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Full Length Article

Attractors for a fluid-structure interaction problem in a time-dependent phase space [☆]

Filippo Gazzola [a], Vittorino Pata [a], Clara Patriarca [a,b,*]

[a] *Dipartimento di Matematica, Politecnico di Milano, Italy*
[b] *Dipartimento di Scienze Matematiche, Politecnico di Torino, Italy* [1]

A R T I C L E I N F O

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A B S T R A C T

This paper is concerned with the long-time dynamics of a fluid-structure interaction problem describing a Poiseuille inflow through a 2D channel containing a rectangular obstacle. Physically, this models the interaction between the wind and the deck of a bridge in a wind tunnel experiment, as time goes to infinity. Due to this interaction, the fluid domain depends on time in an unknown fashion and the problem needs a delicate functional analytic setting. As a result, the solution operator associated to the system acts on a time-dependent phase space, and it cannot be described in terms of a semigroup nor of a process. Nonetheless, we are able to extend the notion of global attractor to this particular setting, and prove its existence and regularity. This provides a strong characterization of the asymptotic behavior of the problem. Moreover, when the inflow is sufficiently small, the attractor reduces to the unique stationary solution of the system, corresponding to a perfectly symmetric configuration.

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* Corresponding author.
E-mail address: clara.patriarca@polito.it (C. Patriarca).
[1] Current address.

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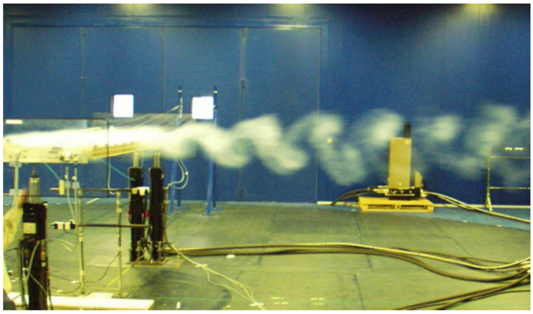

Fig. 1. Wind tunnel experiment at Politecnico di Milano.

1. Introduction

We study the long-time dynamics of a coupled system describing the motion of a fluid in a 2D channel with a rectangular obstacle. We aim at modeling the interaction between the cross-section of the deck of a suspension bridge and the wind as in a wind tunnel experiment where, at the inlet and outlet sections, the velocity field of the fluid has a prescribed parabolic unidirectional profile, called *Poiseuille flow*. See Fig. 1 for a picture taken during a wind tunnel experiment held at Politecnico di Milano. The asymmetry of the flow vortices leeward generates a vertical solicitation (*lift force*) on the plate (deck) [22].

Our analysis is performed on the two-dimensional fluid-structure interaction problem introduced in [5] whose well-posedness has been later established in [44], with the exception that, in the present paper, the channel has finite length. The reason for this choice is that we aim at modeling a wind tunnel (Fig. 1), where the long-time dynamics may also be studied experimentally. The wind tunnel framework allows to introduce several simplifications such as perfect symmetry of the air flow, of the cross-section of the deck, and of the surrounding environment. At the price of much more technical assumptions and proofs, our results may be extended to general asymmetric situations and to the case of more degrees of freedom for the structure, such as torsional movements as in [5] and horizontal translations.

The motion of the fluid is governed by the Navier-Stokes equations. We denote by

$$B = [-d, d] \times [-\delta, \delta]$$

the rectangular rigid body representing the 2D (scaled) cross-section of the deck of a suspension bridge. In fact, also alternative symmetric shapes, such as ellipses, are allowed without altering our results but a rectangular shape is more similar to the cross-section of a bridge. However, the proofs would become much more delicate since the "collision points", that for B coincide with the two segments $[-d, d] \times \{-\delta, \delta\}$, could have very weird behavior (such as union of segments, points, Cantor sets...) and many parameters (curvature, smoothness) would enter into the analysis, see [30,31,35]. Without loss of

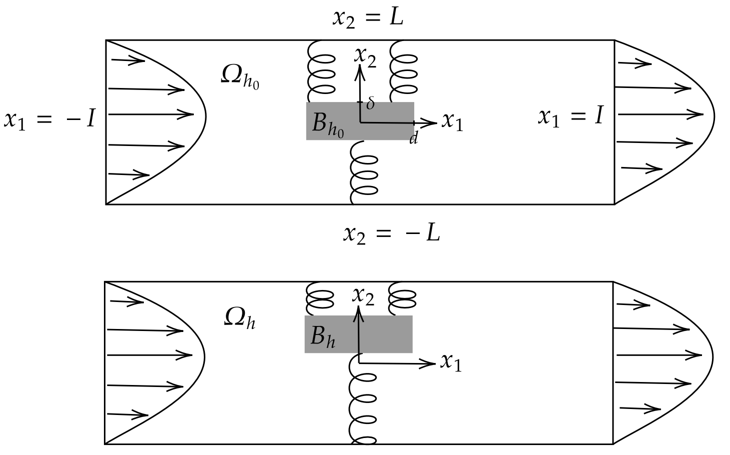

Fig. 2. Channel with the vertically moving obstacle B_h, above at $t = 0$ (taking, e.g., $h_0 = 0$), below at $t > 0$.

generality, we take $d = 1$ so that d becomes the reference length unit. Let $I \gg d = 1$ and $L \gg \delta$ with $\delta < 1$: in particular, the assumption that $I \gg 1$ allows the fluid to "regularize" and recover some symmetry at the outlet section, quite similar to a Poiseuille outflow as wind tunnel experiments seem to confirm. The rigid body is free to move vertically inside the 2D channel (the section of a wind tunnel)

$$\mathcal{R} = (-I, I) \times (-L, L),$$

driven by the action of both a smooth elastic restoring force and the fluid flow, see Fig. 2. The upper and lower boundaries of such a channel are given by $\Gamma = (-I, I) \times \{-L, L\}$, whereas h denotes the vertical displacement of the barycenter of the rigid body from the equilibrium line $x_2 = 0$. Thus,

$$B_h = B + h\hat{e}_2 \quad \forall\, |h| < L - \delta$$

tracks the position of the body after the vertical translation. In particular, when $|h| = L - \delta$ the obstacle collides with Γ. Due to the motion of the rigid body, the domain occupied by the fluid is variable in time and is given by

$$\Omega_h = \mathcal{R} \setminus B_h\,, \qquad \text{where } h = h(t). \tag{1.1}$$

For simplicity, in the sequel we will sometimes omit emphasizing the dependence on $t \in (0, T)$ and, with an abuse of notation, we will denote through a Cartesian product the space-time domain given by

$$\Omega_h \times (0, T) := \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+;\ 0 < t < T,\ x \in \Omega_h = \Omega_{h(t)}\}.$$

We denote by q_λ a stationary Poiseuille flow on the whole channel $\mathcal{R}$ with a prescribed intensity $\lambda > 0$, that is the vector field

$$q_\lambda(x_2) = \lambda(L^2 - x_2^2)\widehat{e}_1 \qquad \forall x_2 \in [-L, L]\,.$$

We observe that q_λ and the associated pressure $\pi_p(x_1, x_2) = -2\mu\lambda\, x_1$, for every $(x_1, x_2) \in \mathcal{R}$, satisfy the steady-state Navier-Stokes equations

$$-\mu\,\Delta q_\lambda + (q_\lambda \cdot \nabla)q_\lambda + \nabla\pi_p = 0\,, \qquad \operatorname{div} q_\lambda = 0 \qquad \text{in } \ \mathcal{R}\,,$$

since $(q_\lambda \cdot \nabla)q_\lambda \equiv 0$ in $\mathcal{R}$. If $m > 0$ is the mass of the body B, and if at the inlet and outlet section of the channel the velocity field reproduces q_λ, the fluid-structure interaction evolution problem on the time-interval is then described by

$$\begin{aligned} &u_t = \mu\,\Delta u - (u\cdot\nabla)u - \nabla p, \quad \operatorname{div} u = 0 \qquad \text{in } \Omega_h \times (0,T),\\ &u = q_\lambda(x_2) \quad \text{on } \partial\mathcal{R}\times(0,T), \qquad u = h'\,\hat{e}_2 \quad \text{on } \partial B_h \times (0,T),\\ &m\,h'' + f(h) = -\hat{e}_2 \cdot \int\limits_{\partial B_h} \mathcal{T}(u,p)\cdot\hat{n}\,d\sigma \qquad \text{in } (0,T), \end{aligned} \tag{1.2}$$

to which we associate the initial conditions $h(0) = h_0$, $h'(0) = k_0$, $u(x,0) = u_0(x)$ in $\Omega_{h_0} = \Omega_{h(0)}$. Here $u : \Omega_h \times (0,T) \to \mathbb{R}^2$ and $p : \Omega_h \times (0,T) \to \mathbb{R}$ are, respectively, the velocity vector field and the scalar pressure, while $\hat{n}$ denotes the outward normal to $\partial\Omega_h$, thus directed towards the interior of ∂B_h. For simplicity the fluid density is normalized to unit, the constant $\mu > 0$ is the (fixed) fluid viscosity, while $\lambda > 0$ in q_λ measures the magnitude of the Poiseuille flow and its variations determine the variations of the Reynolds number. It is understood that λ belongs to a physical range, hence it can not exceed some (possibly very large) value. Since λ remains uniformly bounded, in our estimates all the superlinear powers of λ are controlled by λ times a suitably large constant. The motion of the body is governed by the ODE in (1.2), where $f(h)$ is an elastic smooth restoring force and $\mathcal{T}(u,p)$ is the strain tensor, namely,

$$\mathcal{T}(u,p) = -p\mathbf{I} + 2\mu D(u) \quad \text{with} \quad D(u) = \frac{\nabla u + \nabla^\top u}{2}\,, \tag{1.3}$$

being $\mathbf{I}$ the 2×2-identity matrix, so that the right hand side of the ODE is the lift force exerted by the fluid on the body [28]. Further assumptions on $f(h)$ are given in Section 2.

From a physical (and engineering) point of view, a crucial issue is to prevent structural and areodynamic instabilities [1,2,4,7,10,23], which translates into predicting simple behaviors of the body-fluid system as time goes to infinity. In mathematical terms, this is usually described by means of small (in a suitable sense) subsets of the phase space able to confine the long-time dynamics, namely to substantially reduce the degrees of freedom of the system. To this end, the most effective tool available in the theory of infinite-dimensional dynamical systems is the notion of global attractor [13,15]. But since the fluid domain (1.1) and the phase space for (u, h) are time dependent, for the problem (1.2), the very definition of such an object introduces a major difficulty: there is no way to describe the solutions in terms of a semigroup, and not even in terms of a process in the sense of [13]. For this reason, while well-posedness issues have been abundantly explored (see, e.g., [19,21,33,45]), the existence of a global attractor for the whole fluid-structure interaction problem with a time-dependent fluid domain is extremely delicate, and, to the best of our knowledge, only partial results are known. On the one hand, a part of the literature is devoted to the study of the long-time dynamics of fluid-plate interaction models, see, e.g., [16–18] with a *fixed* fluid domain. On the other hand, the long-time dynamics of the Navier-Stokes equations set on time-varying domains has been studied only when the motion of the domain is prescribed and sufficiently smooth, see [46]; this allows to reformulate the problem on a fixed domain by a coordinate transformation and to apply the techniques for non-autonomous systems, see [11,12,39,42].

Accordingly, one of the main purposes of the present work is to extend the notion of global attractor to cover the case of maps lacking the concatenation property (typical of semigroups or processes), referred to in this paper as semiflows. This allows us to circumvent the main obstruction, leading to a proper definition of global attractor apt to describe the asymptotics of our fluid-structure interaction problem acting on a time-dependent phase space. With this notion at hand, we are able to study the dissipativity properties of (1.2), showing that in the long-time it indeed admits an attractor. As we will see, this is a compact subset of the (variable-in-time) phase space to which all the solutions (u, h) of (1.2) eventually approach. In this respect, the first step is to characterize explicitly the attractor in some particular situation: we will show that if the inflow q_λ is sufficiently small, then the attractor reduces to the unique stationary solution of (1.2).

The symmetric model considered in the present paper illustrates the power of these abstract tools. Not only we believe that our results may be extended to general asymmetric frameworks, but also that semiflows and the related notion of attractor can be used to tackle a much wider class of fluid-structure interaction models with different applications [25,38,43] and to further long-time dynamics evolution problems with variable phase space.

1.1. Plan of the paper

The paper is organized as follows. In Section 2 we introduce the main tools for the analysis of (1.2), and we recall some results about the well-posedness and the existence and uniqueness of equilibrium solutions, the latter holding under smallness assumptions on the flow. In Section 3 we explain why the classical approach does not apply and, particularly, why the description of the dynamics of (1.2) in terms of semigroups or processes seems to be out of reach. In Section 4 we show that, in case of uniqueness, the equilibrium solution is stable. In Section 5 we define what we mean by *semiflow*, and we introduce a time-dependent map which enables us to transform (1.2), which is set in the time-dependent domain (1.1), into a different problem in a *fixed* domain. In Section 6 we state and prove our final result on the existence of a global attractor for (1.2).

2. Weak solutions and well-posedness

Let $D \subset \mathbb{R}^2$ be an open bounded Lipschitz domain. We denote by $L^q(D), q \geq 1$, and $H^m(D), m \geq 0$, respectively the usual Lebesgue and Hilbert spaces with associated norms $\|\cdot\|_{L^q(D)}$ and $\|\cdot\|_{H^m(D)}$, under the convention that $H^0(D) = L^2(D)$.

2.1. Assumptions on the restoring force $\boldsymbol{f = f(h)}$

We assume that

$$f \in C^1(-L+\delta, L-\delta) \quad \text{satisfies} \quad f(0) = 0 \quad \text{and} \quad f'(h) > 0 \quad \forall\, h \in (-L+\delta, L-\delta). \tag{2.1}$$

We point out that the boundary of the channel is somehow artificial in order to restrict to a bounded domain so that our physical model breaks down in case of collision between the obstacle and the boundary. In order to prevent collisions, we require that f be a *strong force*, that is

$$\exists\, r > 0 \qquad \text{s.t.} \qquad \lim_{|h| \to L-\delta} |f(h)| \, \exp\left\{ -\frac{1}{(L-\delta-|h|)^{4+r}} \right\} = +\infty. \tag{2.2}$$

From a mathematical point of view, (2.2) may be probably weakened but it was used in [44] as a *sufficient* condition to avoid collisions and obtain the well-posedness. In any case, here it is not essential to determine the minimal growth condition for f as $|h| \to L - \delta$. The simplest example of function f satisfying (2.2) is

$$f(h) = h \exp \frac{1}{(L-\delta-|h|)^{4+q}},$$

where $q > 0$ (in this case $r = \frac{q}{2}$). Nevertheless, since the restoring force for the deck of a bridge also involves gravity, the function f may not be odd. From (2.1), it follows that $f(h)h > 0$ for all $h \neq 0$ and that there exists ρ such that $f'(h) > \rho > 0$ for all h. Hence, if we put

$$F(h) = \int_0^h f(s)\, ds, \tag{2.3}$$

we obtain

$$f(h)\, h \geq F(h) \geq \frac{\rho}{2} h^2. \tag{2.4}$$

Then, we define the function $M : [0, +\infty) \to (-L+\delta, L-\delta)$ as

$$M(y) := \sup \big\{ |s| : F(s) \leq y \big\}.$$

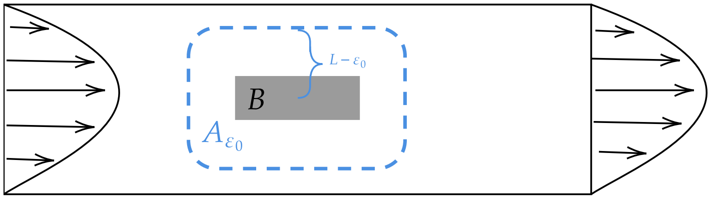

Fig. 3. The smoothened rectangle A_{ε_0}.

Observe that M is a continuous increasing function with $M(0) = 0$. Moreover,

$$\forall\, C \geq 0 \qquad F(h) \leq C \quad \Longrightarrow \quad |h| \leq M(C). \tag{2.5}$$

2.2. Definition of a solenoidal extension

In order to be able to capture the non-homogeneous boundary condition in (1.2), we build a solenoidal extension for the Poiseuille flow, by combining some results from [9,26,28,40]. We need an H^2-solenoidal extension (and not merely H^1) because we need some additional regularity to study the dissipativity properties of our system (1.2). Let $\varepsilon_0 \in (0, L-\delta)$ and consider the "smoothened rectangle"

$$A_{\varepsilon_0} = \left\{(x_1, x_2) \in \mathbb{R}^2\,;\, |x_2| < L - \varepsilon_0, |x_1| < 2 + \sqrt[4]{(L-\varepsilon_0)^4 - x_2^4}\right\}, \tag{2.6}$$

so that $\partial A_{\varepsilon_0} \in C^3$, see Fig. 3. Then, we take the non-simply connected domain

$$\Upsilon_{\varepsilon_0} = \mathcal{R} \setminus \bar{A}_{\varepsilon_0}, \qquad \partial\Upsilon_{\varepsilon_0} = \partial\mathcal{R} \cup \partial A_{\varepsilon_0}, \tag{2.7}$$

and we state

Lemma 2.1. *For any $\eta > 0$, there exist $\varepsilon_0 = \varepsilon_0(\eta)$ and a solenoidal vector field $s = s_{\varepsilon_0} = s_{\varepsilon_0}(\eta)$ such that*

$$s \in H^2(\mathcal{R}) \cap L^\infty(\mathcal{R}), \qquad s = 0 \quad in \quad \bar{A}_{\varepsilon_0}\,, \qquad s = q_\lambda \quad on \quad \partial\mathcal{R}, \tag{2.8}$$

and

$$\left|\int_{\mathcal{R}} (u \cdot \nabla)s \cdot u\, dx\right| \leq \eta\, \|\nabla u\|^2_{L^2(\mathcal{R})} \quad \forall\, u \in H^1_0(\mathcal{R})\,, \tag{2.9}$$

where A_{ε_0} is as in (2.6). Moreover, there exist $c_1, c_2, c_3, c_4 > 0$ such that

$$\begin{aligned} &\|s\|_{L^2(\mathcal{R})} \leq c_1 \lambda \varepsilon_0 e^{2/\varepsilon_0}, \qquad &&\|\nabla s\|_{L^2(\mathcal{R})} \leq c_2 \lambda \varepsilon_0 e^{4/\varepsilon_0}, \\ &\|\Delta s\|_{L^2(\mathcal{R})} \leq c_3 \lambda \varepsilon_0 e^{6/\varepsilon_0} \qquad &&\|s\|_{L^\infty(\mathcal{R})} \leq c_4 \lambda \varepsilon_0 e^{2/\varepsilon_0}. \end{aligned} \tag{2.10}$$

Proof. For any $\varepsilon_0 \in (0, L-\delta)$ to be fixed later, let Υ_{ε_0} be as in (2.7). Consider the Stokes system

$$\begin{cases} -\Delta v + \nabla p = 0 & \text{in } \Upsilon_{\varepsilon_0}, \\ \operatorname{div} v = 0 & \text{in } \Upsilon_{\varepsilon_0}, \\ v = q_\lambda & \text{on } \partial\mathcal{R}, \\ v = 0 & \text{on } \partial A_{\varepsilon_0}. \end{cases} \tag{2.11}$$

Defining weak solutions as in [26, Section IV.1], by [26, Theorem IV.1.1] there exists a unique weak solution $(v, q) \in H^1(\Upsilon_{\varepsilon_0}) \times L^2(\Upsilon_{\varepsilon_0})$ to (2.11) such that

$$\|\nabla v\|_{L^2(\Upsilon_{\varepsilon_0})} + \|p\|_{L^2(\Upsilon_{\varepsilon_0})} \leq c\,\lambda,$$

for some c depending on ε_0. Although $\partial\Upsilon_{\varepsilon_0}$ is not globally of class C^2 (it contains the corners of $\mathcal{R}$), since $\mathcal{R}$ is convex, we may proceed as in [28, Theorem 3.3] to infer that the regularity of the solution can be improved to $(v, q) \in H^2(\Upsilon_{\varepsilon_0}) \times H^1(\Upsilon_{\varepsilon_0})$. Then, we localize the solution of (2.11) in a neighborhood of $\partial\mathcal{R}$. More precisely, let $v = (v_1, v_2)$ be the solution to (2.11), fix $x_0 \in \Upsilon_{\varepsilon_0}$ and let

$$g(x) = \int_{x_0}^{x} (v_1\, dx_2 - v_2\, dx_1) \qquad \forall\, x \in \Upsilon_{\varepsilon_0}$$

be the stream function associated to v (see e.g., [26, Lemma IX.4.1]). As a consequence,

$$v_1 = \frac{\partial g}{\partial x_2}, \qquad v_2 = -\frac{\partial g}{\partial x_1}, \qquad g \in H^3(\Upsilon_{\varepsilon_0}). \tag{2.12}$$

Let $\delta(x) := \operatorname{dist}(x, \partial\mathcal{R})$, $\gamma(\varepsilon_0) := \exp(-1/\varepsilon_0)$, and $\psi_{\varepsilon_0} \in C^\infty(\bar{\Upsilon}_{\varepsilon_0})$ be the cut-off function, introduced in [36] (see also [26, Lemma III.6.2]), satisfying for all $x \in \Upsilon_{\varepsilon_0}$

$$\begin{gathered} |\psi_{\varepsilon_0}(x)| \leq 1, \qquad |\psi_{\varepsilon_0}(x)| = 1 \text{ if } \delta(x) < k_1\gamma^2(\varepsilon_0), \qquad \psi_{\varepsilon_0}(x) = 0 \text{ if } \delta(x) \geq 2\gamma(\varepsilon_0), \\ |\nabla\psi_{\varepsilon_0}(x)| \leq k_2\varepsilon_0/\delta(x), \qquad |D^\alpha\psi_{\varepsilon_0}(x)| \leq k_3\varepsilon_0/\delta^{|\alpha|}(x) \text{ for } |\alpha| \in \{2, 3\}, \end{gathered} \tag{2.13}$$

with $k_1, k_2, k_3 > 0$. We set

$$s = \left(\frac{\partial}{\partial x_2}(g\,\psi_{\varepsilon_0}), -\frac{\partial}{\partial x_1}(g\,\psi_{\varepsilon_0})\right) \quad \text{in } \Upsilon_{\varepsilon_0}, \qquad s = 0 \quad \text{in } \mathcal{R} \setminus \Upsilon_{\varepsilon_0}.$$

From (2.12), we see that s satisfies (2.8). Moreover, we can proceed as in [26, Lemma IX.4.2] to fix $\varepsilon_0 \in (0, L-\delta)$ small enough so as to obtain (2.9). Finally, the estimates in (2.13) imply that

$$|\nabla\psi_{\varepsilon_0}(x)| \leq k_4\varepsilon_0 e^{2/\varepsilon_0}; \qquad |D^2\psi_{\varepsilon_0}| \leq k_5\varepsilon_0 e^{4/\varepsilon_0}; \qquad |D^3\psi_{\varepsilon_0}(x)| \leq k_6\varepsilon_0 e^{6/\varepsilon_0}\,,$$

for all x such that $k_1\gamma^2(\varepsilon_0) < \mathrm{dist}(x, \partial\mathcal{R}) \le 2\gamma(\varepsilon_0)$, for some constants $k_4, k_5, k_6 > 0$. This gives (2.10). $\square$

Remark 2.2. According to the situation, we might not need (2.9). In this case ε_0 can be chosen arbitrarily in the admissible set $(0, L-\delta)$. Except for (2.9), then $s = s_{\varepsilon_0}$ satisfies all other properties stated in Lemma 2.1.

2.3. Steady states

We denote by (u_*, h_*) the steady solutions to problem (1.2), namely the solutions to

$$\begin{aligned} -\mu\,\Delta u_* + (u_* \cdot \nabla) u_* + \nabla p_* = 0, \quad \operatorname{div} u_* = 0 \qquad &\text{in } \Omega_{h_*}, \\ u_* = q_\lambda \quad \text{on } \partial\mathcal{R}, \qquad u_* = 0 \quad &\text{on } \partial B_{h_*}, \end{aligned} \tag{2.14}$$

together with the static fluid-structure interaction condition

$$f(h_*) = -\hat{e}_2 \cdot \int\limits_{\partial B_{h_*}} \mathcal{T}(u_*, p_*) \cdot \hat{n}\, d\sigma. \tag{2.15}$$

Weak solutions $(u_*, h_*) \in H^1(\Omega_{h_*}) \times (-L+\delta, L-\delta)$ to (2.14)-(2.15), whose precise definition is standard [26, Section IX.1], represent equilibrium positions of the body, for a given flow regime of the fluid. In the following theorem we provide a well-posedness result for (2.14)-(2.15).

Theorem 2.3. *Assume that f satisfies (2.1)-(2.2). For any $\lambda > 0$ the problem (2.14)-(2.15) admits a weak solution. Furthermore, there exists $\lambda_* > 0$ such that if $\lambda < \lambda_*$ the problem (2.14)-(2.15) admits a unique weak solution $(u_*, h_*) \in H^1(\Omega_{h_*}) \times (-L+\delta, L-\delta)$ given by $(u_\lambda, 0)$, that is u_*, with $h_* = 0$. Moreover, there exists $C(\lambda) > 0$, with $C(\lambda) \to 0$ as $\lambda \to 0$, such that*

$$\|\nabla u_\lambda\|_{L^2(\Omega_0)} \le C(\lambda).$$

Proof. For any $\tau \in (0, L-\delta)$, build the "smoothened rectangle"

$$\begin{aligned} A_\tau =& (-2,2) \times (-L+\delta+\tau, L) \\ &\cup \{(x_1,x_2) \in \mathbb{R}^2 \,|\, (x_1-2)^4 + (x_2 - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4 < (L - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4, x_1 \ge 2, x_2 \le \tfrac{\delta}{2} + \tfrac{\tau}{2}\} \\ &\cup \{(x_1,x_2) \in \mathbb{R}^2 \,|\, (x_1+2)^4 + (x_2 - \tfrac{\delta}{2} - \tfrac{\tau}{2})^2 < (L - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4, x_1 \le -2, x_2 \le \tfrac{\delta}{2} + \tfrac{\tau}{2}\} \\ &\cup \{(x_1,x_2) \in \mathbb{R}^2 \,|\, (x_1 - 2 - 2L + \delta + \tau)^4 + (x_2 - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4 \\ &\quad < (L - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4, x_1 \ge 2, x_2 > \tfrac{\delta}{2} + \tfrac{\tau}{2}\} \\ &\cup \{(x_1,x_2) \in \mathbb{R}^2 \,|\, (x_1 + 2 + 2L - \delta - \tau)^4 + (x_2 - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4 \\ &\quad < (L - \tfrac{\delta}{2} - \tfrac{\tau}{2})^4, x_1 \le -2, x_2 > \tfrac{\delta}{2} + \tfrac{\tau}{2}\}, \end{aligned}$$

and take the domain

$$\Upsilon_\tau = \mathcal{R}\setminus \bar{A}_\tau, \qquad \partial\Upsilon_\tau = \partial\Upsilon_{\tau,1}\cup\partial\Upsilon_{\tau,2} = \big(\partial\mathcal{R}\setminus\partial A_\tau\big)\cup\big(\partial A_\tau\setminus\{-2\le x_1\le 2\wedge x_2 = L\}\big). \tag{2.16}$$

By symmetry of the problem one can assume that B_{h_*} entirely lies above the horizontal line $x_2 = -L+\delta+\tau$. Then, we can repeat the construction of Lemma 2.1, in which we replace Υ_{ε_0} with Υ_τ as in (2.16) and obtain the existence of a function $s = s_\tau = s_\tau(\eta)$ satisfying (2.8)-(2.9), for any $\eta > 0$.

The proof of existence is similar to the proof of [5, Theorem 1], with some modifications; see also [27] and the revised version in [6]. We define

$$u_* = \hat{u}_* + s.$$

Clearly $\hat{u}_*$ will depend on the particular s chosen, but when we get rid of the solenoidal extension by undoing the change of unknown, we go back to the solution to the original problem u_*. Then, we take as weak formulation of (2.14)-(2.15) the following identity

$$\begin{aligned}\mu\int_{\Omega_{h_*}}\nabla\hat{u}_*\cdot\nabla\phi\,dx &+ \int_{\Omega_{h_*}}(\hat{u}_*\cdot\nabla)\hat{u}_*\cdot\phi\,dx + \int_{\Omega_{h_*}}(\hat{u}_*\cdot\nabla)s\cdot\phi\,dx + \int_{\Omega_{h_*}}(s\cdot\nabla)\hat{u}_*\cdot\phi\,dx\\ &= \int_{\Omega_{h_*}}(s\cdot\nabla)s\cdot\phi\,dx + \int_{\Omega_{h_*}}\nabla s\cdot\nabla\phi\,dx,\end{aligned} \tag{2.17}$$

for any solenoidal test function $\phi\in C_c^\infty(\mathcal{R})$. By a standard Galerkin construction in this context, an apriori bound on $\|\nabla\hat{u}_*\|_{L^2(\Omega_{h_*})}$ is sufficient to have the existence of a weak solution of (2.14)-(2.15) (see for instance [26, Theorem IX.4.1]). Take $\phi = \hat{u}_*$ in (2.17). After using the fact that

$$\int_{\Omega_{h_*}}(\hat{u}_*\cdot\nabla)\hat{u}_*\cdot\hat{u}_*\,dx = \int_{\Omega_{h_*}}(s\cdot\nabla)\hat{u}_*\cdot\hat{u}_*\,dx = 0,$$

we obtain

$$\mu\|\nabla\hat{u}_*\|^2_{L^2(\Omega_{h_*})} + \int_{\Omega_{h_*}}(\hat{u}_*\cdot\nabla)s\cdot\hat{u}_*\,dx = \int_{\Omega_{h_*}}(s\cdot\nabla)s\cdot\hat{u}_*\,dx + \int_{\Omega_{h_*}}\nabla s\cdot\nabla\hat{u}_*\,dx. \tag{2.18}$$

The terms on the right-hand side of (2.18) can then be bounded as

$$\begin{aligned}\int_{\Omega_{h_*}}(s\cdot\nabla)s\cdot\hat{u}_*\,dx &\le \|s\|_{L^4(\Omega_{h_*})}\|\nabla s\|_{L^2(\Omega_{h_*})}\|\hat{u}_*\|_{L^4(\Omega_{h_*})}\\ &\le C\|s\|_{L^4(\Omega_{h_*})}\|\nabla s\|_{L^2(\Omega_{h_*})}\|\nabla\hat{u}_*\|_{L^2(\Omega_{h_*})},\end{aligned}$$

where C is the embedding constant for $H_0^1(\Omega_{h_*}) \subset L^4(\Omega_{h_*})$, and

$$\int_{\Omega_{h_*}} \nabla s \cdot \nabla \hat{u}_* \, dx \leq \|\nabla s\|_{L^2(\Omega_{h_*})} \|\nabla \hat{u}_*\|_{L^2(\Omega_{h_*})}.$$

Finally, by exploiting (2.9) and fixing $\tau \in (0, L-\delta)$ such that η is sufficiently small we obtain the desired uniform bound on $\|\nabla \hat{u}_*\|_{L^2(\Omega_{h_*})}$, which guarantees existence of weak solutions for any value of the parameter λ.

Uniqueness of the solution of (2.17) and its specific form $(u_*, h_*) = (u_\lambda, 0)$ follows by [5, Theorem 1]. $\square$

In [5, Theorem 1], the authors impose a bound both on the Poiseuille flow rate λ and on the Reynolds number $Re = cV/\mu$, where V is a reference speed and $c > 0$ a real constant. In the statement of Theorem 2.3, we joined those two bounds in a unique condition on λ by choosing as reference speed in the Reynolds number precisely the velocity of the Poiseuille flow at the outlets of the channel. As expected, Theorem 2.3 guarantees that the equilibrium position is unique and symmetric, at least for small flow rate of the incoming Poiseuille flow.

To develop our analysis in the subsequent sections, we rewrite problem (2.14)-(2.15) in an equivalent form. For a given $\varepsilon_0 \in (0, L-\delta)$, let

$$s = s_{\varepsilon_0} \tag{2.19}$$

be the function obtained through Lemma 2.1. We emphasize that at this point we are not interested in (2.9), thus we can choose ε_0 arbitrarily in the admissible set $(0, L-\delta)$ (see Remark 2.2). The unique solution $(u_*, h_*) = (u_\lambda, 0)$ to problem (2.14), may be rewritten as

$$(u_*, h_*) = (u_\lambda, 0) = (\hat{u}_\lambda + s, 0).$$

Denoting by

$$\hat{g} := \mu \, \Delta s - (s \cdot \nabla) \, s, \tag{2.20}$$

we have that $(\hat{u}_*, h_*) = (\hat{u}_\lambda, 0)$ satisfies in a weak sense, for any $\lambda < \lambda_*$,

$$\begin{gathered} -\mu \, \Delta \hat{u}_\lambda + (\hat{u}_\lambda \cdot \nabla) \, \hat{u}_\lambda + \nabla p_\lambda + (\hat{u}_\lambda \cdot \nabla) \, s + (s \cdot \nabla) \, \hat{u}_\lambda = \hat{g}, \qquad \operatorname{div} \hat{u}_\lambda = 0 \qquad \text{in } \Omega_0, \\ \hat{u}_\lambda = 0 \quad \text{on } \partial\mathcal{R}, \qquad \hat{u}_\lambda = 0 \quad \text{on } \partial B_0, \end{gathered} \tag{2.21}$$

and

$$0 = f(0) = -\hat{e}_2 \cdot \int_{\partial B_0} \mathcal{T}(\hat{u}_\lambda + s, p_\lambda) \cdot \hat{n} \, d\sigma. \tag{2.22}$$

Finally, we prove a property of the function $\hat{g}$ in (2.20).

Lemma 2.4. *Let $\hat{g}$ be as in (2.20), $s = s_{\varepsilon_0}$ as in Lemma 2.1 for some $\varepsilon_0 \in (0, L-\delta)$. Then $\hat{g} \in L^2(\Omega_h)$ and*

$$\|\hat{g}\|_{L^2(\Omega_h)} \leq \mu \|\Delta s\|_{L^2(\Omega_h)} + \|s\|_{L^4(\Omega_h)} \|\nabla s\|_{L^4(\Omega_h)} \leq c\lambda\, \varepsilon_0 e^{6/\varepsilon_0} \qquad \textit{for some } c > 0\,.$$

Proof. Multiply $\hat{g}$ by $\varphi \in C_c^\infty(\mathcal{R})$ and integrate over Ω_h. We obtain

$$\int_{\Omega_h} \hat{g} \cdot \varphi \, dx = \mu \int_{\Omega_h} \Delta s \cdot \varphi \, dx - \int_{\Omega_h} (s \cdot \nabla) s \cdot \varphi \, dx.$$

We bound the two terms on the right-hand side through the Hölder inequality and we get

$$\left| \int_{\Omega_h} \hat{g} \cdot \varphi \, dx \right| \leq \Big(\mu \|\Delta s\|_{L^2(\Omega_h)} + \|s\|_{L^4(\Omega_h)} \|\nabla s\|_{L^4(\Omega_h)} \Big) \|\varphi\|_{L^2(\Omega_h)} \qquad \forall\, \varphi \in C_c^\infty(\mathcal{R})$$

which, combined with (2.10), proves the statement. □

2.4. Weak solutions to the evolution problem

Notation. The classical functional spaces from fluid mechanics are (see, e.g., [26,49]):

$$\begin{gathered} \mathcal{V}(\mathcal{R}) = \{ v \in C_c^\infty(\mathcal{R}) \,|\, \operatorname{div} v = 0 \, \text{in} \, \mathcal{R} \}, \\ H(\mathcal{R}) = \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\cdot\|_{L^2(\mathcal{R})}, \\ V(\mathcal{R}) = \text{closure of } \mathcal{V} \text{ w.r.t. the norm } \|\nabla \cdot\|_{L^2(\mathcal{R})}. \end{gathered}$$

We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between V and V'. We introduce the product spaces

$$\mathbb{H}(\mathcal{R}) = H(\mathcal{R}) \times \mathbb{R}, \qquad \mathbb{V}(\mathcal{R}) = V(\mathcal{R}) \times \mathbb{R}.$$

In order to define a weak solution to our problem, we also need to define, for every $h \in (-L+\delta, L-\delta)$, the closed subspaces $\mathcal{H}_h \subset \mathbb{H}$ and $\mathcal{H}_h^1 \subset \mathbb{V}$ of *compatible pairs*

$$\mathcal{H}_h = \{ z = (u, l) \in \mathbb{H}(\mathcal{R}) \,|\, u_{B_h} = l\, \hat{e}_2 \}, \qquad \mathcal{H}_h^1 = \{ z = (u, l) \in \mathbb{V}(\mathcal{R}) \,|\, u_{B_h} = l\, \hat{e}_2 \}, \tag{2.23}$$

endowed with the scalar products

$$\langle z_1, z_2 \rangle_{\mathcal{H}_h} = \int_{\Omega_h} u_1 \cdot u_2 \, dx + m\, l_1 l_2, \qquad \langle z_1, z_2 \rangle_{\mathcal{H}_h^1} = \int_{\Omega_h} \nabla u_1 : \nabla u_2 \, dx + m\, l_1 l_2, \tag{2.24}$$

where $z_i = (u_i, l_i)$, and m is the mass of the body B. We call $\|\cdot\|_{\mathcal{H}_h}$ and $\|\cdot\|_{\mathcal{H}_h^1}$ the norms induced by the scalar products in (2.24), and we denote by $\mathcal{H}_h^{-1}$ the dual of $\mathcal{H}_h^1$. The integral in the second formula in (2.24) can be defined on the whole channel $\mathcal{R}$; indeed, $\nabla u_1 = \nabla u_2 = 0$ on B_h, since any element of $\mathcal{H}_h^1$ corresponds to a purely vertical rigid motion on B_h. Recalling (1.3), for all $u_1, u_2 \in V$, we have

$$2\int_{\mathcal{R}} D(u_1) : D(u_2)\, dx = \int_{\mathcal{R}} \nabla u_1 : \nabla u_2\, dx.$$

If $h = h(t)$ is a function from $[0,T]$ to $(-L+\delta, L-\delta)$, we define the following spaces:

$$L^p(0,T;\mathcal{H}^1_{h(t)}) = \left\{ f : [0,T] \to \mathcal{H}^1_{h(t)} \quad \text{s.t.} \quad \|f\|^p_{L^p(0,T;\mathcal{H}^1_{h(t)})} = \int_0^T \|f(t)\|^p_{\mathcal{H}^1_{h(t)}}\, dt < +\infty \right\}$$

for $1 \le p < \infty$, and

$$L^\infty(0,T;\mathcal{H}_{h(t)}) = \left\{ f : [0,T] \to \mathcal{H}_{h(t)} \ \text{s.t.}\ \|f\|_{L^\infty(0,T;\mathcal{H}_{h(t)})} = \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_{\mathcal{H}_{h(t)}} < +\infty \right\}.$$

With the notation at hand, we can move to the evolution problem (1.2). To this end, we assume that

$$h_0 \in [-L+\delta+\hat{\varepsilon}, L-\delta-\hat{\varepsilon}] \qquad \text{and} \qquad (u_0 - q_\lambda, k_0) \in \mathcal{H}_{h_0},$$

where $\hat{\varepsilon} \in (0, L-\delta)$ is arbitrarily fixed. For a given $\varepsilon_0 \in (0, L-\delta)$, let $s = s_{\varepsilon_0}$ be as in Lemma 2.1. The (weak) solutions to the problem (1.2), in the sense of the forthcoming Definition 2.5, have the form

$$u = \hat{u} + s.$$

Again, we point out that $\hat{u}$ depends on the choice of the solenoidal extension s built through Lemma 2.1, but, by undoing the change of variables, one recovers the solution to the original problem. Hence,

$$\text{the solution to the original problem (1.2) does not depend on the solenoidal extension.} \tag{2.25}$$

Given $\hat{g}$ as in (2.20), $\hat{u}$ solves the problem:

$$\begin{aligned}
&\hat{u}_t - \mu\,\Delta\hat{u} + (\hat{u}\cdot\nabla)\,\hat{u} + \nabla p + (\hat{u}\cdot\nabla)\,s + (s\cdot\nabla)\,\hat{u} = \hat{g}, \quad \operatorname{div}\hat{u} = 0 \qquad \text{in } \Omega_h \times (0,T),\\
&\hat{u} = 0 \quad \text{on } \partial\mathcal{R}\times(0,T), \qquad \hat{u} = h'\hat{e}_2 \quad \text{on } \partial B_h \times (0,T),\\
&\hat{u}(x,0) = \hat{u}_0(x) = u_0(x) - s_{\hat{\varepsilon}}(x) \quad \text{for a.e. } x \in \Omega_{h_0}.
\end{aligned} \tag{2.26}$$

According to (1.2), the vertical translation of the obstacle h responds to

$$m\,h'' + f(h) = -\hat{e}_2 \cdot \int_{\partial B_h} \mathcal{T}(\hat{u}+s,p)\cdot \hat{n}\,d\sigma \qquad \text{in } (0,T), \tag{2.27}$$

with some initial conditions $h(0) = h_0$, $h'(0) = k_0$. We observe that $\hat{u}_0 \in L^2(\Omega_{h_0})$ is such that $\hat{u}_0 \cdot \hat{n} = k_0\hat{e}_2 \cdot \hat{n}$ on ∂B_{h_0}. It is worthwhile to emphasize that the knowledge of $h'(t)$ allows to reconstruct the position of the body:

$$B_{h(t)} = B + h(t)\hat{e}_2, \qquad \text{with} \qquad h(t) = h_0 + \int_0^t h'(\tau)\,d\tau. \tag{2.28}$$

We can now define weak solutions to (2.26)-(2.27).

Definition 2.5. A pair $(\hat{u}, h)$ is called a weak solution of (2.26)-(2.27) on $(0,T)$ for all $T > 0$ with initial data $(\hat{u}_0, h_0, k_0)$ if there exists $\varepsilon_0 = \varepsilon_0(\hat{u}_0, h_0, k_0, T) \in (0, L-\delta)$ such that, for $s = s_{\varepsilon_0}$ as in Lemma 2.1,

$$\begin{gathered} h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0]), \\ (\hat{u}, h') \in L^2(0,T;\mathcal{H}^1_{h(t)}) \cap L^\infty(0,T;\mathcal{H}_{h(t)}), \\ \hat{u} \in C([0,T];L^2(\mathcal{R}))\,, \\ (\hat{u}_t, h'') \in L^2(0,T;\mathcal{H}^{-1}_{h(t)}), \end{gathered}$$

and the pair $(\hat{u}(t), h(t))$ verifies, for any $(\phi(t), l(t)) \in \mathcal{H}^1_{h(t)}$ and almost every $t \in (0,T)$,

$$\begin{aligned} &\langle \hat{u}_t(t), \phi(t)\rangle + m\,h''(t)l(t) + f(h(t))\,l(t) + \mu \int_{\mathcal{R}} \nabla\hat{u}(t) : \nabla\phi(t)\,dx \\ &+ \int_{\Omega_h} (\hat{u}(t)\cdot\nabla)\hat{u}(t)\cdot\phi(t)\,dx + \int_{\Omega_h} (\hat{u}(t)\cdot\nabla)s\cdot\phi(t)\,dx + \int_{\Omega_h} (s\cdot\nabla)\hat{u}(t)\cdot\phi(t)\,dx \\ &= \int_{\Omega_h} \hat{g}\cdot\phi(t)\,dx, \end{aligned} \tag{2.29}$$

and $\hat{u}(0) = \hat{u}_0$, $h(0) = h_0$, $h'(0) = k_0$.

Remark 2.6. The requirement $h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0])$ makes Definition 2.5 consistent: it ensures that no collision occurs between the obstacle and the boundary of the channel because there exists a separation strip of size $\varepsilon_0 \in (0, L-\delta)$ for all times, by which one is allowed to build the solenoidal extension s as in Lemma 2.1 precisely by choosing such an ε_0. As the no-collision result is a non trivial issue, it will be recalled explicitly in Corollary 2.9. Also, we point out that the test functions depend on time and on the solution of the problem itself.

Let us show that any classical solution to (2.26)-(2.27) is a weak solution according to Definition 2.5. Incidentally, this also confirms (2.25).

Proposition 2.7. *Let $\hat{g}$ be as in (2.20). If a pair $(\hat{u}, h)$ is a classical solution to (2.26)-(2.27) such that $|h(t)| \leq L - \delta - \varepsilon_0$ for all $t \in [0, T]$ for some $\varepsilon_0 \in (0, L - \delta)$, then it satisfies (2.29) for all $t \in [0, T]$ and for every pair of test functions $(\phi(t), l(t)) \in \mathcal{H}^1_{h(t)}$.*

Proof. In order to obtain (2.29), we choose a test pair $(\phi(t), l(t)) \in \mathcal{H}^1_{h(t)}$. We multiply the first equation in (2.26) by ϕ and integrate by parts over Ω_h. All terms may be treated in a standard manner (see, e.g., [24]). Though, a particular attention must be devoted to the diffusive and pressure terms. Indeed

$$\begin{aligned}
\int_{\Omega_h} (-\mu\Delta\hat{u} + \nabla p) \cdot \phi \, dx &= \int_{\Omega_h} \operatorname{div}\mathcal{T}(\hat{u}, p) \cdot \phi \, dx = -\int_{\partial B_h} (\mathcal{T} \cdot \hat{n}) \cdot \phi \, d\sigma + \int_{\Omega_h} \mathcal{T} : \nabla\phi \, dx \\
&= -l\hat{e}_2 \cdot \int_{\partial B_h} (\mathcal{T}(\hat{u}, p) \cdot \hat{n}) \, d\sigma + \int_{\Omega_h} \mathcal{T}(\hat{u}, p) : \nabla\phi \, dx \\
&= (m\, h'' + f(h))\, l + \mu \int_{\Omega_h} \nabla\hat{u} : \nabla\phi \, dx \\
&= (m\, h'' + f(h))\, l + \mu \int_{\mathcal{R}} \nabla\hat{u} : \nabla\phi \, dx,
\end{aligned}$$

where the last equality holds because $\nabla\phi = 0$ on B_h. Thus, we obtain the weak formulation (2.29). $\square$

2.5. Well-posedness of the evolution problem

We provide the following well-posedness result for (2.26)-(2.27).

Theorem 2.8. *Let $\hat{\varepsilon} \in (0, L - \delta)$ be fixed. For any initial data $h_0 \in [-L + \delta + \hat{\varepsilon}, L - \delta - \hat{\varepsilon}]$, $(\hat{u}_0, k_0)$ in the space $\mathcal{H}_{h_0}$ and every $T > 0$, there exists a unique weak solution $(\hat{u}, h)$ to (2.26)-(2.27), in the sense of Definition 2.5, for some $\varepsilon_0 = \varepsilon_0(h_0, \hat{u}_0, k_0, T) \in (0, L - \delta)$. Moreover, it satisfies the following energy estimate, for every $t \in [0, T]$,*

$$\begin{aligned}
&\|\hat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + m h'(t)^2 + 2F(h(t)) + \mu \int_0^t \|\nabla\hat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})} d\tau \\
&\leq \|\hat{u}_0\|^2_{L^2(\Omega_{h_0})} + m k_0{}^2 + 2F(h_0) + \frac{2}{\mu}\frac{4L^2}{\pi^2}\|\hat{g}\|^2_{L^2(\Omega_h)} t.
\end{aligned} \tag{2.30}$$

Proof. Existence and uniqueness of a weak solution of (2.26)-(2.27) follows from [44], where existence is obtained exploiting a penalization method, while uniqueness is based on the introduction of deformation mappings for the fluid domain allowing to define two different weak solutions on the same domain (see also [19,32]). In order to prove (2.30), we start formally taking $(\phi, l) = (\hat{u}, h')$ in (2.29). We obtain, by the Reynolds Transport Theorem (see, e.g., [34]),

$$\frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2_{L^2(\Omega_h)} + m\,h''h' + f(h)h' + \mu\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} = \int\limits_{\Omega_h} \hat{g}\cdot\hat{u}\,dx - \int\limits_{\Omega_h} (\hat{u}\cdot\nabla)s\cdot\hat{u}\,dx. \tag{2.31}$$

We seek bounds for the two terms on the right-hand side. The first term is bounded by Lemma 2.4, the Hölder inequality, the Young inequality and the Poincaré inequality (with constant $4L^2/\pi^2$). This gives

$$\left|\int\limits_{\Omega_h} \hat{g}\cdot\hat{u}\,dx\right| \le \frac{1}{\mu}\frac{4L^2}{\pi^2}\|\hat{g}\|^2_{L^2(\Omega_h)} + \frac{\mu}{4}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}.$$

In order to bound the second term, by possibly taking ε_0 smaller than $\varepsilon_0 = \varepsilon_0(h_0, \hat{u}_0, k_0, T)$ given in the first statement of the theorem, we exploit (2.9) with $\eta = \mu/4$ and we obtain

$$\left|\int\limits_{\Omega_h} (\hat{u}\cdot\nabla)s\cdot\hat{u}\,dx\right| \le \frac{\mu}{4}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}.$$

From (2.31) and the above bounds we deduce

$$\frac{d}{dt}\Big(\|\hat{u}\|^2_{L^2(\Omega_h)} + m{h'}^2 + 2F(h)\Big) + \mu\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \le \frac{2}{\mu}\frac{4L^2}{\pi^2}\|\hat{g}\|^2_{L^2(\Omega_h)},$$

where $F(h)$ is defined as in (2.3). Integrating on $(0, t)$, we infer (2.30), at least at the formal level. In order to obtain the strong continuity in time of the weak solution with values in $L^2(\mathcal{R})$ and make the previous computations rigorous we can follow step by step [8, Theorem 2.1]. $\square$

In the sequel, we will use the following result, already implicitly stated in Theorem 2.8. It reads as

Corollary 2.9. *For all $T > 0$, there exists $\varepsilon_0 = \varepsilon_0(h_0, \hat{u}_0, k_0, T) \in (0, L - \delta)$ such that*

$$|h(t)| \le L - \delta - \varepsilon_0 \qquad \forall\, t \in [0, T]. \tag{2.32}$$

Moreover, ε_0 decreases as $|h_0|, \|\hat{u}_0\|_{L^2(\Omega_0)}, |k_0|$ and T increase. In particular, the solution to (1.2) is global in time.

Proof. To prove (2.32), we can proceed as in [44, Lemma 3.2]. By contradiction, if the solution of (1.2) was not global in time, then a collision would occur at some finite time $t = T$. But, according to (2.25), the collision is independent of the solenoidal extension and we reach a contradiction by taking $s = s_{\varepsilon_0}$ with $\varepsilon_0 = \varepsilon_0(T)$ ensuring (2.32). $\square$

3. Dissipativity of the solution operator

From Theorem 2.8 we learn that, for every $\hat{\varepsilon} > 0$ small and every $h_0 \in [-L + \delta + \hat{\varepsilon}, L - \delta - \hat{\varepsilon}]$, problem (2.26)-(2.27) generates an operator

$$U(t) : \mathcal{H}_{h_0} \longrightarrow \mathcal{H}_{h(t)},$$

defined by the rule

$$z_0 = (\hat{u}_0, k_0) \longmapsto U(t)z_0 = (\hat{u}(t), h'(t)), \tag{3.1}$$

where, reconstructing h as in (2.28), the pair $(\hat{u}(t), h(t))$ is the unique weak solution at time t to problem (2.26)-(2.27) with initial data

$$\hat{u}(0) = \hat{u}_0, \quad h(0) = h_0, \quad h'(0) = k_0.$$

Remark 3.1. It is worth noting that, although not explicitly written so not to burden the notation, the operator $U(t)$ depends on the particular h_0 chosen. Besides, it acts between different spaces; but this reflects the nature of the fluid-structure interaction problem (2.26)-(2.27), where the functional framework is influenced by the evolution itself.

Let us introduce the proper notion of dissipativity for the dynamical system under consideration. This definition makes sense because all the solutions of (2.26)-(2.27) are global in view of Theorem 2.8 and Corollary 2.9.

Definition 3.2. We call $R_0 > 0$ a *zero-order absorbing radius* if, for any $R > 0$ and $\hat{\varepsilon} > 0$, there exists $t_0 = t_0(R, \hat{\varepsilon})$, called *entering time*, such that, for every

$$h_0 \in [-L + \delta + \hat{\varepsilon}, L - \delta - \hat{\varepsilon}] \qquad \text{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \le R$$

it follows that

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} = \left[\|\hat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2\right]^{\frac{1}{2}} \le R_0 \qquad \forall\, t \ge t_0.$$

We call $R_1 > 0$ a *first-order absorbing radius* if, under the same assumptions, there exists $t_1 = t_1(R, \hat{\varepsilon})$, such that

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} = \left[\|\nabla\hat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2\right]^{\frac{1}{2}} \le R_1 \qquad \forall\, t \ge t_1.$$

Paralleling the classical definition, we may say that the solution operator is dissipative if it admits a zero-order absorbing radius. We now address the dissipativity properties of $U(t)$ in terms of zero-order and first-order absorptions.

Theorem 3.3. *There exists a universal constant $R_0 = R_0(\lambda) > 0$ with the following property: for any $R > 0$ and any $\hat{\varepsilon} \in (0, L-\delta)$, there is an entering time $t_0 = t_0(R, \hat{\varepsilon})$ such that*

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} \leq R_0 \qquad \forall\, t \geq t_0,$$

whenever

$$h_0 \in [-L+\delta+\hat{\varepsilon}, L-\delta-\hat{\varepsilon}] \qquad \text{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \leq R.$$

In compliance with Definition 3.2, the constant R_0 is a zero-order absorbing radius. Moreover, $R_0 \to 0$ as $\lambda \to 0$.

Proof. Define a solenoidal vector field $w \in C^\infty(\mathcal{R}) \cap H^1_0(\mathcal{R})$ such that $w(t) = h(t)\,\hat{e}_2$ in $B_{h(t)}$ as

$$w(x,t) = h(t)\left[-\frac{\partial}{\partial x_2}(\zeta(x)\,x_1), \frac{\partial}{\partial x_1}(\zeta(x)\,x_1)\right] \qquad \forall (x,t) \in \mathcal{R} \times (0,\infty)\,,$$

where, for any $\varepsilon \in (0, L-\delta)$, the function ζ is a C^∞-cut-off equal to 1 in the rectangle $[-2,2] \times [L-\varepsilon, -L+\varepsilon]$ and equal to 0 outside a larger rectangle, both contained in $\mathcal{R}$. The following estimates hold

$$\begin{aligned}
&\|w(t)\|_{L^2(\mathcal{R})} \leq a_1|h(t)|\,, \quad \|\nabla w(t)\|_{L^2(\mathcal{R})} \leq a_2|h(t)|\,,\\
&\|\nabla w(t)\|_{L^\infty(\mathcal{R})} \leq a_3|h(t)|\,, \quad \|w_t(t)\|_{L^2(\mathcal{R})} \leq a_4|h'(t)|\,,
\end{aligned} \tag{3.2}$$

where a_1, a_2, a_3 and a_4 are constants depending on the cut-off ζ. We emphasize that this cut-off function ζ is well-defined due to Corollary 2.9, which guarantees the existence of a separation strip of width ε_0 between the obstacle and the channel. Hence

$$w(x,t) = w_{\varepsilon_0}(x,t)\,. \tag{3.3}$$

Given F as in (2.3), for $\omega \in (0,1)$ to be fixed later, we introduce the energy functionals:

$$\begin{aligned}
E(t) &= \|\hat{u}(t)\|^2_{L^2(\Omega_{h(t)})} + mh'(t)^2 + 2F(h(t)),\\
E_\omega(t) &= E(t) + 2m\,\omega\, h(t)\, h'(t) + 2\int\limits_{\Omega_{h(t)}} \hat{u}(t)\cdot \omega w(t)\, dx.
\end{aligned}$$

From the Young inequality, we have

$$
\begin{aligned}
&2m\omega|h(t)h'(t)| + 2\omega \left| \int_{\Omega_{h(t)}} \hat{u}(t)\cdot w(t)\,dx \right| \\
&\qquad \le \frac{m}{2}h'(t)^2 + \left(2m\omega^2 + 2a_1^2\omega^2\right)h(t)^2 + \frac{1}{2}\|\hat{u}(t)\|^2_{L^2(\Omega_h)}\,.
\end{aligned}
$$

Hence, we obtain the bounds

$$
c_1\,E \le E_\omega \le c_2\,E, \tag{3.4}
$$

for some $c_2 > c_1 > 0$, provided that ω is small enough. So far, we have used an arbitrary $\varepsilon_0 \in (0, L-\delta)$ to build $s = s_{\varepsilon_0}$ in Lemma 2.1, but in view of (2.25), ε_0 may be modified. To this end, we claim that

$$
\exists\,\varepsilon_1 = \varepsilon_1(\lambda) > 0 \quad \text{and} \quad t_0 = t_0(R,\hat{\varepsilon}) \qquad \text{s.t.} \qquad |h(t)| \le L - \delta - \varepsilon_1 \qquad \forall\, t \ge t_0\,. \tag{3.5}
$$

By contradiction, suppose that (3.5) does not hold. Then, by Corollary 2.9, this implies that

$$
\limsup_{t\to\infty} h(t) = L - \delta
$$

and/or similarly for liminf $= -L+\delta$. Then, since $h \in C(\mathbb{R}_+)$, for all $\varepsilon > 0$ there exist two sequences tending to infinity $\{t_1^n\}$ and $\{t_2^n\}$ such that

$$
h(t) \ge L - \delta - \varepsilon \qquad \forall t \in \bigcup_{n=1}^{\infty} [t_1^n, t_2^n]\,.
$$

By (2.2), we can take $\varepsilon > 0$ small enough so that there exists $c > 0$ such that

$$
\begin{aligned}
F\big(h(t)\big) &= \int_0^{h(t)} f(s)ds > \int_{L-\delta-2\varepsilon}^{L-\delta-\varepsilon} f(s)ds > c \int_{L-\delta-2\varepsilon}^{L-\delta-\varepsilon} \exp\frac{1}{(L-\delta-s)^{4+r}}ds \\
&= c\int_{\varepsilon}^{2\varepsilon} \exp\frac{1}{\tau^{4+r}}d\tau \ge c\varepsilon \exp\frac{1}{\varepsilon^{4+r}}
\end{aligned} \tag{3.6}
$$

whenever $t \in \cup_n[t_1^n, t_2^n]$. By (2.25), we can replace the solenoidal extension $s = s_{\varepsilon_0}$ in Lemma 2.1 to the solenoidal extension $s = s_\varepsilon$ and $w = w_\varepsilon$ in (3.3). Given a pair of test functions $(\phi, l) = (\hat{u} + \omega w, h' + \omega h) \in \mathcal{H}_h^1$, we take the scalar product of (2.26) with ϕ and we apply the Reynolds Transport Theorem. By omitting (t), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_\omega - m\omega h'^2 + \omega f(h)h + \mu\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \\
&= \int_{\Omega_h} \hat{g}\cdot\hat{u}\,dx - \int_{\Omega_h} (\hat{u}\cdot\nabla)s\cdot\hat{u}\,dx + \int_{\Omega_h} \hat{u}\cdot\omega w_t\,dx
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
&- \mu \int_{\Omega_h} \nabla \hat{u} : \omega \nabla w \, dx + \int_{\Omega_h} \hat{g} \cdot \omega w \, dx - \int_{\Omega_h} (\hat{u} \cdot \nabla) s \cdot \omega w \, dx \\
&+ \int_{\Omega_h} (\hat{u} \cdot \omega \nabla) w \cdot \hat{u} \, dx - \int_{\Omega_h} (s \cdot \nabla) \hat{u} \cdot \omega w \, dx.
\end{aligned}
$$

We proceed to bound each term on the right-hand side of (3.7). The first term can be controlled by exploiting the Hölder inequality, Lemma 2.4 above, the Young inequality, and the Poincaré inequality. This gives

$$
\int_{\Omega_h} \hat{g} \cdot \hat{u} \, dx \leq \|\hat{g}\|_{L^2(\Omega_h)} \|\hat{u}\|_{L^2(\Omega_h)} \leq \frac{5}{2\mu} \frac{4L^2}{\pi^2} \|\hat{g}\|^2_{L^2(\Omega_h)} + \frac{\mu}{10} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)}.
$$

Similarly for the fifth term

$$
\int_{\Omega_h} \hat{g} \cdot \omega w \, dx \leq \|\hat{g}\|_{L^2(\Omega_h)} \|w\|_{L^2(\Omega_h)} \leq \frac{\mu}{4} \frac{4L^2}{\pi^2} \|\hat{g}\|^2_{L^2(\Omega_h)} + \frac{1}{\mu} \omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})}.
$$

Concerning the second term in the right-hand side, by possibly taking ε smaller, we make use of (2.9) with $\eta = \mu/6$, to get

$$
- \int_{\Omega_h} (\hat{u} \cdot \nabla) s \cdot \hat{u} \, dx \leq \frac{\mu}{10} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)}.
$$

For the third term, we make use of the Hölder inequality, the Poincaré inequality and the Young inequality. We deduce

$$
\int_{\Omega_h} \hat{u} \cdot \omega w_t \, dx \leq \frac{\mu}{10} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} + \frac{5}{2\mu} \frac{4L^2}{\pi^2} \omega^2 \|w_t\|^2_{L^2(\mathcal{R})} .
$$

The fourth term is bounded through the Hölder inequality and the Young inequality as

$$
-\mu \int_{\Omega_h} \nabla \hat{u} : \omega \nabla w \, dx \leq \frac{\mu}{10} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} + \frac{5}{2\mu} \omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})}.
$$

Again, for the sixth term, by possibly taking ε smaller, we exploit (2.9) with $\eta = 1$, the Hölder inequality and the Young inequality. We obtain

$$
- \int_{\Omega_h} (\hat{u} \cdot \nabla) s \cdot \omega w \, dx \leq \omega \eta \|\nabla \hat{u}\|_{L^2(\Omega_h)} \|\nabla w\|_{L^2(\Omega_h)} \leq \frac{\mu}{10} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} + \frac{5}{2\mu} \omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})} .
$$

Finally, by similar methods, we estimate the last two terms as

$$
\int_{\Omega_h} (\hat{u} \cdot \omega \nabla) w \cdot \hat{u} \, dx \leq \omega \frac{4L^2}{\pi^2} \|\nabla w\|_{L^\infty(\mathcal{R})} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} ,
$$

and

$$
\begin{aligned}
-\int_{\Omega_h} (s \cdot \nabla)\hat{u} \cdot \omega w \, dx &\leq \|s\|_{L^\infty(\Omega_h)} \|\nabla \hat{u}\|_{L^2(\Omega_h)} \omega \|w\|_{L^2(\Omega_h)} \\
&\leq \frac{1}{4}\omega^2 \|w\|^2_{L^2(\mathcal{R})} \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} + \|s\|^2_{L^\infty(\Omega_h)} .
\end{aligned}
$$

At this point, by (3.2) and using the fact that $|h| \leq L - \delta$, we define ν through

$$
\begin{aligned}
\frac{\mu}{2} - \omega \frac{4L^2}{\pi^2} \|\nabla w\|_{L^\infty(\mathcal{R})} - \frac{1}{4}\omega^2 \|w\|^2_{L^2(\mathcal{R})} &\geq \mu - a_3 \omega |h| \frac{4L^2}{\pi^2} - \frac{a_1^2}{4}\omega^2 |h|^2 \\
&\geq \mu - a_3 \omega \left(L - \delta\right) \frac{4L^2}{\pi^2} - \frac{a_1^2}{4}\omega^2 (L-\delta)^2 \\
&=: 3\nu > 0
\end{aligned}
$$

if ω is small enough. Inserting all the above inequalities in (3.7), and recalling (2.4), we arrive at

$$
\begin{aligned}
\frac{d}{dt} E_\omega - 2m\omega {h'}^2 + \omega F(h) + 3\nu \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} \leq & \frac{10+\mu^2}{2\mu} \frac{4L^2}{\pi^2} \|\hat{g}\|^2_{L^2(\Omega_h)} + 2\|s\|^2_{L^\infty(\Omega_h)} \\
& + \frac{12}{\mu}\omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})} \\
& + \frac{5}{\mu}\frac{4L^2}{\pi^2}\omega^2 \|w_t\|^2_{L^2(\mathcal{R})} .
\end{aligned}
$$

We apply the following trace inequality, through which we extract a damping term for the obstacle B_h:

$$
\|\nabla \hat{u}\|_{L^2(\Omega_h)} \geq c\|h' \hat{e}_2\|_{L^2(\partial B_h)} = c|\partial B_h||h'|,
$$

for some positive constant c. Moreover we use (3.2) to deduce that

$$
\omega F(h) - \frac{12}{\mu}\omega^2 \|\nabla w\|^2_{L^2(\mathcal{R})} \geq (\omega - a_2^2 \frac{12}{\mu}\omega^2)\, h^2 \geq c\, h^2 ,
$$

where c is a positive constant, if ω is small enough. Thus, using the last estimate in (3.2), we infer

$$
\begin{aligned}
& \frac{d}{dt} E_\omega + (c\nu |\partial B_h|^2 - 2m\omega - a_4^2 \frac{5}{\mu}\frac{4L^2}{\pi^2}\omega^2) {h'}^2 + \omega F(h) + 2\nu \|\nabla \hat{u}\|^2_{L^2(\Omega_h)} \\
& \leq \frac{10+\mu^2}{2\mu} \frac{4L^2}{\pi^2} \|\hat{g}\|^2_{L^2(\Omega_h)} + 2\|s\|^2_{L^\infty(\Omega_h)},
\end{aligned}
$$

where $c\nu|\partial B_h|^2 - 2m\omega - a_4^2 \frac{3}{\mu} \frac{4L^2}{\pi^2}\omega^2 > 0$ if ω is small enough. Finally, applying the Poincaré inequality in the left-hand side, we find

$$\frac{d}{dt}E_\omega + (c\nu|\partial B_h|^2 - 2m\omega - a_4^2\frac{5}{\mu}\frac{4L^2}{\pi^2}\omega^2){h'}^2 + \omega F(h) + \frac{\nu\pi}{4L^2}\|\hat{u}\|^2_{L^2(\Omega_h)} + \nu\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}$$
$$\leq \frac{10+\mu^2}{2\mu}\frac{4L^2}{\pi^2}\|\hat{g}\|^2_{L^2(\Omega_h)} + 2\|s\|^2_{L^\infty(\Omega_h)}.$$

Defining

$$\beta = \min\left\{\frac{c\nu|\partial B_h|^2 - 2m\omega - a_4^2\frac{3}{\mu}\frac{4L^2}{\pi^2}\omega^2}{m}, \frac{\omega}{2}, \frac{\nu\pi}{4L^2}\right\} > 0,$$

we end up with

$$\frac{d}{dt}E_\omega + \beta E + \nu\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \leq \varkappa, \tag{3.8}$$

where

$$\varkappa = \frac{10+\mu^2}{2\mu}\frac{4L^2}{\pi^2}\|\hat{g}\|^2_{L^2(\Omega_h)} + 2\|s\|^2_{L^\infty(\Omega_h)} \tag{3.9}$$

is independent of t due to the very construction of $\hat{g}$ and s. Then, renaming β/c_2 as β, we infer from (3.4) that

$$\frac{d}{dt}E_\omega + \beta E_\omega \leq \varkappa.$$

The Gronwall Lemma yields

$$E_\omega(t) \leq E_\omega(0)e^{-\beta t} + \frac{\varkappa}{\beta}.$$

Inequalities (3.4) imply that

$$\|U(t)z_0\|^2_{\mathcal{H}_{h(t)}} \leq E(t) \leq \frac{c_2}{c_1}\,E(0)e^{-\beta t} + \frac{\varkappa}{\beta c_1}. \tag{3.10}$$

Therefore,

$$\limsup_{t\to\infty}\|U(t)z_0\|_{\mathcal{H}_{h(t)}} \leq \sqrt{\frac{\varkappa}{\beta c_1}}. \tag{3.11}$$

Then, by using (3.6)-(3.10), Lemma 2.1 and Lemma 2.4, we deduce that there exists $c_3 > 0$ such that

$$2\,c\,\varepsilon\,\exp\frac{1}{\varepsilon^{4+r}} \leq 2F\big(h(t)\big) \leq E(t) \leq \frac{c_2}{c_1}\,E(0)e^{-\beta t} + \frac{\varkappa}{\beta c_1} \leq \frac{c_2}{c_1}\,E(0)e^{-\beta t} + c_3\lambda\varepsilon^2\exp\frac{12}{\varepsilon}. \tag{3.12}$$

Choose $\varepsilon > 0$ small enough in such a way that

$$c\,\varepsilon\,\exp\frac{1}{\varepsilon^{4+r}} > c_3\lambda\varepsilon^2\exp\frac{12}{\varepsilon}\,.$$

Then take $t_0 = t_0(R, \hat{\varepsilon})$ such that

$$\frac{c_2}{c_1}E(0)e^{-\beta t} < c_3\lambda\varepsilon^2\exp\frac{12}{\varepsilon} \qquad \forall\, t \geq t_0\,.$$

With these two choices, we contradict (3.12) and we prove (3.5).

We modify once more the solenoidal extension s in Lemma 2.1 and the function w in (3.3) by taking $s = s_{\varepsilon_1}$, $w = w_{\varepsilon_1}$, with ε_1 given by (3.5). This is allowed thanks to (2.25). With this choice (possibly taking ε_1 smaller), we reach again (3.11), which tells us that any ball in $\mathcal{H}_h$ of radius

$$R_0 > \hat{R} := \sqrt{\frac{\varkappa}{\beta c_1}}$$

is absorbing, namely, it captures the dynamics of (2.26)-(2.27) for t large. This translates into the existence of the zero-order absorbing radius R_0 in the sense of Definition 3.2. Summarizing, for any $h_0 \in [-L + \delta + \hat{\varepsilon}, L - \delta - \hat{\varepsilon}]$, given $z_0 = (\hat{u}_0, h_0)$ such that $\|z_0\|_{\mathcal{H}_{h_0}} \leq R$, we have

$$\|U(t)z_0\|_{\mathcal{H}_h} \leq \sqrt{E(t)} \leq R_0, \qquad \forall\, t \geq t_0 = t_0(R, \hat{\varepsilon}) := \frac{1}{\beta}\log\left(\min\left\{1, \frac{c_2}{c_1}\frac{E(0)}{R_0^2 - \hat{R}^2}\right\}\right). \tag{3.13}$$

The proof is finished once we observe that $\varkappa \to 0$ as $\lambda \to 0$. Indeed, from Lemmas 2.1 and 2.4 both $\|s\|_{L^\infty(\Omega_h)} \to 0$ and $\|\hat{g}\|_{L^2(\Omega_h)} \to 0$ as $\lambda \to 0$. $\square$

Note that (3.5) improves Corollary 2.9 by emphasizing a separation strip between the obstacle and the boundary of the channel. Accordingly, throughout the whole section, we take s as in (2.19) by choosing the value ε_1 given by (3.5), i.e. we take $s = s_{\varepsilon_1}$. A consequence of Theorem 3.3 is the existence of a suitable dissipation integral for the solution $(\hat{u}, h)$ to problem (2.26)-(2.27).

Corollary 3.4. *Let $R > 0$ and $\hat{\varepsilon} > 0$ small be arbitrarily given, t_0 as in (3.13) and R_0 as in Theorem 3.3. Assume that $h_0 \in [-L+\delta+\hat{\varepsilon}, L-\delta-\hat{\varepsilon}]$ and $\|z_0\|_{\mathcal{H}_{h_0}} \leq R$. There exists $D = D(R_0) > 0$ such that*

$$\int_t^{t+1} \|\nabla\hat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\,d\tau \leq D \qquad \forall\, t \geq t_0.$$

Moreover, $D \to 0$ as $\lambda \to 0$.

Proof. Integrating inequality (3.8) (recalling (3.9)) on the time-interval $(t, t+1)$, and exploiting (3.4), we get

$$\nu \int_t^{t+1} \|\nabla \hat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\, d\tau \leq \varkappa + E_\omega(t) \leq \varkappa + c_2\, E(t).$$

Hence, in light of (3.13), for $t \geq t_0$ we are led to

$$\int_t^{t+1} \|\nabla \hat{u}(\tau)\|^2_{L^2(\Omega_{h(\tau)})}\, d\tau \leq \frac{\varkappa}{\nu} + \frac{c_2}{\nu} R_0^2.$$

Since we know that $\varkappa, R_0 \to 0$ as $\lambda \to 0$, we are done. $\square$

We have now all the ingredients to proceed to show the existence of the first-order absorbing radius, ensuring dissipativity of higher-order.

Theorem 3.5. *There exists a universal constant $R_1 = R_1(\lambda, L, \delta, d, m, \mu) > 0$ with the following property: for any $R > 0$, and any $\hat{\varepsilon} > 0$ small, it follows that*

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} \leq R_1 \qquad \forall t \geq t_0 + 1,$$

whenever

$$h_0 \in [-L + \delta + \hat{\varepsilon}, L - \delta - \hat{\varepsilon}] \qquad \text{and} \qquad \|z_0\|_{\mathcal{H}_{h_0}} \leq R,$$

where t_0 is given by (3.13). In compliance with Definition 3.2, the constant R_1 is a first-order absorbing radius with entering time $t_1 = t_0 + 1$. Moreover, $R_1 \to 0$ as $\lambda \to 0$.

The proof of the theorem will make use of the uniform Gronwall lemma, that we recall for the reader's convenience (see [48, §III.2, Lemma 1.1])

Lemma 3.6. *Let f_1, f_2, f_3 be three positive locally integrable functions on $(t_0, +\infty)$ such that f_3' is locally integrable on $(t_0, +\infty)$, and which satisfy*

$$\frac{df_3}{dt} \leq f_1 f_3 + f_2, \qquad \int_t^{t+1} f_1(s)\, ds \leq a_1, \qquad \int_t^{t+1} f_2(s)\, ds \leq a_2, \qquad \int_t^{t+1} f_3(s)\, ds \leq a_3$$

for $t \geq t_0$,

where a_1, a_2, a_3 are positive constants. Then

$$f_3(t+1) \leq (a_3 + a_2) \exp(a_1) \qquad \forall\, t \geq t_0.$$

Proof of Theorem 3.5. We proceed similarly to [20] by starting to define some auxiliary functions. Let $\zeta \in C^\infty(\mathbb{R}^2; \mathbb{R})$ be a cut-off function with compact support such that

$\zeta \equiv 1$ in a neighborhood of $\bar{B}_{h_0}$, and set

$$\hat{\zeta}(x,t) = \zeta(x_1, x_2 - h(t) + h_0) \qquad \forall\, (x,t) \in \mathbb{R}^2 \times (0,\infty)\,.$$

We then define the solenoidal vector field $\hat{V}(x,t) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$ by

$$\hat{V}(x,t) = h'(t)\left[-\frac{\partial}{\partial x_2}(\hat{\zeta}(x,t)x_1), \frac{\partial}{\partial x_2}(\hat{\zeta}(x,t)x_1)\right] \qquad \forall\, (x,t) \in \mathbb{R}^2 \times (0,\infty)\,.$$

We notice that $\hat{V}(\cdot,t) \in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for all $t \geq 0$, $\hat{V}(x,\cdot) \in L^\infty(0,T;\mathbb{R}^2)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$, and

$$\|\hat{V}(\cdot,t)\|_{W^{1,\infty}(\Omega_{h(t)})} \leq C|h'(t)| \tag{3.14}$$

for some $C > 0$. At this stage, one proceeds formally (see [24,48]) and assumes that $\hat{u}_t \in L^2(\Omega_h)$. We multiply the first equation in (2.26) with the following function

$$\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V},$$

so as to obtain

$$\begin{aligned}
&\int\limits_{\Omega_h} \hat{u}_t \cdot \left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx - \int\limits_{\Omega_h} \mathrm{div}\mathcal{T}(\hat{u},p)\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx \\
&= -\int\limits_{\Omega_h} (\hat{u}\cdot\nabla)\,\hat{u}\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx \\
&\quad + \int\limits_{\Omega_h} (\hat{u}\cdot\nabla)\, s\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx \\
&\quad + \int\limits_{\Omega_h} (s\cdot\nabla)\,\hat{u}\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx + \int\limits_{\Omega_h} \hat{g}\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx.
\end{aligned} \tag{3.15}$$

Arguing as in [20, Lemma 4.3], we find the identity

$$\begin{aligned}
&-\int\limits_{\Omega_h} \mathrm{div}\mathcal{T}(\hat{u},p)\cdot\left[\hat{u}_t + (\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\right] dx \\
&\quad = \mu\frac{d}{dt}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + m\,h''^2 + f(h)h'' \\
&\qquad + 2\mu\int\limits_{\Omega_h} (D\hat{u}) : \left[(\nabla\hat{u})(\nabla\hat{V}) - D((\hat{u}\cdot\nabla)\hat{V})\right] dx.
\end{aligned}$$

Thus, by plugging the above equality into (3.15), we obtain

$$
\begin{aligned}
&\mu\frac{d}{dt}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + m\,{h''}^2 + \|\hat{u}_t\|^2_{L^2(\Omega_h)} = -\int_{\Omega_h}\hat{u}_t\cdot\left[(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\,dx - f(h)h'' \\
&-2\mu\int_{\Omega_h}(D\hat{u}):\left[(\nabla\hat{u})(\nabla\hat{V})-D((\hat{u}\cdot\nabla)\hat{V})\right]dx-\int_{\Omega_h}(\hat{u}\cdot\nabla)\,\hat{u}\cdot\left[\hat{u}_t+(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\,dx \\
&-\int_{\Omega_h}(\hat{u}\cdot\nabla)\,s\cdot\left[\hat{u}_t+(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\,dx-\int_{\Omega_h}(s\cdot\nabla)\,\hat{u}\cdot\left[\hat{u}_t+(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]dx \\
&+\int_{\Omega_h}\hat{g}\cdot\left[\hat{u}_t+(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\,dx.
\end{aligned}
\tag{3.16}
$$

At this point, some estimates for the terms of the right-hand side of (3.16) are needed, by exploiting in a suitable way the Hölder, the Young and the Poincaré inequalities, together with the properties of the solenoidal extension s. We have the two following inequalities

$$
f(h)h'' \le \frac{m}{2}{h''}^2 + \frac{1}{2m}|f(h)|^2,
$$

and

$$
\left|\int_{\Omega_h}\hat{g}\cdot\hat{u}_t\right| \le \|\hat{g}\|_{L^2(\Omega_h)}\|\hat{u}_t\|_{L^2(\Omega_h)} \le \frac{7}{4}\|\hat{g}\|^2_{L^2(\Omega_h)} + \frac{1}{7}\|\hat{u}_t\|^2_{L^2(\Omega_h)}.
$$

Using the Ladyzhenskaya inequality [48, p.108, (2.16)],

$$
\begin{aligned}
\left|\int_{\Omega_h}(\hat{u}\cdot\nabla)\hat{u}\cdot\hat{u}_t\right| &\le c_1\|\hat{u}\|^{1/2}_{L^2(\Omega_h)}\|\nabla\hat{u}\|_{L^2(\Omega_h)}\|\hat{u}_t\|^{3/2}_{L^2(\Omega_h)} \\
&\le \frac{1}{14}\|\hat{u}_t\|^2_{L^2(\Omega_h)} + c_2\|\hat{u}\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^4_{L^2(\Omega_h)},
\end{aligned}
$$

where c_1 and c_2 are strictly positive constants. For the remaining terms, we argue in a similar manner, finding

$$
\begin{aligned}
\left|\int_{\Omega_h}(\hat{u}\cdot\nabla)s\cdot\hat{u}_t\right| &\le \|\nabla s\|_{L^4(\Omega_h)}\|\hat{u}\|_{L^4(\Omega_h)}\|\hat{u}_t\|_{L^2(\Omega_h)} \\
&\le C\frac{7}{4}\|\nabla s\|^2_{L^4(\Omega_h)}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + \frac{1}{7}\|\hat{u}_t\|^2_{L^2(\Omega_h)},
\end{aligned}
$$

where $C > 0$ depends on the constant describing the Sobolev embedding $H^1_0(\Omega_h) \subset L^4(\Omega_h)$. Exploiting again the Ladyzhenskaya inequality, we deduce

$$
\begin{aligned}
\left|\int_{\Omega_h}(s\cdot\nabla)\hat{u}\cdot\hat{u}_t\right| &\le \|s\|^{1/2}_{L^2(\Omega_h)}\|\nabla\hat{u}\|_{L^2(\Omega_h)}\|\hat{u}_t\|^{3/2}_{L^2(\Omega_h)} \\
&\le c_3\|s\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^4_{L^2(\Omega_h)} + \frac{1}{14}\|\hat{u}_t\|^2_{L^2(\Omega_h)},
\end{aligned}
$$

for some strictly positive constant $c_3 > 0$. For all terms involving the map $\hat{V}(x,t)$, we exploit (3.14). Thus, we have

$$\begin{aligned}
&\left| \int_{\Omega_h} \hat{u}_t \cdot \Big[(\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V} \Big] \right| \\
&\leq \|\hat{V}\|_{W^{1,\infty}(\Omega_h)} \|\hat{u}_t\|_{L^2(\Omega_h)} \|\nabla\hat{u}\|_{L^2(\Omega_h)} + c_4 \|\hat{V}\|_{W^{1,\infty}(\Omega_h)} \|\hat{u}_t\|_{L^2(\Omega_h)} \|\nabla\hat{u}\|_{L^2(\Omega_h)} \\
&\leq c_4 \|\hat{V}\|^2_{W^{1,\infty}(\Omega_h)} \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + \frac{2}{7}\|\hat{u}_t\|^2_{L^2(\Omega_h)} \leq c_4\, {h'}^2 \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + \frac{2}{7}\|\hat{u}_t\|^2_{L^2(\Omega_h)},
\end{aligned}$$

where $c_4 > 0$ changes from line to line, and it depends on the Poincaré constant. Then,

$$\begin{aligned}
\left| \int_{\Omega_h} (D\hat{u}) : \Big[(\nabla\hat{u})(\nabla\hat{V}) - D((\hat{u}\cdot\nabla)\hat{V}) \Big] \right| &\leq c_5 \|\hat{V}\|_{W^{1,\infty}(\Omega_h)} \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \\
&\leq c_5 |h'| \|\nabla\hat{u}\|^2_{L^2(\Omega_h)},
\end{aligned}$$

for some strictly positive constant $c_5 > 0$. Let $c_6, c_7, c_8, c_9, c_{10}$ be some strictly positive constant that might change from line to line. By the Hölder inequality, the Poincaré inequality and (3.14), we obtain

$$\begin{aligned}
&\left| \int_{\Omega_h} (\hat{u}\cdot\nabla)\,\hat{u} \cdot \Big[(\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V} \Big] \right| \\
&\leq \|(\hat{u}\cdot\nabla)\,\hat{u}\|_{L^2(\Omega_h)} \|(\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V}\|_{L^2(\Omega_h)} \\
&\leq \frac{1}{2}\|(\hat{u}\cdot\nabla)\,\hat{u}\|^2_{L^2(\Omega_h)} + \|\hat{V}\|^2_{W^{1,\infty}(\Omega_h)} \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} + c_6 \|\hat{V}\|^2_{W^{1,\infty}(\Omega_h)} \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \\
&\leq \frac{1}{2}\|(\hat{u}\cdot\nabla)\,\hat{u}\|^2_{L^2(\Omega_h)} + c_6 \|\nabla\hat{u}\|^2_{L^2(\Omega_h)}\, {h'}^2 .
\end{aligned}$$

Proceeding step by step as in [20, Theorem 1.2], taking into account for the extra terms, we arrive to an estimate for the term $(\widehat{u}\cdot\nabla)\widehat{u}$:

$$\begin{aligned}
\|(\widehat{u}\cdot\nabla)\widehat{u}\|^2_{L^2(\Omega_h)} \leq c_7 \Big[& \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} (\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)} + \|\nabla s\|^2_{L^\infty(\Omega_h)} + \|s\|^2_{L^\infty(\Omega_h)} \\
& + \|\widehat{u}\|^4_{L^2(\Omega_h)} + \|\widehat{u}\|^2_{L^2(\Omega_h)} \|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}) \\
& + \|\widehat{g}\|^2_{L^2(\Omega_h)} + {h'}^2 \Big] + \frac{2}{7}\|\widehat{u}_t\|^2_{L^2(\Omega_h)} .
\end{aligned}$$

For what concerns the trilinear terms involving the solenoidal extension s, we have

$$\begin{aligned}
\left| \int_{\Omega_h} (\hat{u}\cdot\nabla)\, s \cdot \Big[(\hat{V}\cdot\nabla)\hat{u} - (\hat{u}\cdot\nabla)\hat{V} \Big] \right| &\leq c_7 \|\nabla s\|_{L^4(\Omega_h)} \|\hat{V}\|_{W^{1,\infty}(\Omega_h)} \|\nabla\hat{u}\|^2_{L^2(\Omega_h)} \\
&\leq c_8 \|\nabla s\|_{L^4(\Omega_h)} |h'| \|\nabla\hat{u}\|^2_{L^2(\Omega_h)}
\end{aligned}$$

and

$$\left|\int_{\Omega_h}(s\cdot\nabla)\,\hat{u}\cdot\left[(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\right|\leq c_9\|s\|_{L^\infty(\Omega_h)}|h'|\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}.$$

For what concerns the last term, we have

$$\begin{aligned}\left|\int_{\Omega_h}\hat{g}\cdot\left[(\hat{V}\cdot\nabla)\hat{u}-(\hat{u}\cdot\nabla)\hat{V}\right]\right|&\leq c_9|h'|\|\hat{g}\|_{L^2(\Omega_h)}\|\nabla\hat{u}\|_{L^2(\Omega_h)}\\&\leq\frac{7}{4}\|\hat{g}\|^2_{L^2(\Omega_h)}+c_{10}{h'}^2\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}.\end{aligned}$$

Collecting all together, and dividing by μ, we finally get

$$\begin{aligned}\frac{d}{dt}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}\leq\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}\Big(&\frac{c_2+c_7}{\mu}\|\hat{u}\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}+\frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}\\&+C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)}+\frac{c_4}{\mu}{h'}^2+\frac{c_5}{\mu}|h'|+\frac{c_6}{\mu}{h'}^2+\frac{c_7}{\mu}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}\\&+\frac{c_7}{\mu}\|\nabla s\|^2_{L^\infty(\Omega_h)}+\frac{c_7}{\mu}\|s\|^2_{L^\infty(\Omega_h)}+\frac{c_7}{\mu}\|\widehat{u}\|^4_{L^2(\Omega_h)}\\&+c_8\|\nabla s\|_{L^4(\Omega_h)}|h'|+\frac{c_9}{\mu}\|s\|_{L^\infty(\Omega_h)}|h'|+\frac{c_{10}}{\mu}{h'}^2\Big)\\&+\frac{1}{\mu}(c_7+\frac{7}{4})\|\hat{g}\|^2_{L^2(\Omega_h)}+c_7{h'}^2+\frac{1}{2m\mu}|f(h)|^2.\end{aligned}$$

We are now in a position to apply Lemma 3.6 with the choice

$$\begin{aligned}f_1=&\frac{c_2+c_7}{\mu}\|\hat{u}\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}+\frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)}\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}+C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)}\\&+\frac{c_4}{\mu}{h'}^2+\frac{c_5}{\mu}|h'|+\frac{c_6}{\mu}{h'}^2+\frac{c_7}{\mu}\|\nabla\widehat{u}\|^2_{L^2(\Omega_h)}+\frac{c_7}{\mu}\|\nabla s\|^2_{L^\infty(\Omega_h)}+\frac{c_7}{\mu}\|s\|^2_{L^\infty(\Omega_h)}\\&+\frac{c_7}{\mu}\|\widehat{u}\|^4_{L^2(\Omega_h)}+c_8\|\nabla s\|_{L^4(\Omega_h)}|h'|+\frac{c_9}{\mu}\|s\|_{L^\infty(\Omega_h)}|h'|+\frac{c_{10}}{\mu}{h'}^2,\\f_2=&\frac{1}{\mu}(c_7+\frac{7}{4})\|\hat{g}\|^2_{L^2(\Omega_h)}+c_7{h'}^2+\frac{1}{2m\mu}|f(h)|^2,\\f_3=&\|\nabla\hat{u}\|^2_{L^2(\Omega_h)}.\end{aligned}$$

Indeed, since $t\geq t_0$ with t_0 as in (3.13), from (3.13) we have that $2F(h)\leq R_0^2$, so that by (2.5)

$$|h|\leq\frac{M(R_0^2)}{2},$$

providing in turn a uniform bound for $|f(h)|$. Therefore, with reference to Lemma 3.6, denoting by Q a generic increasing positive function, and exploiting (3.13) and Corollary 3.4, we draw the estimates

$$\int_t^{t+1} f_1(s)\,ds \le a_1, \qquad \int_t^{t+1} f_2(s)\,ds \le a_2, \qquad \int_t^{t+1} f_3(s)\,ds \le a_3 \qquad \text{for } t \ge t_0,$$

with

$$\begin{cases} a_1 = \frac{c_2+c_7}{\mu} R_0^2\, a_3 + \frac{c_3}{\mu}\|s\|^2_{L^2(\Omega_h)} a_3 + C\frac{7}{4\mu}\|\nabla s\|^2_{L^4(\Omega_h)} + \frac{c_4}{\mu}R_0^2 + \frac{c_5}{\mu}R_0 + \frac{c_6}{\mu}R_0^2 + \frac{c_7}{\mu}a_3 \\ \quad + \frac{c_7}{\mu}\|\nabla s\|^2_{L^\infty(\Omega_h)} + \frac{c_7}{\mu}\|s\|^2_{L^\infty(\Omega_h)} + \frac{c_7}{\mu}R_0^4 + \frac{c_8}{\mu}\|\nabla s\|_{L^4(\Omega_h)}R_0 + \frac{c_9}{\mu}\|s\|_{L^\infty(\Omega_h)}R_0 \\ \quad + \frac{c_{10}}{\mu}R_0^2, \\ a_2 = \frac{1}{\mu}(c_7 + \frac{7}{4})\|\hat{g}\|^2_{L^2(\Omega_h)} + Q(R_0), \\ a_3 = D(R_0). \end{cases}$$

The conclusion is

$$\|\nabla \hat{u}\|^2_{L^2(\Omega_h)} \le (a_3 + a_2)\exp(a_1) \qquad \forall t \ge t_0 + 1.$$

The last step is to add mh'^2 to both sides of the inequality above, which allows us to reconstruct the norm of the norm of the solution, to wit,

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} \le \sqrt{(a_3 + a_2)\exp(a_1) + R_0^2} \qquad \forall t \ge t_0 + 1.$$

Here, we leaned on the estimate $mh'^2 \le R_0^2$, coming from (3.13). By calling R_1 the right-hand side, the proof is finished. $\square$

Remark 3.7. Due to the compact embedding

$$\mathcal{H}^1_{h(t)} \Subset \mathcal{H}_{h(t)},$$

which holds true for all t, the closed ball $\mathcal{B}_1(t)$ of radius R_1 in $\mathcal{H}^1_{h(t)}$ is compact in $\mathcal{H}_{h(t)}$. Theorem 3.5 tells that

$$U(t)\mathcal{B} \subset \mathcal{B}_1(t) \qquad \forall\, t \ge t_0 + 1,$$

where $\mathcal{B}$ is the ball of radius R in $\mathcal{H}_{h_0}$, for $R > 0$ arbitrarily given. This shows that the solution operator $U(t)$ defined by the rule (3.1) is not only dissipative in the sense of Definition 3.2, but it has also a regularizing effect.

4. Stability of the unique steady state

In this section, we investigate the convergence of the solutions of (1.2) to the unique steady state, if $\lambda < \lambda_*$, see Theorem 2.3. In particular, we study the convergence of the solutions of (1.2) to those of (2.14)-(2.15), in terms of the convergence of the solutions of (2.26)-(2.27) to the solution of (2.21)-(2.22).

Theorem 4.1. *Let $R > 0$ be arbitrarily fixed and λ_* as in Theorem 2.3. There exists $\lambda_1 = \lambda_1(R) \in (0, \lambda_*)$ such that if $\lambda < \lambda_1$, the weak solution $(\hat{u}, h)$ of problem (2.26)-(2.27), with initial position of the obstacle $h_0 = 0$ and initial velocities $z_0 = (\hat{u}_0, k_0) \in \mathcal{H}_0$ such that $\|z_0\|_{\mathcal{H}_0} \le R$, converges at an exponential rate to the solution $(\hat{u}_\lambda, 0)$ of (2.21)-(2.22) in $\mathcal{H}_0$ as $t \to \infty$.*

Remark 4.2. Theorem 4.1 could be generalized to the case of small $|h_0| > 0$ without much effort. However, we state and prove it for $h_0 = 0$ having in mind the straightforward application of a bridge at its rest position suddenly being invested by a wind gust.

In order to prove Theorem 4.1, we need two preliminary results. We begin by stating the following regularity property on a solution given by Theorem 2.8.

Lemma 4.3. *Let $(\hat{u}, h)$ be the weak solution to (2.26)-(2.27), in the sense of Definition 2.5. There holds*

$$\begin{aligned} t\,\hat{u} \in L^{4/3}(0,T; W^{2,4/3}(\Omega_{h(t)})), \qquad t\,\partial_t \hat{u} \in L^{4/3}(0,T; L^{4/3}(\Omega_{h(t)}))\,, \\ t\,\nabla p \in L^{4/3}(0,T; L^{4/3}(\Omega_{h(t)}))\,. \end{aligned} \tag{4.1}$$

Proof. The proof of (4.1) follows the proof of [32, Proposition 3], up to some slight modification. We report here the main steps. We start by deducing, from a classical interpolation argument [49, Chapter 3, Lemma 3.3] and suitable Sobolev embeddings, that

$$\hat{u} \in L^{4/3}(0,T; L^{4/3}(\Omega_{h(t)}))\,, \qquad (\hat{u}\cdot\nabla)\hat{u} \in L^{4/3}(0,T; L^{4/3}(\Omega_{h(t)}))\,.$$

The second step to prove (4.1) implies introducing the following auxiliary linear system with unknown (U, H):

$$\begin{aligned} \frac{\partial U}{\partial t} - \mu\,\Delta U + \nabla Q = f \qquad & \text{for } x \in \Omega_{h(t)}\,, \\ \operatorname{div} U = 0 \qquad & \text{for } x \in \Omega_{h(t)}\,, \\ U = H'\hat{e}_2 \qquad & \text{for } x \in \partial B_{h(t)}\,, \\ U = 0 \qquad & \text{for } x \in \partial\Gamma_{\mathcal{R}}\,, \end{aligned} \tag{4.2}$$

$$m\,H''(t) = -\hat{e}_2 \cdot \int\limits_{\partial B_{h(t)}} \mathcal{T}(U,Q)\hat{n}\,d\sigma + m\,f_1\,,$$

where f and f_1 are given source terms, and $\Omega_{h(t)}$ and $B_{h(t)}$ are prescribed and not unknown. In particular, they are associated to $(\hat{u},h)$. Following [32, Definition 2], we say that, given $f \in L^{4/3}((0,T)\times\Omega_{h(t)})$, $f_1 \in L^{4/3}(0,T;\mathbb{R})$ and $\varepsilon_0 \in (0, L-\delta)$, then

$$(U,H) \in [L^2(0,T;H_0^1(\Omega_{h(t)})) \cap L^\infty(0,T;L^2(\Omega_{h(t)}))] \times C([0,T];[-L+\delta+\varepsilon_0, L-\delta-\varepsilon_0])$$

is a weak solution to (4.2), with vanishing initial data and source term f, f_1, if U is divergence free, $U(0)=0$, $H(0)=0$ and

$$\int\limits_{\Omega_{h(t)}} \frac{\partial U}{\partial t}\,\phi\,dx + 2\mu\int\limits_{\mathcal{R}} DU : D\phi\,dx + m(H''-f_1)l_\phi = \int\limits_{\Omega_{h(t)}} f\,\phi\,dx$$

for all $\phi \in C_0^\infty([0,T]\times\mathcal{R};\mathbb{R}^2)$ such that $\phi(\cdot,t)|_{B_{h(t)}} = l_\phi(t)\hat{e}_2$, with $l(t)\in\mathbb{R}$. The third step implies showing that weak solutions to (4.2) in the sense given above are unique. This can be done precisely as in [32, Lemma 8], by taking the difference between two weak solutions, which is allowed because the fluid domain is in this case prescribed, thus identical for the two solutions. Then, by [32, Lemma 4] and [29, Theorem 4.1] we know that problem (4.2) has a unique strong solution with vanishing initial data belonging to

$$\begin{aligned} U &\in L^{4/3}(0,T;W^{2,4/3}(\Omega_{h(t)}))\,, & \partial_t U &\in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)})),\\ H &\in W^{2,4/3}(0,T;\mathbb{R})\,, & \nabla Q &\in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))\,, \end{aligned} \tag{4.3}$$

and such that

$$\begin{aligned} &\|U\|_{L^{4/3}(0,T;W^{2,4/3}(\Omega_{h(t)}))} + \|\partial_t U\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))} + \|\nabla Q\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))}\\ &\leq C\left(\|f\|_{L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))} + \|f_1\|_{L^{4/3}(0,T;\mathbb{R})}\right)\,, \end{aligned} \tag{4.4}$$

where C depends on the geometry of the rigid body and on $\mathcal{R}$. Through some integration by parts, it can be shown that any strong solution to (4.2) is also a weak solution to (4.2). The last step implies showing that, given

$$U := t\hat{u}\,, \qquad H := th\,, \qquad Q := tp\,, \tag{4.5}$$

then $(t\hat{u}, th)$ is a weak solution to (4.2) in the sense given above, with source term

$$\begin{aligned} &f := \hat{u} - t(\hat{u}\cdot\nabla)\hat{u} - t(\hat{u}\cdot\nabla)s - t(s\cdot\nabla)\hat{u} + t\hat{g} \in L^{4/3}(0,T;L^{4/3}(\Omega_{h(t)}))\,,\\ &f_1 := h' + t\frac{f(h)}{m} \in L^{4/3}(0,T;\mathbb{R})\,. \end{aligned}$$

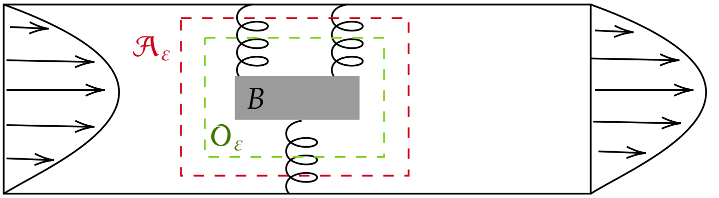

Fig. 4. The subsets $\mathcal{O}_\varepsilon$, $\mathcal{A}_\varepsilon \subset \mathcal{R}$.

Thus, since weak solutions to (4.2) are unique, the solution given by (4.5) must be strong, thus it satisfies the regularly in (4.3) and the estimate in (4.4), which yields the desired result. □

As a second preliminary result, we state and prove a basic proposition allowing to define a change of variables associated to the rigid motion of the obstacle in problem (2.26)-(2.27) in order to be able to compare different solutions; indeed since (2.26)-(2.27) is set on a time-dependent fluid domain, different solutions are defined on different domains. This change of variables depends on time t through h; it was first introduced by Takahashi ([47, Section 4.1]), inspired by Inoue and Wakimoto ([37]). We denote for all $\varepsilon \in (0, L-\delta)$

$$\mathcal{O}_\varepsilon = \left\{x \in R : \mathrm{dist}(x, \Gamma) \geq 2\varepsilon \wedge |x_1| < \tfrac{3}{2}\right\}, \qquad \mathcal{A}_\varepsilon = \left\{x \in R : \mathrm{dist}(x, \Gamma) \leq \varepsilon \vee |x_1| > 2\right\},$$

see Fig. 4 for a representation. Note that, choosing ε to build the function $s = s_\varepsilon$ defined in Lemma 2.1, then

$$s(x) \equiv 0 \qquad \text{on} \qquad \mathcal{R} \setminus \mathcal{A}_\varepsilon. \tag{4.6}$$

Proposition 4.4. *Consider a fixed $h \in W^{1,\infty}(0,T; (-L+\delta, L-\delta))$ with $h(0) = h_0$. For every $t \in [0,T]$ there exists a volume preserving diffeomorphism*

$$\psi(t,\cdot) : \Omega_{h(t)} \longrightarrow \Omega_{h_0}$$

satisfying, for all $\varepsilon > 0$, the following properties:

$$\psi(t, x_1, x_2) = (x_1, x_2 + h(t) - h_0) \quad \forall\, x = (x_1, x_2) \in \mathcal{O}_\varepsilon,$$
$$\psi(t, x_1, x_2) = (x_1, x_2) \quad \forall\, x = (x_1, x_2) \in \mathcal{A}_\varepsilon.$$

Proof. Let $\zeta(x_1, x_2)$ be a smooth cutoff function equal to 0 in $\mathcal{A}_\varepsilon$ and equal to 1 in $\mathcal{O}_\varepsilon$. Then, we define the solenoidal vector field $V : \mathbb{R}^+ \times \Omega_{h(t)} \to \mathbb{R}^2$ as

$$V(t,x) = \nabla \times \{0, 0, -\zeta(x_1, x_2) x_1 h'\}.$$

Notice that

$$V(t,x)=\begin{cases}0 & \text{in } \mathcal{A}_\varepsilon,\\ h'\hat{e}_2 & \text{in } \mathcal{O}_\varepsilon,\end{cases} \tag{4.7}$$

and $V(\cdot,t)\in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ for all $t\geq 0$, $V(x,\cdot)\in L^\infty(0,T;\mathbb{R}^2)$ for all $x\in\mathbb{R}^2$. Then we build the deformation mapping of $\Omega_{h(t)}$ into Ω_{h_0}, $\psi:\mathbb{R}^+\times\Omega_{h(t)}\to\Omega_{h_0}$, as the flow associated to (4.7):

$$\begin{cases}\frac{\partial}{\partial t}\psi(t,x)=V(t,\psi(t,x)),\\ \psi(0,x)=x.\end{cases}$$

Since $\nabla\cdot V=0$, ψ is volume preserving and $\det(\frac{\partial\psi_i}{\partial x_j})_{i,j}=1$ for all $t\geq 0$. The mapping ψ is a smooth function of V. In particular, for some $C>0$,

$$\|\partial_t^j\psi(t,\cdot)\|_{C^k(\bar{\Omega}_{h(t)})}\leq C|h^j(t)| \qquad \forall j=0,1,\forall k=0,1,2. \tag{4.8}$$

Notice that $\psi\in W^{1,\infty}(0,T;C^k(\Omega_{h(t)}))$ for any $k=0,1,2$. $\square$

Through Proposition 4.4, we define $\varphi:\mathbb{R}^+\times\Omega_{h_0}\longrightarrow\Omega_{h(t)}$ by $\varphi=\psi^{-1}$ in the space variables to be the volume preserving diffeomorphism such that, for any $y=(y_1,y_2)\in\mathcal{O}_\varepsilon$,

$$\varphi(t,y_1,y_2)=(y_1,y_2+h_0-h(t))$$

and, for any $y=(y_1,y_2)\in\mathcal{A}_\varepsilon$, $\varphi(t,y_1,y_2)=(y_1,y_2)$. There holds

$$\|\partial_t^j\varphi(t,\cdot)\|_{C^k(\bar{\Omega}_{h_0})}\leq C|h^j(t)| \qquad \forall j=0,1,\forall k=0,1,2.$$

Obviously $\varphi\in W^{1,\infty}(0,T;C^k(\Omega_{h_0}))$ for any $k=0,1,2$. We can now give the

Proof of Theorem 4.1. Let $\varepsilon_1>0$ be given by (3.5) and take $s=s_{\varepsilon_1}$ from Lemma 2.1. Multiply the fluid equation in (2.21) by a function $\phi\in V(\mathcal{R})$ such that $\phi|_{B_0}=\hat{e}_2$ and integrate over Ω_0. After integration by parts it comes

$$\mu\int_{\Omega_0}\nabla\hat{u}_\lambda:\nabla\phi\,dy+\int_{\partial\Omega_0}(-\mu\frac{\partial\hat{u}_\lambda}{\partial\hat{n}}+p_*\hat{n})\cdot\phi\,d\tau+\int_{\Omega_0}(\hat{u}_\lambda\cdot\nabla)\hat{u}_\lambda\cdot\phi\,dy+\int_{\Omega_0}(\hat{u}_\lambda\cdot\nabla)s\cdot\phi\,dy$$
$$+\int_{\Omega_0}(s\cdot\nabla)\hat{u}_\lambda\cdot\phi\,dy=\int_{\Omega_0}\hat{g}\cdot\phi\,dy.$$

Next, by the coupling condition in (2.22), we have that

$$\int_{\partial\Omega_0} (-\mu \frac{\partial \hat{u}_\lambda}{\partial \hat{n}} + p\hat{n}) \cdot \phi \, d\tau = -\hat{e}_2 \cdot \int_{\partial B_0} (\mathcal{T}(\hat{u}_\lambda, p_*) \cdot \hat{n}) \, d\tau = 0.$$

Consequently, we obtain that, for any $\phi \in V(\mathcal{R})$ such that $\phi|_{B_0} = \hat{e}_2$,

$$\mu \int_{\Omega_0} \nabla \hat{u}_\lambda : \nabla \phi \, dy + \int_{\Omega_0} (\hat{u}_\lambda \cdot \nabla)\hat{u}_\lambda \cdot \phi \, dy + \int_{\Omega_0} (\hat{u}_\lambda \cdot \nabla)s \cdot \phi \, dy + \int_{\Omega_0} (s \cdot \nabla)\hat{u}_\lambda \cdot \phi \, dy = \int_{\Omega_0} \hat{g} \cdot \phi \, dy. \tag{4.9}$$

Notice that (4.9) does not see the value of ϕ on B_0, thus we could have taken $\phi|_{B_0} = c\,\hat{e}_2$ with $c \in \mathbb{R}$. Then, let $(\hat{u}, h)$ be the unique solution to problem (2.26)-(2.27) given by Theorem 2.8, with $h_0 = 0$ and some initial velocities $z_0 = (\hat{u}_0, k_0) \in \mathcal{H}_0$ such that $\|z_0\|_{\mathcal{H}_0} \leq R$, for any arbitrary $R > 0$. In order to be able to subtract the weak formulation satisfied by $(\hat{u}, h)$ and that satisfied by $(\hat{u}_*, h_*) = (\hat{u}_\lambda, 0)$, we need to properly map $\hat{u}(t)$ from $\Omega_{h(t)}$ to Ω_0 for every $t > 0$. We follow [32,44]. From (3.5), we infer that $h \in W^{1,\infty}(0, T; [-L + \delta - \varepsilon_1, L - \delta - \varepsilon_1])$. Thus, we can build ψ as in Proposition 4.4 with $h_0 = 0$ and $\varepsilon = \varepsilon_1$; we also define $\varphi = \psi^{-1}$. We introduce

$$v(y,t) = \nabla\psi(\varphi(t,y), t) \cdot \hat{u}(\varphi(t,y), t) \qquad y \in \Omega_0,$$

to be the pullback of $\hat{u}$ by φ, and we set $q(t,y) = p(t, \varphi(y,t))$. We refer to [32, Section 3.2] (see also [44, Section 5]) for the explicit computation of the partial derivatives of v in terms of those of $\hat{u}$, so that the equation satisfied by v reads

$$\begin{aligned} &\langle \partial_t v(t), \phi \rangle + m h''(t) l(t) + f(h(t)) l(t) + \mu \int_{\Omega_0} \nabla v(t) : \nabla \phi \, dy + \int_{\Omega_0} (v(t) \cdot \nabla) v(t) \cdot \phi \, dy \\ &+ \int_{\Omega_0} (v(t) \cdot \nabla) s \cdot \phi \, dy + \int_{\Omega_0} (s \cdot \nabla) v(t) \cdot \phi \, dy = \int_{\Omega_0} \hat{g} \cdot \phi - \int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot \phi \, dy, \end{aligned} \tag{4.10}$$

for any test pair $(\phi, l) \in \mathcal{H}_0^1$ where, using Einstein's summation convention,

$$\begin{aligned} \mathfrak{f}^i = &+ (\partial_k \varphi^i - \delta_{ik}) \partial_t v^k + \partial_k \varphi^i \partial_l v^k (\partial_t \psi^l) + (\partial_k \partial_t \varphi^i) v^k + (\partial^2_{kl} \varphi^i)(\partial_t \psi^l) v^k \\ &+ v^l \partial_l v^k (\partial_k \varphi^i - \delta_{ik}) + (\partial^2_{lk} \varphi^i) v^l v^k + \partial_k q (\partial_i \psi^k - \delta_{ik}) + \mu \Big[- \partial_j \psi^m (\partial^2_{mk} \varphi^i) \partial_l v^k \partial_j \psi^l \\ &- (\partial_k \varphi^i \partial_j \psi^m \partial_j \psi^l - \delta_{ik} \delta_{jm} \delta_{jl}) \partial^2_{ml} v^k - \partial_k \varphi^i \partial_l v^k (\partial^2_{jj} \psi^l) \\ &- \partial_j \psi^m (\partial^3_{mlk} \varphi^i) \partial_j \psi^l v^k - (\partial^2_{lk} \varphi^i) \partial^2_{jj} \psi^l v^k - (\partial^2_{lk} \varphi^i) \partial_j \psi^l \partial_j \psi^m \partial_m v^k \Big]. \end{aligned} \tag{4.11}$$

Then set $w(t) = v(t) - \hat{u}_\lambda$ and subtract (4.9) from (4.10) to obtain:

$$
\begin{aligned}
&\langle \partial_t w(t), \phi\rangle + mh''(t)l + f(h(t))l + \mu \int_{\Omega_0} \nabla w(t) : \nabla \phi \, dy + \int_{\Omega_0} (v(t)\cdot\nabla)w(t)\cdot\phi \, dy \\
&+ \int_{\Omega_0} (w(t)\cdot\nabla)\hat{u}_\lambda \cdot \phi \, dy + \int_{\Omega_0} (w(t)\cdot\nabla)s\cdot\phi \, dy + \int_{\Omega_0} (s\cdot\nabla)w(t)\cdot\phi \, dy \\
&= -\int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t))\cdot\phi \, dy.
\end{aligned}
$$

We follow the same reasoning of the proof of Theorem 3.3. We define

$$
z(x,t) = h(t)\left[-\frac{\partial}{\partial x_2}(\zeta(x)\, x_1), \frac{\partial}{\partial x_1}(\zeta(x)\, x_1)\right] \qquad \forall (x,t)\in \mathcal{R}\times(0,\infty),
$$

where ζ is a C^∞ cut-off function equal to 1 in a small neighborhood of the obstacle B_0 and equal to 0 outside a larger neighborhood. We observe that, for all $t \geq 0$,

$$
z(t)\in C^\infty(\mathcal{R})\cap H^1_0(\mathcal{R}), \qquad \mathrm{div} z(t) = 0\,, \qquad z(t) = h(t)\,\hat{e}_2 \quad \text{in} \quad B_h\,.
$$

Moreover, the following estimates hold:

$$
\begin{gathered}
\|z(t)\|_{L^2(\mathcal{R})} \leq a_1|h(t)|\,, \qquad \|\nabla z(t)\|_{L^2(\mathcal{R})} \leq a_2|h(t)|\,, \qquad \|z(t)\|_{L^\infty(\mathcal{R})} \leq a_3|h(t)| \\
\|\nabla z(t)\|_{L^\infty(\mathcal{R})} \leq a_4|h(t)|\,, \qquad \|z_t(t)\|_{L^2(\mathcal{R})} \leq a_5|h'(t)|\,,
\end{gathered}
$$

where a_1, a_2, a_3, a_4 and a_5 are constants depending on the cut-off function ζ. We introduce, for $\omega \in (0,1)$ to be fixed later, the energy and its perturbation

$$
\begin{aligned}
E(t) &= \|w(t)\|^2_{L^2(\Omega_0)} + m{h'(t)}^2 + 2F(h(t)), \\
E_\omega(t) &= E(t) + 2m\omega\, h(t)\, h'(t) + 2\int_{\Omega_0} w(t)\cdot\omega z(t)\, dy.
\end{aligned}
$$

Such functionals satisfy (3.4), provided that ω is small enough. Then, choosing $(\phi, l) = (w+\omega z, h' + \omega h)$, we infer

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}E_\omega - m\omega {h'}^2 + \omega f(h)h + \mu\|\nabla w\|^2_{L^2(\Omega_0)} \\
&= -\int_{\Omega_0} \mathfrak{f}(v,h,q)\cdot w\, dy - \int_{\Omega_0} (w\cdot\nabla)\hat{u}_\lambda\cdot w\, dy - \int_{\Omega_0} (w\cdot\nabla)s\cdot w\, dy + \int_{\Omega_0} w\cdot\omega z_t\, dy \\
&\quad - \mu\int_{\Omega_0} \nabla w : \omega\nabla z\, dy - \int_{\Omega_0} (w\cdot\nabla)s\cdot\omega z\, dy + \int_{\Omega_0} (w\cdot\omega\nabla)z\cdot w\, dy - \int_{\Omega_h} (s\cdot\nabla)w\cdot\omega z\, dy \\
&\quad - \int_{\Omega_0} (w\cdot\nabla)\hat{u}_\lambda\cdot\omega z\, dy - \int_{\Omega_0} \mathfrak{f}(v,h,q)\cdot\omega z\, dy.
\end{aligned}
$$

Next, we estimate the right-hand side. We start by the first term. Following [32], we divide $\mathfrak{f}$ into pieces

$$\mathfrak{f} = \mathfrak{f}_1 + \mathfrak{f}_2 + \mathfrak{f}_3 + \mathfrak{f}_4 + \mathfrak{f}_5$$

with

$$\begin{aligned}
\mathfrak{f}_1 :=& (\partial_k \partial_t \varphi^i) v^k + (\partial^2_{kl}\varphi^i)(\partial_t \psi^l) v^k - \sum_j \left[\partial_j \psi^m (\partial^3_{mlk}\varphi^i) \partial_j \psi^l v^k + (\partial^2_{lk}\varphi^i) \partial^2_{jj} \psi^l v^k \right], \\
\mathfrak{f}_2 :=& \partial_k \varphi^i \partial_l v^k (\partial_t \psi^l) \\
&- \sum_j \left[\partial_j \psi^m (\partial^2_{mk}\varphi^i) \partial_l v^k \partial_j \psi^l + \partial_k \varphi^i \partial_l v^k (\partial^2_{jj}\psi^l) + (\partial^2_{lk}\varphi^i) \partial_j \psi^l \partial_j \psi^m \partial_m v^k \right], \\
\mathfrak{f}_3 :=& (\partial^2_{lk}\varphi^i) v^l v^k, \quad \mathfrak{f}_4 := v^l \partial_l v^k (\partial_k \varphi^i - \delta_{ik}), \\
\mathfrak{f}_5 :=& (\partial_k \varphi^i - \delta_{ik}) \partial_t v^k + \partial_k \pi_2 (\partial_i \psi^k - \delta_{ik}) - \sum_j (\partial_k \varphi^i \partial_j \psi^m \partial_j \psi^l - \delta_{ik}\delta_{jm}\delta_{jl}) \partial^2_{ml} v^k.
\end{aligned}$$

We have the following estimates, where we use (4.8).

- Concerning the first three terms:

$$\int_{\Omega_0} \mathfrak{f}_1 \cdot w \, dy \le C\|v\|_{L^2(\Omega_0)} \|w\|_{L^2(\Omega_0)} |\hat{h}'| \le C\|v\|_{L^2(\Omega_0)} \left(\|w\|^2_{L^2(\Omega_0)} + m\hat{h}'^2 \right)$$

$$\int_{\Omega_0} \mathfrak{f}_2 \cdot w \, dy| \le C\|\nabla v\|_{L^2(\Omega_0)} \|w\|_{L^2(\Omega_0)} |\hat{h}'| \le C\|\nabla v\|_{L^2(\Omega_0)} \left(\|w\|^2_{L^2(\Omega_0)} + m\hat{h}'^2 \right)$$

$$\int_{\Omega_0} \mathfrak{f}_3 \cdot w \, dy \le C\|v\|^2_{L^4(\Omega_0)} \|w\|_{L^2(\Omega_0)} |\hat{h}'| \le C\|v\|_{L^2(\Omega_0)} \|\nabla v\|_{L^2(\Omega_0)} \left(\|w\|^2_{L^2(\Omega_0)} + m\hat{h}'^2 \right)$$

- For the fourth and fifth terms, following [32], thanks to Lemma 4.3, we have

$$\begin{aligned}
\int_{\Omega_0} \mathfrak{f}_4 \cdot w \, dy &\le C\|v\|_{L^4(\Omega_0)} \|t\,\nabla v\|_{L^4(\Omega_0)} \|\frac{1}{t}(\partial_k \varphi^i - \delta_{ik})\|_{L^\infty(\Omega_0)} \|w\|_{L^2(\Omega_0)} \\
&\le C\|v\|_{L^4(\Omega_0)} \|t\,\nabla v\|_{L^4(\Omega_0)} |\hat{h}'| \|w\|_{L^2(\Omega_0)} \\
&\le C\|\nabla v\|^{1/2}_{L^2(\Omega_0)} \|v\|^{1/2}_{L^2(\Omega_0)} \|t\,\nabla v_2\|_{L^4(\Omega_0)} |\hat{h}'| \|w\|_{L^2(\Omega_0)} \,.
\end{aligned}$$

Next we notice that

$$b_1(t) := \|\nabla v(\cdot,t)\|^{1/2}_{L^2(\Omega_0)} \|t\,\nabla v(\cdot,t)\|_{L^4(\Omega_0)} \in L^1(0,T)\,,$$

due to the Hölder inequality with exponent $p = 4$ and $q = 4/3$. Hence we obtain

$$\int_{\Omega_0} \mathfrak{f}_4 \cdot w \, dy \leq C \|v\|_{L^2(\Omega_0)}^{1/2} b_1 \Big(\|w\|_{L^2(\Omega_0)}^2 + m \hat{h}'^2 \Big)$$

Next, we introduce

$$\begin{aligned} b_2(t) :=& \|t \partial_t v^k(\cdot, t)\|_{L^{4/3}(\tilde{\Omega}^1(t))} + \|\partial_k \pi_2(\cdot, t)\|_{L^{4/3}(\tilde{\Omega}^1(t))} \\ &+ \|t v^k(\cdot, t)\|_{W^{2,4/3}(\tilde{\Omega}^1(t))} \in L^{4/3}(0, T) \,. \end{aligned}$$

We deduce that

$$\int_{\Omega_0} \mathfrak{f}_5 \cdot w \, dy \leq C \, b_2 |\hat{h}'| \|w\|_{L^4(\Omega_0)} \,.$$

Next, we apply the Young inequality twice as follows

$$\begin{aligned} \int_{\Omega_0} \mathfrak{f}_5 \cdot w \, dy \ &\leq C b_2^{2/3} \|w\|_{L^4(\Omega_0)}^2 + C b_2^{4/3} \hat{h}'^2 \\ &\leq C b_2^{4/3} \|w\|_{L^2(\Omega_0)}^2 + C b_2^{4/3} \hat{h}'^2 + \frac{\mu}{8} \|\nabla w\|_{L^2(\Omega_0)}^2, \end{aligned}$$

where $b_2(t)^{4/3} \in L^1(0, T)$.

If we set

$$\begin{aligned} \mathcal{D}_1(t) :=& \|v\|_{L^\infty(0,T;L^2(\Omega_0))} \big(1 + \|\nabla v(\cdot, t)\|_{L^2(\Omega_0)}\big) \\ &+ \|v\|_{L^\infty(0,T;L^2(\Omega_0))}^{1/2} b_1(t) + b_2(t)^{4/3} \in L^1(0, T), \end{aligned} \tag{4.12}$$

we obtain then

$$\int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot w(t) \, dy \leq \mathcal{D}_1(t) \big(\|w(t)\|_{L^2(\Omega_0)}^2 + m \, h'(t)^2 \big) + \frac{\mu}{8} \|\nabla w(t)\|_{L^2(\Omega_0)}^2.$$

Let Q be a positive and increasing function with respect to its variables. We observe that by Lemma 4.3, in particular estimate (4.4), we infer that the function $\mathcal{D}_1$ in (4.12) is such that

$$\mathcal{D}_1(t) = Q(\|v(t)\|_{L^2(\Omega_0)}, \|\nabla v(t)\|_{L^2(\Omega_0)}, \|\hat{g}\|_{L^\infty(\Omega_0)}, \|s\|_{L^\infty(\Omega_0)}, \|\nabla s\|_{L^\infty(\Omega_0)}) \,.$$

Analogous computations bring to the existence of a function

$$\mathcal{D}_2(t) = Q(\|v(t)\|_{L^2(\Omega_0)}, \|\nabla v(t)\|_{L^2(\Omega_0)}, \|\hat{g}\|_{L^\infty(\Omega_0)}, \|s\|_{L^\infty(\Omega_0)}, \|\nabla s\|_{L^\infty(\Omega_0)}) \,,$$

such that

$$\int_{\Omega_0} \mathfrak{f}(v(t), h(t), q(t)) \cdot \omega z(t)\, dy \le \mathcal{D}_2(t)\, m\, h'(t)^2 + \omega^2 \|\nabla z(t)\|^2_{L^2(\Omega_0)}.$$

For the second term and the third term, we exploit [28, (2.26)], the Poincaré inequality and the properties of s defined as in Lemma 2.1. We obtain

$$\begin{aligned}\int_{\Omega_0} (w \cdot \nabla)\hat{u}_\lambda \cdot w\, dy &\le \|w\|^2_{L^4(\Omega_0)} \|\nabla \hat{u}_\lambda\|_{L^2(\Omega_0)} \\ &\le \left(\frac{2}{3\pi}\right)^{1/2} \|\nabla w\|_{L^2(\Omega_0)} \|w\|_{L^2(\Omega_0)} \|\nabla \hat{u}_\lambda\|_{L^2(\Omega_0)} \\ &\le \frac{L}{\sqrt{3}} \left(\frac{2}{\pi}\right)^{3/2} \|\nabla w\|^2_{L^2(\Omega_0)} \|\nabla \hat{u}_\lambda\|_{L^2(\Omega_0)},\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega_0} (w \cdot \nabla)s \cdot w\, dy &\le \|\nabla s\|_{L^4(\Omega_0)} \|w\|_{L^4(\Omega_0)} \|w\|_{L^2(\Omega_0)} \\ &\le \left(\frac{2}{3\pi}\right)^{1/4} \|\nabla s\|_{L^4(\Omega_0)} \|\nabla w\|^{1/2}_{L^2(\Omega_0)} \|w\|^{3/2}_{L^2(\Omega_0)} \\ &\le \left(\frac{2L}{\pi}\right)^{3/2} \left(\frac{2}{3\pi}\right)^{1/4} \|\nabla s\|_{L^4(\Omega_0)} \|\nabla w\|^2_{L^2(\Omega_0)}.\end{aligned}$$

With the same arguments of those in Theorem 3.3, we estimate the terms involving the function z as follows

$$\begin{aligned}\int_{\Omega_0} w \cdot \omega z_t\, dy &\le \frac{\mu}{16} \|\nabla w\|^2_{L^2(\mathcal{R})} + \frac{4}{\mu} \frac{4L^2}{\pi^2} \omega^2 \|z_t\|^2_{L^2(\mathcal{R})}\,, \\ -\mu \int_{\Omega_0} \nabla w : \omega \nabla z\, dy &\le \frac{\mu}{16} \|\nabla w\|^2_{L^2(\mathcal{R})} + \frac{4}{\mu} \omega^2 \|\nabla z\|^2_{L^2(\mathcal{R})}\,; \\ -\int_{\Omega_0} (w \cdot \nabla)s \cdot \omega z\, dy &\le \frac{\mu}{8} \|\nabla w\|^2_{L^2(\mathcal{R})} + \frac{2}{\mu} \omega^2 \|\nabla z\|^2_{L^2(\mathcal{R})}\,, \\ \int_{\Omega_0} (w \cdot \omega \nabla)z \cdot w\, dy &\le \omega \frac{4L^2}{\pi^2} \|\nabla z\|_{L^\infty(\mathcal{R})} \|\nabla w\|^2_{L^2(\mathcal{R})}\,; \\ -\int_{\Omega_0} (s \cdot \nabla)w \cdot \omega z\, dy &\le \|s\|_{L^\infty(\Omega_0)} \|\nabla w\|_{L^2(\Omega_0)} \omega \|z\|_{L^2(\Omega_0)} \\ &\le \frac{\mu}{8} \|s\|^2_{L^\infty(\Omega_0)} \|\nabla w\|^2_{L^2(\Omega_0)} + \frac{2}{\mu} \omega^2 \|z\|^2_{L^2(\mathcal{R})}\,;\end{aligned}$$

and, finally,

$$-\int_{\Omega_0} (w\cdot\nabla)\hat{u}_\lambda\cdot\omega z\,dy \le \frac{2L}{\pi}\|\nabla w\|_{L^2(\Omega_0)}\|\nabla\hat{u}_\lambda\|_{L^2(\Omega_0)}\omega\|z\|_{L^\infty(\Omega_0)}$$
$$\le \frac{4L^2}{\pi^2}\|\nabla w\|^2_{L^2(\Omega_0)}\|\nabla\hat{u}_\lambda\|^2_{L^2(\Omega_0)}+\frac{\omega^2}{4}\|z\|^2_{L^\infty(\mathcal{R})}.$$

Next, with the very same arguments used in Theorem 3.3, provided that ω is small enough we arrive at stating that there exists $c_3>0$ such that

$$\begin{aligned}
&\frac{d}{dt}E_\omega(t)+\omega F(h(t))+\frac{\mu}{4}\|\nabla w(t)\|^2_{L^2(\Omega_0)}+c_3 h'(t)^2 \qquad (4.13)\\
&\le \frac{2L}{\sqrt{3}}\left(\frac{2}{\pi}\right)^{3/2}\|\nabla w(t)\|^2_{L^2(\Omega_0)}\|\nabla\hat{u}_\lambda\|_{L^2(\mathcal{R})}+\frac{2L^{3/2}}{3^{1/4}}\left(\frac{2}{\pi}\right)^{7/4}\|\nabla s\|_{L^4(\Omega_0)}\|\nabla w(t)\|^2_{L^2(\Omega_0)}\\
&\quad+2\mathcal{D}_1(t)\left(\frac{4L^2}{\pi^2}\|\nabla w(t)\|^2_{L^2(\Omega_0)}+m\,h'(t)^2\right)+2\mathcal{D}_2(t)mh'(t)^2\\
&\quad+\frac{\mu}{4}\|s\|^2_{L^\infty(\Omega_0)}\|\nabla w(t)\|^2_{L^2(\Omega_0)}+\frac{8L^2}{\pi^2}\|\nabla w(t)\|^2_{L^2(\Omega_0)}\|\nabla\hat{u}_\lambda\|^2_{L^2(\Omega_0)}.
\end{aligned}$$

From Theorem 2.3 we have that, if $\lambda<\lambda_*$ (the threshold for uniqueness of solutions for the stationary problem), there exists $C_1=C_1(\lambda)>0$ such that $C_1(\lambda)\to 0$ as $\lambda\to 0$ and

$$\|\nabla\hat{u}_\lambda\|_{L^2(\Omega_0)}\le C_1(\lambda).$$

On the other hand, from Theorem 3.3 and Theorem 3.5,

$$\mathcal{D}_1(t),\mathcal{D}_2(t)\to 0 \qquad \text{as} \qquad \lambda\to 0 \qquad \forall\, t\ge t_0+1,$$

with t_0 as in (3.13). Thus, with t sufficiently large, every term on the right-hand side of (4.13) multiplying $\|\nabla w(t)\|^2_{L^2(\Omega_0)}$ and $h'(t)^2$ tends to 0 as $\lambda_*\to 0$. As a consequence, one can find $\lambda_1>0$ such that, if $\lambda<\lambda_1$, there exist $c_4,c_5>0$ such that

$$\frac{d}{dt}E_\omega+\omega F(h)+c_4\|w\|^2_{L^2(\Omega_0)}+c_5{h'}^2\le 0,$$

where we have used implicitly the Poincaré inequality. Use (2.4) and the definition of E to obtain

$$\frac{d}{dt}E_\omega+\gamma E\le 0,$$

with $\gamma=\min\{2\omega,2c_4,\frac{2c_5}{m}\}>0$. Then, renaming γ/c_2 as γ, we find from (3.4) that $\frac{d}{dt}E_\omega+\gamma E_\omega\le 0$, which, together with (3.4), implies that $c_1E(t)\le c_2E(0)e^{-\gamma t}$. Thus there exists $c>0$ such that

$$\|v(t) - \hat{u}_\lambda\|^2_{L^2(\Omega_0)} + mh'(t)^2 + 2F(h(t)) \le c\big(\|\hat{u}_0 - \hat{u}_*\|^2_{L^2(\Omega_0)} + mk_0{}^2\big)e^{-\gamma t}, \tag{4.14}$$

since $F(0) = 0$. From (4.14), the convergence of the solutions in $\mathcal{H}_0$ is established as $t \to \infty$, which proves the claim. □

5. The dynamical system approach

5.1. Semiflow vs semigroup

We now want to revisit the results of Section 3 within the framework of infinite-dimensional dynamical systems, where the solution is viewed as a trajectory in a suitable phase space. Let us begin with the abstract definition of a strongly continuous semiflow.

Definition 5.1. Let $(\mathcal{X}, d)$ be a complete metric space. A family of one-parameter maps $S(t) : \mathcal{X} \to \mathcal{X}$ is called a *(strongly continuous) semiflow* on $\mathcal{X}$ if

(i) $S(0) = \mathrm{id}_{\mathcal{X}}$ (the identity map in $\mathcal{X}$);
(ii) the map $t \mapsto S(t)x$ is continuous for all $x \in \mathcal{X}$;
(iii) the map $x \mapsto S(t)x$ is continuous for all $t \ge 0$.

If in addition the concatenation property holds, that is, $S(t + \tau) = S(t)S(\tau)$ for all $t, \tau \ge 0$, then $S(t)$ is called a *strongly continuous semigroup* (see, e.g., [48]).

Obviously, a semigroup would simplify the analysis of the dynamics but, as already mentioned, the evolution maps $S(t)$ of (2.26)-(2.27) do not satisfy the concatenation property. The main reason relies on peculiarity of (2.26)-(2.27), where at each time step the pair $(\hat{u}(t), h'(t))$ belongs to a different functional space $\mathcal{H}_{h(t)}$, for the domain of fluid $\Omega_{h(t)}$ depends itself on the solution. The key idea to overcome this difficulty, for a given initial position h_0 of the obstacle, is to map at every time t the cylindrical domain $\Omega_{h(t)} \times (0, T)$ onto $\Omega_{h_0} \times (0, T)$, via a suitable change of variables.

Throughout the whole section, let then

$$h_0 \in (-L + \delta, L - \delta)$$

be fixed, and denote

$$\mathcal{H} = \{z = (v, l) \in \mathbb{H}(\mathcal{R}) \,|\, v_{B_{h_0}} = l\hat{e}_2\}, \qquad \mathcal{H}^1 = \{z = (v, l) \in \mathbb{V}(\mathcal{R}) \,|\, v_{B_{h_0}} = l\hat{e}_2\},$$

to which we associate the norms

$$\|z\|^2_{\mathcal{H}} = \int_{\Omega_{h_0}} |v|^2 \, dx + m\, l^2, \qquad \|z\|^2_{\mathcal{H}^1} = \int_{\Omega_{h_0}} |\nabla v|^2 \, dx + m\, l^2,$$

where $z = (v, l)$, and m is the mass of the body as in (2.27). The spaces $\mathcal{H}$ and $\mathcal{H}^1$ are exactly the ones defined in (2.23), where the dependence of h_0 is dropped, since the position h_0 of the obstacle is now fixed. In particular, we have the compact embedding $\mathcal{H}^1 \Subset \mathcal{H}$. Given $z_0 = (\hat{u}_0, k_0) \in \mathcal{H}$, we consider the solution operator $U(t) : \mathcal{H} \to \mathcal{H}_{h(t)}$ of Section 3, recalling that $h(t)$ is the second component of the weak solution to (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$. Hence,

$$U(t)z_0 = (\hat{u}(t), h'(t)).$$

Let $\varepsilon_0 \in (0, L - \delta)$ be such that

$$\min_{t\in[0,T]} \operatorname{dist}(\partial B_{h(t)}, \Gamma) \geq \varepsilon_0. \tag{5.1}$$

The existence of such an ε_0 comes from Corollary 2.9. For this $h(t)$, we can build for any $t > 0$ the map $\psi(t, \cdot)$ of Proposition 4.4, where we take $\varepsilon = \varepsilon_0$, and define its inverse with respect to the space variables, that we denote by $\varphi(t, \cdot) = \psi^{-1}(t, \cdot)$. In order to recast our results in the semiflow language, the main ingredient is the introduction of the family of maps, depending on $h(t)$,

$$\Phi_t : \mathcal{H}_{h(t)} \to \mathcal{H},$$

given by

$$\Phi_t(\hat{u}(t), h'(t)) = (\nabla\psi(\varphi(t, y), t) \cdot \hat{u}(\varphi(t, y), t), h'(t)),$$

whose properties will be described later in this section. Then, we define the one-parameter family of operators

$$S(t) : \mathcal{H} \to \mathcal{H},$$

acting by the rule

$$z_0 \mapsto S(t)z_0 = \Phi_t(U(t)z_0). \tag{5.2}$$

Roughly speaking, what we do is to think the obstacle as fixed during the whole evolution. Accordingly, the variable h loses its physical meaning, since it does not represent any longer the position of the obstacle, but its effects appear inside the equation through the map Φ_t.

Theorem 5.2. *The map $S(t)$ fulfills the semiflow axioms* (i)-(iii) *of Definition 5.1 on the complete metric space $\mathcal{H}$, endowed with the distance induced by the norm $\|\cdot\|_{\mathcal{H}}$.*

As it will be clear, it is however false that $S(t)$ is a semigroup. The proof of Theorem 5.2 is carried out in the remaining of the section. In particular, in the next Subsection 5.2, we state and prove some preliminary results in order to properly characterize the action of the map Φ_t on problem (2.26)-(2.27). The conclusion of the proof will be given in Subsection 5.3, by verifying the semiflow properties of Definition 5.1.

5.2. Properties of the map Φ_t

Recalling that $h_0 \in (-L+\delta, L-\delta)$ has been fixed once for all, throughout this subsection, we consider a given $\varepsilon \in (0, L-\delta)$ and a given function

$$h \in W^{1,\infty}(0,T;[-L+\delta+\varepsilon, L-\delta-\varepsilon]) \qquad \forall T>0 \tag{5.3}$$

such that

$$h(0)=h_0, \qquad B_{h(t)} \subset \mathcal{O}_\varepsilon \qquad \forall t\in[0,T].$$

With this choice, let $s=s_\varepsilon$ be a function obtained through Lemma 2.1. Moreover, we can build the volume preserving diffeomorphism ψ of Proposition 4.4, along with its inverse $\varphi=\psi^{-1}$. Then, we denote

$$g_{ij}=\frac{\partial\varphi^k}{\partial y^i}\frac{\partial\varphi^k}{\partial y^j}, \qquad g^{ij}=\frac{\partial\psi^i}{\partial x^k}\frac{\partial\psi^j}{\partial x^k}, \qquad \Gamma^i_{kj}=g^{il}\left(\frac{\partial g_{kl}}{\partial y^j}+\frac{\partial g_{jl}}{\partial y^k}-\frac{\partial g_{kj}}{\partial y^l}\right)=\frac{\partial\psi^i}{\partial x^l}\frac{\partial^2\varphi^l}{\partial y^k\partial y^j}, \tag{5.4}$$

where g_{ij} defines a metric on $\mathbb{R}^2$ since $\det(\frac{\partial\psi_i}{\partial x_j})_{i,j}=1$. Call (now the space variable is y)

$$v(y,t)=\nabla\psi(\varphi(t,y),t)\cdot\hat{u}(\varphi(t,y),t) \qquad y\in\Omega_{h_0}, \tag{5.5}$$

the pullback of $\hat{u}$ by φ, and set

$$q(y,t)=p(\varphi(y,t),t).$$

We follow the procedure in [47, paragraph 4.2] to transform the Navier-Stokes equation (2.26) in the cylindrical domain $\Omega_{h_0}\times(0,T)$. Thanks to (4.6), for each term involving s, the maps ψ and φ correspond to the identity. Thus, we obtain the (weak) problem with variable coefficients in the new unknown v (at this stage, the function $h(t)$ is prescribed)

$$\begin{aligned}
&v_t+\mathcal{M}v-\mu\,\mathcal{L}v+\mathcal{N}v+(v\cdot\nabla)\,s+(s\cdot\nabla)\,v+\mathcal{G}q=\hat{g} \qquad \text{in } \Omega_{h_0}\times(0,T)\\
&\operatorname{div} v=0 \qquad \text{in } \Omega_{h_0}\times(0,T)\\
&v=0 \qquad \text{on } \Gamma\times(0,T)\\
&v=h'\hat{e}_2 \qquad \text{on } \partial B_h\times(0,T)\\
&\lim_{|y_1|\to\infty} v(y_1,y_2)=0\\
&v(0)=\hat{u}_0.
\end{aligned} \tag{5.6}$$

The operators $\mathcal{M}, \mathcal{L}, \mathcal{N}$ appearing in (5.6) are defined here below (the exponent i stands for the i-th component, and we use the Einstein notation).

$$
\begin{aligned}
(\mathcal{M}v)^i =& \partial_l v^i \partial_t \psi^l + \partial_k \psi^i (\partial_k \partial_t \varphi^i) v^k + \partial_k \psi^i \partial^2_{kl} \varphi^i \partial_t \psi^l v^k, \\
(\mathcal{L}v)^i =& \partial_k \psi^i \partial_j \psi^m (\partial^2_{mk} \varphi^i) \partial_l v^k \partial_j \psi^l + \partial_j \psi^m \partial^2_{ml} v^i \partial_j \psi^l + \partial_l v^i (\partial^2_{jj} \psi^l) \\
& + \partial_k \psi^i \partial_j \psi^m (\partial^3_{mlk} \varphi^i) \partial_j \psi^l v^k + \partial_k \psi^i (\partial^2_{lk} \varphi^i) \partial^2_{jj} \psi^l v^k \\
& + \partial_k \psi^i (\partial^2_{lk} \varphi^i) \partial_j \psi^l \partial_j \psi^m \partial_m v^k, \\
(\mathcal{N}v)^i =& v^l \partial_l v^i + \partial_k \psi^i v^l (\partial^2_{lk} \varphi^i) v^k, \\
(\mathcal{G}q)^i =& g^{ij} \partial_j q.
\end{aligned}
\tag{5.7}
$$

Remark 5.3. Note that:

- $(\partial_t + \mathcal{M})v$ corresponds to the original time derivative $\hat{u}_t$;
- $\mathcal{L}v$ corresponds to $\Delta \hat{u}$;
- $\mathcal{N}v$ corresponds to $(\hat{u} \cdot \nabla)\hat{u}$;
- $\mathcal{G}q$ corresponds to ∇p.

In particular, in $\mathcal{A}_\varepsilon$ these operators coincide with the original ones; the same is true in $\mathcal{O}_\varepsilon$, except for

$$(\partial_t + \mathcal{M})v = (\partial_t - h' \hat{e}_2 \cdot \nabla)v.$$

The first equation in (5.6) can be rewritten as

$$v_t - \mu \, \Delta v + (v \cdot \nabla) \, v + \nabla q + (v \cdot \nabla) \, s + (s \cdot \nabla) \, v = \hat{g} + \mathcal{F}(v, h, q),$$

where

$$\mathcal{F}(v, h, q) = \mu(\mathcal{L} - \Delta)v - \mathcal{M}v - (\mathcal{N}v - (v \cdot \nabla)v) - (\mathcal{G} - \nabla)q.$$

Observe that

$$\mathcal{F}(v, h, q) = \begin{cases} 0 & \text{in} \quad \mathcal{A}_\varepsilon \\ h' \hat{e}_2 \cdot \nabla v & \text{in} \quad \bar{\mathcal{O}}_\varepsilon, \end{cases}$$

thus $\mathcal{F}$ has compact support in Ω_{h_0}. The introduction of the maps ψ and φ allows to remove the dependence on time from the fluid domain, with a consequent strengthening of the coupling between the equations governing the motion of the fluid and the one governing the motion of the obstacle. Such a strengthening appears in the fictitious force $\mathcal{F} = \mathcal{F}(v, h, q)$, where the dependence on h is hidden in ψ and φ. This renders the dynamics structurally non-autonomous, and this is the reason why we do not end up with a semigroup.

Remark 5.4. If $h(t)$ is not just any prescribed function, but it is exactly the second component of the weak solution to (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$, and $\varepsilon = \varepsilon_0$ with ε_0 as in (5.1), we have the equivalence between (5.6) and the original equation (2.26), in terms of *strong solutions*. This was proven in [47, Propositions 4.5, 4.6], which in turn refers to [37, Theorem 2.5]

Here, we are interested in the construction of *weak solutions*. To this aim, leaning on some ideas of [41], we introduce for any fixed $t > 0$ the scalar products

$$\begin{aligned} \langle v_1, v_2 \rangle_t &= \int\limits_{\Omega_{h_0}} g_{ij}(y,t) v_1^i(y) v_2^j(y)\, dy, \\ \langle D_g v_1, D_g v_2 \rangle_t &= \int\limits_{\Omega_{h_0}} g_{ij}(y,t) g^{kl}(y,t) \nabla_k v_1^i \nabla_l v_2^j\, dy, \end{aligned} \tag{5.8}$$

where

$$\nabla_k v^i = \frac{\partial v^i}{\partial y^k} + \Gamma^i_{kj} v^j,$$

and we denote by

$$\|v\|_t^2 = \langle v, v \rangle_t \qquad \text{and} \qquad \|D_g v\|_t^2 = \langle D_g v, D_g v \rangle_t$$

the induced (square) norms. We emphasize that the scalar products in (5.8) explicitly depend on the choice of the function h in (5.3), that for the moment is understood to be given. Under the change of variables induced by φ, for any $t \geq 0$ we have the equalities

$$\langle v_1, v_2 \rangle_t = \int\limits_{\Omega_{h(t)}} \hat{u}_1 \cdot \hat{u}_2\, dx, \qquad \langle D_g v_1, D_g v_2 \rangle_t = \int\limits_{\Omega_{h(t)}} \nabla \hat{u}_1 : \nabla \hat{u}_2\, dx.$$

Moreover, since g_{ij} is a positive definite invertible matrix and the spatial derivatives of $\varphi(\cdot, t)$ are bounded functions (see also [37, Section 3]), there exist $C_1, C_2 > 0$ (depending on $T > 0$), such that, for any fixed $t \in [0, T]$,

$$C_1 \|v\|_{L^2(\Omega_{h_0})} \leq \|v\|_t \leq C_2 \|v\|_{L^2(\Omega_{h_0})}. \tag{5.9}$$

Analogously, there exist two positive constants C_3 and C_4 such that

$$C_3 \|\nabla v\|_{L^2(\Omega_{h_0})} \leq \|D_g v\|_t \leq C_4 \|\nabla v\|_{L^2(\Omega_{h_0})}. \tag{5.10}$$

This allows us to introduce the norms on $\mathcal{H}$ and $\mathcal{H}^1$

$$|z|_{t,\mathcal{H}} = \sqrt{\|v\|_t^2 + m\, l^2}\,, \qquad |z|_{t,\mathcal{H}^1} = \sqrt{\|D_g v\|_t^2 + m\, l^2}\,,$$

equivalent to the original ones. Again, we point out that such an equivalence is uniform for a fixed $T > 0$. Now we give the rigorous definition of a weak solution to problem (5.6).

Definition 5.5. Let the given function h comply with (5.3). A function v is a weak solution to (5.6), with initial value $v(0) = \hat{u}_0$, if

$$(v, h') \in L^2(0, T; \mathcal{H}^1) \cap L^\infty(0, T; \mathcal{H}), \qquad (\partial_t v, h'') \in L^2(0, T; \mathcal{H}^{-1}),$$

and, for any pair $(\tilde{\phi}, l) \in \mathcal{H}^1$ and almost every $t \geq 0$,

$$\begin{aligned} &\langle \partial_t v(t), \tilde{\phi} \rangle_t + \langle \mathcal{M} v(t), \tilde{\phi} \rangle_t + m\, h''(t) l + f(h(t))\, l - \mu \langle \mathcal{L} v(t), \tilde{\phi} \rangle_t + \langle \mathcal{N} v(t), \tilde{\phi} \rangle_t \\ &+ \int_{\Omega_{h_0}} (v(t) \cdot \nabla)\, s \cdot \tilde{\phi}\, dy + \int_{\Omega_{h_0}} (s \cdot \nabla)\, v(t) \cdot \tilde{\phi}\, dy = \int_{\Omega_{h_0}} \hat{g} \cdot \tilde{\phi}\, dy. \end{aligned} \tag{5.11}$$

We are ready to prove the equivalence between problem (5.6) and the original problem (2.26)-(2.27) in terms of weak solutions.

Proposition 5.6. *Let $(\hat{u}, h)$ be the weak solution to problem (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$, and let v be a weak solution to (5.6) with the same h and initial datum $\hat{u}_0$, in the sense of Definition 5.5. Then $\hat{u}$ and v are related by (5.5), that is,*

$$v(\cdot, t) = \nabla \psi(\varphi(t, \cdot), t) \cdot \hat{u}(\varphi(t, \cdot), t).$$

Proof. The proposition is proven by establishing a correspondence among each term in (5.11) and in (2.29). The function h is now the second component of the weak solution $(\hat{u}, h)$ to problem (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$, and ε_0 is as in (5.1). Then, we can build the map ψ of Proposition 4.4, where we take $\varepsilon = \varepsilon_0$, and define its inverse with respect to the space variables, $\varphi = \psi^{-1}$. From (5.5), we obtain that

$$\hat{u}(x, t) = \nabla \varphi(\psi(t, x), t) \cdot v(\psi(t, x), t). \tag{5.12}$$

Concerning the test function ϕ and $\tilde{\phi}$ appearing in the two definitions of solution, applying the change of variable we produce a bijection $\phi \leftrightarrow \tilde{\phi}$ given by

$$\phi(x, t) = \nabla \varphi(\psi(t, x), t) \cdot \tilde{\phi}(\psi(t, x)). \tag{5.13}$$

Indeed, as ψ and φ are volume preserving, we do not lose the divergence-free property of the functions (see for instance [37, Proposition 2.4]). Thus, by plugging (5.12)-(5.13) into (2.29), after integrating by parts and using the fact that $y = \psi(t, x) \in \Omega_{h_0}$, we obtain

$$\begin{aligned}
&\int_{\Omega_{h_0}} \partial_t[\nabla\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\,dy + m\,h''(t)l(t) + f(h(t))\,l(t) \\
&\quad -\mu\int_{\Omega_{h_0}} \Delta[\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\,dy \\
&\quad +\int_{\Omega_{h_0}} [(v(y,t)\cdot\nabla)\,s\cdot\tilde{\phi}(y) + (s\cdot\nabla)\,v(y,t)\cdot\tilde{\phi}(y)]\,dy \\
&\quad +\int_{\Omega_{h_0}} (\nabla\varphi(y,t)\cdot v(y,t)\cdot\nabla)\,[\nabla\varphi(y,t)\cdot v(y,t)]\cdot\nabla\varphi(y,t)\cdot\tilde{\phi}(y)\,dy = \int_{\Omega_{h_0}} \hat{g}\cdot\tilde{\phi}(y)\,dy.
\end{aligned} \tag{5.14}$$

We remark that in the equality above we have used the properties of a function s of Lemma 2.1, which is nonzero whenever φ, ψ are the identity, together with the function $\hat{g}$ of (2.20). From (5.7), we have that

$$\begin{aligned}
\partial_t[\nabla\varphi(y,t)\cdot v(y,t)] &= \partial_k\varphi^i\partial_t v^k + \partial_k\varphi^i\partial_l v^k\partial_t\psi^l + (\partial_k\partial_t\varphi^i)v^k + \partial^2_{kl}\varphi^i\partial_t\psi^l v^k \\
&= \partial_k\varphi^i\partial_t v^k + \partial_k\varphi^i(\mathcal{M}v)^i, \\
\Delta[\varphi(y,t)\cdot v(y,t)] &= \partial_j\psi^m(\partial^2_{mk}\varphi^i)\partial_l v^k\partial_j\psi^l + \partial_k\varphi^i\partial_j\psi^m\partial^2_{ml}v^k\partial_i\psi^l + \partial_k\varphi^i\partial_l v^k(\partial^2_{jj}\psi^l) \\
&\quad + \partial_j\psi^m(\partial^3_{mlk}\varphi^i)\partial_j\psi^l v^k + (\partial^2_{lk}\varphi^i)\partial^2_{jj}\psi^l v^k + (\partial^2_{lk}\varphi^i)\partial_j\psi^l\partial_j\psi^m\partial_m\hat{v}^k \\
&= \partial_k\varphi^i(\mathcal{L}v)^i, \\
(\nabla\varphi(y,t)\cdot v(y,t)\cdot\nabla)\,[\nabla\varphi(y,t)\cdot v(y,t)] &= \partial_k\varphi^i v^l\partial_l v^k + v^l(\partial^2_{lk}\varphi^i)v^k = \partial_k\varphi^i(\mathcal{N}v)^i.
\end{aligned}$$

Thus, through the definition of the scalar products in (5.8), we obtain that (5.14) is equivalent to (5.11), which completes the proof. □

5.3. Proof of Theorem 5.2

On account of (5.5), we rewrite the map Φ_t as

$$\Phi_t(\hat{u}(t), h'(t)) = (v(t), h'(t)),$$

where now v is defined by choosing the function $h(t)$ to be the second component of the weak solution to (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$, and $\varepsilon = \varepsilon_0$, with ε_0 as in (5.1). Then, point (i) of Definition 5.1 follows directly from the properties of ψ and φ. Point (ii) is a consequence of Theorem 2.8 and Proposition 5.6, from which we learn that (v, h') is equal almost everywhere to a continuous function from $[0,T]$ to $\mathcal{H}$ with respect to the norm $|\cdot|_{t,\mathcal{H}}$. By the equivalence relation between the norms given in (5.9), this implies the continuity with respect to $\|\cdot\|_{\mathcal{H}}$ as well. The next proposition proves point (iii).

Proposition 5.7. *Let $R > 0$ be arbitrarily fixed, and let $n = 1, 2$. For any pair of initial velocities*

$$z_{0,n} = (\hat{u}_{0,n}, k_{0,n}) \in \mathcal{H} \qquad \text{such that} \qquad \|z_{0,n}\|_{\mathcal{H}} \leq R,$$

the estimate

$$\|S(t)z_{0,1} - S(t)z_{0,2}\|_{\mathcal{H}} \leq K\|z_{0,1} - z_{0,2}\|_{\mathcal{H}}$$

holds for every $t \in [0, T]$, for some positive constant $K = K(R, T)$.

Proof. Let $z_{0,n} = (\hat{u}_{0,n}, k_{0,n}) \in \mathcal{H}$ be such that $\|z_{0,n}\|_{\mathcal{H}} \leq R$. Setting further $h_n(0) = h_0$, there exists a unique weak solution $(\hat{u}_n, h_n)$ to problem (2.26)-(2.27). From Corollary 2.9, there is ε_0, depending on R and T, such that $B_{h_n} \subset \mathcal{O}_{\varepsilon_0}$. Thus, through Lemma 2.1 we can build $s = s_{\varepsilon_0}$ as well as ψ_n as in Proposition 4.4 and $\varphi_n = \psi_n^{-1}$, where the subscript $n = 1, 2$ depends on whether we consider h_1 or h_2. In order to estimate the distance between $S(t)z_{0,1} = (v_1, h_1)$ and $S(t)z_{0,2} = (v_2, h_2)$ in terms of the distance between $z_{0,1}$ and $z_{0,2}$, we exploit again the result and the procedure implemented in [44, Section 5], where it is proven the uniqueness for solutions to problem (2.26)-(2.27). The same procedure has been used Section 4, and it can be recast here step-by-step, up to the obvious changes. In order to make the proof of the theorem self-contained, let us briefly describe the procedure: we introduce the two maps

$$\mathsf{F} : \mathbb{R}^+ \times \Omega_{h_2(t)} \longrightarrow \Omega_{h_1(t)} \qquad \text{and} \qquad \mathsf{G} : \mathbb{R}^+ \times \Omega_{h_1(t)} \longrightarrow \Omega_{h_2(t)},$$

defined as

$$\mathsf{F} = \varphi_1(t, \psi_2(t, x)) \qquad \text{and} \qquad \mathsf{G} = \varphi_2(t, \psi_1(t, x)).$$

This is possible since $h_1(0) = h_2(0) = h_0$. Following [44, Section 5], let

$$\hat{\mathfrak{u}}_2(x, t) = \nabla \mathsf{F}(\mathsf{G}(x), t) \cdot \hat{u}_2(\mathsf{G}(x), t) \qquad x \in \Omega_{h_1(t)}$$

be the pullback of $\hat{u}_2$ by G, and $q_2(x, t) = p_2(\mathsf{G}(x), t)$. Next, we call

$$w = \hat{u}_1 - \hat{\mathfrak{u}}_2, \qquad h = h_1 - h_2,$$

and we take the difference between the weak formulation satisfied by $(\hat{u}_1, h_1)$ and that satisfied by $(\hat{\mathfrak{u}}_2, h_2)$. We obtain

$$\langle \partial_t w, \phi \rangle + m\, h'' l + [f(h_1) - f(h_2)]\, l + \mu \int_{\mathcal{R}} \nabla w : \nabla \phi \, dx + \int_{\Omega_{h_1}} (\hat{u}_1 \cdot \nabla) w \cdot \phi \, dx$$
$$+ \int_{\Omega_{h_1}} (w \cdot \nabla) \hat{\mathfrak{u}}_2 \cdot \phi \, dx + \int_{\Omega_{h_1}} (s \cdot \nabla) w \cdot \phi \, dx + \int_{\Omega_{h_1}} (w \cdot \nabla) s \cdot \phi \, dx = \int_{\Omega_{h_1}} \mathfrak{f} \cdot \phi \, dx,$$

where the expression of $\mathfrak{f}$ reads as in (4.11), once we substitute v with $\hat{\mathfrak{u}}_2$, Ω_0 with Ω_{h_1}, φ with G and ψ with F. Then, we take $(\phi, l) = (w, h')$ and we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega_{h_1})} + m{h'}^2 + 2\int_{h_2}^{h_1} f(s)\,ds\right) + \mu\|\nabla w\|^2_{L^2(\Omega_{h_1})} \\
&\quad = -\int_{\Omega_{h_1}} (w\cdot\nabla)\hat{\mathfrak{u}}_2\cdot w\,dx - \int_{\Omega_{h_1}} (w\cdot\nabla)s\cdot w\,dx \\
&\qquad + \int_{\Omega_{h_1}} \mathfrak{f}\cdot\phi\,dx.
\end{aligned}
\tag{5.15}
$$

Next, we estimate each term on the right-hand side. The first two terms can be bounded by suitably exploiting the Hölder inequality, [28, (2.26)], the Poincaré inequality and the Young inequality. We obtain

$$
\begin{aligned}
\left|\int_{\Omega_{h_1}} (w\cdot\nabla)\hat{\mathfrak{u}}_2\cdot w\,dx\right| &\le \|w\|^2_{L^4(\Omega_{h_1})}\|\nabla\hat{\mathfrak{u}}_2\|_{L^2(\Omega_{h_1})} \\
&\le \left(\frac{2}{3\pi}\right)^{1/2}\|w\|_{L^2(\Omega_{h_1})}\|\nabla w\|_{L^2(\Omega_{h_1})}\|\nabla\hat{\mathfrak{u}}_2\|_{L^2(\Omega_{h_1})} \\
&\le \frac{2}{3\pi\mu}\|\nabla\hat{\mathfrak{u}}_2\|^2_{L^2(\Omega_{h_1})}\|w\|^2_{L^2(\Omega_{h_1})} + \frac{\mu}{4}\|\nabla w\|^2_{L^2(\Omega_{h_1})},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\int_{\Omega_{h_1}} (w\cdot\nabla)s\cdot w\,dx\right| &\le \|w\|^2_{L^4(\Omega_{h_1})}\|\nabla s\|_{L^2(\Omega_{h_1})} \\
&\le \left(\frac{2}{3\pi}\right)^{1/2}\|w\|_{L^2(\Omega_{h_1})}\|\nabla w\|_{L^2(\Omega_{h_1})}\|\nabla s\|_{L^2(\Omega_{h_1})} \\
&\le \frac{2}{3\pi\mu}\|\nabla s\|^2_{L^2(\Omega_{h_1})}\|w\|^2_{L^2(\Omega_{h_1})} + \frac{\mu}{4}\|\nabla w\|^2_{L^2(\Omega_{h_1})}.
\end{aligned}
$$

For what concerns the last term, analogously to what we did in Section 4, we can proceed step by step, up to the obvious changes, to obtain the existence of a function

$$
\begin{aligned}
&\mathcal{D}_1 = \mathcal{D}_1(t) \\
&\quad = Q(\|\hat{\mathfrak{u}}_2(t)\|_{L^2(\Omega_{h_1})}, \|\nabla\hat{\mathfrak{u}}_2(t)\|_{L^2(\Omega_{h_1})}, \|\hat{g}\|_{L^\infty(\mathcal{R})}, \|s\|_{L^\infty(\mathcal{R})}, \|\nabla s\|_{L^\infty(\mathcal{R})}) \in L^1(0,T),
\end{aligned}
$$

the L^1-bound depending only on R, such that

$$\int_{\Omega_{h_1}} \mathfrak{f} \cdot \phi \, dx \leq \mathcal{D}_1(\|w\|^2_{L^2(\Omega_{h_1})} + m{h'}^2) + \frac{\mu}{2}\|\nabla w\|^2_{L^2(\Omega_{h_1})}.$$

Then, calling

$$\Lambda(t) = \mathcal{D}_1(t) + \frac{2}{3\pi\mu}\|\nabla s\|^2_{L^2(\Omega_{h_1(t)})} + \frac{2}{3\pi\mu}\|\nabla \hat{\mathfrak{u}}_2(t)\|^2_{L^2(\Omega_{h_1(t)})},$$

and by inserting all the above inequalities in (5.15), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2_{L^2(\Omega_{h_1})} + m{h'}^2 + 2\int_{h_2}^{h_1} f(s)\, ds\right) \leq \Lambda(\|w\|^2_{L^2(\Omega_{h_1})} + m{h'}^2).$$

Moreover, defining the functions

$$\begin{aligned}\Theta(t) &= \|w(x,t)\|^2_{L^2(\Omega_{h_1(t)})} + mh'(t)^2 \\ &= \|\hat{u}_1(x,t) - \nabla \mathsf{F}(\mathsf{G}(x),t) \cdot \hat{u}_2(\mathsf{G}(x),t)\|^2_{L^2(\Omega_{h_1(t)})} + m(h_1'(t) - h_2'(t))^2,\end{aligned}$$

and observing that

$$\Theta(0) = \|\hat{u}_{0,1} - \hat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2,$$

we obtain

$$\Theta(t) \leq \Theta(0) + \int_0^t 2\,\Lambda(\tau)\Theta(\tau)\, d\tau \qquad \forall\, t \in [0,T].$$

The Gronwall Lemma (integral form) then gives

$$\Theta(t) \leq K\Theta(0) \qquad \forall\, t \in [0,T], \tag{5.16}$$

having set $K = \exp\left[\int_0^T 2\,\Lambda(\tau)\, d\tau\right]$. The final step is to rewrite (5.16) on Ω_{h_0} by applying the coordinate transformation $x = \varphi_1(t,y)$. Given

$$v_1(y,t) = \nabla\psi_1(\varphi_1(t,y),t) \cdot \hat{u}_1(\varphi_1(t,y),t) \quad \text{and} \quad v_2(y,t) = \nabla\psi_2(\varphi_2(t,y),t) \cdot \hat{u}_2(\varphi_2(t,y),t),$$

for all $t \in [0,T]$ we obtain

$$\begin{aligned}&\|\nabla\varphi_1(t,y) \cdot v_1(y,t) - \nabla\varphi_1(t,y) \cdot v_2(y,t)\|^2_{L^2(\Omega_{h_0})} + m(h_1'(t) - h_2'(t))^2 \\ &\leq K\Big(\|\hat{u}_{0,1} - \hat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2\Big),\end{aligned}$$

which in turn can be rewritten as

$$\|v_1(t) - v_2(t)\|_t^2 + m(h_1'(t) - h_2'(t))^2 \le K\Big(\|\hat{u}_{0,1} - \hat{u}_{0,2}\|^2_{L^2(\Omega_{h_0})} + m(k_{0,1} - k_{0,2})^2\Big).$$

Note that the norm $\|\cdot\|_t$ above is constructed by taking φ_1 in (5.4). Therefore, recalling the definition of the norm $|\cdot|_{t,\mathcal{H}}$, we arrive at

$$|S(t)z_{0,1} - S(t)z_{0,2}|_{t,\mathcal{H}} \le K\|z_{0,1} - z_{0,2}\|_{\mathcal{H}} \qquad \forall\, t \in [0, T].$$

The desired conclusion follows by applying (5.9), up to redefining the constant K. □

6. The global attractor of the semiflow

The further step is to translate the dissipative features of our system in the semiflow language. Let us begin by recalling some classical notions (see, e.g., [14,48,3]). In what follows, $S(t)$ is a strongly continuous semiflow acting on a complete metric space $(\mathcal{X}, d)$.

Definition 6.1. A set $\mathcal{B}_0 \subset \mathcal{X}$ is called an *absorbing set* for $S(t)$ if for every bounded set $\mathcal{B} \subset \mathcal{X}$ there exists an *entering time* $t_{\mathcal{B}} \ge 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0 \qquad \forall\, t \ge t_{\mathcal{B}}.$$

The existence of a bounded absorbing set witnesses the dissipative character of a semiflow, since the dynamics is eventually confined in a bounded subset of the phase space. And indeed, in the recent literature, the definition of a dissipative semiflow is exactly the one of a semiflow possessing a bounded absorbing set. Nonetheless, in spite of its boundedness, an absorbing set can be to some extent a very large object. For instance, if $\mathcal{X}$ is a (closed) subset of a Banach space, an absorbing set might share the same dimension of the whole space (think to a ball). For this reason, one would like to exhibit a stronger form of dissipativity. The natural way to do that is to invoke compactness, since this is the correct notion to translate the fact that the dynamics loses degrees of freedom. Accordingly, the strategy is to look for the existence of compact sets, hence meager in the space, able to attract (in a suitable sense) all the trajectories of the semiflow in the long-time. This attraction property is expressed in terms of Hausdorff semidistance in $\mathcal{X}$: given two (nonempty) sets $\mathcal{B}, \mathcal{C} \subset X$, their Hausdorff semidistance is defined as

$$\boldsymbol{\delta}(\mathcal{B}, \mathcal{C}) = \sup_{x \in \mathcal{B}} d(x, \mathcal{C}) = \sup_{x \in \mathcal{B}} \inf_{y \in \mathcal{C}} d(x, y).$$

In a completely equivalent manner, we can write

$$\boldsymbol{\delta}(\mathcal{B}, \mathcal{C}) = \inf\big\{\varepsilon > 0 : \mathcal{B} \subset \mathcal{O}_\varepsilon(\mathcal{C})\big\},$$

where $\mathcal{O}_\varepsilon(\mathcal{C}) = \bigcup_{y \in \mathcal{C}} \big\{x \in \mathcal{X} : d(x, y) < \varepsilon\big\}$ is the ε-neighborhood of $\mathcal{C}$.

Definition 6.2. A set $\mathcal{K} \subset \mathcal{X}$ is called an *attracting set* for $S(t)$ if, for every bounded set $\mathcal{B} \subset \mathcal{X}$,

$$\lim_{t\to\infty} \boldsymbol{\delta}(S(t)\mathcal{B}, \mathcal{K}) = 0.$$

Whenever there exists a compact attracting set the semiflow is said to be *asymptotically compact*.

Remark 6.3. Clearly, an absorbing set is in particular an attracting set. It is also apparent that if the semiflow is asymptotically compact, then it is dissipative, in the sense that it possesses a bounded absorbing set.

Once the existence of a compact absorbing set is established, one might ask if there is the best possible one among those sets. This leads to our last definition.

Definition 6.4. The *global attractor* $\mathcal{A}$ of $S(t)$ is the smallest compact attracting set.

In the literature, the notion of global attractor is usually given in the context of semigroups, and not just semiflows (see, e.g., [48,3]). In particular, with the only exception of [14], the classical definition differs from the one given above since, besides the attraction property, one requires also the invariance, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$. Unfortunately, when dealing with semiflows (and not semigroups), the invariance seems to be out of reach. Nonetheless, our definition makes perfectly sense. The only problem is the existence of such a set. To this aim, we state the following result.

Theorem 6.5. *An asymptotically compact semiflow possesses the global attractor in the sense of Definition 6.4.*

Proof. The idea of the proof is somehow already contained in [14], although in that paper $S(t)$ is a semigroup. In fact, the theorem remains true if $S(t)$ is a one-parameter selfmap of $\mathcal{X}$, without requiring any of the axioms (i)-(iii) of Definition 5.1. Consider the family of sets

$$\mathbb{K} = \big\{\mathcal{K} \subset \mathcal{X} : \mathcal{K} \text{ is compact and attracting}\big\},$$

which, due to the hypothesis, is nonempty. Besides, let $\mathfrak{C}$ be the collection of all possible sequences of the form

$$y_n = S(t_n)x_n,$$

where x_n is a bounded sequence in $\mathcal{X}$ and $t_n \to \infty$. For any $y_n \in \mathfrak{C}$ we denote

$$\mathfrak{L}(y_n) = \big\{w \in \mathcal{X} : y_n \to w \text{ up to a subsequence}\big\}.$$

Note that $\mathfrak{L}(y_n) \neq \emptyset$. Indeed, let $\mathcal{K} \in \mathbb{K}$. Then there exists $w_n \in \mathcal{K}$ such that

$$d(y_n, w_n) \to 0.$$

Invoking the compactness of $\mathcal{K}$, there is $w \in \mathcal{K}$ and a subsequence w_{n_i} converging to w. Hence,

$$d(y_{n_i}, w) \le d(y_{n_i}, w_{n_i}) + d(w_{n_i}, w) \to 0.$$

Finally, define the set

$$\mathcal{A}^\star = \bigcup_{y_n \in \mathfrak{C}} \mathfrak{L}(y_n).$$

We claim that $\mathcal{A}^\star$ is attracting: if not, there exist a bounded set $\mathcal{B} \subset \mathcal{X}$, a sequence $t_n \to \infty$ and $\varepsilon > 0$ such that

$$\boldsymbol{\delta}(S(t_n)\mathcal{B}, \mathcal{A}^\star) \ge 2\varepsilon.$$

From the definition of Hausdorff semidistance, this implies the existence of a sequence $x_n \in \mathcal{B}$, hence bounded, for which

$$d(S(t_n)x_n, \mathcal{A}^\star) \ge \varepsilon.$$

But, as we saw, $y_n = S(t_n)x_n$ has limit points, which belong to $\mathcal{A}^\star$ by construction. This yields the claim. It is also apparent that $\mathcal{A}^\star$ is contained in any closed attracting set. Accordingly, the set

$$\mathcal{A} = \overline{\mathcal{A}^\star} \quad \text{(closure in } \mathcal{X})$$

is the smallest element of $\mathbb{K}$. An equivalent way to define $\mathcal{A}$ is to put

$$\mathcal{A} = \bigcap_{\mathcal{K} \in \mathbb{K}} \mathcal{K},$$

noting that the (compact) sets $\mathcal{K} \in \mathbb{K}$ fulfill the finite intersection property, for they all contain $\mathcal{A}^\star$. $\square$

We can now go back to our particular semiflow $S(t)$ on $\mathcal{H}$ associated to problem (2.26)-(2.27), and defined in (5.2). The main result of this section reads as follows.

Theorem 6.6. *The semiflow $S(t) : \mathcal{H} \to \mathcal{H}$ possesses the global attractor.*

Remark 6.7. In the situation of Theorem 4.1, the attractor reduces to the unique stationary point, and the convergence rate to the attractor is exponential. But in the general case, as it happens for the simpler situation of semigroups, the structure of the attractor might be very complicated, and the convergence rate is not predictable: in principle it could be arbitrarily slow. An interesting issue would be to consider a different object, relaxing the minimality condition characterizing the global attractor, which remains compact and attracts the trajectories at an exponential rate. Paralleling the terminology of the classical theory, this object would be an exponential attractor.

Before entering the details of the proof, let us recall once again that we are working under the hypothesis that $h_0 \in (-L + \delta, L - \delta)$ is fixed. Theorem 6.6 makes use of the technical results of Section 5.2. In view of the definition of the semiflow $S(t)$, the function $h(t)$ will always be the second component of the weak solution to (2.26)-(2.27) with initial data $(\hat{u}_0, h_0, k_0)$. Our purpose is to investigate the long-time dynamics of $S(t)z_0$ as $z_0 = (\hat{u}_0, k_0)$ is allowed to run in a bounded set of $\mathcal{H}$. To this aim, we need to improve (and make uniform) the equivalence relations between the norms given in (5.9) and (5.10). This can be done by exploiting the dissipativity properties of the solution operator $U(t)$ of Section 3.

Proof of Theorem 6.6. In the light of Theorem 6.5, all we need to do is showing that $S(t)$ is asymptotically compact. In fact, we will obtain a stronger result, namely, the existence of a compact absorbing set $\mathcal{B}_1 \subset \mathcal{H}$. Indeed, given a bounded set $\mathcal{B} \subset \mathcal{H}$, we know from Theorem 3.3 and Theorem 3.5 that there exist two universal constants $R_0, R_1 > 0$ and two entering times

$$t_0 = t_0(\mathcal{B}) \qquad \text{and} \qquad t_1 = t_1(\mathcal{B}) = t_0 + 1$$

such that, for every $z_0 \in \mathcal{B}$,

$$\|U(t)z_0\|_{\mathcal{H}_{h(t)}} = \sqrt{\|\hat{u}(t)\|_{L^2(\Omega_{h(t)})} + mh'(t)^2} \le R_0 \qquad \forall\, t \ge t_0, \tag{6.1}$$

and

$$\|U(t)z_0\|_{\mathcal{H}^1_{h(t)}} = \sqrt{\|\nabla\hat{u}(t)\|_{L^2(\Omega_{h(t)})} + mh'(t)^2} \le R_1 \qquad \forall\, t \ge t_1. \tag{6.2}$$

Inequality (6.1), together with (3.5), implies the existence of a constant $C = C(R_0)$ such that

$$\|h\|_{W^{1,\infty}(t_0,\infty;\mathbb{R})} \le C.$$

Thus, for every $t \ge t_0$, relations (5.9)-(5.10) improve into

$$C_1\|v\|_{L^2(\Omega_{h_0})} \le \|v\|_t \le C_2\|v\|_{L^2(\Omega_{h_0})}, \tag{6.3}$$

$$C_3\|\nabla v\|_{L^2(\Omega_{h_0})} \le \|D_g v\|_t \le C_4\|\nabla v\|_{L^2(\Omega_{h_0})}, \tag{6.4}$$

where now the constants C_1, C_2, C_3, C_4 depend only on R_0 (and on t_0). Invoking the coordinate transformation φ, we have the equality

$$\|\nabla\hat{u}(t)\|_{L^2(\Omega_{h(t)})} = \|D_g v(t)\|_t.$$

Looking at (6.2) and to the definition of $|\cdot|_{t,\mathcal{H}^1}$, this yields

$$|S(t)\mathcal{B}|_{t,\mathcal{H}^1} = \|U(t)\mathcal{B}\|_{\mathcal{H}^1_{h(t)}} \le R_1 \qquad \forall\, t \ge t_1.$$

Hence, taking $t \ge t_1 > t_0$, from the (uniform) equivalence of the norms established in (6.3) and (6.4), up to redefining the universal constant R_1, we conclude that

$$\|S(t)\mathcal{B}\|_{\mathcal{H}^1} \le R_1 \qquad \forall\, t \ge t_1.$$

This means that the ball $\mathcal{B}_1$ of $\mathcal{H}_1$ of radius R_1, which is compact in $\mathcal{H}$ in view of the compact embedding $\mathcal{H}^1 \Subset \mathcal{H}$, is an absorbing set for $S(t)$. $\square$

Declaration of competing interest

All authors declare that they have no conflicts of interest.

Data availability

No data was used for the research described in the article.

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