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Calculus of Variations

Steady-state Navier–Stokes flow in an obstructed pipe under mixed boundary conditions and with a prescribed transversal flux rate

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Abstract

The steady motion of a viscous incompressible fluid in an obstructed finite pipe is modeled through the Navier–Stokes equations with mixed boundary conditions involving the Bernoulli pressure and the tangential velocity on the inlet and outlet of the tube, while a transversal flux rate F is prescribed along the pipe. Existence of a weak solution to such Navier–Stokes system is proved without any restriction on the data by means of the Leray–Schauder Principle, in which the required a priori estimate is obtained by a contradiction argument based on Bernoulli's law. Through variational techniques and with the use of an exact flux carrier, an explicit upper bound on F (in terms of the viscosity, diameter and length of the tube) ensuring the uniqueness of such weak solution is given. This upper bound is shown to converge to zero at a given rate as the length of the pipe goes to infinity. In an axially symmetric framework, we also prove the existence of a weak solution displaying rotational symmetry.

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Contents

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1 Introduction

In 1838, the French physicist Jean Léonard Marie Poiseuille gave a preliminary oral report to the Philomatic Society of Paris [65] concerning the effects of pressure, tube length and diameter on the flow of water through glass tubes. In 1839, the Prussian hydraulic engineer Gotthilf Heinrich Ludwig Hagen conducted experiments on the flow of water in cylindrical brass pipes, the outcome of which were then published in the paper [33]. Independently from each other [73], both Poiseuille and Hagen empirically derived a physical law that gives the pressure drop in an incompressible and Newtonian fluid flowing in laminar regime through a sufficiently long, straight, circular pipe of constant cross section. Precisely, the celebrated Hagen–Poiseuille law [53, Chapter II] states

$$\Delta p = \frac{16\mu h F}{\pi R^4},$$

where:

- Δp is the pressure difference between the outflow and inflow;
- μ is the dynamic viscosity of the fluid;
- $2h$ is the length of the pipe;
- F is the (volumetric) transversal flow rate;
- R is the radius of the pipe.

Theoretically justified in 1845 by George Gabriel Stokes [71], the Hagen–Poiseuille law has ever since played a fundamental role in the development of Fluid Mechanics. Not only it has provided conclusive evidence for the use of the no-slip boundary condition on solid boundaries (see [52, Chapter XI]), it also yields an (axisymmetric) exact solution for the steady-state Navier–Stokes equations governing the motion of a viscous incompressible fluid along a finite straight circular pipe (see identity (3.3) below). In fact, such exact solution is uniquely determined by the flow rate F, see again [53, Chapter II]. This raises a first natural question:

If the pipe contains an obstruction (fixed obstacle) and we prescribe a flow rate F along the pipe, can we compare the resulting Navier–Stokes flow with the Hagen–Poiseuille flow determined by F?

Of course, the previous inquiry makes sense only if the corresponding Navier–Stokes system with a prescribed transversal flux rate is solvable. One is therefore led to impose boundary conditions on the different parts of the boundary of the pipe and of the obstacle as well. This is also motivated by the fact that a large variety of problems in Mathematical Fluid Mechanics are studied in unbounded pipes or in a system of unbounded pipes (that is, domains having non-compact boundaries); see in particular the celebrated *Leray problem* [64], the works of Ladyzhenskaya & Solonnikov [48, 51], Amick [4, 5], Pileckas et al. [37, 69] and the recent articles [76, 77]. Nevertheless, not only the numerical approximation of these problems must be set in bounded domains (finite pipes or conjunction of finite pipes) [35], also theoretical approaches have been devised in regions with compact boundaries (for example, Leray's argument on the *invading domains* [45, 55]) in order to tackle the original problem, thereby introducing artificial boundary conditions on truncating surfaces, see the articles by Blazy, Nazarov & Specovius-Neugebauer [9, 59] and references therein. A second natural question then arises:

Which boundary conditions should be imposed on the different parts of the boundary of the pipe and of the obstacle?

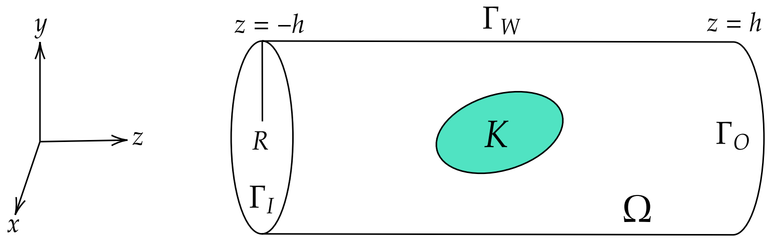

Fig. 1 Representation of the domain Ω

We follow the model studied by Korobkov, Pileckas & Russo in [46], previously discussed briefly in [35], that considers non-standard boundary conditions in the sense of [14, 18, 38, 66]. In the space $\mathbb{R}^3$ we use a system of cylindrical coordinates $(\rho, \theta, z) \in [0, \infty) \times [0, 2\pi] \times \mathbb{R}$, in which any spatial point will be denoted by $\xi = \rho\widehat{\rho} + z\widehat{k}$, with $\rho \geq 0$, $z \in \mathbb{R}$ and $\left\{\widehat{\rho}, \widehat{\theta}, \widehat{k}\right\} \subset \mathbb{R}^3$ the orthonormal basis in this geometry. Given $h > R > 1$, we consider an open straight cylinder $\mathcal{M}$ of radius R and length $2h$ whose axis of symmetry is directed along the z-axis:

$$\mathcal{M} = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R, \ -h < z < h\right\}.$$

Let $K \subset \mathcal{M}$ be an open, bounded and simply connected set with a Lipschitz boundary such that $\partial K \cap \partial\mathcal{M} = \emptyset$, and define the domain

$$\Omega = \mathcal{M} \backslash \overline{K}. \tag{1.1}$$

We decompose the boundary of Ω as $\partial\Omega = \Gamma_I \cup \Gamma_W \cup \Gamma_O$, where

$$\begin{gathered}\Gamma_I = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R\,, \ z = -h\right\}, \qquad \Gamma_O = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R\,, \ z = h\right\}, \\ \Gamma_W = \left\{\xi \in \mathbb{R}^3 \mid \rho = R\,, \ -h < z < h\right\} \cup \partial K.\end{gathered} \tag{1.2}$$

The outward unit normal to $\partial\Omega$ is denoted by ν. Henceforth we will refer to Γ_I and Γ_O in (1.2) as the *inlet* and *outlet* of Ω, respectively, while Γ_W includes all the *physical walls* of Ω (Fig. 1).

In the present article we study the steady-state Navier–Stokes equations with mixed boundary conditions on the different parts of $\partial\Omega$, that is, the following system of partial differential equations:

$$\begin{cases} -\eta\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \Gamma_W, \\ u \times \nu = 0, \quad p + \dfrac{1}{2}|u|^2 = p_- \quad \text{on} \quad \Gamma_I, \\ u \times \nu = 0, \quad p + \dfrac{1}{2}|u|^2 = p_+ \quad \text{on} \quad \Gamma_O, \\ \displaystyle\int_{\Sigma(s)} u \cdot \widehat{k} = F \qquad \forall s \in [-h, h]. \end{cases} \tag{1.3}$$

In (1.3), $\eta > 0$ is the (constant) kinematic viscosity of the fluid, $u : \Omega \longrightarrow \mathbb{R}^3$ is the velocity vector field, $p : \Omega \longrightarrow \mathbb{R}$ is the scalar pressure and $f : \Omega \longrightarrow \mathbb{R}^3$ represents an external force acting on the fluid. While $(1.3)_2$ describes the usual no-slip boundary condition on the physical walls Γ_W, the first equality in $(1.3)_3$–$(1.3)_4$ dictates that the fluid flow must enter and leave the domain Ω orthogonal to the inlet and outlet walls. The second identity in (1.3)$_3$–(1.3)$_4$ imposes that, respectively on the inlet Γ_I and outlet Γ_O, the *Bernoulli pressure* defined as $\Phi \doteq p + |u|^2/2$ must equal some constants $p_{\mp} \in \mathbb{R}$ that represent the *unknown pressure drop* $p_- - p_+$ along the pipe (therefore, $p_{\mp}$ are unknown, not prescribed, constants that depend on the solution). Finally, (1.3)$_5$ dictates that the transversal flow rate of the velocity field must be constant along the pipe, given by a quantity $F \in \mathbb{R}$, where we have defined

$$\Sigma(s) \doteq \{\xi \in \Omega \mid 0 < \rho < R\,,\ z = s\} \qquad \forall s \in [-h, h].$$

In the literature concerning the steady-state Navier–Stokes equations (or related models) with mixed boundary conditions in bounded domains, existence of weak solutions is usually ensured under a smallness assumption on the data, see [2, 3, 13, 14, 25, 40, 70]. In the case of an unobstructed pipe ($K = \emptyset$), the authors of [46] show the existence of a generalized solution to (1.3) (for *any* external force and flow rate F) by means of the Leray–Schauder Theorem, making use of the additional regularity provided by the smoothness of the lateral boundary of $\mathcal{M}$. Since K has merely a Lipschitz boundary, such approach cannot be directly applied to our model. We nevertheless prove the existence of a generalized solution to (1.3) (without any restriction on the data, see Theorem 3.4) through the following procedure:

- Firstly, by assuming that K has a $\mathcal{C}^2$-boundary, in Theorem 3.1 we provide a smooth flux carrier of F, which enables us to show in Theorem 3.2 that any weak solution of (1.3) is, in fact, a strong solution. Then, in Theorem 3.3 we apply the Leray–Schauder Principle in order to prove the existence of (at least) one weak solution to (1.3): in this context, the required a priori estimates are obtained through a contradiction argument that employs Bernoulli's law [41] for solutions of the Euler equation (3.54).
- Secondly, whenever K has a Lipschitz boundary, in Theorem 3.4 (the most important of the present article) we take a family of smooth domains $(K_n)_{n\in\mathbb{N}}$ that *outer approximates* K in such a way that the Lipschitz character of K_n remains uniformly bounded with respect to $n \in \mathbb{N}$, giving us a uniform control on domain dependence of the constants involved. Such approximation scheme was first designed by Nečas in [60] and subsequently applied by Verchota [74, 75], Daners [16, 17] and many other authors. Theorem 3.3, combined with a contradiction argument that again employs Bernoulli's law, enables us to obtain a sequence of weak solutions $\{(u_n, p_n)\}_{n\in\mathbb{N}} \subset H^1(\Omega_n) \times L^2(\Omega_n)$ to (1.3) in $\Omega_n \doteq \mathcal{M}\backslash\overline{K_n}$ whose norms are uniformly bounded. Since the elements of this sequence are defined on different domains, a suitable extension allows us to extract a sub-sequence that converges (in a sense made precise in Theorem 3.4) to a weak solution $(u, p) \in H^1(\Omega) \times L^2(\Omega)$ of (1.3) in Ω.

In [46, Section 5] the authors prove the unique solvability of (1.3) under a smallness assumption on the data. Since such smallness assumptions are ubiquitous in Mathematical Fluid Mechanics (also for the existence of some Navier–Stokes flows, see again [2, 3, 13, 14, 25, 40, 70] or [27, Chapter XIII]), interest in the possibility of quantifying them was already shown in the work of Amick [4] (for the existence of generalized solutions to Leray's problem) and then continued in [29, 30] for the unique solvability of Navier–Stokes equations with non-homogeneous Dirichlet boundary conditions in some particular domains. In Theorem 3.4 we show how the smallness assumption for unique solvability of (1.3) depends strongly on the size of a *given* flux carrier, on the Poincaré constant of Ω and on the Sobolev constant for the embedding of some closed subspace of $H^1(\Omega)$ into $L^4(\Omega)$. Through the use of variational techniques (namely, symmetrization arguments and the concentration-compactness principle), in Sect. 2.1 we give lower and upper bounds for these Sobolev constants in terms of R and h, while in Theorem 3.1 and Corollary 3.1 we construct a flux carrier of the flow rate F whose Dirichlet norm can be explicitly estimated in terms of the *relative capacity* of K (inside $\mathcal{M}$) and of the *Bogovskii constant* of Ω. This justifies the purpose of Sect. 2.2, where we provide a lower bound for the relative capacity of K which is independent of the shape and position of K inside $\mathcal{M}$, and Sect. 2.3, where a *universal* lower bound for the Bogovskii constant of any bounded Lipschitz domain in $\mathbb{R}^3$ is given. These results allow us to write in Corollary 3.3 an explicit upper bound (in terms of η, $|K|$, R and h) on the flow rate F that guarantees the unique solvability of (1.3). Moreover, in the case of zero external force and when K is the unit ball of $\mathbb{R}^3$, we prove that such upper bound decays like $h^{-1/2}$ as $h \to \infty$ (here $R > 1$ is fixed). It is left as an open question the possibility of improving this asymptotic behavior.

Symmetry results concerning the Navier–Stokes equations have been the subject of many mathematical studies, due to their connection with regularity issues [1, 62, 63] and the solvability of the renowned *Leray-flux problem* [42–44], among others. Whenever the (smooth) obstacle K and the external force f are *axisymmetric*, in Sect. 3.3 we address the question of existence of axisymmetric solutions to (1.3), that is, solutions displaying rotational symmetry with respect to the z-axis. This result is indeed achieved by firstly assembling an axisymmetric flux carrier of the flow rate, see Theorem 3.5, which can be considered as the axially symmetric version of Theorem 3.1. Then Theorem 3.6 guarantees the existence of a weak axisymmetric solution to (1.3) (without further restrictions on the data) as a consequence of the Leray–Schauder Theorem, properly adapted to spaces containing axisymmetric vector fields. In particular, in this symmetric framework, Corollary 3.3 yields an explicit upper bound on F ensuring that the unique weak solution of (1.3) is axisymmetric.

2 Functional inequalities

We emphasize that, for the sake of simplicity, no distinction will be made for the notation of functional spaces of scalars, vectors or matrices.

2.1 Explicit bounds for some Sobolev embedding constants

We consider the following Sobolev space of functions vanishing on Γ_W:

$$S(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_W\}.$$

Since $|\Gamma_W| > 0$, the Poincaré inequality holds in $S(\Omega)$, which means that $v \mapsto \|\nabla v\|_{L^2(\Omega)}$ is indeed a norm on $S(\Omega)$. Moreover, in view of the embedding $H^1(\Omega) \subset L^p(\Omega)$ for every $p \in [2, 6]$, the Sobolev constant for the embeddings $S(\Omega) \subset L^p(\Omega)$ admits a variational definition:

$$\mathcal{S}_p \doteq \min_{w \in S(\Omega)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^p(\Omega)}} \qquad \forall p \in [2, 6]. \tag{2.1}$$

For every $p \in [2, 6]$ we have $\mathcal{S}_p > 0$ and, by (2.1),

$$\mathcal{S}_p \, \|v\|^2_{L^p(\Omega)} \leq \|\nabla v\|^2_{L^2(\Omega)} \qquad \forall v \in S(\Omega). \tag{2.2}$$

We emphasize that the inequality in (2.2) is equally valid for scalar or vector functions (with the same constant). Indeed, if $v = (v_1, v_2, v_3) \in S(\Omega)$ is a vector field, by the Minkowski inequality we get

$$
\begin{aligned}
\|v\|^p_{L^p(\Omega)} &= \big\| \, |v_1|^2 + |v_2|^2 + |v_3|^2 \, \big\|^{p/2}_{L^{p/2}(\Omega)} \\
&\leq \left(\|v_1\|^2_{L^p(\Omega)} + \|v_2\|^2_{L^p(\Omega)} + \|v_3\|^2_{L^p(\Omega)} \right)^{p/2} \\
&\leq \left(\frac{1}{\mathcal{S}_p} \right)^{p/2} \left(\|\nabla v_1\|^2_{L^2(\Omega)} + \|\nabla v_2\|^2_{L^2(\Omega)} + \|\nabla v_3\|^2_{L^2(\Omega)} \right)^{p/2} \\
&= \left(\frac{1}{\mathcal{S}_p} \right)^{p/2} \|\nabla v\|^p_{L^2(\Omega)}.
\end{aligned}
\tag{2.3}
$$

The purpose of this section is to provide explicit lower and upper bounds for the constants $\mathcal{S}_2$ and $\mathcal{S}_4$. We firstly treat the case $p = 2$ (Poincaré inequality). Let $J_0 : [0, \infty) \longrightarrow \mathbb{R}$ be the Bessel function of the first kind of order zero, whose first zero is given by

$$
\mu_0 \approx 2.40483. \tag{2.4}
$$

Theorem 2.1 *Let Ω be as in (1.1) and $\mu_0 > 0$ as in (2.4). For any vector field $u \in S(\Omega)$ one has*

$$
\|u\|_{L^2(\Omega)} \leq \frac{\sqrt{3}}{\left(\max \left\{ \pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}| - |K|)}}, \dfrac{\sqrt{16h^2\mu_0^2 + \pi^2 R^2}}{4Rh} \right\}^2 - \dfrac{2}{h^2} \right)^{1/2}} \|\nabla u\|_{L^2(\Omega)}. \tag{2.5}
$$

In particular,

$$
\mathcal{S}_2 \geq \frac{1}{3} \left(\max \left\{ \pi \sqrt[3]{\frac{2\pi}{3(|\mathcal{M}| - |K|)}}, \frac{\sqrt{16h^2\mu_0^2 + \pi^2 R^2}}{4Rh} \right\}^2 - \frac{2}{h^2} \right). \tag{2.6}
$$

Proof In view of (2.3), it suffices to prove (2.5) for any scalar function $u \in S(\Omega)$. We start by defining the following subspace of $S(\Omega)$:

$$
S_*(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_I \cup \Gamma_W \}. \tag{2.7}
$$

Let us denote by $\Omega_+ \subset \mathbb{R}^3$ the reflection of Ω with respect to the plane $z = h$, that is,

$$
\Omega_+ = \{ (x, y, 2h - z) \mid (x, y, z) \in \Omega \}. \tag{2.8}
$$

In the same way $K_+ \subset \mathcal{M}_+ \subset \mathbb{R}^3$ are defined as the reflections of K and $\mathcal{M}$, correspondingly, with respect to the plane $z = h$:

$$
K_+ = \{ (x, y, 2h - z) \mid (x, y, z) \in K \}, \qquad \mathcal{M}_+ = \left\{ \xi \in \mathbb{R}^3 \mid 0 < \rho < R, \ h < z < 3h \right\}. \tag{2.9}
$$

We then write $\Omega_\sharp = \Omega \cup \Omega_+$ and, given a function $u \in S_*(\Omega)$ (scalar or vector), we define its even extension to $\Omega_\sharp$ according to the formula

$$
u_\sharp(x, y, z) = \begin{cases} u(x, y, z) & \text{if } (x, y, z) \in \Omega \\ u(x, y, 2h - z) & \text{if } (x, y, z) \in \Omega_+, \end{cases} \tag{2.10}
$$

so that $u_\sharp \in H_0^1(\Omega_\sharp)$. Now, let $\Omega^* \subset \mathbb{R}^3$ be a ball having the same measure as $\Omega_\sharp$, and thus its radius is

$$R_0 = \sqrt[3]{\frac{3|\Omega_\sharp|}{4\pi}} = \sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)}.$$

Since the Poincaré constant in the unit ball is given by π (the first zero of the spherical Bessel function of order zero), by rescaling, the Poincaré constant of Ω^* is π^2/R_0^2. In view of the Faber-Krahn inequality [22, 47], this implies that

$$\min_{w\in H_0^1(\Omega_\sharp)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega_\sharp)}}{\|w\|_{L^2(\Omega_\sharp)}} \geq \min_{w\in H_0^1(\Omega^*)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega^*)}}{\|w\|_{L^2(\Omega^*)}} = \frac{\pi}{R_0},$$

and therefore

$$\|u_\sharp\|_{L^2(\Omega_\sharp)} \leq \frac{R_0}{\pi}\|\nabla u_\sharp\|_{L^2(\Omega_\sharp)} = \frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)}\,\|\nabla u_\sharp\|_{L^2(\Omega_\sharp)}. \tag{2.11}$$

On the other hand, defining $\mathcal{M}_\sharp \doteq \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R,\ -h < z < 3\,h\right\}$, a direct computation shows

$$-\Delta\left[J_0\left(\frac{\mu_0}{R}\rho\right)\cos\left(\frac{\pi}{4h}(z-h)\right)\right] = \left(\frac{\mu_0^2}{R^2} + \frac{\pi^2}{16h^2}\right) J_0\left(\frac{\mu_0}{R}\rho\right)\cos\left(\frac{\pi}{4h}(z-h)\right) \qquad \forall\xi \in \mathcal{M}_\sharp\,;$$

such eigenfunction is positive in $\mathcal{M}_\sharp$ and vanishes on $\partial\mathcal{M}_\sharp$. Thus, the Poincaré inequality in $\mathcal{M}_\sharp$ reads

$$\|w\|_{L^2(\mathcal{M}_\sharp)} \leq \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2R^2}}\,\|\nabla w\|_{L^2(\mathcal{M}_\sharp)} \qquad \forall w \in H_0^1(\mathcal{M}_\sharp).$$

Since $u_\sharp$ can be extended by 0 in K and in K_+, it becomes an element of $H_0^1(\mathcal{M}_\sharp)$ that satisfies

$$\|u_\sharp\|_{L^2(\Omega_\sharp)} \leq \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2R^2}}\,\|\nabla u_\sharp\|_{L^2(\Omega_\sharp)}. \tag{2.12}$$

Now, it can be easily seen that

$$\|u_\sharp\|^2_{L^2(\Omega_\sharp)} = 2\,\|u\|^2_{L^2(\Omega)}, \qquad \|u_\sharp\|^4_{L^4(\Omega_\sharp)} = 2\,\|u\|^4_{L^4(\Omega)}, \qquad \|\nabla u_\sharp\|^2_{L^2(\Omega_\sharp)} = 2\,\|\nabla u\|^2_{L^2(\Omega)}, \tag{2.13}$$

which, inserted respectively into (2.11) and (2.12), yields the inequality

$$\|u\|_{L^2(\Omega)} \leq \min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2R^2}}\right\}\|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in S_*(\Omega). \tag{2.14}$$

Consider now a scalar function $u \in S(\Omega)$, and define $U_1, U_2 \in H^1(\Omega)$ according to

$$U_1(x,y,z) = \frac{h+z}{2h}u(x,y,z), \qquad U_2(x,y,z) = \frac{h-z}{2h}u(x,y,z) \qquad \text{for a.e. } (x,y,z) \in \Omega. \tag{2.15}$$

Since U_1 vanishes on $\Gamma_I \cup \Gamma_W$ and U_2 vanishes on $\Gamma_O \cup \Gamma_W$, the previous argument (of reflecting Ω with respect to the planes $z = \pm h$) can be applied to deduce that both U_1 and U_2 satisfy (2.14). Moreover, since $u = U_1 + U_2$ in Ω, we then obtain

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} &= \|U_1\|^2_{L^2(\Omega)} + \|U_2\|^2_{L^2(\Omega)} + \frac{1}{2h^2}\int_\Omega (h^2 - z^2)|u|^2 \\ &\le \|U_1\|^2_{L^2(\Omega)} + \|U_2\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\|^2_{L^2(\Omega)},\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{2}\|u\|^2_{L^2(\Omega)} &\le \|U_1\|^2_{L^2(\Omega)} + \|U_2\|^2_{L^2(\Omega)} \\ &\le \min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2 R^2}}\right\}^2 \\ &\quad \times \left(\|\nabla U_1\|^2_{L^2(\Omega)} + \|\nabla U_2\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{2.16}$$

On the other hand, after applying Young's inequality we get

$$\begin{aligned}\|\nabla U_1\|^2_{L^2(\Omega)} + \|\nabla U_2\|^2_{L^2(\Omega)} &= \frac{1}{2h^2}\|u\|^2_{L^2(\Omega)} + \frac{1}{h^2}\int_\Omega zu\frac{\partial u}{\partial z} + \frac{1}{2h^2}\int_\Omega (h^2 + z^2)|\nabla u|^2 \\ &\le \frac{1}{2h^2}\|u\|^2_{L^2(\Omega)} + \frac{1}{h}\left(\frac{1}{2h}\|u\|^2_{L^2(\Omega)} + \frac{h}{2}\|\nabla u\|^2_{L^2(\Omega)}\right) \\ &\quad + \|\nabla u\|^2_{L^2(\Omega)} \\ &= \frac{1}{h^2}\|u\|^2_{L^2(\Omega)} + \frac{3}{2}\|\nabla u\|^2_{L^2(\Omega)},\end{aligned} \tag{2.17}$$

which, once inserted into (2.16), yields (2.5) after noticing that

$$\min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2 R^2}}\right\}^2 < \frac{h^2}{2},$$

since $h > R$. This concludes the proof. □

Remark 2.1 Since $\Omega \subset \{(x, y, z) \in \mathbb{R}^3 \mid -R \le y \le R\}$, the standard proof of the Poincaré inequality for functions in $H^1_0(\Omega)$ (see, for example, [27, Theorem II.5.1]) yields

$$\|w\|_{L^2(\Omega)} \le R\,\|\nabla w\|_{L^2(\Omega)} \qquad \forall w \in H^1_0(\Omega),$$

and thus, a larger upper bound than the one given in (2.5). In fact, from (2.5) we infer

$$\|w\|_{L^2(\Omega)} \le \frac{\sqrt{3}R}{\mu_0^2 - 2}\|\nabla w\|_{L^2(\Omega)} \le 0.46R\|\nabla w\|_{L^2(\Omega)} \qquad \forall w \in S(\Omega).$$

Remark 2.2 In the sequel, we will always apply the Sobolev inequalities (2.2) to divergence-free vector fields. Going back to the proof of Theorem 2.1, let us define the following subspace of $H^1_0(\Omega_\sharp)$:

$$H^1_{0,\sigma}(\Omega_\sharp) = \{v \in H^1_0(\Omega_\sharp) \mid \nabla \cdot v = 0 \ \text{ in } \ \Omega_\sharp\},$$

so that

$$\min_{w\in H^1_{0,\sigma}(\Omega_\sharp)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega_\sharp)}}{\|w\|_{L^2(\Omega_\sharp)}} \geq \min_{w\in H^1_0(\Omega_\sharp)\setminus\{0\}} \frac{\|\nabla w\|_{L^2(\Omega_\sharp)}}{\|w\|_{L^2(\Omega_\sharp)}}. \tag{2.18}$$

The Rayleigh quotient

$$\min_{w\in H^1_{0,\sigma}(\Omega_\sharp)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega_\sharp)}}{\|w\|^2_{L^2(\Omega_\sharp)}} \tag{2.19}$$

corresponds to the first eigenvalue of the Stokes operator in $\Omega_\sharp$ under Dirichlet boundary conditions, see [15, Chapter 4]. If a strict inequality holds in (2.18), and one could be able to explicitly compute the Rayleigh quotient (2.19), this naturally would improve the estimate given in (2.12). In fact, it is shown in [78, Theorem 1.2] that a strict inequality holds in (2.18) whenever the three-dimensional bounded domain under consideration has a boundary of class $\mathcal{C}^1$ (whereas the strict inequality holds for any bounded domain in $\mathbb{R}^2$ with a locally Lipschitz boundary, see [39, Theorem 1.1]). Extensive research on the computation of the eigenvalues of the Stokes operator, for special different domains, was performed by Rummler et al. in [54, 67, 68]. Symmetrization techniques do not seem to provide any help in this context, since the component-wise Schwarz rearrangement [34, Chapter 2] of a vector field does not necessarily preserve the divergence-free condition.

We now prove the following:

Theorem 2.2 *Let Ω be as in (1.1) and $\mu_0 > 0$ as in (2.4). For any vector field $u \in S(\Omega)$ one has*

$$\|u\|_{L^4(\Omega)} \leq \sqrt{\frac{3}{\pi}}\, \frac{h \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^{3/4}}{\left(h^2 \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^2 - 2\right)^{1/2}} \|\nabla u\|_{L^2(\Omega)}. \tag{2.20}$$

In particular,

$$\mathcal{S}_4 \geq \frac{\pi}{3}\, \frac{h^2 \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^2 - 2}{h^2 \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^{3/2}}. \tag{2.21}$$

Proof In view of (2.3), it suffices to prove (2.20) for scalar functions $u \in S(\Omega)$. As in the proof of Theorem 2.1, we introduce the reflected domains (with respect to the plane $z = h$) $\Omega_+ \subset \mathbb{R}^3$ and $K_+ \subset \mathcal{M}_+ \subset \mathbb{R}^3$ given by (2.8) and (2.9). We then write $\Omega_\sharp = \Omega \cup \Omega_+$ and, given a scalar function $u \in S_*(\Omega)$ (see (2.7)), we define its extension $u_\sharp \in H^1_0(\Omega_\sharp)$

according to (2.10). Next, we recall that del Pino-Dolbeault [19, Theorem 1] obtained the following (optimal) Gagliardo-Nirenberg inequality in $\mathbb{R}^3$:

$$\|w\|_{L^4(\Omega_\sharp)} \leq \left(\frac{1}{2\pi^2}\right)^{1/6} \|\nabla w\|_{L^2(\Omega_\sharp)}^{1/2} \|w\|_{L^3(\Omega_\sharp)}^{1/2} \qquad \forall w \in H_0^1(\Omega_\sharp). \tag{2.22}$$

Since functions in $H_0^1(\Omega_\sharp)$ may be extended by zero outside $\Omega_\sharp$, they can be seen as functions defined over the whole space. Therefore, $u_\sharp$ also verifies (2.22). An application of Hölder's inequality gives

$$\|u_\sharp\|_{L^3(\Omega_+)}^3 = \int_{\Omega_\sharp} |u_\sharp|^2 |u_\sharp| \leq \|u_\sharp\|_{L^4(\Omega_\sharp)}^2 \|u_\sharp\|_{L^2(\Omega_\sharp)}$$

which, inserted into (2.22) (replacing w by $u_\sharp$), yields

$$\|u_\sharp\|_{L^4(\Omega_\sharp)} \leq \left(\frac{1}{2\pi^2}\right)^{1/6} \|\nabla u_\sharp\|_{L^2(\Omega_\sharp)}^{1/2} \|u_\sharp\|_{L^4(\Omega_\sharp)}^{1/3} \|u_\sharp\|_{L^2(\Omega_\sharp)}^{1/6},$$

or equivalently

$$\|u_\sharp\|_{L^4(\Omega_\sharp)}^{2/3} \leq \left(\frac{1}{2\pi^2}\right)^{1/6} \|\nabla u_\sharp\|_{L^2(\Omega_\sharp)}^{1/2} \|u_\sharp\|_{L^2(\Omega_\sharp)}^{1/6}. \tag{2.23}$$

As in (2.13), it can be easily seen that $\|u_\sharp\|_{L^4(\Omega_\sharp)}^4 = 2\,\|u\|_{L^4(\Omega)}^4$, so that (2.23) becomes

$$\|u\|_{L^4(\Omega)}^{2/3} \leq \left(\frac{1}{\pi}\right)^{1/3} \|\nabla u\|_{L^2(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/6}. \tag{2.24}$$

After inserting (2.14) into (2.24) and taking the $\frac{2}{3}$-roots of the resulting inequality we conclude that

$$\|u\|_{L^4(\Omega)} \leq \frac{1}{\sqrt{\pi}} \min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}| - |K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2 + \pi^2 R^2}}\right\}^{1/4} \|\nabla u\|_{L^2(\Omega)} \qquad \forall u \in S_*(\Omega). \tag{2.25}$$

Consider now a scalar function $u \in S(\Omega)$, and define $U_1, U_2 \in H^1(\Omega)$ according to (2.15), so that U_1 vanishes on $\Gamma_I \cup \Gamma_W$, U_2 vanishes on $\Gamma_O \cup \Gamma_W$ and $u = U_1 + U_2$ in Ω. Therefore, both U_1 and U_2 satisfy (2.25). In order to prove (2.20) we apply Minkowski's and Hölder's inequality in the following way:

$$\begin{aligned}
\|u\|_{L^4(\Omega)}^4 &= \left\| \, |U_1|^2 + 2U_1U_2 + |U_2|^2 \, \right\|_{L^2(\Omega)}^2 \\
&\leq \left(\|U_1\|_{L^4(\Omega)}^2 + 2\|U_1U_2\|_{L^2(\Omega)} + \|U_2\|_{L^4(\Omega)}^2\right)^2 \\
&\leq \left(\|U_1\|_{L^4(\Omega)}^2 + 2\|U_1\|_{L^4(\Omega)}\|U_2\|_{L^4(\Omega)} + \|U_2\|_{L^4(\Omega)}^2\right)^2 \\
&\leq 4\left(\|U_1\|_{L^4(\Omega)}^2 + \|U_2\|_{L^4(\Omega)}^2\right)^2.
\end{aligned}$$

Therefore, in view of (2.17)–(2.25) we have

$$
\begin{aligned}
\|u\|_{L^4(\Omega)} &\le \sqrt{\frac{2}{\pi}} \min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}|-|K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2+\pi^2R^2}}\right\}^{1/4} \\
&\quad \times \left(\|\nabla U_1\|_{L^2(\Omega)}^2+\|\nabla U_2\|_{L^2(\Omega)}^2\right)^{1/2} \\
&\le \sqrt{\frac{2}{\pi}} \min\left\{\frac{1}{\pi}\sqrt[3]{\frac{3}{2\pi}(|\mathcal{M}|-|K|)},\ \frac{4Rh}{\sqrt{16h^2\mu_0^2+\pi^2R^2}}\right\}^{1/4} \\
&\quad \times \left(\frac{1}{h^2}\|u\|_{L^2(\Omega)}^2+\frac{3}{2}\|\nabla u\|_{L^2(\Omega)}^2\right)^{1/2},
\end{aligned}
$$

from where we derive (2.20) after applying (2.5) in the right-hand side of this last inequality. □

We are also interested in obtaining explicit upper bounds for the Sobolev embedding constants $\mathcal{S}_2$ and $\mathcal{S}_4$. For this, let us assume that

$$
K \subseteq \mathcal{P} \subset \mathcal{M} \quad \text{where} \quad \mathcal{P} = \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < 1,\ -1 < z < 1\right\}, \tag{2.26}
$$

that is, the obstacle K can be enclosed by a open straight cylinder $\mathcal{P}$ of radius 1 and length 2 whose axis of symmetry is directed along the z-axis. The precision of the lower bounds given in Theorems 2.1 and 2.2 is evaluated in the following result:

Proposition 2.1 *Let Ω be as in (1.1) and assume (2.26), with*

$$
h \ge \frac{R}{3}\left(\frac{\pi}{\mu_0}\right)^3 + \frac{1}{R^2}. \tag{2.27}
$$

Then

$$
\begin{aligned}
&\frac{1}{3}\left(\frac{\mu_0^2}{R^2}-\frac{2}{h^2}+\frac{\pi^2}{16h^2}\right) \\
&\quad \le \mathcal{S}_2 \le \frac{10(R^2(h-1)^3(5R^2+3h^2-6h+3)+4(R+1)(R-1)^3(R^2-2R+5))}{5R^4(h-1)^5+16(R+1)(R-1)^5}
\end{aligned} \tag{2.28}
$$

and

$$
\frac{\pi}{3}\frac{\left(\frac{\mu_0}{R}\right)^2-\frac{1}{h^2}\left(2-\frac{\pi^2}{16}\right)}{\left(\left(\frac{\mu_0}{R}\right)^2+\left(\frac{\pi}{4h}\right)^2\right)^{3/4}} \le \mathcal{S}_4 \le \frac{7\sqrt{\pi}(8R^4-2R^3-R^2+12R+20)}{\sqrt{(R-1)^3(21R^6+128R+128)}}. \tag{2.29}
$$

Proof Since (2.26) and (2.27) hold, it can be easily noticed that

$$
\max\left\{\pi\sqrt[3]{\frac{2\pi}{3(|\mathcal{M}|-|K|)}},\ \frac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\} = \frac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh},
$$

and the lower bounds in (2.28)–(2.29) follow, respectively, from (2.6)–(2.21).

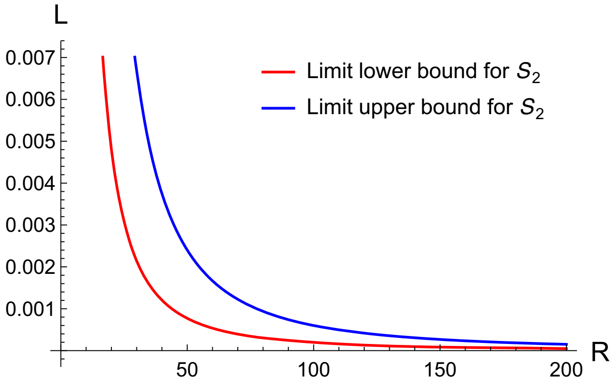

Fig. 2 Comparison between the lower and upper bounds for $L \doteq \lim_{h\to\infty} \mathcal{S}_2$, for $R > 1$

Consider now the function $X_0 : \Omega\backslash\overline{\mathcal{P}} \longrightarrow \mathbb{R}$ defined by

$$X_0(\xi) \doteq \begin{cases} (R-\rho)(h+z)(z+1) & \text{if } \ 0 \leq \rho < R, \quad -h \leq z \leq -1 \\ (R-\rho)(\rho-1)(1-z^2) & \text{if } \ 1 \leq \rho < R, \quad -1 < z < 1 \\ (R-\rho)(h-z)(z-1) & \text{if } \ 0 \leq \rho < R, \quad 1 \leq z \leq h, \end{cases}$$

which, if extended by zero inside $\mathcal{P}$, is an element of $H^1_0(\Omega) \subset S(\Omega)$. We also define $X_1 : \Omega\backslash\overline{\mathcal{P}} \longrightarrow \mathbb{R}$ as

$$X_1(\xi) \doteq \begin{cases} (R-\rho)(R+z)(z+1) & \text{if } \ 0 \leq \rho < R, \quad -R \leq z \leq -1 \\ (R-\rho)(\rho-1)(1-z^2) & \text{if } \ 1 \leq \rho < R, \quad -1 < z < 1 \\ (R-\rho)(R-z)(z-1) & \text{if } \ 0 \leq \rho < R, \quad 1 \leq z \leq R, \end{cases}$$

which, if extended by zero inside $\mathcal{P}$ and for $|z| \geq R$, becomes an element of $S(\Omega)$ (recall that $h > R$). Therefore, both X_0 and X_1 can be tested in the quotient (2.1), yielding respectively the upper bounds in (2.28) and (2.29). $\square$

Remark 2.3 It can be inferred from Proposition 2.1 that

$$\frac{\mu_0^2}{3R^2} \leq \lim_{h\to\infty} \mathcal{S}_2 \leq \frac{6}{R^2} \qquad \forall R > 1. \tag{2.30}$$

The ratio between the upper and lower bounds in (2.30) equals $18/\mu_0^2 \approx 3.1124$, for every $R > 1$. A comparison between the lower and upper bounds in (2.30), as functions of $R > 1$, is shown in Fig. 2.

On the other hand we have

$$\frac{\pi}{3}\sqrt{\frac{\mu_0}{R}} \leq \lim_{h\to\infty} \mathcal{S}_4 \leq \frac{7\sqrt{\pi}(8R^4 - 2R^3 - R^2 + 12R + 20)}{\sqrt{(R-1)^3(21R^6 + 128R + 128)}} \qquad \forall R > 1. \tag{2.31}$$

The ratio between the upper and lower bounds in (2.31) tends to $8\sqrt{21}/\sqrt{\pi\mu_0} \approx 13.3377$ as $R \to \infty$. A finer upper bound for $\lim_{h\to\infty} \mathcal{S}_4$ will be given in the next result.

In the spirit of [31, Proposition 3.1] we now prove:

Theorem 2.3 *Let Ω be as in (1.1) and assume (2.26)–(2.27). We then have*

$$\frac{\pi}{3}\sqrt{\frac{\mu_0}{R}} \le \lim_{h\to\infty} \mathcal{S}_4 \le \frac{10.4528}{\sqrt{R}} \qquad \forall R > 1. \tag{2.32}$$

Proof For the sake of clarity, in this proof we will denote by Ω_h the domain defined in (1.1), and by $\mathcal{S}_4(h) > 0$ the Sobolev embedding constant (2.1) for $p = 4$.

Consider the infinite nozzle of radius R given by

$$\mathcal{M}_\infty \doteq \left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R\right\}$$

and put

$$\mathcal{S}_4^\infty = \inf_{w\in H_0^1(\mathcal{M}_\infty)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\mathcal{M}_\infty)}}{\|w\|^2_{L^4(\mathcal{M}_\infty)}}, \tag{2.33}$$

so that

$$\mathcal{S}_4^\infty \|v\|^2_{L^4(\mathcal{M}_\infty)} \le \|\nabla v\|^2_{L^2(\mathcal{M}_\infty)} \qquad \forall v \in H_0^1(\mathcal{M}_\infty). \tag{2.34}$$

Since for $h_2 > h_1 > 1$ we have the inclusions $H_0^1(\Omega_{h_1}) \subset H_0^1(\Omega_{h_2}) \subset H_0^1(\mathcal{M}_\infty)$ induced by trivial extension, it follows that

$$\mathcal{Z}_4(h_1) \ge \mathcal{Z}_4(h_2) \ge \mathcal{S}_4^\infty \quad \text{for} \quad h_2 > h_1 > 1, \tag{2.35}$$

where we have defined

$$\mathcal{Z}_4(h) \doteq \min_{w\in H_0^1(\Omega_h)\setminus\{0\}} \frac{\|\nabla w\|^2_{L^2(\Omega_h)}}{\|w\|^2_{L^4(\Omega_h)}} \qquad \forall h > 1. \tag{2.36}$$

We claim that

$$\text{the infimum in (2.33) is attained at a function } v \in H_0^1(\mathcal{M}_\infty) \text{ with } \|v\|_{L^4(\mathcal{M}_\infty)} = 1. \tag{2.37}$$

To prove (2.37) we consider a sequence $(u_n)_{n\in\mathbb{N}} \subset H_0^1(\mathcal{M}_\infty)$ such that $\|u_n\|_{L^4(\mathcal{M}_\infty)} = 1$, for every $n \in \mathbb{N}$, and $\|\nabla u_n\|^2_{L^2(\mathcal{M}_\infty)} \to \mathcal{S}_4^\infty$ as $n \to \infty$. It then follows from the Poincaré inequality that $(u_n)_{n\in\mathbb{N}}$ is bounded in $H_0^1(\mathcal{M}_\infty)$. By Lions' Lemma [56], after passing to a subsequence, there exist points $\xi_n = \rho_n\widehat{\rho} + z_n\widehat{k} \in \mathbb{R}^3$, with $\rho_n \ge 0$ and $z_n \in \mathbb{R}$ for every $n \in \mathbb{N}$, with

$$v_n \doteq \xi_n * u_n \rightharpoonup v \qquad \text{weakly in } H^1(\mathbb{R}^3) \text{ for some function } v \in H^1(\mathbb{R}^3)\setminus\{0\}, \text{ as } n \to \infty,$$

in which we use the notation $\xi * u \doteq u(\cdot - \xi)$ to denote the translation of a function $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ with respect to $\xi \in \mathbb{R}^3$. Since $\mathcal{M}_\infty$ is bounded in the radial direction and $v \neq 0$, also the sequence $(\rho_n)_{n\in\mathbb{N}}$ is bounded and we may therefore assume that $\rho_n = 0$ for every $n \in \mathbb{N}$. Consequently, $v \in H_0^1(\mathcal{M}_\infty)$ and $v_n \rightharpoonup v$ in $H_0^1(\mathcal{M}_\infty)$ as $n \to \infty$. By the Brezis-Lieb Lemma [11], we then have

$$1 = \lim_{n\to\infty} \|u_n\|^4_{L^4(\mathcal{M}_\infty)} = \lim_{n\to\infty} \|v_n\|^4_{L^4(\mathcal{M}_\infty)} = \|v\|^4_{L^4(\mathcal{M}_\infty)} + c^4 \qquad \text{with}$$
$$c \doteq \lim_{n\to\infty} \|v_n - v\|_{L^4(\mathcal{M}_\infty)}.$$

Consequently, $\|v\|^2_{L^4(\mathcal{M}_\infty)} + c^2 \geq 1$ and from (2.34) we get

$$\begin{aligned} \mathcal{S}_4^\infty \leq \mathcal{S}_4^\infty \left(\|v\|^2_{L^4(\mathcal{M}_\infty)} + c^2 \right) &\leq \|\nabla v\|^2_{L^2(\mathcal{M}_\infty)} + \lim_{n\to\infty} \|\nabla(v_n - v)\|^2_{L^2(\mathcal{M}_\infty)} \\ &= \lim_{n\to\infty} \|\nabla v_n\|^2_{L^2(\mathcal{M}_\infty)} = \lim_{n\to\infty} \|\nabla u_n\|^2_{L^2(\mathcal{M}_\infty)} = \mathcal{S}_4^\infty . \end{aligned} \tag{2.38}$$

From this we deduce that

$$\|v\|^2_{L^4(\mathcal{M}_\infty)} + c^2 = 1 = \|v\|^4_{L^4(\mathcal{M}_\infty)} + c^4 .$$

Since $v \neq 0$, we infer that $c = 0$ and then $\|v\|_{L^4(\mathcal{M}_\infty)} = 1$. Therefore, $\|\nabla v\|^2_{L^2(\mathcal{M}_\infty)} \geq \mathcal{S}_4^\infty$ by definition of $\mathcal{S}_4^\infty$ in (2.33). Then it follows from (2.38) that $\|\nabla v\|^2_{L^2(\mathcal{M}_\infty)} = \mathcal{S}_4^\infty$ and $\lim\limits_{n\to\infty} \|\nabla(v_n - v)\|_{L^2(\mathcal{M}_\infty)} = 0$, and consequently $\lim\limits_{n\to\infty} \|v_n - v\|_{L^2(\mathcal{M}_\infty)} = 0$ by the Poincaré inequality. Hence $v_n \to v$ in $H^1_0(\mathcal{M}_\infty)$ as $n \to \infty$, and v is a minimizer of the of the Sobolev quotient (2.33). This proves the claim (2.37).

We next claim that

$$\lim_{h\to\infty} \mathcal{Z}_4(h) = \mathcal{S}_4^\infty . \tag{2.39}$$

To show this, let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ be a non-negative function with $\phi \equiv 1$ on $B_{\frac{1}{2}}$ and $\phi \equiv 0$ on $\mathbb{R}^3 \setminus B_1$ (here $B_r \subset \mathbb{R}^3$ denotes the ball of radius $r > 0$ centered at the origin). Moreover, let

$$\phi_n(x) \doteq \phi\left(\frac{x}{n}\right) \qquad \forall x \in \mathbb{R}^3 , \ n \geq 1,$$

so that $\phi_n \in \mathcal{C}_0^\infty(B_n)$. By (2.37), there exists a function $v \in H^1_0(\mathcal{M}_\infty)$ with $\|v\|_{L^4(\mathcal{M}_\infty)} = 1$ and $\|\nabla v\|^2_{L^2(\mathcal{M}_\infty)} = \mathcal{S}_4^\infty$. It is then standard to see that the sequence $v_n \doteq \phi_n v \in H^1_0(\mathcal{M}_\infty)$, $n \in \mathbb{N}$, satisfies $v_n \to v$ in $H^1_0(\mathcal{M}_\infty)$ as $n \to \infty$ and, hence,

$$\|v_n\|_{L^4(\mathcal{M}_\infty)} \to 1 \quad \text{and} \quad \|\nabla v_n\|^2_{L^2(\mathcal{M}_\infty)} \to \mathcal{S}_4^\infty \qquad \text{as} \ \ n \to \infty. \tag{2.40}$$

Since $v_n = 0$ on $\mathcal{M}_\infty \setminus B_n$ we have

$$u_n \doteq \xi_n * v_n \in H^1_0(\Omega_{h_n}) \quad \text{for} \ n \in \mathbb{N} \ \text{with} \ \xi_n \doteq (n+1)\widehat{k} \ \text{and} \ h_n \doteq 2n+1.$$

It thus follows from (2.35)–(2.40) that

$$\mathcal{S}_4^\infty \leq \lim_{h\to\infty} \mathcal{Z}_4(h) = \lim_{n\to\infty} \mathcal{Z}_4(h_n) \leq \lim_{n\to\infty} \frac{\|\nabla u_n\|^2_{L^2(\Omega_{h_n})}}{\|u_n\|^2_{L^4(\Omega_{h_n})}} = \lim_{n\to\infty} \frac{\|\nabla v_n\|^2_{L^2(\mathcal{M}_\infty)}}{\|v_n\|^2_{L^4(\mathcal{M}_\infty)}} = \mathcal{S}_4^\infty .$$

which yields the equality in (2.39). Then, in view of (2.39) and the inclusion $H^1_0(\Omega_h) \subset S(\Omega_h)$ for every $h > 1$, we also deduce the inequality

$$\lim_{h\to\infty} \mathcal{S}_4(h) \leq \mathcal{S}_4^\infty . \tag{2.41}$$

An upper bound for $\mathcal{S}_4^\infty = \mathcal{S}_4^\infty(R)$ will be given by seeking the minimum in (2.33) among separated-variable functions, that is, having the form $V(\rho)W(z)$ for $\rho \in [0, R]$ and $z \in \mathbb{R}$. Given $V_R \in H^1(0, R)\backslash\{0\}$ such that $V_R(R) = 0$, (2.37) implies that

$$\mathcal{S}_4^\infty(R) \leq \sqrt{2\pi} \min_{W\in H^1(\mathbb{R})\backslash\{0\}} \frac{\|W\|^2_{L^2(\mathbb{R})}\left(\int_0^R \rho|V_R'(\rho)|^2 d\rho\right) + \|W'\|^2_{L^2(\mathbb{R})}\left(\int_0^R \rho|V_R(\rho)|^2 d\rho\right)}{\|W\|^2_{L^4(\mathbb{R})}\sqrt{\int_0^R \rho|V_R(\rho)|^4 d\rho}}. \tag{2.42}$$

A simple rescaling argument shows that

$$\mathcal{S}_4^\infty(R) = \sqrt{\frac{R_0}{R}}\,\mathcal{S}_4^\infty(R_0) \qquad \forall R, R_0 > 1,$$

which, once inserted in (2.42), yields

$$\mathcal{S}_4^\infty(R) \leq \sqrt{\frac{2\pi R_0}{R}} \min_{W\in H^1(\mathbb{R})\backslash\{0\}} \frac{\|W\|^2_{L^2(\mathbb{R})}\left(\int_0^{R_0} \rho|V_{R_0}'(\rho)|^2 d\rho\right) + \|W'\|^2_{L^2(\mathbb{R})}\left(\int_0^{R_0} \rho|V_{R_0}(\rho)|^2 d\rho\right)}{\|W\|^2_{L^4(\mathbb{R})}\sqrt{\int_0^{R_0} \rho|V_{R_0}(\rho)|^4 d\rho}}. \tag{2.43}$$

Let $\mathrm{cn} : \mathbb{R} \longrightarrow [-1, 1]$ be the Jacobian elliptic cosine function with modulus $k = 1/\sqrt{2}$, which satisfies

$$\mathrm{cn}''(t) + \mathrm{cn}(t)^3 = 0 \qquad \forall t \in \mathbb{R}, \tag{2.44}$$

and whose first zero is given by

$$\alpha \doteq \sqrt{2}\int_0^{\pi/2} \frac{1}{\sqrt{2-\sin(t)^2}}\, dt \approx 1.85407,$$

see [7] for more details. Then the function

$$V_R(\rho) \doteq \frac{1}{\beta_R}\mathrm{cn}\left(\frac{\alpha}{R}\rho\right) \qquad \forall \rho \in [0, R]$$

vanishes at $\rho = R$, where $\beta_R > 0$ is a normalization constant such that

$$\int_0^R \rho|V_R(\rho)|^4 d\rho = 1.$$

Let $R_0 > 1$ be such that

$$\int_0^{R_0} \rho|V_{R_0}'(\rho)|^2 d\rho = 1\,;$$

numerically we find $R_0 \approx 2.80143$. Then, the Euler-Lagrange equation associated to the minimization problem in (2.43) reads

$$-\left(\int_0^{R_0} \rho|V_{R_0}(\rho)|^2 d\rho\right) W''(z) + W(z) = \lambda W(z)^3 \qquad \forall z \in \mathbb{R}, \tag{2.45}$$

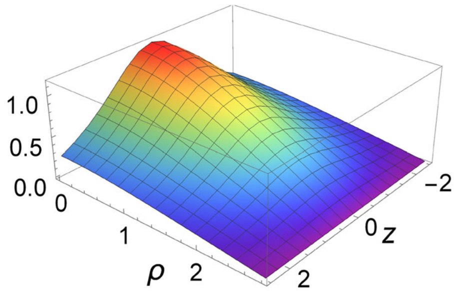

Fig. 3 Graph of the function $V_{R_0}(\rho)W_{R_0}(z)$ for $(\rho, z) \in [0, R_0] \times [-2.5, 2.5]$

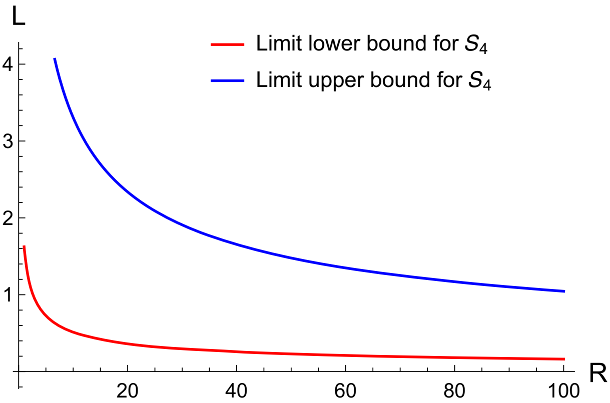

Fig. 4 Comparison between the lower and upper bounds for $L \doteq \lim_{h\to\infty} \mathcal{S}_4$, for $R > 1$

with $\lambda \in \mathbb{R}$ being a Lagrange multiplier. By direct substitution we deduce that the function

$$W_{R_0}(z) \doteq \left[\cosh\left(\frac{z}{\sqrt{\int_0^{R_0} \rho|V_{R_0}(\rho)|^2 d\rho}}\right)\right]^{-1} \qquad \forall z \in \mathbb{R}$$

is a solution of (2.45) with $\lambda = 2$. The function $V_{R_0}W_{R_0}$ minimizes the ratio in (2.43) (Fig. 3). Once inserting their expressions in (2.43) we obtain

$$\mathcal{S}_4^{\infty}(R) \leq \frac{10.4528}{\sqrt{R}} \qquad \forall R > 1.$$

The proof is then concluded as a consequence of (2.31)–(2.41). □

Two remarks concerning Theorem 2.3 are in order.

Remark 2.4 The ratio between the upper and lower bounds in (2.32) is approximately 6.4366 for every $R > 1$, which therefore improves by around 52% the ratio given in (2.31). A comparison between the lower and upper bounds in (2.32), as functions of $R > 1$, is shown in Fig. 4.

Remark 2.5 The limit in (2.39) can be understood by stating that the mass of the Sobolev minimizer in (2.36) tends to concentrate on one side of the obstacle (which somehow "disappears") as $h \to \infty$. The function $v \in H_0^1(\mathcal{M}_\infty)$ with $\|v\|_{L^4(\mathcal{M}_\infty)} = 1$ that attains the minimum in (2.33) is a nontrivial weak solution of the following semilinear elliptic equation (see [72, Chapter I]):

$$\begin{cases} -\Delta v = \mathcal{S}_4^\infty v^3 & \textit{in} \quad \mathcal{M}_\infty, \\ v = 0 & \textit{on} \quad \partial\mathcal{M}_\infty. \end{cases} \tag{2.46}$$

Notice that the Jacobian elliptic cosine function solves (2.44), which can be viewed as the one-dimensional version of (2.46). This motivates the choice of the (radial) function V_R in the proof of Theorem 2.3.

2.2 Relative capacity of *K* inside $\mathcal{M}$

Let Ω be as in (1.1). The *relative capacity* of K with respect to $\mathcal{M}$ is defined by

$$\mathrm{Cap}_{\mathcal{M}}(K) \doteq \min_{v \in H_0^1(\mathcal{M})} \left\{ \int_{\mathcal{M}} |\nabla v|^2 \;\middle|\; v = 1 \;\text{ in }\; \overline{K} \right\} \tag{2.47}$$

and the *relative capacity potential* of K with respect to $\mathcal{M}$, that is, the function $\phi \in H_0^1(\mathcal{M})$ achieving the minimum in (2.47), satisfies

$$\Delta\phi = 0 \;\text{ in }\; \Omega, \qquad \phi = 0 \;\text{ on }\; \partial\mathcal{M}, \qquad \phi = 1 \;\text{ in }\; \overline{K}, \qquad \mathrm{Cap}_{\mathcal{M}}(K) = \|\nabla\phi\|^2_{L^2(\Omega)}, \tag{2.48}$$

see [58, Chapter 2] for further references. In this section we firstly give a lower bound for $\mathrm{Cap}_{\mathcal{M}}(K)$ which is independent of the shape and position of K inside $\mathcal{M}$. More precisely, we have:

Theorem 2.4 *Let Ω be as in* (1.1)*. Then*

$$\mathrm{Cap}_{\mathcal{M}}(K) \geq \sqrt[3]{\frac{3}{2}} \, \frac{4\pi}{\sqrt[3]{\dfrac{2\pi}{|K|}} - \sqrt[3]{\dfrac{1}{R^2 h}}}. \tag{2.49}$$

Proof We apply again a symmetrization argument. For this, let $\mathcal{M}^* \subset \mathbb{R}^3$ be the ball centered at the origin of radius

$$R_2 = \sqrt[3]{\frac{3}{2} R^2 h},$$

and let $K^* \subset \mathbb{R}^3$ be the ball centered at the origin of radius

$$R_1 = \sqrt[3]{\frac{3}{4\pi} |K|},$$

so that $R_1 < R_2$, $|\mathcal{M}^*| = |\mathcal{M}|$ and $|K^*| = |K|$. The relative capacity potential of K^* with respect to $\mathcal{M}^*$, denoted by $\phi_0 \in H_0^1(\mathcal{M}^*)$, is the function

$$\phi_0(\xi) = \frac{\dfrac{1}{\sqrt{\rho^2 + z^2}} - \dfrac{1}{R_2}}{\dfrac{1}{R_1} - \dfrac{1}{R_2}} \qquad \forall \xi \in \mathcal{M}^* \setminus \overline{K^*},$$

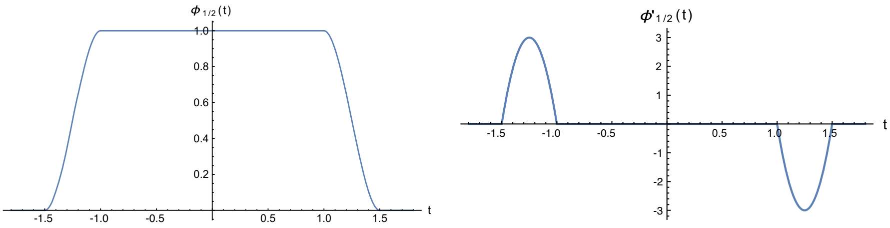

Fig. 5 Graph of the function ϕ_ε (left) and of its derivative (right), for $\varepsilon = 1/2$

so that

$$\mathrm{Cap}_{\mathcal{M}^*}(K^*) = \|\nabla \phi_0\|^2_{L^2(\mathcal{M}^*)} = \frac{4\pi}{\dfrac{1}{R_1} - \dfrac{1}{R_2}}. \tag{2.50}$$

Now, the *symmetric decreasing rearrangement* $\phi^* \in H^1_0(\mathcal{M}^*)$ of ϕ is such that

$$\phi^* = 1 \quad \text{in} \quad \overline{K^*} \qquad \text{and} \qquad \|\nabla \phi^*\|_{L^2(\mathcal{M}^*)} \leq \|\nabla \phi\|_{L^2(\mathcal{M})},$$

see [34, Chapter 2] for more details, so that

$$\mathrm{Cap}_{\mathcal{M}^*}(K^*) \leq \|\nabla \phi^*\|^2_{L^2(\mathcal{M}^*)} \leq \mathrm{Cap}_{\mathcal{M}}(K). \tag{2.51}$$

Inequality (2.49) follows directly from (2.50)–(2.51). □

In order to assess the precision of (2.49), we prove the following:

Proposition 2.2 *Let* Ω *be as in* (1.1) *and assume* (2.26). *Then*

$$\sqrt[3]{\frac{3}{2}}\,\frac{4\pi}{\sqrt[3]{\dfrac{2\pi}{|K|}} - \sqrt[3]{\dfrac{1}{R^2 h}}} \leq \mathrm{Cap}_{\mathcal{M}}(K) \leq \frac{12\pi}{175}\left[\frac{(22+13h)(R+1)}{R-1} + \frac{2(3R^2+7R-10)}{h-1}\right]. \tag{2.52}$$

Proof We just have to prove the upper bound in (2.52). To do this, given any $\varepsilon > 0$ we define the function $\phi_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$\phi_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t \in (-\infty, -1-\varepsilon] \cup [1+\varepsilon, \infty) \\ \dfrac{1}{\varepsilon^3}\left[2|t|^3 - 3(\varepsilon+2)t^2 + 6(1+\varepsilon)|t| + \varepsilon^3 - 3\varepsilon - 2\right] & \text{if} \quad t \in (-1-\varepsilon, -1) \cup (1, 1+\varepsilon) \\ 1 & \text{if} \quad t \in [-1, 1], \end{cases}$$

see also [70, Section 4], whose plot for $\varepsilon = 1/2$ is displayed above (Fig. 5). Then, $\phi_\varepsilon \in \mathcal{C}^1(\mathbb{R})$, $\mathrm{supp}(\phi_\varepsilon) = [-1-\varepsilon, 1+\varepsilon]$, $\phi'_\varepsilon(1) = \phi'_\varepsilon(-1) = 0$, so that $\phi_\varepsilon \in H^2(\mathbb{R})$. In particular, by selecting $\varepsilon_0 = R - 1$ or $\varepsilon_0 = h - 1$ we notice that

$$\phi_{R-1}(R) = 0\,; \qquad \phi_{R-1}(\rho) = 1 \quad \forall \rho \in [0, 1]\,; \qquad \phi'_{R-1}(1) = \phi'_{R-1}(R) = 0. \tag{2.53}$$

and

$$\phi_{h-1}(\pm h) = 0\,; \qquad \phi_{h-1}(z) = 1 \quad \forall z \in [-1, 1]\,; \qquad \phi'_{h-1}(\pm 1) = \phi'_{h-1}(\pm h) = 0. \tag{2.54}$$

We then define the function $X(\xi) = \phi_{R-1}(\rho)\phi_{h-1}(z)$, for every $\xi \in \mathcal{M}$. In view of (2.53)–(2.54) we have that $X \in H^1_0(\mathcal{M})$ and $X \equiv 1$ on $\partial\mathcal{P}$. Therefore, (2.47) implies that

$$\mathrm{Cap}_{\mathcal{M}}(K) \leq \int_{\mathcal{M}} |\nabla X|^2,$$

which, after an explicit calculation, gives the upper bound in (2.52). □

2.3 The Bogovskii constant of Ω

Let $Q \subset \mathbb{R}^3$ be any bounded Lipschitz domain, and consider the space of p-integrable functions in Q (with $p \geq 1$) having zero mean value:

$$L^p_0(Q) = \left\{ g \in L^p(Q) \;\Big|\; \int_Q g = 0 \right\}. \tag{2.55}$$

We define the Bogovskii constant of Q as

$$C_B(Q) \doteq \sup_{g \in L^2_0(Q)\setminus\{0\}} \inf \left\{ \frac{\|\nabla v\|_{L^2(Q)}}{\|g\|_{L^2(Q)}} \;\Bigg|\; v \in H^1_0(Q),\ \nabla\cdot v = g \ \text{ in } \ Q \right\}, \tag{2.56}$$

see [24, Section 2]. Bogovskii [10] showed that, given any $q \in L^2_0(Q)$, there exists a vector field $X \in H^1_0(Q)$ such that $\nabla \cdot X = q$ in Q and

$$\|\nabla X\|_{L^2(Q)} \leq C_B(Q)\|q\|_{L^2(Q)}.$$

Then we obtain the bound

$$\|\nabla q\|_{H^{-1}(Q)} = \sup_{\substack{X \in H^1_0(Q) \\ \|\nabla X\|_{L^2(Q)}=1}} \left| \int_Q q\,(\nabla \cdot X) \right| \geq \frac{1}{C_B(Q)} \sup_{\substack{g \in L^2_0(Q) \\ \|g\|_{L^2(Q)}=1}} \left| \int_Q qg \right| = \frac{1}{C_B(Q)} \|q\|_{L^2(Q)}$$

that is,

$$\|q\|_{L^2(Q)} \leq C_B(Q)\|\nabla q\|_{H^{-1}(Q)} \qquad \forall q \in L^2_0(Q). \tag{2.57}$$

In this section we provide a *universal* lower bound for the Bogovskii constant of any bounded Lipschitz domain in $\mathbb{R}^3$ (in fact, the same result holds in any dimension).

Theorem 2.5 *For every bounded Lipschitz domain $Q \subset \mathbb{R}^3$ it holds that*

$$C_B(Q) \geq 1.$$

Proof Given any $q \in L^2_0(Q)$, it suffices to show that

$$\|\nabla X\|_{L^2(Q)} \geq \|q\|_{L^2(Q)}, \tag{2.58}$$

for every $X = (X_1, X_2, X_3) \in H^1_0(Q)$ such that $\nabla \cdot X = q$ in Q. If we denote by $\widetilde{X}$ and $\widetilde{q}$ the extensions by zero of X and q outside Q, respectively, it holds that $\widetilde{X} \in H^1_0(\mathbb{R}^3)$ and $\widetilde{q} \in L^2_0(\mathbb{R}^3)$. We then apply the Fourier Transform to the divergence equation $\nabla \cdot X = q$ in Q, thereby obtaining

$$\sum_{j=1}^{3} i\,\xi_j \widehat{X_j}(\xi) = \widehat{q}(\xi) \qquad \forall \xi \in \mathbb{R}^3,$$

or equivalently

$$\xi \cdot \widehat{X}(\xi) = -i\,\widehat{q}(\xi) \qquad \forall \xi \in \mathbb{R}^3\,, \tag{2.59}$$

where, with some abuse of notation, we have defined $\widehat{X} \doteq \mathcal{F}\left\{\widetilde{X}\right\}$ (component-wise) and $\widehat{q} \doteq \mathcal{F}\{\widetilde{q}\}$. Now, in view of Plancherel's identity and (2.59) we have

$$\begin{aligned}
\int_Q |\nabla X|^2 &= \int_{\mathbb{R}^3} |\nabla \widetilde{X}|^2 = \sum_{k,j=1}^{3} \int_{\mathbb{R}^3} \left| \frac{\partial \widetilde{X}_k}{\partial x_j} \right|^2 = \frac{1}{(2\pi)^3} \sum_{k,j=1}^{3} \int_{\mathbb{R}^3} \left| \mathcal{F}\left\{ \frac{\partial \widetilde{X}_k}{\partial x_j} \right\} \right|^2 \\
&= \frac{1}{(2\pi)^3} \sum_{k,j=1}^{3} \int_{\mathbb{R}^3} |i\,\xi_j \widehat{X_k}(\xi)|^2 d\xi = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\xi|^2 |\widehat{X}(\xi)|^2 d\xi \\
&\geq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\xi \cdot \widehat{X}(\xi)|^2 d\xi = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\widehat{q}|^2 = \int_Q |q|^2\,,
\end{aligned}$$

so that (2.58) follows. □

We are also interested in giving upper bounds for the Bogovskii constant of the domain Ω defined in (1.1). The assumption on the obstacle K becomes much more involved than (2.26) because the method introduced in [24, Section 2] requires that the domain Ω satisfies the so-called *cone property*, see [58, Chapter 1] for further details. In fact, any such domain can be decomposed as the union of a finite number of domains, each one being star-shaped with respect to an open ball strictly contained in it. In order to simplify the presentation, here we prove:

Theorem 2.6 *Let Ω be as in (1.1) and assume that $K = \mathcal{B}$ (the unit ball of $\mathbb{R}^3$). Then, there exists a constant $C_B > 0$, independent of K, R and h, such that*

$$1 \leq C_B(\Omega) \leq C_B \sqrt{1 + \frac{3R^2 h - 2}{\sigma_1(h-1)}\,\frac{\sqrt{R^2+h^2}}{R-1}} \left(\frac{\sigma_2}{(R-1)^3} \right)^{1/4} \left(\log\left(\frac{\sigma_2}{(R-1)^3} \right) \right)^{3/4}, \tag{2.60}$$

where $\sigma_1 \doteq R^2 \arctan\left(\sqrt{R^2-1}\right) - \sqrt{R^2-1}$ *and* $\sigma_2 \doteq \pi R^2(h-1) + \sigma_1(h+1)$.

Proof We just have to prove the upper bound in (2.60). To do this, we write $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, where

$$\begin{aligned}
\Omega_1 \doteq &\left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R\,,\ \ 1 < z < h\right\} \cup \\
&\left\{\xi \in \mathbb{R}^3 \mid 1 < \rho < R\,,\ \ \rho\sin(\theta) > 1\,,\ \ -h < z < 1\right\}\,,
\end{aligned}$$

and this domain is star-shaped with respect to the open ball (described in Cartesian coordinates)

$$x^2 + \left(y - \frac{R+1}{2} \right)^2 + (z - (h-R+1))^2 < \left(\frac{R-1}{2} \right)^2.$$

Similarly,

$$\begin{aligned}
\Omega_2 \doteq &\left\{\xi \in \mathbb{R}^3 \mid 0 < \rho < R\,,\ \ -h < z < -1\right\} \cup \\
&\left\{\xi \in \mathbb{R}^3 \mid 1 < \rho < R\,,\ \ \rho\sin(\theta) < -1\,,\ \ -1 < z < h\right\}\,,
\end{aligned}$$

and this domain is star-shaped with respect to the open ball (described in Cartesian coordinates)

$$x^2 + \left(y + \frac{R+1}{2}\right)^2 + (z + (h - R + 1))^2 < \left(\frac{R-1}{2}\right)^2 .$$

We clearly have

$$\mathrm{diam}(\Omega_1) = \mathrm{diam}(\Omega_2) = 2\sqrt{R^2 + h^2}\,, \qquad |\Omega_1| = |\Omega_2| = \sigma_2\,, \qquad |\Omega_1 \cap \Omega_2| = 2\sigma_1(h-1)\,,$$

so that from [21, Theorem 3.2] we obtain the bound

$$C_B(\Omega_i) \le C_B \frac{\sqrt{R^2+h^2}}{R-1}\left(\frac{\sigma_2}{(R-1)^3}\right)^{1/4}\left(\log\left(\frac{\sigma_2}{(R-1)^3}\right)\right)^{3/4} \quad \text{for } i \in \{1, 2\}.$$

Then, by following exactly the proof of [24, Theorem 5.1] we obtain the upper bound in (2.60). □

Remark 2.6 By combining [24, Proposition 5.1] with [24, Theorem 5.1] we can, alternatively, provide the following explicit bound for the Bogovskii constant of Ω when $K = \mathcal{B}$:

$$1 \le C_B(\Omega) \le \sqrt{12\left(1 + \frac{2\pi}{3}\frac{3R^2h-2}{\sigma_1(h-1)}\right)}\sqrt{328 + \frac{447\sqrt{\sigma_2}}{(R-1)^{3/2}} + \frac{154\sigma_2}{(R-1)^3} + \frac{48(R^2+h^2)}{(R-1)^2}\left(23 + \frac{16\sqrt{\sigma_2}}{(R-1)^{3/2}}\right)^2},$$

which, if compared to (2.60), grows faster as $h \to \infty$.

3 The Navier–Stokes equations with mixed boundary conditions

3.1 Construction of a flux carrier and weak formulation of the problem

It is well-known that a crucial step in the search for generalized solutions to the Navier–Stokes equations $(1.3)_1$ under non-homogeneous Dirichlet boundary conditions lies in the construction of a flux carrier that satisfies the Leray-Hopf inequality, see [26], which is usually obtained in the way of Ladyzhenskaya-Solonnikov [50] by means of the Hopf cut-off function [36]. The mixed boundary conditions used in our model (1.3) prevent us to follow such approach, and therefore, a flux carrier verifying the Leray–Hopf inequality seems out of reach in our setting. Instead, we prove the following result:

Theorem 3.1 *Let Ω be as in (1.1), K having a $\mathcal{C}^2$-boundary. Given $F \in \mathbb{R}$, there exists a vector field $\Psi_* \in H^2(\Omega)$ such that*

$$\begin{cases} \nabla \cdot \Psi_* = 0 \ \ in \ \ \Omega\,; & \Psi_* \times \nu = 0 \ \ on \ \ \Gamma_I \cup \Gamma_O\,; \\ \Psi_* = 0 \ \ on \ \ \Gamma_W\,; & \displaystyle\int_{\Sigma(s)} \Psi_* \cdot \widehat{k} = F \quad \forall s \in [-h, h]. \end{cases} \tag{3.1}$$

Moreover, there holds the estimate

$$\|\nabla \Psi_*\|_{L^2(\Omega)} \le \frac{2|F|}{\pi R^2}\,(1 + 3C_B(\Omega))\left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K)}\right). \tag{3.2}$$

Proof Consider a Hagen–Poiseuille flow having flow rate F in $\mathcal{M}$, that is,

$$U_0(\xi) \doteq \frac{2F}{\pi R^4}(R^2 - \rho^2)\widehat{k} \qquad \forall \xi \in \mathcal{M}. \tag{3.3}$$

It is straightforward to observe that $U_0 \in \mathcal{C}^\infty(\overline{\mathcal{M}})$ is divergence-free, it vanishes on the lateral boundary of $\mathcal{M}$ and $U_0 \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$. Moreover,

$$\int_{\Gamma_I} U_0 \cdot \widehat{k} = \int_{\Gamma_O} U_0 \cdot \widehat{k} = F. \tag{3.4}$$

Let $\phi \in H_0^1(\mathcal{M})$ be the relative capacity potential of K with respect to $\mathcal{M}$, as in Sect. 2.2. Since K has a boundary of class $\mathcal{C}^2$, $\mathcal{M}$ is convex and its lateral boundary is smooth, standard elliptic regularity arguments show that $\phi \in H^2(\Omega)$. We then define

$$\Psi_1(\xi) \doteq (1-\phi(\xi))U_0(\xi) = \frac{2F}{\pi R^4}(R^2-\rho^2)(1-\phi(\xi))\widehat{k} \qquad \forall \xi \in \Omega\,, \tag{3.5}$$

which is an element of $H^2(\Omega)$ that vanishes of Γ_W and such that $\Psi_1 \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$. Its restriction to $\partial\Omega$, denoted by $\gamma(\Psi_1)$, is an element of $H^{3/2}(\partial\Omega)$ such that

$$\int_{\partial\Omega} \gamma(\Psi_1)\cdot\nu = \int_{\partial\mathcal{M}} \gamma(\Psi_1)\cdot\nu + \int_{\partial K} \gamma(\Psi_1)\cdot\nu = -F + F + 0 = 0. \tag{3.6}$$

Therefore $\nabla \cdot \Psi_1 \in H^1(\Omega) \cap L_0^2(\Omega)$ (see (2.55)), and so from [27, Theorem III.3.3] we deduce the existence of another vector field $X_1 \in H_0^1(\Omega)$ satisfying

$$\nabla \cdot X_1 = -\nabla \cdot \Psi_1 \quad \text{in} \quad \Omega \qquad \text{and} \qquad \|\nabla X_1\|_{L^2(\Omega)} \leq C_B(\Omega)\|\nabla \cdot \Psi_1\|_{L^2(\Omega)}, \tag{3.7}$$

see (2.56) as well. Additionally, in view of (3.6), [27, Theorem IV.1.1] ensures the existence of a unique weak solution $(\Psi_*, \Pi_*) \in H^1(\Omega) \times L_0^2(\Omega)$ to the Stokes problem

$$\begin{cases} -\Delta\Psi_* + \nabla\Pi_* = 0\,, \quad \nabla\cdot\Psi_* = 0 \quad \text{in} \quad \Omega\,, \\ \Psi_* = \gamma(\Psi_1) \quad \text{on} \quad \partial\Omega. \end{cases} \tag{3.8}$$

A simple integration by parts shows that

$$\int_\Omega \nabla\Psi_* \cdot \nabla\varphi = 0 \qquad \forall \varphi \in H_{0,\sigma}^1(\Omega) \doteq \{v \in H_0^1(\Omega) \mid \nabla\cdot v = 0 \ \text{ in } \ \Omega\}.$$

As $\Psi_* - \Psi_1 - X_1 \in H_{0,\sigma}^1(\Omega)$, we can set $\varphi = \Psi_* - \Psi_1 - X_1$ in the last identity and use (3.7) to get

$$\|\nabla\Psi_*\|_{L^2(\Omega)}^2 = \int_\Omega \nabla\Psi_*\cdot\nabla\Psi_1 + \int_\Omega \nabla\Psi_*\cdot\nabla X_1 \leq (3C_B(\Omega)+1)\|\nabla\Psi_*\|_{L^2(\Omega)}\|\nabla\Psi_1\|_{L^2(\Omega)}\,,$$

that is,

$$\|\nabla\Psi_*\|_{L^2(\Omega)} \leq (3C_B(\Omega)+1)\|\nabla\Psi_1\|_{L^2(\Omega)}. \tag{3.9}$$

Moreover, since $\gamma(\Psi_1) \in H^{3/2}(\partial\Omega)$, K is of class $\mathcal{C}^2$, the cylinder $\mathcal{M}$ is convex and its lateral boundary is smooth, by merging the well-known regularity results for the solutions of the steady-state Stokes equations under non-homogeneous Dirichlet boundary conditions (see [12, Teorema, page 311]) with a localization argument through a partition of unity (as in [14, Theorem A.1]) we may establish that $(\Psi_*, \Pi_*) \in H^2(\Omega) \times H^1(\Omega)$. As $\Psi_* = \Psi_1$ on $\partial\Omega$, we have

$$\nabla\cdot\Psi_* = 0 \ \text{ in } \ \Omega\,; \qquad \Psi_* = 0 \ \text{ on } \ \Gamma_W\,; \qquad \Psi_* \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O. \tag{3.10}$$

Now, given $s \in (-h, h]$, we define the region $\Omega(s) \subset \mathbb{R}^3$ by

$$\Omega(s) \doteq \{\xi \in \Omega \mid -h < z < s\} \qquad \Longrightarrow \qquad \partial\Omega(s) = \Gamma_I \cup \{\xi \in \Gamma_W \mid -h < z < s\} \cup \Sigma(s). \tag{3.11}$$

Since $\Psi_* = U_0$ on Γ_I, from (3.4)–(3.10) and the Divergence Theorem we infer that

$$0 = \int_{\Omega(s)} \nabla \cdot \Psi_* = \int_{\partial\Omega(s)} \Psi_* \cdot \nu = -F + \int_{\Sigma(s)} \Psi_* \cdot \widehat{k},$$

that is,

$$\int_{\Sigma(s)} \Psi_* \cdot \widehat{k} = F \qquad \forall s \in [-h, h].$$

Therefore, Ψ_* satisfies all the properties in (3.1). Applying the Maximum Principle we may write

$$\begin{aligned}\|\nabla\Psi_1\|_{L^2(\Omega)} &\le \|(\nabla\phi)^\top U_0\|_{L^2(\Omega)} + \|(1-\phi)\nabla U_0\|_{L^2(\Omega)} \\ &\le \|U_0\|_{L^\infty(\mathcal{M})}\|\nabla\phi\|_{L^2(\Omega)} + \|\nabla U_0\|_{L^2(\mathcal{M})} \\ &= \frac{2|F|}{\pi R^2}\left(\sqrt{\mathrm{Cap}_{\mathcal{M}}(K)} + 2\sqrt{\pi h}\right),\end{aligned}$$

so that, from (3.9), the estimate in (3.2) follows. □

Remark 3.1 Notice that, by definition of (3.3), $(U_0 \cdot \nabla)U_0 \equiv 0$ in $\mathcal{M}$ but $(\Psi_* \cdot \nabla)\Psi_*$ is not (necessarily) identically zero in Ω.

Remark 3.2 In the proof of Theorem 3.1, the condition $\nabla\cdot\Psi_1 \in H^1(\Omega)\cap L^2_0(\Omega)$ is not enough to ensure that $X_1 \in H^2(\Omega) \cap H^1_0(\Omega)$. In fact, the continuity of the divergence operator from $H^1(\Omega) \cap L^2_0(\Omega)$ to $H^2(\Omega) \cap H^1_0(\Omega)$ has been proved for $\mathcal{C}^{1,1}$-domains in [23]. By setting $\Psi_2 \doteq \Psi_* - \Psi_1$, we see from (3.8) that $\Psi_2 \in H^2(\Omega) \cap H^1_0(\Omega)$ is a (strong) solution to the generalized Stokes problem

$$\begin{cases} -\Delta\Psi_2 + \nabla\Pi_* = \Delta\Psi_1\,, \quad \nabla\cdot\Psi_2 = -\nabla\cdot\Psi_1 \ \ in \ \ \Omega\,, \\ \Psi_2 = 0 \ \ on \ \ \partial\Omega. \end{cases} \tag{3.12}$$

In the case of a merely Lipschitz obstacle K, Theorem 3.1 can be rephrased as follows:

Corollary 3.1 *Let Ω be as in (1.1), K having a Lipschitz boundary. Given $F \in \mathbb{R}$, there exists a vector field $\Psi_* \in H^1(\Omega)$ satisfying (3.1) and the estimate (3.2).*

Proof Consider the Hagen–Poiseuille flow having flow rate F in $\mathcal{M}$, defined as in (3.3). Let $\phi \in H^1_0(\mathcal{M})$ be the relative capacity potential of K with respect to $\mathcal{M}$, as in Sect. 2.2. We then define $\Psi_1 \in H^1(\Omega)$ as in (3.5), which vanishes of Γ_W and is such that $\Psi_1 \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$. Its restriction to $\partial\Omega$, denoted by $\gamma(\Psi_1)$, is an element of $H^{1/2}(\partial\Omega)$ such that (3.6) holds. Therefore $\nabla\cdot\Psi_1 \in L^2_0(\Omega)$, and so from [27, Theorem III.3.3] we deduce the existence of another vector field $X_1 \in H^1_0(\Omega)$ satisfying (3.7). The vector field $\Psi_* \doteq \Psi_1 + X_1$ is an element of $H^1(\Omega)$ satisfying (3.1) and the estimate (3.2). □

In view of the identity

$$\nabla\left(\frac{1}{2}|u|^2\right) = (\nabla u)^\top u \quad \text{in} \quad \Omega\,,$$

it is customary (see, for example, [35, 46]) to add the term $(\nabla u)^\top u$ to both sides of the equation of conservation of momentum $(1.3)_1$, thereby resulting in the problem

$$\begin{cases} -\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi = f\,, \quad \nabla\cdot u = 0 \ \text{ in } \ \Omega\,, \\ u = 0 \ \text{ on } \ \Gamma_W\,, \\ u\times\nu = 0\,, \quad \Phi = p_- \ \text{ on } \ \Gamma_I\,, \\ u\times\nu = 0\,, \quad \Phi = p_+ \ \text{ on } \ \Gamma_O\,, \\ \displaystyle\int_{\Sigma(s)} u\cdot\widehat{k} = F \qquad \forall s\in[-h,h]. \end{cases} \tag{3.13}$$

Now, given $f\in\mathcal{C}^1(\Omega)$, assume that $u=(u_1,u_2,u_3)\in\mathcal{C}^2(\overline{\Omega})$ and $\Phi\in\mathcal{C}^1(\overline{\Omega})$ solve (3.13) in the classical sense. We take a vector function $\varphi=(\varphi_1,\varphi_2,\varphi_3)\in H^1(\Omega)$ and integrate by parts the equation of conservation of momentum $(3.13)_1$ in the following way:

$$\begin{aligned} \int_\Omega f\cdot\varphi &= \int_\Omega \Big[-\eta\Delta u + (u\cdot\nabla)u - (\nabla u)^\top u + \nabla\Phi\Big]\cdot\varphi \\ &= \eta\int_\Omega \nabla u\cdot\nabla\varphi - \eta\int_{\partial\Omega}\frac{\partial u}{\partial\nu}\cdot\varphi + \int_\Omega\Big[\nabla u - (\nabla u)^\top\Big]u\cdot\varphi - \int_\Omega \Phi(\nabla\cdot\varphi) + \int_{\partial\Omega}\Phi\nu\cdot\varphi \\ &= \eta\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega\Big[\nabla u - (\nabla u)^\top\Big]u\cdot\varphi - \int_\Omega \Phi(\nabla\cdot\varphi) + \int_{\partial\Omega}\left(\Phi\nu - \eta\frac{\partial u}{\partial\nu}\right)\cdot\varphi. \end{aligned} \tag{3.14}$$

If, in addition, we assume that φ is divergence-free and vanishes on Γ_W, from (3.14) we obtain

$$\begin{aligned} \int_\Omega f\cdot\varphi = \eta\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega\Big[\nabla u - (\nabla u)^\top\Big]u\cdot\varphi + \int_{\Gamma_I}\left(\Phi\nu - \eta\frac{\partial u}{\partial\nu}\right)\cdot\varphi \\ + \int_{\Gamma_O}\left(\Phi\nu - \eta\frac{\partial u}{\partial\nu}\right)\cdot\varphi. \end{aligned} \tag{3.15}$$

Notice that $\nu = \mp\widehat{k}$ on Γ_I and Γ_O, respectively, thus $u_1=u_2=0$ on $\Gamma_I\cup\Gamma_O$, in view of $(3.13)_3$–$(3.13)_4$. The regularity and incompressibility condition of u then imply that

$$\frac{\partial u_3}{\partial z} = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) = 0 \quad \text{on} \quad \Gamma_I\cup\Gamma_O.$$

If we further impose that $\varphi\times\nu=0$ on $\Gamma_I\cup\Gamma_O$ (so that $\varphi_1=\varphi_2=0$ on $\Gamma_I\cup\Gamma_O$), we get

$$\frac{\partial u}{\partial\nu}\cdot\varphi = \mp\frac{\partial u_3}{\partial z}\varphi_3 = 0 \quad \text{on} \quad \Gamma_I\cup\Gamma_O. \tag{3.16}$$

In order to get rid of all boundary terms in (3.15), one must also suppose that

$$\int_{\Gamma_I}\varphi\cdot\widehat{k} = 0 \qquad \text{or} \qquad \int_{\Gamma_O}\varphi\cdot\widehat{k} = 0$$

which, combined with the fact that φ is divergence-free and vanishes on Γ_W, yields

$$\int_{\Sigma(s)}\varphi\cdot\widehat{k} = 0 \qquad \forall s\in[-h,h]. \tag{3.17}$$

By inserting (3.16)–(3.17) into (3.15) we finally obtain

$$\eta\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega\Big[\nabla u - (\nabla u)^\top\Big]u\cdot\varphi = \int_\Omega f\cdot\varphi\,,$$

where it suffices to have $u \in H^1(\Omega)$ and the Bernoulli pressure Φ is no longer present. This motivates the introduction of the following functional spaces (of vector fields) that will be employed hereafter:

$$\mathcal{V}_*(\Omega) = \left\{ v \in H^1(\Omega) \;\middle|\; \begin{array}{ll} \nabla \cdot v = 0 \ \text{ in } \ \Omega\,; & v \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\,; \\ v = 0 \ \text{ on } \ \Gamma_W\,; & \displaystyle\int_{\Sigma(s)} v \cdot \widehat{k} = 0 \quad \forall s \in [-h, h] \end{array} \right\}$$

and

$$\mathcal{V}(\Omega) = \left\{ v \in H^1(\Omega) \mid \nabla \cdot v = 0 \ \text{ in } \ \Omega\,; \qquad v \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\,; \qquad v = 0 \ \text{ on } \ \Gamma_W \right\},$$

which are Hilbert spaces if endowed with the Dirichlet scalar product of the gradients, denoted by

$$[v, w]_{\mathcal{V}(\Omega)} \doteq \int_\Omega \nabla v \cdot \nabla w \qquad \forall v, w \in \mathcal{V}(\Omega). \tag{3.18}$$

With respect to the boundary-value problem (3.13), throughout this section we assume that $F \in \mathbb{R}$ and $f \in L^2(\Omega)$ are a given transversal flux rate and external forcing term, respectively. We can now give the following definition for the weak solutions of problem (3.13) (or, equivalently, of problem (1.3)):

Definition 3.1 Given $\Psi \in \mathcal{V}(\Omega)$ satisfying (3.1), we say that a vector field $u \in \mathcal{V}(\Omega)$ is a **weak solution** of (3.13) if $u - \Psi \in \mathcal{V}_*(\Omega)$ and

$$\eta \int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega \left[\nabla u - (\nabla u)^\top \right] u \cdot \varphi = \int_\Omega f \cdot \varphi \qquad \forall \varphi \in \mathcal{V}_*(\Omega). \tag{3.19}$$

It is important to establish a precise connection between the boundary-value problem (3.13) and the weak formulation given in Definition 3.1. This is done firstly in the case of a smooth obstacle, where one is able to prove that any weak solution of (3.13) has additional regularity. Indeed:

Theorem 3.2 *Let Ω be as in (1.1), K having a $\mathcal{C}^2$-boundary. If $u \in \mathcal{C}^2(\overline{\Omega})$ and $\Phi \in \mathcal{C}^1(\overline{\Omega})$ solve (3.13) in the classical sense, then u is a solution of the variational problem (3.19). Conversely, given $\Psi \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ satisfying (3.1), if $u \in \mathcal{V}(\Omega)$ is a weak solution of (3.13), then $u \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ and there exists a unique $\Phi \in H^1(\Omega) \cap L^2_0(\Omega)$ such that the pair (u, Φ) solves $(3.13)_1$ point-wise almost everywhere in Ω. The boundary conditions for u in $(3.13)_2$–$(3.13)_3$–$(3.13)_4$ are verified in the sense of $H^{3/2}(\partial\Omega)$ (and also the condition on the transversal flow rate $(3.13)_5$), while the boundary conditions for Φ in $(3.13)_3$–$(3.13)_4$ are satisfied in the sense of $H^{1/2}(\partial\Omega)$.*

Proof In order to simplify the presentation of this proof, given a bounded domain $Q \subset \mathbb{R}^3$ we denote

$$\mathcal{C}^\infty_{0,\sigma}(Q) \doteq \{ v \in \mathcal{C}^\infty_0(Q) \mid \nabla \cdot v = 0 \ \text{ in } \ Q \}.$$

If $u \in \mathcal{C}^2(\overline{\Omega})$ and $\Phi \in \mathcal{C}^1(\overline{\Omega})$ solve (3.13) in the classical sense, the computations in (3.14) show that u is a solution of the variational problem (3.19). Now, given $\Psi \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ satisfying (3.1), suppose that $u \in \mathcal{V}(\Omega)$ is a weak solution of (3.13). Since $u - \Psi \in \mathcal{V}_*(\Omega)$, the boundary conditions for u in $(3.13)_2$–$(3.13)_3$–$(3.13)_4$ are satisfied in the $H^{1/2}(\partial\Omega)$-trace sense (and also the condition on the transversal flow rate $(3.13)_5$). Let $\varphi \in \mathcal{C}^\infty_{0,\sigma}(\Omega)$, so that $\varphi \in \mathcal{V}_*(\Omega)$; in fact, it suffices to check that φ has a zero transversal flow rate. Given $s \in (-h, h]$, we define the region $\Omega(s) \subset \mathbb{R}^3$ as in (3.11). Since φ vanishes on $\partial\Omega$, after applying the Divergence Theorem we infer

$$\int_{\Sigma(s)} \varphi \cdot \widehat{k} = \int_{\partial\Omega(s)} \varphi \cdot \nu = \int_{\Omega(s)} \nabla \cdot \varphi = 0.$$

By definition of distributional derivative and integration by parts we obtain

$$\langle \Delta u, \varphi \rangle_{\mathcal{D}(\Omega)} = \int_{\Omega} u \cdot \Delta\varphi = -\int_{\Omega} \nabla u \cdot \nabla\varphi \,, \tag{3.20}$$

which, once inserted into (3.19), yields

$$-\eta\langle \Delta u, \varphi \rangle_{\mathcal{D}(\Omega)} + \int_{\Omega} \Big[\nabla u - (\nabla u)^{\top}\Big] u \cdot \varphi - \int_{\Omega} f \cdot \varphi = 0 \qquad \forall \varphi \in \mathcal{C}^{\infty}_{0,\sigma}(\Omega).$$

Since Ω is connected, in virtue of [32, Theorem 2.3] there exists a unique $\Phi \in L^2(\Omega)/\mathbb{R}$ (Φ is uniquely determined up to an additive constant) such that

$$-\eta\Delta u + (u \cdot \nabla)u - (\nabla u)^{\top} u + \nabla\Phi = f \quad \text{in distributional sense in } \Omega.$$

We choose the constant component of the Bernoulli pressure Φ in such a way that $\Phi \in L^2_0(\Omega)$, and thus it is uniquely defined. Moreover, since $f \in L^2(\Omega)$, the usual interior regularity results for the steady-state Navier–Stokes equations (see [27, Theorem IX.5.1]) imply that $u \in H^2_{\mathrm{loc}}(\Omega)$, $\Phi \in H^1_{\mathrm{loc}}(\Omega)$ and that

$$-\eta\Delta u + (u \cdot \nabla)u - (\nabla u)^{\top} u + \nabla\Phi = f \quad \text{almost everywhere in } \Omega. \tag{3.21}$$

In view of the embedding $H^1(\Omega) \subset L^6(\Omega)$ we have $(u \cdot \nabla)u - (\nabla u)^{\top} u \in L^{3/2}(\Omega)$ and therefore, (3.21) implies that $\operatorname{div}(-\eta\nabla u + \Phi\mathbb{I}_3) \in L^{3/2}(\Omega)$, where $\mathbb{I}_3$ is the 3×3-identity matrix. Then, [27, Theorem III.2.2] guarantees that

$$(-\eta\nabla u + \Phi\mathbb{I}_3) \cdot \nu \in W^{-\frac{2}{3},\frac{3}{2}}(\partial\Omega)\,,$$

and the validity of the Green identity

$$\begin{aligned} &\int_{\Omega} \operatorname{div}(-\eta\nabla u + \Phi\mathbb{I}_3) \cdot \varphi - \eta\int_{\Omega} \nabla u \cdot \nabla\varphi + \int_{\Omega} \Phi(\nabla \cdot \varphi) \\ &\quad = \left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, \varphi \right\rangle_{\partial\Omega} \qquad \forall \varphi \in W^{1,3}(\Omega)\,, \end{aligned} \tag{3.22}$$

where the "boundary term" $\langle \cdot, \cdot \rangle_{\partial\Omega}$ represents the duality product between $W^{-\frac{2}{3},\frac{3}{2}}(\partial\Omega)$ and $W^{\frac{2}{3},3}(\partial\Omega)$. Notice that, as a straightforward consequence of (3.19)–(3.21)–(3.22), we have

$$\left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, \varphi \right\rangle_{\partial\Omega} = 0 \qquad \forall \varphi \in W^{1,3}(\Omega) \cap \mathcal{V}_*(\Omega). \tag{3.23}$$

Now, as in the proof of Theorem 2.1 we introduce the reflected domains (with respect to the planes $z = \pm h$) $\Omega_{\pm} \subset \mathbb{R}^3$ defined by

$$\Omega_- = \{(x, y, -2h - z) \mid (x, y, z) \in \Omega\} \quad \text{and} \quad \Omega_+ = \{(x, y, 2h - z) \mid (x, y, z) \in \Omega\}\,,$$

and set $\Omega_\sharp \doteq \Omega_- \cup \Omega \cup \Omega_+$, see Fig. 6.

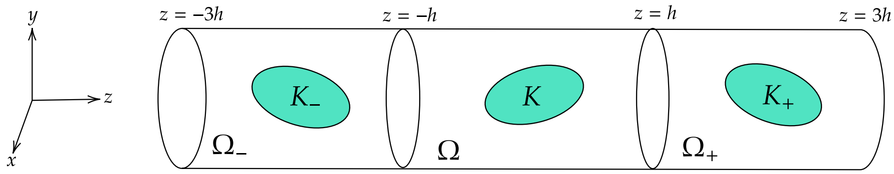

Fig. 6 Representation of the extended domain $\Omega_\sharp$

We extend the functions $u=(u_1,u_2,u_3)$ and $f=(f_1,f_2,f_3)$ to $\Omega_\sharp$ by the formulas

$$
\begin{aligned}
&u_\sharp(x,y,z)\\
&=\begin{cases} u_-(x,y,z)\doteq(-u_1(x,y,-2h-z),-u_2(x,y,-2h-z),u_3(x,y,-2h-z)) & \text{if } (x,y,z)\in\Omega_-\\ u(x,y,z) & \text{if } (x,y,z)\in\Omega\\ u_+(x,y,z)\doteq(-u_1(x,y,2h-z),-u_2(x,y,2h-z),u_3(x,y,2h-z)) & \text{if } (x,y,z)\in\Omega_+ \end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
&f_\sharp(x,y,z)\\
&=\begin{cases} f_-(x,y,z)\doteq(-f_1(x,y,-2h-z),-f_2(x,y,-2h-z),f_3(x,y,-2h-z)) & \text{if } (x,y,z)\in\Omega_-\\ f(x,y,z) & \text{if } (x,y,z)\in\Omega\\ f_+(x,y,z)\doteq(-f_1(x,y,2h-z),-f_2(x,y,2h-z),f_3(x,y,2h-z)) & \text{if } (x,y,z)\in\Omega_+. \end{cases}
\end{aligned}
$$

It is clear that $u_\pm\in H^1(\Omega_\pm)$ is divergence-free in $\Omega_\pm$, $f_\pm\in L^2(\Omega_\pm)$ and $f_\sharp\in L^2(\Omega_\sharp)$. Moreover, since $u\times\nu=0$ on $\Gamma_I\cup\Gamma_O$, we also have $u_\sharp\in H^1(\Omega_\sharp)$. Now, take any scalar function $\varphi\in\mathcal{C}_0^\infty(\Omega\cup\Omega_+)$ such that $\mathrm{supp}(\varphi)\cap\Gamma_O\neq\emptyset$ and integrate in the following way (recall that ν is the outward unit normal to Ω, and therefore it is directed to the interior of Ω_+):

$$
\begin{aligned}
\int_{\Omega_\sharp}u_\sharp\cdot\nabla\varphi&=\int_{\mathrm{supp}(\varphi)\cap\Omega}u\cdot\nabla\varphi+\int_{\mathrm{supp}(\varphi)\cap\Omega_+}u_+\cdot\nabla\varphi\\
&=-\int_{\mathrm{supp}(\varphi)\cap\Omega}\varphi(\nabla\cdot u)+\int_{\partial(\mathrm{supp}(\varphi)\cap\Omega)}\varphi(u\cdot\nu)-\int_{\mathrm{supp}(\varphi)\cap\Omega_+}\varphi(\nabla\cdot u_+)\\
&\quad+\int_{\partial(\mathrm{supp}(\varphi)\cap\Omega_+)}\varphi(u_+\cdot\nu_+)\\
&=\int_{\mathrm{supp}(\varphi)\cap\Gamma_O}\varphi(u\cdot\nu)-\int_{\mathrm{supp}(\varphi)\cap\Gamma_O}\varphi(u_+\cdot\nu)=0\,,
\end{aligned}
$$

where $\nu_\pm$ denotes the outward unit normal to $\Omega_\pm$, so that $\nu=-\nu_+$ on Γ_O. By choosing any scalar function $\varphi\in\mathcal{C}_0^\infty(\Omega\cup\Omega_-)$ such that $\mathrm{supp}(\varphi)\cap\Gamma_I\neq\emptyset$ and performing the same integration, we conclude

$$
\int_{\Omega_\sharp}\varphi(\nabla\cdot u_\sharp)=-\int_{\Omega_\sharp}u_\sharp\cdot\nabla\varphi=0\qquad\forall\varphi\in\mathcal{C}_{0,\sigma}^\infty(\Omega_\sharp)\,,
$$

that is, $u_\sharp$ is divergence-free in the whole $\Omega_\sharp$. We then claim that

$$
\eta\int_{\Omega_\sharp}\nabla u_\sharp\cdot\nabla\varphi+\int_{\Omega_\sharp}\Big[\nabla u_\sharp-(\nabla u_\sharp)^\top\Big]u_\sharp\cdot\varphi=\int_{\Omega_\sharp}f_\sharp\cdot\varphi\qquad\forall\varphi\in\mathcal{C}_{0,\sigma}^\infty(\Omega_\sharp).\tag{3.24}
$$

To prove (3.24), let $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{C}^\infty_{0,\sigma}(\Omega_\sharp)$. Since φ vanishes on Γ_W, from (3.21)–(3.22) we get

$$\begin{aligned}
&\eta \int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega \Big[\nabla u - (\nabla u)^\top\Big] u \cdot \varphi \\
&\quad = \int_\Omega \Big[\mathrm{div}(-\eta \nabla u + \Phi \mathbb{I}_3) + (u \cdot \nabla) u - (\nabla u)^\top u\Big] \cdot \varphi - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi \Big\rangle_{\partial\Omega} \\
&\quad = \int_\Omega f \cdot \varphi - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi \Big\rangle_{\Gamma_I} - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi \Big\rangle_{\Gamma_O}
\end{aligned} \tag{3.25}$$

Now, define $\varphi_\pm \in \mathcal{C}^\infty(\Omega)$ according to

$$\begin{aligned}
&\varphi_+(x, y, z) \doteq (-\varphi_1(x, y, 2h - z), -\varphi_2(x, y, 2h - z), \varphi_3(x, y, 2h - z)) \qquad \forall (x, y, z) \in \Omega\,, \\
&\varphi_-(x, y, z) \doteq (-\varphi_1(x, y, -2h - z), -\varphi_2(x, y, -2h - z), \varphi_3(x, y, -2h - z)) \qquad \forall (x, y, z) \in \Omega.
\end{aligned}$$

We clearly have

$$\begin{aligned}
&\varphi_+ \in W^{1,3}(\Omega)\,; \qquad \nabla \cdot \varphi_+ = 0 \quad \text{in} \quad \Omega\,; \qquad \varphi_+ = 0 \quad \text{on} \quad \Gamma_W \cup \Gamma_I. \\
&\varphi_- \in W^{1,3}(\Omega)\,; \qquad \nabla \cdot \varphi_- = 0 \quad \text{in} \quad \Omega\,; \qquad \varphi_- = 0 \quad \text{on} \quad \Gamma_W \cup \Gamma_O.
\end{aligned}$$

Successive applications of the changes of variables

$$(x, y, z) \in \Omega \longmapsto (x, y, 2h - z) \in \Omega_+ \qquad \text{and} \qquad (x, y, z) \in \Omega \longmapsto (x, y, -2h - z) \in \Omega_-$$

allow us to show that

$$\begin{gathered}
\int_\Omega \nabla u \cdot \nabla \varphi_\pm = \int_{\Omega_\pm} \nabla u_\pm \cdot \nabla \varphi\,, \qquad \int_\Omega \Big[\nabla u - (\nabla u)^\top\Big] u \cdot \varphi_\pm = \int_{\Omega_\pm} \Big[\nabla u_\pm - (\nabla u_\pm)^\top\Big] u_\pm \cdot \varphi\,, \\
\int_\Omega f \cdot \varphi_\pm = \int_{\Omega_\pm} f_\pm \cdot \varphi.
\end{gathered} \tag{3.26}$$

From (3.21)–(3.22)–(3.26) we then obtain the identity

$$\begin{aligned}
&\eta \int_{\Omega_+} \nabla u_+ \cdot \nabla \varphi + \int_{\Omega_+} \Big[\nabla u_+ - (\nabla u_+)^\top\Big] u_+ \cdot \varphi = \eta \int_\Omega \nabla u \cdot \nabla \varphi_+ + \int_\Omega \Big[\nabla u - (\nabla u)^\top\Big] u \cdot \varphi_+ \\
&= \int_\Omega \Big[\mathrm{div}(-\eta \nabla u + \Phi \mathbb{I}_3) + (u \cdot \nabla) u - (\nabla u)^\top u\Big] \cdot \varphi_+ - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi_+ \Big\rangle_{\partial\Omega} \\
&= \int_\Omega f \cdot \varphi_+ - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi_+ \Big\rangle_{\Gamma_O} = \int_{\Omega_+} f_+ \cdot \varphi - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi_+ \Big\rangle_{\Gamma_O}\,,
\end{aligned} \tag{3.27}$$

and similarly,

$$\eta \int_{\Omega_-} \nabla u_- \cdot \nabla \varphi + \int_{\Omega_-} \Big[\nabla u_- - (\nabla u_-)^\top\Big] u_- \cdot \varphi = \int_{\Omega_-} f_- \cdot \varphi - \Big\langle \Phi \nu - \eta \frac{\partial u}{\partial \nu}, \varphi_- \Big\rangle_{\Gamma_I}. \tag{3.28}$$

A combination of (3.25)–(3.27)–(3.28) allows us to write

$$\begin{aligned}
&\eta \int_{\Omega_\sharp} \nabla u_\sharp \cdot \nabla \varphi + \int_{\Omega_\sharp} \left[\nabla u_\sharp - (\nabla u_\sharp)^\top\right] u_\sharp \cdot \varphi \\
&\quad = \eta \int_{\Omega_-} \nabla u_- \cdot \nabla \varphi + \int_{\Omega_-} \left[\nabla u_- - (\nabla u_-)^\top\right] u_- \cdot \varphi + \eta \int_{\Omega} \nabla u \cdot \nabla \varphi + \int_{\Omega} \left[\nabla u - (\nabla u)^\top\right] u \cdot \varphi \\
&\qquad + \eta \int_{\Omega_+} \nabla u_+ \cdot \nabla \varphi + \int_{\Omega_+} \left[\nabla u_+ - (\nabla u_+)^\top\right] u_+ \cdot \varphi \qquad (3.29)\\
&\quad = \int_{\Omega_\sharp} f_\sharp \cdot \varphi - 2\left(\left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, \varphi_3 \widehat{k}\right\rangle_{\Gamma_I} + \left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, \varphi_3 \widehat{k}\right\rangle_{\Gamma_O}\right) \\
&\quad = \int_{\Omega_\sharp} f_\sharp \cdot \varphi - 2\left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, (\varphi\cdot\nu)\nu\right\rangle_{\partial\Omega}.
\end{aligned}$$

In view of the Divergence Theorem we clearly have

$$\int_{\partial\Omega} (\varphi\cdot\nu)\nu\cdot\nu = \int_\Omega \nabla\cdot\varphi = 0\,,$$

so that [27, Theorem IV.1.1] guarantees the existence of a unique weak solution $(w, q) \in H^1(\Omega) \times L^2_0(\Omega)$ to the Stokes problem

$$\begin{cases} -\Delta w + \nabla q = 0\,, \quad \nabla\cdot w = 0 \quad \text{in} \quad \Omega\,, \\ w = (\varphi\cdot\nu)\nu \quad \text{on} \quad \partial\Omega\,, \end{cases} \qquad (3.30)$$

which can be equivalently written as

$$\begin{cases} -\Delta w + \nabla q = 0\,, \quad \nabla\cdot w = 0 \quad \text{in} \quad \Omega\,, \\ w = 0 \quad \text{on} \quad \Gamma_W\,, \quad w = \varphi_3 \widehat{k} \quad \text{on} \quad \Gamma_I\,, \quad w = \varphi_3 \widehat{k} \quad \text{on} \quad \Gamma_O. \end{cases}$$

As in the proof of Theorem 3.1 we can argue that $(w, q) \in H^2(\Omega) \times H^1(\Omega)$. For a given $s \in (-h, h]$, we define the region $\Omega(s) \subset \mathbb{R}^3$ as in (3.11). We clearly have

$$\int_{\Sigma(s)} w\cdot\widehat{k} = \int_{\partial\Omega(s)} w\cdot\nu + \int_{\Gamma_I} w\cdot\widehat{k} = \int_{\Omega(s)} \nabla\cdot w + \int_{\partial\Omega_-} \varphi\cdot\nu_- = 0 + \int_{\Omega_-} \nabla\cdot\varphi = 0.$$

Therefore, $w \in H^2(\Omega) \cap \mathcal{V}_*(\Omega)$ and from (3.23) we deduce that

$$\left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, (\varphi\cdot\nu)\nu\right\rangle_{\partial\Omega} = \left\langle \Phi\nu - \eta\frac{\partial u}{\partial \nu}, w\right\rangle_{\partial\Omega} = 0\,,$$

which, once inserted into (3.29), proves (3.24). As $u_\sharp$ is divergence-free in $\Omega_\sharp$, [27, Theorem IX.5.1] can be invoked again to deduce that $u_\sharp \in H^2_{\text{loc}}(\Omega_\sharp)$. Therefore, $u \in H^2(\Omega)$, thus showing that the boundary conditions for u in $(3.13)_2$–$(3.13)_3$–$(3.13)_4$ are verified in the sense of $H^{3/2}(\partial\Omega)$. Furthermore, from (3.21) we obtain $\Phi \in H^1(\Omega)$, so that $\Phi|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$ and the Green identity (3.22) holds in strong form for every $\varphi \in H^1(\Omega)$. Given any $\varphi \in \mathcal{V}_*(\Omega)$, (3.23) then implies that

$$\int_{\partial\Omega} \Phi(\varphi\cdot\nu) = \eta\int_{\partial\Omega} \frac{\partial u}{\partial\nu}\cdot\varphi = -\eta\int_{\Gamma_I} \frac{\partial u_3}{\partial z}\varphi_3 + \eta\int_{\Gamma_O} \frac{\partial u_3}{\partial z}\varphi_3 = 0\,, \qquad (3.31)$$

because $u_1 = u_2 = 0$ on $\Gamma_I \cup \Gamma_O$, and so the regularity and incompressibility condition of u imply that

$$\frac{\partial u_3}{\partial z} = -\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right) = 0 \quad \text{on} \quad \Gamma_I \cup \Gamma_O.$$

Identity (3.31) states that

$$\int_{\Gamma_I} \Phi(\varphi \cdot \nu) + \int_{\Gamma_O} \Phi(\varphi \cdot \nu) = 0 \qquad \forall \varphi \in \mathcal{V}_*(\Omega). \tag{3.32}$$

We claim that

$$\Phi = p_- \quad \text{a.e. on } \Gamma_I \qquad \text{and} \qquad \Phi = p_+ \quad \text{a.e. on } \Gamma_O\,,$$

for some constants $p_\pm \in \mathbb{R}$. As in [35, Section 4] and [46, Section 2], we take any two points $\xi_1, \xi_2 \in \Gamma_I$ and connect them by a curve $\Upsilon \subset \overline{\Omega}$ which is normal to Γ_I at each of them. Then we consider a vector field $\varphi_0 \in \mathcal{V}_*(\Omega)$ having support confined to a small tube about Υ, with a unit net flux into Ω near ξ_1, and out of Ω near ξ_2. Taking $\varphi = \varphi_0$ in (3.32) and letting the radius of the tube to zero implies that $\Phi(\xi_1) = \Phi(\xi_2)$. Thus, by repeating this argument we deduce that Φ must be constant on Γ_I (and analogously, that Φ must be constant on Γ_O). □

Whenever K has a Lipschitz boundary, the extension argument of Theorem 3.2 cannot be directly applied, as we are not able to guarantee that the solution of (3.30) belongs to $W^{1,3}(\Omega)$, see [23, 28] for further details. As a consequence, we simply state the following:

Corollary 3.2 *Let Ω be as in (1.1), K having a Lipschitz boundary. If $u \in \mathcal{C}^2(\overline{\Omega})$ and $\Phi \in \mathcal{C}^1(\overline{\Omega})$ solve (3.13) in the classical sense, then u is a solution of the variational problem (3.19). Conversely, if $u \in \mathcal{V}(\Omega)$ is a weak solution of (3.13), then there exists a unique $\Phi \in L^2_0(\Omega)$ such that the pair (u, Φ) solves $(3.13)_1$ in distributional sense in Ω. The boundary conditions for u in $(3.13)_2$-$(3.13)_3$-$(3.13)_4$ are satisfied in the sense of $H^{1/2}(\partial\Omega)$ (and also the condition on the transversal flow rate $(3.13)_5$).*

3.2 Existence of solutions and explicit bound on *F* for unique solvability

As already stated, in order to prove the existence of a weak solution to (3.13) in Ω, we firstly study the corresponding problem whenever the obstacle K has a smooth boundary. Here we prove the following:

Theorem 3.3 *Let Ω be as in (1.1), K having a $\mathcal{C}^2$-boundary. For any $F \in \mathbb{R}$ and $f \in L^2(\Omega)$, there exists at least one weak solution $u \in \mathcal{V}(\Omega)$ of problem (3.13).*

Proof Throughout this proof, $C > 0$ will denote a generic constant that depends on Ω, η and F, but that may change from line to line. We follow closely the proof of [46, Theorem 3.1].

Let $\Psi_* \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ be the vector field that arises from Theorem 3.1. To prove the existence of a weak solution $u \in \mathcal{V}(\Omega)$ of (3.13) amounts to show the existence of $\widehat{u} \in \mathcal{V}_*(\Omega)$ such that

$$\eta \int_\Omega \nabla\widehat{u} \cdot \nabla\varphi + \int_\Omega \mathcal{E}(\widehat{u} + \Psi_*)(\widehat{u} + \Psi_*) \cdot \varphi = \int_\Omega f \cdot \varphi - \eta \int_\Omega \nabla\Psi_* \cdot \nabla\varphi \qquad \forall \varphi \in \mathcal{V}_*(\Omega)\,, \tag{3.33}$$

so that the solution will be given by $u = \widehat{u} + \Psi_*$. In (3.33) we have denoted by $\mathcal{E}(w) = \nabla w - (\nabla w)^\top$ the skew-symmetric gradient of any $w \in H^1(\Omega)$. For a fixed $\widehat{u} \in \mathcal{V}_*(\Omega)$, the applications

$$\varphi \in \mathcal{V}_*(\Omega) \longmapsto \int_\Omega \mathcal{E}(\widehat{u} + \Psi_*)(\widehat{u} + \Psi_*) \cdot \varphi \quad \text{and} \quad \varphi \in \mathcal{V}_*(\Omega) \longmapsto \int_\Omega (f \cdot \varphi - \eta \nabla\Psi_* \cdot \nabla\varphi)$$

clearly define linear continuous functions on $\mathcal{V}_*(\Omega)$. Then, in view of the Riesz Representation Theorem, the identity (3.33) may be written as

$$[\eta\,\widehat{u} + \mathcal{P}(\widehat{u}) - \mathcal{F}, \varphi]_{\mathcal{V}(\Omega)} = 0 \qquad \forall \varphi \in \mathcal{V}_*(\Omega)\,,$$

see (3.18), for some (unique) elements $\mathcal{P}(\widehat{u}), \mathcal{F} \in \mathcal{V}_*(\Omega)$ such that

$$\begin{aligned} [\mathcal{P}(\widehat{u}), \varphi]_{\mathcal{V}(\Omega)} &= \int_\Omega \mathcal{E}(\widehat{u} + \Psi_*)(\widehat{u} + \Psi_*) \cdot \varphi \quad \text{and} \\ [\mathcal{F}, \varphi]_{\mathcal{V}(\Omega)} &= \int_\Omega (f \cdot \varphi - \eta \nabla \Psi_* \cdot \nabla \varphi) \qquad \forall \varphi \in \mathcal{V}_*(\Omega). \end{aligned}$$

We have so defined a linear operator $\mathcal{P} : \mathcal{V}_*(\Omega) \longrightarrow \mathcal{V}_*(\Omega)$ and we are led to find a solution $\widehat{u} \in \mathcal{V}_*(\Omega)$ of the following the nonlinear operator equation:

$$\widehat{u} + \frac{1}{\eta}(\mathcal{P}(\widehat{u}) - \mathcal{F}) = 0 \quad \text{in} \;\; \mathcal{V}_*(\Omega). \tag{3.34}$$

Exactly as in [49, Chapter 5, Theorem 1] one can show that the operator $\mathcal{P}$ is compact. Therefore, as a consequence of the Leray–Schauder Principle, in order to prove that (3.34) possesses at least one solution, it suffices to guarantee that any $v^\lambda \in \mathcal{V}_*(\Omega)$ such that

$$v^\lambda + \frac{\lambda}{\eta}(\mathcal{P}(v^\lambda) - \mathcal{F}) = 0 \quad \text{in} \;\; \mathcal{V}_*(\Omega)\,, \tag{3.35}$$

is uniformly bounded with respect to $\lambda \in [0, 1]$. Given $\lambda \in [0, 1]$ and $v^\lambda \in \mathcal{V}_*(\Omega)\backslash\{0\}$ such that (3.35) holds, we clearly have

$$\begin{aligned} &\eta \int_\Omega \nabla v^\lambda \cdot \nabla \varphi + \lambda \int_\Omega \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*) \cdot \varphi \\ &= \lambda \int_\Omega f \cdot \varphi - \lambda\, \eta \int_\Omega \nabla \Psi_* \cdot \nabla \varphi \qquad \forall \varphi \in \mathcal{V}_*(\Omega). \end{aligned} \tag{3.36}$$

Notice that $\mathcal{E}(v)w \cdot w \equiv 0$ in Ω, for all $v, w \in H^1(\Omega)$. By putting $\varphi = v^\lambda$ in (3.36) we obtain

$$\eta \|\nabla v^\lambda\|^2_{L^2(\Omega)} + \lambda \int_\Omega \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*) \cdot v^\lambda = \lambda \int_\Omega f \cdot v^\lambda - \lambda\, \eta \int_\Omega \nabla \Psi_* \cdot \nabla v^\lambda. \tag{3.37}$$

As in the proof of Theorem 3.2, it can be shown that $v^\lambda \in H^2(\Omega) \cap \mathcal{V}_*(\Omega)$ and the existence of a scalar pressure $\Phi^\lambda \in H^1(\Omega) \cap L^2_0(\Omega)$ such that

$$\begin{cases} -\eta \Delta v^\lambda + \nabla \Phi^\lambda = \lambda \left[f + \eta \Delta \Psi_* - \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*) \right] \quad \text{almost everywhere in } \Omega\,, \\ \Phi^\lambda = p^\lambda_- \quad \text{on} \quad \Gamma_I\,, \quad \Phi^\lambda = p^\lambda_+ \quad \text{on} \quad \Gamma_O\,, \end{cases} \tag{3.38}$$

for some constants $p^\lambda_\pm \in \mathbb{R}$. Notice that, since $\Psi_* \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$, the regularity and divergence-free condition of v^λ imply that

$$\int_{\partial\Omega} \frac{\partial v^\lambda}{\partial \nu} \cdot \Psi_* = \int_{\Gamma_I} \frac{\partial v^\lambda}{\partial \nu} \cdot \Psi_* + \int_{\Gamma_O} \frac{\partial v^\lambda}{\partial \nu} \cdot \Psi_* = 0. \tag{3.39}$$

Likewise, $(3.38)_2$ implies that

$$\int_\Omega \nabla \Phi^\lambda \cdot \Psi_* = \int_{\partial\Omega} \Phi^\lambda (\Psi_* \cdot \nu) = F(p^\lambda_+ - p^\lambda_-) \doteq F p^\lambda_*\,, \tag{3.40}$$

and also

$$p_-^\lambda = \frac{1}{\pi R^2} \int_{\Gamma_I} \Phi^\lambda \qquad \text{and} \qquad p_+^\lambda = \frac{1}{\pi R^2} \int_{\Gamma_O} \Phi^\lambda. \tag{3.41}$$

We multiply $(3.38)_1$ by Ψ_* and integrate by parts in Ω taking (3.39)–(3.40) into account, obtaining

$$\begin{aligned} &\eta \int_\Omega \nabla v^\lambda \cdot \nabla \Psi_* + \lambda \int_\Omega \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*) \cdot \Psi_* + F p_*^\lambda \\ &= \lambda \int_\Omega f \cdot \Psi_* + \lambda\, \eta \int_\Omega \Delta \Psi_* \cdot \Psi_*. \end{aligned} \tag{3.42}$$

Adding the identities (3.37) and (3.42) gives us

$$\begin{aligned} \eta \|\nabla v^\lambda\|^2_{L^2(\Omega)} = &-F p_*^\lambda - \eta(1+\lambda) \int_\Omega \nabla v^\lambda \cdot \nabla \Psi_* \\ &+ \lambda \int_\Omega f \cdot (v^\lambda + \Psi_*) + \lambda\, \eta \int_\Omega \Delta \Psi_* \cdot \Psi_*. \end{aligned} \tag{3.43}$$

On the other hand, observe that $f + \eta \Delta \Psi_* - \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*) \in L^{3/2}(\Omega)$. Moreover, from (2.2), Hölder's and Young's inequalities we obtain the estimate

$$\begin{aligned} &\|f + \eta \Delta \Psi_* - \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*)\|_{L^{3/2}(\Omega)} \\ &\quad \leq \|f\|_{L^{3/2}(\Omega)} + \eta \|\Delta \Psi_*\|_{L^{3/2}(\Omega)} + \|\mathcal{E}(v^\lambda + \Psi_*)\|_{L^2(\Omega)} \|v^\lambda + \Psi_*\|_{L^6(\Omega)} \\ &\quad \leq \|f\|_{L^{3/2}(\Omega)} + \eta \|\Delta \Psi_*\|_{L^{3/2}(\Omega)} + \frac{4}{\sqrt{\mathcal{S}_6}} \left(\|\nabla v^\lambda\|^2_{L^2(\Omega)} + \|\nabla \Psi_*\|^2_{L^2(\Omega)} \right). \end{aligned}$$

The pair $(v^\lambda, \Phi^\lambda) \in W^{2,3/2}(\Omega) \times W^{1,3/2}(\Omega)$ is also a strong solution to the Stokes system $(3.38)_1$ in Ω, with a right-hand side given by $f + \eta \Delta \Psi_* - \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*)$. If we apply the same extension argument of the proof of Theorem 3.2 we can then invoke the usual local regularity results for the Stokes equations (see, for example, [27, Theorem IV.4.1]) to obtain the estimate

$$\begin{aligned} &\|v^\lambda\|_{W^{2,3/2}(\Omega)} + \|\Phi^\lambda\|_{W^{1,3/2}(\Omega)} \\ &\quad \leq C \|f + \eta \Delta \Psi_* - \mathcal{E}(v^\lambda + \Psi_*)(v^\lambda + \Psi_*)\|_{L^{3/2}(\Omega)} \\ &\quad \leq C \left[\|f\|_{L^{3/2}(\Omega)} + \eta \|\Delta \Psi_*\|_{L^{3/2}(\Omega)} + \frac{4}{\sqrt{\mathcal{S}_6}} \left(\|\nabla v^\lambda\|^2_{L^2(\Omega)} + \|\nabla \Psi_*\|^2_{L^2(\Omega)} \right) \right]. \end{aligned} \tag{3.44}$$

From the trace inequality and (3.41)–(3.44) we then get

$$|p_\pm^\lambda| \leq C \left[\|f\|_{L^{3/2}(\Omega)} + \eta \|\Delta \Psi_*\|_{L^{3/2}(\Omega)} + \frac{4}{\sqrt{\mathcal{S}_6}} \left(\|\nabla v^\lambda\|^2_{L^2(\Omega)} + \|\nabla \Psi_*\|^2_{L^2(\Omega)} \right) \right]. \tag{3.45}$$

By contradiction, suppose now that the norms $\|\nabla v^\lambda\|_{L^2(\Omega)}$ are not uniformly bounded with respect to $\lambda \in [0, 1]$. Then, there must exist $\lambda_0 \in [0, 1]$ and a sequence $(\lambda_k)_{k \in \mathbb{N}} \subset [0, 1]$ such that

$$\lim_{k \to \infty} \lambda_k = \lambda_0 \qquad \text{and} \qquad \lim_{k \to \infty} J_k = +\infty\,, \quad \text{with } J_k \doteq \|\nabla v^{\lambda_k}\|_{L^2(\Omega)} \quad \forall k \in \mathbb{N}.$$

Given $k \in \mathbb{N}$, if we divide the identity (3.43) (with $\lambda = \lambda_k$) by J_k^2, we obtain

$$\eta = -\widehat{p}_*^{\lambda_k} F - \eta \frac{1+\lambda_k}{J_k} \int_\Omega \nabla \widehat{v}^{\lambda_k} \cdot \nabla \Psi_* + \frac{\lambda_k}{J_k} \int_\Omega f \cdot \left(\widehat{v}^{\lambda_k} + \frac{\Psi_*}{J_k} \right)$$
$$+\eta \frac{\lambda_k}{J_k^2} \int_\Omega \Delta \Psi_* \cdot \Psi_* \qquad \forall k \in \mathbb{N}, \tag{3.46}$$

where we have defined

$$\widehat{v}^{\lambda_k} \doteq \frac{v^{\lambda_k}}{J_k}, \qquad \widehat{\Phi}^{\lambda_k} \doteq \frac{\Phi^{\lambda_k}}{J_k^2}, \qquad \widehat{p}_\pm^{\lambda_k} \doteq \frac{p_\pm^{\lambda_k}}{J_k^2}, \qquad \widehat{p}_*^{\lambda_k} \doteq \frac{p_*^{\lambda_k}}{J_k^2} \qquad \forall k \in \mathbb{N}.$$

The sequence $(\widehat{v}^{\lambda_k})_{k\in\mathbb{N}}$ is obviously bounded in $\mathcal{V}_*(\Omega)$. The estimates (3.44)–(3.45) show that, respectively, $(\widehat{\Phi}^{\lambda_k})_{k\in\mathbb{N}}$ is bounded in $W^{1,3/2}(\Omega)$ and $(\widehat{p}_\pm^{\lambda_k})_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}$. Therefore, there exist $\widehat{v} \in \mathcal{V}_*(\Omega)$, $\widehat{\Phi} \in W^{1,3/2}(\Omega)$ and $\widehat{p}_\pm \in \mathbb{R}$ such that the following convergences hold as $k \to \infty$:

$$\begin{aligned}
&\widehat{v}^{\lambda_k} \rightharpoonup \widehat{v} \quad \text{weakly in } \mathcal{V}_*(\Omega)\,; \qquad \widehat{v}^{\lambda_k} \to \widehat{v} \quad \text{strongly in } L^p(\Omega) \text{ for every } p \in [1,6)\,;\\
&\widehat{\Phi}^{\lambda_k} \rightharpoonup \widehat{\Phi} \quad \text{weakly in } W^{1,3/2}(\Omega)\,; \qquad \widehat{\Phi}^{\lambda_k} \to \widehat{\Phi} \quad \text{strongly in } L^p(\partial\Omega) \text{ for every } p \in [1,2)\,;\\
&\lambda_k \to \lambda_0 \quad \text{and} \quad \widehat{p}_\pm^{\lambda_k} \to \widehat{p}_\pm \quad \text{in } \mathbb{R}\,,
\end{aligned} \tag{3.47}$$

along sub-sequences that are not being relabeled, see also [61, Theorem 6.2]. By taking the limit as $k \to \infty$ in (3.46) (along the sequences satisfying (3.47)) we obtain

$$\eta = -(\widehat{p}_+ - \widehat{p}_-) F. \tag{3.48}$$

A contradiction will be reached in (3.48) after proving that $\widehat{p}_+ = \widehat{p}_-$. In order to do so, recall from (3.38) that for every $k \in \mathbb{N}$ we have

$$-\eta \Delta v^{\lambda_k} + \lambda_k\, \mathcal{E}(v^{\lambda_k} + \Psi_*)(v^{\lambda_k} + \Psi_*) + \nabla \Phi^{\lambda_k} = \lambda_k\, (f + \eta \Delta \Psi_*) \quad \text{almost everywhere in } \Omega. \tag{3.49}$$

We multiply both sides of identity (3.49) by a vector field $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (not necessarily divergence-free), integrate by parts in Ω and then divide the resulting equality by J_k^2 in order to obtain

$$\frac{\eta}{J_k} \int_\Omega \nabla \widehat{v}^{\lambda_k} \cdot \nabla \varphi + \lambda_k \int_\Omega \mathcal{E}\left(\widehat{v}^{\lambda_k} + \frac{\Psi_*}{J_k} \right) \left(\widehat{v}^{\lambda_k} + \frac{\Psi_*}{J_k} \right) \cdot \varphi - \int_\Omega \widehat{\Phi}^{\lambda_k} (\nabla \cdot \varphi)$$
$$= \frac{\lambda_k}{J_k^2} \int_\Omega (f + \eta \Delta \Psi_*) \cdot \varphi\,, \tag{3.50}$$

for every $k \in \mathbb{N}$. In order to handle the nonlinear term appearing in (3.50), we write

$$\int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k}) \widehat{v}^{\lambda_k} \cdot \varphi = \int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k}) \widehat{v} \cdot \varphi + \int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k})(\widehat{v}^{\lambda_k} - \widehat{v}) \cdot \varphi \qquad \forall k \in \mathbb{N}. \tag{3.51}$$

On one hand, for a fixed $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have that the application

$$\psi \in \mathcal{V}_*(\Omega) \longmapsto \int_\Omega \mathcal{E}(\psi) \widehat{v} \cdot \varphi$$

clearly defines a continuous functional on $\mathcal{V}_*(\Omega)$, so that the weak convergences in $(3.47)_1$–$(3.47)_2$ imply

$$\lim_{k\to\infty}\int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k})\widehat{v}\cdot\varphi = \int_\Omega \mathcal{E}(\widehat{v})\widehat{v}\cdot\varphi \quad \text{and}$$
$$\lim_{k\to\infty}\int_\Omega \widehat{\Phi}^{\lambda_k}(\nabla\cdot\varphi) = \int_\Omega \widehat{\Phi}(\nabla\cdot\varphi) = -\int_\Omega \nabla\widehat{\Phi}\cdot\varphi. \tag{3.52}$$

On the other hand, we notice that

$$\left|\int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k})(\widehat{v}^{\lambda_k}-\widehat{v})\cdot\varphi\right| \le 2\|\widehat{v}^{\lambda_k}-\widehat{v}\|_{L^4(\Omega)}\|\varphi\|_{L^4(\Omega)} \qquad \forall k\in\mathbb{N}\,,$$

so that the strong convergence in $(3.47)_1$ implies

$$\lim_{k\to\infty}\int_\Omega \mathcal{E}(\widehat{v}^{\lambda_k})(\widehat{v}^{\lambda_k}-\widehat{v})\cdot\varphi = 0. \tag{3.53}$$

By taking the limit in (3.50) as $k\to\infty$, and observing (3.47)–(3.52)–(3.53), we get

$$\lambda_0\int_\Omega \mathcal{E}(\widehat{v})\widehat{v}\cdot\varphi + \int_\Omega \nabla\widehat{\Phi}\cdot\varphi = 0 \qquad \forall\varphi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)\,,$$

that is, the pair $(\widehat{v},\widehat{\Phi})\in\mathcal{V}_*(\Omega)\times W^{1,3/2}(\Omega)$ satisfies in strong form the following Euler-type equation:

$$\lambda_0\left[\nabla\widehat{v}-(\nabla\widehat{v})^\top\right]\widehat{v}+\nabla\widehat{\Phi}=0\,,\quad \nabla\cdot\widehat{v}=0 \ \text{ in } \ \Omega. \tag{3.54}$$

Moreover, from $(3.38)_2$ (replacing λ by λ_k and dividing both identities in $(3.38)_2$ by J_k^2, for any $k\in\mathbb{N}$), the strong convergences in $(3.47)_2$–$(3.47)_3$ imply that

$$\widehat{\Phi}=\widehat{p}_- \ \text{ on } \ \Gamma_I \qquad \text{and} \qquad \widehat{\Phi}=\widehat{p}_+ \ \text{ on } \ \Gamma_O. \tag{3.55}$$

We set $\widehat{v}_0 \doteq \sqrt{\lambda_0}\,\widehat{v}$ and $\widehat{\Phi}_0 \doteq \widehat{\Phi}-|\widehat{v}_0|^2/2$, so that the pair $(\widehat{v}_0,\widehat{\Phi}_0)\in\mathcal{V}_*(\Omega)\times W^{1,3/2}(\Omega)$ satisfies in strong form the Euler equation

$$(\widehat{v}_0\cdot\nabla)\widehat{v}_0+\nabla\widehat{\Phi}_0=0\,,\quad \nabla\cdot\widehat{v}_0=0 \ \text{ in } \ \Omega.$$

Since $\widehat{v}_0 = 0$ on Γ_W, the Bernoulli law [37, Lemma 4] (see [6, Theorem 2.2] and [41, Theorem 1] as well) states that $\widehat{\Phi}_0$ must be constant on each of the connected components of Γ_W. More precisely, if we denote the lateral boundary of $\mathcal{M}$ by

$$\mathcal{L}\doteq\left\{\xi\in\mathbb{R}^3 \mid \rho=1\,,\ -h<z<-h\right\}\,,$$

there exist constants $\widehat{p}_{\mathcal{L}},\widehat{p}_K\in\mathbb{R}$ such that $\widehat{\Phi}_0=\widehat{p}_{\mathcal{L}}$ on $\mathcal{L}$ and $\widehat{\Phi}_0=\widehat{p}_K$ on ∂K. Thus,

$$\widehat{\Phi}=\widehat{p}_{\mathcal{L}} \ \text{ on } \ \mathcal{L} \qquad \text{and} \qquad \widehat{\Phi}=\widehat{p}_K \ \text{ on } \ \partial K. \tag{3.56}$$

Since $\partial K\cap\partial\mathcal{M}=\emptyset$, there also exist $\rho_*\in(0,R)$ and $h_1,h_2\in(-h,h)$ with $h_1<h_2$ and such that

$$\overline{K}\subsetneq\left\{\xi\in\mathbb{R}^3 \mid 0\le\rho<\rho_*\,,\ h_1<z<h_2\right\}. \tag{3.57}$$

In view of (3.55) we have $\widehat{\Phi}(\rho,\theta,\pm h)=\widehat{p}_\pm$ for almost every $(\rho,\theta)\in(0,R)\times[0,2\pi]$. Thus, for almost every $(\rho,\theta,z)\in(\rho_*,R)\times[0,2\pi]\times(-h,0)$ we may write

$$\widehat{p}_- - \widehat{\Phi}(\rho,\theta,z)=\widehat{\Phi}(\rho,\theta,-h)-\widehat{\Phi}(\rho,\theta,z)=-\int_{-h}^{z}\frac{\partial\widehat{\Phi}}{\partial z_0}(\rho,\theta,z_0)\,dz_0.$$

Integrating this last equality with respect to $(\theta, z) \in [0, 2\pi] \times (-h, 0)$ gives us

$$\rho \int_{-h}^{0} \int_{0}^{2\pi} |\widehat{p}_- - \widehat{\Phi}(\rho, \theta, z)| \, d\theta \, dz \leq \rho \int_{-h}^{0} \int_{0}^{2\pi} \int_{-h}^{z} |\nabla \widehat{\Phi}(\rho, \theta, z_0)| \, dz_0 \, d\theta \, dz$$
$$\leq \rho h \int_{-h}^{0} \int_{0}^{2\pi} |\nabla \widehat{\Phi}(\rho, \theta, z_0)| \, d\theta \, dz_0. \tag{3.58}$$

Since $\nabla \widehat{\Phi} \in L^1(\Omega)$, given any integer $n > n_0 \doteq [1/(R - \rho_*)]$, the Mean Value Theorem for Lebesgue integrals can be applied to deduce that

$$\left| \left\{ \rho \in \left(R - \frac{1}{n}, R \right) \ \Big| \ \rho \int_{-h}^{0} \int_{0}^{2\pi} |\nabla \widehat{\Phi}(\rho, \theta, z_0)| \, d\theta \, dz_0 \leq n \|\nabla \widehat{\Phi}\|_{L^1(\Omega_n)} \right\} \right| > 0 \,,$$

where $\Omega_n \doteq \left\{ \xi \in \Omega \mid R - \frac{1}{n} < \rho < R \right\}$ for every integer $n > n_0$. Therefore, we can find a sequence of numbers $(\rho_n^-)_{n \geq n_0} \subset (\rho_*, R)$ such that $\rho_n^- \to R$ as $n \to \infty$ and

$$\rho_n^- \int_{-h}^{0} \int_{0}^{2\pi} |\nabla \widehat{\Phi}(\rho_n^-, \theta, z_0)| \, d\theta \, dz_0 \leq n \|\nabla \widehat{\Phi}\|_{L^1(\Omega_n)} \qquad \forall n \geq n_0. \tag{3.59}$$

In view of the Euler-type equation (3.54), Hölder's and Poincaré's inequality we have

$$\|\nabla \widehat{\Phi}\|_{L^1(\Omega_n)} = \lambda_0 \|(\widehat{v} \cdot \nabla)\widehat{v} - (\nabla \widehat{v})^\top \widehat{v}\|_{L^1(\Omega_n)}$$
$$\leq 2\lambda_0 \|\nabla \widehat{v}\|_{L^2(\Omega_n)} \|\widehat{v}\|_{L^2(\Omega_n)} \leq \frac{2\lambda_0}{n} \|\nabla \widehat{v}\|^2_{L^2(\Omega_n)} \,,$$

which, once inserted into (3.59), gives

$$\rho_n^- \int_{-h}^{0} \int_{0}^{2\pi} |\nabla \widehat{\Phi}(\rho_n^-, \theta, z_0)| \, d\theta \, dz_0 \leq 2\lambda_0 \|\nabla \widehat{v}\|^2_{L^2(\Omega_n)} \qquad \forall n \geq n_0.$$

Since $\nabla \widehat{v} \in L^2(\Omega)$ and $|\Omega_n| \to 0$ as $n \to \infty$, the last inequality and (3.58) imply that

$$\lim_{n \to \infty} \int_{-h}^{0} \int_{0}^{2\pi} |\widehat{p}_- - \widehat{\Phi}(\rho_n^-, \theta, z)| \, d\theta \, dz = 0. \tag{3.60}$$

Similarly we can prove the existence of a sequence $(\rho_n^+)_{n \geq n_0} \subset (\rho_*, R)$ such that $\rho_n^+ \to R$ as $n \to \infty$ and

$$\lim_{n \to \infty} \int_{0}^{h} \int_{0}^{2\pi} |\widehat{p}_+ - \widehat{\Phi}(\rho_n^+, \theta, z)| \, d\theta \, dz = 0. \tag{3.61}$$

Also, in view of (3.56)–(3.57), for almost any $(\rho, \theta, z) \in (\rho_*, R) \times [0, 2\pi] \times (-h, h)$ we have

$$\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho, \theta, z) = \widehat{\Phi}(R, \theta, z) - \widehat{\Phi}(\rho, \theta, z) = \int_{\rho}^{R} \frac{\partial \widehat{\Phi}}{\partial \rho_0}(\rho_0, \theta, z) \, d\rho_0 \,,$$

so that

$$\rho \int_{-h}^{h} \int_{0}^{2\pi} |\widehat{\Phi}(\rho, \theta, z) - \widehat{p}_{\mathcal{L}}| \, d\theta \, dz \leq \rho \int_{-h}^{h} \int_{0}^{2\pi} \int_{\rho}^{R} |\nabla \widehat{\Phi}(\rho_0, \theta, z)| \, d\rho_0 \, d\theta \, dz$$
$$\leq \int_{-h}^{h} \int_{0}^{2\pi} \int_{\rho}^{R} \rho_0 |\nabla \widehat{\Phi}(\rho_0, \theta, z)| \, d\rho_0 \, d\theta \, dz \,,$$

and since $\nabla \widehat{\Phi} \in L^1(\Omega)$, the last inequality implies that

$$\lim_{\rho \to R} \int_{-h}^{h} \int_{0}^{2\pi} |\widehat{\Phi}(\rho, \theta, z) - \widehat{p}_{\mathcal{L}}| \, d\theta \, dz = 0. \tag{3.62}$$

Given any integer $n \geq n_0$ and $(\theta, z) \in [0, 2\pi] \times (-h, 0)$ we can therefore write

$$|\widehat{p}_{\mathcal{L}} - \widehat{p}_{-}| \leq |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_n^-, \theta, z)| + |\widehat{\Phi}(\rho_n^-, \theta, z) - \widehat{p}_{-}|.$$

By integrating this last inequality for $(\theta, z) \in [0, 2\pi] \times (-h, 0)$ we obtain

$$|\widehat{p}_{\mathcal{L}} - \widehat{p}_{-}| \leq \frac{1}{2\pi h} \left(\int_{-h}^{0} \int_{0}^{2\pi} |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_n^-, \theta, z)| \, d\theta \, dz + \int_{-h}^{0} \int_{0}^{2\pi} |\widehat{\Phi}(\rho_n^-, \theta, z) - \widehat{p}_{-}| \, d\theta \, dz \right)$$
$$\leq \frac{1}{2\pi h} \left(\int_{-h}^{h} \int_{0}^{2\pi} |\widehat{p}_{\mathcal{L}} - \widehat{\Phi}(\rho_n^-, \theta, z)| \, d\theta \, dz + \int_{-h}^{0} \int_{0}^{2\pi} |\widehat{\Phi}(\rho_n^-, \theta, z) - \widehat{p}_{-}| \, d\theta \, dz \right) ,$$

so that, by taking the limit as $n \to \infty$ in the last inequality and observing (3.60)–(3.62) we deduce that $\widehat{p}_{\mathcal{L}} = \widehat{p}_{-}$. In a similar fashion, as a consequence of (3.61)–(3.62) we obtain $\widehat{p}_{\mathcal{L}} = \widehat{p}_{+}$. Therefore, $\widehat{p}_{-} = \widehat{p}_{+}$ and a contradiction is reached in (3.48), so that the norms $\|\nabla v^\lambda\|_{L^2(\Omega)}$ are uniformly bounded with respect to $\lambda \in [0, 1]$. This concludes the proof. □

Remark 3.3 In order to reach a contradiction in the proof of Theorem 3.3 we proved that $\widehat{p}_{+} = \widehat{p}_{-} = \widehat{p}_{\mathcal{L}}$, but no information is shed about the value of the constant $\widehat{p}_K \in \mathbb{R}$ appearing in (3.56). Nevertheless, as pointed out in [45, Section 2] and according to the counterexample by Amick [6, Example 3.1], generally one cannot claim that $\widehat{p}_K = \widehat{p}_{+} = \widehat{p}_{-} = \widehat{p}_{\mathcal{L}}$. What can be said about the value of $\widehat{p}_K$?

We are now in position to prove the main result of the present article:

Theorem 3.4 *Let Ω be as in (1.1), K having a Lipschitz boundary. For any $F \in \mathbb{R}$ and $f \in L^2(\Omega)$, there exists at least one weak solution $u \in \mathcal{V}(\Omega)$ of problem (3.13). Moreover, if there exists a vector field $\Psi \in \mathcal{V}(\Omega)$ satisfying (3.1) and the inequality*

$$\frac{\mathcal{S}_4}{2} \eta^2 > 2\|\Psi\|_{L^4(\Omega)} \left(\frac{\|\nabla \Psi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_4}} - \|\Psi\|_{L^4(\Omega)} \right)$$
$$+\eta \left(\|\nabla \Psi\|_{L^2(\Omega)} + 2\sqrt{\mathcal{S}_4} \|\Psi\|_{L^4(\Omega)} \right) + \frac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}} \,, \tag{3.63}$$

then the weak solution $u \in \mathcal{V}(\Omega)$ of (3.13) is unique and admits the estimate

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{2\eta\|\nabla\Psi\|_{L^2(\Omega)} + \dfrac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}}}{\eta - \dfrac{2\|\Psi\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_4}}} < \frac{\mathcal{S}_4}{2}\eta + \|\nabla\Psi\|_{L^2(\Omega)} - \sqrt{\mathcal{S}_4}\|\Psi\|_{L^4(\Omega)}\,, \tag{3.64}$$

where $\mathcal{S}_2$, $\mathcal{S}_4 > 0$ are as in (2.1).

Proof Following [60] (see [20, Theorem 5.1] for a simplified presentation and [8, Theorem 5.1] as well) we know that there exists a family of open domains $(K_n)_{n\in\mathbb{N}}$ such that

- $\overline{K} \subset K_n \subset \mathcal{M}$ and $\partial K_n \cap \partial\mathcal{M} = \emptyset$ for every $n \in \mathbb{N}$;
- K_n has a boundary of class $\mathcal{C}^2$;
- $\overline{K_{n+1}} \subset K_n$ for every $n \in \mathbb{N}$ and

$$K = \bigcap_{n\in\mathbb{N}} K_n\,; \qquad \lim_{n\to\infty} |K_n\backslash K| = 0.$$

- For every n, the domains $\overline{K_n}$ and $\overline{K}$ are homeomorphic through a bi-Lipschitz map having a Lipschitz constant bounded independently of n (see [8, Remark 5.3]). In particular, the sequence formed by the Lipschitz constants of the family $(\partial K_n)_{n\in\mathbb{N}}$ is uniformly bounded $(\star)$.

In what follows, $C > 0$ will denote a generic constant that depends only on the Lipschitz character of $\partial\Omega$, but that may change from line to line.

We set $\Omega_n \doteq \mathcal{M}\backslash\overline{K_n}$, so that $\Omega_n \subset \Omega$ for every $n \in \mathbb{N}$. Then, given any $F \in \mathbb{R}$, $f \in L^2(\Omega)$ and $n \in \mathbb{N}$, Theorem 3.3 ensures the existence of at least one weak solution $(u_n, \Phi_n) \in \mathcal{V}(\Omega_n) \times L^2_0(\Omega_n)$ of

$$\begin{cases} -\eta\Delta u_n + \mathcal{E}(u_n)u_n + \nabla\Phi_n = f\,, \quad \nabla\cdot u_n = 0 \ \text{ in } \ \Omega_n\,, \\ u_n = 0 \ \text{ on } \ \Gamma_W^{(n)} \\ u_n \times \nu = 0\,, \quad \Phi_n = p_n^- \ \text{ on } \ \Gamma_I\,, \\ u_n \times \nu = 0\,, \quad \Phi_n = p_n^+ \ \text{ on } \ \Gamma_O\,, \\ \displaystyle\int_{\Sigma_n(s)} u_n \cdot \widehat{k} = F \qquad \forall s \in [-h,h]\,, \end{cases} \tag{3.65}$$

for some (unknown) constants $p_n^\pm \in \mathbb{R}$, where we have defined

$$\Gamma_W^{(n)} \doteq \mathcal{L} \cup \partial K_n \quad \text{and} \quad \Sigma_n(s) \doteq \{\xi \in \Omega_n \mid 0 < \rho < R\,,\ z = s\} \qquad \forall s \in [-h,h]\,,\ n \in \mathbb{N}.$$

Furthermore we have $(u_n, \Phi_n) \in H^2(\Omega_n) \times H^1(\Omega_n)$. For every $n \in \mathbb{N}$ we define the functions

$$\widetilde{u_n} \doteq \begin{cases} u_n & \text{in } \Omega_n, \\ 0 & \text{in } K_n\backslash\overline{K}, \end{cases} \quad \text{and} \quad \widetilde{\Phi_n} \doteq \begin{cases} \Phi_n & \text{in } \Omega_n\,, \\ 0 & \text{in } K_n\backslash\overline{K}\,, \end{cases}$$

so that $\widetilde{\Phi_n} \in L^2_0(\Omega)$, $\widetilde{u_n} \in S_\star(\Omega)$ is divergence-free separately in Ω_n and $K_n\backslash\overline{K}$, where we have introduced

$$S_\star(\Omega) \doteq \left\{v \in H^1(\Omega) \mid v = 0 \ \text{ on } \ \Gamma_W\,; \quad v \times \nu = 0 \ \text{ on } \ \Gamma_I \cup \Gamma_O\right\}\,,$$

which is a closed subspace of $H^1(\Omega)$. Moreover,

$$\|\nabla \widetilde{u_n}\|_{L^2(\Omega)} = \|\nabla u_n\|_{L^2(\Omega_n)} \quad \text{and} \quad \|\widetilde{\Phi_n}\|_{L^2(\Omega)} = \|\Phi_n\|_{L^2(\Omega_n)} \qquad \forall n \in \mathbb{N}. \tag{3.66}$$

Given $n \in \mathbb{N}$, two essential observations are in order. Firstly, since $\Phi_n \in L^2_0(\Omega_n)$, there exists $X_n \in H^1_0(\Omega_n)$ such that

$$\nabla \cdot X_n = \Phi_n \quad \text{in} \quad \Omega_n \qquad \text{and} \qquad \|\nabla X_n\|_{L^2(\Omega_n)} \leq C_B(\Omega_n)\|\Phi_n\|_{L^2(\Omega_n)}. \tag{3.67}$$

From [10] we know that $C_B(\Omega_n)$ (the Bogovskii constant of Ω_n, see Sect. 2.3) depends on the Lipschitz character of $\partial\Omega_n$ (therefore, on the Lipschitz nature of ∂K_n), see also [27, Section III.3]. Therefore, property ($\star$) ensures that $C_B(\Omega_n) \leq C$ for every $n \in \mathbb{N}$. We multiply the first identity in (3.65)$_1$ by X_n and integrate by parts in Ω_n to obtain

$$\eta \int_{\Omega_n} \nabla u_n \cdot \nabla X_n + \int_{\Omega_n} \mathcal{E}(u_n)u_n \cdot X_n - \|\Phi_n\|^2_{L^2(\Omega_n)} = \int_{\Omega_n} f \cdot X_n \qquad \forall n \in \mathbb{N}.$$

Let us denote by $\widetilde{X_n} \in H^1_0(\Omega)$ the zero extension of X_n inside K_n. We apply Hölder's inequality and (2.2)–(3.66)–(3.67) in order to write

$$\begin{aligned}
\|\Phi_n\|^2_{L^2(\Omega_n)} &= \eta \int_{\Omega_n} \nabla u_n \cdot \nabla X_n + \int_{\Omega} \mathcal{E}(\widetilde{u_n})\widetilde{u_n} \cdot \widetilde{X_n} - \int_{\Omega} f \cdot \widetilde{X_n} \\
&\leq \eta \|\nabla u_n\|_{L^2(\Omega_n)} \|\nabla X_n\|_{L^2(\Omega_n)} + \frac{2}{\mathcal{S}_4} \|\nabla \widetilde{u_n}\|^2_{L^2(\Omega)} \|\nabla \widetilde{X_n}\|_{L^2(\Omega)} \\
&\quad + \frac{1}{\sqrt{\mathcal{S}_2}} \|f\|_{L^2(\Omega)} \|\nabla \widetilde{X_n}\|_{L^2(\Omega)} \\
&= \left(\eta \|\nabla u_n\|_{L^2(\Omega_n)} + \frac{2}{\mathcal{S}_4} \|\nabla u_n\|^2_{L^2(\Omega_n)} + \frac{1}{\sqrt{\mathcal{S}_2}} \|f\|_{L^2(\Omega)} \right) \|\nabla X_n\|_{L^2(\Omega_n)} \\
&\leq C \left(\eta \|\nabla u_n\|_{L^2(\Omega_n)} + \frac{2}{\mathcal{S}_4} \|\nabla u_n\|^2_{L^2(\Omega_n)} + \frac{1}{\sqrt{\mathcal{S}_2}} \|f\|_{L^2(\Omega)} \right) \|\Phi_n\|_{L^2(\Omega_n)},
\end{aligned}$$

thereby yielding

$$\|\Phi_n\|_{L^2(\Omega_n)} \leq C \left(\eta \|\nabla u_n\|_{L^2(\Omega_n)} + \frac{2}{\mathcal{S}_4} \|\nabla u_n\|^2_{L^2(\Omega_n)} + \frac{1}{\sqrt{\mathcal{S}_2}} \|f\|_{L^2(\Omega)} \right) \qquad \forall n \in \mathbb{N}. \tag{3.68}$$

Secondly, observe that $f - \mathcal{E}(u_n)u_n \in L^{3/2}(\Omega_n)$, so from (2.2)–(3.66) and Hölder's inequality we estimate

$$\begin{aligned}
\|f - \mathcal{E}(u_n)u_n\|_{L^{3/2}(\Omega_n)} &\leq \|f\|_{L^{3/2}(\Omega)} + \|\mathcal{E}(u_n)\|_{L^2(\Omega_n)} \|u_n\|_{L^6(\Omega_n)} \\
&= \|f\|_{L^{3/2}(\Omega)} + \|\mathcal{E}(\widetilde{u_n})\|_{L^2(\Omega)} \|\widetilde{u_n}\|_{L^6(\Omega)} \\
&\leq \|f\|_{L^{3/2}(\Omega)} + \frac{2}{\sqrt{\mathcal{S}_6}} \|\nabla u_n\|^2_{L^2(\Omega_n)} \qquad \forall n \in \mathbb{N}.
\end{aligned}$$

The pair $(u_n, \Phi_n) \in W^{2,3/2}(\Omega_n) \times W^{1,3/2}(\Omega_n)$ is also a strong solution to the Stokes system (3.65)$_1$ in Ω_n, with a right-hand side given by $f - \mathcal{E}(u_n)u_n$. If we apply the same extension argument of the proof of Theorem 3.2 we can then invoke the usual local regularity results for the Stokes equations (see, for example, [27, Theorem IV.4.1]) to obtain the estimate

$$\begin{aligned}
\|u_n\|_{W^{2,3/2}(\Omega_n)} + \|\Phi_n\|_{W^{1,3/2}(\Omega_n)} &\leq C_n \|f - \mathcal{E}(u_n)u_n\|_{L^{3/2}(\Omega_n)} \\
&\leq C_n \left(\|f\|_{L^{3/2}(\Omega)} + \frac{2}{\sqrt{\mathcal{S}_6}} \|\nabla u_n\|^2_{L^2(\Omega_n)} \right),
\end{aligned}$$

for some constant $C_n > 0$ that depends on the Lipschitz character of Ω_n (therefore, on the Lipschitz nature of ∂K_n), see [12, Teorema, page 311]. Property ($\star$) ensures the uniform boundedness of such family of constants, so that

$$\|u_n\|_{W^{2,3/2}(\Omega_n)} + \|\Phi_n\|_{W^{1,3/2}(\Omega_n)} \leq C\left(\|f\|_{L^{3/2}(\Omega)} + \frac{2}{\sqrt{\mathcal{S}_6}}\|\nabla u_n\|^2_{L^2(\Omega_n)}\right) \qquad \forall n \in \mathbb{N}. \tag{3.69}$$

In the first part of the proof we will show that the norms $\|\nabla u_n\|_{L^2(\Omega_n)}$ are uniformly bounded with respect to $n \in \mathbb{N}$. For this, given $n \in \mathbb{N}$, we multiply the first identity in $(3.65)_1$ by u_n and integrate by parts, each term separately, in the following way:

$$\begin{aligned} -\int_{\Omega_n} \Delta u_n \cdot u_n &= \|\nabla u_n\|^2_{L^2(\Omega_n)} - \int_{\partial\Omega_n} \frac{\partial u_n}{\partial \nu} \cdot u_n \\ &= \|\nabla u_n\|^2_{L^2(\Omega_n)} - \left(\int_{\Gamma_I} \frac{\partial u_n}{\partial \nu} \cdot u_n + \int_{\Gamma_O} \frac{\partial u_n}{\partial \nu} \cdot u_n\right). \end{aligned}$$

Since $u_n \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$, the regularity and incompressibility condition of u_n in Ω_n allow us to prove

$$\int_{\Gamma_I} \frac{\partial u_n}{\partial \nu} \cdot u_n = \int_{\Gamma_O} \frac{\partial u_n}{\partial \nu} \cdot u_n = 0\,,$$

so that

$$-\int_{\Omega_n} \Delta u_n \cdot u_n = \|\nabla u_n\|^2_{L^2(\Omega_n)}. \tag{3.70}$$

Concerning the nonlinear term, we have

$$\int_{\Omega_n} \mathcal{E}(u_n) u_n \cdot u_n = 0. \tag{3.71}$$

Regarding the pressure term, from $(3.65)_3$–$(3.65)_4$–$(3.65)_5$ we infer

$$\int_{\Omega_n} \nabla\Phi_n \cdot u_n = \int_{\partial\Omega_n} \Phi_n (u_n \cdot \nu) = F(p_n^+ - p_n^-). \tag{3.72}$$

By adding the identities (3.70)–(3.71)–(3.72) we obtain

$$\eta\|\nabla u_n\|^2_{L^2(\Omega_n)} + F(p_n^+ - p_n^-) = \int_\Omega f \cdot \widetilde{u_n} \qquad \forall n \in \mathbb{N}. \tag{3.73}$$

Notice that the boundary conditions for the pressure in $(3.65)_3$–$(3.65)_4$ imply that

$$p_n^- = \frac{1}{\pi R^2}\int_{\Gamma_I} \Phi_n \qquad \text{and} \qquad p_n^+ = \frac{1}{\pi R^2}\int_{\Gamma_O} \Phi_n \qquad \forall n \in \mathbb{N}. \tag{3.74}$$

From the trace inequality (recall again property ($\star$)) and (3.69)–(3.74) we then get

$$|p_n^\pm| \leq \frac{C}{\sqrt{\pi} R}\left(\|f\|_{L^{3/2}(\Omega)} + \frac{2}{\sqrt{\mathcal{S}_6}}\|\nabla u_n\|^2_{L^2(\Omega_n)}\right) \qquad \forall n \in \mathbb{N}. \tag{3.75}$$

By contradiction, suppose now that the norms $\|\nabla u_n\|_{L^2(\Omega_n)}$ are not uniformly bounded with respect to $n \in \mathbb{N}$. Then, there must exists a sub-sequence (not being relabeled) such that

$$\lim_{n\to\infty} J_n = +\infty \quad \text{with} \quad J_n \doteq \|\nabla u_n\|_{L^2(\Omega_n)} \quad \forall n \in \mathbb{N}. \tag{3.76}$$

The estimates in (3.68)–(3.75) (see also (3.66)) enable us to establish that, along this divergent sub-sequence, the following sequences are all uniformly bounded with respect to $n \in \mathbb{N}$:

$$(\widehat{u_n})_{n\in\mathbb{N}} \doteq \left(\frac{\widetilde{u_n}}{J_n}\right)_{n\in\mathbb{N}} \subset S_\star(\Omega)\,; \quad (\widehat{\Phi_n})_{n\in\mathbb{N}} \doteq \left(\frac{\widetilde{\Phi_n}}{J_n^2}\right)_{n\in\mathbb{N}} \subset L^2(\Omega)\,;$$
$$(\widehat{p_n}^{\pm})_{n\in\mathbb{N}} \doteq \left(\frac{p_n^{\pm}}{J_n^2}\right)_{n\in\mathbb{N}} \subset \mathbb{R}.$$

There must exist $\widehat{u} \in S_\star(\Omega)$, $\widehat{\Phi} \in L^2(\Omega)$ and $\widehat{p}_\pm \in \mathbb{R}$ such that the following convergences hold:

$$\begin{aligned} &\widehat{u_n} \rightharpoonup \widehat{u} \quad \text{weakly in } S_\star(\Omega)\,; && \widehat{u_n} \to \widehat{u} \quad \text{strongly in } L^p(\Omega) \text{ for every } p \in [1,6)\,; \\ &\widehat{\Phi_n} \rightharpoonup \widehat{\Phi} \quad \text{weakly in } L^2(\Omega)\,; && \widehat{p_n}^{\pm} \to \widehat{p}_\pm \quad \text{in } \mathbb{R}\,, \end{aligned} \tag{3.77}$$

as $n \to \infty$, along sub-sequences that are not being relabeled. Notice that

$$\left|\frac{1}{J_n^2}\int_\Omega f\cdot\widetilde{u_n}\right| \leq \frac{1}{\sqrt{\mathcal{S}_2}\,J_n}\|f\|_{L^2(\Omega)}\|\nabla\widehat{u_n}\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad n\to\infty. \tag{3.78}$$

If we then divide identity (3.73) by J_n^2 and let $n \to \infty$ along the sub-sequences in (3.77), we obtain

$$\eta = -F(\widehat{p}_+ - \widehat{p}_-). \tag{3.79}$$

A contradiction will be reached in (3.79) after proving that $\widehat{p}_+ = \widehat{p}_-$. Firstly, given any scalar function $\phi \in \mathcal{C}_0^\infty(\Omega)$, an integration by parts and the divergence-free condition in $(3.65)_1$ imply that

$$\int_\Omega \widehat{u_n}\cdot\nabla\phi = \frac{1}{J_n}\int_{\Omega_n} u_n\cdot\nabla\phi = \frac{1}{J_n}\int_{\partial\Omega_n}\phi(u_n\cdot\nu) = 0 \qquad \forall n\in\mathbb{N}\,,$$

since u_n vanishes on ∂K_n and so does ϕ on $\partial\mathcal{M}$. Then, along the subsequences (3.77), the weak convergence in $(3.77)_1$ yields

$$\int_\Omega \widehat{u}\cdot\nabla\phi = -\int_\Omega \phi(\nabla\cdot\widehat{u}) = 0 \qquad \forall\phi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R})\,,$$

thus proving that $\nabla\cdot\widehat{u} = 0$ almost everywhere in Ω, that is, $\widehat{u} \in \mathcal{V}(\Omega)$. Secondly, we multiply both sides of the first identity in $(3.65)_1$ by a vector field $\varphi \in \mathcal{C}_0^\infty(\Omega)$ (not necessarily divergence-free), integrate by parts in Ω_n, each term separately, in the following way:

$$-\int_{\Omega_n}\Delta u_n\cdot\varphi = \int_{\Omega_n}\nabla u_n\cdot\nabla\varphi - \int_{\partial\Omega_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi = \int_\Omega \nabla\widetilde{u_n}\cdot\nabla\varphi - \int_{\partial K_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi. \tag{3.80}$$

Concerning the nonlinear term, we simply put

$$\int_{\Omega_n}\mathcal{E}(u_n)u_n\cdot\varphi = \int_\Omega \mathcal{E}(\widetilde{u_n})\widetilde{u_n}\cdot\varphi. \tag{3.81}$$

Regarding the pressure term, from $(3.65)_3$–$(3.65)_4$ we infer

$$\int_{\Omega_n}\nabla\Phi_n\cdot\varphi = -\int_{\Omega_n}\Phi_n(\nabla\cdot\varphi) + \int_{\partial\Omega_n}\Phi_n(\varphi\cdot\nu) = -\int_\Omega \widetilde{\Phi_n}(\nabla\cdot\varphi) + \int_{\partial K_n}\Phi_n(\varphi\cdot\nu). \tag{3.82}$$

By adding the identities (3.80)–(3.81)–(3.82), and then dividing the result by J_n^2, we obtain

$$\begin{aligned}&\frac{\eta}{J_n}\int_\Omega \nabla\widehat{u_n}\cdot\nabla\varphi+\int_\Omega \mathcal{E}(\widehat{u_n})\widehat{u_n}\cdot\varphi-\int_\Omega \widehat{\Phi_n}(\nabla\cdot\varphi)\\&\quad+\frac{1}{J_n^2}\int_{\partial K_n}\left(\Phi_n\nu-\eta\frac{\partial u_n}{\partial\nu}\right)\cdot\varphi=\frac{1}{J_n^2}\int_{\Omega_n} f\cdot\varphi\qquad\forall n\in\mathbb{N},\end{aligned}\tag{3.83}$$

along the sub-sequences (3.77). The convergences in (3.77) (see again (3.51)–(3.52)–(3.53)–(3.78)) imply

$$\begin{aligned}&\lim_{n\to\infty}\frac{\eta}{J_n}\int_\Omega \nabla\widehat{u_n}\cdot\nabla\varphi=0\,,\qquad \lim_{n\to\infty}\int_\Omega \mathcal{E}(\widehat{u_n})\widehat{u_n}\cdot\varphi=\int_\Omega \mathcal{E}(\widehat{u})\widehat{u}\cdot\varphi\,,\\&\lim_{n\to\infty}\int_\Omega \widehat{\Phi_n}(\nabla\cdot\varphi)=\int_\Omega \widehat{\Phi}(\nabla\cdot\varphi)\,,\qquad \lim_{n\to\infty}\frac{1}{J_n^2}\int_{\Omega_n} f\cdot\varphi=0.\end{aligned}\tag{3.84}$$

In view of Hölder's inequality, the trace inequality (recall property ($\star$)) and (3.69), the boundary integral appearing in (3.83) can be treated as follows:

$$\begin{aligned}&\left|\frac{1}{J_n^2}\int_{\partial K_n}\left(\Phi_n\nu-\eta\frac{\partial u_n}{\partial\nu}\right)\cdot\varphi\right|\\&\quad\le\frac{1}{J_n^2}\left\|\Phi_n\nu-\eta\frac{\partial u_n}{\partial\nu}\right\|_{L^2(\partial K_n)}\|\varphi\|_{L^2(\partial K_n)}\\&\quad\le\frac{1}{J_n^2}\left(\|\Phi_n\|_{L^2(\partial K_n)}+\eta\|\nabla u_n\|_{L^2(\partial K_n)}\right)\|\varphi\|_{L^2(\partial(K_n\setminus\overline{K}))}\\&\quad\le\frac{C}{J_n^2}\left(\|\Phi_n\|_{W^{1,3/2}(\Omega_n)}+\eta\|u_n\|_{W^{2,3/2}(\Omega_n)}\right)\|\nabla\varphi\|_{L^2(K_n\setminus\overline{K})}\\&\quad\le C(1+\eta)\left(\frac{1}{J_n^2}\|f\|_{L^{3/2}(\Omega)}+\frac{2}{\sqrt{\mathcal{S}_6}}\|\nabla\widehat{u_n}\|^2_{L^2(\Omega)}\right)\|\nabla\varphi\|_{L^2(K_n\setminus\overline{K})}\to 0\quad\text{as } n\to\infty,\end{aligned}\tag{3.85}$$

because $\varphi\in H^1(\Omega)$, $(\widehat{u_n})_{n\in\mathbb{N}}\subset S_\star(\Omega)$ is uniformly bounded and $|K_n\setminus K|\to 0$ as $n\to\infty$. By taking the limit in (3.83) as $n\to\infty$, and observing (3.84)–(3.85), we get

$$\int_\Omega \mathcal{E}(\widehat{u})\widehat{u}\cdot\varphi-\int_\Omega \widehat{\Phi}(\nabla\cdot\varphi)=0\qquad\forall\varphi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}^3)\,,$$

that is, the pair $(\widehat{u},\widehat{\Phi})\in\mathcal{V}(\Omega)\times L^2(\Omega)$ satisfies in distributional form the following Euler-type equation:

$$\left[\nabla\widehat{u}-(\nabla\widehat{u})^\top\right]\widehat{u}+\nabla\widehat{\Phi}=0\,,\quad \nabla\cdot\widehat{u}=0\quad\text{in}\quad\Omega.\tag{3.86}$$

Since $\mathcal{E}(\widehat{u})\widehat{u}\in L^{3/2}(\Omega)$, (3.86) proves that actually $\widehat{\Phi}\in W^{1,3/2}(\Omega)$. Now, from the properties of the approximating family $(K_n)_{n\in\mathbb{N}}$ we deduce the existence of $\ell\in(0,h)$ such that

$$\overline{K_n}\subsetneq\{\xi\in\mathcal{M}\mid -\ell<z<\ell\}\qquad\forall n\in\mathbb{N}.$$

We introduce the following sub-domains of $\mathcal{M}$ (see Fig. 7 below):

$$\Omega_I\doteq\{\xi\in\mathcal{M}\mid -h<z<-\ell\}\qquad\text{and}\qquad\Omega_O\doteq\{\xi\in\mathcal{M}\mid \ell<z<h\}\,.$$

Along the weakly convergent subsequence $(3.77)_2$, and in the light of (3.69)–(3.76), we infer that the sequences $(\widehat{\Phi_n})_{n\in\mathbb{N}}\subset W^{1,3/2}(\Omega_I)$ and $(\widehat{\Phi_n})_{n\in\mathbb{N}}\subset W^{1,3/2}(\Omega_O)$ are uniformly

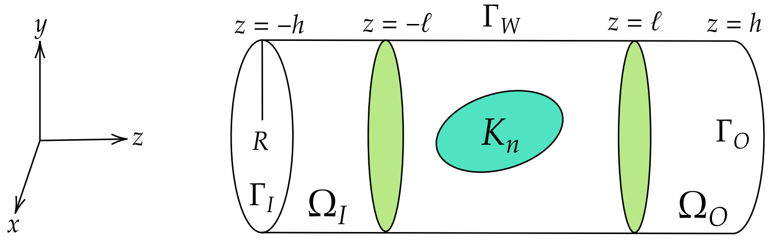

Fig. 7 Representation of the domains Ω_I and Ω_O

bounded. Therefore, there exist $\widehat{\Phi_I} \in W^{1,3/2}(\Omega_I)$ and $\widehat{\Phi_O} \in W^{1,3/2}(\Omega_O)$ such that the following convergences holds as $n \to \infty$:

$$\begin{aligned} &\widehat{\Phi_n} \rightharpoonup \widehat{\Phi_I} \quad \text{weakly in } W^{1,3/2}(\Omega_I)\,; && \widehat{\Phi_n} \rightharpoonup \widehat{\Phi_O} \quad \text{weakly in } W^{1,3/2}(\Omega_O)\,; \\ &\widehat{\Phi_n} \to \widehat{\Phi_I} \quad \text{strongly in } L^2(\Omega_I)\,; && \widehat{\Phi_n} \to \widehat{\Phi_O} \quad \text{strongly in } L^2(\Omega_O)\,; \\ &\widehat{\Phi_n} \to \widehat{\Phi_I} \quad \text{strongly in } L^1(\partial\Omega_I)\,; && \widehat{\Phi_n} \to \widehat{\Phi_O} \quad \text{strongly in } L^1(\partial\Omega_O)\,, \end{aligned} \tag{3.87}$$

along sequences that are not being relabeled. In view of $(3.65)_3$–$(3.65)_4$, the strong convergences in $(3.87)_3$ imply that $\widehat{\Phi_I} = \widehat{p}_-$ on Γ_I and $\widehat{\Phi_O} = \widehat{p}_+$ on Γ_O. But since we also have that $\widehat{\Phi_n} \rightharpoonup \Phi_I$ weakly in $L^2(\Omega_I)$ and $\widehat{\Phi_n} \rightharpoonup \Phi_O$ weakly in $L^2(\Omega_O)$ as $n \to \infty$, by uniqueness of the weak limit there must hold $\widehat{\Phi} = \widehat{\Phi_I}$ on Ω_I and $\widehat{\Phi} = \widehat{\Phi_O}$ on Ω_O. Therefore, since $\widehat{\Phi} \in W^{1,3/2}(\Omega)$,

$$\widehat{\Phi} = \widehat{p}_- \quad \text{on} \quad \Gamma_I\,; \qquad \widehat{\Phi} = \widehat{p}_+ \quad \text{on} \quad \Gamma_O. \tag{3.88}$$

We set $\widehat{\Phi}_* \doteq \widehat{\Phi} - |\widehat{u}|^2/2$, so that the pair $(\widehat{u}, \widehat{\Phi}_*) \in \mathcal{V}(\Omega) \times W^{1,3/2}(\Omega)$ satisfies the Euler equation

$$(\widehat{u} \cdot \nabla)\widehat{u} + \nabla\widehat{\Phi}_* = 0\,, \quad \nabla \cdot \widehat{u} = 0 \quad \text{in} \quad \Omega.$$

Since $\widehat{u} = 0$ on $\mathcal{L}$, the Bernoulli law [37, Lemma 4] can be again applied to deduce the existence of a constant $\widehat{p}_{\mathcal{L}} \in \mathbb{R}$ such that $\widehat{\Phi}_* = \widehat{p}_{\mathcal{L}}$ on $\mathcal{L}$. Thus, $\widehat{\Phi} = \widehat{p}_{\mathcal{L}}$ on $\mathcal{L}$. Exactly as in the proof of Theorem 3.3 one can show that $\widehat{p}_{\mathcal{L}} = \widehat{p}_+ = \widehat{p}_-$, so we will not repeat the argument here. This yields a contradiction in (3.79) and, therefore, enables us to state the existence of a constant $M > 0$ such that

$$\|\nabla u_n\|_{L^2(\Omega_n)} \le M \qquad \forall n \in \mathbb{N}. \tag{3.89}$$

In the second part of the proof we construct a weak solution to (3.13). From (3.66)–(3.68)–(3.89) we deduce that the sequences $(\widetilde{u_n})_{n\in\mathbb{N}} \subset S_\star(\Omega)$ and $(\widetilde{\Phi_n})_{n\in\mathbb{N}} \subset L^2_0(\Omega)$ are uniformly bounded. Thus, there exist $u \in S_\star(\Omega)$ and $\Phi \in L^2_0(\Omega)$ such that the following convergences hold as $n \to \infty$:

$$\begin{aligned} &\widetilde{u_n} \rightharpoonup u \quad \text{weakly in } S_\star(\Omega)\,; && \widetilde{u_n} \to u \quad \text{strongly in } L^p(\Omega) \text{ for every } p \in [1,6)\,; \\ &\widetilde{u_n} \to u \quad \text{strongly in } L^p(\partial\Omega) \text{ for every } p \in [1,4)\,; && \widetilde{\Phi_n} \rightharpoonup \Phi \quad \text{weakly in } L^2(\Omega)\,, \end{aligned} \tag{3.90}$$

along sub-sequences that are not being relabeled. Now, given any scalar function $\phi \in \mathcal{C}_0^\infty(\Omega)$, an integration by parts and the divergence-free condition in $(3.65)_1$ imply that

$$\int_\Omega \widetilde{u_n} \cdot \nabla\phi = \int_{\Omega_n} u_n \cdot \nabla\phi = \int_{\partial\Omega_n} \phi(u_n \cdot \nu) = 0 \qquad \forall n \in \mathbb{N}\,,$$

since u_n vanishes on ∂K_n and so does ϕ on $\partial\mathcal{M}$. Then, along the subsequence (3.90), the weak convergence in $(3.90)_1$ yields

$$\int_\Omega u\cdot\nabla\phi = 0 \qquad \forall \phi\in\mathcal{C}_0^\infty(\Omega;\mathbb{R}),$$

thus proving that $\nabla\cdot u = 0$ almost everywhere in Ω, that is, $u\in\mathcal{V}(\Omega)$. From $(3.65)_5$ we also deduce

$$\int_{\Gamma_I}\widetilde{u_n}\cdot\widehat{k} = F \qquad \forall n\in\mathbb{N},$$

so that the strong convergence in $(3.90)_2$ gives

$$\left|\int_{\Gamma_I} u\cdot\widehat{k} - F\right| = \left|\int_{\Gamma_I}(u-\widetilde{u_n})\cdot\widehat{k}\right| \le \|\widetilde{u_n}-u\|_{L^1(\Gamma_I)} \le \|\widetilde{u_n}-u\|_{L^1(\partial\Omega)} \to 0 \quad \text{as } n\to\infty.$$

Since $u\in\mathcal{V}(\Omega)$, the previous computation combined with the Divergence Theorem allow us to conclude

$$\int_{\Sigma(s)} u\cdot\widehat{k} = F \qquad \forall s\in[-h,h]. \tag{3.91}$$

Now, given any $\varphi\in\mathcal{V}_*(\Omega)$ and $n\in\mathbb{N}$, we multiply the first identity in $(3.65)_1$ by φ and integrate by parts, each term separately, in the following way:

$$\begin{aligned}-\int_{\Omega_n}\Delta u_n\cdot\varphi &= \int_{\Omega_n}\nabla u_n\cdot\nabla\varphi - \int_{\partial\Omega_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi \\ &= \int_{\Omega_n}\nabla u_n\cdot\nabla\varphi - \int_{\partial K_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi - \left(\int_{\Gamma_I}\frac{\partial u_n}{\partial\nu}\cdot\varphi + \int_{\Gamma_O}\frac{\partial u_n}{\partial\nu}\cdot\varphi\right).\end{aligned}$$

Since $\varphi\times\nu = 0$ on $\Gamma_I\cup\Gamma_O$, the regularity and incompressibility condition of u_n in Ω_n allow us to prove

$$\int_{\Gamma_I}\frac{\partial u_n}{\partial\nu}\cdot\varphi = \int_{\Gamma_O}\frac{\partial u_n}{\partial\nu}\cdot\varphi = 0,$$

so that

$$-\int_{\Omega_n}\Delta u_n\cdot\varphi = \int_{\Omega_n}\nabla u_n\cdot\nabla\varphi - \int_{\partial K_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi = \int_\Omega\nabla\widetilde{u_n}\cdot\nabla\varphi - \int_{\partial K_n}\frac{\partial u_n}{\partial\nu}\cdot\varphi. \tag{3.92}$$

Concerning the nonlinear term, we simply put

$$\int_{\Omega_n}\mathcal{E}(u_n)u_n\cdot\varphi = \int_\Omega\mathcal{E}(\widetilde{u_n})\widetilde{u_n}\cdot\varphi. \tag{3.93}$$

Regarding the pressure term, from $(3.65)_3$–$(3.65)_4$ we infer

$$\int_{\Omega_n}\nabla\Phi_n\cdot\varphi = \int_{\partial\Omega_n}\Phi_n(\varphi\cdot\nu) = \int_{\partial K_n}\Phi_n(\varphi\cdot\nu). \tag{3.94}$$

By adding the identities (3.92)–(3.93)–(3.94) we obtain

$$\eta\int_\Omega\nabla\widetilde{u_n}\cdot\nabla\varphi + \int_\Omega\mathcal{E}(\widetilde{u_n})\widetilde{u_n}\cdot\varphi + \int_{\partial K_n}\left(\Phi_n\nu - \eta\frac{\partial u_n}{\partial\nu}\right)\cdot\varphi = \int_{\Omega_n} f\cdot\varphi \qquad \forall n\in\mathbb{N}, \tag{3.95}$$

along the sub-sequences given in (3.90). With the help of both convergences in $(3.90)_1$ (see again (3.51)–(3.52)–(3.53)–(3.78)) we can easily prove that

$$\lim_{n\to\infty}\int_\Omega \nabla\widetilde{u_n}\cdot\nabla\varphi = \int_\Omega \nabla u\cdot\nabla\varphi \qquad \text{and} \qquad \lim_{n\to\infty}\int_\Omega \mathcal{E}(\widetilde{u_n})\widetilde{u_n}\cdot\varphi = \int_\Omega \mathcal{E}(u)u\cdot\varphi. \tag{3.96}$$

Also, notice that

$$\left|\int_{\Omega_n} f\cdot\varphi - \int_\Omega f\cdot\varphi\right| = \left|\int_{\Omega\setminus\Omega_n} f\cdot\varphi\right| = \left|\int_{K_n\setminus K} f\cdot\varphi\right| \to 0 \quad \text{as } n\to\infty\,, \tag{3.97}$$

because $f\cdot\varphi\in L^1(\Omega)$ and $|K_n\setminus K|\to 0$ as $n\to\infty$. In order to handle the boundary term appearing in (3.95), we employ Hölder's inequality, the trace inequality (recall property ($\star$)) and (3.69)–(3.89):

$$\begin{aligned}
&\left|\int_{\partial K_n}\left(\Phi_n\nu - \eta\frac{\partial u_n}{\partial\nu}\right)\cdot\varphi\right| \\
&\leq \left\|\Phi_n\nu - \eta\frac{\partial u_n}{\partial\nu}\right\|_{L^2(\partial K_n)} \|\varphi\|_{L^2(\partial K_n)} \\
&\leq \left(\|\Phi_n\|_{L^2(\partial K_n)} + \eta\|\nabla u_n\|_{L^2(\partial K_n)}\right)\|\varphi\|_{L^2(\partial(K_n\setminus\overline{K}))} \\
&\leq C\left(\|\Phi_n\|_{W^{1,3/2}(\Omega_n)} + \eta\|u_n\|_{W^{2,3/2}(\Omega_n)}\right)\|\nabla\varphi\|_{L^2(K_n\setminus\overline{K})} \\
&\leq C(1+\eta)\left(\|f\|_{L^{3/2}(\Omega)} + \frac{2M^2}{\sqrt{\mathcal{S}_6}}\right)\|\nabla\varphi\|_{L^2(K_n\setminus\overline{K})} \to 0 \quad \text{as } n\to\infty\,,
\end{aligned} \tag{3.98}$$

because $\varphi\in H^1(\Omega)$ and $|K_n\setminus K|\to 0$ as $n\to\infty$. By taking the limit as $n\to\infty$ in (3.95), observing (3.96)–(3.97)–(3.98), we finally conclude that

$$\eta\int_\Omega \nabla u\cdot\nabla\varphi + \int_\Omega \mathcal{E}(u)u\cdot\varphi = \int_\Omega f\cdot\varphi \qquad \forall\varphi\in\mathcal{V}_*(\Omega)\,,$$

that is, $u\in\mathcal{V}(\Omega)$ is a weak solution of problem (3.13) in Ω (recall also (3.91)).

For the final part of the proof, suppose there exists $\Psi\in\mathcal{V}(\Omega)$ satisfying (3.1)–(3.63). As a consequence of (2.2) we readily see that

$$\frac{\|\nabla\Psi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_4}} \geq \|\Psi\|_{L^4(\Omega)}\,,$$

so that we also have

$$\|\Psi\|_{L^4(\Omega)} < \frac{\sqrt{\mathcal{S}_4}}{2}\eta.$$

We then claim that every weak solution $u\in\mathcal{V}(\Omega)$ of (3.13) admits the estimate

$$\|\nabla u - \nabla\Psi\|_{L^2(\Omega)} \leq \frac{\eta\|\nabla\Psi\|_{L^2(\Omega)} + \dfrac{2}{\sqrt{\mathcal{S}_4}}\|\nabla\Psi\|_{L^2(\Omega)}\|\Psi\|_{L^4(\Omega)} + \dfrac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}}}{\eta - \dfrac{2\|\Psi\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_4}}}. \tag{3.99}$$

Since $u-\Psi\in\mathcal{V}_*(\Omega)$, we represent any such solution as $u=v+\Psi$, for some $v\in\mathcal{V}_*(\Omega)$. Upon substitution of this representation into the weak formulation (3.19) and then by testing with $\varphi = v$, we get

$$\eta\|\nabla v\|^2_{L^2(\Omega)} = -\eta\int_\Omega \nabla v\cdot\nabla\Psi - \int_\Omega \mathcal{E}(\Psi)\Psi\cdot v + \int_\Omega f\cdot v - \int_\Omega \mathcal{E}(v)\Psi\cdot v.$$

After applying Hölder's inequality and the Sobolev inequalities in (2.2) we deduce

$$\eta\|\nabla v\|^2_{L^2(\Omega)} \le \left(\eta\|\nabla\Psi\|_{L^2(\Omega)} + \frac{2\|\nabla\Psi\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_4}}\|\Psi\|_{L^4(\Omega)} + \frac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}}\right)\|\nabla v\|_{L^2(\Omega)} + \frac{2\|\Psi\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_4}}\|\nabla v\|^2_{L^2(\Omega)},$$

from where the estimate (3.99) follows. Now, assume that there exist two weak solutions $u_1, u_2 \in \mathcal{V}(\Omega)$ of (3.13), that are represented as $u_1 = v_1 + \Psi$ and $u_2 = v_2 + \Psi$ for some $v_1, v_2 \in \mathcal{V}_*(\Omega)$. We set $w \doteq v_2 - v_1 = u_2 - u_1 \in \mathcal{V}_*(\Omega)$ so that, by taking the difference between the weak formulations (3.19) satisfied by u_1 and u_2 we deduce

$$\begin{aligned}\eta\int_\Omega \nabla w\cdot\nabla\varphi &= \int_\Omega [(\varphi\cdot\nabla)v_2\cdot w + (\varphi\cdot\nabla)w\cdot v_1 - (w\cdot\nabla)v_2\cdot\varphi - (v_1\cdot\nabla)w\cdot\varphi]\\ &\quad + \int_\Omega [(\varphi\cdot\nabla)\Psi\cdot w + (\varphi\cdot\nabla)w\cdot\Psi - (\Psi\cdot\nabla)w\cdot\varphi - (w\cdot\nabla)\Psi\cdot\varphi]\\ &\quad \forall\varphi\in\mathcal{V}_*(\Omega).\end{aligned}$$

We put $\varphi = w$ in this last identity, apply Hölder's inequality and the estimates in (2.2)–(3.99) to get

$$\begin{aligned}\eta\|\nabla w\|^2_{L^2(\Omega)} &= \int_\Omega [(w\cdot\nabla)w\cdot v_1 - (v_1\cdot\nabla)w\cdot w - (\Psi\cdot\nabla)w\cdot w + (w\cdot\nabla)w\cdot\Psi]\\ &\le 2\left(\|v_1\|_{L^4(\Omega)} + \|\Psi\|_{L^4(\Omega)}\right)\|\nabla w\|_{L^2(\Omega)}\|w\|_{L^4(\Omega)}\\ &\le \frac{2}{\sqrt{\mathcal{S}_4}}\left(\frac{1}{\sqrt{\mathcal{S}_4}}\|\nabla v_1\|_{L^2(\Omega)} + \|\Psi\|_{L^4(\Omega)}\right)\|\nabla w\|^2_{L^2(\Omega)}\\ &\le \frac{2}{\sqrt{\mathcal{S}_4}}\left(\frac{\eta\|\nabla\Psi\|_{L^2(\Omega)} + \frac{2}{\sqrt{\mathcal{S}_4}}\|\nabla\Psi\|_{L^2(\Omega)}\|\Psi\|_{L^4(\Omega)} + \frac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}}}{\eta\sqrt{\mathcal{S}_4} - 2\|\Psi\|_{L^4(\Omega)}} + \|\Psi\|_{L^4(\Omega)}\right)\\ &\quad\times\|\nabla w\|^2_{L^2(\Omega)},\end{aligned}$$

which proves that $w = 0$ (and, therefore, unique weak solvability for (3.13)) provided that (3.63) is observed. Finally, the estimate in (3.64) follows directly from (3.63)–(3.99), noticing that

$$\begin{aligned}\|\nabla u\|_{L^2(\Omega)} &\le \|\nabla u - \nabla\Psi\|_{L^2(\Omega)} + \|\nabla\Psi\|_{L^2(\Omega)} \le \frac{2\eta\|\nabla\Psi\|_{L^2(\Omega)} + \frac{\|f\|_{L^2(\Omega)}}{\sqrt{\mathcal{S}_2}}}{\eta - \frac{2\|\Psi\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_4}}}\\ &< \frac{\eta\|\nabla\Psi\|_{L^2(\Omega)} + \frac{\mathcal{S}_4}{2}\eta^2 - \frac{2}{\sqrt{\mathcal{S}_4}}\|\nabla\Psi\|_{L^2(\Omega)}\|\Psi\|_{L^4(\Omega)} + 2\|\Psi\|^2_{L^4(\Omega)} - 2\eta\sqrt{\mathcal{S}_4}\|\Psi\|_{L^4(\Omega)}}{\eta - \frac{2\|\Psi\|_{L^4(\Omega)}}{\sqrt{\mathcal{S}_4}}}\\ &= \frac{\mathcal{S}_4}{2}\eta + \|\nabla\Psi\|_{L^2(\Omega)} - \sqrt{\mathcal{S}_4}\|\Psi\|_{L^4(\Omega)}.\end{aligned}$$

□

Remark 3.4 Theorems 3.1, 3.2, 3.3 and 3.4 remain valid if, instead of a circular tube, we consider a container of arbitrary cross-section, that is, if we set

$$\mathcal{M} = \left\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in \Theta \,, \; -h < z < h\right\} ,$$

with $\Theta \subset \mathbb{R}^2$ being any open bounded domain having a smooth boundary. In this case, the corresponding Hagen–Poiseuille flow (3.3) is defined as (we use Cartesian coordinates):

$$U_0(x, y, z) = \frac{F}{\ell_0} v_0(x, y)\widehat{k} \qquad \forall (x, y, z) \in \overline{\mathcal{M}} \,,$$

where $v_0 \in H^1_0(\Theta; \mathbb{R})$ is a weak solution of the following torsion problem:

$$-\Delta v_0 = 1 \quad \text{in} \quad \Theta \,, \qquad v_0 = 0 \quad \text{on} \quad \partial\Theta \,,$$

and

$$\ell_0 \doteq \int_\Theta v_0 = \int_\Theta |\nabla v_0|^2 \neq 0.$$

Remark 3.5 Let Ω be as in (1.1), K having a Lipschitz boundary. Given $F \in \mathbb{R}$ and the approximation scheme $(K_n)_{n\in\mathbb{N}}$ described at the beginning of the proof of Theorem 3.4, from Theorem 3.1 we deduce the existence of a vector field $\Psi_n \in H^2(\Omega_n)$ satisfying

$$\begin{cases} \nabla \cdot \Psi_n = 0 \;\text{ in }\; \Omega_n \,; & \Psi_n \times \nu = 0 \;\text{ on }\; \Gamma_I \cup \Gamma_O \,; \\ \Psi_n = 0 \;\text{ on }\; \Gamma_W^{(n)} \,; & \displaystyle\int_{\Sigma_n(s)} \Psi_n \cdot \widehat{k} = F \quad \forall s \in [-h, h] \,, \end{cases} \tag{3.100}$$

together with the estimate

$$\|\nabla \Psi_n\|_{L^2(\Omega_n)} \leq \frac{2|F|}{\pi R^2} \left(1 + 3C_B(\Omega_n)\right) \left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K_n)}\right) \qquad \forall n \in \mathbb{N}. \tag{3.101}$$

From [10] we know that $C_B(\Omega_n)$ depends on the Lipschitz character of $\partial\Omega_n$ (therefore, on the Lipschitz nature of ∂K_n), see also [27, Section III.3]. Therefore, property ($\star$) ensures that $C_B(\Omega_n) \leq C$ for every $n \in \mathbb{N}$, where $C > 0$ is a constant determined by the Lipschitz character of $\partial\Omega$. On the other hand, since $\overline{K_n} \subseteq K_0$ for every $n \in \mathbb{N}$, we clearly have

$$\mathrm{Cap}_{\mathcal{M}}(K_n) \leq \mathrm{Cap}_{\mathcal{M}}(K_0) \qquad \forall n \in \mathbb{N}.$$

Therefore, the vector field

$$\widetilde{\Psi_n} \doteq \begin{cases} \Psi_n & \text{in } \; \Omega_n \,, \\ 0 & \text{in } \; K_n \setminus \overline{K} \,, \end{cases}$$

is an element of $S_\star(\Omega)$ such that

$$\|\nabla \widetilde{\Psi_n}\|_{L^2(\Omega)} \leq \frac{2|F|}{\pi R^2} (1 + 3C) \left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K_0)}\right) \qquad \forall n \in \mathbb{N}.$$

As a consequence, there exists $\Psi \in S_\star(\Omega)$ for which the following convergences hold as $n \to \infty$:

$$\begin{aligned} &\widetilde{\Psi_n} \rightharpoonup \Psi \quad \text{weakly in } S_\star(\Omega) \,; \qquad \widetilde{\Psi_n} \to \Psi \quad \text{strongly in } L^p(\Omega) \text{ for every } p \in [1, 6) \,; \\ &\widetilde{\Psi_n} \to \Psi \quad \text{strongly in } L^p(\partial\Omega) \text{ for every } p \in [1, 4). \end{aligned} \tag{3.102}$$

along a (not relabeled) sub-sequence. As in the proof of Theorem 3.4 we can show that $\Psi \in \mathcal{V}(\Omega)$ and

$$\int_{\Sigma(s)} \Psi \cdot \widehat{k} = F \qquad \forall s \in [-h, h],$$

so that Ψ is a flux carrier of F in the sense of Corollary 3.1.

In the absence of an external force, and combined with Theorems 2.2-2.4-2.5, Theorem 3.4 can be formulated in a quite explicit way. For this we define

$$\mathcal{S}_* \doteq \frac{\pi}{3} \frac{h^2 \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^2 - 2}{h^2 \max\left\{\pi \sqrt[3]{\dfrac{2\pi}{3(|\mathcal{M}|-|K|)}}, \dfrac{\sqrt{16h^2\mu_0^2+\pi^2R^2}}{4Rh}\right\}^{3/2}},$$

which corresponds to the lower bound for $\mathcal{S}_4$ given in Theorem 2.2, and prove the following:

Corollary 3.3 *Let Ω be as in (1.1), K having a Lipschitz boundary, and suppose $f = 0$. For any $F \in \mathbb{R}$, there exists at least one weak solution $u \in \mathcal{V}(\Omega)$ of problem (3.13). Moreover, if*

$$|F| < \frac{\sqrt{13}-3}{64} \frac{\sqrt{\pi}\, R^2 \mathcal{S}_*}{\sqrt{h} + \dfrac{\sqrt[6]{3}}{\sqrt{\sqrt[3]{\dfrac{4\pi}{|K|}} - \sqrt[3]{\dfrac{2}{R^2 h}}}}} \eta, \tag{3.103}$$

there exists a unique weak solution $u \in \mathcal{V}(\Omega)$ of (3.13) which, moreover, admits the estimate

$$\|\nabla u\|_{L^2(\Omega)} \leq \frac{8}{5-\sqrt{13}} \frac{1}{\pi R^2} (1 + 3C_B(\Omega)) \left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K)}\right) |F|. \tag{3.104}$$

Proof We can select $\Psi = \Psi_*$ in the proof of Theorem 3.4, $\Psi_* \in H^1(\Omega)$ being the flux carrier built in Corollary 3.1. In view of (2.2), the condition for unique solvability (3.63) is certainly satisfied if

$$\frac{\mathcal{S}_4}{2}\eta^2 > \frac{2}{\mathcal{S}_4} \|\nabla \Psi_*\|^2_{L^2(\Omega)} + 3\eta \|\nabla \Psi_*\|_{L^2(\Omega)},$$

which is equivalent to

$$\|\nabla \Psi_*\|_{L^2(\Omega)} < \frac{\sqrt{13}-3}{4} \mathcal{S}_4\, \eta. \tag{3.105}$$

In turn, from (3.2) we observe that (3.105) will be satisfied if

$$\frac{2|F|}{\pi R^2} (1 + 3C_B(\Omega)) \left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K)}\right) < \frac{\sqrt{13}-3}{4} \mathcal{S}_4\, \eta. \tag{3.106}$$

By inserting into (3.106) the lower bounds given in (2.49) and Theorem 2.5 (as well as $\mathcal{S}_*$) we obtain (3.103). Since (2.2)–(3.105) also imply that

$$\|\Psi_*\|_{L^4(\Omega)} < \frac{\sqrt{13}-3}{4} \sqrt{\mathcal{S}_4}\, \eta, \tag{3.107}$$

the estimate (3.104) follows directly from (3.2)–(3.64)–(3.107). □

Remark 3.6 Theorem 3.4 gives an upper bound on the "size" of the transversal flow rate F and external force f that guarantees unique solvability for (3.13). Corollary 3.3 expresses this upper bound in terms of the viscosity, the volume of K, the radius and diameter of $\mathcal{M}$, but is **independent** of the shape and position of K inside $\mathcal{M}$. Both results should be compared with [46, Theorem 5.1].

Remark 3.7 Given $F \in \mathbb{R}$ and $f \in L^2(\Omega)$ such that (3.63) holds, from Theorem 3.4 we know that there exists a unique weak solution $u \in \mathcal{V}(\Omega)$ of (3.13) which, moreover, admits the estimate (3.64). Then, Corollary 3.2 ensures the existence of a unique $\Phi \in L^2_0(\Omega)$ such that the pair (u, Φ) solves $(3.13)_1$ in distributional sense in Ω. Exactly as in (3.68) we recover the estimate

$$\|\Phi\|_{L^2(\Omega)} \leq C_B(\Omega)\left(\eta\|\nabla u\|_{L^2(\Omega)} + \frac{2}{\mathcal{S}_4}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{\sqrt{\mathcal{S}_2}}\|f\|_{L^2(\Omega)}\right). \tag{3.108}$$

Assuming that K is the unit ball of $\mathbb{R}^3$, an explicit upper bound for $C_B(\Omega)$ is given in Remark 2.6. This, once inserted into (3.108) and combined with (2.6)–(2.21)–(2.52)–(3.64), would provide an explicit upper bound on the $L^2(\Omega)$-norm of the Bernoulli pressure, in terms of η, the length and radius of $\mathcal{M}$.

Remark 3.8 Suppose that $f = 0$ and that K has a $\mathcal{C}^2$-boundary. Given $F \in \mathbb{R}\backslash\{0\}$, from Theorems 3.2-3.3-3.4 we deduce the existence of a unique pair $(u, \Phi) \in (H^2(\Omega) \cap \mathcal{V}_*(\Omega)) \times (H^1(\Omega) \cap L^2_0(\Omega))$ that satisfies the system (3.13) point-wise almost everywhere in $\overline{\Omega}$, for some unknown constants $p_\pm \in \mathbb{R}$. After multiplying the first identity in $(3.13)_1$ by u and integrating by parts in Ω we obtain

$$\eta\|\nabla u\|^2_{L^2(\Omega)} + F(p_+ - p_-) = 0\,, \tag{3.109}$$

so that the (unknown) pressure drop $p_- - p_+$ has the same sign as F (similarly to what is expressed in the Hagen–Poiseuille law, see [53, Chapter II]). From $(3.13)_5$ and the trace inequality we easily obtain

$$|F| \leq \int_{\Gamma_I} |u| \leq \sqrt{\pi}R\,\|u\|_{L^2(\Gamma_I)} \leq C\|\nabla u\|_{L^2(\Omega)}\,,$$

for some constant $C > 0$ that depends only on Ω (and that may change from line to line). By inserting the above inequality into (3.109) we get

$$\eta\|\nabla u\|_{L^2(\Omega)} \leq C|p_+ - p_-|.$$

If we further suppose that (3.103) holds, upon substitution of (3.104) into (3.109) we may also estimate the (unknown) pressure drop in terms of the transversal flow rate as follows:

$$|p_- - p_+| \leq \frac{64\eta}{(5-\sqrt{13})^2}\frac{1}{\pi^2 R^4}\,(1+3C_B(\Omega))^2\left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K)}\right)^2|F|.$$

The Author warmly expresses his gratitude to Professor Giovanni Paolo Galdi (Pittsburgh) for his fruitful insight and useful comments that led to the inclusion of Remark 3.8 in the present article.

Remark 3.9 Suppose that $f = 0$, K is the unit ball of $\mathbb{R}^3$ and that (3.103) holds. As a consequence of Proposition 2.2, Theorem 2.6 and (3.104), for sufficiently large $h > 1$ we then have

$$\|\nabla u\|_{L^2(\Omega)} \leq C(\eta, R)\, h^{7/4} \log(h)^{3/4}\, |F| \,,$$

where $C(\eta, R) > 0$ depends only on η and R. Such growth rate (with respect to h) must be confronted with the one given by Ladyzhenskaya & Solonnikov in [51, Equation 3.4]. Similarly, from (2.29)–(3.99) we deduce that

$$\|\nabla u - \nabla \Psi_*\|_{L^2(\Omega)} < C(\eta, R)\, h^{7/2} \log(h)^{3/2}\, |F|^2 \,,$$

which should be compared to [46, Lemma 5.1].

3.3 Axial symmetry

Given $\phi \in [0, 2\pi]$, we denote by

$$R_z(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the rotation matrix about the z-axis by an angle ϕ. It can be easily verified that $R_z(\phi)^{-1} = R_z(\phi)^\top$, and therefore $\det(R_z(\phi)) = 1$, for every $\phi \in [0, 2\pi]$. In order to determine the existence of generalized solutions to problem (3.13) displaying rotational symmetry with respect to the z-axis, the following definition is given (see [1]):

Definition 3.2 – We say that a domain $D \subset \mathbb{R}^3$ is **axisymmetric** (with respect to the z-axis) if

$$\xi \in D \quad \Longleftrightarrow \quad R_z(\phi)\xi \in D \quad \forall \phi \in [0, 2\pi].$$

– If $D \subset \mathbb{R}^3$ is an axisymmetric domain, we say that a scalar function $g : D \longrightarrow \mathbb{R}$ is **axisymmetric** (with respect to the z-axis) if

$$g(\xi) = g(R_z(\phi)\xi) \qquad \forall \xi \in D,\ \phi \in [0, 2\pi].$$

Then, given a scalar function $g : D \longrightarrow \mathbb{R}$ and $\phi \in [0, 2\pi]$, we denote by

$$g^\phi(\xi) \doteq g(R_z(\phi)\xi) \qquad \forall \xi \in D \,,$$

its "axially rotated" transform by an angle ϕ.

– If $D \subset \mathbb{R}^3$ is an axisymmetric domain, we say that a vector field $G : D \longrightarrow \mathbb{R}^3$ is **axisymmetric** (with respect to the z-axis) if

$$G(\xi) = R_z(\phi)^\top G(R_z(\phi)\xi) \qquad \forall \xi \in D,\ \phi \in [0, 2\pi].$$

Then, given a vector field $g : D \longrightarrow \mathbb{R}^3$ and $\phi \in [0, 2\pi]$, we denote by

$$G^\phi(\xi) \doteq R_z(\phi)^\top G(R_z(\phi)\xi) \qquad \forall \xi \in D \,,$$

its "axially rotated" transform by an angle ϕ.

It is clear from Definition 3.2 that if Ω is as in (1.1), then Ω is axisymmetric if and only if the obstacle K is axisymmetric. In order to study the existence of axisymmetric weak solutions of problem (3.13), we must firstly build an axisymmetric flux carrier. We accompany Theorem 3.1 with the following result:

Theorem 3.5 *Let Ω be as in (1.1), K being axisymmetric and having a $\mathcal{C}^2$-boundary. Given $F \in \mathbb{R}$, there exists an axisymmetric vector field $X_* \in H^2(\Omega)$ such that*

$$\begin{cases} \nabla \cdot X_* = 0 \;\; in \;\; \Omega\,; & X_* \times \nu = 0 \;\; on \;\; \Gamma_I \cup \Gamma_O\,; \\ X_* = 0 \;\; on \;\; \Gamma_W\,; & \displaystyle\int_{\Sigma(s)} X_* \cdot \widehat{k} = F \quad \forall s \in [-h, h]. \end{cases} \tag{3.110}$$

Moreover, there holds the estimate

$$\|\nabla X_*\|_{L^2(\Omega)} \leq \frac{2|F|}{\pi R^2}\,(1 + 3C_B(\Omega))\left(2\sqrt{\pi h} + \sqrt{\mathrm{Cap}_{\mathcal{M}}(K)}\right). \tag{3.111}$$

Proof Let $\Psi_* \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ be the vector field that arises from Theorem 3.1, which can represented in cylindrical coordinates as

$$\Psi_*(\xi) = \Psi_*^\rho(\rho,\theta,z)\widehat{\rho} + \Psi_*^\theta(\rho,\theta,z)\widehat{\theta} + \Psi_*^z(\rho,\theta,z)\widehat{z} \qquad \forall \xi \in \Omega.$$

Then, inspired by [57, Lemma 12], we define the axisymmetric vector field

$$\begin{aligned} X_*(\xi) \;\doteq\; & \frac{1}{2\pi}\left[\left(\int_0^{2\pi} \Psi_*^\rho(\rho,\theta,z)\,d\theta\right)\widehat{\rho} + \left(\int_0^{2\pi} \Psi_*^\theta(\rho,\theta,z)\,d\theta\right)\widehat{\theta}\right. \\ & \left. + \left(\int_0^{2\pi} \Psi_*^z(\rho,\theta,z)\,d\theta\right)\widehat{z}\right] \qquad \forall \xi \in \Omega\,, \end{aligned}$$

which is an element of $H^2(\Omega)$. After differentiating under the integral sign we observe that

$$\begin{aligned} (\nabla \cdot X_*)(\xi) &= \frac{1}{2\pi}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\int_0^{2\pi} \Psi_*^\rho(\rho,\theta,z)\,d\theta\right) + \frac{\partial}{\partial z}\left(\int_0^{2\pi} \Psi_*^z(\rho,\theta,z)\,d\theta\right)\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\Psi_*^\rho)(\rho,\theta,z) + \frac{\partial \Psi_*^z}{\partial z}(\rho,\theta,z)\right)d\theta \\ &= -\frac{1}{2\pi\rho}\int_0^{2\pi}\frac{\partial \Psi_*^\theta}{\partial\theta}(\rho,\theta,z)\,d\theta \\ &= \frac{1}{2\pi\rho}\left[\Psi_*^\theta(\rho,0,z) - \Psi_*^\theta(\rho,2\pi,z)\right] = 0 \quad \text{for a.e. } \xi \in \Omega\,, \end{aligned}$$

that is, X_* is divergence-free. Since K is axisymmetric and

$$\Psi_* \times \nu = 0 \;\; \text{on} \;\; \Gamma_I \cup \Gamma_O\,; \qquad \Psi_* = 0 \;\; \text{on} \;\; \Gamma_W\,,$$

we easily deduce that also

$$X_* \times \nu = 0 \;\; \text{on} \;\; \Gamma_I \cup \Gamma_O\,; \qquad X_* = 0 \;\; \text{on} \;\; \Gamma_W\,,$$

thus implying that $X_* \in \mathcal{V}(\Omega)$. By noticing that

$$\begin{aligned} X_*(\rho,-h) \cdot \widehat{k} &= \frac{1}{2\pi}\int_0^{2\pi} \Psi_*^z(\rho,\theta,-h)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} U_0(\rho,\theta,-h)\cdot\widehat{k}\,d\theta \\ &= \frac{2F}{\pi R^4}(R^2 - \rho^2) \qquad \forall \rho \in [0,R]\,, \end{aligned}$$

see (3.3), we obtain

$$\int_{\Sigma(-h)} X_* \cdot \widehat{k} = F\,,$$

but as $X_* \in \mathcal{V}(\Omega)$, this immediately implies

$$\int_{\Sigma(s)} X_* \cdot \widehat{k} = F \qquad \forall s \in [-h, h].$$

In order to prove that X_* does not increase the Dirichlet norm of Ψ_*, we start by noticing that

$$(\nabla X_*)(\rho, z) = \frac{1}{2\pi} \int_0^{2\pi} \nabla \Psi_*(\rho, \theta, z)\, d\theta \quad \text{for a.e. } \xi \in \Omega\,,$$

(some elements of the Jacobian matrix ∇X_* are identically zero), so that an application of Jensen's inequality yields

$$|(\nabla X_*)(\rho, z)|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\nabla \Psi_*(\rho, \theta, z)|^2\, d\theta \quad \text{for a.e. } \xi \in \Omega. \tag{3.112}$$

Since Ω is axisymmetric, there exists a domain $\widetilde{\Omega} \subset \mathbb{R}^2$ such that the following representation holds:

$$\Omega = \{\xi \in \mathbb{R}^3 \mid (\rho, z) \in \widetilde{\Omega}\,, \quad \theta \in [0, 2\pi]\}.$$

Subsequently, from (3.112) we obtain

$$\begin{aligned} \|\nabla X_*\|^2_{L^2(\Omega)} &= 2\pi \int_{\widetilde{\Omega}} \rho\, |(\nabla X_*)(\rho, z)|^2 d\rho\, dz \\ &\leq \int_0^{2\pi} \int_{\widetilde{\Omega}} \rho |\nabla \Psi_*(\rho, \theta, z)|^2\, d\rho\, dz\, d\theta = \|\nabla \Psi_*\|^2_{L^2(\Omega)}\,, \end{aligned}$$

which gives the estimate (3.111) in view of (3.2), and also concludes the proof. □

The main goal of this section is to complement, in an axisymmetric framework, Theorem 3.3 with the following result; see also [30, Theorem 3.4] and [63, 76] for related works in unbounded domains (the whole $\mathbb{R}^3$ and an infinitely long nozzle, respectively).

Theorem 3.6 *Let Ω be as in (1.1), K being axisymmetric and having a $\mathcal{C}^2$-boundary. Assume also that $f \in L^2(\Omega)$ is an axisymmetric external force. Then, given any flux rate $F \in \mathbb{R}$,*

- *there exists (at least) one axisymmetric weak solution $u \in \mathcal{V}(\Omega)$ of (3.13), that is, it satisfies (3.19) for every axisymmetric vector field $\varphi \in \mathcal{V}_*(\Omega)$;*
- *if $u \in \mathcal{V}(\Omega)$ is a weak solution of (3.13), then $u^\phi \in \mathcal{V}(\Omega)$ and it is also a weak solution of (3.13), for every $\phi \in [0, 2\pi]$;*
- *if there exists an axisymmetric vector field $\Psi \in \mathcal{V}(\Omega)$ satisfying (3.1)–(3.63), then the unique weak solution of (3.13) is axisymmetric.*

Proof Let $X_* \in H^2(\Omega) \cap \mathcal{V}(\Omega)$ be the vector field arising from Theorem 3.5. We introduce the space

$$\mathcal{Z}_*(\Omega) = \{v \in \mathcal{V}_*(\Omega) \mid v \text{ is axisymmetric}\}\,,$$

which is a closed subspace of $\mathcal{V}_*(\Omega)$ and therefore it constitutes a Hilbert space under the Dirichlet scalar product of the gradients, see (3.18). To prove the existence of an axisymmetric weak solution $u \in \mathcal{V}(\Omega)$ of (3.13) amounts to show the existence of $\widehat{u} \in \mathcal{Z}_*(\Omega)$ such that

$$\eta \int_\Omega \nabla \widehat{u} \cdot \nabla \varphi + \int_\Omega \mathcal{E}(\widehat{u} + X_*)(\widehat{u} + X_*) \cdot \varphi = \int_\Omega f \cdot \varphi - \eta \int_\Omega \nabla X_* \cdot \nabla \varphi \qquad \forall \varphi \in \mathcal{Z}_*(\Omega)\,, \tag{3.113}$$

so that the solution will be given by $u = \widehat{u} + X_*$. For a fixed $\widehat{u} \in \mathcal{Z}_*(\Omega)$, the applications

$$\varphi \in \mathcal{Z}_*(\Omega) \longmapsto \int_\Omega \mathcal{E}(\widehat{u} + X_*)(\widehat{u} + X_*) \cdot \varphi \qquad \text{and} \qquad \varphi \in \mathcal{Z}_*(\Omega) \longmapsto \int_\Omega (f \cdot \varphi - \eta \nabla X_* \cdot \nabla \varphi)$$

clearly define linear continuous functions on $\mathcal{Z}_*(\Omega)$. Then, in view of the Riesz Representation Theorem, the identity (3.113) may be written as

$$[\eta \widehat{u} + \mathcal{P}(\widehat{u}) - \mathcal{F}, \varphi]_{\mathcal{V}(\Omega)} = 0 \qquad \forall \varphi \in \mathcal{Z}_*(\Omega)\,,$$

for some (unique) elements $\mathcal{P}(\widehat{u}), \mathcal{F} \in \mathcal{Z}_*(\Omega)$ such that

$$[\mathcal{P}(\widehat{u}), \varphi]_{\mathcal{V}(\Omega)} = \int_\Omega \mathcal{E}(\widehat{u} + X_*)(\widehat{u} + X_*) \cdot \varphi \quad \text{and}$$
$$[\mathcal{F}, \varphi]_{\mathcal{V}(\Omega)} = \int_\Omega (f \cdot \varphi - \eta \nabla X_* \cdot \nabla \varphi) \qquad \forall \varphi \in \mathcal{Z}_*(\Omega).$$

We have so defined a linear operator $\mathcal{P} : \mathcal{Z}_*(\Omega) \longrightarrow \mathcal{Z}_*(\Omega)$ and we are led to find a solution $\widehat{u} \in \mathcal{Z}_*(\Omega)$ of the following the nonlinear operator equation:

$$\widehat{u} + \frac{1}{\eta}(\mathcal{P}(\widehat{u}) - \mathcal{F}) = 0 \quad \text{in} \;\; \mathcal{Z}_*(\Omega). \tag{3.114}$$

Exactly as in [49, Chapter 5, Theorem 1] one can show that the operator $\mathcal{P}$ is compact. Therefore, as a consequence of the Leray–Schauder Principle, in order to prove that (3.114) possesses at least one solution, it suffices to guarantee that any $v^\lambda \in \mathcal{Z}_*(\Omega)$ such that

$$v^\lambda + \frac{\lambda}{\eta}(\mathcal{P}(v^\lambda) - \mathcal{F}) = 0 \quad \text{in} \;\; \mathcal{Z}_*(\Omega)\,, \tag{3.115}$$

is uniformly bounded with respect to $\lambda \in [0, 1]$. This can be achieved imitating the corresponding part of the proof of Theorem 3.3 (restricting ourselves to spaces of axisymmetric scalar or vector functions), and therefore is omitted here. In conclusion, (3.13) has at least one axisymmetric weak solution.

Now, let $u \in \mathcal{V}(\Omega)$ be a weak solution of (3.13). For $\phi \in [0, 2\pi]$, as is Definition 3.2 we set

$$u^\phi(\xi) = R_z(\phi)^\top u(R_z(\phi)\xi) \qquad \forall \xi \in \Omega.$$

It is then clear that $u^\phi \in H^1(\Omega)$, $u^\phi = 0$ on Γ_W and $u^\phi \times \nu = 0$ on $\Gamma_I \cup \Gamma_O$. Moreover, given $s \in [-h, h]$, the change of variables $\xi \in \Sigma(s) \longmapsto R_z(\phi)\xi \in \Sigma(s)$ yields

$$\int_{\Sigma(s)} u^\phi \cdot \widehat{k} = \int_{\Sigma(s)} u \cdot \widehat{k} = F \qquad \forall s \in [-h, h].$$

On the other hand, we have

$$\nabla u^\phi(\xi) = R_z(\phi)^\top \, \nabla u(R_z(\phi)\xi)\, R_z(\phi) \qquad \text{for a.e.} \;\; \xi \in \Omega. \tag{3.116}$$

Given any $\varphi \in \mathcal{V}_*(\Omega)$, by properties of the matrix trace operator and the Frobenius inner product (in particular, cyclic permutations) we deduce from (3.116) the following identities for a.e. $\xi \in \Omega$:

- $(\nabla \cdot u^\phi)(\xi) = \mathrm{tr}(\nabla u^\phi(\xi)) = \mathrm{tr}((\nabla u(R_z(\phi)\xi)) = (\nabla \cdot u)(R_z(\phi)\xi) = 0$, so that u^ϕ is also divergence-free. This already proves that $u^\phi \in \mathcal{V}(\Omega)$;
- $\nabla u^\phi(\xi) \cdot \nabla \varphi^\phi(\xi) = \nabla u(R_z(\phi)\xi) \cdot \nabla \varphi(R_z(\phi)\xi)$;
- $\left[\nabla u^\phi(\xi) - (\nabla u^\phi(\xi))^\top\right] u^\phi(\xi) \cdot \varphi^\phi(\xi) = \left[\nabla u(R_z(\phi)\xi) - (\nabla u(R_z(\phi)\xi))^\top\right] u(R_z(\phi)\xi) \cdot \varphi(R_z(\phi)\xi)$;

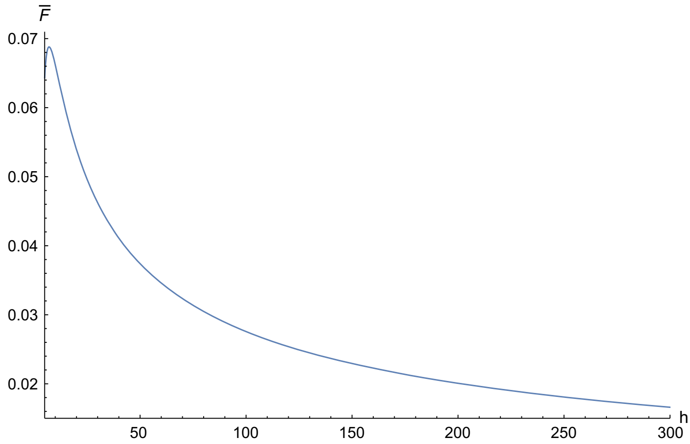

Fig. 8 Upper bound for the Reynolds number (3.117) as a function of $h > R$, for $R = 5$

- $f(\xi) \cdot \varphi^\phi(\xi) = f(R_z(\phi)\xi) \cdot \varphi(R_z(\phi)\xi)$, since f is axisymmetric.

Since $u \in \mathcal{V}(\Omega)$ is a weak solution of (3.13), successive applications of the change of variables

$$\xi \in \Omega \longmapsto R_z(\phi)\xi \in \Omega$$

allow us then to conclude that

$$\eta \int_\Omega \nabla u^\phi \cdot \nabla \varphi^\phi + \int_\Omega \left[\nabla u^\phi - (\nabla u^\phi)^\top\right] u^\phi \cdot \varphi^\phi = \int_\Omega f \cdot \varphi^\phi \qquad \forall \varphi \in \mathcal{V}_*(\Omega).$$

In particular,

$$\eta \int_\Omega \nabla u^\phi \cdot \nabla \varphi + \int_\Omega \left[\nabla u^\phi - (\nabla u^\phi)^\top\right] u^\phi \cdot \varphi = \int_\Omega f \cdot \varphi \qquad \forall \varphi \in \mathcal{Z}_*(\Omega)\,,$$

thus showing that u^ϕ is also a weak solution of (3.13), for every $\phi \in [0, 2\pi]$. In order to conclude the proof, we observe that if there exists an axisymmetric vector field $\Psi \in \mathcal{V}(\Omega)$ satisfying (3.1)–(3.63), then necessarily $u = u^\phi$ for every $\phi \in [0, 2\pi]$, where $u \in \mathcal{V}(\Omega)$ is the unique weak solution of (3.13). This proves that u is axisymmetric. $\square$

If we maintain the radius R of the cylinder $\mathcal{M}$ fixed, the result of Corollary 3.3 yields an explicit upper bound for the Reynolds number (here understood as the ratio between the transversal flux rate and the viscosity) that ensures the existence of a unique weak solution of problem (1.3) (in the absence of external forcing) which, moreover, is axisymmetric. In Fig. 8 we plot the quantity

$$\overline{F} \doteq \frac{\sqrt{13}-3}{64} \frac{\sqrt{\pi}\, R^2 \mathcal{S}_*}{\sqrt{h} + \dfrac{\sqrt[6]{3}}{\sqrt{\sqrt[3]{\dfrac{4\pi}{|K|}} - \sqrt[3]{\dfrac{2}{R^2 h}}}}}\, \eta\,, \tag{3.117}$$

as a function of $h > R$, for $\eta = 1$, $R = 5$ and assuming that K is the unit ball of $\mathbb{R}^3$.

Furthermore, from Theorem 2.3 we know that

$$\lim_{h\to\infty} \mathcal{S}_* = \frac{\pi}{3}\sqrt{\frac{\mu_0}{R}}.$$

Therefore,

$$\overline{F} \sim \frac{1}{\sqrt{h}} \quad \text{as} \quad h \to \infty\,,$$

which should be compared with [31, Remark 4.2]. Is it possible to improve this asymptotic behavior?

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