(0,T)$, where $\widetilde{F'}(s) = F'(s)$ if $s \in (-1,1)$ and $\widetilde{F'}(\pm1) = \pm\infty$. An application of the Fatou lemma, together with (4.31), entails that $\widetilde{F'}(\phi) \in L^2(0,T;L^2(\Omega))$. In turn, it also implies that $|\phi(x,t)| < 1$ almost everywhere in $\Omega\times(0,T)$. In addition, this is sufficient to conclude that $\xi = F'(\phi) \in L^2(0,T;H^1(\Omega))$. Finally, passing to the limit as $k \to \infty$ in (4.18), we obtain that ϕ is a weak solution to (4.1) fulfilling (4.2), (4.3), while corresponding to $\phi_0 \in H^1(\Omega)\cap L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)} \le 1$ and $|\overline{\phi_0}| < 1$.

In order to conclude this part, we are left to deal the general case where the initial datum ϕ_0 only belongs to $L^\infty(\Omega)$ with $\|\phi_0\|_{L^\infty(\Omega)} \le 1$ and $|\overline{\phi_0}| < 1$. To this aim, by classical mollification there exists a sequence $\{\phi_0^m\}_{m\in\mathbb{N}}$ such that $\phi_0^m \in C^\infty(\overline{\Omega})$: $-1 \le \phi_0^m(x) \le 1$ for all $x \in \overline{\Omega}$, for any $m \in \mathbb{N}$, and $\phi_0^m \to \phi_0$ strongly in $L^r(\Omega)$, for any $r \in [1,\infty)$, $\phi_0^m \rightharpoonup \phi_0$ weakly* in $L^\infty(\Omega)$, and $|\overline{\phi_0^m}| < 1$. By the previous analysis, there exists a weak solution ϕ^m for any $m \in \mathbb{N}$. Then, since $\mathcal{E}(\phi_0^m)$ is uniformly bounded in m and the lower semicontinuity of the norm with respect to the weak convergence, it is straightforward to deduce (4.28) and (4.31) by replacing ϕ^k and μ^k with ϕ^m and μ^m. Thus, arguing as before, the sequence ϕ^m converges as in (4.32) to a limit function ϕ satisfying (4.2) and (4.3), as well as $\phi(0) = \phi_0$ in Ω.

Finally, concerning the energy identity, the convexity of F, the assumption (H_2), the regularity (4.2) and [18, Proposition 4.2] entail that

$$\mathcal{E}_{\text{nloc}}(\phi(t)) - \mathcal{E}_{\text{nloc}}(\phi_0) = \int_0^t \langle \partial_t\phi(\tau), \mu(\tau)\rangle \, \mathrm{d}\tau, \qquad \text{for all } t \in [0,T].$$

Owing to this, (4.4) directly follows from choosing $v = \mu$ in (4.3) and integrating the resulting equation in $[0,t]$ for any $0 \le t \le T$.

Sobolev estimates. We first observe that the regularity of the approximated solutions $\{\phi^k, \mu^k\}_{k\in\mathbb{N}}$ in (4.17) (in particular, the strict separation property) allows us to compute the time and the spatial derivatives of $(4.18)_2$, which gives

$$\partial_t\mu^k = F''(\phi^k)\partial_t\phi^k - K * \partial_t\phi^k, \quad \nabla\mu^k = F''(\phi^k)\nabla\phi^k - \nabla K * \phi^k, \quad \text{a.e. in } \Omega\times(0,T). \tag{4.33}$$

In addition, the map $t \to \|\nabla\mu^k(t)\|^2_{L^2(\Omega)}$ belongs to $AC([0,T])$ and the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\mu^k\|^2_{L^2(\Omega)} = \left\langle \partial_t\mu^k, \Delta\mu^k\right\rangle_{H^1_{(0)}(\Omega)}$$

holds almost everywhere in $(0,T)$. Thus, multiplying $(4.18)_1$ by $\partial_t\mu^k$, integrating over Ω, and exploiting (4.33), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu^k\|^2_{L^2(\Omega)} + \int_\Omega F''(\phi^k)|\partial_t\phi^k|^2\,\mathrm{d}x - \int_\Omega K * \partial_t\phi^k\,\partial_t\phi^k\,\mathrm{d}x + \int_\Omega \mathbf{u}^k\cdot\nabla\phi^k\,\partial_t\mu^k\,\mathrm{d}x = 0. \tag{4.34}$$

We rewrite the key term $(\mathbf{u}^k\cdot\nabla\phi^k, \partial_t\mu^k)$. By using (4.33) and the fact that $\mathbf{u}^k \in C_0^\infty((0,T); C^\infty_{0,\sigma}(\Omega;\mathbb{R}^2))$, we observe that

$$\begin{aligned}\int_\Omega \mathbf{u}^k\cdot\nabla\phi^k\,\partial_t\mu^k\,\mathrm{d}x &= \int_\Omega \left(\mathbf{u}^k\cdot\nabla\phi^k\right)F''(\phi^k)\,\partial_t\phi^k\,\mathrm{d}x - \int_\Omega\left(\mathbf{u}^k\cdot\nabla\phi^k\right)K*\partial_t\phi^k\,\mathrm{d}x\\ &= \int_\Omega \mathbf{u}^k\cdot\nabla\left(F'(\phi^k)\right)\partial_t\phi^k\,\mathrm{d}x - \int_\Omega\left(\mathbf{u}^k\cdot\nabla\phi^k\right)K*\partial_t\phi^k\,\mathrm{d}x\\ &= \int_\Omega\left(\mathbf{u}^k\cdot\nabla\mu^k\right)\partial_t\phi^k\,\mathrm{d}x + \int_\Omega\left(\mathbf{u}^k\cdot\left(\nabla K*\phi^k\right)\right)\partial_t\phi^k\,\mathrm{d}x\\ &\quad - \int_\Omega\left(\mathbf{u}^k\cdot\nabla\phi^k\right)K*\partial_t\phi^k\,\mathrm{d}x\\ &= \int_\Omega\left(\mathbf{u}^k\cdot\nabla\mu^k\right)\partial_t\phi^k\,\mathrm{d}x + \int_\Omega\left(\mathbf{u}^k\cdot\left(\nabla K*\phi^k\right)\right)\partial_t\phi^k\,\mathrm{d}x\\ &\quad + \int_\Omega\left(\mathbf{u}^k\cdot\nabla(K*\partial_t\phi^k)\right)\phi^k\,\mathrm{d}x.\end{aligned} \tag{4.35}$$

By exploiting the assumption (H_2) and the uniform L^∞ bound of ϕ^k, we have

$$\begin{aligned}\left|\int_\Omega\left(\mathbf{u}^k\cdot\left(\nabla K*\phi^k\right)\right)\partial_t\phi^k\,\mathrm{d}x\right| &\le \|\mathbf{u}^k\|_{L^2(\Omega)}\|\nabla K*\phi^k\|_{L^\infty(\Omega)}\|\partial_t\phi^k\|_{L^2(\Omega)}\\ &\le \|\mathbf{u}^k\|_{L^2(\Omega)}\|K\|_{W^{1,1}(\mathbb{R}^2)}\|\phi^k\|_{L^\infty(\Omega)}\|\partial_t\phi^k\|_{L^2(\Omega)}\\ &\le \frac{\alpha}{8}\|\partial_t\phi^k\|_{L^2(\Omega)}^2 + C\|\mathbf{u}^k\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.36}$$

Similarly, we also find

$$\begin{aligned}\left|\int_\Omega\left(\mathbf{u}^k\cdot\nabla(K*\partial_t\phi^k)\right)\phi^k\,\mathrm{d}x\right| &\le \|\mathbf{u}^k\|_{L^2(\Omega)}\|\nabla K*\partial_t\phi^k\|_{L^2(\Omega)}\|\phi^k\|_{L^\infty(\Omega)}\\ &\le \|\mathbf{u}^k\|_{L^2(\Omega)}\|K\|_{W^{1,1}(\mathbb{R}^2)}\|\partial_t\phi^k\|_{L^2(\Omega)}\|\phi^k\|_{L^\infty(\Omega)}\\ &\le \frac{\alpha}{8}\|\partial_t\phi^k\|_{L^2(\Omega)}^2 + C\|\mathbf{u}^k\|_{L^2(\Omega)}^2.\end{aligned} \tag{4.37}$$

In order to control the first term $(\mathbf{u}^k\cdot\nabla\mu^k,\partial_t\phi^k)$ on the right-hand side in (4.35), we need a preliminary estimate of the H^1 norm of $\nabla\mu^k$. To this end, let us first observe from (4.18) that $\mu^k-\overline{\mu^k}=\mathcal{N}(\partial_t\phi^k+\mathbf{u}\cdot\nabla\phi^k)$. Then, in light of (2.1), we find

$$\|\nabla\mu^k\|_{H^1(\Omega)} \le C\left(\|\partial_t\phi^k\|_{L^2(\Omega)} + \|\mathbf{u}^k\cdot\nabla\phi^k\|_{L^2(\Omega)}\right). \tag{4.38}$$

In order to estimate the second term on the right-hand side in (4.38), we deduce from (4.33) that

$$\mathbf{u}^k\cdot\nabla\phi^k = \frac{1}{F''(\phi^k)}\left(\mathbf{u}^k\cdot\nabla\mu^k + \mathbf{u}^k\cdot\left(\nabla K*\phi^k\right)\right),\quad \text{a.e. in } \Omega\times(0,T). \tag{4.39}$$

By the strict convexity of F, we notice that $F''(s)^{-1}\le\alpha^{-1}$ for any $s\in(-1,1)$. By using (H_2), (2.2) and the uniform L^∞ bound of ϕ^k, we obtain

$$
\begin{aligned}
\|\mathbf{u}^k\cdot\nabla\phi^k\|_{L^2(\Omega)} &\le \frac{2}{\alpha}\left(\|\mathbf{u}^k\cdot\nabla\mu^k\|_{L^2(\Omega)}+\|\mathbf{u}\cdot(\nabla K*\phi^k)\|_{L^2(\Omega)}\right)\\
&\le C\|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^4(\Omega)}+C\|\mathbf{u}^k\|_{L^2(\Omega)}\|\nabla K*\phi^k\|_{L^\infty(\Omega)}\\
&\le C\|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^2(\Omega)}^{\frac12}\|\nabla\mu^k\|_{H^1(\Omega)}^{\frac12}+C\|\mathbf{u}^k\|_{L^2(\Omega)}\|K\|_{W^{1,1}(\mathbb{R}^2)}\|\phi^k\|_{L^\infty(\Omega)}\\
&\le C\|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^2(\Omega)}^{\frac12}\|\nabla\mu^k\|_{H^1(\Omega)}^{\frac12}+C\|\mathbf{u}^k\|_{L^2(\Omega)}.
\end{aligned}
\tag{4.40}
$$

Then, combining (4.38) and (4.40), we arrive at

$$
\|\nabla\mu^k\|_{H^1(\Omega)}\le C\left(\|\partial_t\phi^k\|_{L^2(\Omega)}+\|\mathbf{u}^k\|_{L^4(\Omega)}^2\|\nabla\mu^k\|_{L^2(\Omega)}+\|\mathbf{u}^k\|_{L^2(\Omega)}\right). \tag{4.41}
$$

Now, by using (2.2) and (4.41), we find

$$
\begin{aligned}
\left|\int_\Omega\left(\mathbf{u}^k\cdot\nabla\mu^k\right)\partial_t\phi^k\,\mathrm{d}x\right| &\le \|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^4(\Omega)}\|\partial_t\phi^k\|_{L^2(\Omega)}\\
&\le \|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^2(\Omega)}^{\frac12}\|\nabla\mu^k\|_{H^1(\Omega)}^{\frac12}\|\partial_t\phi^k\|_{L^2(\Omega)}\\
&\le \|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^2(\Omega)}^{\frac12}\|\partial_t\phi^k\|_{L^2(\Omega)}^{\frac32}+\|\mathbf{u}^k\|_{L^4(\Omega)}^2\|\nabla\mu^k\|_{L^2(\Omega)}\|\partial_t\phi^k\|_{L^2(\Omega)}\\
&\quad+\|\mathbf{u}^k\|_{L^2(\Omega)}^{\frac12}\|\mathbf{u}^k\|_{L^4(\Omega)}\|\nabla\mu^k\|_{L^2(\Omega)}^{\frac12}\|\partial_t\phi^k\|_{L^2(\Omega)}\\
&\le \frac{\alpha}{8}\|\partial_t\phi^k\|_{L^2(\Omega)}^2+C\|\mathbf{u}^k\|_{L^4(\Omega)}^4\|\nabla\mu^k\|_{L^2(\Omega)}^2+C\|\mathbf{u}^k\|_{L^2(\Omega)}^2.
\end{aligned}
\tag{4.42}
$$

Concerning the last term $(K*\partial_t\phi^k,\partial_t\phi^k)$ in (4.34), by exploiting (H_2), the properties of the Laplace operator A_0 and (4.26), we are led to

$$
\begin{aligned}
\int_\Omega K*\partial_t\phi^k\,\partial_t\phi^k\,\mathrm{d}x &\le \|\nabla K*\partial_t\phi^k\|_{L^2(\Omega)}\|\nabla\mathcal{N}\partial_t\phi\|_{L^2(\Omega)}\\
&\le \frac{\alpha}{8}\|\partial_t\phi\|_{L^2(\Omega)}^2+C\left(\|\mathbf{u}^k\|_{L^2(\Omega)}^2+\|\nabla\mu^k\|_{L^2(\Omega)}^2\right).
\end{aligned}
\tag{4.43}
$$

Inserting the estimates (4.36), (4.37), (4.42) and (4.43) in (4.34), and recalling (H_3), we eventually deduce that

$$
\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mu^k\|_{L^2(\Omega)}^2+\frac{\alpha}{2}\|\partial_t\phi^k\|_{L^2(\Omega)}^2\le C\left(1+\|\mathbf{u}^k\|_{L^4(\Omega)}^4\right)\|\nabla\mu^k\|_{L^2(\Omega)}^2+C\|\mathbf{u}^k\|_{L^2(\Omega)}^2. \tag{4.44}
$$

Therefore, the Gronwall lemma entails that

$$
\|\nabla\mu^k(t)\|_{L^2(\Omega)}^2\le\left(\|\nabla\mu^k(0)\|_{L^2(\Omega)}^2+C\int_0^t\|\mathbf{u}^k(\tau)\|_{L^2(\Omega)}^2+\|\nabla\mu^k(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau\right)\mathrm{e}^{C\int_0^t\|\mathbf{u}(\tau)\|_{L^4(\Omega)}^4\,\mathrm{d}\tau}, \tag{4.45}
$$

for all $t\in[0,T]$. Furthermore, integrating in time (4.41) and (4.44) on $[0,T]$, and using (4.45), we also obtain

$$\begin{aligned}\int_0^T \|\partial_t\phi^k(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu^k(\tau)\|_{H^1(\Omega)}^2 \,\mathrm{d}\tau \\ \le C\left(\|\nabla\mu^k(0)\|_{L^2(\Omega)}^2 + C\int_0^T \|\mathbf{u}^k(\tau)\|_{L^2(\Omega)}^2 + \|\nabla\mu^k(\tau)\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau\right) \\ \times\left(1+\int_0^T \|\mathbf{u}^k(\tau)\|_{L^4(\Omega)}^4 \,\mathrm{d}\tau\right)\exp\left(C\int_0^T \|\mathbf{u}^k(\tau)\|_{L^4(\Omega)}^4 \,\mathrm{d}\tau\right).\end{aligned} \tag{4.46}$$

Before concluding this step, we derive a further estimate for the L^p norms of $\nabla\phi^k$. We know that $\mu^k \in L^2(0,T;H^2(\Omega))$. Then it follows by comparison in $(4.33)_2$ and by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p \in [2,\infty)$ that $F''(\phi^k)\nabla\phi^k \in L^2(0,T;L^p(\Omega))$. This allows us to rigorously multiply $(4.33)_2$ by $|\nabla\phi^k|^{p-2}\nabla\phi^k$ and integrate over Ω. As a result, we get

$$\int_\Omega F''(\phi^k)|\nabla\phi^k|^p \,\mathrm{d}x = \int_\Omega |\nabla\phi^k|^{p-2}\nabla\phi^k\cdot\nabla\mu^k \,\mathrm{d}x + \int_\Omega |\nabla\phi^k|^{p-2}\nabla\phi^k\cdot\nabla K*\phi^k \,\mathrm{d}x.$$

By using the assumption (H_2), the Hölder inequality and the Young inequalities, together with uniform L^∞ bound of ϕ^k, it is easily seen that

$$\int_\Omega F''(\phi^k)|\nabla\phi^k|^p \,\mathrm{d}x \le C\left(1+\|\nabla\mu^k\|_{L^p(\Omega)}^p\right), \tag{4.47}$$

for some C depending on p. This entails, in particular, that

$$\|\nabla\phi^k\|_{L^p(\Omega)} \le C\left(1+\|\nabla\mu^k\|_{L^p(\Omega)}\right). \tag{4.48}$$

Finally, we highlight that all the constants C in (4.45), (4.46) and (4.47) are also independent of the velocity field $\mathbf{u}^k$ and the initial condition ϕ_0^k. In fact, they only depend on α, K and Ω. This completes the proof of Theorem 4.1.

Regularity. The case (i). By definition of ϕ_0^k, it is easily seen that $\phi_0^k \to \phi_0$ and $\nabla\phi_0^k \to \nabla\phi$ almost everywhere in Ω. In light of (4.15), we also have that $F''(\phi_0^k)|\nabla\phi_0^k| \le F''(\phi_0)|\nabla\phi_0|$ almost everywhere in Ω. Since $F''(\phi_0)\nabla\phi_0 \in L^2(\Omega;\mathbb{R}^2)$ by assumption, we simply deduce that

$$\int_\Omega |F''(\phi_0^k)\nabla\phi_0^k - F''(\phi_0)\nabla\phi_0|^2 \,\mathrm{d}x \to 0,$$

namely, $F''(\phi_0^k)\nabla\phi_0^k \to F''(\phi_0)\nabla\phi_0$ in $L^2(\Omega;\mathbb{R}^2)$. On the other hand, it is straightforward to prove that $K*\phi_0^k \to K*\phi_0$ in $H^1(\Omega)$. In addition, it is possible to show from (4.17) and (4.18) that $\nabla\mu^k(0) = F''(\phi_0^k)\nabla\phi_0^k - \nabla K*\phi_0^k$ in Ω. Therefore, we deduce that $\nabla\mu^k(0) \to F''(\phi_0)\nabla\phi_0 - \nabla K*\phi_0$ in $L^2(\Omega;\mathbb{R}^2)$. As an immediate consequence, we get

$$\|\nabla\mu^k(0)\|_{L^2(\Omega)} \to \|F''(\phi_0)\nabla\phi_0 - \nabla K*\phi_0\|_{L^2(\Omega)}, \quad \text{as } k\to\infty. \tag{4.49}$$

Next, recalling that $\mathbf{u}^k \to \mathbf{u}$ in $L^4(0,T;L^4_\sigma)$ and the uniform estimate (4.22), we infer from (4.45) and (4.46) that

$$\|\nabla\mu^k\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t\phi^k\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\mu^k\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{4.50}$$

Thanks to (4.30), we obtain

$$\|\mu^k\|_{L^\infty(0,T;H^1(\Omega))} + \|\mu^k\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{4.51}$$

Concerning the concentration ϕ^k, we deduce from (4.48), (4.51) and the interpolation inequality

$$\|u\|_{L^q(0,T;L^p(\Omega))} \le C\|u\|_{L^\infty(0,T;L^2(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))},$$

where $q = \dfrac{2p}{p-2}$ and $p \in (2,\infty)$, that

$$\|\phi^k\|_{L^\infty(\Omega\times(0,T))} \le 1, \quad \|\phi^k\|_{L^\infty(0,T;H^1(\Omega))} + \|\phi^k\|_{L^q(0,T;W^{1,p}(\Omega))} \le C. \tag{4.52}$$

In a similar way, by comparison in (4.18), we are led to

$$\|F'(\phi^k)\|_{L^\infty(0,T;H^1(\Omega))} + \|F'(\phi^k)\|_{L^q(0,T;W^{1,p}(\Omega))} \le C. \tag{4.53}$$

Furthermore, recalling (4.26), we obtain from $\mathbf{u} \in L^4(0,T;L^4_\sigma(\Omega))$ and (4.51) that

$$\|\partial_t \phi^k\|_{L^4(0,T;H^1(\Omega)')} \le C. \tag{4.54}$$

Also, by the assumption (H_4), an application of the Trudinger-Moser inequality and the estimate (4.53) (cf. [28, Theorem 5.2]) entails that

$$\|F''(\phi^k)\|_{L^\infty(0,T;L^p(\Omega))} \le C_p, \quad \forall\, p \in [2,\infty). \tag{4.55}$$

Notice that $\partial_t F'(\phi^k) = F''(\phi^k)\partial_t\phi^k$. Owing to this, for any $v \in H^1(\Omega)$, we have

$$\big|\langle \partial_t F'(\phi^k), v\rangle\big| \le \|F''(\phi^k)\|_{L^4(\Omega)}\|\partial_t\phi^k\|_{L^2(\Omega)}\|v\|_{L^4(\Omega)}, \tag{4.56}$$

which, in turn, implies that

$$\|\partial_t F'(\phi^k)\|_{L^2(0,T;H^1(\Omega)')} \le C. \tag{4.57}$$

Thus, in light of (4.33) and (4.50), we immediately deduce that

$$\|\partial_t \mu^k\|_{L^2(0,T;H^1(\Omega)')} \le C. \tag{4.58}$$

Exploiting the above uniform estimates (4.50)-(4.58), by a compactness argument (simpler than the one for the existence of weak solutions), we pass to the limit in (4.18) as $k \to \infty$ obtaining that the limit function ϕ is a strong solution to (4.1) satisfying (4.6). In particular, $(4.1)_1$ holds almost everywhere in $\Omega \times (0,T)$, $(4.1)_2$ holds almost everywhere in $\partial\Omega \times (0,T)$. Since the map $t \to \{x \in \Omega : |\phi(x,t)| = 1\}$ is continuous in $[0,T]$ and $F'(\phi) \in L^\infty(0,T;H^1(\Omega))$, we observe that $|\{x \in \Omega : |\phi(x,t)| = 1\}| = 0$ for all $t \in [0,T]$. Besides, the estimates (4.7), (4.8) and (4.9) follow from (4.30), (4.45), (4.46), (4.48), (4.49), (4.56) and the lower semicontinuity of the norm with respect to the weak convergence. In particular, to get the estimates (4.7)-(4.8), we need that, up to subsequences,

$$\int_0^T \|\nabla\mu^k(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \to \int_0^T \|\nabla\mu(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \quad \text{as } k \to \infty. \tag{4.59}$$

This easily follows from (4.51) and (4.58) by means of the Aubin-Lions Lemma. Nevertheless, in more general cases, such as when $\Omega \subset \mathbb{R}^3$, one cannot rely on the summability on $\partial_t \mu$. Therefore, we propose a more general and direct proof of (4.59), avoiding the use of Aubin-Lions Lemma, by exploiting the energy identity (4.20). First notice that, by the above convergences,

$$\int_0^T \int_\Omega \mathbf{u}^k \cdot \nabla \mu^k \, \phi^k \, \mathrm{d}x\mathrm{d}\tau \to \int_0^T \int_\Omega \mathbf{u} \cdot \nabla \mu \, \phi \, \mathrm{d}x\mathrm{d}\tau,$$

and

$$\mathcal{E}_{\mathrm{nloc}}(\phi(T)) \le \liminf_{k\to\infty} \mathcal{E}_{\mathrm{nloc}}(\phi^k(T)).$$

Therefore, by using (4.20), we deduce that

$$\begin{aligned}\mathcal{E}_{\mathrm{nloc}}(\phi(T)) + \liminf_{k\to\infty} \int_0^T \|\nabla \mu^k(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau &\le \liminf_{k\to\infty} \left(\mathcal{E}_{\mathrm{nloc}}(\phi^k(T)) + \int_0^T \|\nabla \mu^k(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau \right) \\ &\le \liminf_{k\to\infty} \left(\mathcal{E}_{\mathrm{nloc}}(\phi^k_0) + \int_0^T \int_\Omega \mathbf{u}^k \cdot \nabla \mu^k \, \phi^k \, \mathrm{d}x\mathrm{d}\tau \right) \\ &= \mathcal{E}_{\mathrm{nloc}}(\phi_0) + \int_0^T \int_\Omega \mathbf{u} \cdot \nabla \mu \, \phi \, \mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

On the other hand, we also have the following energy identity

$$\mathcal{E}_{\mathrm{nloc}}(\phi(T)) + \int_0^T \|\nabla \mu(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau = \mathcal{E}_{\mathrm{nloc}}(\phi_0) + \int_0^T \int_\Omega \mathbf{u} \cdot \nabla \mu \, \phi \, \mathrm{d}x\mathrm{d}\tau.$$

By the chain of inequalities above, we infer that

$$\int_0^T \|\nabla \mu(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau \ge \liminf_{k\to\infty} \int_0^T \|\nabla \mu^k(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau$$

which implies, together with the convergence $\nabla \mu^k \rightharpoonup \nabla \mu$ weakly in $L^2(0,T; L^2(\Omega;\mathbb{R}^2))$, up to a subsequence, that

$$\int_0^T \|\nabla \mu(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau = \liminf_{k\to\infty} \int_0^T \|\nabla \mu^k(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau.$$

From this, we clearly obtain (4.59) up to a subsequence.

Finally, if we also assume $\mathbf{u} \in L^\infty(0,T; L^2_\sigma(\Omega))$ then, by comparison in (4.1) (cf. (4.26)), it is easily seen that $\partial_t \phi \in L^\infty(0,T; H^1(\Omega)')$, which concludes the proof related to the case (i).

Separation property and further regularity. The case (ii). We may first argue as in the proof of [29, Theorem 4.1] to conclude with (4.11) through a direct argument (see Remark 4.4). Then, following [54, Corollary 4.5], we can also modify that proof slightly upon eliminating the cut-off function η_n, instead by testing (4.12) with

$v := \phi_n(t) = (\phi(t) - k_n)_+$, for the increasing sequence $k_n := 1 - \delta - \delta 2^{-n} \overset{n\to\infty}{\to} 1 - \delta$, $1 - 2\delta < k_n < 1 - \delta$, where $0 < \delta < \delta_0/2$ (this implies that $\phi_n(0) = 0$, for all $n \in \mathbb{N}_0$). Moreover, the drift term becomes

$$\mathcal{Z} := \int_\Omega \phi\, \mathbf{u} \cdot \nabla \phi_n \mathrm{d}x = \int_\Omega \mathbf{u} \cdot \nabla (\phi_n)^2 \, \mathrm{d}x = 0, \tag{4.60}$$

so that one has[3]

$$\frac{1}{2}\|\phi_n(t)\|^2_{L^2(\Omega)} + F''(1-2\delta) \int_0^t \|\nabla \phi_n\|^2_{L^2(\Omega)} \, \mathrm{d}s \le \int_0^t \int_{A_n(s)} (\nabla K * \phi) \cdot \nabla \phi_n \mathrm{d}x \, \mathrm{d}s,$$

for all $t \in [0,1]$, assuming $T \ge 1$ without loss of generality. This inequality allows us to make minor changes in the arguments employed in [29, Theorem 4.1] to deduce that $\|\phi\|_{L^\infty([0,1]\times\Omega)} \le 1 - \delta$, and to exploit (4.11) to obtain (4.10).

Next, setting the difference quotient $\partial_t^h f(t) = h^{-1}(f(t+h) - f(t))$ for $0 < t < T - h$, we write

$$\partial_t^h \mu(t) = \partial_t^h \phi(t) \left(\int_0^1 F''(s\phi(t+h) + (1-s)\phi(t)) \, \mathrm{d}s \right) - K * \partial_t^h \phi(t), \quad \text{for a.a. } t \text{ in } (0,T).$$

In light of (4.10), it easily follows that $\|s\phi(\cdot + h) - (1-s)\phi\|_{L^\infty(\Omega\times(0,T-h))} \le 1 - \delta$ for all $s \in (0,1)$. Owing to this, and using the properties of K (see (H_2)) and the basic inequality $\|\partial_t^h \phi\|_{L^2(0,T-h;L^2(\Omega))} \le \|\partial_t \phi\|_{L^2(0,T;L^2(\Omega))}$, we deduce that

$$\|\partial_t^h \mu\|_{L^2(0,T-h;L^2(\Omega))} \le C,$$

where C is independent of h. This entails that $\partial_t \mu \in L^2(0,T;L^2(\Omega))$. In turn, recalling that $\mu \in L^2(0,T;H^2(\Omega))$, we also obtain $\mu \in C([0,T];H^1(\Omega))$. The proof of Theorem 4.1 is thus concluded. □

5. Proof of Theorem 1.5

First of all, we rewrite the nonlocal AGG model (1.8) in the non-conservative form as

$$\begin{cases} \rho(\phi)\partial_t \mathbf{u} + \rho(\phi)(\mathbf{u}\cdot\nabla)\mathbf{u} - \rho'(\phi)(\nabla\mu\cdot\nabla)\mathbf{u} - \operatorname{div}(\nu(\phi)D\mathbf{u}) + \nabla\Pi = \mu\nabla\phi, \\ \operatorname{div}\mathbf{u} = 0, \\ \partial_t\phi + \mathbf{u}\cdot\nabla\phi = \Delta\mu, \\ \mu = F'(\phi) - K * \phi, \end{cases} \tag{5.1}$$

in $\Omega \times (0,T)$, which is endowed with the boundary and initial conditions (1.9).

5.1. The approximate problem: the semi-Galerkin scheme

Let us consider the family of eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and corresponding eigenfunctions $\{\mathbf{w}_j\}_{j=1}^\infty$ of the Stokes operator $\mathbf{A}$. For any integer $m \ge 1$, let $\mathbf{V}_m$ denote the finite-dimensional subspaces of $L^2_\sigma(\Omega)$ defined as $\mathbf{V}_m = \operatorname{span}\{\mathbf{w}_1, \cdots, \mathbf{w}_m\}$. The orthogonal projection on $\mathbf{V}_m$ with respect to the inner product in $L^2_\sigma(\Omega)$ is

[3] Here $A_n(t) := \{x \in \Omega : \phi(x,t) \ge k_n\}$, $\quad t \in [0,1]$.

denoted by $\mathbb{P}_m$. Recalling that Ω is a C^2-domain, we have that $\mathbf{w}_j \in V^2_{0,\sigma}(\Omega)$ for all $j \in \mathbb{N}$. In addition, the following inequalities hold

$$\|\mathbf{v}\|_{H^1(\Omega)} \le C_m \|\mathbf{v}\|_{L^2(\Omega)}, \quad \|\mathbf{v}\|_{H^2(\Omega)} \le C_m \|\mathbf{v}\|_{L^2(\Omega)} \quad \forall\, \mathbf{v} \in \mathbf{V}_m. \tag{5.2}$$

Let us fix $T > 0$. For any $m \in \mathbb{N}$, we claim that there exists an approximate solution $(\mathbf{u}_m, \phi_m)$ to system (1.8)-(1.9) in the following sense:

$$\begin{cases} \mathbf{u}_m \in C([0,T]; \mathbf{V}_m) \cap H^1(0,T; \mathbf{V}_m), \\ \phi_m \in L^\infty(\Omega \times (0,T)) : \quad |\phi_m(x,t)| < 1 \ \text{ a.e. in } \Omega \times (0,T), \\ \phi_m \in L^\infty(0,T; H^1(\Omega)) \cap L^q(0,T; W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \partial_t \phi_m \in L^\infty(0,T; H^1(\Omega)') \cap L^2(0,T; L^2(\Omega)), \\ \mu_m \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap H^1(0,T; H^1(\Omega)'), \\ F'(\phi_m) \in L^\infty(0,T; H^1(\Omega)), \quad F''(\phi_m) \in L^\infty(0,T; L^p(\Omega)), \quad \forall\, p \in [2,\infty), \end{cases} \tag{5.3}$$

such that

$$\begin{aligned} &(\rho(\phi_m)\partial_t \mathbf{u}_m, \mathbf{w}) + (\rho(\phi_m)(\mathbf{u}_m \cdot \nabla)\mathbf{u}_m, \mathbf{w}) + (\nu(\phi_m) D\mathbf{u}_m, \nabla \mathbf{w}) \\ &\quad - \frac{\rho_1 - \rho_2}{2} \left((\nabla \mu_m \cdot \nabla)\mathbf{u}_m, \mathbf{w} \right) = - \left(\phi_m \nabla \mu_m, \mathbf{w} \right), \end{aligned} \tag{5.4}$$

for all $\mathbf{w} \in \mathbf{V}_m$, in $[0,T]$, and

$$\partial_t \phi_m + \mathbf{u}_m \cdot \nabla \phi_m = \Delta \mu_m, \quad \mu_m = F'(\phi_m) - K * \phi_m, \quad \text{a.e. in } \Omega \times (0,T). \tag{5.5}$$

In addition, the approximate solution $(\mathbf{u}_m, \phi_m)$ satisfies the boundary and initial conditions

$$\begin{cases} \mathbf{u}_m = \mathbf{0}, \quad \partial_{\mathbf{n}} \mu_m = 0 & \text{on } \partial\Omega \times (0,T), \\ \mathbf{u}_m(\cdot,0) = \mathbb{P}_m \mathbf{u}_0, \quad \phi(\cdot,0) = \phi_0 & \text{in } \Omega. \end{cases} \tag{5.6}$$

5.2. Existence of the approximate solutions

We perform a fixed point argument to determine the existence of the approximate solutions satisfying (5.4)-(5.6) as in [37,38]. To this aim, we suppose that $\mathbf{v} \in C([0,T]; \mathbf{V}_m)$ is given. Then the corresponding convective nonlocal Cahn-Hilliard system reads as

$$\partial_t \phi_m + \mathbf{v} \cdot \nabla \phi_m = \Delta \mu_m, \quad \mu_m = F'(\phi_m) - K * \phi_m \quad \text{in } \Omega \times (0,T), \tag{5.7}$$

with boundary and initial conditions

$$\partial_{\mathbf{n}} \mu_m = 0 \quad \text{on } \partial\Omega \times (0,T), \quad \phi_m(\cdot,0) = \phi_0 \quad \text{in } \Omega \times (0,T). \tag{5.8}$$

Thanks to the case (i) of Theorem 4.1, there exists a unique solution ϕ_m to (5.7)-(5.8) such that

$$\begin{cases} \phi_m \in L^\infty(\Omega \times (0,T)) : \quad |\phi_m| < 1 \text{ a.e. in } \Omega \times (0,T), \\ \phi_m \in L^\infty(0,T;H^1(\Omega)) \cap L^q(0,T;W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \partial_t \phi_m \in L^\infty(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega)), \\ \mu_m \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)'), \\ F'(\phi_m) \in L^\infty(0,T;H^1(\Omega)), \quad F''(\phi_m) \in L^\infty(0,T;L^p(\Omega)), \quad \forall\, p \in [2,\infty). \end{cases} \tag{5.9}$$

Moreover, on account of (4.4), by repeating line by line the proof of Theorem 4.1 (cf. energy estimates), we find

$$\int_0^T \|\nabla \mu_m(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau \le C + \int_0^T \|\mathbf{v}(\tau)\|^2_{L^2(\Omega)} \, \mathrm{d}\tau. \tag{5.10}$$

where C depends only on Ω, F and K, but is independent of m as all the other constants C in the sequel of this proof.

We now make the ansatz

$$\mathbf{u}_m(x,t) = \sum_{j=1}^m a_j^m(t)\mathbf{w}_j(x), \quad \forall\,(x,t) \in \Omega \times [0,T],$$

as the solution to the Galerkin approximation of (5.4), that is,

$$\begin{aligned} &(\rho(\phi_m)\partial_t \mathbf{u}_m, \mathbf{w}_l) + (\rho(\phi_m)(\mathbf{v}\cdot\nabla)\mathbf{u}_m, \mathbf{w}_l) + (\nu(\phi_m)D\mathbf{u}_m, \nabla \mathbf{w}_l) \\ &\quad - \frac{\rho_1-\rho_2}{2}\left((\nabla\mu_m\cdot\nabla)\mathbf{u}_m, \mathbf{w}_l\right) = -\left(\phi_m \nabla\mu_m, \mathbf{w}_l\right), \quad \forall\, l = 1,\dots,m, \end{aligned} \tag{5.11}$$

satisfying the initial condition $\mathbf{u}_m(\cdot,0) = \mathbb{P}_m \mathbf{u}_0$.

Arguing as in [37, Section 4], we introduce $\mathbf{A}^m(t) = (a_1^m(t),\dots,a_m^m(t))^T$ and we observe that (5.11) is equivalent to the system of differential equations

$$\mathbf{M}^m(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}^m + \mathbf{L}^m(t)\mathbf{A}^m = \mathbf{G}^m(t),$$

where the matrices $\mathbf{M}^m(t), \mathbf{L}^m(t)$ and the vector $\mathbf{G}^m(t)$ are defined as follows:

$$(\mathbf{M}^m(t))_{lj} = \int_\Omega \rho(\phi_m(t))\mathbf{w}_j\cdot\mathbf{w}_l \, \mathrm{d}x,$$

$$\begin{aligned} (\mathbf{L}^m(t))_{lj} &= \int_\Omega \rho(\phi_m(t))(\mathbf{v}(t)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l + \nu(\phi_m(t))D\mathbf{w}_j : \nabla\mathbf{w}_l \mathrm{d}x \\ &\quad - \int_\Omega \left(\frac{\rho_1-\rho_2}{2}\right)(\nabla\mu_m(t)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l, \end{aligned}$$

$$(\mathbf{G}(t))_l = -\int_\Omega \phi_m(t)\nabla\mu_m(t)\cdot\mathbf{w}_l \, \mathrm{d}x,$$

as well as $\mathbf{A}^m(0) = ((\mathbf{u}_0,\mathbf{w}_1),\dots,(\mathbf{u}_0,\mathbf{w}_m))^T$. The regularity properties in (5.9) imply that both ϕ_m and μ_m belongs to $C_{\mathrm{w}}([0,T];H^1(\Omega)) \cap C([0,T];L^p(\Omega))$ for any $p \in [1,\infty)$ (cf. [58]). In turn, since $\rho(\cdot)$ is a linear function and ν is globally Lipschitz, $\rho(\phi_m)$ and $\nu(\phi_m)$ also belong to $C([0,T];L^p(\Omega))$ for any $p \in [1,\infty)$. As such, we immediately observe that, for any $s,t \in [0,T]$,

$$|(\mathbf{M}^m(t))_{lj} - (\mathbf{M}^m(s))_{lj}| \le \|\mathbf{w}_j\|_{L^\infty(\Omega)}\|\mathbf{w}_l\|_{L^\infty(\Omega)} \int_\Omega |\rho(\phi_m(t)) - \rho(\phi_m(s))| \,\mathrm{d}x \underset{s\to t}{\longrightarrow} 0.$$

Since $\mathbf{V}_m \subset V^2_{0,\sigma}(\Omega)$, we observe that

$$(\mathbf{G}^m(t))_l - (\mathbf{G}^m(s))_l = -\int_\Omega (\phi_m(t) - \phi_m(s))\,\nabla\mu_m(t)\cdot\mathbf{w}_l \,\mathrm{d}x - \int_\Omega (\mu_m(t) - \mu_m(s))\,\nabla\phi_m(s)\cdot\mathbf{w}_l \,\mathrm{d}x.$$

Then, recalling that ϕ_m and μ_m belongs to $C_{\mathrm{w}}([0,T];H^1(\Omega))$, we infer that

$$\begin{aligned}|(\mathbf{G}^m(t))_l - (\mathbf{G}^m(s))_l| &\le \|\mathbf{w}_l\|_{L^\infty(\Omega)}\|\nabla\mu_m(t)\|_{L^2(\Omega)}\|\phi_m(t) - \phi_m(s)\|_{L^2(\Omega)} \\ &\quad + \|\mathbf{w}_l\|_{L^\infty(\Omega)}\|\nabla\phi_m(s)\|_{L^2(\Omega)}\|\mu_m(t) - \mu_m(s)\|_{L^2(\Omega)} \underset{s\to t}{\longrightarrow} 0.\end{aligned}$$

Furthermore, exploiting once again that $\mathbf{V}_m \subset V^2_{0,\sigma}(\Omega)$, we notice that

$$\begin{aligned}&\left|\int_\Omega \rho(\phi_m(t))(\mathbf{v}(t)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x - \int_\Omega \rho(\phi_m(s))(\mathbf{v}(s)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x\right| \\ &= \left|\int_\Omega (\rho(\phi_m(t)) - \rho(\phi_m(s)))\,(\mathbf{v}(t)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x + \int_\Omega \rho(\phi_m(s))((\mathbf{v}(t) - \mathbf{v}(s))\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x\right| \\ &\le \|\rho(\phi_m(t)) - \rho(\phi_m(s))\|_{L^3(\Omega)}\|\mathbf{v}(t)\|_{L^2(\Omega)}\|\nabla\mathbf{w}_j\|_{L^6(\Omega)}\|\mathbf{w}_l\|_{L^\infty(\Omega)} \\ &\quad + \|\rho(\phi_m(s))\|_{L^6(\Omega)}\|\mathbf{v}(t) - \mathbf{v}(s)\|_{L^2(\Omega)}\|\nabla\mathbf{w}_j\|_{L^3(\Omega)}\|\mathbf{w}_l\|_{L^\infty(\Omega)} \underset{s\to t}{\longrightarrow} 0\end{aligned}$$

and

$$\left|\int_\Omega (\nu(\phi_m(t)) - \nu(\phi_m(s)))\,D\mathbf{w}_j : \nabla\mathbf{w}_l \,\mathrm{d}x\right| \le \|\nu(\phi_m(t)) - \nu(\phi_m(s))\|_{L^2(\Omega)}\|D\mathbf{w}_j\|_{L^4(\Omega)}\|\nabla\mathbf{w}_l\|_{L^4(\Omega)} \underset{s\to t}{\longrightarrow} 0.$$

On the other hand, integrating by parts and exploiting the boundary conditions, we have

$$\int_\Omega (\nabla\mu_m(t)\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x = -\int_\Omega \mu(t)\Delta\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x - \int_\Omega \mu(t)\nabla\mathbf{w}_j : \nabla\mathbf{w}_l \,\mathrm{d}x.$$

Thus, we find

$$\left|\int_\Omega (\nabla(\mu_m(t) - \mu_m(s))\cdot\nabla)\mathbf{w}_j\cdot\mathbf{w}_l \,\mathrm{d}x\right| \le C\|\mathbf{w}_j\|_{H^2(\Omega)}\|\mathbf{w}_l\|_{H^2(\Omega)}\|\mu_m(s) - \mu_m(t)\|_{L^2(\Omega)} \underset{s\to t}{\longrightarrow} 0.$$

Thus, we derive that $\mathbf{M}^m$ and $\mathbf{L}^m$ belong to $C([0,T];\mathbb{R}^{m\times m})$ and $\mathbf{G}^m \in C([0,T];\mathbb{R}^m)$. Furthermore, being ρ strictly positive, we also have that $\mathbf{M}^m$ is positive definite and thus the inverse $(\mathbf{M}^m)^{-1} \in C([0,T];\mathbb{R}^{m\times m})$. Thus, the existence and uniqueness theorem for systems of linear ODEs guarantees that there exists a unique solution $\mathbf{u}_m \in C^1([0,T];\mathbf{V}_m)$.

Next, multiplying (5.11) by a_l^m and summing over l, we obtain

$$\int_\Omega \rho(\phi_m)\partial_t\left(\frac{|\mathbf{u}_m|^2}{2}\right)\mathrm{d}x + \int_\Omega \rho(\phi_m)\mathbf{v}\cdot\nabla\left(\frac{|\mathbf{u}_m|^2}{2}\right)\mathrm{d}x + \int_\Omega \nu(\phi_m)|D\mathbf{u}_m|^2\,\mathrm{d}x$$
$$-\int_\Omega \left(\frac{\rho_1-\rho_2}{2}\right)\nabla\mu_m\cdot\nabla\left(\frac{|\mathbf{u}_m|^2}{2}\right)\mathrm{d}x = -\int_\Omega \phi_m\nabla\mu_m\cdot\mathbf{u}_m\,\mathrm{d}x.$$

Arguing exactly as in [37, Section 4.2] and exploiting (5.5), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_m)\frac{|\mathbf{u}_m|^2}{2}\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|D\mathbf{u}_m|^2\,\mathrm{d}x = -\int_\Omega \phi_m\nabla\mu_m\cdot\mathbf{u}_m\,\mathrm{d}x. \tag{5.12}$$

By the Poincaré-Korn (see (2.5)) and the Young inequalities, as well as $|\phi_m|<1$ almost everywhere in $\Omega\times(0,T)$, we infer by the divergence theorem that

$$\begin{aligned}
-\int_\Omega \phi_m\nabla\mu_m\cdot\mathbf{u}_m\,\mathrm{d}x &= \int_\Omega \mu_m\nabla\phi_m\cdot\mathbf{u}_m\,\mathrm{d}x\\
&= \int_\Omega F'(\phi_m)\nabla\phi_m\cdot\mathbf{u}_m\,\mathrm{d}x - \int_\Omega K*\phi_m\nabla\phi_m\cdot\mathbf{u}_m\,\mathrm{d}x\\
&= \underbrace{\int_\Omega \nabla(F(\phi_m))\cdot\mathbf{u}_m\,\mathrm{d}x}_{=0} + \int_\Omega \phi_m\nabla K*\phi_m\cdot\mathbf{u}_m\,\mathrm{d}x\\
&\le \|\phi_m\|_{L^\infty(\Omega)}\|\nabla K\|_{L^1(\mathbb{R}^2)}\|\phi_m\|_{L^2(\Omega)}\|\mathbf{u}_m\|_{L^2(\Omega)}\\
&\le \sqrt{\frac{2}{\lambda_1}}\|\nabla K\|_{L^1(\mathbb{R}^2)}|\Omega|^{\frac12}\|D\mathbf{u}_m\|_{L^2(\Omega)}\\
&\le \frac{\nu_*}{2}\|D\mathbf{u}_m\|^2 + \frac{\|\nabla K\|^2_{L^1(\mathbb{R}^2)}|\Omega|}{\nu_*\lambda_1}.
\end{aligned} \tag{5.13}$$

Here we have used that $\nabla F(\phi_m) = F'(\phi_m)\nabla\phi_m$ almost everywhere in $\Omega\times(0,T)$ (see Remark 4.3). We are thus led to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_m)\frac{|\mathbf{u}_m|^2}{2}\,\mathrm{d}x + \frac{\nu_*}{2}\|D\mathbf{u}_m\|^2 \le \frac{\|\nabla K\|^2_{L^1(\mathbb{R}^2)}|\Omega|}{\nu_*\lambda_1}.$$

Integrating the above inequality in time, with $s\in[0,T]$ and exploiting the upper and lower bounds of ρ, we get

$$\max_{t\in[0,T]}\|\mathbf{u}_m(t)\|^2_{L^2(\Omega)} \le \frac{\rho^*}{\rho_*}\|\mathbf{u}_0\|^2_{L^2(\Omega)} + \frac{2\|\nabla K\|^2_{L^1(\mathbb{R}^2)}|\Omega|T}{\nu_*\lambda_1} =: M^2. \tag{5.14}$$

Let us now introduce the closed ball

$$X := \left\{\mathbf{u}\in C([0,T];\mathbf{V}_m) : \|\mathbf{u}\|_{C([0,T];\mathbf{V}_m)} \le M\right\},$$

and define the map

$$\mathcal{S} : X \to X, \quad \mathcal{S}(\mathbf{v}) := \mathbf{u}_m.$$

We need now to show that $\mathcal{S}$ is compact. To this aim, we control the time derivative of $\mathbf{u}_m$. In particular, multiplying (5.11) by $\mathrm{d}\left(a_l^m\right)/\mathrm{d}t$ and summing over l, we have

$$\begin{aligned}\rho_* \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)}^2 \le & -(\rho(\phi_m)(\mathbf{v}\cdot\nabla)\mathbf{u}_m, \partial_t \mathbf{u}_m) - (\nu(\phi_m) D\mathbf{u}_m, \nabla \partial_t \mathbf{u}_m) \\ & + \frac{\rho_1 - \rho_2}{2}((\nabla \mu_m \cdot \nabla)\mathbf{u}_m, \partial_t \mathbf{u}_m) - (\phi_m \nabla K * \phi_m, \partial_t \mathbf{u}_m).\end{aligned}$$

Here, we have used that $-(\phi\nabla\mu_m, \partial_t\mathbf{u}_m) = -(\phi_m \nabla K * \phi_m, \partial_t \mathbf{u}_m)$ (cf. (5.13)). By exploiting (5.2) and the global bound $|\phi_m| \le 1$ almost everywhere in $\Omega \times (0,T)$, we obtain

$$\begin{aligned}\rho_* \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)}^2 \le\ & \rho^* \|\mathbf{v}\|_{L^4(\Omega)} \|\nabla \mathbf{u}_m\|_{L^4(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} + \nu^* \|D\mathbf{u}_m\|_{L^2(\Omega)} \|\nabla \partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ & + \left|\frac{\rho_1 - \rho_2}{2}\right| \|\nabla \mu_m\|_{L^2(\Omega)} \|\nabla \mathbf{u}_m\|_{L^4(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^4(\Omega)} \\ & + \|\phi_m\|_{L^\infty(\Omega)} \|\nabla K * \phi_m\|_{L^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ \le\ & C\|\nabla \mathbf{v}\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{H^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} + C\|D\mathbf{u}_m\|_{L^2(\Omega)} \|\nabla \partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ & + C\|\nabla \mu_m\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{H^2(\Omega)} \|\nabla \partial_t \mathbf{u}_m\|_{L^2(\Omega)} + \|\nabla K\|_{L^1(\mathbb{R}^2)} \|\phi_m\|_{L^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ \le\ & C_m \|\mathbf{v}\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{L^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} + C_m \|\mathbf{u}_m\|_{L^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ & + C_m \|\nabla \mu_m\|_{L^2(\Omega)} \|\mathbf{u}_m\|_{L^2(\Omega)} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} + \|\nabla K\|_{L^1(\mathbb{R}^2)} |\Omega|^{\frac{1}{2}} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} \\ \le\ & C_m \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)} \left(M^2 + M\left(1 + \|\nabla \mu_m\|_{L^2(\Omega)}\right) + \|\nabla K\|_{L^1(\mathbb{R}^2)} |\Omega|^{\frac{1}{2}}\right) \\ \le\ & \frac{\rho_*}{2} \|\partial_t \mathbf{u}_m\|_{L^2(\Omega)}^2 + C_m \left(M^4 + M^2\left(1 + \|\nabla \mu_m\|_{L^2(\Omega)}^2\right) + \|\nabla K\|_{L^1(\mathbb{R}^2)}^2 |\Omega|\right).\end{aligned}$$

Then, integrating over $[0,T]$ and using (5.10), we deduce that

$$\begin{aligned}\int_0^T \|\partial_t \mathbf{u}_m(\tau)\|^2 \,\mathrm{d}\tau \le\ & \frac{2}{\rho_*}\left[C_m(M^2 + M^4)T + C_m M^2\left(C + \|\mathbf{v}\|_{L^2(0,T;L^2(\Omega))}^2\right) + C_m \|\nabla K\|_{L^1(\mathbb{R}^2)}^2 |\Omega| T\right] \\ \le\ & \frac{2}{\rho_*}\left[C_m(M^2 + M^4)T + C_m M^2\left(C + M^2 T\right) + C_m \|\nabla K\|_{L^1(\mathbb{R}^2)}^2 |\Omega| T\right] =: \widetilde{M}^2,\end{aligned}$$

namely

$$\|\partial_t \mathbf{u}_m\|_{L^2(0,T;\mathbf{V}_m)}^2 \le \widetilde{M}, \tag{5.15}$$

Recalling that $\mathbf{V}_m$ is finite dimensional, the Aubin-Lions Lemma entails $C([0,T];\mathbf{V}_m) \cap H^1(0,T;\mathbf{V}_m) \overset{c}{\hookrightarrow} C([0,T];\mathbf{V}_m)$. Therefore, since $\mathcal{S} : X \to Y$, where $Y = \{\mathbf{u} \in X : \|\partial_t \mathbf{u}_m\|_{L^2(0,T;\mathbf{V}_m)}^2 \le \widetilde{M}\}$, it follows that the map $\mathcal{S}$ is compact (more precisely, $\overline{\mathcal{S}(X)}$ is compact in $C([0,T];\mathbf{V}_m)$).

In order to complete our fixed point argument, we are left to show that $\mathcal{S} : X \to X$ is continuous. To this aim, we consider a sequence $\{\mathbf{v}_n\}_{n=1}^\infty \subset X$ such that $\mathbf{v}_n \to \mathbf{v}^\star$ in $C([0,T];\mathbf{V}_m)$; consequently, there exists a sequence $\{(\phi_n, \mu_n)\}_{n=1}^\infty$ and $(\phi^\star, \mu^\star)$ that solve the convective nonlocal Cahn-Hilliard equation (5.7)-(5.8), where $\mathbf{v}$ is replaced by $\mathbf{v}_n$ and $v^\star$, respectively. Following the uniqueness argument performed in the proof of Theorem 4.1, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_n-\phi^\star\right\|_*^2+\frac{3\alpha}{4}\left\|\phi_n-\phi^\star\right\|_{L^2(\Omega)}^2$$
$$\leq C\left\|\phi_n-\phi^\star\right\|_*^2+(\phi_n(\mathbf{v}_n-\mathbf{v}^\star),\nabla\mathcal{N}(\phi_n-\phi^\star))+(\mathbf{v}^\star(\phi_n-\phi^\star),\nabla\mathcal{N}(\phi_n-\phi^\star)).$$

Here we have used that $\overline{\phi_n}=\overline{\phi^\star}=\overline{\phi_0}$. In light of $\mathbf{v}^\star\in C([0,T];\mathbf{V}_m)$ and (5.2), we notice that

$$\begin{aligned}|(\mathbf{v}^\star(\phi_n-\phi^\star),\nabla\mathcal{N}(\phi_n-\phi^\star))|&\leq C\|\mathbf{v}^\star\|_{L^\infty(\Omega)}\|\phi\|_{L^2(\Omega)}\left\|\nabla\mathcal{N}(\phi_n-\phi^\star)\right\|_{L^2(\Omega)}\\&\leq\frac{\alpha}{4}\|\phi\|_{L^2(\Omega)}^2+C_mM^2\left\|\phi_n-\phi^\star\right\|_*^2.\end{aligned}$$

Since $|\phi_n|<1$ almost everywhere in $\Omega\times(0,T)$, we also find

$$\begin{aligned}\left|\left(\phi_n(\mathbf{v}_n-\mathbf{v}^\star),\nabla\mathcal{N}(\phi_n-\phi^\star)\right)\right|&\leq\|\phi_n\|_{L^\infty(\Omega)}\|\mathbf{v}_n-\mathbf{v}^\star\|_{L^2(\Omega)}\left\|\nabla\mathcal{N}(\phi_n-\phi^\star)\right\|_{L^2(\Omega)}\\&\leq C\|\phi_n-\phi^\star\|_*^2+C\|\mathbf{v}_n-\mathbf{v}^\star\|_{L^2(\Omega)}^2.\end{aligned}$$

Thus, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\phi_n-\phi^\star\right\|_*^2+\frac{\alpha}{2}\left\|\phi_n-\phi^\star\right\|_{L^2(\Omega)}^2\leq C\left\|\phi_n-\phi^\star\right\|_*^2+C\left\|\mathbf{v}_n-\mathbf{v}^\star\right\|_{L^2(\Omega)}^2$$

and the Gronwall Lemma yields

$$\left\|\phi_n-\phi^\star\right\|_{L^\infty(0,T;H^1(\Omega)')}^2+\left\|\phi_n-\phi^\star\right\|_{L^2(0,T;L^2(\Omega))}^2\leq C\mathrm{e}^{CT}\left(T+T^2\right)\left\|\mathbf{v}_n-\mathbf{v}^\star\right\|_{C([0,T];\mathbf{V}_m)}^2\underset{n\to\infty}{\longrightarrow}0. \quad (5.16)$$

On the other hand, recalling that $\{\mathbf{v}_n\}_n$ and $\mathbf{v}^\star$ belong to X, by the regularity (i) in Theorem 4.1 (more precisely, (4.7)-(4.8)) we infer that

$$\|\partial_t\phi_n\|_{L^\infty(0,T;H^1(\Omega)')}+\|\partial_t\phi_n\|_{L^2(0,T;L^2(\Omega))}+\|\mu_n\|_{L^\infty(0,T;H^1(\Omega))}+\|\nabla\mu_n\|_{L^2(0,T;H^1(\Omega))}\leq C, \quad (5.17)$$
$$\|\partial_t\phi^\star\|_{L^\infty(0,T;H^1(\Omega)')}+\|\partial_t\phi^\star\|_{L^2(0,T;L^2(\Omega))}+\|\mu^\star\|_{L^\infty(0,T;H^1(\Omega))}+\|\nabla\mu^\star\|_{L^2(0,T;H^1(\Omega))}\leq C. \quad (5.18)$$

On account of the estimates (4.30), (4.48), by repeating the argument used to obtain (4.53), (4.55) and (4.58), we find that

$$\|\phi_n\|_{L^\infty(0,T;H^1(\Omega))}+\|\partial_t\mu_n\|_{L^2(0,T;H^1(\Omega)')}+\|F'(\phi_n)\|_{L^\infty(0,T;H^1(\Omega))}+||F''(\phi_n)\|_{L^\infty(0,T;L^p(\Omega))}\leq C, \quad (5.19)$$
$$\|\phi^\star\|_{L^\infty(0,T;H^1(\Omega))}+\|\partial_t\mu^\star\|_{L^2(0,T;H^1(\Omega)')}+\|F'(\phi^\star)\|_{L^\infty(0,T;H^1(\Omega))}+||F''(\phi^\star)\|_{L^\infty(0,T;L^p(\Omega))}\leq C, \quad (5.20)$$

for any $p\in[2,\infty)$. Here, the C depends on p, but it is independent of n. Then, we first observe from Lebesgue's interpolation, the global bound in $L^\infty(\Omega\times(0,T))$ of ϕ_n and $\phi^\star$, and (5.16) that

$$\|\phi_n-\phi^*\|_{L^4(0,T;L^4(\Omega))}\underset{n\to\infty}{\longrightarrow}0. \quad (5.21)$$

Furthermore, in light of the above estimates, the Aubin-Lions lemma ensures that (up to subsequences) $\mu_n-\mu^\star\to\mu^\infty$ as $n\to\infty$ in $L^2(0,T;L^2(\Omega))$. We claim that $\mu^\infty\equiv0$. In fact, since $\phi_n\to\phi^\star$ (up to a subsequence) almost everywhere in $\Omega\times(0,T)$, we deduce from (5.19) and (5.20) that $F'(\phi_n)\rightharpoonup F'(\phi^\star)$ weakly in $L^2(\Omega\times(0,T))$. Also, it is easily seen that $K*\phi_n\to K*\phi^\star$ in $L^2(\Omega\times(0,T))$. Thus, we immediately infer that $\mu^\infty\equiv0$. More precisely, we have

$$\|\mu_n-\mu^\star\|_{L^2(0,T;L^2(\Omega))}\underset{n\to\infty}{\longrightarrow}0. \quad (5.22)$$

We now define $\mathbf{u}_n = \mathcal{S}(\mathbf{v}_n) \in Y$, for any $n \in \mathbb{N}$, and $\mathbf{u}^\star = \mathcal{S}(\mathbf{v}^\star) \in Y$. We set $\mathbf{u} = \mathbf{u}_n - \mathbf{u}^\star$, $\Phi = \phi_n - \phi^\star$, $\mathbf{V} = \mathbf{v}_n - \mathbf{v}^\star$, $\Theta = \mu_n - \mu^\star$, and we observe that

$$
\begin{aligned}
&(\rho(\phi_n)\partial_t \mathbf{u}, \mathbf{w}) + ((\rho(\phi_n) - \rho(\phi^\star))\partial_t \mathbf{u}^\star, \mathbf{w}) + (\rho(\phi_n)(\mathbf{v}_n \cdot \nabla)\mathbf{u}_n - \rho(\phi^\star)(\mathbf{v}^\star \cdot \nabla)\mathbf{u}^\star, \mathbf{w}) + (\nu(\phi_n)D\mathbf{u}, \nabla \mathbf{w}) \\
&+ ((\nu(\phi_n) - \nu(\phi^\star))D\mathbf{u}^\star, \nabla \mathbf{w}) - \frac{\rho_1 - \rho_2}{2}((\nabla \mu_n \cdot \nabla)\mathbf{u}_n - (\nabla \mu^\star \cdot \nabla)\mathbf{u}^\star, \mathbf{w}) = (\mu_n \nabla \phi_n - \mu^\star \nabla \phi^\star, \mathbf{w}),
\end{aligned}
$$

for all $\mathbf{w} \in \mathbf{V}_m$ and in $[0, T]$. Choosing then $\mathbf{w} = \mathbf{u}$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_n)|\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega \nu(\phi_n)|D\mathbf{u}|^2 \,\mathrm{d}x \\
&= \frac{\rho_1 - \rho_2}{4}\int_\Omega \partial_t \phi_n |\mathbf{u}|^2 \,\mathrm{d}x - \frac{\rho_1 - \rho_2}{2}\int_\Omega \Phi(\partial_t \mathbf{u}^\star \cdot \mathbf{u}) \,\mathrm{d}x \\
&\quad - \int_\Omega (\rho(\phi_n)(\mathbf{v}_n \cdot \nabla)\mathbf{u}_n - \rho(\phi^\star)(\mathbf{v}^\star \cdot \nabla)\mathbf{u}^\star) \cdot \mathbf{u} \,\mathrm{d}x - \int_\Omega (\nu(\phi_n) - \nu(\phi^\star))D\mathbf{u}^\star : \nabla \mathbf{u} \,\mathrm{d}x \\
&\quad + \frac{\rho_1 - \rho_2}{2}\int_\Omega ((\nabla \mu_n \cdot \nabla)\mathbf{u}_n - (\nabla \mu^\star \cdot \nabla)\mathbf{u}^\star) \cdot \mathbf{u} \,\mathrm{d}x + \int_\Omega (\mu_n \nabla \phi_n - \mu^\star \nabla \phi^\star) \cdot \mathbf{u} \,\mathrm{d}x.
\end{aligned}
$$

Thanks to the embedding $H^1_{0,\sigma}(\Omega) \hookrightarrow L^4(\Omega; \mathbb{R}^2)$, by exploiting (5.2), we have

$$
\left| \frac{\rho_1 - \rho_2}{4}\int_\Omega \partial_t \phi_n |\mathbf{u}|^2 \,\mathrm{d}x \right| \le C\|\partial_t \phi_n\|_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^4(\Omega)} \le C_m \|\partial_t \phi_n\|_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^2(\Omega)},
$$

and

$$
\begin{aligned}
\left| \frac{\rho_1 - \rho_2}{2}\int_\Omega \Phi(\partial_t \mathbf{u}^\star \cdot \mathbf{u}) \,\mathrm{d}x \right| &\le C\|\Phi\|_{L^4(\Omega)}\|\partial_t \mathbf{u}^\star\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\le C_m \|\partial_t \mathbf{u}^\star\|^2_{L^2(\Omega)}\|\mathbf{u}\|^2_{L^2(\Omega)} + C_m \|\Phi\|^2_{L^4(\Omega)}.
\end{aligned}
$$

Similarly, recalling that $\mathbf{v}_n, \mathbf{v}^\star, \mathbf{u}_n, \mathbf{u}^\star \in X$,

$$
\begin{aligned}
&- \int_\Omega (\rho(\phi_n)(\mathbf{v}_n \cdot \nabla)\mathbf{u}_n - \rho(\phi^\star)(\mathbf{v}^\star \cdot \nabla)\mathbf{u}^\star) \cdot \mathbf{u} \,\mathrm{d}x \\
&= -\frac{\rho_1 - \rho_2}{2}\int_\Omega \Phi((\mathbf{v}_n \cdot \nabla)\mathbf{u}_n) \cdot \mathbf{u} \,\mathrm{d}x - \int_\Omega \rho(\phi^\star)(\mathbf{v} \cdot \nabla)\mathbf{u}_n) \cdot \mathbf{u} \,\mathrm{d}x - \int_\Omega \rho(\phi^\star)(\mathbf{v}^\star \cdot \nabla)\mathbf{u}) \cdot \mathbf{u} \,\mathrm{d}x \\
&\le C\|\Phi\|_{L^4(\Omega)}\|\mathbf{v}_n\|_{L^\infty(\Omega)}\|\nabla \mathbf{u}_n\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} + C\|\mathbf{v}\|_{L^4(\Omega)}\|\nabla \mathbf{u}_n\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\quad + C\|\mathbf{v}^\star\|_{L^4(\Omega)}\|\nabla \mathbf{u}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\le C_m M^2 \|\Phi\|_{L^4(\Omega)} \; \|\mathbf{u}\|_{L^2(\Omega)} + C_m M \|\mathbf{v}\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)} + C_m M \|\mathbf{u}\|^2_{L^2(\Omega)} \\
&\le C_m \left(1 + M^2\right) \|\mathbf{u}\|^2_{L^2(\Omega)} + C_m M^2 \left(\|\Phi\|^2_{L^4(\Omega)} + \|\mathbf{V}\|^2_{L^2(\Omega)} \right).
\end{aligned}
$$

Since $\nu \in W^{1,\infty}(\mathbb{R})$, we get

$$
\begin{aligned}
-\int_\Omega (\nu(\phi_n)-\nu(\phi^\star))D\mathbf{u}^\star : \nabla\mathbf{u}\,\mathrm{d}x &\le C\|\Phi\|_{L^4(\Omega)}\|D\mathbf{u}^\star\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|_{L^4(\Omega)}\\
&\le C_m M^2\|\mathbf{u}\|^2_{L^2(\Omega)} + C_m\|\Phi\|^2_{L^4(\Omega)}.
\end{aligned}
$$

On the other hand, observing that

$$
\begin{aligned}
&\frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_n\cdot\nabla)\mathbf{u}_n - (\nabla\mu^\star\cdot\nabla)\mathbf{u}^\star)\cdot\mathbf{u}\,\mathrm{d}x\\
&= \frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_n\cdot\nabla)\mathbf{u} + (\nabla\Theta\cdot\nabla)\mathbf{u}^\star)\cdot\mathbf{u}\,\mathrm{d}x\\
&= -\frac{\rho_1-\rho_2}{2}\left((\mu_n\nabla\mathbf{u},\nabla\mathbf{u}) + (\mu_n\mathbf{u},\Delta\mathbf{u})\right) - \frac{\rho_1-\rho_2}{2}\left((\Theta\nabla\mathbf{u}^\star,\nabla\mathbf{u}) + (\Theta\mathbf{u},\Delta\mathbf{u}^\star)\right),
\end{aligned}
$$

we infer from (5.2) and (5.19) that

$$
\begin{aligned}
&\left|\frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_n\cdot\nabla)\mathbf{u}_n - (\nabla\mu^\star\cdot\nabla)\mathbf{u}^\star)\cdot\mathbf{u}\,\mathrm{d}x\right|\\
&\le C\|\mu_n\|_{L^2(\Omega)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega)} + C\|\mu_n\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty}\|\Delta\mathbf{u}\|_{L^2(\Omega)}\\
&\quad + C\|\Theta\|_{L^2(\Omega)}\|\nabla\mathbf{u}^\star\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^4(\Omega)} + C\|\Theta\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\|\Delta\mathbf{u}^\star\|_{L^2(\Omega)}\\
&\le C_m\left(1+M^2\right)\|\mathbf{u}\|^2_{L^2(\Omega)} + C_m\|\Theta\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Finally, using again (5.2) and (5.19)-(5.20), we get

$$
\begin{aligned}
\int_\Omega (\mu_n\nabla\phi_n - \mu^\star\nabla\phi^\star)\cdot\mathbf{u}\,\mathrm{d}x &= \int_\Omega (-\Phi\nabla\mu_n + \Theta\nabla\phi^\star)\cdot\mathbf{u}\,\mathrm{d}x\\
&\le \|\nabla\mu_n\|_{L^2(\Omega)}\|\Phi\|_{L^4(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} + \|\Theta\|_{L^2(\Omega)}\|\nabla\phi^\star\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)}\\
&\le C_m\|\mathbf{u}\|^2_{L^2(\Omega)} + C_m\left(\|\Theta\|^2_{L^2(\Omega)} + \|\Phi\|^2_{L^4(\Omega)}\right).
\end{aligned}
$$

Combining the above inequalities and recalling that $\rho \ge \rho_*$, we are thus led to the differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi_n)|\mathbf{U}|^2\,\mathrm{d}x \le H_1(t)\int_\Omega \rho(\phi_n)|U|^2\,\mathrm{d}x + H_2(t),
$$

where

$$
H_1 := C_m\left(1+\|\partial_t\phi_n\|^2_{L^2(\Omega)} + \|\partial_t\mathbf{u}^\star\|^2_{L^2(\Omega)}\right), \quad H_2 := C_m\left(1+\|\Phi\|^2_{L^4(\Omega)} + \|\Theta\|^2_{L^2(\Omega)} + \|\mathbf{v}\|^2_{L^2(\Omega)}\right).
$$

Hence, the Gronwall lemma entails

$$
\max_{t\in[0,T]}\|\mathbf{u}(t)\|^2_{L^2(\Omega)} \le \frac{1}{\rho_*}\mathrm{e}^{\int_0^T H_1(\tau)\,\mathrm{d}\tau}\int_0^T H_2(\tau)\,\mathrm{d}\tau. \tag{5.23}
$$

Note that $H_1 \in L^1(0,T)$ and $H_2 \in L^1(0,T)$, thanks to (5.15) and (5.17)-(5.20). In addition, in light of $\mathbf{v}_n \to \widetilde{\mathbf{v}}$ in $C([0,T];\mathbf{V}_m)$ and (5.21)-(5.22), we deduce that from (5.23) that $\mathbf{u}_n \to \mathbf{u}^\star$ in $C([0,T];\mathbf{V}_m)$, implying that the map $\mathcal{S}$ is continuous.

In conclusion, we can apply the Schauder fixed point theorem to $\mathcal{S}$. This gives the existence of an approximate solution $(\mathbf{u}_m, \phi_m)$ in $[0, T]$ satisfying (5.3)-(5.6).

5.3. Uniform estimates independent of the approximation parameter

Integrating $(5.5)_1$ over Ω, we find

$$\overline{\phi_m}(t) = \frac{1}{|\Omega|}\int_\Omega \phi_m(t)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \phi_0\,\mathrm{d}x, \quad \forall\, t \in [0, T].$$

Taking $\mathbf{w} = \mathbf{u}_m$ in (5.4) and arguing as above, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\rho(\phi_m)|\mathbf{u}_m|^2\,\mathrm{d}x + \int_\Omega \nu(\phi_m)|D\mathbf{u}_m|^2\,\mathrm{d}x = \int_\Omega \mu_m \nabla\phi_m \cdot \mathbf{u}_m\,\mathrm{d}x.$$

Let us recall that ϕ_m satisfies the energy identity (4.4), i.e.,

$$\mathcal{E}_{\mathrm{nloc}}(\phi_m(t)) + \int_0^t \|\nabla\mu_m(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau + \int_0^t\int_\Omega \phi_m\,\mathbf{u}_m \cdot \nabla\mu_m\,\mathrm{d}x\,\mathrm{d}\tau = \mathcal{E}_{\mathrm{nloc}}((\phi_0), \quad \forall\, t \in [0, T].$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\mathbf{u}_m, \phi_m) + \int_\Omega \nu(\phi_m)|D\mathbf{u}_m|^2\,\mathrm{d}x + \int_\Omega |\nabla\mu_m|^2\,\mathrm{d}x = 0, \tag{5.24}$$

where

$$E(\mathbf{u}_m, \phi_m) = \int_\Omega \frac{1}{2}\rho(\phi_m)|\mathbf{u}_m|^2\,\mathrm{d}x + \mathcal{E}_{\mathrm{nloc}}(\phi_m).$$

Notice that, being $|\phi_m| < 1$ almost everywhere in $\Omega \times (0, T)$, $\mathcal{E}_{\mathrm{nloc}}(\phi_m) \geq -C_e$ almost everywhere in $(0, T)$, where C_e is independent of m. Then, we can define $\widehat{E}(\mathbf{u}_m, \phi_m) = E(\mathbf{u}_m, \phi_m) + C_e \geq 0$. We now integrate (5.24) with respect to time in $[0, T]$ and we obtain

$$\widehat{E}(\mathbf{u}_m(t), \phi_m(t)) + \int_0^t\int_\Omega \nu(\phi_m)|D\mathbf{u}_m(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega |\nabla\mu_m(\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau = \widehat{E}(\mathbb{P}_m\mathbf{u}_0, \phi_0). \tag{5.25}$$

By the properties of $\mathbb{P}_m$, we immediately deduce that

$$\widehat{E}(\mathbb{P}_m\mathbf{u}_0, \phi_0) \leq C_e + \frac{\rho_*}{2}\|\mathbf{u}_0\|^2_{L^2(\Omega)} + \mathcal{E}_{\mathrm{nloc}}(\phi_0).$$

Therefore, we conclude that

$$\|\mathbf{u}_m\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}_m\|_{L^2(0,T;H^1(\Omega))} \leq C_E, \quad \|\nabla\mu_m\|_{L^2(0,T;H)} \leq C_E, \tag{5.26}$$

where C_E is independent of m. Owing to (5.26), the embedding $L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; H^1_{0,\sigma}(\Omega)) \hookrightarrow L^4(0, T; L^4_\sigma(\Omega))$ and the assumptions on ϕ_0, we can apply (i) of Theorem 4.1. In particular, (4.7)-(4.9) entails that

$$\begin{cases} \|\phi_m\|_{L^\infty(0,T;H^1(\Omega))} + \|F'(\phi_m)\|_{L^\infty(0,T;H^1(\Omega))} \le C, \\ \|\phi_m\|_{L^q(0,T;W^{1,p}(\Omega))} \le C_p, \quad q = \dfrac{2p}{p-2}, \quad \forall\, p \in (2,\infty), \\ \|\partial_t\phi_m\|_{L^\infty(0,T;H^1(\Omega)')} + \|\partial_t\phi_m\|_{L^2(0,T;L^2(\Omega))} \le C, \\ \|\mu_m\|_{L^\infty(0,T;H^1(\Omega))} + \|\mu_m\|_{L^2(0,T;H^2(\Omega))} + \|\mu_m\|_{H^1(0,T;H^1(\Omega)')} \le C, \\ \|F''(\phi_m)\|_{L^\infty(0,T;L^p(\Omega))} \le C_p, \quad \forall\, p \in [2,\infty). \end{cases} \tag{5.27}$$

Next, taking $\mathbf{w} = \partial_t\mathbf{u}_m$ in (5.4), we find

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \nu(\phi_m)|D\mathbf{u}_m|^2\,\mathrm{d}x + \int_\Omega \rho(\phi_m)|\partial_t\mathbf{u}_m|^2\,\mathrm{d}x \\ &= -\int_\Omega \rho(\phi_m)((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m)\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x + \int_\Omega \nu'(\phi_m)\partial_t\phi_m|D\mathbf{u}_m|^2\,\mathrm{d}x \\ &\quad + \frac{\rho_1-\rho_2}{2}\int_\Omega ((\nabla\mu_m\cdot\nabla)\mathbf{u}_m)\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x + \int_\Omega \mu_m\nabla\phi_m\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x. \end{aligned} \tag{5.28}$$

In addition, following [37,39], we can choose $\mathbf{w} = \mathbf{A}\mathbf{u}_m$ in (5.4), obtaining

$$\begin{aligned} -\frac{1}{2}(\nu(\phi_m)\Delta\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) &= -(\rho(\phi_m)\partial_t\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) - (\rho(\phi_m)(\mathbf{u}_m\cdot\nabla)\mathbf{A}\mathbf{u}_m) \\ &\quad + \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) \\ &\quad + (\mu_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m) + (\nu'(\phi_m)D\mathbf{u}_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m). \end{aligned}$$

By the regularity theory of the Stokes operator, there exists $\pi_m \in C([0,T];H^1)$ such that $-\Delta\mathbf{u}_m + \nabla\pi_m = \mathbf{A}\mathbf{u}_m$ almost everywhere in $\Omega\times(0,T)$. Furthermore, Lemma 3.1 implies that

$$\|\pi_m\|_{L^4(\Omega)} \le C\|\nabla\mathbf{u}_m\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Since $(\nu(\phi_m)\nabla\pi_m, \mathbf{A}\mathbf{u}_m) = -(\nu'(\phi_m)\pi_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m)$, we arrive at

$$\begin{aligned} \frac{1}{2}(\nu(\phi_m)\mathbf{A}\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) &= -(\rho(\phi_m)\partial_t\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) - (\rho(\phi_m)(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) \\ &\quad + \frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\mathbf{u}_m, \mathbf{A}\mathbf{u}_m) + (\mu_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m) \\ &\quad + (\nu'(\phi_m)D\mathbf{u}_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m) - \frac{1}{2}(\nu'(\phi_m)\pi_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m). \end{aligned} \tag{5.29}$$

Let us now estimate the terms on the right-hand side in (5.28) and (5.29). Set ω_1 a positive constant whose value will be determined later on. By using (2.2) and (5.26), we have

$$\begin{aligned} \left|\int_\Omega \rho(\phi_m)((\mathbf{u}_m\cdot\nabla)\mathbf{u}_m)\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x\right| &\le \rho^*\|\mathbf{u}_m\|_{L^4(\Omega)}\|\nabla\mathbf{u}_m\|_{L^4(\Omega)}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)} \\ &\le \frac{\rho_*}{8}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}^2 + \frac{\nu_*\omega_1}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|D\mathbf{u}_m\|_{L^2(\Omega)}^4 \end{aligned}$$

and

$$\left|\int_\Omega \nu'(\phi_m)\partial_t\phi_m|D\mathbf{u}_m|^2\,\mathrm{d}x\right| \le C\|\partial_t\phi_m\|_{L^2(\Omega)}\|D\mathbf{u}_m\|_{L^4(\Omega)}^2$$
$$\le \frac{\nu_*\omega_1}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\partial_t\phi_m\|_{L^2(\Omega)}^2\|D\mathbf{u}_m\|_{L^2(\Omega)}^2.$$

Exploiting (2.2) once again, together with (5.27), we obtain

$$\left|\frac{\rho_1-\rho_2}{2}\int_\Omega((\nabla\mu_m\cdot\nabla)\mathbf{u}_m)\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x\right|$$
$$\le C\|\nabla\mu_m\|_{L^4(\Omega)}\|\nabla\mathbf{u}_m\|_{L^4(\Omega)}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}$$
$$\le C\|\nabla\mu_m\|_{L^2(\Omega)}^{\frac12}\|\nabla\mu_m\|_{H^1(\Omega)}^{\frac12}\|D\mathbf{u}_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^{\frac12}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}$$
$$\le \frac{\rho_*}{8}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}^2 + \frac{\nu_*\omega_1}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\mu_m\|_{H^1(\Omega)}^2\|D\mathbf{u}_m\|_{L^2(\Omega)}^2$$

and

$$\left|\int_\Omega \mu_m\nabla\phi_m\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x\right| = \left|\int_\Omega \phi_m\nabla\mu_m\cdot\partial_t\mathbf{u}_m\,\mathrm{d}x\right| \le \frac{\rho_*}{8}\|\partial_t\mathbf{u}_m\|^2 + C\|\nabla\mu_m\|_{L^2(\Omega)}^2.$$

Arguing as in the proof of [37, Section 4] related to the terms I_8 and I_9, we find

$$|(\rho(\phi_m)\partial_t\mathbf{u}_m,\mathbf{A}\mathbf{u}_m)| \le \frac{2\omega_1(\rho^*)^2}{\rho_*}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + \frac{\rho_*}{8\omega_1}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}^2$$

and

$$|(\rho(\phi_m)(\mathbf{u}_m\cdot\nabla)\mathbf{u}_m,\mathbf{A}\mathbf{u}_m)| \le \frac{\nu_*}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|D\mathbf{u}_m\|_{L^2(\Omega)}^4.$$

Proceeding as above, we get

$$\left|\frac{\rho_1-\rho_2}{2}((\nabla\mu_m\cdot\nabla)\mathbf{u}_m,\mathbf{A}\mathbf{u}_m)\right| \le C\|\nabla\mu_m\|_{L^4(\Omega)}\|\nabla\mathbf{u}_m\|_{L^4(\Omega)}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}$$
$$\le C\|\nabla\mu_m\|_{L^2(\Omega)}^{\frac12}\|\nabla\mu_m\|_{H^1(\Omega)}^{\frac12}\|Du_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^{\frac32}$$
$$\le \frac{\nu_*}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\mu_m\|_{H^1(\Omega)}^2\|D\mathbf{u}_m\|_{L^2(\Omega)}^2$$

and

$$|(\mu_m\nabla\phi_m,\mathbf{A}\mathbf{u}_m)| = |(\phi_m\nabla\mu_m,\mathbf{A}\mathbf{u}_m)| \le \frac{\nu_*}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\mu_m\|_{L^2(\Omega)}^2.$$

By $\nu' \in W^{1,\infty}(\mathbb{R})$ and (5.27), it follows that

$$|(\nu'(\phi_m)D\mathbf{u}_m\nabla\phi_m,\mathbf{A}\mathbf{u}_m)| \le C\|D\mathbf{u}_m\|_{L^4(\Omega)}\|\nabla\phi_m\|_{L^4(\Omega)}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}$$
$$\le C\|D\mathbf{u}_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^{\frac32}\|\nabla\phi_m\|_{L^4(\Omega)}$$
$$\le \frac{\nu_*}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\phi_m\|_{L^4(\Omega)}^4\|D\mathbf{u}_m\|_{L^2(\Omega)}^2.$$

Lastly, by (2.2) and (5.27), we infer that

$$\begin{aligned}|(\nu'(\phi_m)\pi_m\nabla\phi_m, \mathbf{A}\mathbf{u}_m)| &\le C\|\pi_m\|_{L^4(\Omega)}\|\nabla\phi_m\|_{L^4(\Omega)}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}\\ &\le C\|\nabla\mathbf{u}_m\|_{L^2(\Omega)}^{\frac12}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^{\frac32}\|\nabla\phi_m\|_{L^4(\Omega)}\\ &\le \frac{\nu_*}{32}\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 + C\|\nabla\phi_m\|_{L^4(\Omega)}^4\|D\mathbf{u}_m\|_{L^2(\Omega)}^2.\end{aligned}$$

Adding up (5.28) with (5.29) multiplied by ω_1, and taking into account the previous estimates, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}H_m + \frac{\rho_*}{2}\|\partial_t\mathbf{u}_m\|_{L^2(\Omega)}^2 + \left(\frac{\nu_*\omega_1}{4} - \frac{2\omega_1^2(\rho^*)^2}{\rho_*}\right)\|\mathbf{A}\mathbf{u}_m\|_{L^2(\Omega)}^2 \le D_m H_m + Q_m,$$

where

$$\begin{aligned}&H_m(t) := \frac12\int_\Omega \nu(\phi_m(t))|D\mathbf{u}_m(t)|^2\,\mathrm{d}x,\\ &D_m(t) := C\left(1 + \|D\mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \|\partial_t\phi_m(t)\|_{L^2(\Omega)}^2 + \|\nabla\mu_m(t)\|_{H^1(\Omega)}^2 + \|\nabla\phi_m(t)\|_{L^4(\Omega)}^4\right),\\ &Q_m(t) := C\|\nabla\mu_m(t)\|_{L^2(\Omega)}^2.\end{aligned}$$

In turn, setting $\omega_1 = \dfrac{\nu_*\rho_*}{16(\rho^*)^2} > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}H_m + \frac{\rho_*}{2}\|\partial_t\mathbf{u}_m(t)\|_{L^2(\Omega)}^2 + \frac{\nu_*\omega_1}{8}\|\mathbf{A}\mathbf{u}_m(t)\|_{L^2(\Omega)}^2 \le D_m H_m + Q_m. \tag{5.30}$$

Observe now that, by (5.26) and (5.27), $D_m \in L^1(0,T)$ and $Q_m \in L^1(0,T)$. Thus, an application of the Gronwall lemma gives

$$H_m(t) \le \left(H_m(0) + \int_0^T Q_m(\tau)\,\mathrm{d}\tau\right)\exp\left(\int_0^T D_m(\tau)\,\mathrm{d}\tau\right), \quad \forall t \in [0,T]. \tag{5.31}$$

By the properties of the projector $\mathbb{P}_m$ and (5.26) and (5.27), we observe that

$$H_m(0) \le C\|\mathbf{u}_0\|_{H^1_{0,\sigma}(\Omega)}^2, \quad \int_0^T Q_m(\tau)\,\mathrm{d}\tau \le C, \quad \int_0^T D_m(\tau)\,\mathrm{d}\tau \le C.$$

Thus, we conclude from (5.30) and (5.31) that

$$\|\mathbf{u}_m\|_{L^\infty(0,T;H^1_{0,\sigma}(\Omega))} + \|\partial_t\mathbf{u}_m\|_{L^2(0,T;L^2_\sigma(\Omega))} + \|\mathbf{u}_m\|_{L^2(0,T;V^2_{0,\sigma}(\Omega))} \le C. \tag{5.32}$$

5.4. Passage to the limit and existence of global strong solutions

Thanks to the estimates (5.26), (5.27) and (5.32) (which are uniform with respect to the parameter m), we deduce the following convergences (up to subsequences)

$$\begin{aligned}
&\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weakly}^\star \text{ in } L^\infty(0,T;H^1_{0,\sigma}(\Omega)),\\
&\mathbf{u}_m \rightharpoonup \mathbf{u} \quad \text{weakly in } L^2(0,T;V^2_{0,\sigma}(\Omega)) \cap H^1(0,T;L^2_\sigma(\Omega)),\\
&\phi_m \rightharpoonup \phi \quad \text{weakly}^\star \text{ in } L^\infty(0,T;H^1(\Omega)) \cap L^\infty(\Omega \times (0,T)),\\
&\phi_m \rightharpoonup \phi \quad \text{weakly in } L^q(0,T;W^{1,p}(\Omega)), \quad q=\frac{2p}{p-2}, \quad \forall\, p \in (2,\infty),\\
&\phi_m \rightharpoonup \phi \quad \text{weakly in } H^1(0,T;L^2(\Omega) \cap W^{1,\infty}(0,T;H^1(\Omega)'),\\
&\mu_m \rightharpoonup \mu \quad \text{weakly}^\star \text{ in } L^\infty(0,T;H^1(\Omega)),\\
&\mu_m \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)').
\end{aligned} \tag{5.33}$$

By means of Aubin-Lions Lemma, we have the following strong convergences,

$$\begin{aligned}
&\mathbf{u}_m \to \mathbf{u} \quad \text{strongly in } L^2(0,T;H^1_{0,\sigma}(\Omega)),\\
&\phi_m \to \phi \quad \text{strongly in } C([0,T];L^p(\Omega)), \quad \forall\, p \in [2,\infty),\\
&\mu_m \to \mu \quad \text{strongly in } L^2(0,T;H^1(\Omega)).
\end{aligned} \tag{5.34}$$

As an immediate consequence, we infer that

$$\begin{aligned}
&\rho(\phi_m) \to \rho(\phi) \quad \text{strongly in } C([0,T];L^p(\Omega)), \quad \forall\, p \in [2,\infty),\\
&\nu(\phi_m) \to \nu(\phi) \quad \text{strongly in } C([0,T];L^p(\Omega)), \quad \forall\, p \in [2,\infty).
\end{aligned} \tag{5.35}$$

On the other hand, we only know so far that $\phi \in L^\infty(\Omega \times (0,T))$ is such that $\|\phi\|_{L^\infty(\Omega\times(0,T))} \le 1$. But, due to the convergence (up to a subsequence) $\phi_m \to \phi$ almost everywhere in $\Omega \times (0,T)$ and (5.27), the Fatou lemma entails that $F'(\phi)^2 \in L^2(0,T;L^2(\Omega))$. In turn, this gives that $|\phi| < 1$ almost everywhere in $\Omega \times (0,T)$. Owing to this, it is possible to show that

$$F'(\phi_m) \rightharpoonup F'(\phi) \quad \text{weakly}^\star \text{ in } L^\infty(0,T;H^1(\Omega)). \tag{5.36}$$

The above properties are sufficient to show the convergence of the nonlinear terms in (5.4)-(5.5). Then, in a standard way, we pass to the limit as $m \to \infty$ in (5.4)-(5.5). Reasoning now as in [37], we infer the existence of a pressure $\Pi \in L^2(0,T;H^1_{(0)}(\Omega))$, such that

$$\nabla \Pi = -\rho(\phi)\partial_t \mathbf{u} - \rho(\phi)(\mathbf{u}\cdot\nabla)\mathbf{u} + \rho'(\phi)(\nabla\mu\cdot\nabla)\mathbf{u} + \operatorname{div}(\nu(\phi)D\mathbf{u}) + \mu\nabla\phi,$$

almost everywhere in $\Omega \times (0,T)$.

Concerning the separation property, thanks to the regularity (1.17) on $[0,T]$, we infer from Remark 4.4 (cf. also Theorem 4.1) that, for any $0 < \tau \le T$, there exists $\delta = \delta(\tau) \in (0,1)$ such that it holds

$$\sup_{t\in[\tau,T]} \|\phi(t)\|_{L^\infty(\Omega)} \le 1-\delta. \tag{5.37}$$

Instead, if we additionally assume $\|\phi_0\| \le 1-\delta_0$, for some $\delta_0 \in (0,1)$, then an application of Theorem 4.1, case (ii) implies that there exists $\delta^\star > 0$ (depending also on δ_0) such that the solution ϕ satisfies (1.19).

In order to complete the proof of the existence, we are left to discuss the *globality* of the solution $(\mathbf{u},\Pi,\phi)$. In fact, we have only shown so far the existence of a solution $(\mathbf{u},\Pi,\phi)$ to (1.8)-(1.9) defined on a given time interval $[0,T]$ for any fixed $T > 0$. Nevertheless, we can easily construct a global solution $(\mathbf{u},\Pi,\phi)$ to (1.8)-(1.9) defined on the time interval $[0,\infty)$ and satisfying (i)-(iv). Indeed, we first consider the solution $(\mathbf{u}_1,\Pi_1,\phi_1)$ defined on $[0,1]$ originating from $(\mathbf{u}_0,\phi_0)$. Next, we notice that $\mathbf{u}_1(1) \in H^1_{0,\sigma}(\Omega)$ and $\phi_1(1) \in H^1(\Omega) \cap L^\infty(\Omega)$ with $|\overline{\phi_1(1)}| < 1$. In addition, in light of (5.37), $\|\phi_1(1)\|_{L^\infty(\Omega)} \le 1-\delta$, for some $\delta > 0$. Thanks to the proof above, for any $n \in \mathbb{N}$ with $n \geq 2$, there exists a solution $(\mathbf{u}_n, \Pi_n, \phi_n)$ to (1.8)-(1.9) defined in the time interval $[1, n]$. In particular, by the uniqueness property proven in the (next) Subsection 5.5, we have that $\mathbf{u}_n = \mathbf{u}_k$, $\Pi_n = \Pi_k$, $\phi_n = \phi_k$ in $[1, n]$, provided that $n < k$. Therefore, we obtain the solution $(\mathbf{u}, \Pi, \phi)$ defined in $[0, \infty)$ by setting

$$(\mathbf{u}(t), \Pi(t), \phi(t)) = \begin{cases} (\mathbf{u}_1(t), \Pi_1(t), \phi_1(t)), & t \in [0,1], \\ (\mathbf{u}_n(t), \Pi_n(t), \phi_n(t)), & t \in [1,n], \quad \forall n \geq 2. \end{cases} \tag{5.38}$$

Finally, we observe that $(\mathbf{u}, \Pi, \phi)$ satisfies the energy equality

$$E(\mathbf{u}(t), \phi(t)) + \int_\tau^t \left\| \sqrt{\nu(\phi(s))} |D\mathbf{u}(s)\right\|_{L^2(\Omega)}^2 + \|\nabla \mu(s)\|_{L^2(\Omega))}^2 \, \mathrm{d}s = E(\mathbf{u}(\tau), \phi(\tau))$$

for every $0 < \tau \leq t < \infty$, which clearly follows from the regularity in each interval $[0, T]$ (cf. (5.33)-(5.35)). This implies that $\mathbf{u} \in L^\infty(0, \infty; L^2_\sigma(\Omega)) \cap L^2(0, \infty; H^1_{0,\sigma}(\Omega))$ and $\nabla \mu \in L^2(0, \infty; L^2(\Omega; \mathbb{R}^2))$. By interpolation, it follows that $\mathbf{u} \in L^4(0, \infty; L^4_\sigma(\Omega))$. Then, in light of Theorem 4.1, we deduce from the estimates (4.7)-(4.8) as $T \to \infty$ that $\nabla \mu \in L^\infty(0, \infty; L^2(\Omega; \mathbb{R}^2))$ and $\partial_t \phi \in L^2(0, \infty; L^2(\Omega))$. By (4.9), we also infer that

$$\begin{aligned} &\phi \in L^\infty(0, \infty; H^1(\Omega)) \cap L^q_{\mathrm{uloc}}([0, \infty); W^{1,p}(\Omega)), \quad q = \frac{2p}{p-2}, \quad p \in (2, \infty), \\ &\partial_t \phi \in L^\infty(0, \infty; H^1(\Omega)'), \quad F'(\phi) \in L^\infty(0, \infty; H^1(\Omega)), \quad F''(\phi) \in L^\infty(0, \infty; L^p(\Omega)), \quad p \in [2, \infty), \\ &\mu \in BC_{\mathrm{w}}([0, \infty); H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([0, \infty); H^2(\Omega)) \cap H^1_{\mathrm{uloc}}([0, \infty); H^1(\Omega)'). \end{aligned}$$

Moreover, by Remark (4.4), there exists $\delta > 0$ such that $\sup_{t \in [\tau, \infty)} \|\phi(t)\|_{L^\infty(\Omega)} \leq 1 - \delta$. On the other hand, recalling that $\mathbf{u}$ is the limit of the approximate solutions $\mathbf{u}_m$ in $[0, n]$ for each $n \in \mathbb{N}$ and that each $\mathbf{u}_m$ satisfies (5.30), it is easily seen from the uniform Gronwall lemma that

$$\mathbf{u} \in L^\infty(0, \infty; H^1_{0,\sigma}(\Omega)) \cap H^1_{\mathrm{uloc}}([0, \infty); L^2_\sigma(\Omega)) \cap L^2_{\mathrm{uloc}}([0, \infty); V^2_{0,\sigma}(\Omega)),$$

and, in turn, $\Pi \in L^2_{\mathrm{uloc}}([0, \infty); H^1_{(0)}(\Omega))$. The proof of the existence of global strong solutions in the statement of Theorem 1.5 is thus concluded.

5.5. Continuous dependence estimate for "separated" strong solutions

Consider two sets of initial data $(\mathbf{u}_0^1, \phi_0^1)$ and $(\mathbf{u}_0^2, \phi_0^2)$ satisfying the assumptions of Theorem 1.5. In particular, we consider "separated" initial data, i.e. $\|\phi_0^i\|_{L^\infty(\Omega)} < 1$ for $i = 1, 2$. We denote by $(\mathbf{u}_j, \Pi_j, \phi_j)$, $j = 1, 2$, the strong solutions to (1.8)-(1.9) originating from $(\mathbf{u}_0^j, \phi_0^j)$. Clearly both the solutions satisfy (1.17) and the statement (iv) of Theorem 1.5. Let us set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $P = \Pi_1 - \Pi_2$, $\Phi = \phi_1 - \phi_2$ and $\Theta = F'(\phi_1) - F'(\phi_2) - K * \Phi$. These functions satisfy the system

$$\begin{aligned} &\rho(\phi_1)\partial_t \mathbf{u} + (\rho(\phi_1) - \rho(\phi_2)\partial_t \mathbf{u}_2 + \rho(\phi_1)(\mathbf{u}_1 \cdot \nabla)\mathbf{u} + \rho(\phi_1)(\mathbf{u} \cdot \nabla)\mathbf{u}_2 + (\rho(\phi_1) - \rho(\phi_2))(\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 \\ &\quad - \frac{\rho_1 - \rho_2}{2}(\nabla \mu_1 \cdot \nabla)\mathbf{u} - \frac{\rho_1 - \rho_2}{2}(\nabla \Theta \cdot \nabla)\mathbf{u}_2 - \operatorname{div}(\nu(\phi_1) D\mathbf{u}) - \operatorname{div}(\nu((\phi_1) - \nu(\phi_2)) D\mathbf{u}_2) + \nabla P \\ &= \mu_1 \nabla \Phi + \Theta \nabla \phi_2, \\ &\partial_t \Phi + \mathbf{u}_1 \cdot \nabla \Phi + \mathbf{u} \cdot \nabla \phi_2 = \Delta \Theta, \end{aligned} \tag{5.39}$$

almost everywhere in $\Omega \times (0, T)$. We observe that

$$-\int_\Omega \partial_t \rho(\phi_1)\frac{|\mathbf{u}|^2}{2}\,\mathrm{d}x + \int_\Omega \rho(\phi_1)(\mathbf{u}_1\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega (\nabla\mu_1\cdot\nabla)\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}x = 0, \tag{5.40}$$

$$\int_\Omega (\nabla\Theta\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x = -\int_\Omega \Theta\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \Theta\nabla\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x, \tag{5.41}$$

and

$$\int_\Omega (\mu_1\nabla\Phi + \Theta\nabla\phi_2)\cdot\mathbf{u}\,\mathrm{d}x = -\int_\Omega \Phi(\nabla K * \phi_1)\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \phi_2(\nabla K*\Phi)\cdot\mathbf{u}\,\mathrm{d}x. \tag{5.42}$$

Multiplying (5.39)$_1$ by $\mathbf{u}$ and integrating over Ω, we find (cf. [37, Equation 6.3])

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \int_\Omega \nu(\phi_1)|D\mathbf{U}|^2\,\mathrm{d}x \\
&= -\int_\Omega (\rho(\phi_1)-\rho(\phi_2)\partial_t\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \rho(\phi_1)(\mathbf{u}\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega (\rho(\phi_1)-\rho(\phi_2))(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x \\
&\quad - \int_\Omega (\nu(\phi_1)-\nu(\phi_2))D\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\nabla\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x \\
&\quad - \int_\Omega \Phi(\nabla K*\phi_1)\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \phi_2(\nabla K*\Phi)\cdot\mathbf{U}\,\mathrm{d}x.
\end{aligned} \tag{5.43}$$

By the strict convexity of F, we notice that

$$\begin{aligned}
\int_\Omega \nabla\Theta\cdot\nabla\Phi\,\mathrm{d}x &= \int_\Omega F''(\phi_1)|\nabla\Phi|^2\,\mathrm{d}x + \int_\Omega (F''(\phi_1)-F''(\phi_2))\nabla\phi_2\cdot\nabla\phi\,\mathrm{d}x - \int_\Omega \nabla K*\Phi\cdot\nabla\Phi\,\mathrm{d}x \\
&\geq \alpha\|\nabla\Phi\|^2_{L^2(\Omega)} + \int_\Omega (F''(\phi_1)-F''(\phi_2))\nabla\phi_2\cdot\nabla\phi\,\mathrm{d}x - \int_\Omega \nabla K*\Phi\cdot\nabla\Phi\,\mathrm{d}x.
\end{aligned}$$

Then, multiplying (5.39)$_2$ by Φ and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|^2_{L^2(\Omega)} + \alpha\|\nabla\Phi\|^2_{L^2(\Omega)} \\
&= \int_\Omega \phi_2(\mathbf{u}\cdot\nabla\Phi)\,\mathrm{d}x + \int_\Omega \nabla K*\Phi\cdot\nabla\Phi\,\mathrm{d}x - \int_\Omega (F''(\phi_1)-F''(\phi_2))\nabla\phi_2\cdot\nabla\phi\,\mathrm{d}x.
\end{aligned} \tag{5.44}$$

Adding together (5.43) and (5.44) and exploiting the hypothesis (H_6), we arrive at

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Phi\|^2_{L^2(\Omega)}\right) + \nu_*\|D\mathbf{u}\|^2_{L^2(\Omega)} + \alpha\|\nabla\Phi\|^2_{L^2(\Omega)} \\
&\le -\int_\Omega (\rho(\phi_1) - \rho(\phi_2)\partial_t \mathbf{u}_2 \cdot \mathbf{u}\,\mathrm{d}x - \int_\Omega \rho(\phi_1)(\mathbf{u}\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega (\rho(\phi_1)-\rho(\phi_2))(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x \\
&\quad - \int_\Omega (\nu(\phi_1)-\nu(\phi_2))D\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\nabla\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x \qquad (5.45)\\
&\quad - \int_\Omega \Phi(\nabla K * \phi_1)\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \phi_2(\nabla K*\Phi)\cdot\mathbf{U}\,\mathrm{d}x + \int_\Omega \phi_2(\mathbf{u}\cdot\nabla\Phi)\,\mathrm{d}x \\
&\quad + \int_\Omega \nabla K*\Phi\cdot\nabla\Phi\,\mathrm{d}x - \int_\Omega (F''(\phi_1)-F''(\phi_2))\nabla\phi_2\cdot\nabla\phi\,\mathrm{d}x.
\end{aligned}
$$

By using (1.4), (2.2) and (2.5), we obtain

$$
\begin{aligned}
\left|\int_\Omega (\rho(\phi_1) - \rho(\phi_2)\partial_t\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x\right| &\le C\|\Phi\|_{L^4(\Omega)}\|\partial_t\mathbf{u}_2\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\le C\|\partial_t\mathbf{u}_2\|_{L^2(\Omega)}\|\Phi\|^{\frac12}_{L^2(\Omega)}\left(\|\Phi\|^{\frac12}_{L^2(\Omega)} + \|\nabla\Phi\|^{\frac12}_{L^2(\Omega)}\right)\|\mathbf{u}\|^{\frac12}_{L^2(\Omega)}\|D\mathbf{u}\|^{\frac12}_{L^2(\Omega)} \\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\alpha}{10}\|\nabla\Phi\|^2_{L^2(\Omega)} \\
&\quad + C\left(1+\|\partial_t\mathbf{u}_2\|^2_{L^2(\Omega)}\right)\left(\|\Phi\|^2_{L^2(\Omega)} + \|\mathbf{u}\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

In a similar way, by (1.17) and (2.2), we have

$$
\begin{aligned}
\left|\int_\Omega \rho(\phi_1)(\mathbf{u}\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x\right| &\le C\|\mathbf{u}\|_{L^4(\Omega)}\|\nabla\mathbf{u}_2\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|^2_{L^2(\Omega)} + C\|\mathbf{u}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Thanks to (1.4), (1.17) and (2.3), it follows that

$$
\begin{aligned}
\left|\int_\Omega (\rho(\phi_1)-\rho(\phi_2))(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x\right| &\le C\|\Phi\|_{L^4(\Omega)}\|\mathbf{u}_2\|_{L^\infty(\Omega)}\|\nabla\mathbf{u}_2\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)} \\
&\le C(\|\Phi\|_{L^2(\Omega)} + \|\nabla\Phi\|_{L^2(\Omega)})\|\mathbf{u}_2\|^{\frac12}_{H^2(\Omega)}\|\mathbf{u}\|^{\frac12}_{L^2(\Omega)}\|D\mathbf{u}\|^{\frac12}_{L^2(\Omega)} \\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|^2_{L^2(\Omega)} + \frac{\alpha}{10}\|\nabla\Phi\|^2_{L^2(\Omega)} \\
&\quad + C\|\mathbf{u}_2\|^2_{H^2(\Omega)}\|\mathbf{u}\|^2_{L^2(\Omega)} + C\|\Phi\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Next, recalling that the separation property of ϕ_1 and ϕ_2 implies that $|F'(\phi_1) - F'(\phi_2)| \le C|\phi|$ almost everywhere in $\Omega\times(0,T)$ for some universal constant C (depending only on the norms of the initial data), and by using the assumption on K, we infer that

$$
\left|\frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x\right| = \left|\frac{\rho_1-\rho_2}{2}\right|\left|\int_\Omega (F'(\phi_1)-F'(\phi_2))\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega (K*\Phi)\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x\right|
$$

$$
\begin{aligned}
&\le C\|\Phi\|_{L^4(\Omega)}\|\Delta \mathbf{u}_2\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^4(\Omega)}\\
&\le C\|\Phi\|_{L^2(\Omega)}^{\frac12}\left(\|\Phi\|_{L^2(\Omega)}^{\frac12}+\|\nabla\Phi\|_{L^2(\Omega)}^{\frac12}\right)\|\Delta \mathbf{u}_2\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}^{\frac12}\|D\mathbf{u}\|_{L^2(\Omega)}^{\frac12}\\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|_{L^2(\Omega)}^2+\frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2\\
&\quad + C\left(1+\|\Delta \mathbf{u}_2\|_{L^2(\Omega)}^2\right)\left(\|\Phi\|_{L^2(\Omega)}^2+\|\mathbf{u}\|_{L^2(\Omega)}^2\right).
\end{aligned}
$$

Similarly, we find

$$
\begin{aligned}
\left|\frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\nabla\mathbf{u}_2:\nabla\mathbf{u}\,\mathrm{d}x\right| &= \left|\frac{\rho_1-\rho_2}{2}\right|\left|\int_\Omega (F'(\phi_1)-F'(\phi_2))\nabla\mathbf{u}_2:\nabla\mathbf{u}\,\mathrm{d}x-\int_\Omega (K*\Phi)\nabla\mathbf{u}_2:\nabla\mathbf{u}\,\mathrm{d}x\right|\\
&\le C\|\Phi\|_{L^4(\Omega)}\|\nabla\mathbf{u}_2\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\\
&\le C\|\Phi\|_{L^2(\Omega)}^{\frac12}\left(\|\Phi\|_{L^2(\Omega)}^{\frac12}+\|\nabla\Phi\|_{L^2(\Omega)}^{\frac12}\right)\|\nabla\mathbf{u}_2\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|_{L^2(\Omega)}^2+\frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2+C\left(1+\|\nabla\mathbf{u}_2\|_{L^4(\Omega)}^4\right)\|\Phi\|_{L^2(\Omega)}^2
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_\Omega (\nu(\phi_1)-\nu(\phi_2))D\mathbf{u}_2:\nabla\mathbf{u}\,\mathrm{d}x\right| &\le C\|\Phi\|_{L^4(\Omega)}\|D\mathbf{u}_2\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\\
&\le \frac{\nu_*}{12}\|D\mathbf{u}\|_{L^2(\Omega)}^2+\frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2+C\left(1+\|\nabla\mathbf{u}_2\|_{L^4(\Omega)}^4\right)\|\Phi\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Due to the boundedness property $\|\phi_j\|_{L^\infty(\Omega)}\le 1$ and (H_2), we also have

$$
\begin{aligned}
\left|\int_\Omega \Phi(\nabla K*\phi_1)\cdot\mathbf{u}\,\mathrm{d}x+\int_\Omega \phi_1(\nabla K*\Phi)\cdot\mathbf{u}\mathrm{d}x\right| &\le C\left(\|\phi_1\|_{L^\infty(\Omega)}+\|\phi_2\|_{L^\infty(\Omega)}\right)\|\Phi\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}\\
&\le C\left(\|\Phi\|_{L^2(\Omega)}^2+\|\mathbf{u}\|_{L^2(\Omega)}^2\right),
\end{aligned}
$$

$$
\left|\int_\Omega \phi_2(\mathbf{u}\cdot\nabla\Phi)\,\mathrm{d}x\right|\le \|\phi_2\|_{L^\infty(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\Phi\|_{L^2(\Omega)}\le\frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2+C\|\mathbf{u}\|_{L^2(\Omega)}^2,
$$

and

$$
\left|\int_\Omega \nabla K*\Phi\cdot\nabla\Phi\,\mathrm{d}x\right|\le\frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2+C\|\Phi\|_{L^2(\Omega)}^2.
$$

Lastly, using the separation property, we observe that $|F''(\phi_1)-F''(\phi_2)|\le C|\phi|$ almost everywhere in $\Omega\times(0,T)$ for some universal constant C. Combining this fact and (2.2), we get

$$
\begin{aligned}
\left|\int_\Omega (F''(\phi_1)-F''(\phi_2))\nabla\phi_2\cdot\nabla\phi\,\mathrm{d}x\right| &\le C\|\Phi\|_{L^4(\Omega)}\|\nabla\phi_2\|_{L^4(\Omega)}\|\nabla\Phi\|_{L^2(\Omega)}\\
&\le C\|\Phi\|_{L^2(\Omega)}^{\frac12}\left(\|\Phi\|_{L^2(\Omega)}^{\frac12}+\|\nabla\Phi\|_{L^2(\Omega)}^{\frac12}\right)\|\nabla\phi_2\|_{L^4(\Omega)}\|\nabla\Phi\|_{L^2(\Omega)}
\end{aligned}
$$

$$\leq \frac{\alpha}{10}\|\nabla\Phi\|_{L^2(\Omega)}^2 + C\left(1+\|\nabla\phi_2\|_{L^4(\Omega)}^4\right)\|\Phi\|_{L^2(\Omega)}^2.$$

Therefore, adding up the above estimates, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Phi\|_{L^2(\Omega)}^2\right) \leq H(\cdot)\left(\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Phi\|_{L^2(\Omega)}^2\right),$$

where $H(\cdot)$ is defined by (1.21). Note that $H \in L^1(0,T)$ owing to (1.17). In conclusion, an application of the Gronwall lemma gives uniqueness of strong solutions as well as (1.20). The proof of Theorem 1.5 is hereby complete.

6. Proof of Theorem 1.7: Propagation of regularity for weak solutions

Let $(\mathbf{u},\phi)$ be a weak solution on $[0,T]$ satisfying (i)-(iv) as ensured by Theorem 1.3 and let $\tau \in (0,T)$ be fixed. Since $F'(\phi) \in L^2(0,T;H^1(\Omega))$, exploiting the conservation of mass and (1.12), there exists $\tau_1 \in (0,\tau)$ such that

$$\phi(\tau_1) \in H^1(\Omega), \qquad \left|\overline{\phi(\tau_1)}\right| < 1, \quad \text{and} \quad F'(\phi(\tau_1)) \in H^1(\Omega).$$

Recalling now that $C_{\mathrm{w}}([0,T];L^2_\sigma) \cap L^2(0,T;H^1_{0,\sigma}(\Omega)) \hookrightarrow L^4(0,T;L^4_\sigma(\Omega))$, an application of Theorem 4.1 (see also Remark 4.3) entails that

$$\begin{cases} \phi \in L^\infty(\tau_1,T;L^\infty(\Omega)): \quad |\phi(x,t)|<1 \ \text{ for a.a. } x\in\Omega,\ \forall\, t\in[\tau_1,T],\\ \phi \in L^\infty(\tau_1,T;H^1(\Omega)) \cap L^q(\tau_1,T;W^{1,p}(\Omega)), \quad q=\dfrac{2p}{p-2}, \quad \forall\, p\in(2,\infty),\\ \partial_t\phi \in L^\infty(\tau_1,T;H^1(\Omega)') \cap L^2(\tau_1,T;L^2(\Omega)),\\ \mu \in L^\infty(\tau_1,T;H^1(\Omega)) \cap L^2(\tau_1,T;H^2(\Omega)) \cap H^1(\tau_1,T;H^1(\Omega)'),\\ F'(\phi) \in L^\infty(\tau_1,T;H^1(\Omega)), \quad F''(\phi)\in L^\infty(\tau_1,T;L^p(\Omega)), \quad \forall\, p\in[2,\infty), \end{cases} \tag{6.1}$$

which satisfies

$$\partial_t\phi + \mathbf{u}\cdot\nabla\phi = \Delta\mu, \quad \mu = F'(\phi) - K*\phi, \quad \text{a.e. in } \Omega\times(\tau_1,T), \tag{6.2}$$

as well as

$$\mathcal{E}(\phi(t)) + \int_{\tau_1}^t \|\nabla\mu(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau + \int_{\tau_1}^t\int_\Omega \phi\,\mathbf{u}\cdot\nabla\mu\,\mathrm{d}x\,\mathrm{d}\tau = \mathcal{E}(\phi_{\tau_1}), \quad \forall\, t\in[\tau_1,T]. \tag{6.3}$$

In addition, there exists $\tau_2 \in (\tau_1,\tau)$ such that

$$\sup_{t\in[\tau_2,T]} \|\phi(t)\|_{L^\infty(\Omega)} \leq 1-\delta. \tag{6.4}$$

Furthermore, we also have

$$\partial_t\mu \in L^2(\tau_2,T;L^2(\Omega)) \quad \text{and} \quad \mu\in C([\tau_2,T];H^1(\Omega)). \tag{6.5}$$

It is worth pointing out that the uniqueness of weak solutions in Theorem 4.1 guarantees that ϕ and the solution originating from $\phi(\tau_1)$ coincide.

Next, in light of the above propagation of regularity of the concentration, we improve the regularity of the restriction of $\partial_t(\rho(\phi)\mathbf{u})$ to divergence free test functions (not the full distributional time derivative of $\rho(\phi)\mathbf{u}$). To this end, we first recall from (1.14) that

$$\langle \partial_t(\rho(\phi)\mathbf{u}), \mathbf{v}\rangle_{V^2_{0,\sigma}(\Omega)} = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in V^2_{0,\sigma}(\Omega), \tag{6.6}$$

for almost any $t \in (\tau_1, T)$, where

$$(\mathbf{f}, \mathbf{v}) = (\rho(\phi)\mathbf{u} \otimes \mathbf{u}, \nabla \mathbf{v}) - (\nu(\phi) D\mathbf{u}, \nabla \mathbf{v}) - \frac{\rho_1 - \rho_2}{2}(\mathbf{u} \otimes \nabla \mu, \nabla \mathbf{v}) + (\mu \nabla \phi, \mathbf{v}).$$

Thanks to (6.1), we find that

$$\begin{aligned} |(\mathbf{f}, \mathbf{v})| &\le C \left(\|\mathbf{u}\|^2_{L^4(\Omega)} + \|D\mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mu\|_{L^4(\Omega)} \|\mathbf{u}\|_{L^4(\Omega)} + \|\mu\|_{L^4(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \right) \|\mathbf{v}\|_{H^1_{0,\sigma}(\Omega)} \\ &\le C \left(1 + \|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\nabla \mu\|_{H^1(\Omega)} \right) \|\mathbf{v}\|_{H^1_{0,\sigma}(\Omega)}, \end{aligned}$$

for any $\mathbf{v} \in V^2_{0,\sigma}(\Omega)$ and almost any $t \in (\tau_1, T)$. Since $V^2_{0,\sigma}(\Omega)$ is dense in $H^1_{0,\sigma}(\Omega)$, the functional $\mathbf{f}$ has a unique extension to $H^1_{0,\sigma}(\Omega)$. As a result, we deduce that $\mathbf{f} \in L^2(\tau_1, T; H^1_{0,\sigma}(\Omega)')$. By definition of the weak time derivative, this clearly entails that $\partial_t(\rho(\phi)\mathbf{u})|_{H^1_{0,\sigma}(\Omega)} \in L^2(\tau_1, T; H^1_{0,\sigma}(\Omega)')$ and

$$\langle \partial_t(\rho(\phi)\mathbf{u}), \mathbf{v}\rangle_{H^1_{0,\sigma}(\Omega)} = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in H^1_{0,\sigma}(\Omega), \tag{6.7}$$

almost everywhere in (τ_1, T). As a consequence, we can apply [23, Lemma 5.3] which gives that the chain rule

$$\langle \partial_t(\rho(\phi)\mathbf{u}), \mathbf{u}\rangle_{H^1_{0,\sigma}(\Omega)'} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \partial_t \rho(\phi)|\mathbf{u}|^2\,\mathrm{d}x$$

holds almost everywhere in (τ_1, T). Then, since $\mathbf{u} \in L^2(0, T; H^1_{0,\sigma}(\Omega))$ is now allowed as a test function in (6.7), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \partial_t\rho(\phi)|\mathbf{u}|^2\,\mathrm{d}x - \int_\Omega \rho(\phi)\mathbf{u} \otimes \mathbf{u} : \nabla \mathbf{u}\,\mathrm{d}x \\ &\qquad + \int_\Omega \nu(\phi)|D\mathbf{u}|^2\,\mathrm{d}x + \frac{\rho_1 - \rho_2}{2}\int_\Omega \mathbf{u} \cdot (\nabla \mu \cdot \nabla)\,\mathbf{u}\,\mathrm{d}x = \int_\Omega \mu \nabla \phi \cdot \mathbf{u}\,\mathrm{d}x, \end{aligned}$$

almost everywhere in (τ_1, T). Thanks to (6.2), we observe that

$$\frac{\rho_1 - \rho_2}{2}\int_\Omega \partial_t \phi\, \frac{|\mathbf{u}|^2}{2}\,\mathrm{d}x - \int_\Omega \rho(\phi)\mathbf{u} \cdot \nabla\left(\frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}x + \frac{\rho_1 - \rho_2}{2}\int_\Omega \nabla \mu \cdot \nabla\left(\frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}x = 0.$$

Thus, after an integration in time, we reach

$$\begin{aligned} &\frac{1}{2}\int_\Omega \rho(\phi(t))|\mathbf{u}(t)|^2\,\mathrm{d}x + \int_0^t \left\| \sqrt{\nu(\phi(s))} D\mathbf{u}(s) \right\|^2_{L^2(\Omega)} \mathrm{d}s - \int_0^t \int_\Omega \mu \nabla \phi \cdot \mathbf{u}\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{2}\int_\Omega \rho(\phi(\tau_1))|\mathbf{u}(\tau_1)|^2\,\mathrm{d}x, \end{aligned}$$

for all $t \in [\tau_1, T]$. In light of (6.3), we find the energy identity

$$E(\mathbf{u}(t), \phi(t)) + \int_{\tau_1}^{t} \left\| \sqrt{\nu(\phi(s))} D\mathbf{u}(s) \right\|_{L^2(\Omega)}^2 + \|\nabla \mu(s)\|_{L^2(\Omega)}^2 \, \mathrm{d}s = E(\mathbf{u}(\tau_1), \phi(\tau_1)), \quad \forall \tau_1 \le t \le T.$$

Next, owing to the energy identity, and exploiting (6.1) and (6.4), there exists $\tau_3 \in (\tau_2, \tau)$, such that

$$\mathbf{u}(\tau_3) \in H^1_{0,\sigma}(\Omega), \quad \phi(\tau_3) \in H^1(\Omega), \quad F'(\phi(\tau_3)) \in H^1(\Omega), \quad \|\phi(\tau_3)\|_{L^\infty(\Omega)} \le 1 - \delta.$$

An application of Theorem 1.5 yields the existence of a unique global strong solution $\left(\widetilde{\mathbf{u}}, \widetilde{\Pi}, \widetilde{\phi}\right)$ to (1.8)-(1.9) on $[\tau_3, \infty)$ departing from the initial datum $(\mathbf{u}(\tau_3), \phi(\tau_3))$. Our aim is to show that actually $\left(\widetilde{\mathbf{u}}(t), \widetilde{\phi}(t)\right)$ coincides with $(\mathbf{u}(t), \phi(t))$ on $[\tau_3, T]$. To achieve it, we argue similarly to the proof of the uniqueness for strong "separated" solutions given in Subsection 5.5. In particular, we will only show the main differences. For the clarity of presentation, we set $(\mathbf{u}_1(t), \phi_1(t)) = (\mathbf{u}(t + \tau_3), \phi(t + \tau_3))$ for $t \in [0, T - \tau_3]$ and $(\mathbf{u}_2(t+\tau_3), \phi_2(t+\tau_3)) = (\widetilde{\mathbf{u}}(t), \widetilde{\phi}(t))$ for $t \in [0, \infty)$. The initial data become $(\mathbf{u}_1(0), \phi_1(0)) = (\mathbf{u}_2(0), \phi_2(0)) = (\mathbf{u}(\tau_3), \phi(\tau_3))$. Furthermore, we recall that

$$\begin{cases} \mathbf{u}_1 \in C_{\mathrm{w}}([0, T - \tau_3]; L^2_\sigma) \cap L^2(0, T - \tau_3; H^1_{0,\sigma}(\Omega)), \quad \partial_t(\rho(\phi_1)\mathbf{u}_1) \in L^2(0, T - \tau_3; H^1_{0,\sigma}(\Omega)'), \\ \phi_1 \in L^\infty(0, T - \tau_3; H^1(\Omega)) \cap L^q(0, T - \tau_3; W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad \forall p \in (2, \infty), \\ \partial_t \phi_1 \in L^\infty(0, T - \tau_3; H^1(\Omega)') \cap L^2(0, T - \tau_3; L^2(\Omega)), \\ \mu_1 \in L^\infty(0, T - \tau_3; H^1(\Omega)) \cap L^2(0, T - \tau_3; H^2(\Omega)) \cap H^1(0, T - \tau_3; H^1(\Omega)'), \\ F'(\phi_1) \in L^\infty(0, T - \tau_3; H^1(\Omega)). \end{cases} \tag{6.8}$$

Thanks to (6.2) and (6.8), (6.7) can be rewritten as follows

$$\begin{aligned} &\langle \partial_t(\rho(\phi)\mathbf{u}), \mathbf{w} \rangle_{H^1_{0,\sigma}(\Omega)} - \int_\Omega \partial_t \rho(\phi_1)\, \mathbf{u}_1 \cdot \mathbf{w} \, \mathrm{d}x + \int_\Omega (\rho(\phi_1)\mathbf{u}_1 \cdot \nabla)\, \mathbf{u}_1 \cdot \mathbf{w} \, \mathrm{d}x \\ &\quad - \int_\Omega (\rho'(\phi_1)\nabla\mu_1 \cdot \nabla)\, \mathbf{u}_1 \cdot \mathbf{w} \, \mathrm{d}x + \int_\Omega \nu(\phi_1) D\mathbf{u}_1 : D\mathbf{w} \, \mathrm{d}x = - \int_\Omega \phi_1 (\nabla K * \phi_1) \cdot \mathbf{w} \, \mathrm{d}x, \end{aligned} \tag{6.9}$$

for any $\mathbf{w} \in H^1_{0,\sigma}(\Omega)$ and almost any $t \in (\tau_1, T)$. At the same time, the pair $(\mathbf{u}_2, \phi_2)$ satisfies (1.17) as well as

$$\begin{aligned} &\rho(\phi_2)\partial_t \mathbf{u}_2 + \rho(\phi_2)(\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 - \rho'(\phi_2)(\nabla\mu_2 \cdot \nabla)\mathbf{u}_2 - \mathrm{div}\,(\nu(\phi_2) D\mathbf{u}_2) + \nabla \Pi_2 = \mu_2 \nabla \phi_2, \\ &\partial_t \phi_2 + \mathbf{u}_2 \cdot \nabla \phi_2 = \Delta \mu_2, \quad \mu_2 = F'(\phi_2) - K * \phi_2, \end{aligned} \tag{6.10}$$

almost everywhere in $\Omega \times (0, T - \tau_3)$. Moreover, it is essential to notice that both ϕ_1 and ϕ_2 are strictly separated since the initial concentration $\phi(\tau_2)$ is strictly separated, i.e. $\|\phi_i(t)\|_{L^\infty(\Omega)} \le 1 - \delta$, for all $t \in [0, T - \tau_3]$, $i = 1, 2$, for some $\delta \in (0, 1)$.

We now set $(\mathbf{u}, \Phi) = (\mathbf{u}_1 - \mathbf{u}_2, \phi_1 - \phi_2)$ in $[0, T - \tau_3]$. It follows from (6.9) and (6.10) that this pair satisfies the weak formulation:

$$
\begin{aligned}
&\langle \partial_t(\rho(\phi_1)\mathbf{u}), \mathbf{w}\rangle_{H^1_{0,\sigma}(\Omega)} - (\partial_t\rho(\phi_1)\,\mathbf{u}_1, \mathbf{w}) + (\partial_t(\rho(\phi_1)\mathbf{u}_2), \mathbf{w}) - (\rho(\phi_2)\partial_t\mathbf{u}_2, \mathbf{w}) \\
&\quad + (\rho(\phi_1)(\mathbf{u}_1\cdot\nabla)\mathbf{u}, \mathbf{w}) + (\rho(\phi_1)(\mathbf{u}\cdot\nabla)\mathbf{u}_2, \mathbf{w}) + ((\rho(\phi_1)-\rho(\phi_2))(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2, \mathbf{w}) \\
&\quad - \frac{\rho_1-\rho_2}{2}((\nabla\mu_1\cdot\nabla)\mathbf{u}, \mathbf{w}) - \frac{\rho_1-\rho_2}{2}((\nabla\Theta\cdot\nabla)\mathbf{u}_2, \mathbf{w}) + (\nu(\phi_1)D\mathbf{u}, \nabla\mathbf{w}) \\
&\quad + ((\nu(\phi_1)-\nu(\phi_2))D\mathbf{u}_2, \nabla\mathbf{w}) = (\mu_1\nabla\Phi, \mathbf{w}) + (\Theta\nabla\phi_2, \mathbf{w}),
\end{aligned}
\tag{6.11}
$$

for any $\mathbf{w} \in H^1_{0,\sigma}(\Omega)$, in $(0, T-\tau_3)$, and

$$
\partial_t\Phi + \mathbf{u}_1\cdot\nabla\Phi + \mathbf{u}\cdot\nabla\phi_2 = \Delta\Theta, \quad \Theta = F'(\phi_1) - F'(\phi_2) - K * \Phi \quad \text{a.e. in } \Omega\times(0, T-\tau_3). \tag{6.12}
$$

As next step, we take $\mathbf{w} = \mathbf{u}$ in (6.11) and apply the chain rule formula in [23, Lemma 5.3] with $\widehat{\rho} = \rho(\phi_1)$ and $\mathbf{u}$ on the interval $(0, T-\tau_3)$. Clearly, we have $\widehat{\rho} \in H^1(0, T-\tau_3; L^2(\Omega))$ and $\mathbf{u} \in L^\infty(0, T-\tau_3; L^2_\sigma(\Omega)) \cap L^2(0, T-\tau_3; H^1_{0,\sigma}(\Omega))$. We now claim that $\partial_t(\widehat{\rho}\mathbf{u}) \in L^2(0, T; H^1_{0,\sigma}(\Omega)')$. In fact, by definition, we have

$$
\partial_t(\widehat{\rho}\mathbf{u}) = \partial_t(\rho(\phi_1)\mathbf{u}_1) - \partial_t(\rho(\phi_1)\mathbf{u}_2).
$$

Observe that $\partial_t(\rho(\phi_1)\mathbf{u}_1) \in L^2(0, T; H^1_{0,\sigma}(\Omega)')$ by the first part of the proof. Moreover, $\partial_t(\rho(\phi_1)\mathbf{u}_2) \in L^2(0, T; H^1_{0,\sigma}(\Omega)')$ by (6.8) and $\mathbf{u}_2 \in L^2(0, \infty; V^2_{0,\sigma}(\Omega)) \cap H^1(0, \infty; L^2_\sigma(\Omega))$. Therefore, by using [23, Lemma 5.3], the chain rule

$$
\langle \partial_t(\rho(\phi_1)\mathbf{u}), \mathbf{u}\rangle_{H^1_{0,\sigma}(\Omega)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho(\phi)_1|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega \partial_t\rho(\phi_1)|\mathbf{u}|^2\,\mathrm{d}x
$$

holds almost everywhere in $(0, T-\tau_3)$. Also, we observe that

$$
-(\partial_t\rho(\phi_1)\mathbf{u}_1, \mathbf{u}) + (\partial_t(\rho(\phi_1)\mathbf{u}_2), \mathbf{u}) = -\int_\Omega \partial_t\rho(\phi_1)|\mathbf{u}|^2\,\mathrm{d}x + (\rho(\phi_1)\partial_t\mathbf{u}_2, \mathbf{u}).
$$

Thus, exploiting the above relations and (5.40)-(5.42), we find (cf. (5.43))

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \int_\Omega \nu(\phi_1)|D\mathbf{u}|^2\,\mathrm{d}x \\
&= -\int_\Omega (\rho(\phi_1) - \rho(\phi_2)\partial_t\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \rho(\phi_1)(\mathbf{u}\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega (\rho(\phi_1)-\rho(\phi_2))(\mathbf{u}_2\cdot\nabla)\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x \\
&\quad - \int_\Omega (\nu(\phi_1)-\nu(\phi_2))D\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\Delta\mathbf{u}_2\cdot\mathbf{u}\,\mathrm{d}x - \frac{\rho_1-\rho_2}{2}\int_\Omega \Theta\nabla\mathbf{u}_2 : \nabla\mathbf{u}\,\mathrm{d}x \\
&\quad - \int_\Omega \Phi(\nabla K * \phi_1)\cdot\mathbf{u}\,\mathrm{d}x - \int_\Omega \phi_2(\nabla K * \Phi)\cdot\mathbf{u}\,\mathrm{d}x.
\end{aligned}
\tag{6.13}
$$

The rest of the argument follows by repeating line by line the proof of the continuous dependence estimate for “separated” strong solutions given in Subsection 5.5. As a result, we obtain the following differential inequality

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Phi\|^2_{L^2(\Omega)}\right) \le K(t)\left(\int_\Omega \frac{\rho(\phi_1)}{2}|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\|\Phi\|^2_{L^2(\Omega)}\right)
$$

almost everywhere in $(0, T-\tau_3)$, where K is defined as in (1.21).

By the regularity of the strong solution $\mathbf{u}_2$, it immediately follows that $\widetilde{K} \in L^1(0, T-\tau_3)$. Then, we conclude from the Gronwall lemma that $(\mathbf{u}_1(t), \phi_1(t)) = (\mathbf{u}_2(t), \phi_2(t))$ on $[0, T-\tau_3]$, and thereby $(\mathbf{u}(t), \phi(t)) = (\widetilde{\mathbf{u}}(t), \widetilde{\phi}(t))$ on $[\tau_3, T]$. So, setting $\Pi(t) = \widetilde{\Pi}(t)$ on $[\tau_3, \infty)$, we have that $(\mathbf{u}, \Pi, \phi)$ is a "separated" strong solution on $[\tau, \infty) \subset [\tau_3, \infty)$.

In the last part, we demonstrate that any weak solution converges to an equilibrium, i.e., a minimum of the nonlocal Helmholtz free energy (1.7). To this end, we first observe from the previous part and Theorem 4.1-(ii) that

$$
\begin{cases}
\mathbf{u} \in BUC([1,\infty); H^1_{0,\sigma}(\Omega)) \cap L^2_{\mathrm{uloc}}([1,\infty); V^2_{0,\sigma}(\Omega)) \cap H^1_{\mathrm{uloc}}([1,\infty); L^2_\sigma(\Omega)),\\
\phi \in L^\infty(1,\infty; L^\infty(\Omega)) \text{ such that } \sup\limits_{t\in[1,+\infty)} \|\phi(t)\|_{L^\infty(\Omega)} \le 1-\delta,\\
\phi \in C_{\mathrm{w}}([1,\infty); H^1(\Omega)) \cap L^q_{\mathrm{uloc}}([1,\infty); W^{1,p}(\Omega)), \quad q = \dfrac{2p}{p-2}, \quad p \in (2,\infty),\\
\partial_t \phi \in L^\infty(1,\infty; H^1(\Omega)') \cap L^2(1,\infty; L^2(\Omega)),\\
\mu \in BUC([1,\infty); H^1(\Omega)) \cap L^2_{\mathrm{uloc}}([1,\infty); H^2(\Omega)) \cap H^1_{\mathrm{uloc}}([1,\infty); L^2(\Omega)).
\end{cases}
\tag{6.14}
$$

In addition, the energy identity

$$
E(\mathbf{u}(t), \phi(t)) + \int_1^t \left\| \sqrt{\nu(\phi(s))} D\mathbf{u}(s) \right\|^2_{L^2(\Omega)} + \|\nabla \mu(s)\|^2_{L^2(\Omega))} \,\mathrm{d}s = E(\mathbf{u}(1), \phi(1)) \tag{6.15}
$$

holds for every $t \ge 1$. Thanks to the separation property, the classical theory for second-order parabolic semilinear equations (cf. [28, Corollary 5.6] and the references therein) entails that there exists $\gamma \in (0,1)$ such that

$$
\phi \in BUC([2,\infty), C^\gamma(\overline{\Omega})). \tag{6.16}
$$

Now we define the ω-limit set of $(\mathbf{u}, \phi)$ as

$$
\omega(\mathbf{u}, \phi) = \left\{ (\mathbf{v}, \Phi) \in L^2_\sigma(\Omega) \times L^\infty(\Omega) : \exists\, t_n \nearrow \infty \text{ s.t. } (\mathbf{u}(t_n), \phi(t_n)) \to (\mathbf{v}, \Phi) \text{ in } L^2_\sigma(\Omega) \times L^\infty(\Omega) \right\}.
$$

In light of (6.14) and (6.16), it follows that $\omega(\mathbf{u}, \phi)$ is non-empty, compact and connected in $L^2_\sigma(\Omega) \times L^\infty(\Omega)$. Also, we observe that any $(\mathbf{v}, \Phi) \in \omega(\mathbf{u}, \phi)$ is such that $\|\Phi\|_{C(\overline{\Omega})} \le 1-\delta$.

Next, we claim that

$$
\omega(\mathbf{u}, \phi) \subset \left\{ (\mathbf{0}, \phi_\infty) : \phi_\infty \in C^\gamma(\overline{\Omega}) \text{ solves } (1.25) \right\}. \tag{6.17}
$$

Arguing as in [8, Section 3], subtracting the Helmholtz free energy equation (cf. (4.4))

$$
\mathcal{E}_{\mathrm{nloc}}(\phi(t)) + \int_1^t \|\nabla \mu(s)\|^2_{L^2(\Omega)} \,\mathrm{d}\tau + \int_1^t \int_\Omega \phi\, \mathbf{u} \cdot \nabla \mu \,\mathrm{d}x \,\mathrm{d}s = \mathcal{E}_{\mathrm{nloc}}(\phi(1)), \quad \forall\, t \in [1,\infty), \tag{6.18}
$$

from (6.15) we have, for all $t \in (1,\infty)$,

$$
E_{\mathrm{kin}}(\mathbf{u}(t), \phi(t)) + \int_1^t \left\| \sqrt{\nu(\phi(s))} D\mathbf{u}(s) \right\|^2_{L^2(\Omega)} \,\mathrm{d}s = E_{\mathrm{kin}}(\mathbf{u}(1), \phi(1)) + \int_1^t \int_\Omega \phi\, \mathbf{u} \cdot \nabla \mu \,\mathrm{d}x \,\mathrm{d}\tau. \tag{6.19}
$$

Let us set $\varepsilon > 0$. We observe from (6.15) that $\mathbf{u} \in L^2(0,\infty; H^1_{0,\sigma}(\Omega))$ and $\nabla\mu \in L^2(0,\infty; L^2(\Omega;\mathbb{R}^2))$, there exists $T > 0$ such that $\|\mathbf{u}(T)\|_{L^2(\Omega)} \le \varepsilon$ and $\|\nabla\mu\|_{L^2((T,\infty;L^2(\Omega))} \le \varepsilon$. Then, we infer that

$$
\begin{aligned}
\max_{t\in[T,\infty)} \int_\Omega \frac{1}{2}\rho(\phi(t))|\mathbf{u}(t)|^2 \,\mathrm{d}x + \nu_* \int_T^\infty \|D\mathbf{u}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \int_\Omega \frac{1}{2}\rho(\phi(T))|\mathbf{u}(T)|^2\,\mathrm{d}x + \int_T^\infty \|\phi(s)\|_{L^\infty(\Omega)}\|\mathbf{u}(s)\|_{L^2(\Omega)}\|\nabla\mu(s)\|_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \frac{\rho^*}{2}\|\mathbf{u}(T)\|^2_{L^2(\Omega)} + \frac{\nu_*}{2}\int_T^\infty \|D\mathbf{u}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\int_T^\infty \|\nabla\mu(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s \\
&\le \frac{\rho^*}{2}\varepsilon^2 + \frac{\nu_*}{2}\int_T^\infty \|D\mathbf{u}(s)\|^2_{L^2(\Omega)}\,\mathrm{d}s + C\varepsilon^2,
\end{aligned}
$$

which gives that

$$
\max_{t\in[T,\infty)} \|\mathbf{u}(t)\|_{L^2(\Omega)} \le 2C\varepsilon,
$$

where C is independent of T and ε. Thus, $\mathbf{u}(t) \to \mathbf{0}$ as $t \nearrow \infty$.

Let us now consider $t_n \nearrow \infty$ and let $(\mathbf{u}(t_n), \phi(t_n) \to (\mathbf{0}, \phi_\infty)$ in $L^2_\sigma(\Omega) \times L^\infty(\Omega)$ as $n \to \infty$. We now set $(\mathbf{u}_n(t), \phi_n(t)) = (\mathbf{u}(t+t_n), \phi(t+t_n))$ for $t \in [1,\infty)$. Clearly, $\mathbf{u}_n(t) \to \mathbf{0}$ in $L^2_\sigma(\Omega)$ as $n \to \infty$. Also, since $\mathbf{u}_n$ is uniformly bounded in $L^\infty(0,\infty; L^2_\sigma(\Omega)) \cap L^2(0,\infty; H^1_\sigma(\Omega))$, and exploiting Theorem 4.1, (6.14), (6.16) and (6.18), it is easy to deduce that

$$
\|\phi_n\|_{L^\infty(0,T;C(\overline{\Omega}))} \le 1-\delta, \quad \|\phi_n\|_{L^2(0,T;H^1(\Omega))} \le C, \quad \|\partial_t\phi_n\|_{L^2(0,T;H^1(\Omega)')} \le C, \quad \|\mu_n\|_{L^2(0,T;H^1(\Omega))} \le C
$$

for some C independent of n and for any $T > 0$, where $\mu_n(t) = \mu(t+t_n)$. Then, ϕ_n converges to ϕ' strongly in $L^2(0,T;L^2(\Omega))$ for any $T > 0$ and μ_n converges to μ' weakly in $L^2(0,T;H^1(\Omega))$ for any $T > 0$. It follows that ϕ' is a weak solution to (4.1) in the sense of Theorem 4.1 with chemical potential μ', divergence-free drift $\mathbf{v} = \mathbf{0}$, and initial datum $\phi'(0) = \phi_\infty$. In addition, we have $E_{\text{nloc}}(\phi_n(t)) \to E_{\text{nloc}}(\phi'(t))$ for almost every $t \in [1,\infty)$ as $n \to \infty$. However, since $\mathbf{u} \in L^2(0,\infty; H^1_{0,\sigma}(\Omega))$ and $\nabla\mu \in L^2(0,\infty; L^2(\Omega;\mathbb{R}^2))$, $\phi\mathbf{u}\cdot\nabla\mu \in L^1(0,\infty; L^1(\Omega))$. Then, the limit $E_\infty := \lim_{t\to\infty} E_{\text{nloc}}(\phi(t))$ exists and is unique. Therefore, we infer that $E_{\text{nloc}}(\phi'(t)) = E_\infty$ almost everywhere in $[1,\infty)$. We conclude from the energy equality of ϕ' that $\nabla\mu' = 0$ for almost every $t \in [1,\infty)$, and thereby $\partial_t\phi'(t) = 0$ for almost every $t \in [1,\infty)$. As such, $\phi'(t) \equiv \phi_\infty$ for all $t \in [1,\infty)$ and

$$
F'(\phi_\infty) - K * \phi_\infty = \mu_\infty \quad \text{in } \Omega, \quad \text{for some } \mu_\infty \in \mathbb{R}.
$$

This proves (6.17). We are left to show that the whole weak solution converges to $(\mathbf{0}, \phi_\infty)$ as t goes to $+\infty$. We know that, thanks to (6.15), the limit energy E_∞ is constant on $\omega(\mathbf{u},\phi)$. Thus, we deduce from (1.24) that, for all $t > 0$,

$$
E_\infty + \int_t^{+\infty} \left\|\sqrt{\nu(\phi(s))}D\mathbf{u}(s)\right\|^2_{L^2(\Omega)} + \|\nabla\mu(s)\|^2_{L^2(\Omega))}\,\mathrm{d}s = E(\mathbf{u}(t),\phi(t))
$$

from which we deduce (see (1.16))

$$
\begin{aligned}
\int_t^{+\infty} \|\nabla\mu(s)\|^2_{L^2(\Omega))}\,\mathrm{d}s &\le E(\mathbf{u}(t),\phi(t)) - E_\infty \\
&= E(\mathbf{u}(t),\phi(t)) - E_{\mathrm{nloc}}(\phi_\infty) \\
&= \frac{1}{2}\int_\Omega \rho(\phi)|\mathbf{u}|^2 + E_{\mathrm{nloc}}(\phi(t)) - E_{\mathrm{nloc}}(\phi_\infty).
\end{aligned}
\tag{6.20}
$$

To conclude, we now need the real analyticity of the potential F in order to apply a suitable version of the Łojasiewicz-Simon inequality (see, for instance [30, Lemma 2.20]). This amounts to say that there is $\theta \in (0,1/2]$ and $T_0 > 0$ sufficiently large such that

$$
|E_{\mathrm{nloc}}(\phi(t)) - E_{\mathrm{nloc}}(\phi_\infty)|^{1-\theta} \le C\|\mu - \overline{\mu}\|_{L^2(\Omega))} \le C\|\nabla\mu\|_{L^2(\Omega))}, \quad \forall\, t \ge T_0, \tag{6.21}
$$

for some $C > 0$. Therefore, we get from (6.20) and (6.21) that (see also (1.4))

$$
|E(\mathbf{u}(t),\phi(t)) - E_\infty|^{1-\theta} \le C(\|\mathbf{u}\|_{L^2(\Omega))} + \|\nabla\mu\|_{L^2(\Omega))}), \quad \forall\, t \ge T_0.
$$

We can now argue as in [8, Sec. 6] to infer that $(\mathbf{u}(t),\phi(t))$ converges to $(\mathbf{0},\phi_\infty)$ in $L^2_\sigma(\Omega) \times L^\infty(\Omega)$ as $t \to +\infty$. The proof of Theorem 1.7 is hereby complete.

7. Proof of Theorem 1.9: Improved continuous dependence estimate for matched densities

We consider two sets of initial data $(\mathbf{u}_0^1,\phi_0^1)$ and $(\mathbf{u}_0^2,\phi_0^2)$ which satisfy the assumptions of Theorem 1.5, respectively, with constant density $\rho = \rho_1 = \rho_2 > 0$ (i.e., we consider the nonlocal Model H). We denote by $(\mathbf{u}_1,\Pi_1,\phi_1)$ and $(\mathbf{u}_2,\Pi_2,\phi_2)$ the corresponding strong solutions provided by Theorem 1.5. Let us set $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $P = \Pi_1 - \Pi_2$, $\Phi = \phi_1 - \phi_2$, $\Theta = \mu_1 - \mu_2 = F'(\phi_1) - F'(\phi_2) - K * \Phi$, which solve

$$
\begin{aligned}
&\rho\partial_t\mathbf{u} + \rho\mathrm{div}\,(\mathbf{u}_1 \otimes \mathbf{u}) + \rho\mathrm{div}\,(\mathbf{u} \otimes \mathbf{u}_2) - \mathrm{div}\,(\nu(\phi_1)D\mathbf{u}) - \mathrm{div}\,((\nu(\phi_1) - \nu(\phi_2))D\mathbf{u}_2) + \nabla P \\
&\quad = \mu_1\nabla\Phi + \Theta\nabla\phi_2, \\
&\partial_t\Phi + \mathbf{u}_1\cdot\nabla\Phi + \mathbf{u}\cdot\nabla\phi_2 = \Delta\Theta,
\end{aligned}
\tag{7.1}
$$

almost everywhere in $\Omega \times (0,\infty)$. Multiplying $(7.1)_1$ by $\mathbf{A}^{-1}\mathbf{u}$ and $(7.1)_2$ by $\mathcal{N}(\Phi - \overline{\Phi})$ (notice that, by the conservation of mass, $\overline{\Phi}$ is constant), integrating over Ω and adding the resulting equations together, we find the identity

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2}\|\mathbf{u}\|^2_\sharp + \frac{1}{2}\|\Phi - \overline{\Phi}\|^2_*\right) + (\Theta, \Phi - \overline{\Phi}) + (\nu(\phi_1)D\mathbf{u}, \nabla\mathbf{A}^{-1}\mathbf{u}) \\
&= \rho(\mathbf{u}_1 \otimes \mathbf{u}, \nabla\mathbf{A}^{-1}\mathbf{u}) + \rho(\mathbf{u} \otimes \mathbf{u}_2, \nabla\mathbf{A}^{-1}\mathbf{u}) - \big((\nu(\phi_1) - \nu(\phi_2))D\mathbf{u}_2, \nabla\mathbf{A}^{-1}\mathbf{u}\big) \\
&\quad - (\mathbf{u}_1\cdot\nabla\Phi, \mathcal{N}(\Phi - \overline{\Phi})) - (\mathbf{u}\cdot\nabla\phi_2, \mathcal{N}(\Phi - \overline{\Phi})) + (\mu_1\nabla\Phi, \mathbf{A}^{-1}\mathbf{u}) + (\Theta\nabla\phi_2, \mathbf{A}^{-1}\mathbf{u}).
\end{aligned}
\tag{7.2}
$$

Arguing as in [39, proof of Theorem 3.1], we observe that

$$
\begin{aligned}
\big(\nu(\phi_1)D\mathbf{u}, \nabla\mathbf{A}^{-1}\mathbf{u}\big) &= \big(\nu(\phi_1)\nabla\mathbf{u}, D\mathbf{A}^{-1}\mathbf{u}\big) = -\,\big(\mathbf{u}, \mathrm{div}\,(\nu(\phi_1)D\mathbf{A}^{-1}\mathbf{u})\big) \\
&= -\,\big(\mathbf{u}, \nu'(\phi_1)D\mathbf{A}^{-1}\mathbf{u}\nabla\phi_1\big) - \frac{1}{2}\big(\mathbf{u}, \nu(\phi_1)\Delta\mathbf{A}^{-1}\mathbf{u}\big) \\
&= -\,\big(\mathbf{u}, \nu'(\phi_1)D\mathbf{A}^{-1}\mathbf{u}\nabla\phi_1\big) + \frac{1}{2}\big(\mathbf{u}, \nu(\phi_1)\mathbf{u}\big) - \frac{1}{2}\big(\mathbf{u}, \nu(\phi_1)\nabla\pi\big),
\end{aligned}
$$

where the artificial pressure $\pi \in L^\infty(0,T;H^1_{(0)}(\Omega))$ is associated to the Stokes problem $-\Delta \mathbf{A}^{-1}\mathbf{U} + \nabla \pi = \mathbf{u}$ in $\Omega \times (0,\infty)$. Thanks to the above relation and by (4.13) and (5.42), we obtain the differential inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2}\|\mathbf{u}\|_\sharp^2 + \frac{1}{2}\|\Phi - \overline{\Phi}\|_*^2\right) + \frac{\nu_*}{2}\|\mathbf{u}\|_{L^2(\Omega)}^2 + \frac{3\alpha}{4}\|\phi\|_{L^2(\Omega)}^2 \\
&\le C\|\phi - \overline{\phi}\|_*^2 + \left|\overline{\phi^1} - \overline{\phi^2}\right| \left(\|F'(\phi^1)\|_{L^1(\Omega)} + \|F'(\phi^2)\|_{L^1(\Omega)}\right) + \rho(\mathbf{u}_1 \otimes \mathbf{u}, \nabla \mathbf{A}^{-1}\mathbf{u}) \\
&\quad - \rho(\mathbf{u} \otimes \mathbf{u}_2, \nabla \mathbf{A}^{-1}\mathbf{u}) - \big((\nu(\phi_1) - \nu(\phi_2))D\mathbf{u}_2, \nabla \mathbf{A}^{-1}\mathbf{u}\big) + \big(\mathbf{u}, \nu'(\phi_1) D\mathbf{A}^{-1}\mathbf{u}\nabla\phi_1\big) \\
&\quad + \frac{1}{2}(\mathbf{u}, \nu(\phi_1)\nabla\pi) - (\mathbf{u}_1 \cdot \nabla\Phi, \mathcal{N}(\Phi - \overline{\Phi})) - (\mathbf{u} \cdot \nabla\phi_2, \mathcal{N}(\Phi - \overline{\Phi})) \\
&\quad - (\Phi(\nabla K * \phi_1), \mathbf{A}^{-1}\mathbf{u}) - (\phi_2(\nabla K * \Phi), \mathbf{A}^{-1}\mathbf{u}).
\end{aligned} \tag{7.3}$$

By using (2.2) and (2.6), we have

$$\begin{aligned}
\left|\rho \int_\Omega \mathbf{u}_1 \otimes \mathbf{u} : \nabla \mathbf{A}^{-1}\mathbf{u}\,\mathrm{d}x\right| &\le C\|\mathbf{u}_1\|_{L^4(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^4(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\mathbf{u}_1\|_{L^4(\Omega)}^4\|\mathbf{u}\|_\sharp^2
\end{aligned}$$

and

$$\begin{aligned}
\left|\rho \int_\Omega \mathbf{u} \otimes \mathbf{u}_2 : \nabla \mathbf{A}^{-1}\mathbf{u}\,\mathrm{d}x\right| &\le \|\mathbf{u}_2\|_{L^4(\Omega)}\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^4(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\mathbf{u}_2\|_{L^4(\Omega)}^4\|\mathbf{u}\|_\sharp^2.
\end{aligned}$$

In a similar way, recalling the assumption (H_6), we find

$$\begin{aligned}
\left|\int_\Omega (\nu(\phi_1) - \nu(\phi_2))\, D\mathbf{u}_2 : \nabla \mathbf{A}^{-1}\mathbf{u}\,\mathrm{d}x\right| &\le C\|\Phi\|_{L^2(\Omega)}\|D\mathbf{u}_2\|_{L^4(\Omega)}\|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^4(\Omega)} \\
&\le \frac{\alpha}{16}\|\Phi\|_{L^2(\Omega)}^2 + \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|D\mathbf{u}_2\|_{L^4(\Omega)}^4\|\mathbf{u}\|_\sharp^2
\end{aligned}$$

and

$$\begin{aligned}
\left|\int_\Omega \nu'(\phi_1)\mathbf{u} \cdot \left(D\mathbf{A}^{-1}\mathbf{u}\right)\nabla\phi_1\,\mathrm{d}x\right| &\le C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^4(\Omega)}\|\nabla\phi_1\|_{L^4(\Omega)} \\
&\le \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\nabla\phi_1\|_{L^4(\Omega)}^4\|\mathbf{u}\|_\sharp^2.
\end{aligned}$$

Exploiting now Lemma 3.1, we obtain

$$\begin{aligned}
\left|\frac{1}{2}\int_\Omega \nu(\phi_1)\mathbf{u} \cdot \nabla\pi\,\mathrm{d}x\right| &= \left|\frac{1}{2}\int_\Omega \nu'(\phi_1)\mathbf{u} \cdot \nabla\phi_1\pi\,\mathrm{d}x\right| \\
&\le C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\phi_1\|_{L^4(\Omega)}\|\pi\|_{L^4(\Omega)} \\
&\le C\|\mathbf{u}\|_{L^2(\Omega)}\|\nabla\phi_1\|_{L^4(\Omega)}\|\nabla \mathbf{A}^{-1}\mathbf{u}\|_{L^2(\Omega)}^{\frac{1}{2}}\|\mathbf{u}\|_{L^2(\Omega)}^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\nabla\phi_1\|_{L^4(\Omega)}^4\|\mathbf{u}\|_\sharp^2.$$

Next, arguing exactly as in (4.14), we get

$$\left|\int_\Omega \mathbf{u}_1 \cdot \nabla\Phi \cdot \mathcal{N}(\Phi - \overline{\Phi}))\,\mathrm{d}x\right| \leq \frac{\alpha}{16}\|\Phi\|_{L^2(\Omega)}^2 + C\|\mathbf{u}_1\|_{L^4(\Omega)}^4\|\Phi - \overline{\Phi}\|_*^2 + C\left|\overline{\Phi}\right|^2.$$

Since $\|\phi_2\|_{L^\infty(\Omega\times(0,\infty))} \leq 1$, we infer that

$$\left|\int_\Omega \mathbf{u}\cdot\nabla\phi_2\,\mathcal{N}(\Phi - \overline{\Phi}))\,\mathrm{d}x\right| = \left|\int_\Omega \phi_2\mathbf{u}\cdot\nabla\mathcal{N}(\Phi - \overline{\Phi}))\,\mathrm{d}x\right| \leq \frac{\nu_*}{24}\|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\Phi - \overline{\Phi}\|_*^2.$$

Lastly, by (H_2) and $\|\phi_i\|_{L^\infty(\Omega\times(0,\infty))} \leq 1$ for $i = 1, 2$, we deduce that

$$\left|\int_\Omega \Phi(\nabla K * \phi_1)\cdot\mathbf{A}^{-1}\mathbf{u}\,\mathrm{d}x\right| \leq \|\nabla K * \phi_1\|_{L^\infty(\Omega)}\|\Phi\|_{L^2(\Omega)}\|\mathbf{A}^{-1}\mathbf{u}\|_{L^2(\Omega)} \leq \frac{\alpha}{16}\|\Phi\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_\sharp^2$$

and

$$\left|\int_\Omega \phi_2(\nabla K * \Phi)\cdot\mathbf{A}^{-1}\mathbf{u}\,\mathrm{d}x\right| \leq \|\phi_2\|_{L^\infty(\Omega)}\|\nabla K * \Phi\|_{L^2(\Omega)}\|\mathbf{A}^{-1}\mathbf{u}\|_{L^2(\Omega)} \leq \frac{\alpha}{16}\|\Phi\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_\sharp^2.$$

Combining (7.3) with above inequalities, we are led to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2}\|\mathbf{u}\|_\sharp^2 + \frac{1}{2}\|\Phi - \overline{\Phi}\|_*^2\right) \leq \Lambda_1(t)\left(\frac{\rho}{2}\|\mathbf{u}\|_\sharp^2 + \frac{1}{2}\|\Phi - \overline{\Phi}\|_*^2\right) + \Lambda_2(t)\left|\overline{\Phi}\right| + C\left|\overline{\phi}\right|^2,$$

where

$$\Lambda_1(t) := C\left(1 + \|\mathbf{u}_1(t)\|_{L^4(\Omega)}^4 + \|\mathbf{u}_2(t)\|_{L^4(\Omega)}^4 + \|D\mathbf{u}_2(t)\|_{L^4(\Omega)}^4 + \|\nabla\phi_1(t)\|_{L^4(\Omega)}^4\right)$$

and

$$\Lambda_2(t) := \|F'(\phi_1(t))\|_{L^1(\Omega)} + \|F'(\phi_2(t))\|_{L^1(\Omega)}.$$

Owing to (1.17), it is easily seen that $\Lambda_j \in L^1(0,T)$ for $j = 1, 2$. Thus, it follows from the Gronwall lemma that (1.26) holds. The proof of Theorem 1.9 is thus concluded.

8. Proof of Theorem 1.10: Matched versus unmatched density

Let us fix $T > 0$. Consider $(\mathbf{u}, \Pi, \phi)$ and $(\mathbf{u}_H, \Pi_H, \phi_H)$ the strong solutions to the nonlocal AGG model with density $\rho(\phi)$ and to the nonlocal Model H with constant density $\overline{\rho} > 0$ (i.e. (1.8)-(1.9)) with $\overline{\rho} = \rho_1 = \rho_2$, respectively. We assume that both $(\mathbf{u}, \Pi, \phi)$ and $(\mathbf{u}_H, \Pi_H, \phi_H)$ originate from the same initial datum $(\mathbf{u}_0, \phi_0)$. Therefore, setting $\mathbf{v} = \mathbf{u} - \mathbf{u}_H$, $Q = \Pi - \Pi_H$, $\Phi = \phi - \phi_H$, we have

$$
\begin{aligned}
&\left(\frac{\rho_1+\rho_2}{2}\right)\partial_t \mathbf{v} + \left(\frac{\rho_1-\rho_2}{2}\phi\right)\partial_t \mathbf{u} + \left(\frac{\rho_1+\rho_2}{2} - \overline{\rho}\right)\partial_t \mathbf{u}_H + \rho(\phi)(\mathbf{u}\cdot\nabla)\mathbf{u} - \overline{\rho}(\mathbf{u}_H\cdot\nabla)\mathbf{u}_H \\
&\quad - \left(\frac{\rho_1-\rho_2}{2}\right)((\nabla\mu\cdot\nabla)\mathbf{u}) - \operatorname{div}(\nu(\phi)D\mathbf{v}) - \operatorname{div}((\nu(\phi)-\nu(\phi_H)D\mathbf{u}_H) + \nabla Q \\
&= \mu\nabla\phi - \mu_H\nabla\phi_H, \\
&\partial_t\Phi + \mathbf{u}\cdot\nabla\Phi + \mathbf{v}\cdot\nabla\phi_H = \Delta M,
\end{aligned}
\tag{8.1}
$$

almost everywhere in $\Omega\times(0,T)$ where $M = \mu - \mu_H = F'(\phi) - F'(\phi_H) - K * \Phi$. Multiplying $(8.1)_1$ by $\mathbf{A}^{-1}\mathbf{V}$ and integrating over Ω, we find

$$
\begin{aligned}
&\left(\frac{\rho_1+\rho_2}{4}\right)\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}\|_\sharp^2 + (\nu(\phi)D\mathbf{v}, \nabla\mathbf{A}^{-1}\mathbf{v}) = -\frac{\rho_1-\rho_2}{2}(\phi\partial_t\mathbf{u}, \mathbf{A}^{-1}\mathbf{v}) \\
&\quad - \left(\frac{\rho_1-\rho_2}{2} - \overline{\rho}\right)(\partial_t\mathbf{u}_H, \mathbf{A}^{-1}\mathbf{v}) - ((\rho(\phi)(\mathbf{u}\cdot\nabla)\mathbf{u} - \overline{\rho}(\mathbf{u}_H\cdot\nabla)\mathbf{u}_H), \mathbf{A}^{-1}\mathbf{v}) \\
&\quad + \frac{\rho_1-\rho_2}{2}((\nabla\mu\cdot\nabla)\mathbf{u}, \mathbf{A}^{-1}\mathbf{v}) - ((\nu(\phi)-\nu(\phi_H))D\mathbf{u}_H, \nabla\mathbf{A}^{-1}\mathbf{v}) \\
&\quad - (\Phi(\nabla K * \phi), \mathbf{A}^{-1}\mathbf{v}) - (\phi_H(\nabla K * \Phi), \mathbf{A}^{-1}\mathbf{v}).
\end{aligned}
\tag{8.2}
$$

Observe now that (cf. [39, Section 3])

$$
(\nu(\phi)D\mathbf{v}, \nabla\mathbf{A}^{-1}\mathbf{v}) = -\left(\mathbf{v}, \nu'(\phi)D\mathbf{A}^{-1}\mathbf{v}\nabla\phi\right) + \frac{1}{2}(\mathbf{v}, \nu(\phi)\mathbf{v}) + \frac{1}{2}\left(\mathbf{v}, \nu'(\phi)\tilde{\Pi}\nabla\phi\right), \tag{8.3}
$$

where the artificial pressure is determined by the Stokes problem $-\Delta\mathbf{A}^{-1}\mathbf{v} + \nabla\tilde{\Pi} = \mathbf{v}$ almost everywhere in $\Omega\times(0,T)$. On the other hand, multiplying $(8.1)_2$ by $\mathcal{N}\Phi$ (notice that $\overline{\Phi}\equiv 0$ by the conservation of mass, since the two solutions originate from the same initial data) and integrating over Ω, we obtain (cf. (4.13))

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Phi\|_*^2 + \frac{3\alpha}{4}\|\Phi\|_{L^2(\Omega)}^2 \le C\|\Phi\|_*^2 + (\Phi\mathbf{u}, \nabla\mathcal{N}\Phi) + (\phi_H\mathbf{v}, \nabla\mathcal{N}\varphi). \tag{8.4}
$$

Here C stands for a generic positive constant which may depend on given quantities and which may vary even within the same line. Adding (8.2) and (8.4) together, and exploiting (8.3), we end up with

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\rho_1+\rho_2}{4}\right)\|\mathbf{v}\|_\sharp^2 + \frac{1}{2}\|\Phi\|_*^2\right) + \frac{\nu_*}{2}\|\mathbf{v}\|_{L^2(\Omega)}^2 + \frac{3\alpha}{4}\|\Phi\|_{L^2(\Omega)}^2 \\
&\le -\frac{\rho_1-\rho_2}{2}(\phi\partial_t\mathbf{u}, \mathbf{A}^{-1}\mathbf{v}) - \left(\frac{\rho_1-\rho_2}{2} - \overline{\rho}\right)(\partial_t\mathbf{u}_H, \mathbf{A}^{-1}\mathbf{v}) \\
&\quad - ((\rho(\phi)(\mathbf{u}\cdot\nabla)\mathbf{u} - \overline{\rho}(\mathbf{u}_H\cdot\nabla)\mathbf{u}_H), \mathbf{A}^{-1}\mathbf{v}) + \frac{\rho_1-\rho_2}{2}((\nabla\mu\cdot\nabla)\mathbf{u}, \mathbf{A}^{-1}\mathbf{v}) \\
&\quad - ((\nu(\phi)-\nu(\phi_H))D\mathbf{u}_H, \nabla\mathbf{A}^{-1}\mathbf{v}) + \left(\mathbf{v}, \nu'(\phi)D\mathbf{A}^{-1}\mathbf{v}\nabla\phi\right) - \frac{1}{2}\left(\mathbf{v}, \nu'(\phi)\tilde{\Pi}\nabla\phi\right) \\
&\quad + (\Phi\mathbf{u}, \nabla\mathcal{N}\Phi) + (\phi_H\mathbf{v}, \nabla\mathcal{N}\Phi) - (\Phi(\nabla K * \phi), \mathbf{A}^{-1}\mathbf{v}) - (\phi_H(\nabla K * \Phi), \mathbf{A}^{-1}\mathbf{v}).
\end{aligned}
\tag{8.5}
$$

Since $\|\phi\|_{L^\infty(\Omega\times(0,T))} \le 1$, we have

$$
\left|\frac{\rho_1-\rho_2}{2}\int_\Omega \phi\partial_t\mathbf{u}\cdot\mathbf{A}^{-1}\mathbf{v}\right| \le C\left|\frac{\rho_1-\rho_2}{2}\right|^2 \|\partial_t\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\mathbf{v}\|_\sharp^2,
$$

and

$$\left| \left(\frac{\rho_1 - \rho_2}{2} - \overline{\rho} \right) \int\limits_\Omega \partial_t \mathbf{u}_H \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x \right| \leq C \|\mathbf{v}\|_\sharp^2 + C \left| \frac{\rho_1 - \rho_2}{2} - \overline{\rho} \right|^2 \|\partial_t \mathbf{u}_H\|_{L^2(\Omega)}^2.$$

Integrating by parts, we find

$$\begin{aligned}
&- \int\limits_\Omega (\rho(\phi)(\mathbf{u} \cdot \nabla)\mathbf{u} - \overline{\rho}(\mathbf{u}_H \cdot \nabla)\mathbf{u}_H) \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x \\
&= - \int\limits_\Omega \rho(\phi)(\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x - \int\limits_\Omega (\rho(\phi) - \overline{\rho})(\mathbf{u}_H \cdot \nabla)\mathbf{u}_H \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x - \int\limits_\Omega \rho(\phi)(\mathbf{u}_H \cdot \nabla)\mathbf{v} \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x \\
&= - \int\limits_\Omega \rho(\phi)(\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x - \int\limits_\Omega (\rho(\phi) - \overline{\rho})(\mathbf{u}_H \cdot \nabla)\mathbf{u}_H \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x + \int\limits_\Omega \rho(\phi)(\mathbf{u}_H \cdot \nabla)\mathbf{A}^{-1} \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x \\
&\quad + \frac{\rho_1 - \rho_2}{2} \int\limits_\Omega (\nabla \phi \cdot \mathbf{u}_H) \left(\mathbf{v} \cdot \mathbf{A}^{-1} \mathbf{v} \right) \, \mathrm{d}x.
\end{aligned}$$

Then, recalling that $\mathbf{u}_H \in L^\infty(0, \infty; H^1_{0,\sigma}(\Omega))$, we obtain

$$\begin{aligned}
\left| \int\limits_\Omega \rho(\phi)(\mathbf{v} \cdot \nabla)\mathbf{u} \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x \right| &\leq C \|\mathbf{v}\|_{L^2(\Omega)} \|\nabla \mathbf{u}\|_{L^4(\Omega)} \|\mathbf{A}^{-1} \mathbf{v}\|_{L^4(\Omega)} \\
&\leq \frac{\nu_*}{28} \|\mathbf{v}\|_{L^2(\Omega)}^2 + C \|\nabla \mathbf{u}\|_{L^4(\Omega)}^2 \|\mathbf{v}\|_\sharp^2
\end{aligned}$$

and

$$\begin{aligned}
\left| - \int\limits_\Omega (\rho(\phi) - \overline{\rho})(\mathbf{u}_H \cdot \nabla)\mathbf{u}_H \cdot \mathbf{A}^{-1} \mathbf{v} \, \mathrm{d}x \right| &\leq \|\rho(\phi) - \overline{\rho}\|_{L^\infty(\Omega)} \|\mathbf{u}_H\|_{L^4(\Omega)} \|\nabla \mathbf{u}_H\|_{L^2(\Omega)} \|\mathbf{A}^{-1} \mathbf{v}\|_{L^4(\Omega)} \\
&\leq C \left(\left| \frac{\rho_1 - \rho_2}{2} \right| + \left| \frac{\rho_1 + \rho_2}{2} - \overline{\rho} \right| \right)^2 + C \|\mathbf{v}\|_\sharp^2.
\end{aligned}$$

On the other hand, by (2.2), (2.3), (2.4) and (2.6), we infer that

$$\begin{aligned}
\left| \int\limits_\Omega \rho(\phi)(\mathbf{u}_H \cdot \nabla)\mathbf{A}^{-1} \mathbf{v} \cdot \mathbf{v} \, \mathrm{d}x \right| &\leq C \|\mathbf{u}_H\|_{L^4(\Omega)} \|\nabla \mathbf{A}^{-1} \mathbf{v}\|_{L^4(\Omega)} \|\mathbf{v}\|_{L^2(\Omega)} \\
&\leq \frac{\nu_*}{28} \|\mathbf{v}\|_{L^2(\Omega)}^2 + C \|\mathbf{v}\|_\sharp^2
\end{aligned}$$

and

$$\begin{aligned}
\left| \int\limits_\Omega (\nabla \phi \cdot \mathbf{u}_H) \left(\mathbf{v} \cdot \mathbf{A}^{-1} \mathbf{v} \right) \, \mathrm{d}x \right| &\leq \left| \frac{\rho_1 - \rho_2}{2} \right| \|\mathbf{u}_H\|_{L^4(\Omega)} \|\mathbf{v}\|_{L^2(\Omega)} \|\mathbf{A}^{-1} \mathbf{v}\|_{L^\infty(\Omega)} \|\nabla \phi\|_{L^4(\Omega)} \\
&\leq C \left| \frac{\rho_1 - \rho_2}{2} \right| \|\mathbf{v}\|_{L^2(\Omega)}^{\frac{3}{2}} \|\mathbf{v}\|_\sharp^{\frac{1}{2}} \|\nabla \phi\|_{L^4(\Omega)} \\
&\leq \frac{\nu_*}{28} \|\mathbf{v}\|_{L^2(\Omega)}^2 + C \|\nabla \phi\|_{L^4(\Omega)}^4 \|\mathbf{v}\|_\sharp^2,
\end{aligned}$$

as well as

$$\left|\int_\Omega (\nu(\phi)-\nu(\phi_H))D\mathbf{u}_H : \nabla \mathbf{A}^{-1}\mathbf{v}\,\mathrm{d}x\right| \le C\|\Phi\|_{L^2(\Omega)}\|D\mathbf{u}_H\|_{L^4(\Omega)}\|\nabla\mathbf{A}^{-1}\mathbf{v}\|_{L^4(\Omega)}$$
$$\le C\|\Phi\|_{L^2(\Omega)}\|D\mathbf{u}_H\|_{L^4(\Omega)}\|\mathbf{v}\|_\sharp^{\frac12}\|\mathbf{v}\|_{L^2(\Omega)}^{\frac12}$$
$$\le \frac{\alpha}{12}\|\Phi\|_{L^2(\Omega)}^2 + \frac{\nu_*}{28}\|\mathbf{v}\|_{L^2(\Omega)}^2 + C\|D\mathbf{u}_H\|_{L^4(\Omega)}^4\|\mathbf{v}\|_\sharp^2.$$

In a similar way, by using assumption (H_6), we also get

$$\left|\int_\Omega \nu'(\phi)\mathbf{v}\cdot(D\mathbf{A}^{-1}\mathbf{v}\nabla\phi)\,\mathrm{d}x\right| \le C\|\mathbf{v}\|_{L^2(\Omega)}\|\nabla\mathbf{A}^{-1}\mathbf{v}\|_{L^4(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}$$
$$\le \frac{\nu_*}{28}\|\mathbf{v}\|_{L^2(\Omega)}^2 + C\|\nabla\phi\|_{L^4(\Omega)}^4\|\mathbf{v}\|_\sharp^2.$$

Since $\mathbf{u}\in L^\infty(0,\infty;H^1_{0,\sigma}(\Omega))$, it is easily seen that

$$\left|\frac{\rho_1-\rho_2}{2}\int_\Omega (\nabla\mu\cdot\nabla)\mathbf{u}\cdot\mathbf{A}^{-1}\mathbf{v}\,\mathrm{d}x\right| \le \left|\frac{\rho_1-\rho_2}{2}\right| \|\nabla\mu\|_{L^4(\Omega)}\|\nabla\mathbf{u}\|_{L^2(\Omega)}\|\mathbf{A}^{-1}\mathbf{v}\|_{L^4(\Omega)}$$
$$\le C\|\mathbf{v}\|_\sharp^2 + C\left|\frac{\rho_1-\rho_2}{2}\right|^2\|\nabla\mu\|_{L^4(\Omega)}^2.$$

Now, exploiting Lemma 3.1, we get

$$\left|\int_\Omega \nu'(\phi)\mathbf{v}\cdot\nabla\phi\tilde{\Pi}\,\mathrm{d}x\right| \le C\|\mathbf{V}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}\|\tilde{\Pi}\|_{L^4(\Omega)}$$
$$\le C\|\mathbf{v}\|_{L^2(\Omega)}\|\nabla\phi\|_{L^4(\Omega)}\|\nabla\mathbf{A}^{-1}\mathbf{v}\|_{L^2(\Omega)}^{\frac12}\|\mathbf{v}\|_{L^2(\Omega)}^{\frac12}$$
$$\le \frac{\nu_*}{28}\|\mathbf{v}\|_{L^2(\Omega)}^2 + C\|\nabla\phi\|_{L^4(\Omega)}^4\|\mathbf{v}\|_\sharp^2.$$

Finally, as in the proof of Theorem 1.9, we have

$$\left|\int_\Omega \Phi\mathbf{u}\cdot\nabla\mathcal{N}\Phi\,\mathrm{d}x\right| \le \frac{\alpha}{12}\|\Phi\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{L^4(\Omega)}^4\|\Phi\|_*^2$$

and

$$\left|\int_\Omega \phi_H\mathbf{v}\cdot\nabla\mathcal{N}\Phi\,\mathrm{d}x\right| \le \frac{\nu_*}{28}\|\mathbf{v}\|_{L^2(\Omega)}^2 + C\|\Phi\|_*^2,$$

as well as

$$\left|\int_\Omega \Phi(\nabla K*\phi)\cdot\mathbf{A}^{-1}\mathbf{v}\,\mathrm{d}x\right| + \left|\int_\Omega \phi_H(\nabla K*\Phi)\cdot\mathbf{A}^{-1}\mathbf{v}\,\mathrm{d}x\right| \le \frac{\alpha}{12}\|\Phi\|_{L^2(\Omega)}^2 + C\|\mathbf{v}\|_\sharp^2.$$

Combining the above estimates, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(\frac{\rho_1+\rho_2}{4}\right)\|\mathbf{v}\|_\sharp^2+\frac{1}{2}\|\Phi\|_*^2\right)+\frac{\nu_*}{4}\|\mathbf{v}\|_{L^2(\Omega)}^2+\frac{\alpha}{2}\|\Phi\|_{L^2(\Omega)}^2$$
$$\leq R_1\left(\left(\frac{\rho_1+\rho_2}{4}\right)\|\mathbf{v}\|_\sharp^2+\frac{1}{2}\|\Phi\|_*^2\right)+R_2\left(\left|\frac{\rho_1-\rho_2}{2}\right|^2+\left|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\right|^2\right),$$

where

$$R_1:=C\left(1+\|\nabla\mathbf{u}\|_{L^4(\Omega)}^2+\|\nabla\mathbf{u}_H\|_{L^4(\Omega)}^4+\|\nabla\phi\|_{L^4(\Omega)}^4\right),$$
$$R_2:=C\left(1+\|\partial_t\mathbf{u}_H\|_{L^2(\Omega)}^2+\|\partial_t\mathbf{u}\|_{L^2(\Omega)}^2+\|\nabla\mu\|_{L^4(\Omega)}^2\right).$$

Notice that C depends on the norm of the initial data and the time T. An application of the Gronwall lemma yields

$$\|\mathbf{v}(t)\|_\sharp^2+\|\Phi(t)\|_*^2\leq\frac{\left(\left|\frac{\rho_1-\rho_2}{2}\right|^2+\left|\frac{\rho_1+\rho_2}{2}-\overline{\rho}\right|^2\right)}{\min\{\frac{\rho_1+\rho_2}{4},\frac{1}{2}\}}\int_0^t \mathrm{e}^{\int_s^t R_1(\tau)\,\mathrm{d}\tau}R_2(s)\,\mathrm{d}s,\quad\forall\,t\in[0,T].$$

Therefore, in light of (1.17), the above inequality implies the desired conclusion (1.27). The proof of Theorem 1.10 is finished.

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Appendix A. Global "separated" solutions to (4.1) with smooth divergence free drift

In this Appendix, we establish the existence of global *regular* solutions to the nonlocal Cahn-Hilliard equation with *smooth* divergence-free drift. More precisely, we aim to construct solutions to (4.1) satisfying the separation property for all times.

Theorem A.1. *Let the assumptions* (H_1)*-*(H_5) *hold and let* $T>0$ *be given. If* $\mathbf{u}\in\mathcal{D}(0,T;C^\infty_{0,\sigma}(\Omega;\mathbb{R}^2))$ *and* $\phi_0\in H^1(\Omega)\cap L^\infty(\Omega)$ *with* $\|\phi_0\|_{L^\infty(\Omega)}<1$ *and* $|\overline{\phi_0}|<1$*, then there exists a solution* ϕ *to* (4.1) *such that*

$$\begin{cases}\phi\in L^\infty(0,T;H^1(\Omega)\cap L^\infty(\Omega)): \quad \sup_{t\in[0,T]}\|\phi(t)\|_{L^\infty(\Omega)}\leq 1-\delta,\\ \phi\in L^q(0,T;W^{1,p}(\Omega)),\quad q=\dfrac{2p}{p-2},\quad\forall\,p\in(2,\infty),\\ \partial_t\phi\in L^\infty(0,T;H^1(\Omega)')\cap L^2(0,T;L^2(\Omega)),\\ \mu\in C([0,T];H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega)),\end{cases}\tag{A.1}$$

where $\delta\in(0,1)$ *depends on* T*,* $\mathbf{u}$ *and* ϕ_0*. Any solution satisfying the above properties is a strong solution, that is,*

$$\begin{aligned} &\partial_t \phi + \mathbf{u} \cdot \nabla \phi = \Delta \mu, \quad \mu = F'(\phi) - K * \phi, \qquad a.e. \ in \ \Omega \times (0,T), \\ &\partial_{\mathbf{n}} \mu = 0, \qquad a.e. \ on \ \partial\Omega \times (0,T), \quad \phi(\cdot,0) = \phi_0, \qquad a.e. \ in \ \Omega. \end{aligned} \tag{A.2}$$

Proof. Let us first introduce the Yosida approximation of the singular potential F. For any $\lambda > 0$, we define $F_\lambda : \mathbb{R} \to \mathbb{R}$ such that $F_\lambda(s) = \frac{\lambda}{2}|A_\lambda s|^2 + F(J_\lambda(s))$ where $J_\lambda = (I + \lambda F')^{-1}$ and $A_\lambda = \frac{1}{\lambda}(I - J_\lambda)$. We report the following main properties (see [17] and [28, Section 3]):

(a1) for any $\lambda > 0$, $F_\lambda \in C^{2,1}_{\text{loc}}(\mathbb{R})$ such that $F_\lambda(0) = F'_\lambda(0) = 0$;
(a2) for any $0 < \lambda^\sharp \le 1$, there exists $C_\sharp > 0$ such that

$$F_\lambda(s) \ge \frac{1}{4\lambda^\sharp} s^2 - C_\sharp, \quad \forall\, s \in \mathbb{R},\ \forall\, \lambda \in (0, \lambda^\sharp]; \tag{A.3}$$

(a3) F_λ is convex with

$$F''_\lambda(s) \ge \frac{\alpha}{1+\alpha}, \quad \forall\, s \in \mathbb{R};$$

(a4) for any $\lambda > 0$, F'_λ is Lipschitz on $\mathbb{R}$ with constant $\frac{1}{\lambda}$;
(a5) as $\lambda \to 0$, $F_\lambda(s) \to F(s)$ for all $s \in \mathbb{R}$, $|F'_\lambda(s)| \to |F'(s)|$ for $s \in (-1,1)$ and F'_λ converges uniformly to F' on any compact subset of $(-1,1)$; furthermore, $|F'_\lambda(s)| \to +\infty$ for every $|s| \ge 1$.

Let us now fix $\lambda^\star$ to be positive and sufficiently small. We will choose $\lambda^\star$ will be defined later on. We claim that, for any $\lambda \in (0, \lambda^\star)$, there exists a function ϕ_λ such that

$$\phi_\lambda \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^1(\Omega)'), \tag{A.4}$$

which satisfies the variational formulation

$$\langle \partial_t \phi_\lambda, v \rangle - (\phi_\lambda\, \mathbf{u}, \nabla v) + (\nabla \mu_\lambda, \nabla v) = 0, \quad \forall\, v \in H^1(\Omega),\ \text{a.e. in } (0,T), \tag{A.5}$$

where $\mu_\lambda = F'_\lambda(\phi_\lambda) - K * \phi_\lambda \in L^2(0,T; H^1(\Omega))$, as well as $\phi(\cdot,0) = \phi_0(\cdot)$ in Ω. The proof of the existence of the approximating solution ϕ_λ is carried out by the Galerkin scheme. The argument is rather standard and we refer the reader to [21,28].

Conservation of mass and energy estimates. First, taking $v = 1$ in (A.5), we obtain that $\overline{\phi_\lambda}(t) = \overline{\phi_0}$ for all $t \in [0,T]$. Since $\|\phi_0\|_{L^\infty(\Omega)} < 1$ by assumption, we clearly infer that $|\overline{\phi_\lambda}(t)| = |\overline{\phi_0}| < 1$ for all $t \in [0,T]$. Next, we define the energy functional $\mathcal{E}_\lambda : L^2(\Omega) \to \mathbb{R}$ as follows

$$\mathcal{E}_\lambda(u) := \int_\Omega F_\lambda(u)\, \mathrm{d}x - \frac{1}{2} \int_\Omega (K * u)\ u\, \mathrm{d}x.$$

In light of (a1) and (a4), it is easily seen that $|F_\lambda(s)| \le \frac{1}{\lambda} s^2$ for all $s \in \mathbb{R}$. In turn, this gives that

$$\mathcal{E}_\lambda(u) \le \left(\frac{1}{\lambda} + \frac{\|K\|_{W^{1,1}(\mathbb{R}^2)}}{2} \right) \|u\|^2_{L^2(\Omega)}, \tag{A.6}$$

thus $\mathcal{E}_\lambda$ is well defined in $L^2(\Omega)$. Moreover, by the assumption on the kernel K in (H_2) and (a2), for any $\lambda < \lambda^\star$, we have

$$\begin{aligned}\mathcal{E}_\lambda(u) &\geq \frac{1}{4\lambda^\star}\|u\|^2_{L^2(\Omega)} - C_{\lambda^\star}|\Omega| - \frac{1}{2}\|K*u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\geq \left(\frac{1}{4\lambda^\star} - \frac{\|K\|_{W^{1,1}(\mathbb{R}^2)}}{2}\right)\|u\|^2_{L^2(\Omega)} - C_{\lambda^\star}|\Omega|.\end{aligned} \tag{A.7}$$

Hence, setting $\lambda^\star \leq \left[4\left(1 + \frac{\|K\|_{W^{1,1}(\mathbb{R}^2)}}{2}\right)\right]^{-1}$, we are led to

$$\mathcal{E}_\lambda(u) \geq \|u\|^2_{L^2(\Omega)} - C_b, \quad \forall\, u \in L^2(\Omega), \quad \forall\, \lambda \in (0, \lambda^\star], \tag{A.8}$$

where $C_b > 0$ is a constant independent of λ as well as any other constant in the sequel unless it is explicitly pointed out. Let us now take $v = \mu_\lambda$ in (A.5). By using (A.4), (A.6), [18, Proposition 4.2] and the definition of μ_λ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\phi_\lambda) + \|\nabla\mu_\lambda\|^2_{L^2(\Omega)} + \int_\Omega \phi_\lambda\, \mathbf{u}\cdot\nabla\mu_\lambda\,\mathrm{d}x = 0.$$

Thanks to (A.8), we easily get

$$\begin{aligned}\left|\int_\Omega \phi_\lambda\, \mathbf{u}\cdot\nabla\mu_\lambda\,\mathrm{d}x\right| &\leq \|\mathbf{u}\|_{L^\infty(\Omega)}\|\phi_\lambda\|_{L^2(\Omega)}\|\nabla\mu_\lambda\|_{L^2(\Omega)} \\ &\leq \frac{1}{2}\|\nabla\mu_\lambda\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathbf{u}\|^2_{L^\infty(\Omega)}\left(C_b + \mathcal{E}_\lambda(\phi_\lambda)\right).\end{aligned}$$

Then, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\lambda(\phi_\lambda) + \frac{1}{2}\|\nabla\mu_\lambda\|^2_{L^2(\Omega)} \leq \frac{1}{2}\|\mathbf{u}\|^2_{L^\infty(\Omega)}\left(C_b + \mathcal{E}_\lambda(\phi_\lambda)\right). \tag{A.9}$$

In light of the general properties of the Yosida approximation of a convex function (see, in particular, [57, Proposition 1.8, Chapter IV]), we recall that $F_\lambda(s)$ is increasing in λ towards $F(s)$ for all $s \in \mathbb{R}$. Since $\|\phi_0\|_{L^\infty(\Omega)}$, we infer that $\mathcal{E}_\lambda(\phi_0) \leq \mathcal{E}(\phi_0) < \infty$. Therefore, it follows from the Gronwall lemma applied to (A.9) that

$$\mathcal{E}_\lambda(\phi_\lambda(t)) \leq \left(\mathcal{E}(\phi_0) + \frac{C_b}{2}\int_0^T \|\mathbf{u}(\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau\right)\exp\left(\int_0^T \mathbf{u}(\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau\right), \quad \forall\, t \in [0,T]. \tag{A.10}$$

Combining (A.8) with (A.10), and integrating (A.9) on $[0,T]$, we obtain

$$\begin{aligned}&\max_{t\in[0,T]}\|\phi_\lambda(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^T \|\nabla\mu_\lambda(\tau)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau \\ &\leq 2\left(C_b + \mathcal{E}(\phi_0) + \frac{C_b}{2}\int_0^T \|\mathbf{u}(\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau\right)\exp\left(\int_0^T \mathbf{u}(\tau)\|^2_{L^\infty(\Omega)}\,\mathrm{d}\tau\right).\end{aligned} \tag{A.11}$$

Next, recalling that F'_λ is Lipschitz and $\phi_\lambda(t) \in H^1(\Omega)$ for almost every $t \in [0,T]$, it follows from [52] that $\nabla F'_\lambda(\phi_\lambda(t)) = F''_\lambda(\phi_\lambda(t))\nabla\phi_\lambda(t)$ for almost every $x \in \Omega$ and $t \in [0,T]$. By definition of μ_λ and (a3), we clearly have

$$\left(\frac{\alpha}{1+\alpha}\right)^2 \int_\Omega |\nabla \phi_\lambda|^2 \,\mathrm{d}x \le 2\|\nabla \mu_\lambda\|^2_{L^2(\Omega)} + 2\|K\|^2_{W^{1,1}(\mathbb{R}^2)}\|\phi_\lambda\|^2_{L^2(\Omega)}. \tag{A.12}$$

Let us now control $\overline{\mu_\lambda}$. Multiplying μ_λ by $\phi_\lambda - \overline{\phi_\lambda}$ and integrating over Ω, we find

$$\int_\Omega F'_\lambda(\phi_\lambda)\left(\phi_\lambda - \overline{\phi_\lambda}\right) \mathrm{d}x = \int_\Omega \mu_\lambda \left(\phi_\lambda - \overline{\phi_\lambda}\right) \mathrm{d}x + \int_\Omega K * \phi_\lambda \left(\phi_\lambda - \overline{\phi_\lambda}\right) \mathrm{d}x.$$

Observing that $(\overline{\mu_\lambda}, \phi_\lambda - \overline{\phi}_\lambda) = 0$, we infer from the properties of K, the Poincaré inequality, (A.16) and the conservation of mass that

$$\begin{aligned}\int_\Omega F'_\lambda(\phi_\lambda)\left(\phi_\lambda - \overline{\phi}_\lambda\right) \mathrm{d}x &= \int_\Omega \left(\mu_\lambda - \overline{\mu_\lambda}\right)\left(\phi_\lambda - \overline{\phi_\lambda}\right) \mathrm{d}x + \int_\Omega K * \phi_\lambda \left(\phi_\lambda - \overline{\phi_\lambda}\right) \mathrm{d}x \\ &\le C\left(1 + \|\nabla \mu_\lambda\|_{L^2(\Omega)}\right).\end{aligned}$$

Now, we recall from [28, Proof of Theorem 3.4] (which is inspired by [53]) that

$$\|F'_\lambda(\phi_\lambda)\|_{L^1(\Omega)} \le C^1 \left|\int_\Omega F'_\lambda(\phi_\lambda)\big(\phi_\lambda - \overline{\phi_\lambda}\big)\,\mathrm{d}x\right| + C^2, \tag{A.13}$$

where C^j, $j = 1, 2$, are positive constants that only depend on F, Ω and $\overline{\phi_0}$. Then, combining the above estimates with (A.13), we have

$$\begin{aligned}|\overline{\mu_\lambda}| &\le \frac{1}{|\Omega|}\left(\int_\Omega |F'_\lambda(\phi_\lambda)|\,\mathrm{d}x + \left|\int_\Omega K * \phi_\lambda \,\mathrm{d}x\right|\right) \\ &\le \frac{C^1}{|\Omega|}\left|\int_\Omega F'_\lambda(\phi_\lambda)\big(\phi_\lambda - \overline{\phi_\lambda}\big)\,\mathrm{d}x\right| + \frac{C^2}{|\Omega|} + \frac{C\|K\|_{L^1(\mathbb{R}^2)}}{|\Omega|}\|\phi_\lambda\|_{L^2(\Omega)} \\ &\le C\left(1 + \|\nabla \mu_\lambda\|_{L^2(\Omega)}\right),\end{aligned} \tag{A.14}$$

where C depends on F, Ω, $\overline{\phi_0}$, $\mathcal{E}(\phi_0)$ and $\|\mathbf{u}\|_{L^2(0,T;L^\infty(\Omega))}$. On the other hand, concerning $\partial_t \phi_\lambda$, it is immediate to check that

$$\|\partial_t \phi_\lambda\|_{H^1(\Omega)'} \le \|\phi_\lambda\|_{L^2(\Omega)}\|\mathbf{u}\|_{L^\infty(\Omega)} + \|\nabla \mu_\lambda\|_{L^2(\Omega)}. \tag{A.15}$$

Therefore, owing to (A.11), (A.12), (A.14) and (A.15), we obtain

$$\|\phi_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla \phi_\lambda\|_{L^2(0,T;L^2(\Omega))} + \|\partial_t \phi_\lambda\|_{L^2(0,T;H^1(\Omega)')} + \|\mu_\lambda\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{A.16}$$

In addition, we get by comparison that

$$\|F'_\lambda(\phi_\lambda)\|_{L^2(0,T;H^1(\Omega))} \le C. \tag{A.17}$$

Sobolev estimates. We derive higher-order Sobolev estimates following the argument used in [21]. To this aim, we introduce the difference quotient $\partial_t^h f(t) = h^{-1}\left(f(t+h) - f(t)\right)$ and the shift $S^h f(t) = f(t+h)$ for $0 < t < T - h$. Subtracting now the weak formulation (A.5) evaluated at time t from the one at time $t + h$, dividing by h and choosing $\mathcal{N}\partial_t^h \phi_\lambda$ as test function, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\phi_\lambda\|_*^2 - \int_\Omega S^h\phi_\lambda\,\partial_t^h\mathbf{u}\cdot\nabla\mathcal{N}\partial_t\phi_\lambda\,\mathrm{d}x - \int_\Omega \partial_t^h\phi_\lambda\,\mathbf{u}\cdot\nabla\mathcal{N}\partial_t^h\phi_\lambda\,\mathrm{d}x + \int_\Omega \partial_t^h\mu_\lambda\partial_t^h\phi_\lambda\,\mathrm{d}x = 0. \tag{A.18}$$

By (a3), we have

$$\begin{aligned}\int_\Omega \partial_t^h\mu_\lambda\partial_t^h\phi_\lambda\,\mathrm{d}x &= \int_\Omega \frac{1}{h}\left(F_\lambda'(S^h\phi_\lambda) - F_\lambda'(\phi_\lambda)\right)\partial_t^h\phi_\lambda\,\mathrm{d}x - \int_\Omega \left(K*\partial_t^h\phi_\lambda\right)\partial_t^h\phi_\lambda\,\mathrm{d}x \\ &\geq \frac{\alpha}{1+\alpha}\|\partial_t^h\phi_\lambda\|_{L^2(\Omega)}^2 - \int_\Omega \left(K*\partial_t^h\phi_\lambda\right)\partial_t^h\phi_\lambda\,\mathrm{d}x.\end{aligned} \tag{A.19}$$

Also, we observe that (cf. (4.13))

$$\begin{aligned}(K*\partial_t^h\phi_\lambda, \partial_t^h\phi_\lambda) &\leq \|\nabla K*\partial_t^h\phi_\lambda\|_{L^2(\Omega)}\|\nabla\mathcal{N}\partial_t^h\phi_\lambda\|_{L^2(\Omega)} \\ &\leq \frac{\alpha}{4(1+\alpha)}\|\partial_t^h\phi_\lambda\|_{L^2(\Omega)}^2 + C\|\partial_t^h\phi_\lambda\|_*^2.\end{aligned} \tag{A.20}$$

On the other hand, we infer from (2.1), (2.2) and (A.11) that

$$\begin{aligned}\left|\int_\Omega S^h\phi_\lambda\,\partial_t^h\mathbf{u}\cdot\nabla\mathcal{N}\partial_t\phi_\lambda\,\mathrm{d}x\right| &\leq \|\partial_t^h\mathbf{u}\|_{L^4(\Omega)}\|S^h\phi_\lambda\|_{L^2(\Omega)}\|\nabla\mathcal{N}\partial_t^h\phi_\lambda\|_{L^4(\Omega)} \\ &\leq \frac{\alpha}{4(1+\alpha)}\|\partial_t^h\phi_\lambda\|_{L^2(\Omega)}^2 + C\|\partial_t^h\mathbf{u}\|_{L^4(\Omega)}^2\end{aligned}$$

and

$$\left|\int_\Omega \partial_t^h\phi_\lambda\,\mathbf{u}\cdot\nabla\mathcal{N}\partial_t^h\phi_\lambda\,\mathrm{d}x\right| \leq \frac{\alpha}{4(1+\alpha)}\|\partial_t^h\phi_\lambda\|_{L^2(\Omega)}^2 + C\|\mathbf{u}\|_{L^\infty(\Omega)}^2\|\partial_t^h\phi_\lambda\|_*^2.$$

Therefore, we derive from (A.18) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_t^h\phi_\lambda\|_*^2 + \frac{\alpha}{4(1+\alpha)}\|\partial_t^h\phi_\lambda\|_{L^2(\Omega)}^2 \leq C\left(1+\|\mathbf{u}\|_{L^\infty(\Omega)}^2\right)\|\partial_t^h\phi_\lambda^k\|_*^2 + C\|\partial_t^h\mathbf{u}\|_{L^4(\Omega)}^2. \tag{A.21}$$

Recalling that $\overline{\phi_\lambda}(t) \equiv \overline{\phi_0}$, and thereby $\overline{\partial_t\phi}(t) \equiv 0$, for all $t \in [0, T]$, we infer from (A.5) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi_\lambda - \phi_0\|_*^2 - \int_\Omega \phi_\lambda\,\mathbf{u}\cdot\nabla\mathcal{N}(\phi_\lambda - \phi_0)\,\mathrm{d}x + \int_\Omega \mu\,(\phi_\lambda - \phi_0)\,\mathrm{d}x = 0. \tag{A.22}$$

Observing that

$$\int_\Omega \left(F_\lambda'(\phi_\lambda) - F_\lambda'(\phi_0)\right)(\phi_\lambda - \phi_0)\,\mathrm{d}x \geq 0,$$

and exploiting (A.11), we obtain

$$-\int_\Omega \mu\,(\phi_\lambda - \phi_0)\,\mathrm{d}x = -\int_\Omega F_\lambda'(\phi_\lambda)\,(\phi_\lambda - \phi_0)\,\mathrm{d}x + \int_\Omega K*\phi_\lambda\,(\phi_\lambda - \phi_0)\,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \int_\Omega F_\lambda'(\phi_0)\,(\phi_\lambda - \phi_0)\,\mathrm{d}x + \int_\Omega K * \phi_\lambda\,(\phi_\lambda - \phi_0)\,\mathrm{d}x \\
&\leq \|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)} \|\nabla \mathcal{N}(\phi_\lambda - \phi_0)\|_{L^2(\Omega)} \\
&+ \|\nabla K\|_{L^1(\mathbb{R}^2)} \|\phi_\lambda\|_{L^2(\Omega)} \|\nabla \mathcal{N}(\phi_\lambda - \phi_0)\|_{L^2(\Omega)} \\
&\leq C(1 + \|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)}) \|\phi_\lambda - \phi_0\|_*.
\end{aligned}
$$

Similarly, we have

$$
\left| \int_\Omega \phi_\lambda\,\mathbf{u} \cdot \nabla \mathcal{N}(\phi_\lambda - \phi_0)\,\mathrm{d}x \right| \leq \|\mathbf{u}\|_{L^\infty(\Omega)} \|\phi_\lambda\|_{L^2(\Omega)} \|\phi_\lambda - \phi_0\|_* \leq C \|\mathbf{u}\|_{L^\infty(\Omega)} \|\phi_\lambda - \phi_0\|_*.
$$

In conclusion, integrating (A.22) in $(0,t)$ for $t \in (0,T)$, we find

$$
\frac{1}{2}\|\phi_\lambda(t) - \phi_0\|_*^2 \leq C\left(1 + \|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}\right) \int_0^t \|\phi_\lambda^k(s) - \phi_{0,k}\|_*\,\mathrm{d}s \tag{A.23}
$$

and a well-known version of the Gronwall lemma (see [17, Lemma A.5]) implies that

$$
\frac{1}{2}\|\phi_\lambda(t) - \phi_0\|_* \leq t\left(1 + \|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^\infty(0,T;L^\infty(\Omega))}\right), \quad \forall\, t \in (0,T).
$$

In order to obtain an uniform estimate in λ, we are left to control $\|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)}$. To this aim, we first recall from [28, Lemma 3.10] that

$$
F_\lambda''(s) = \frac{1}{\lambda}\left[1 - \frac{1}{1 + \lambda F''(J_\lambda(s))}\right], \quad \forall\, s \in (-1,1),
$$

where J_λ is the resolvent operator. In light of [57, Chapter IV, Proposition 1.7], $J_\lambda(s) \to s$ for all $s \in (-1,1)$, which entails that $F_\lambda''(s) \to F''(s)$ for all $s \in (-1,1)$. Furthermore, since $J_\lambda(0) = 0$ (cf. $F'(0) = 0$) and J_λ is a contraction, J_λ is bounded on compact subset of $(-1,1)$ independently of λ. Observing that $F_\lambda''(s) \leq F''(J_\lambda(s))$, it follows that $F_\lambda''(s)$ is also bounded on compact subset of $(-1,1)$ independently of λ. In particular, since $\|\phi_0\|_{L^\infty(\Omega)} < 1$, we have that $\|F_\lambda''(\phi_0)\|_{L^\infty(\Omega)} \leq C_F$, where C_F is independent of λ. By Lebesgue's dominated convergence theorem, we infer that

$$
\lim_{\lambda \to 0} \|\nabla F_\lambda'(\phi_0)\|_{L^2(\Omega)} = \lim_{\lambda \to 0} \|F_\lambda''(\phi_0)\nabla\phi_0\|_{L^2(\Omega)} = \|F''(\phi_0)\nabla\phi_0\|_{L^2(\Omega)} \leq C_F \|\phi_0\|_{H^1(\Omega)}. \tag{A.24}
$$

Therefore, choosing $t = h$ in (A.23) and exploiting (A.24) and $\mathbf{u} \in C_0^\infty((0,T); C_{0,\sigma}^\infty(\Omega;\mathbb{R}^2))$, we conclude that $\|\partial_t^h \phi_\lambda(0)\|_* \leq C$. Now, an application of Gronwall's lemma to (A.21) entails that

$$
\begin{aligned}
&\max_{t \in [0,T-h]} \|\partial_t^h \phi_\lambda(t)\|_*^2 + \int_0^T \|\partial_t^h \phi_\lambda(\tau)\|_{L^2(\Omega)}^2\,\mathrm{d}\tau \\
&\quad \leq C\left(\|\partial_t^h \phi_\lambda(0)\|_*^2 + \int_0^T \|\partial_t^h \mathbf{u}(\tau)\|_{L^4(\Omega)}^2\,\mathrm{d}\tau\right) \exp\left(CT + C\int_0^T \|\mathbf{u}(\tau)\|_{L^\infty(\Omega)}^2\,\mathrm{d}\tau\right).
\end{aligned}
$$

Recalling the inequality $\|\partial_t^h \mathbf{u}\|_{L^2(0,T-h;L^4(\Omega))} \leq \|\partial_t \mathbf{u}\|_{L^2(0,T;L^4(\Omega))}$, we conclude that

$$\|\partial_t^h \phi_\lambda\|_{L^\infty(0,T;H^1(\Omega)')} + \|\partial_t^h \phi_\lambda\|_{L^2(0,T;L^2(\Omega))} \le C,$$

where C is also independent of h. Passing then to the limit as $h \to 0$, this gives

$$\|\partial_t \phi_\lambda\|_{L^\infty(0,T;H^1(\Omega)')} + \|\partial_t \phi_\lambda\|_{L^2(0,T;L^2(\Omega))} \le C. \tag{A.25}$$

Next, by comparison in (A.5) and using (A.14), we easily obtain that

$$\|\mu_\lambda\|_{L^\infty(0,T;H^1(\Omega))} + \|F_\lambda'(\phi_\lambda)\|_{L^\infty(0,T;H^1(\Omega))} \le C.$$

In addition, by elliptic regularity, we have

$$\|\mu_\lambda\|_{H^2(\Omega)} \le C\left(\|\partial_t\phi_\lambda\|_{L^2(\Omega)} + \|\mathbf{u}\cdot\nabla\phi_\lambda\|_{L^2(\Omega)}\right) \le C\left(\|\partial_t\phi_\lambda\|_{L^2(\Omega)} + \|\mathbf{u}\|_{L^\infty(\Omega)}\|\nabla\phi_\lambda\|_{L^2(\Omega)}\right).$$

Thus, thanks to (A.16) and (A.25), we also infer that

$$\|\mu_\lambda\|_{L^2(0,T;H^2(\Omega))} \le C, \tag{A.26}$$

Finally, recalling that $F_\lambda''(\phi_\lambda)\nabla\phi_\lambda = \nabla\mu_\lambda + \nabla K * \phi_\lambda$ almost everywhere in $\Omega\times(0,T)$, we find (cf. also (4.48))

$$\|\nabla\phi_\lambda\|_{L^p(\Omega)} \le C\left(1 + \|\nabla\mu_\lambda\|_{L^p(\Omega)}\right). \tag{A.27}$$

Then, by making use of the interpolation inequality $\|u\|_{L^q(0,T;L^p(\Omega))} \le C\|u\|_{L^\infty(0,T;L^2(\Omega))}\|u\|_{L^2(0,T;H^1(\Omega))}$, where $q = \dfrac{2p}{p-2}$ and $p \in (2,\infty)$, and by using (A.27) and (A.26), we are led to

$$\|\phi_\lambda\|_{L^q(0,T;W^{1,p}(\Omega))} \le C, \quad q = \frac{2p}{p-2}, \quad \forall\, p \in (2,\infty). \tag{A.28}$$

Passage to the limit and further regularities. Thanks to the above estimates (A.16)-(A.17), (A.25)-(A.26) and to the convergence properties (a5), we deduce by standard compactness arguments and by passing to the limit as $\lambda \to 0$ in (A.5) that there exist $\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^q(0,T;W^{1,p}(\Omega))$, where $q = \dfrac{2p}{p-2}$ and $p \in (2,\infty)$, such that $\partial_t\phi \in L^\infty(0,T;H^1(\Omega)') \cap L^2(0,T;L^2(\Omega))$ and $\mu \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, which satisfy the problem (A.2). Furthermore, by a classical argument for singular potentials (see, e.g., the proof of [28, Theorem 3.15]), we deduce that $\phi \in L^\infty(\Omega\times(0,T))$ such that $|\phi| < 1$ almost everywhere in $\Omega\times(0,T)$. By comparison in $(A.2)_1$, we infer that $F'(\phi) \in L^\infty(0,T;H^1(\Omega))$. In light of assumption (H_4), arguing as in [29] we find $F''(\phi) \in L^\infty(0,T;L^p(\Omega))$ for all $p \in [2,\infty)$. Owing to this regularity, we can recast the argument in [29, Section 4.1] for the advective case by observing that the corresponding drift term vanishes once again (i.e., $\mathcal{Z} = 0$, cf. (4.60), see the proof of Theorem 4.1, (ii)). This yields the existence of a constant $\delta > 0$ such that $\|\phi\|_{L^\infty(\Omega\times(0,T))} \le 1-\delta$. To conclude this proof, we are left to show an estimate for $\partial_t\mu$. We observe that

$$\partial_t^h\mu = \partial_t^h\phi\left(\int_0^1 F''(sS^h\phi + (1-s)\phi)\,\mathrm{d}s\right) - K * \partial_t^h\phi, \quad 0 < t \le T-h. \tag{A.29}$$

By the separation property, $\|sS^h\phi + (1-s)\phi)\|_{L^\infty(\Omega\times(0,T-h))} \le 1-\delta$ for all $s \in (0,1)$. Then, by the properties of K and exploiting that $\|\partial_t^h\phi\|_{L^2(0,T-h;L^2(\Omega))} \le \|\partial_t\phi\|_{L^2(0,T;L^2(\Omega))}$, we obtain that $\|\partial_t^h\mu^k\|_{L^2(0,T-h;L^2(\Omega))} \le C$, where $C > 0$ is independent of h. This implies that $\partial_t\mu \in L^2(0,T;L^2(\Omega))$. The proof of Theorem A.1 is now completed. □

References

[1] H. Abels, Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities, Commun. Math. Phys. 289 (2009) 45–73, https://doi.org/10.1007/s00220-009-0806-4.

[2] H. Abels, On a diffuse interface model for two-phase flows of viscous, incompressible fluids with matched densities, Arch. Ration. Mech. Anal. 194 (2009) 463–506, https://doi.org/10.1007/s00205-008-0160-2.

[3] H. Abels, Strong well-posedness of a diffuse interface model for a viscous, quasi-incompressible two-phase flow, SIAM J. Math. Anal. 44 (2012) 316–340, https://doi.org/10.1137/110829246.

[4] H. Abels, D. Breit, Weak solutions for a non-Newtonian diffuse interface model with different densities, Nonlinearity 29 (2016) 3426–3453, https://doi.org/10.1088/0951-7715/29/11/3426.

[5] H. Abels, D. Depner, H. Garcke, Existence of weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities, J. Math. Fluid Mech. 15 (2013) 453–480, https://doi.org/10.1007/s00021-012-0118-x.

[6] H. Abels, D. Depner, H. Garcke, On an incompressible Navier-Stokes/Cahn-Hilliard system with degenerate mobility, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 30 (2013) 1175–1190, https://doi.org/10.1016/j.anihpc.2013.01.002.

[7] H. Abels, H. Garcke, Weak solutions and diffuse interface models for incompressible two-phase flows, in: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer, Cham, 2018, pp. 1267–1327.

[8] H. Abels, H. Garcke, A. Giorgini, Global regularity and asymptotic stabilization for the incompressible Navier–Stokes-Cahn–Hilliard model with unmatched densities, Math. Ann. (2023), https://doi.org/10.1007/s00208-023-02670-2.

[9] H. Abels, H. Garcke, G. Grün, Thermodynamically consistent, frame indifferent diffuse interface models for incompressible two-phase flows with different densities, Math. Models Methods Appl. Sci. 22 (2012) 1150013, https://doi.org/10.1142/S0218202511500138.

[10] H. Abels, Y. Terasawa, Weak solutions for a diffuse interface model for two-phase flows of incompressible fluids with different densities and nonlocal free energies, Math. Methods Appl. Sci. 43 (2020) 3200–3219, https://doi.org/10.1002/mma.6111.

[11] H. Abels, Y. Terasawa, Convergence of a nonlocal to a local diffuse interface model for two-phase flow with unmatched densities, Discrete Contin. Dyn. Syst., Ser. S 15 (2022) 1871–1881, https://doi.org/10.3934/dcdss.2022117.

[12] H. Abels, J. Weber, Local well-posedness of a quasi-incompressible two-phase flow, J. Evol. Equ. 21 (2021) 3477–3502, https://doi.org/10.1007/s00028-020-00646-2.

[13] J. Bedrossian, N. Rodríguez, A.L. Bertozzi, Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion, Nonlinearity 24 (2011) 1683–1714, https://doi.org/10.1088/0951-7715/24/6/001.

[14] A.L. Bertozzi, S. Esedoḡlu, A. Gillette, Inpainting of binary images using the Cahn-Hilliard equation, IEEE Trans. Image Process. 16 (2007) 285–291, https://doi.org/10.1109/TIP.2006.887728.

[15] F. Boyer, Nonhomogeneous Cahn-Hilliard fluids, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 18 (2001) 225–259, https://doi.org/10.1016/S0294-1449(00)00063-9.

[16] F. Boyer, A theoretical and numerical model for the study of incompressible mixture flows, Comput. Fluids 31 (2002) 41–68, https://doi.org/10.1016/S0045-7930(00)00031-1.

[17] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert, North-Holland Mathematics Studies, vol. 5, North-Holland Publishing Co./American Elsevier Publishing Co., Inc., Amsterdam-London/New York, 1973.

[18] P. Colli, P. Krejčí, E. Rocca, J. Sprekels, Nonlinear evolution inclusions arising from phase change models, Czechoslov. Math. J. 57 (132) (2007) 1067–1098, https://doi.org/10.1007/s10587-007-0114-0.

[19] E. Davoli, L. Scarpa, L. Trussardi, Local asymptotics for nonlocal convective Cahn-Hilliard equations with $W^{1,1}$ kernel and singular potential, J. Differ. Equ. 289 (2021) 35–58, https://doi.org/10.1016/j.jde.2021.04.016.

[20] E. Davoli, L. Scarpa, L. Trussardi, Nonlocal-to-local convergence of Cahn-Hilliard equations: Neumann boundary conditions and viscosity terms, Arch. Ration. Mech. Anal. 239 (2021) 117–149, https://doi.org/10.1007/s00205-020-01573-9.

[21] F. Della Porta, A. Giorgini, M. Grasselli, The nonlocal Cahn-Hilliard-Hele-Shaw system with logarithmic potential, Nonlinearity 31 (2018) 4851–4881, https://doi.org/10.1088/1361-6544/aad52a.

[22] S. Frigeri, Global existence of weak solutions for a nonlocal model for two-phase flows of incompressible fluids with unmatched densities, Math. Models Methods Appl. Sci. 26 (2016) 1955–1993, https://doi.org/10.1142/S0218202516500494.

[23] S. Frigeri, On a nonlocal Cahn-Hilliard/Navier-Stokes system with degenerate mobility and singular potential for incompressible fluids with different densities, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 38 (2021) 647–687, https://doi.org/10.1016/j.anihpc.2020.08.005.

[24] S. Frigeri, C.G. Gal, M. Grasselli, On nonlocal Cahn-Hilliard-Navier-Stokes systems in two dimensions, J. Nonlinear Sci. 26 (2016) 847–893, https://doi.org/10.1007/s00332-016-9292-y.

[25] S. Frigeri, C.G. Gal, M. Grasselli, Regularity results for the nonlocal Cahn-Hilliard equation with singular potential and degenerate mobility, J. Differ. Equ. 287 (2021) 295–328, https://doi.org/10.1016/j.jde.2021.03.052.

[26] S. Frigeri, M. Grasselli, Nonlocal Cahn-Hilliard-Navier-Stokes systems with singular potentials, Dyn. Partial Differ. Equ. 9 (2012) 273–304, https://doi.org/10.4310/DPDE.2012.v9.n4.a1.

[27] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, J. Math. Anal. Appl. 286 (2003) 11–31, https://doi.org/10.1016/S0022-247X(02)00425-0.

[28] C.G. Gal, A. Giorgini, M. Grasselli, The nonlocal Cahn-Hilliard equation with singular potential: well-posedness, regularity and strict separation property, J. Differ. Equ. 263 (2017) 5253–5297, https://doi.org/10.1016/j.jde.2017.06.015.

[29] C.G. Gal, A. Giorgini, M. Grasselli, The separation property for 2D Cahn-Hilliard equations: local, nonlocal and fractional energy cases, Discrete Contin. Dyn. Syst. 43 (2023) 2270–2304, https://doi.org/10.3934/dcds.2023010.

[30] C.G. Gal, M. Grasselli, Longtime behavior of nonlocal Cahn-Hilliard equations, Discrete Contin. Dyn. Syst. 34 (2014) 145–179, https://doi.org/10.3934/dcds.2014.34.145.

[31] C.G. Gal, M. Grasselli, H. Wu, Global weak solutions to a diffuse interface model for incompressible two-phase flows with moving contact lines and different densities, Arch. Ration. Mech. Anal. 234 (2019) 1–56, https://doi.org/10.1007/s00205-019-01383-8.

[32] G.P. Galdi, An introduction to the Navier-Stokes initial-boundary value problem, in: Fundamental Directions in Mathematical Fluid Mechanics, in: Adv. Math. Fluid Mech., Birkhäuser, Basel, 2000, pp. 1–70.

[33] G. Giacomin, J.L. Lebowitz, Exact macroscopic description of phase segregation in model alloys with long range interactions, Phys. Rev. Lett. 76 (1996) 1094–1097, https://doi.org/10.1103/PhysRevLett.76.1094.

[34] G. Giacomin, J.L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits, J. Stat. Phys. 87 (1997) 37–61, https://doi.org/10.1007/BF02181479.

[35] G. Giacomin, J.L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions II: Interface motion, SIAM J. Appl. Math. 58 (1998) 1707–1729, https://doi.org/10.1137/S0036139996313046.

[36] M.-H. Giga, A. Kirshtein, C. Liu, Variational modeling and complex fluids, in: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer, Cham, 2018, pp. 73–113.

[37] A. Giorgini, Well-posedness of the two-dimensional Abels-Garcke-Grün model for two-phase flows with unmatched densities, Calc. Var. Partial Differ. Equ. 60 (2021) 100, https://doi.org/10.1007/s00526-021-01962-2.

[38] A. Giorgini, Existence and stability of strong solutions to the Abels-Garcke-Grün model in three dimensions, Interfaces Free Bound. 24 (2022) 565–608, https://doi.org/10.4171/ifb/482.

[39] A. Giorgini, A. Miranville, R. Temam, Uniqueness and regularity for the Navier-Stokes-Cahn-Hilliard system, SIAM J. Math. Anal. 51 (2019) 2535–2574, https://doi.org/10.1137/18M1223459.

[40] A. Giorgini, R. Temam, Weak and strong solutions to the nonhomogeneous incompressible Navier-Stokes-Cahn-Hilliard system, J. Math. Pures Appl. (9) 144 (2020) 194–249, https://doi.org/10.1016/j.matpur.2020.08.009.

[41] M. Grmela, V. Klika, M. Pavelka, Dynamic and renormalization-group extensions of the Landau theory of critical phenomena, Entropy 22 (2020) 978, https://doi.org/10.3390/e22090978.

[42] G. Grün, F. Guillén-González, S. Metzger, On fully decoupled, convergent schemes for diffuse interface models for two-phase flow with general mass densities, Commun. Comput. Phys. 19 (2016) 1473–1502, https://doi.org/10.4208/cicp.scpde14.39s.

[43] M.E. Gurtin, D. Polignone, J. Viñals, Two-phase binary fluids and immiscible fluids described by an order parameter, Math. Models Methods Appl. Sci. 6 (1996) 815–831, https://doi.org/10.1142/S0218202596000341.

[44] J. He, H. Wu, Global well-posedness of a Navier-Stokes-Cahn-Hilliard system with chemotaxis and singular potential in 2D, J. Differ. Equ. 297 (2021) 47–80, https://doi.org/10.1016/j.jde.2021.06.022.

[45] M. Heida, J. Málek, K.R. Rajagopal, On the development and generalizations of Cahn-Hilliard equations within a thermodynamic framework, Z. Angew. Math. Phys. 63 (2012) 145–169, https://doi.org/10.1007/s00033-011-0139-y.

[46] H. Hemmerich, The Diffuse Interface Approach in Material Science, Springer, Berlin, Heidelberg, 2003.

[47] M. Kotschote, R. Zacher, Strong solutions in the dynamical theory of compressible fluid mixtures, Math. Models Methods Appl. Sci. 25 (2015) 1217–1256, https://doi.org/10.1142/S0218202515500311.

[48] P. Krejčí, E. Rocca, J. Sprekels, A nonlocal phase-field model with nonconstant specific heat, Interfaces Free Bound. 9 (2007) 285–306, https://doi.org/10.4171/IFB/165.

[49] S.-O. Londen, H. Petzeltová, Convergence of solutions of a non-local phase-field system, Discrete Contin. Dyn. Syst., Ser. S 4 (2011) 653–670, https://doi.org/10.3934/dcdss.2011.4.653.

[50] S.-O. Londen, H. Petzeltová, Regularity and separation from potential barriers for a non-local phase-field system, J. Math. Anal. Appl. 379 (2011) 724–735, https://doi.org/10.1016/j.jmaa.2011.02.003.

[51] J. Lowengrub, L. Truskinovsky, Quasi-incompressible Cahn-Hilliard fluids and topological transitions, Proc. R. Soc. Lond. Ser. A, Math. Phys. Eng. Sci. 454 (1998) 2617–2654, https://doi.org/10.1098/rspa.1998.0273.

[52] M. Marcus, V.J. Mizel, Absolute continuity on tracks and mappings of Sobolev spaces, Arch. Ration. Mech. Anal. 45 (1972) 294–320, https://doi.org/10.1007/BF00251378.

[53] A. Miranville, S. Zelik, Robust exponential attractors for Cahn-Hilliard type equations with singular potentials, Math. Methods Appl. Sci. 27 (2004) 545–582, https://doi.org/10.1002/mma.464.

[54] A. Poiatti, The 3D strict separation property for the nonlocal Cahn-Hilliard equation with singular potential, https://doi.org/10.48550/arXiv.2303.07745, 2022.

[55] J.S. Rowlinson, Translation of J. D. van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density", J. Stat. Phys. 20 (1979) 197–244, https://doi.org/10.1007/BF01011513.

[56] M. Shokrpour Roudbari, G. Şimşek, E.H. van Brummelen, K.G. van der Zee, Diffuse-interface two-phase flow models with different densities: a new quasi-incompressible form and a linear energy-stable method, Math. Models Methods Appl. Sci. 28 (2018) 733–770, https://doi.org/10.1142/S0218202518500197.

[57] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Mathematical Surveys and Monographs, vol. 49, American Mathematical Society, Providence, RI, 1997.

[58] W.A. Strauss, On continuity of functions with values in various Banach spaces, Pac. J. Math. 19 (1966) 543–551, http://projecteuclid.org/euclid.pjm/1102993723.

[59] Y. Xie, O. Wodo, B. Ganapathysubramanian, A diffuse interface model for incompressible two-phase flow with large density ratios, in: Advances in Computational Fluid-Structure Interaction and Flow Simulation, in: Model. Simul. Sci. Eng. Technol., Birkhäuser/Springer, Cham, 2016, pp. 203–215.