

THE GEOMETRY OF (NON-ABELIAN) LANDAU LEVELS

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Abstract. The purpose of this paper is threefold: First of all the topological aspects of the Landau Hamiltonian are reviewed in the light (and with the jargon) of theory of topological insulators. In particular it is shown that the Landau Hamiltonian has a generalized even time-reversal symmetry (TRS). Secondly, a new tool for the computation of the topological numbers associated with each Landau level is introduced. The latter is obtained by combining the Dixmier trace and the (resolvent of the) harmonic oscillator. Finally, these results are extended to models with non-Abelian magnetic fields. Two models are investigated in details: the Jaynes-Cummings model and the “Quaternionic” model.

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*“Que vivan los estudiantes
 Jardín de nuestra alegría
 Son aves que no se asustan
 De animal ni policía.”*

Violeta Parra

1. Introduction

The quantum motion of a two-dimensional electron subjected to a perpendicular and uniform magnetic field B is determined by the differential operator

$$H_B := \frac{\epsilon_B}{2} \left[\left(-i\ell_B \frac{\partial}{\partial x_1} - \frac{x_2}{2\ell_B} \right)^2 + \left(-i\ell_B \frac{\partial}{\partial x_2} + \frac{x_1}{2\ell_B} \right)^2 \right] \quad (1.1)$$

where $\epsilon_B \propto B$ fixes the magnetic energy scale and $\ell_B \propto B^{-\frac{1}{2}}$ defines the typical magnetic length. The equation (1.1) defines a self-adjoint operator on the (position) Hilbert space $L^2(\mathbb{R}^2)$. The operator H_B is known as *Landau Hamiltonian*. The spectral theory of H_B is well established since the dawn of Quantum Mechanics [Foc, La] especially for its connection with the elementary theory of the harmonic oscillator (see also [AHS] for a more modern point of view). The spectrum of H_B is pure point and is given by

$$\sigma(H_B) = \left\{ E_j := \epsilon_B \left(j + \frac{1}{2} \right) \mid j \in \mathbb{N}_0 \right\}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The energy E_j is usually called the j -th Landau level. Each Landau level is infinitely degenerate due to the high symmetry of H_B . Indeed, the Landau Hamiltonian commutes with the perpendicular component of the angular momentum L_3 and with the generators of the magnetic translations G_1 and G_2 (see Section 3.1). The infinite dimensional eigenspace associated to each Landau level E_j is completely characterized by the related spectral projection Π_j , called the j -th Landau projection. The topology of the Landau Hamiltonian is encoded exactly in the Landau projections.

As it is well known, the magnetic field breaks the time reversal symmetry (TRS). In Quantum Mechanics the “standard” time reversal symmetry is implemented by the complex conjugation C and a simple computation shows that $CH_B C = H_{-B}$. In the context of the classification scheme for topological insulators [AZ, SRFL, Kit, RSFL] the Landau Hamiltonian is usually considered as the prototype model for *class A* systems. The latter are the systems which break all the fundamental (pseudo-)symmetries. As a consequence, the topological phases of H_B are predicted to be labelled by integers according to the celebrated periodic table for topological insulators. From a physical point of view these topological invariants are associated with the quantized values of the transverse Hall conductivity [TKNN, BES]. On the other hand they are mathematically associated with the Chern numbers of suitable vector bundles [DN1, DN2, Nov, Lys] that are constructed from the Landau projections Π_j by exploiting the invariance of H_B under the magnetic translations [Zak1, Zak2]. Indeed the invariance under the \mathbb{Z}^2 -action induced by the magnetic translations allows

to define a magnetic version of the Bloch-Floquet-Zak transform [Kuc] which decomposes H_B , along with all its spectral functions, into a fibered operator over the Brillouin (two dimensional) torus $\mathbb{B}_B \simeq \mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ (see Section 3.3 for more details). At level of the projections Π_j this procedure defines continuous maps $\mathbb{T}^2 \ni k \mapsto \Pi_j(k)$ which provide complex vector bundles $\mathcal{E}_j \rightarrow \mathbb{T}^2$. The construction of these (so called) *spectral bundles* is classical and depicts an incarnation of the Serre-Swan duality [Ser, Swa]. There is a rich literature devoted to a rigorous definition of the spectral bundles (see e. g. [Pan] or [DL, Lemma 4.5] or [DG1, Section 2]) and the main aspects of the construction will be summarized in Section 3.4. The crucial point we want to emphasize here is that the topology of the Landau levels E_j is encoded in the spectral bundles \mathcal{E}_j associated with the Landau projections Π_j . In absence of extra symmetries these vector bundles must be classified in the category of complex vector bundles over the two-dimensional torus \mathbb{T}^2 and a classical result by Peterson [Pet] provides the isomorphism

$$\text{Vec}_{\mathbb{C}}^m(\mathbb{T}^2) \stackrel{c_1}{\simeq} H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z} \quad (1.2)$$

where $\text{Vec}_{\mathbb{C}}^m(\mathbb{T}^2)$ is the set of equivalence classes of rank m complex vector bundles over \mathbb{T}^2 , $H^2(\mathbb{T}^2, \mathbb{Z})$ is the second cohomology group of \mathbb{T}^2 (with integer coefficients) and the map c_1 is the (first) Chern class.

The chain of isomorphisms (1.2), that provides the vector bundle interpretation of the topological classification of the Landau projections Π_j , is justified on the basis of the absence of further fundamental symmetries. Nevertheless, a deeper look at the structure of the operator H_B shows that it is not totally correct to claim that the Landau Hamiltonian does not have (pseudo-)symmetries. To this end we need the following notion:

Definition 1.1 (Generalized TRS). Let \mathcal{H} be a complex Hilbert space endowed with an anti-unitary involution C which defines a complex structure. A (non necessary bounded) linear operator H has a *generalized* time reversal symmetry (TRS) if there is an auxiliary unitary operator F such that

$$\Theta H \Theta^{-1} = H, \quad \Theta := FC.$$

The symmetry Θ is called “*Real*” (or even) if $\Theta^2 = +\mathbf{1}$ and “*Quaternionic*” (or odd) if $\Theta^2 = -\mathbf{1}$.

At the algebraic level a generalized symmetry Θ of the type described in Definition 1.1 implements the inversion of the time evolution generated by the operator H , and for this reason it is fair to refer to Θ as a TRS. Indeed, this is mainly due to the anti-linearity of Θ in accordance with the Wigner’s theorem [Wig]. Interestingly, the Wigner’s theorem point of view has been used in various recent works [FM, Thi, Kub] to generalize and “algebrize” the classification program for topological insulators to abstract Hilbert spaces endowed with suitable group actions. Within such algebraic framework, Definition 1.1 provides the natural formulation for a “genuine” TRS. It is also worth mentioning that the generalized structure of the TRS presented in Definition 1.1 has already been considered since the first works on the classification of topological insulators (see e. g. [SRFL, p.

3] or [RSFL, eq. (3)]). On the other hand, in application to model of physical inspiration [AZ, SRFL, RSFL] the choice of a “physical” TRS is often subordinate to (usually undeclared) extra conditions like the invariance of the position operators under the time inversion. However, these additional constraints are not usually introduced in the abstract algebraic setting, especially when the notion of spatial coordinates is not definable from the context. This fact opens a space of discussion about the best possible definition of TRS in the context of topological insulators. We do not pretend to solve this semantic and/or philosophic diatribe in this paper. And, independently of the terminology used, it is a fact that the Landau Hamiltonian admits a generalized symmetry of the type described in Definition 1.1.

Proposition 1.1. *The Landau Hamiltonian H_B admits a generalized TRS of “Real” type implemented by $\Theta = FC$ where F is the flip operator defined by*

$$(F\psi)(x_1, x_2) = \psi(x_2, x_1), \quad \psi \in L^2(\mathbb{R}^2)$$

and C is the usual complex conjugation $C\psi = \overline{\psi}$.

The proof of Proposition 1.1 will be discussed in Section 3.2. Even though the proof of the latter result does not present major complications, it deserves some considerations. As already mentioned, the Landau Hamiltonian H_B is usually considered as the prototypical example for topological insulators in class A, that is the class of systems that break all the fundamental symmetries, and two dimensional systems in this class have \mathbb{Z} distinguished topological phases (interpreted as Chern numbers). However, such classification is based on the (tacit!) assumption that the complex conjugation provides the natural, and the unique, realization for the TRS. On the other hand, Proposition 1.1 tells us that the Landau Hamiltonian H_B possesses the generalized “Real” TRS Θ , hence it may be considered as a topological system of *class AI* (according to the accepted nomenclature) in the classification scheme that chooses Θ as fundamental symmetry. This fact generates an apparent contradiction, since 2-dimensional systems in class AI are predicted to exist only in the topologically trivial phase [SRFL, Kit, RSFL]. Even though it may seem that this contradiction exists only at a level of use (or abuse) of terminology, it inspires a related question: Is the symmetry Θ responsible for a different, maybe finer, topological classification of the Landau Hamiltonian? The answer to the latter question is the first main result of this paper (Theorem 1.1) and it is obtained by an accurate analysis of the correct category of vector bundles underlying the Landau Hamiltonian. As a payoff, we get also the reconciliation of the (apparent) contradiction stated above. Our analysis aims also to point out the importance of the (a priori) choice of the fixed fundamental symmetries used for the classification of topological systems. Without extra constraints (e.g. the action of the symmetries on a bunch of special operators), different choices of fundamental symmetries are possible and appropriate choices can lead in principle to a finer classification tables. Before passing to the description of the first main result, two brief final comments are necessary: First of all the TRS Θ in Proposition

1.1 is pretty “fragile” in the sense that it can be broken by the effect of a generic background potential (cf. Remark 3.1); Secondly, the study of topological systems with generalized symmetries of the type of Θ can be included in the more general analysis of topological systems with *point group symmetries* (see [Gom1, CC, GT] and reference therein).

The operator Θ described in Proposition 1.1 combines together with the Bloch-Floquet-Zak transform and endows the spectral bundles \mathcal{E}_j associated with the projections Π_j with an additional “Real” structure. In a nutshell, this means that the vector bundle $\mathcal{E}_j \rightarrow \mathbb{T}^2$ acquires an involutive and fiberwise anti-linear map on the total space $\Theta : \mathcal{E}_j \rightarrow \mathcal{E}_j$ which covers an involution $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ on the base space. The notion of “Real” vector bundle has been introduced for the first time in [Ati1]. The construction of “Real” vector bundles from gapped systems with an even TRS is explained in detail in [DG1] (see also Section 3.4 for a more concise description) and the topological (cohomology based) classification of these structures has been investigated in [DG1, DG3]. In order to properly describe the type of “Real” vector bundle associated with each pair (Π_j, Θ) one needs to specify the type of involution that Θ induces on the Brillouin torus $\mathbb{B}_B \simeq \mathbb{T}^2$. It turns out that (see Section 3.4) the relevant involution is the following:

Definition 1.2 (Flip involution). Let \mathbb{T}^2 be a two dimensional torus with coordinates (k_1, k_2) . The *flip involution* is the involutive homeomorphism $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined by $f : (k_1, k_2) \mapsto (-k_2, -k_1)$. The pair (\mathbb{T}^2, f) defines an *involutive space* with a non-empty fixed-point set $(\mathbb{T}^2)^f := \{k \in \mathbb{T}^2 \mid f(k) = k\} \simeq \mathbb{S}^1$ made by anti-diagonal points $(k, -k)$.

In summary, each pair (Π_j, Θ) defines a “Real” vector bundle (\mathcal{E}_j, Θ) over the involutive space (\mathbb{T}^2, f) . The latter claim is proved in detail in Proposition 3.1. Let $\text{Vec}_{\Re}^m(\mathbb{T}^2, f)$ be the set of isomorphism classes of rank m “Real” vector bundles over (\mathbb{T}^2, f) . The topological classification of (Π_j, Θ) amounts to the classification of $\text{Vec}_{\Re}^m(\mathbb{T}^2, f)$ and the latter boils down to the following chain of isomorphisms

$$\text{Vec}_{\Re}^1(\mathbb{T}^2, f) \xrightarrow{c_1^{\Re}} H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \xrightarrow{\iota} H^2(\mathbb{T}^2, \mathbb{Z}). \quad (1.3)$$

In the first isomorphism, $H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1))$ is the second equivariant cohomology group of the involutive space (\mathbb{T}^2, f) with local system of coefficients $\mathbb{Z}(1)$ (cf. [DG1, Section 5.1] and references therein for a brief introduction to the equivariant cohomology) and the map c_1^{\Re} is called (first) “Real” Chern class. This first isomorphism is known as Kahn’s isomorphism and has been proved in [Kah]; see also [DG1, Section 5.2] and [Gom1, Appendix A]. The second isomorphism in (1.3) is the content of Proposition 3.2. in this case ι is the map which forgets the \mathbb{Z}_2 action induced by the involution f . Since $c_1 = \iota \circ c_1^{\Re}$ coincides with the (usual) Chern class it follows from (1.3) that the “Real” *line bundles*¹ over (\mathbb{T}^2, f) are completely specified by the Chern class of the underlying complex line bundle. This fact is preparatory to present the first main result of this paper.

¹Line bundle is used as a synonym for rank-one vector bundle.

Theorem 1.1. *At each Landau level E_j of the Hamiltonian H_B it is associated a “Real” line bundle (\mathcal{E}_j, Θ) over the involutive (Brillouin) torus $(\mathbb{T}_B^2, \mathfrak{f})$. The topology of this line bundle is completely classified by its Chern class $c_1(\mathcal{E}_j) \in \mathbb{Z}$. It turns out that*

$$c_1(\mathcal{E}_j) = 1, \quad \forall j \in \mathbb{N}_0$$

implying accordingly that all the Landau levels are topologically equivalent.

A complete proof of Theorem 1.1 will be presented in Section 3.5. Let us point out here that Theorem 1.1 adds two new information to the isomorphisms (1.3): First of all the rank of the (complex) vector bundles \mathcal{E}_j is $\text{rk}(\mathcal{E}_j) = 1$; Secondly the Chern number associated to each vector bundle \mathcal{E}_j can be computed and is $c_1(\mathcal{E}_j) = 1$. These information can be extracted from the Bloch-Floquet-Zak representation of the Landau projections Π_j , but it is often useful (especially for generalizations beyond the periodic case) to have available formulas that allow the computation of the same quantities in the position space $L^2(\mathbb{R}^2)$ directly in terms of the projections Π_j . The existing formulas of this type involve the use of the *trace per unit volume* (see Section 3.6). However, it is also possible to compute the same quantities in a more (noncommutative) “geometric” way by using the *Dixmier trace* (see Appendix B): This is the second main result of this work.

Let us consider the harmonic oscillator

$$Q_B = \sum_{j=1}^2 \left(-\ell_B^2 \frac{\partial^2}{\partial x_j^2} + \frac{1}{4} \frac{x_j^2}{\ell_B^2} \right). \quad (1.4)$$

The relation between Q_B and the Landau operator H_B , made manifest by the presence of the magnetic length ℓ_B^2 , will be clarified in Section 3.6. The operator (1.4) has a pure point spectrum given by

$$\sigma(Q_B) = \{ \lambda_j := j + 2 \mid j \in \mathbb{N}_0 \}$$

and each eigenvalue has finite degeneracy $\text{Mult}[\lambda_j] = j + 1$. The resolvent

$$Q_{B,\xi}^{-1} := (Q_B + 2\xi \mathbf{1})^{-1} \quad (1.5)$$

is certainly well defined for $\xi \geq 0$ and is a compact operator. It turns out that (Corollary 3.2 and Corollary 3.3)

$$\begin{aligned} \text{rk}(\mathcal{E}_j) &= \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} \Pi_j) \\ c_1(\mathcal{E}_j) &= \frac{i}{\ell_B^2} \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} \Pi_j [\partial_1(\Pi_j), \partial_2(\Pi_j)]) \end{aligned} \quad (1.6)$$

where Tr_{Dix} denotes the Dixmier trace and $\partial_i(A)$ is a shorthand for the commutators (when defined)

$$\partial_i(A) := -i [X_i, A], \quad i = 1, 2$$

where X_1 and X_2 are the position operators. The Dixmier trace is the prototype of a singular (non-normal) trace. It was introduced for the first time by Dixmier in [Dix1]. The theory of Dixmier trace is reviewed in Appendix B along with a list of selected references. Formulas (1.6) provide the link

between the topology of the spectral bundle associated with the Landau projection Π_j and some “numerical indexes” of Π_j calculated directly in the position space through the functional $T \mapsto \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1}T)$. These equations are indeed obtained as a special application of the following more general result:

Theorem 1.2. *For all $T \in \mathcal{M}_B^1$ the following equality*

$$\frac{1}{2\Omega_B} \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1}T) = \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\Lambda_n} T \chi_{\Lambda_n}) \quad (1.7)$$

holds true independently of $\xi \geq 0$.

Theorem 1.2 is proved in full detail in Appendix B.3. In order to make the claim understandable, let us add few information. The quantity $\Omega_B := \pi \ell_B^2$ provides the area of the *magnetic disk* of radius ℓ_B . The von Neumann algebra generated by the spectral projections of H_B is denoted with \mathcal{M}_B , and $\mathcal{M}_B^1 \subset \mathcal{M}_B$ is a suitable (weakly dense) ideal. The $\Lambda_n \subseteq \mathbb{R}^2$ must provide an increasing sequence of compact subsets such that $\Lambda_n \nearrow \mathbb{R}^2$ and which satisfies the *Følner condition* (see e. g. [Gre] for more details). The (Lebesgue) volume of Λ_n is denoted with $|\Lambda_n|$ and χ_{Λ_n} is the characteristic function of the set Λ_n which acts as a (multiplication) self-adjoint projection on $L^2(\mathbb{R}^2)$. The right-hand side of (1.7) defines the trace per unit of volume of the operator T and Theorem 1.2 provides the link between the Dixmier trace and the trace per unit of volume through the mediation of the regularizing operator $Q_{B,\xi}^{-1}$.

The use of the Dixmier trace combined with the resolvent of the harmonic oscillator Q_B offers some advantages with respect to the use of the trace per unit of volume. First of all, the computation of the trace per unit volume always implies the choice (a priori) of a suitable approximating sequence $\{\Lambda_n\}$ of bounded regions of the plane to obtain the desired thermodynamic limit. This election could in principle affect the result depending of the nature of the operator whose trace is to be calculated. One of the main question in this business is to understand the class of operators that admit a well-defined trace per unit of volume independently of the election of the approximating sequence $\{\Lambda_n\}$. On the other hand, the use of the left-hand side of (1.7) circumvents this problem by proposing an intrinsic way to calculate the thermodynamic quantities. From a physical point of view the operator Q_B is responsible for a “natural” quantization of the space as a consequence of the quantization of the oscillation frequencies (or equivalently the quantization of the *cyclotron orbits*). The thermodynamic limit is then recovered through the computation of the Dixmier trace. This argument provides the “physical justification” for equality (1.7). Even from a computation point of view, the use of the left-hand side of (1.7) presents some advantages. Indeed, the Dixmier trace can be easily computed on the basis which diagonalizes Q_B (in view of Lemma B.2) and the latter is known explicitly: For instance it can be described in terms of the Laguerre functions (3.11). Finally, the content of Theorem 1.2 has also a noncommutative geometric flavor. Indeed, the left-hand side of (1.7) has the structure of the Connes’ noncommutative integral (cf. [GBVF, eq. (7.83)]) which provides the noncommutative version

of the Wodzicki's residue (see *e.g.* [GBVF, Section 7.6] for more details). Moreover, equation (1.7) also provides the extension to the continuum of an analogous formula proved by Bellissard for two-dimensional discrete systems [Bel, Section 2.6]. These analogies suggest that the resolvent $Q_{B,\xi}^{-1}$ should be related to a Dirac operator. Indeed, it can be proven that $Q_{B,\xi}^{-1}$ is proportional to $|D_B|^{-d}$ (with $d = 2$) where D_B is a suitable *magnetic Dirac operator*². The magnetic Dirac operator D_B enters in the construction of a *bounded* (magnetic) spectral triple for the Landau operator which provides the “natural” geometric object for the noncommutative geometry of the Quantum Hall effect in the continuum. An accurate analysis of these aspects, motivated by the exigency to conclude the program started in [BES] twenty-five years ago, will be presented in a separated work [DM]. Finally, let us cite the recent work [AMSZ] where a similar use of the Dixmier trace has been proposed.

The last achievement of this paper concerns the extension of the topological analysis performed for the Landau Hamiltonian H_B to models with *non-Abelian* magnetic fields. A sufficiently detailed explanation of what is meant with non-Abelian magnetic field is postponed to Section 4. Let us only anticipate here that we will focus mainly on models for particles with spin $s = \frac{1}{2}$. In this case, the resulting non-Abelian magnetic Hamiltonians act on the space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ and have the typical structure

$$\mathcal{H}_{B,b} := H_B \otimes \mathbf{1}_2 + c_b \epsilon_B \mathcal{W} + C_{B,b} \mathbf{1} \quad (1.8)$$

where H_B is the Landau Hamiltonian, $c_b := \frac{b}{B\ell_B}$ is a coupling constant for the non-Abelian magnetic field, $C_{B,b} \propto c_b^2 \epsilon_B$ is a suitable energy constant and \mathcal{W} is the term that specifies the type of coupling between the particle and the non-Abelian magnetic field. We will focus on two models in particular. In Section 4.2 we will study the *Jaynes-Cummings model* specified by the (Rashba-type) coupling potential

$$\mathcal{W}_{JC} := (K_1 \otimes \sigma_2 - K_2 \otimes \sigma_1).$$

Here $\sigma_1, \sigma_2, \sigma_3$ denote the three Pauli matrices and K_1, K_2 are the kinetic momenta (3.4) associated with the Landau Hamiltonian H_B . It turns out that the Jaynes-Cummings model has a pure point spectrum which can be calculated explicitly, as shown by (4.7). Moreover, this model possesses a generalized TRS of even type (denoted by Ξ). Mimicking the analysis of the Landau Hamiltonian we will prove in Section 4.2 the following result:

Theorem 1.3. *At each energy level E_j^\pm of the Jaynes-Cummings model it is associated a “Real” line bundle (\mathcal{E}_j^\pm, Ξ) over the involutive (Brillouin) torus $(\mathbb{T}_B^2, \mathfrak{f})$. The topology of this line bundle is completely classified by*

² For the interested readers we anticipate that D_B acts on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^4$ and is given by

$$D_B := \frac{1}{\sqrt{2}}(K_1 \otimes \gamma_1 + K_2 \otimes \gamma_2 + G_1 \otimes \gamma_3 + G_1 \otimes \gamma_4)$$

where the operators K_1, K_2, G_1, G_2 are defined in Section 3.1 and the 4×4 matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are Clifford generators which satisfy the canonical anti-commutation relations (CAR) on \mathbb{C}^4 .

its Chern class $c_1(\mathcal{E}_j^\pm) \in \mathbb{Z}$. It turns out that

$$c_1(\mathcal{E}_j^\pm) = 1, \quad \forall j \in \mathbb{N}_0$$

implying accordingly that all the energy levels are topologically equivalent.

It is worth pointing out that also in this case the topological numbers (the rank and the Chern number) associated with the energy levels can be computed with the help of the Dixmier trace.

The second model of non-Abelian magnetic Hamiltonian studied in Section 4.3 is specified by the coupling potential

$$\mathcal{W}_Q := r_0(K_1 - K_2) \otimes \mathbf{1}_2 + (K_1 + K_2) \otimes (r_1\sigma_1 + r_2\sigma_3) \quad (1.9)$$

where r_0, r_1, r_2 are real constants subject to the normalization $r_0^2 + r_1^2 + r_2^2 = 1$. The detailed study of the spectrum of this operator is beyond the scope of this work and this is left for future investigations. However, only on the basis of the analysis of the symmetries of this model it is possible to anticipate some interesting topological properties. First of all, let us note that the Hamiltonian \mathcal{H}_Q associated with the potential \mathcal{W}_Q has a positive spectrum as a consequence of the general structure (4.5). Secondly, the potential \mathcal{W}_Q endows \mathcal{H}_Q with generalized TRS of odd type. If one assumes the existence of a gap around the energy $E > 0$ in the spectrum of \mathcal{H}_Q , then the related spectral projection $\mathcal{P}_E := \chi_{(-\infty, E]}(\mathcal{H}_Q)$ will be endowed with a ‘‘Quaternionic’’ structure according to [DG2]. This fact justifies the choice of the name ‘‘Quaternionic’’ model for \mathcal{H}_Q . Moreover, it permits to relate the topological properties of \mathcal{P}_E to the classification of ‘‘Quaternionic’’ vector bundles over the involutive space $(\mathbb{T}^2, \mathfrak{f})$, namely to the description of the set $\text{Vec}_\Omega^{2m}(\mathbb{T}^2, \mathfrak{f})$ of isomorphism classes of rank $2m$ ‘‘Quaternionic’’ vector bundles over $(\mathbb{T}^2, \mathfrak{f})$. The description of $\text{Vec}_\Omega^{2m}(\mathbb{T}^2, \mathfrak{f})$ boils down to the following chain of maps

$$\text{Vec}_\Omega^{2m}(\mathbb{T}^2, \mathfrak{f}) \xrightarrow{\kappa} H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^\mathfrak{f}, \mathbb{Z}(1)) \xrightarrow{j} H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \xrightarrow{c_1} \mathbb{Z} \quad (1.10)$$

where the first isomorphism is induced by the *FKMM-invariant* κ [DG2, DG4, DG5], the second map j amounts to the injection $j : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $j : n \mapsto 2n$ (Lemma A.2 and Lemma A.3) and the last isomorphism is the same as described in (1.3). In conclusion, one has that $\text{Vec}_\Omega^{2m}(\mathbb{T}^2, \mathfrak{f}) \simeq 2\mathbb{Z}$ and the isomorphism classes are completely determined by the (even) values of the Chern classes of the underlying complex vector bundles (Corollary A.1). The latter general result is central to prove:

Theorem 1.4. *Assume that the spectrum of the ‘‘Quaternionic’’ model \mathcal{H}_Q has a gap around the energy $E > 0$ and let \mathcal{P}_E be the associated Fermi projection. To \mathcal{P}_E is associated an even rank ‘‘Quaternionic’’ vector bundle (\mathcal{E}_E, Ξ') over the involutive (Brillouin) torus $(\mathbb{T}^2, \mathfrak{f})$. Moreover the topology of this vector bundle is completely classified by its Chern class $c_1(\mathcal{E}_j) \in 2\mathbb{Z}$ which can only take even values.*

Section 4.3 is devoted to the proof of Theorem 1.4. Also in this case the relevant topological invariants can be computed by formulas involving the Dixmier trace of the projection \mathcal{P}_E .

In conclusion this paper contains a detailed study of the topology and the geometry of the Landau levels which takes into account the role of extra structures deriving from possible generalized TRS. It is shown that these structures can be of “Real” type or even of “Quaternionic” type in the regime of non-Abelian magnetic fields. This implies that the topology of these systems should be studied inside the correct category of “Real” or “Quaternionic” vector bundles. As additional, but not less relevant result, we proved that the topological numbers which specify the topology of the energy spectrum of the various models under analysis can be computed directly in the position space using the resolvent of the harmonic oscillator and the Dixmier trace. This paper contains several new results but the investigation initiated here is far from being considered completed. The case of magnetic systems in presence of background potentials (periodic, aperiodic, random, . . .) is already subject of ongoing investigations.

Structure of the paper. Section 2 is devoted to the connection between the flip involution and the particle exchange symmetry. Instead, the geometry and the topology associated with the energy levels of the Landau Hamiltonian are studied in Section 3. Then, the results obtained in this section are generalized to the case of non-Abelian magnetic fields in Section 4. In particular the Jaynes-Cummings model and the “Quaternionic” model are studied in detail here. Appendix A contains the computations of the equivariant cohomology groups of the two-dimensional torus endowed with the flip involution. Finally, Appendix B provides a “crash course” on the Dixmier trace as well as the proof of the crucial Theorem 1.2.

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2. Flip operator and particle exchange symmetry

The aim of this preliminary section is to introduce the notion of *flip operator* which will play a crucial role in the subsequent part of the work. We find instructive to relate this operator to a fundamental physical symmetry: the *particle exchange symmetry*.

Let $\mathcal{H} := L^2(\mathbb{R}) \otimes \mathbb{C}^\ell$ be the Hilbert space of a one-dimensional particle of spin $(\ell - 1)/2$. A system made of *two* of these particles is described in

the Hilbert space

$$\mathcal{H}^{(2)} := \bigotimes_{j=1,2} \mathcal{H}_j \simeq L^2(\mathbb{R}^2) \otimes \mathbb{C}^{\ell^2}$$

where the label $j = 1, 2$ is used to distinguish the two particles. Let

$$\Psi(x_1, x_2) := \begin{pmatrix} \psi_1(x_1, x_2) \\ \vdots \\ \psi_{\ell^2}(x_1, x_2) \end{pmatrix}$$

be a generic vector of $\mathcal{H}^{(2)}$.

The *flip operator* $F : \mathcal{H}^{(2)} \rightarrow \mathcal{H}^{(2)}$ is defined by

$$(F\Psi)(x_1, x_2) := \Psi(x_2, x_1), \quad \Psi \in \mathcal{H}^{(2)}.$$

It is a unitary involution, *i. e.* $F = F^{-1} = F^*$. Clearly the role of F is to exchange particle 1 with particle 2 and vice versa. Sometimes, in the physical literature F is known as the *particle exchange operator*.

The *complex conjugation operator* is naturally defined on $\mathcal{H}^{(2)}$ by

$$(C\Psi)(x_1, x_2) := \overline{\Psi}(x_1, x_2) = \begin{pmatrix} \overline{\psi_1}(x_1, x_2) \\ \vdots \\ \overline{\psi_{\ell^2}}(x_1, x_2) \end{pmatrix}, \quad \Psi \in \mathcal{H}^{(2)}.$$

It is anti-linear and verifies $C = C^{-1} = C^*$. Moreover one can easily check that

$$FC = CF.$$

In the physics literature the operator C implements the time reversal symmetry for bosonic particles.

Let

$$p_j := -i \frac{\partial}{\partial x_j} \otimes \mathbf{1}_{\ell^2} = \begin{pmatrix} -i \frac{\partial}{\partial x_j} & & \\ & \ddots & \\ & & -i \frac{\partial}{\partial x_j} \end{pmatrix}, \quad j = 1, 2$$

be the momentum operator of the j -th particle. In absence of any interaction between the two particles the kinetic (total) energy of the system is described (in a suitable system of physical units) by the Hamiltonian

$$H_0 := p_1^2 + p_2^2.$$

Simple calculations show that

$$F p_j F = p_{j+1}, \quad C p_j C = -p_j, \quad j = 1, 2$$

where $j + 1$ is meant modulo 2. As a consequence, one has that

$$F H_0 F = H_0 = C H_0 C,$$

namely H_0 is left invariant under the independent action of F and C .

Let us suppose now that the kinetic momentum of the particle 1 is changed by a gauge potential A_1 produced by the particle 2 according to $p_1 \mapsto p_1 + A_1$ where $A_1 \in \mathcal{C}(\mathbb{R}) \otimes \text{Mat}_{\ell^2}(\mathbb{C})$ acts as

$$(A_1\Psi)(x_1, x_2) := \begin{pmatrix} a_{1,1}(x_2) & \dots & a_{1,\ell^2}(x_2) \\ \vdots & \ddots & \vdots \\ a_{\ell^2,1}(x_2) & \dots & a_{\ell^2,\ell^2}(x_2) \end{pmatrix} \begin{pmatrix} \psi_1(x_1, x_2) \\ \vdots \\ \psi_{\ell^2}(x_1, x_2) \end{pmatrix},$$

namely as the matrix-valued operator of multiplication by functions $a_{n,m}$ in the *only* variable x_2 . Let us assume that also the kinetic momentum of the particle 2 is changed in a similar way by a gauge potential A_2 produced by the particle 2. The new total energy of the system is then given by

$$H_A := (p_1 + A_1(x_2))^2 + (p_2 + A_2(x_1))^2.$$

The action of F on the gauge potentials is given by

$$FA_1(x_2)F := A_1(x_1), \quad FA_2(x_1)F := A_2(x_2),$$

while C acts as

$$CA_1(x_2)C := \overline{A_1}(x_2), \quad CA_2(x_1)C := \overline{A_2}(x_1).$$

In general H_A is not invariant under the separate action of F and C . However, we are interested in the case in which H_A is left invariant by the composed operator

$$\Theta := FC = CF.$$

By definition Θ is an anti-unitary involution, namely $\Theta = \Theta^{-1} = \Theta^*$. Moreover, a direct calculation shows that

$$\Theta H_A \Theta = (p_1 - \overline{A_2}(x_2))^2 + (p_2 - \overline{A_1}(x_1))^2$$

hence the symmetry condition

$$\Theta H_A \Theta = H_A \tag{2.1}$$

is guaranteed by the constraint

$$A_1 = -\overline{A_2}.$$

In the next sections we will see how a two-dimensional particle in a uniform magnetic field provides a particular example of a system where the symmetry (2.1) is realized.

3. The geometry of the Landau levels

The quantum dynamics of a particle of mass m and charge q (for electrons $q = -e$ with $e > 0$) is generated by the *magnetic* Schrödinger operator

$$H_A := \frac{1}{2m} \left(-i\hbar\nabla - \frac{q}{c} A \right)^2 \tag{3.1}$$

defined on the Hilbert space $L^2(\mathbb{R}^d)$. The Hamiltonian (3.1) is expressed in the CGS system of units: here c is the speed of light and $\hbar := 2\pi\hbar$ is the Planck constant. The vector potential A is responsible for the coupling of the particle with the magnetic field $B := \nabla \times A$. Under quite general assumptions on the vector potential A the operator (3.1) turns out to be self-adjoint with cores $\mathcal{C}_c(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^2)$ given by the compactly

supported continuous functions and the Schwartz functions, respectively [LS, Theorem 3].

3.1. The Landau Hamiltonian. The Landau Hamiltonian H_B is the magnetic Schrödinger operator on $L^2(\mathbb{R}^2)$ with vector potential describing a uniform perpendicular magnetic field B . It is well known that such a vector potential is not unique but needs a choice of gauge. In this work we will use the vector potential in the *symmetric gauge*, namely

$$A_L(x) := \frac{B}{2} e_\perp \wedge x = \frac{B}{2} (-x_2, x_1), \quad x := (x_1, x_2) \in \mathbb{R}^2 \quad (3.2)$$

where $e_\perp := (0, 0, 1)$ is the unit vector orthogonal to the plan \mathbb{R}^2 where the particle is confined and $B \in \mathbb{R}$ describes the strength and the orientation with respect to e_\perp of the magnetic field. The *Landau Hamiltonian* $H_B := H_{A_L}$ is then defined as the two-dimensional magnetic Schrödinger operator (3.1) with the vector potential (3.2). In the following we will assume $B > 0$ which means that the magnetic field is positively oriented with respect to the direction of e_\perp .

From the constants which appear in the definition of H_A and A_L one can define the *magnetic energy*

$$\epsilon_B := \frac{|q|B\hbar}{mc},$$

and the *magnetic length*

$$\ell_B := \sqrt{\frac{c\hbar}{|q|B}}.$$

Henceforward, we will assume $q < 0$ which corresponds to the case of electrons.

With the notation introduced above the Landau Hamiltonian can be written as

$$H_B := \frac{\epsilon_B}{2} (K_1^2 + K_2^2) \quad (3.3)$$

where the (magnetic) *kinetic momenta* K_1 and K_2 are defined by

$$K_1 := \left(-i\ell_B \frac{\partial}{\partial x_1} - \frac{x_2}{2\ell_B} \right), \quad K_2 := \left(-i\ell_B \frac{\partial}{\partial x_2} + \frac{x_1}{2\ell_B} \right). \quad (3.4)$$

The expressions (3.4) define essentially self-adjoint operators with core $\mathcal{C}_c(\mathbb{R}^d)$ (see e. g. [Hal, Proposition 9.40]) Therefore, we will use the symbols K_1 and K_2 to denote the unique self-adjoint extensions of the operator initially defined by (3.3).

The spectral theory of the Landau Hamiltonian H_B is a well established topic [AHS] and it is strictly related with the elementary theory of the harmonic oscillator [Foc, La]. In order to compute the spectrum of H let us introduce the *dual momenta*

$$G_1 := \left(-i\ell_B \frac{\partial}{\partial x_2} - \frac{x_1}{2\ell_B} \right), \quad G_2 := \left(-i\ell_B \frac{\partial}{\partial x_1} + \frac{x_2}{2\ell_B} \right). \quad (3.5)$$

Also the (3.5) defines a pair of self-adjoint operators with core $\mathcal{C}_c(\mathbb{R}^2)$. Moreover, the commutation relations

$$\begin{aligned} [K_1, K_2] &= -i\mathbf{1} = [G_1; G_2] \\ [K_i, G_j] &= 0 \quad i, j = 1, 2 \end{aligned} \quad (3.6)$$

can be easily proved by a direct computation on the core $\mathcal{C}_c(\mathbb{R}^2)$. In view of the (3.6) one can define two pairs of *creation-annihilation operators*

$$a^\pm := \frac{1}{\sqrt{2}} (K_1 \pm iK_2), \quad b^\pm := \frac{-1}{\sqrt{2}} (G_1 \pm iG_2). \quad (3.7)$$

The a^\pm and b^\pm , are closable operators initially defined on the dense domains $\mathcal{C}_c(\mathbb{R}^2) \subset \mathcal{S}(\mathbb{R}^2)$. Moreover, a^- and b^- are the adjoint of a^+ and b^+ , respectively. These operators are subjected to the following commutation rules:

$$[a^\pm, b^\pm] = 0, \quad [a^-, a^+] = \mathbf{1} = [b^-, b^+]. \quad (3.8)$$

Let $\psi_0 \in \mathcal{S}(\mathbb{R}^2)$ be the normalized function

$$\psi_0(x) := \frac{1}{\ell_B \sqrt{2\pi}} e^{-\frac{|x|^2}{4\ell_B^2}}. \quad (3.9)$$

A direct computation shows that $a^-\psi_0 = 0 = b^-\psi_0$. Acting on ψ_0 with the creation operators one defines

$$\psi_n := \frac{1}{\sqrt{n_1!n_2!}} (a^+)^{n_1} (b^+)^{n_2} \psi_0, \quad n := (n_1, n_2) \in \mathbb{N}_0^2. \quad (3.10)$$

Evidently $\psi_n \in \mathcal{S}(\mathbb{R}^2)$ for any $n \in \mathbb{N}_0^2$. Moreover, by a recursive application of the commutation rules (3.8), one can prove that $(\psi_n, \psi_{n'}) = \delta_{n,n'}$. The set $\{\psi_n \mid n \in \mathbb{N}_0^2\} \subset \mathcal{S}(\mathbb{R}^2)$ provides a complete orthonormal system for $L^2(\mathbb{R}^2)$ called (magnetic) *Laguerre basis*. The set of the finite linear combinations of elements of this basis defines a dense subspace $\mathcal{L}_B \subset \mathcal{S}(\mathbb{R}^2)$ which is left invariant by the action of a^\pm and b^\pm . Moreover, a^\pm and b^\pm are closable on \mathcal{L}_B . The normalized eigenfunctions (3.10) can be expressed as [JL, RW]

$$\psi_n(x) := \psi_0(x) \sqrt{\frac{n_1!}{n_2!}} \left[\frac{x_1 + ix_2}{\ell_B \sqrt{2}} \right]^{n_2 - n_1} L_{n_1}^{(n_2 - n_1)} \left(\frac{|x|^2}{2\ell_B^2} \right) \quad (3.11)$$

where

$$L_m^{(\alpha)}(\zeta) := \sum_{j=0}^m \frac{(\alpha + m)(\alpha + m - 1) \dots (\alpha + j + 1)}{j!(m - j)!} (-\zeta)^j, \quad \alpha, \zeta \in \mathbb{R}$$

are the *generalized Laguerre polynomial* of degree m (with the usual convention $0! = 1$) [GR, Sect. 8.97].

By using the definitions (3.7) and the commutation relations (3.6) one obtains after a straightforward calculation that

$$H_B = \epsilon_B \left(a^+ a^- + \frac{1}{2} \mathbf{1} \right) = \epsilon_B \left(a^- a^+ - \frac{1}{2} \mathbf{1} \right). \quad (3.12)$$

The first consequence is that any Laguerre vector ψ_n is an eigenvector of H_B . This implies that the Laguerre basis provides an orthonormal system which diagonalizes H_B according to

$$H_B \psi_n = \epsilon_B \left(n_1 + \frac{1}{2} \right) \psi_n, \quad n = (n_1, n_2) \in \mathbb{N}_0^2.$$

Hence, the spectrum of H_B is a sequence of eigenvalues given by

$$\sigma(H_B) = \left\{ E_j := \epsilon_B \left(j + \frac{1}{2} \right) \mid j \in \mathbb{N}_0 \right\} \quad (3.13)$$

and H_B turns out to be essentially self-adjoint also on the core \mathcal{L}_B ³. We refer to the eigenvalue E_j as the j -th *Landau level*.

Each Landau level is infinitely degenerate. A simple computation shows that the orthogonal component of the angular momentum can be written as

$$\begin{aligned} L_3 &:= -i\hbar \left(x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1} \right) \\ &= \hbar \left(\frac{K_1^2 + K_2^2}{2} - \frac{G_1^2 + G_2^2}{2} \right) = \hbar (a^+ a^- - b^+ b^-). \end{aligned}$$

This implies that $[H_B, L_3] = 0$. Then, the Laguerre functions $\psi_{(n_1, n_2)}$ are simultaneous eigenfunctions of H_B and L_3 with eigenvalues E_{n_1} and $l := \hbar(n_1 - n_2)$, respectively. This shows that the possible eigenvalues of the angular momentum L_3 for a particle in the energy level n_1 are $\ell = \hbar m$ with $-\infty < m \leq n_1$.

Let $\mathcal{H}_j \subset L^2(\mathbb{R}^2)$ be the eigenspace relative to the j -th eigenvalue of H_B . Clearly, \mathcal{H}_j is spanned by $\psi_{(j, m)}$ with $m \in \mathbb{N}_0$ and the spectral projection $\Pi_j : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}_j$ is described (in Dirac notation) by

$$\Pi_j := \sum_{m=0}^{\infty} |\psi_{(j, m)}\rangle \langle \psi_{(j, m)}|. \quad (3.14)$$

One infers from (3.10) the recursive relations

$$\Pi_j = \frac{1}{j} a^+ \Pi_{j-1} a^-, \quad \Pi_j = \frac{1}{j+1} a^- \Pi_{j+1} a^+ \quad (3.15)$$

and after an iteration one gets

$$\Pi_j = \frac{1}{j!} (a^+)^j \Pi_0 (a^-)^j.$$

The *Landau projections* Π_j are integral kernel operators with kernel given by [RW]

$$\Pi_j(x, y) := \frac{1}{2\pi\ell_B^2} e^{-\frac{|x-y|^2}{4\ell_B^2}} e^{-i\frac{x \wedge y}{2\ell_B^2}} L_j^{(0)} \left(\frac{|x-y|^2}{2\ell_B^2} \right) \quad (3.16)$$

where $x \wedge y := x_1 y_2 - x_2 y_1$.

³The closure of the cores $\mathcal{L}_B \subset \mathcal{S}(\mathbb{R}^2)$ or $C_c^\infty(\mathbb{R}^2)$ with respect to the operator graph norm of H_B defines the domain $\mathcal{D}(H_B) \subset L^2(\mathbb{R}^2)$ which is called (*second*) *magnetic Sobolev space* [LL].

3.2. Discrete symmetries of the Landau Hamiltonian. We have already mentioned in Section 2 that the “standard” time reversal symmetry of a spinless particle is implemented by the complex conjugation $C\psi = \bar{\psi}$.

The effect of the operator C on the dynamics of a two-dimensional charged particle subjected to a uniform magnetic field can be deduced by observing that

$$C K_1 C = -G_2, \quad C K_2 C = -G_1 \quad (3.17)$$

which in turn implies

$$C a^\pm C = \pm i b^\mp. \quad (3.18)$$

From the above relations it follows that the Landau Hamiltonian and the angular momentum are not left invariant by the action of C . In particular one has that

$$\begin{aligned} C H_B C &= H_{-B}, \\ C L_3 C &= -L_3, \end{aligned}$$

namely the effect of the transformation implemented by C is to invert the sign of the magnetic field B and of the angular momentum L_3 .

The Landau Hamiltonian, although is not left invariant by the complex conjugation C , admits a generalized TRS in the sense of Definition 1.1. Consider the *flip operator* (cf. Section 2) $F : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ defined by

$$(F\psi)(x_1, x_2) := \psi(x_2, x_1), \quad \psi \in L^2(\mathbb{R}^2). \quad (3.19)$$

Clearly $F = F^{-1} = F^*$, namely F is a linear unitary involution. Moreover F is real in the sense that $CF = FC$. The operator F implements the exchange of the components of position and momentum. This implies that

$$F K_1 F = G_1, \quad F K_2 F = G_2 \quad (3.20)$$

and in turn

$$F a^\pm F = -b^\pm.$$

Let $\Theta := FC$. This is an anti-unitary involution since $\Theta^2 = FCFC = F^2 = \mathbf{1}$. Moreover, by combining together (3.17) and (3.20) one gets

$$\Theta K_1 \Theta = -K_2, \quad \Theta K_2 \Theta = -K_1. \quad (3.21)$$

Equations (3.20) along with (3.3) imply

$$\Theta H_B \Theta = H_B \quad (3.22)$$

and this completes the proof of Proposition 1.1.

Remark 3.1 (Perturbation by potentials). In usual physical applications the Landau Hamiltonian H_B is perturbed by an *electrostatic potential* V (a multiplication operator on $L^2(\mathbb{R}^2)$) which describes the interaction of the electrons with the nuclei arranged in the space. The perturbed Hamiltonian is usually denoted by $H_{B,V} := H_B + V$. Since $H_{B,V}$ is required to be self-adjoint, V has to be real, namely $CV = VC$. Therefore, $H_{B,V}$ has the odd TRS implemented by Θ if and only if V commutes with the flip operator, *i. e.* $FV = VF$. The latter condition is quite strong. For instance a potential of the type

$$V(x_1, x_2) = a \cos(x_1) + b \cos(x_2), \quad a, b \in \mathbb{R}$$

commutes with F if and only if $a = b$. As soon as $a \neq b$ the associated perturbed operator $H_{B,V}$ breaks the generalized TRS induced by Θ . ◀

Remark 3.2 (Spectral projections and noncommutative vector bundles). The spectral projection Π_j associated to the j -th Landau level inherits the same symmetries of the Landau operator. In particular one has that $\Theta\Pi_j\Theta = \Pi_j$ for all $j \in \mathbb{N}_0$. For reasons that will be clarified in Section 3.4, it is appropriate to refer to the pair (Π_j, Θ) as the (non-commutative) “Real” line bundle associated to the j -th Landau level E_j . This description still survives if a suitable, sufficiently weak perturbation V is added. More precisely, let us assume that the Landau Hamiltonian is perturbed by a bounded electrostatic potential V of norm $\|V\| < +\infty$. By observing that the separation between two consecutive Landau levels (gap size) is

$$\Delta E := E_{j+1} - E_j = \epsilon_B$$

one concludes with a simple perturbative argument [Kat, V §-3, Theorem 4.10] that if

$$2\|V\| < \epsilon_B$$

then the points $g_j := j\epsilon_B$, $j \in \mathbb{N}_0$ are still in the resolvent set of $H_{B,V}$. This allows to define a perturbed spectral projection by means of the Riesz-Dunford integral

$$\Pi_j^V := \frac{i}{2\pi} \oint_{C_j} dz \frac{1}{H_{B,V} - z\mathbf{1}}$$

where $C_j := \{g_j + \frac{\epsilon_B}{2}(e^{i\theta} - 1) \mid \theta \in [0, 2\pi]\}$ is the circle in the complex plane which crosses the real axis in g_{j-1} and g_j . When the perturbation V meets the conditions described in Remark 3.1 to ensure that $\Theta H_{B,V} \Theta = H_{B,V}$ it also follows that $\Theta \Pi_j^V \Theta = \Pi_j^V$ and the pair (Π_j^V, Θ) still defines a (non-commutative) “Real” line bundle associated to the j -th perturbed Landau level. However, it is worth to emphasize that the case of admissible electrostatic perturbations is beyond the scope of this work. ◀

3.3. The magnetic Bloch-Floquet-Zak transform. Let us define the family of unitary operators

$$T_m := (-1)^{m_1 m_2} e^{-i\sqrt{2\pi}(m_1 G_2 + m_2 G_1)}, \quad m := (m_1, m_2) \in \mathbb{Z}^2.$$

The commutations relations (3.6) imply that

$$[T_m, H_B] = 0, \quad \forall m \in \mathbb{Z}^2$$

namely the operators T_m are symmetries of the Landau Hamiltonian H_B . Moreover, an application of the Baker-Campbell-Hausdorff formula shows that

$$T_{m+m'} = T_m T_{m'}, \quad \forall m, m' \in \mathbb{Z}^2,$$

namely the mapping $m \mapsto T_m$ provides a unitary representation of \mathbb{Z}^2 on $L^2(\mathbb{R}^2)$ which leaves invariant H_B . An explicit computation provides

$$(T_m \psi)(x) = (-1)^{\sigma(m)} e^{-i\sqrt{\frac{\pi}{2}} \frac{m \wedge x}{\ell_B}} \psi(x - \sqrt{2\pi} \ell_B m), \quad \psi \in L^2(\mathbb{R}^2)$$

where $m \wedge x := m_1 x_2 - m_2 x_1$ and $\sigma(m) = m_1 m_2$. The operators T_1 and T_2 are therefore called *magnetic translations* [Zak1, Zak2].

The \mathbb{Z}^2 -action implemented by the magnetic translations can be used to define the *magnetic* Bloch-Floquet-Zak transform which is a (natural) generalization of the usual Bloch-Floquet transform [Kuc]. For that we need to introduce the magnetic *unit cell*

$$\mathbb{Y}_B := \left[0, \sqrt{2\pi}\ell_B\right]^2$$

and the magnetic *Brillouin torus*

$$\mathbb{B}_B := \mathbb{R}^2 / \left(\frac{\sqrt{2\pi}}{\ell_B}\mathbb{Z}\right)^2 \simeq \left[0, \frac{\sqrt{2\pi}}{\ell_B}\right]^2.$$

Topologically, \mathbb{B}_B is a rescaled version of the standard torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. For every $\psi \in C_c(\mathbb{R}^2)$ let us define the transform

$$(\mathcal{U}_B\psi)_k(\mathbf{y}) := \sum_{m \in \mathbb{Z}^2} e^{-i\mathbf{k} \cdot (\mathbf{y} - \sqrt{2\pi}\ell_B m)} (T_m\psi)(\mathbf{y}) \quad (3.23)$$

where $\mathbf{k} := (k_1, k_2)$, $\mathbf{y} := (y_1, y_2)$ and $\mathbf{k} \cdot \mathbf{y} := k_1 y_1 + k_2 y_2$. From (3.23), one immediately infers the (pseudo-)periodicity properties

$$\begin{cases} T_n(\mathcal{U}_B\psi)_k(\mathbf{y}) = (\mathcal{U}_B\psi)_k(\mathbf{y}) \\ (\mathcal{U}_B\psi)_{k+n\frac{\sqrt{2\pi}}{\ell_B}}(\mathbf{y}) = e^{-i\frac{\sqrt{2\pi}}{\ell_B}n \cdot \mathbf{y}} (\mathcal{U}_B\psi)_k(\mathbf{y}) \end{cases} \quad \forall n \in \mathbb{Z}^2. \quad (3.24)$$

The first of (3.24) can be equivalently written as

$$(\mathcal{U}_B\psi)_k(\mathbf{y} - n\sqrt{2\pi}\ell_B) = e^{i\sqrt{\frac{\pi}{2}}\frac{n_1 y_2 - n_2 y_1}{\ell_B}} (\mathcal{U}_B\psi)_k(\mathbf{y}), \quad \forall n \in \mathbb{Z}^2$$

and shows that the function $(\mathcal{U}_B\psi)_k$ is completely determined by its restriction on the unit cell \mathbb{Y}_B . Equivalently, one can think to $(\mathcal{U}_B\psi)_k$ as an element of the *fiber* Hilbert space

$$\mathfrak{h}_B := \{\phi \in L^2_{\text{loc}}(\mathbb{R}^2) \mid T_n\phi = \phi, \quad \forall n \in \mathbb{Z}^2\}.$$

endowed with the scalar product

$$\langle \phi_1, \phi_2 \rangle_{\mathfrak{h}_B} := \int_{\mathbb{Y}_B} d\mathbf{y} \overline{\phi_1(\mathbf{y})} \phi_2(\mathbf{y}).$$

Clearly, one has the isomorphism of Hilbert spaces

$$\mathfrak{h}_B \simeq L^2(\mathbb{Y}_B). \quad (3.25)$$

From the second of (3.24) one can argue that \mathcal{U}_B establishes a unitary transformation $\mathcal{U}_B : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}_{\text{eq}}^B$ where the space of equivariant functions

$$\mathcal{H}_{\text{eq}}^B := \left\{ \varphi \in L^2_{\text{loc}}(\mathbb{R}^2, \mathfrak{h}_B) \mid \varphi_{k+n\frac{\sqrt{2\pi}}{\ell_B}}(\mathbf{y}) = e^{-i\frac{\sqrt{2\pi}}{\ell_B}n \cdot \mathbf{y}} \varphi_k(\mathbf{y}), \quad \forall n \in \mathbb{Z}^2 \right\}$$

is made into a Hilbert space by the scalar product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_{\text{eq}}^B} := \int_{\mathbb{B}_B} d\mu(k) \langle \varphi_{1k}, \varphi_{2k} \rangle_{\mathfrak{h}_B}$$

where $d\mu(k) := \frac{\ell_B^2}{2\pi} dk$ is the normalized (Haar) measure of \mathbb{B}_B . The Hilbert space $\mathcal{H}_{\text{eq}}^B$ is indeed a direct integral $\int_{\mathbb{B}_B}^{\oplus} d\mu(k) \mathfrak{h}_B$ (see [Dix2, Part II, Chapter 6] for the theory of direct integrals) and in view of the isomorphism

(3.25) one infers that the magnetic Bloch-Floquet-Zak transform defines a unitary map

$$\mathcal{U}_B : L^2(\mathbb{R}^2) \longrightarrow \int_{\mathbb{B}_B}^{\oplus} d\mu(k) L^2(\mathbb{Y}_B) .$$

The main feature of the magnetic Bloch-Floquet-Zak transform is that it diagonalizes the magnetic translations, *i. e.*

$$(\mathcal{U}_B T^m \psi)_k(\mathbf{y}) = e^{-i\sqrt{2\pi}\ell_B k \cdot m} \psi_k(\mathbf{y}) , \quad m \in \mathbb{Z}^2 .$$

Said differently, the operators T^m acquire a fibered representation on the direct integral given by

$$T^m \longmapsto \mathcal{U}_B T^m \mathcal{U}_B^{-1} := \int_{\mathbb{B}_B}^{\oplus} d\mu(k) T^m(k)$$

where $T^m(k)$ acts on the fiber Hilbert space $L^2(\mathbb{Y}_B)$ as the multiplication by the phase $e^{-i\sqrt{2\pi}\ell_B k \cdot m}$.

It is worth noting that the generators G_1 and G_2 of the magnetic translations do not commute with the magnetic translations; indeed $[T_j, G_j] = (-1)^{j+1} \sqrt{2\pi} T_j$, $j = 1, 2$. This implies that the generators G_1 and G_2 are not decomposable operators in the sense that they do not respect the direct integral decomposition induced by the magnetic Bloch-Floquet-Zak transform. The same happens to the operators b^+ and b^- .

3.4. "Real" line bundles associated to Landau levels. The operators K_1 and K_2 , and in turn the raising-lowering operators a^{\pm} and the Landau Hamiltonian H_B , commute with the magnetic translations and so they admit a direct integral decomposition. Let

$$K_j \longrightarrow \mathcal{U}_B K_j \mathcal{U}_B^{-1} := \int_{\mathbb{B}_B}^{\oplus} d\mu(k) K_j(k) , \quad j = 1, 2$$

be the direct integral decomposition of K_j . A direct computation shows that the operators $K_j(k)$ act on a suitable dense domain of $L^2(\mathbb{Y}_B)$ as

$$\begin{aligned} K_1(k) &:= -i\ell_B \frac{\partial}{\partial y_1} - \frac{y_2}{2\ell_B} + \ell_B k_1 , \\ K_2(k) &:= -i\ell_B \frac{\partial}{\partial y_2} + \frac{y_1}{2\ell_B} + \ell_B k_2 . \end{aligned}$$

In view of the linear dependence of $K_j(k)$ by k , it follows that the domain of definition of $K_j(k)$ is independent of k . A similar description holds for the Landau Hamiltonian H_B . One has that

$$H_B \longrightarrow \mathcal{U}_B H_B \mathcal{U}_B^{-1} := \int_{\mathbb{B}_B}^{\oplus} d\mu(k) H_B(k) , \quad j = 1, 2$$

where

$$H_B(k) := \frac{\epsilon_B}{2} (K_1(k)^2 + K_2(k)^2) . \quad (3.26)$$

The domain of $H_B(k)$ turns out to be independent of k and the map

$$\mathbb{B}_B \ni k \longmapsto \frac{1}{H_B(k) - z\mathbf{1}} \in \mathcal{K}(L^2(\mathbb{Y}_B)) , \quad z \in \mathbb{C} \setminus \mathbb{R}$$

is norm-continuous⁴ with values in the algebra of the compact operators $\mathcal{K}(L^2(\mathbb{Y}_B))$. This immediately implies that the spectrum of $H_B(k)$ is purely discrete and changes continuously with respect to k . A general result in the theory of fibered operators states that

$$\sigma(H_B) = \bigcup_{k \in \mathbb{B}_B} \sigma(H_B(k)) .$$

This equality, along with the description of $\sigma(H_B)$ given by (3.13) and the continuity of the spectra of $H_B(k)$ implies that

$$\sigma(H_B(k)) = \sigma(H_B) = \{E_j \mid j \in \mathbb{N}_0\} , \quad \forall k \in \mathbb{T}_B^2 .$$

In other words the energy bands $k \mapsto E_j(k)$ associated to the fibered Hamiltonian $H_B(k)$ are *flat*, namely they are constant and coincide with the Landau levels, *i. e.* $E_j(k) = E_j$. The only missing information concerns the multiplicity of these eigenvalues. To answer this question we need to examine the Landau projections.

The continuity of the resolvent of $H_B(k)$ leads to a continuous family of spectral projections

$$\mathbb{B}_B \ni k \longmapsto \Pi_j(k) := \frac{i}{2\pi} \oint_{C_j} dz \frac{1}{H_B(k) - z\mathbf{1}} \in \mathcal{K}(L^2(\mathbb{Y}_B)) ,$$

where the circles C_j have been defined in Remark 3.2. Moreover, the compactness of $(H_B(k) - z\mathbf{1})^{-1}$ implies that $\Pi_j(k)$ is finite rank, and therefore trace class, for all $k \in \mathbb{B}_B$. By applying functional calculation one gets

$$\Pi_j \longrightarrow \mathcal{U}_B \Pi_j \mathcal{U}_B^{-1} := \int_{\mathbb{B}_B}^{\oplus} d\mu(k) \Pi_j(k) , \quad j \in \mathbb{N}_0 \quad (3.27)$$

where Π_j is the Landau projection (3.14).

Lemma 3.1. *Let $k \mapsto \Pi_j(k)$ be the continuous family of projections associated to the Landau projection Π_j by the Bloch-Floquet-Zak transform. It holds true that*

$$\mathrm{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = 1 , \quad \forall k \in \mathbb{B}_B .$$

Proof. Since the $\Pi_j(k)$ are trace class it follows that $\mathrm{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = r(k) \in \mathbb{N}$ for all $k \in \mathbb{T}_B^2$. However, the continuity of $k \mapsto \Pi_j(k)$ implies the continuity of $k \mapsto r(k)$ which in turn implies $r(k) = r_0 \in \mathbb{N}$ for all $k \in \mathbb{T}_B^2$. Since the measure $d\mu$ is normalized one gets

$$r_0 = \int_{\mathbb{B}_B} d\mu(k) \mathrm{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = 1$$

where the second equality will be proved in Lemma 3.2 and Corollary 3.2. \square

Since the multiplicity of $E_j(k)$ is measured by the trace of $\Pi_j(k)$ one immediately gets:

Corollary 3.1. *For each $k \in \mathbb{B}_B$ and $j \in \mathbb{N}_0$ the Landau level $E_j(k) = E_j$ is a non degenerate eigenvalue of the fiber Hamiltonian $H_B(k)$.*

⁴Indeed one can prove that the mapping $k \mapsto H_B(k)$ defines an entire analytic family in the sense of Kato with compact resolvent. For more details we refer to [MPPT, Section 3.3].

The the map $k \mapsto \Pi_j(k)$ defines a complex vector bundle $\mathcal{E}_j \rightarrow \mathbb{T}^2 \simeq \mathbb{B}_B$ with total space given by

$$\mathcal{E}_j := \bigsqcup_{k \in \mathbb{T}_B^2} \text{Ran}[\Pi_j(k)] . \quad (3.28)$$

The construction of the *spectral bundle* (also called Bloch-bundle [Pan]) $\mathcal{E}_j \rightarrow \mathbb{T}^2$ is standard (see [DL, Lemma 4.5] or [DG1, Section 2] for more details) and provides a manifestation of the Serre-Swan duality. In view of Lemma 3.1 one infers that

$$\dim(\text{Ran}[\Pi_j(k)]) = \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = 1 ,$$

namely the \mathcal{E}_j are *line bundles*. We will refer to \mathcal{E}_j as the j -th *Landau line bundle*.

The complex conjugation C and the flip operator F do not have a nice behavior under the Bloch-Floquet-Zak transform \mathcal{U}_B . However, their composition $\Theta := FC$ acts in a nice way. Indeed from (3.17) and (3.20) one deduces $\Theta T_1 \Theta = T_2$. By combining this relation with the definition (3.23) one gets

$$\Theta[(\mathcal{U}_B \psi)_k] = (\mathcal{U}_B \Theta \psi)_{\mathfrak{f}(k)} = (\mathcal{U}_B F \overline{\psi})_{\mathfrak{f}(k)} , \quad \psi \in L^2(\mathbb{R}^2)$$

where $\mathfrak{f} : \mathbb{B}_B \rightarrow \mathbb{B}_B$ is the *flip-involution* on the torus $\mathbb{B}_B \simeq \mathbb{T}^2$ described in Definition 1.2. Then, with a little abuse of notation, we can think of Θ as an anti-unitary map which acts on the direct integral intertwining the fiber over k with the fiber over $\mathfrak{f}(k)$.

From the TRS invariance (3.22) one can reconstruct the action of Θ on the fiber Hamiltonians $H_B(k)$ defined by (3.26). A straightforward calculation shows that

$$\Theta H_B(k) \Theta = H_B(\mathfrak{f}(k)) , \quad \forall k \in \mathbb{B}_B . \quad (3.29)$$

By functional calculus one concludes that the same relation is inherited by the Landau projections, *i. e.*

$$\Theta \Pi_j(k) \Theta = \Pi_j(\mathfrak{f}(k)) , \quad \forall k \in \mathbb{B}_B . \quad (3.30)$$

This symmetry translates at level of the Landau line bundle \mathcal{E}_j and defines a homeomorphism of the total space still (with a little abuse of notation) denoted by $\Theta : \mathcal{E}_j \rightarrow \mathcal{E}_j$ with the property that Θ is an anti-linear involution between the conjugate fibers over k and $\mathfrak{f}(k)$, *i. e.*

$$\begin{cases} \Theta : \mathcal{E}_j|_k \longrightarrow \mathcal{E}_j|_{\mathfrak{f}(k)} \\ \Theta^2|_k = \text{Id}_{\mathcal{E}_j} \end{cases} , \quad \forall k \in \mathbb{B}_B .$$

A complex vector bundle endowed with such a symmetry is called “Real” [Ati1, DG1]. Summing up the considerations stated above, and using the topological identification $\mathbb{B}_B \simeq \mathbb{T}^2$, we can conclude that:

Proposition 3.1. *To each Landau level E_j of the Landau Hamiltonian H_B is associated a “Real” line bundle (\mathcal{E}_j, Θ) over the involutive torus $(\mathbb{T}^2, \mathfrak{f})$.*

3.5. The topology of the Landau levels. In view of Proposition 3.1 the topological properties of the j -th Landau level E_j can be read from the “Real” line bundle (\mathcal{E}_j, Θ) .

Let $\text{Vec}_{\mathfrak{R}}^m(\mathbb{T}^2, \mathfrak{f})$ be the set of equivalence classes of rank m “Real” vector bundles over the involutive torus $(\mathbb{T}^2, \mathfrak{f})$. Proposition 3.1 implies that each Landau level E_j defines an element of $\text{Vec}_{\mathfrak{R}}^1(\mathbb{T}^2, \mathfrak{f})$. Therefore, the study of the topological properties of E_j amounts to the topological classification of $\text{Vec}_{\mathfrak{R}}^m(\mathbb{T}^2, \mathfrak{f})$. The latter is provided by the following crucial result:

Proposition 3.2. *Let \mathbb{T}^2 be a two dimensional torus endowed with the flip involution \mathfrak{f} . There are isomorphisms*

$$\text{Vec}_{\mathfrak{R}}^1(\mathbb{T}^2, \mathfrak{f}) \xrightarrow{c_1^{\mathfrak{R}}} H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \xrightarrow{\iota} H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$$

where ι is the map which forgets the \mathbb{Z}_2 action of the involution and $c_1 = \iota \circ c_1^{\mathfrak{R}}$ coincides with the (usual) Chern class.

The first isomorphism $\text{Vec}_{\mathfrak{R}}^1(\mathbb{T}^2, \mathfrak{f}) \simeq H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1))$ is known as Kahn’s isomorphism. It has been proved in [Kah]; see also [DG1, Section 5.2] and [Gom1, Appendix A]. The map $c_1^{\mathfrak{R}}$ establishing the isomorphism is called (first) “Real” Chern class and the target space $H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1))$ is the second equivariant cohomology group of the involutive space $(\mathbb{T}^2, \mathfrak{f})$ with local system of coefficient $\mathbb{Z}(1)$ (for an introduction to the equivariant cohomology we refer to [DG1, Section 5.1] and references therein). The last isomorphism $H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$ is very well known in literature. The equality $c_1 = \iota \circ c_1^{\mathfrak{R}}$ follows from the definition of $c_1^{\mathfrak{R}}$ [Kah]. The only missing new ingredient to complete the proof of Proposition 3.2 is the justification of the second isomorphism established by the forgetting map ι . This is supplied by Lemma A.1.

Proposition 3.2 says that each complex line bundle over \mathbb{T}^2 admits a unique (up to isomorphisms) “Real” structure compatible with the flip involution \mathfrak{f} . As a consequence, since the “Real” structure does not introduce any additional topological information, the (usual) Chern class provides a complete topological characterization for the line bundle.

We are now in position to sketch the proof of Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.1 we know that each Landau level E_j defines a “Real” line bundle (\mathcal{E}_j, Θ) over $(\mathbb{T}_B^2, \mathfrak{f})$, and therefore an element of $\text{Vec}_{\mathfrak{R}}^1(\mathbb{T}^2, \mathfrak{f})$. Proposition 3.2 clarifies that the topology of the “Real” line bundle (\mathcal{E}_j, Θ) is completely described by the Chern class $c_1(\mathcal{E}_j) \in \mathbb{Z}$ of the underlying complex line bundle. The computation $c_1(\mathcal{E}_j) = 1$ for all $j \in \mathbb{N}_0$ is not new in literature (see e.g. [BES, Lemma 5] or [Kun]). However, we will present a different computation in Section 3.7. \square

3.6. The rank of the Landau projections. The purpose of this section is to complete the proof of Lemma 3.1 which provides the computation of the rank of the (complex) vector bundle \mathcal{E}_j associated to the Landau

projection Π_j by the construction described in Section 3.4. Indeed, we will do a little more, proving the first of formulas (1.6), *i. e.*

$$\text{rk}(\mathcal{E}_j) = \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1}\Pi_j) = 1. \quad (3.31)$$

The symbol Tr_{Dix} in (3.31) denotes the *Dixmier trace* (see Appendix B, and references therein for a crash course on the subject). The operator $Q_{B,\xi}^{-1}$ is the resolvent of

$$Q_B := \frac{1}{2} (K_1^2 + K_2^2 + G_1^2 + G_2^2) \quad (3.32)$$

where the operators K_j and G_j are defined by (3.4). From (3.7) one obtains that

$$Q_B = a^+a^- + b^+b^- + 2\mathbf{1}.$$

The last equality suggests that Q_B is a self-adjoint operator diagonalized by the Laguerre basis (3.10). Indeed one has that

$$Q_B\psi_{(n,m)} = (n+m+2)\psi_{(n,m)}, \quad \forall (n,m) \in \mathbb{N}_0^2.$$

Then, Q_B has a pure point positive spectrum given by

$$\sigma(Q_B) = \{\lambda_j := j+2 \mid j \in \mathbb{N}_0\}$$

and every eigenvector λ_j has a finite multiplicity $\text{Mult}[\lambda_j] = j+1$. The eigenspace associated to λ_j is spanned by $\{\psi_{(n,m)} \mid n+m=j\}$. Finally

$$Q_{B,\xi}^{-s} := (Q_B + 2\xi\mathbf{1})^{-s} \in \mathcal{K}(L^2(\mathbb{R}^2)) \quad (3.33)$$

is a compact operator for every $s > 0$ and $\xi \geq 0$.

Remark 3.3 (Relation with the harmonic oscillator). Starting from (3.4) and (3.5), one can rewrite Q_B in the form (1.4), namely as a two-dimensional harmonic oscillators in the dimensionless variable $x_j/\sqrt{2}\ell_B$ and frequency $\omega = 1/\sqrt{2}$. \blacktriangleleft

Lemma B.4 states that $Q_{B,\xi}^{-s}$ is trace class for for all $s > 2$ and that $Q_{B,\xi}^{-2}$ is a measurable operator in the Dixmier ideal. However, this measurability properties change when Q_B^{-s} is multiplied by the Landau projection Π_j . Lemma B.5 shows that $Q_{B,\xi}^{-s}\Pi_j$ is trace class for for all $s > 1$ and that $Q_{B,\xi}^{-1}\Pi_j$ is a measurable operator in the Dixmier ideal. In particular Lemma B.5 (ii) provides the proof of the second equality in (3.31). Therefore, the remaining part of this section will be devoted to the proof of the first equality in (3.31).

The rank of the vector bundle \mathcal{E}_j can be computed as

$$\text{rk}(\mathcal{E}_j) = \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = \int_{\mathbb{B}_B} d\mu(k) \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k))$$

in view of the independence of $\text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k))$ and the normalization of the measure $d\mu(k)$. The next result needs the trace per unit volume \mathcal{T}_B defined in Appendix B.3.

Lemma 3.2. *Let Π_j be the j -th Landau projection. It holds true that*

$$\int_{\mathbb{Y}_B} d\mu(k) \operatorname{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) = |\mathbb{Y}_B| \mathcal{T}_B(\Pi_j),$$

where the proportionality factor is the volume of the unit cell \mathbb{Y}_B .

Proof. Let $\chi_{\mathbb{Y}_B}$ be the characteristic function of the unit cell \mathbb{Y}_B . The set of functions $g_m(x) := \frac{1}{2\sqrt{\pi}\ell_B} \chi_{\mathbb{Y}_B}(x) e^{i\frac{\sqrt{\pi}}{\ell_B} m \cdot x}$ provides an orthonormal basis of $L^2(\mathbb{Y}_B)$. Moreover, one has that

$$\operatorname{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\mathbb{Y}_B} \Pi_j \chi_{\mathbb{Y}_B}) = \sum_{m \in \mathbb{Z}^2} \langle g_m, \Pi_j g_m \rangle_{L^2(\mathbb{R}^2)}.$$

in view of Lemma B.7. Since the Bloch-Floquet-Zak is unitary it follows that

$$\langle g_m, \Pi_j g_m \rangle_{L^2(\mathbb{R}^2)} = \int_{\mathbb{B}_B} d\mu(k) \langle (\mathcal{U}_B g_m)_k, \Pi_j(k) (\mathcal{U}_B g_m)_k \rangle_{L^2(\mathbb{Y}_B)}$$

with $(\mathcal{U}_B g_m)_k(y) := \frac{1}{2\sqrt{\pi}\ell_B} e^{i(\frac{\sqrt{\pi}}{\ell_B} m - k) \cdot y}$, after a straightforward calculation. For every fixed $k \in \mathbb{T}_B^2$ the vectors $(\mathcal{U}_B g_m)_k$ provides an orthonormal basis of $L^2(\mathbb{Y}_B)$. Since $\Pi_j(k)$ is finite rank, hence trace class, for all $k \in \mathbb{T}_B^2$ (see Section 3.4) one gets the equality

$$\operatorname{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\mathbb{Y}_B} \Pi_j \chi_{\mathbb{Y}_B}) = \int_{\mathbb{B}_B} d\mu(k) \operatorname{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)). \quad (3.34)$$

The claim follows by observing that

$$\mathcal{T}_B(\Pi_j) = \frac{1}{|\chi_{\mathbb{Y}_B}|} \operatorname{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\mathbb{Y}_B} \Pi_j \chi_{\mathbb{Y}_B})$$

in view of Lemma B.7 and definition B.20. \square

Corollary 3.2. *The rank of the vector bundle \mathcal{E}_j associated to the Landau projection Π_j is given by de formula (3.31).*

Proof. From Lemma 3.2 it follows that $\operatorname{rk}(\mathcal{E}_j) = |\mathbb{Y}_B| \mathcal{T}_B(\Pi_j)$ and from Theorem B.2 one gets $\operatorname{rk}(\mathcal{E}_j) = \operatorname{Tr}_{\operatorname{Dix}}(Q_{B,\xi}^{-1} \Pi_j)$ since $|\mathbb{Y}_B| = 2\Omega_B$. The last equality follows from Lemma B.5 (ii). \square

3.7. The Chern numbers of the Landau projections. In this section we want to compute the Chern class of the (complex) line bundle \mathcal{E}_j associated to the Landau projection Π_j . More precisely, we will prove the second of formulas (1.6), *i. e.*

$$c_1(\mathcal{E}_j) = \frac{i}{\ell_B^2} \operatorname{Tr}_{\operatorname{Dix}}(Q_{B,\xi}^{-1} \Pi_j [\partial_1(\Pi_j), \partial_2(\Pi_j)]) = 1 \quad (3.35)$$

completing, in this way, the proof of Theorem 1.1.

The spectral line bundle \mathcal{E}_j is defined through the family of projections $\kappa \mapsto \Pi_j$ and in this case it is well known that the two-form which represents the Chern class of \mathcal{E}_j is given by

$$\tilde{c}_1(\mathcal{E}_j) := \frac{i}{2\pi} R_j(k) dk$$

where $dk := dk_1 \wedge dk_2$ is the (non normalized) volume-form and

$$R_j(k) := \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k) [\partial_{k_1} \Pi_j(k), \partial_{k_2} \Pi_j(k)]) \quad (3.36)$$

is the trace of the Grassmann-Berry curvature (see [DG3, Section II.F] and [Tau, Section 12.5]). The Chern number of \mathcal{E}_j is then given by

$$c_1(\mathcal{E}_j) = \int_{\mathbb{B}_B} \tilde{c}_1(\mathcal{E}_j) = \frac{i2\pi}{|\mathbb{Y}_B|} \int_{\mathbb{B}_B} d\mu(k) R_j(k) \quad (3.37)$$

where in the last equality the normalized measure $d\mu(k)$ and the relation $|\mathbb{B}_B| |\mathbb{Y}_B| = (2\pi)^2$ have been used.

The next task is to rewrite the formula for $c_1(\mathcal{E}_j)$ in the position space $L^2(\mathbb{R}^2)$. Let X_1 and X_2 be the position operators on $L^2(\mathbb{R}^2)$ and Π_j the j -th Landau projection. The two commutators $\partial_i(\Pi_j) := -i[X_i, \Pi_j]$, $i = 1, 2$, define integral operators with kernels

$$\partial_i(\Pi_j)(x, y) = -i(x_i - y_i) \Pi_j(x, y) \quad (3.38)$$

where $\Pi_j(x, y)$ is the kernel (3.16). It turns out that $\partial_i(\Pi_1)$ and $\partial_i(\Pi_2)$ are bounded operators that commute with the magnetic translations. Therefore, they can be decomposed by means of transformation the magnetic Bloch-Floquet-Zak transform. An explicit calculation provides

$$\partial_i(\Pi_j) \longmapsto \mathcal{U}_B \partial_i(\Pi_j) \mathcal{U}_B^{-1} := \int_{\mathbb{B}_B}^{\oplus} d\mu(k) \partial_{k_i} \Pi_j(k). \quad (3.39)$$

Lemma 3.3. *Let Π_j be the j -th Landau projection. The bounded operator $\Pi_j[\partial_1(\Pi_j), \partial_2(\Pi_j)]$ admits the trace per unit volume and*

$$c_1(\mathcal{E}_j) = i2\pi \mathcal{T}_B(\Pi_j[\partial_1(\Pi_j), \partial_2(\Pi_j)]). \quad (3.40)$$

Proof. Let $\mathcal{R}_j := \Pi_j[\partial_1(\Pi_j), \partial_2(\Pi_j)]$. Since $\partial_1(\Pi_j)$ and $\partial_2(\Pi_j)$ are bounded and invariant under magnetic translations and the definition it follows that also \mathcal{R}_j meets the same properties. From Lemma B.7 one deduces that $\chi_{\mathbb{Y}_B} \Pi_j[\partial_1(\Pi_j), \partial_2(\Pi_j)] \chi_{\mathbb{Y}_B}$ is trace class. The invariance under magnetic translations, along with the definition (B.20), justifies the equality

$$\mathcal{T}_B(\mathcal{R}_j) = \frac{1}{|\mathbb{Y}_B|} \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\mathbb{Y}_B} \mathcal{R}_j \chi_{\mathbb{Y}_B}).$$

The same argument used to prove equation (3.34) can be used to obtain

$$\text{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\mathbb{Y}_B} \mathcal{R}_j \chi_{\mathbb{Y}_B}) = \int_{\mathbb{B}_B} d\mu(k) \text{Tr}_{L^2(\mathbb{Y}_B)}(\mathcal{R}_j(k))$$

where $\mathcal{R}_j(k)$ is the Bloch-Floquet-Zak decomposition of the operator \mathcal{R}_j . A comparison between the definition of \mathcal{R}_j , the Bloch-Floquet-Zak decomposition of $\partial_i(\Pi_j)$ provided by (3.3), and the definition of $R_j(k)$ in (3.36) gives that $R_j(k) = \text{Tr}_{L^2(\mathbb{Y}_B)}(\mathcal{R}_j(k))$. This implies that

$$\mathcal{T}_B(\mathcal{R}_j) = \frac{1}{|\mathbb{Y}_B|} \int_{\mathbb{B}_B} d\mu(k) R_j(k).$$

By putting together the latter equation with (3.37) one obtains equation (3.40). \square

Corollary 3.3. *The Chern class of the vector bundle \mathcal{E}_j associated to the Landau projection Π_j is given by formula (3.35).*

Proof. Let us start simplifying the expression of the operator \mathcal{R}_j introduced in the proof of Lemma 3.40. The position operators can be expressed as $X_1 = \ell_B(K_2 - G_1)$ and $X_2 = \ell_B(G_2 - K_1)$. Since the Landau projections commute with G_1 and G_2 one gets that $\partial_1(\Pi_j) = -i\ell_B[K_2, \Pi_j]$ and $\partial_2(\Pi_j) = i\ell_B[K_1, \Pi_j]$. Since $K_1 = \frac{1}{\sqrt{2}}(a^+ + a^-)$ and $K_2 = \frac{1}{i\sqrt{2}}(a^+ - a^-)$ one obtains

$$\begin{aligned}\partial_1(\Pi_j) &= -\frac{\ell_B}{\sqrt{2}} ([a^+, \Pi_j] - [a^-, \Pi_j]) \\ \partial_2(\Pi_j) &= i\frac{\ell_B}{\sqrt{2}} ([a^+, \Pi_j] + [a^-, \Pi_j])\end{aligned}\tag{3.41}$$

and in turn

$$[\partial_1(\Pi_j), \partial_2(\Pi_j)] = -i\ell_B^2 [a^+, \Pi_j], [a^-, \Pi_j] .$$

The relations (3.15) along with the orthogonality of the Landau projections and the commutation relations $[a^-, a^+] = \mathbf{1}$ provide

$$[\partial_1(\Pi_j), \partial_2(\Pi_j)] = -i\ell_B^2 (\Pi_j + (j-1)\Pi_{j-1} - (j+1)\Pi_{j+1}) .$$

Exploiting again the orthogonality of the projections one gets $\mathcal{R}_j = -i\ell_B^2 \Pi_j$. It turns out that \mathcal{R}_j is in the ideal where Theorem B.2 holds true. Therefore, from Lemma 3.40 and Lemma B.5 (ii) we deduce formula (3.35). \square

Remark 3.4 (An integral identity). The operator \mathcal{R}_j defined in the proof of Lemma 3.3 has an integral kernel which can be explicitly computed from (3.16) and (3.38). Along the diagonal $x = y$ the kernel reads

$$\varrho_j(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} dy dz [x \wedge z + z \wedge y + y \wedge x] \Pi_j(x, y) \Pi_j(y, z) \Pi_j(z, x) .$$

On the other hand, in view of the equality $\mathcal{R}_j = -i\ell_B^2 \Pi_j$, one gets $\varrho_j(x) = \frac{-i}{2\pi}$. This leads to the following integral identity

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} dy dz f_x(y, z) e^{i f_x(y, z)} \Psi_j(x - y) \Psi_j(y - z) \Psi_j(z - x) = \frac{\pi^2}{2i}$$

where $f_x(y, z) := x \wedge z + z \wedge y + y \wedge x$ and $\Psi_j(x) := e^{-\frac{|x|^2}{2}} L_j^{(0)}(|x|)$. \blacktriangleleft

4. The geometry of the non-Abelian Landau levels

The magnetic Hamiltonian with non-Abelian magnetic field acts on the Hilbert space $L^2(\mathbb{R}^2) \otimes \mathbb{C}^{2s+1}$ which describes particles of spin s and is defined in a similar way to (3.1) by

$$\mathcal{H}_{\mathcal{A}} := \frac{1}{2m} \left(-i\hbar \nabla \otimes \mathbf{1}_{2s+1} - \frac{q}{c} \mathcal{A} \right)^2 \tag{4.1}$$

where $\mathbf{1}_{2s+1}$ is the identity matrix acting on \mathbb{C}^{2s+1} and \mathcal{A} is a *non-abelian* vector potential. In this paper we will focus on vector potential of the form

$$\mathcal{A}(x_1, x_2) := A(x_1, x_2) \otimes \mathbf{1}_{2s+1} + b \mathbf{1}_{L^2} \otimes (\gamma_1, \gamma_2) , \tag{4.2}$$

where A is a usual vector potential which generates an orthogonal magnetic field $B = \nabla \times A$, $\mathbf{1}_{L^2}$ is the identity operator on the coordinate space $L^2(\mathbb{R}^2)$, b a coupling constant (with the dimension of a magnetic field times a length), and $\gamma_1, \gamma_2 \in \text{Mat}_{2s+1}(\mathbb{C})$ two hermitian matrices. We will

use the short notation $\mathbf{1} := \mathbf{1}_{L^2} \otimes \mathbf{1}_{2s+1}$ for the identity operator on the full space. The non-abelian magnetic field (or curvature) associated with \mathcal{A} is given by the equation

$$\mathcal{B} := \nabla \times \mathcal{A} - \frac{i}{\hbar} \mathcal{A} \times \mathcal{A}.$$

We will refer to [EHS] (and references therein) for more details about non-Abelian magnetic fields. We just notice that with the choice (4.2) The non-abelian magnetic field turns out to be orthogonal, *i. e.* $\mathcal{B} = (0, 0, \mathcal{B}_\perp)$ with

$$\mathcal{B}_\perp := B \otimes \mathbf{1}_2 - i \frac{b^2}{\hbar} \mathbf{1}_{L^2} \otimes [\gamma_1, \gamma_2]. \quad (4.3)$$

4.1. The non-Abelian Landau Hamiltonian. Hereafter we will assume that the Abelian part of the vector potential which enters in the definition of the non-Abelian magnetic Hamiltonian (4.1) is given by the potential A_L defined in (3.2). We will denote with the symbol $H_{B,b}(\gamma_1, \gamma_2)$ the related non-Abelian Landau Hamiltonian.

The introduction of the *non-Abelian kinetic momenta*

$$\begin{aligned} \mathcal{K}_1 &:= K_1 \otimes \mathbf{1}_{2s+1} - c_B \mathbf{1}_{L^2} \otimes \gamma_1 \\ \mathcal{K}_2 &:= K_2 \otimes \mathbf{1}_{2s+1} - c_B \mathbf{1}_{L^2} \otimes \gamma_2, \end{aligned} \quad (4.4)$$

with K_1 and K_2 being defined by (3.4) and

$$c_b := \frac{b}{Bl_B}$$

allows to write

$$\mathcal{H}_{B,b}(\gamma_1, \gamma_2) = \frac{\epsilon_B}{2} (\mathcal{K}_1^2 + \mathcal{K}_2^2). \quad (4.5)$$

A simple computation shows that

$$\mathcal{H}_{B,b}(\gamma_1, \gamma_2) = H_B \otimes \mathbf{1}_{2s+1} + c_b \mathcal{W}_1 + c_b^2 \mathcal{W}_2$$

where H_B is the Landau Hamiltonian described in Section 3.1, and

$$\mathcal{W}_1 := -\epsilon_B (K_1 \otimes \gamma_1 + K_2 \otimes \gamma_2), \quad \mathcal{W}_2 := \frac{\epsilon_B}{2} \mathbf{1}_{L^2} \otimes (\gamma_1^2 + \gamma_2^2)$$

are perturbations.

The non-Abelian kinetic momenta obey the following commutation relation

$$[\mathcal{K}_1, \mathcal{K}_2] = -i \mathbf{1} + c_b^2 \mathbf{1}_{L^2} \otimes [\gamma_1, \gamma_2].$$

They have a “canonical” commutation relation when

$$[\gamma_1, \gamma_2] = -i \delta \mathbf{1}_{2s+1}, \quad \delta \in \mathbb{R}.$$

The dual momenta $G_1 \otimes \mathbf{1}_{2s+1}$ and $G_2 \otimes \mathbf{1}_{2s+1}$ commute with \mathcal{K}_1 , \mathcal{K}_2 and consequently with $H_{B,b}(\gamma_1, \gamma_2)$. This implies that the magnetic translations $T_m \otimes \mathbf{1}_{2s+1}$ are symmetries of the Hamiltonian $\mathcal{H}_{B,b}(\gamma_1, \gamma_2)$. As a consequence, the magnetic Bloch-Floquet-Zak transform $\mathcal{U}_B \otimes \mathbf{1}_{2s+1}$ decomposes $\mathcal{H}_{B,b}(\gamma_1, \gamma_2)$ in a family of Hamiltonians acting on the fiber Hilbert space $L^2(\mathbb{Y}_B) \otimes \mathbb{C}^{2s+1}$ and parametrized by the points of the Brillouin torus \mathbb{B}_B . The details of the construction can be recovered from Section 3.3.

Even the construction of Section 3.4 can be extended to the non-Abelian case. More precisely every isolated spectral region $\Sigma \subset \sigma(\mathcal{H}_{B,b}(\gamma_1, \gamma_2))$ separated from the rest of the spectrum by gaps defines a spectral projection \mathcal{P}_Σ and in turn a projection-valued map $\mathbb{B}_B \ni k \mapsto \mathcal{P}_\Sigma(k)$. The latter provides a vector bundle $\mathcal{E}_\Sigma \rightarrow \mathbb{B}_B$ according to the prescription (3.28). In the next sections we will study the topology of the spectral bundles obtained from two distinct models of non-Abelian Landau Hamiltonians.

In the following we will focus our attention on the special (but interesting) case of particles with spin $s = \frac{1}{2}$. In this case a basis of the algebra $\text{Mat}_2(\mathbb{C})$ is given by the identity $\mathbf{1}_2$ and the three Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

4.2. The Jaynes-Cummings model. This model corresponds to the choice of

$$\gamma_1 = -\sigma_2, \quad \gamma_2 = +\sigma_1$$

in the non-abelian vector potential (4.2). The associated orthogonal part of the non-Abelian magnetic field is given by

$$\mathcal{B}_\perp = B \otimes \mathbf{1}_2 - i \frac{b^2}{\hbar} \mathbf{1}_{L^2} \otimes [\sigma_1, \sigma_2] = \mathbf{1}_{L^2} \otimes \left(B \mathbf{1}_2 + 2 \frac{b^2}{\hbar} \sigma_z \right).$$

according to formula (4.3). This magnetic field is a $\mathbb{U}(2)$ matrix, and is a superposition of a $\mathbb{U}(1)$ field $B \mathbf{1}_2$ and a $\mathbb{S}\mathbb{U}(2)$ field $2 \frac{b^2}{\hbar} \sigma_z$ and for this reason one refers to it as a $\mathbb{U}(1) \times \mathbb{S}\mathbb{U}(2)$ gauge field (see e. g. [PP]).

The resulting non-Abelian Landau Hamiltonian is given by

$$\mathcal{H}_{JC} = H_B \otimes \mathbf{1}_2 + c_b \epsilon_B \mathcal{W}_{JC} + c_b^2 \epsilon_B \mathbf{1} \quad (4.6)$$

with the perturbation given by

$$\begin{aligned} \mathcal{W}_{JC} &= (K_1 \otimes \sigma_2 - K_2 \otimes \sigma_1) \\ &= i a^+ \otimes \left(\frac{\sigma_1 - i \sigma_2}{\sqrt{2}} \right) - i a^- \otimes \left(\frac{\sigma_1 + i \sigma_2}{\sqrt{2}} \right). \end{aligned}$$

The first equality says that \mathcal{W}_{JC} is the *Rashba spin-orbit coupling* [WV, Zha] while the second equality shows that \mathcal{W}_{JC} can be interpreted as the celebrated *Jaynes-Cummings potential* [Sho]. The latter observation justifies the use of the expression Jaynes-Cummings model for the Hamiltonian H_{JC} .

The Jaynes-Cummings model can be solved exactly [WV, Zha, EHS, JZM]. In matricial form the Hamiltonian (4.6) reads

$$\mathcal{H}_{JC} = \epsilon_B \begin{pmatrix} a^+ a^- + \frac{1}{2} (1 + 2c_b^2) \mathbf{1} & -i \sqrt{2} c_b a^- \\ i \sqrt{2} c_b a^+ & a^+ a^- + \frac{1}{2} (1 + 2c_b^2) \mathbf{1} \end{pmatrix}.$$

Let us introduce the family of vectors

$$\Phi_{(j,m)}^\pm := \begin{pmatrix} \sin(\theta_j^\pm) \psi_{(j-1,m)} \\ i \cos(\theta_j^\pm) \psi_{(j,m)} \end{pmatrix}, \quad (j, m) \in \mathbb{N}_0^2$$

where the $\psi_{(j,m)}$ are the elements of the Laguerre basis (3.11) (with the convention $\psi_{(-1,m)} = 0$) and the angles θ_j^\pm are defined by the relation

$$\tan(\theta_j^\pm) := \frac{\sqrt{8c_b^2 j}}{1 \pm \sqrt{1 + 8c_b^2 j}}.$$

A direct computation shows that the family $\Phi_{(j,m)}^\pm$ defines a complete orthonormal system in $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ which diagonalizes the Jaynes-Cummings model. Indeed, it holds that $\mathcal{H}_{JC} \Phi_{(j,m)}^\pm = E_j^\pm \Phi_{(j,m)}^\pm$ where the eigenvalues are given by the formula

$$\begin{cases} E_0 := \epsilon_B \left(\frac{1}{2} + c_b^2 \right) & \text{if } j = 0 \\ E_j^\pm := \epsilon_B \left(j \pm \frac{1}{2} \sqrt{1 + j8c_b^2} + c_b^2 \right) & \text{if } j > 0. \end{cases} \quad (4.7)$$

Therefore, the spectrum of \mathcal{H}_{JC} is pure point and each eigenvalue is infinitely degenerate as a consequence of the commutation relation with the dual momenta $G_1 \otimes \mathbf{1}_2$ and $G_2 \otimes \mathbf{1}_2$. Every eigenvalue E_j^\pm defines a spectral projections \mathcal{P}_j^\pm given by the formula

$$\mathcal{P}_j^\pm = \begin{pmatrix} \sin(\theta_j^\pm)^2 \Pi_{j-1} & -i \frac{\sin(\theta_j^\pm) \cos(\theta_j^\pm)}{\sqrt{j}} a^- \Pi_j \\ i \frac{\sin(\theta_j^\pm) \cos(\theta_j^\pm)}{\sqrt{j}} \Pi_j a^+ & \cos(\theta_j^\pm)^2 \Pi_j \end{pmatrix} \quad (4.8)$$

where the Π_j are the Landau projections (3.14).

The Hamiltonian \mathcal{H}_{JC} has a relevant discrete symmetry. Let us define the *twisted flip operator*

$$\mathcal{F} := F \otimes \vartheta = \begin{pmatrix} F & 0 \\ 0 & -iF \end{pmatrix}$$

where F is the flip operator defined by (3.19) and $\vartheta \in \text{Mat}_2(\mathbb{C})$ is (up to a phase) the unitary operator that meets the relations $\vartheta \sigma_1 \vartheta^{-1} = -\sigma_2$ and $\vartheta \sigma_2 \vartheta^{-1} = \sigma_1$. Starting from the matricial form of the kinetic momenta

$$\mathcal{K}_1 = \begin{pmatrix} K_1 & -i c_b \\ +i c_b & K_1 \end{pmatrix}, \quad \mathcal{K}_2 = \begin{pmatrix} K_2 & -c_b \\ -c_b & K_2 \end{pmatrix},$$

and with the help of the relations (3.20), one easily verifies that

$$\mathcal{F} \mathcal{K}_1 \mathcal{F}^{-1} = \begin{pmatrix} G_1 & c_b \\ c_b & G_1 \end{pmatrix} \quad \mathcal{F} \mathcal{K}_2 \mathcal{F}^{-1} = \begin{pmatrix} G_2 & -i c_b \\ i c_b & G_2 \end{pmatrix}. \quad (4.9)$$

Let $\Xi := \mathcal{F} C$ be the composition of the twisted flip operator \mathcal{F} and the complex conjugation C on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. By combining the relations (3.17) and (4.9), one gets

$$\Xi \mathcal{K}_1 \Xi^{-1} = -\mathcal{K}_2, \quad \Xi \mathcal{K}_2 \Xi^{-1} = -\mathcal{K}_1 \quad (4.10)$$

that in turn implies

$$\Xi \mathcal{H}_{CJ} \Xi^{-1} = \mathcal{H}_{CJ} \quad (4.11)$$

in view of the general structure (4.5) for non-Abelian Landau Hamiltonians. Moreover, a direct computation shows that $\Xi^2 = \mathcal{F} \mathcal{F}^{-1} = \mathbf{1}$, showing that

Ξ is an anti-linear involution on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ or a generalized even TRS in the sense of Definition 1.1.

As a consequence of the invariance of \mathcal{H}_{JC} under the magnetic translations, and the consequent possibility of defining the magnetic Bloch-Floquet-Zak transform, a vector bundle $\mathcal{E}_j^\pm \rightarrow \mathbb{B}_B$ can be associated to each spectral projector \mathcal{P}_j^\pm (see the construction of Section 3.4). Equation (4.11) implies that the anti-unitary operator Ξ endows the vector bundle \mathcal{E}_j^\pm with a ‘‘Real’’ structure over the involutive torus $(\mathbb{T}^2, \mathfrak{f})$. The resulting ‘‘Real’’ vector bundle will be denoted with (\mathcal{E}_j^\pm, Ξ) . The rank r of \mathcal{E}_j^\pm can be deduced with the same argument used in the proof of Lemma 3.1. Let $k \mapsto \mathcal{P}_j^\pm(k)$ be the fiber-decomposition of \mathcal{P}_j^\pm obtained via the magnetic Bloch-Floquet-Zak transform. The rank r_0 of the vector bundle \mathcal{E}_j^\pm must match the dimension of the range of $\mathcal{P}_j^\pm(k)$ which is computed by the trace on the space $L^2(\mathbb{Y}_B) \otimes \mathbb{C}^2$. The latter is given by $\text{Tr}_{L^2(\mathbb{Y}_B) \otimes \mathbb{C}^2} = \text{Tr}_{L^2(\mathbb{Y}_B)} \otimes \text{Tr}_{\mathbb{C}^2}$. A direct computation shows that

$$\begin{aligned} r &= \text{Tr}_{L^2(\mathbb{Y}_B) \otimes \mathbb{C}^2}(\mathcal{P}_j^\pm(k)) \\ &= \sin(\theta_j^\pm)^2 \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_{j-1}(k)) + \cos(\theta_j^\pm)^2 \text{Tr}_{L^2(\mathbb{Y}_B)}(\Pi_j(k)) \end{aligned} \quad (4.12)$$

where (the fibered version of) equation (4.8) has been used for the computation of the trace $\text{Tr}_{\mathbb{C}^2}$. In view of Lemma 3.1 one immediately gets $r = 1$, namely each (\mathcal{E}_j^\pm, Ξ) is a ‘‘Real’’ line bundle over $(\mathbb{T}^2, \mathfrak{f})$.

The rank of \mathcal{E}_j^\pm can be also computed in terms of the Dixmier trace. Indeed, Lemma (B.3) provides

$$\begin{aligned} \text{Tr}_{\text{Dix}} \left((Q_{B,\xi}^{-1} \otimes \mathbf{1}_2) \mathcal{P}_j^\pm \right) &= \sin(\theta_j^\pm)^2 \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} \Pi_{j-1}) \\ &\quad + \cos(\theta_j^\pm)^2 \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} \Pi_j) \end{aligned}$$

and comparing the latter equation with (4.12) and (3.31) one gets

$$\text{rk}(\mathcal{E}_j^\pm) = \text{Tr}_{\text{Dix}} \left((Q_{B,\xi}^{-1} \otimes \mathbf{1}_2) \mathcal{P}_j^\pm \right) = 1.$$

To classify the topological phases of the Jaynes-Cummings model we can invoke Proposition 3.2 which ensures that the topology of (\mathcal{E}_j^\pm, Ξ) is completely characterized by the Chern class of \mathcal{E}_j^\pm as a complex vector bundle over \mathbb{T}^2 . Therefore, to complete the proof of Theorem 1.3 we need to compute $c_1(\mathcal{E}_j^\pm)$. After repeating step by step the arguments used in Section 3.7, and in particular in Lemma 3.3, one can prove that

$$c_1(\mathcal{E}_j^\pm) = \frac{i2\pi}{|\mathbb{Y}_B|} \text{Tr}_{L^2(\mathbb{R}^2) \otimes \mathbb{C}^2}(\chi_{\mathbb{Y}_B} \mathcal{R}_j^\pm \chi_{\mathbb{Y}_B}) = i2\pi \mathcal{T}_B(\text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm)) \quad (4.13)$$

where $\mathcal{R}_j^\pm := \mathcal{P}_j^\pm[\partial_1(\mathcal{P}_j^\pm), \partial_2(\mathcal{P}_j^\pm)]$ and $\text{Tr}_{\mathbb{C}^2}$ is meant as a map from the bounded operators on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ into the bounded operators on $L^2(\mathbb{R}^2)$. Let us assume for the moment that $\text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm)$ meets the condition of Theorem B.2. By combining the latter with Lemma B.3 one can rewrite

equation (4.13) as follows

$$c_1(\mathcal{E}_j^\pm) = \frac{i}{\ell_B^2} \text{Tr}_{\text{Dix}} \left((Q_{B,\xi}^{-1} \otimes \mathbf{1}_2) \mathcal{R}_j^\pm \right). \quad (4.14)$$

Let us study the operator $\text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm)$. By setting

$$[\partial_1(\mathcal{P}_j^\pm), \partial_2(\mathcal{P}_j^\pm)] = -i \begin{pmatrix} D_1 & L \\ L^* & D_2 \end{pmatrix}$$

one gets

$$\begin{aligned} \text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm) &= -i \sin(\theta_j^\pm)^2 \Pi_{j-1} D_1 - i \cos(\theta_j^\pm)^2 \Pi_j D_2 \\ &\quad + \frac{\sin(\theta_j^\pm) \cos(\theta_j^\pm)}{\sqrt{j}} (\Pi_j \mathfrak{a}^+ L - \mathfrak{a}^- \Pi_j L^*). \end{aligned}$$

A tedious calculation provides

$$\begin{aligned} D_1 &:= i \sin(\theta_j^\pm)^4 [\partial_1(\Pi_{j-1}), \partial_2(\Pi_{j-1})] \\ &\quad + \ell_B^2 \sin(\theta_j^\pm)^2 \cos(\theta_j^\pm)^2 ((j-1)\Pi_{j-2} + \Pi_{j-1} - j\Pi_j) \end{aligned}$$

and

$$\begin{aligned} D_2 &:= i \cos(\theta_j^\pm)^4 [\partial_1(\Pi_j), \partial_2(\Pi_j)] \\ &\quad + \ell_B^2 \sin(\theta_j^\pm)^2 \cos(\theta_j^\pm)^2 (j\Pi_{j-1} + \Pi_j - (j+1)\Pi_{j+1}) \end{aligned}$$

for the diagonal elements and

$$L := \frac{\sin(\theta_j^\pm) \cos(\theta_j^\pm)}{\sqrt{j}} (\mathfrak{a}^- [\partial_1(\Pi_j), \partial_2(\Pi_j)] - i \ell_B^2 \Pi_j \mathfrak{a}^-)$$

for the off-diagonal one. By putting all the pieces together, and after some more algebraic manipulations, one gets

$$\begin{aligned} \text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm) &= \sin(\theta_j^\pm)^6 (-i \ell_B^2 \Pi_{j-1}) + \cos(\theta_j^\pm)^6 (-i \ell_B^2 \Pi_j) \\ &\quad - i \ell_B^2 \sin(\theta_j^\pm)^2 \cos(\theta_j^\pm)^2 (\sin(\theta_j^\pm)^2 \Pi_{j-1} + \cos(\theta_j^\pm)^2 \Pi_j) \\ &\quad + \sin(\theta_j^\pm)^2 \cos(\theta_j^\pm)^2 ((-i \ell_B^2 \Pi_j) + (-i \ell_B^2 \Pi_{j-1})) \end{aligned}$$

where the identity $\Pi_j [\partial_1(\Pi_j), \partial_2(\Pi_j)] = -i \ell_B^2 \Pi_j$ has been repeatedly used. Finally, by using basic trigonometric identities, one obtains

$$\text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm) = -i \ell_B^2 (\sin(\theta_j^\pm)^2 \Pi_{j-1} + \cos(\theta_j^\pm)^2 \Pi_j).$$

The last equation shows that $\text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm)$ is in the right algebra for the application of Theorem B.2. Then, equation (4.14) can be used and one immediately gets

$$c_1(\mathcal{E}_j^\pm) = \frac{i}{\ell_B^2} \text{Tr}_{\text{Dix}} \left(Q_{B,\xi}^{-1} \text{Tr}_{\mathbb{C}^2}(\mathcal{R}_j^\pm) \right) = 1. \quad (4.15)$$

4.3. The ‘‘Quaternionic’’ model. This model corresponds to the elections of

$$\gamma_1 = -\alpha, \quad \gamma_2 = \sigma_2 \alpha \sigma_2,$$

in the non-abelian vector potential (4.2), with

$$\alpha = \alpha^* = \bar{\alpha}$$

a real and hermitian element of $\text{Mat}_2(\mathbb{C})$. The matrices γ_1 and γ_2 can be parametrized by three real parameters $r_0, r_1, r_2 \in \mathbb{R}$ as follows

$$\gamma_1 = \begin{pmatrix} -r_0 - r_2 & -r_1 \\ -r_1 & -r_0 + r_2 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} r_0 - r_2 & -r_1 \\ -r_1 & r_0 + r_2 \end{pmatrix}.$$

Since $[\gamma_1, \gamma_2] = 0$ it follows from equation (4.3) that

$$\mathcal{B}_\perp = B \otimes \mathbf{1}_2.$$

This magnetic field corresponds to a $\mathbb{U}(1)$ gauge field. As a consequence the non-Abelian kinetic momenta meet the canonical commutation relation $[\mathcal{K}_1, \mathcal{K}_2] = -i\mathbf{1}$.

The resulting non-Abelian Landau Hamiltonian \mathcal{H}_Q is given by (4.5) or equivalently by

$$\mathcal{H}_Q := \epsilon_B \left(\mathfrak{A}^+ \mathfrak{A}^- + \frac{1}{2} \mathbf{1} \right) \quad (4.16)$$

where $\mathfrak{A}^\pm := \frac{1}{\sqrt{2}}(\mathcal{K}_1 \pm i\mathcal{K}_2)$ are explicitly given by

$$\begin{aligned} \mathfrak{A}^+ &= \begin{pmatrix} a^+ + c_b(e^{-i\frac{\pi}{4}} r_0 + e^{i\frac{\pi}{4}} r_2) & c_b e^{i\frac{\pi}{4}} r_1 \\ c_b e^{i\frac{\pi}{4}} r_1 & a^+ + c_b(e^{-i\frac{\pi}{4}} r_0 - e^{i\frac{\pi}{4}} r_2) \end{pmatrix} \\ \mathfrak{A}^- &= \begin{pmatrix} a^- + c_b(e^{i\frac{\pi}{4}} r_0 + e^{-i\frac{\pi}{4}} r_2) & c_b e^{-i\frac{\pi}{4}} r_1 \\ c_b e^{-i\frac{\pi}{4}} r_1 & a^- + c_b(e^{i\frac{\pi}{4}} r_0 - e^{-i\frac{\pi}{4}} r_2) \end{pmatrix}. \end{aligned}$$

It turns out that

$$\mathcal{H}_Q = H_B \otimes \mathbf{1}_2 + c_b \epsilon_B \mathcal{W}_Q + c_b^2 \epsilon_B |r|^2 \mathbf{1} \quad (4.17)$$

with $|r|^2 := r_0^2 + r_1^2 + r_2^2 = 1$. The perturbation \mathcal{W}_Q is given by (1.9) or can be equivalently expressed by

$$\begin{aligned} \mathcal{W}_Q &= e^{i\frac{\pi}{4}} a^+ \otimes (r_0 \mathbf{1}_2 - i(r_1 \sigma_1 + r_2 \sigma_3)) \\ &\quad + e^{-i\frac{\pi}{4}} a^- \otimes (r_0 \mathbf{1}_2 + i(r_1 \sigma_1 + r_2 \sigma_3)). \end{aligned}$$

The equation (4.16), along with the commutation relation $[\mathfrak{A}^-, \mathfrak{A}^+] = \mathbf{1}$, might suggest at first sight to use the technique of ladder operators to compute the spectrum of \mathcal{H}_Q . However, this simple approach does not work because the operator \mathfrak{A}^+ has no ground state (*i. e.* the kernel is empty). More specifically consider the orthonormal basis of $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ given by

$$\Phi_{(j,m)}^\pm := \frac{1}{\sqrt{2(r_1^2 + r_2^2) \pm 2r_2 \sqrt{r_1^2 + r_2^2}}} \begin{pmatrix} r_1 \psi_{(j,m)} \\ -(r_2 \pm \sqrt{r_1^2 + r_2^2}) \psi_{(j,m)} \end{pmatrix}.$$

A direct computation shows that

$$\mathfrak{A}^- \Phi_{(0,m)}^\pm = c_b e^{i\frac{\pi}{4}} \left(r_0 \pm i\sqrt{r_1^2 + r_2^2} \right) \Phi_{(0,m)}^\pm.$$

and $\Phi_{(0,m)}^\pm$ are the only eigenvectors of \mathfrak{A}^- .

The calculation of the spectrum of \mathcal{H}_Q is beyond the scope of this work and will be left for future investigations. However, from the general structure (4.5) we know that \mathcal{H}_Q has a positive spectrum and we can denote with $\mathcal{P}_E := \chi_{(-\infty, E]}(\mathcal{H}_Q)$ the Fermi at energy $E > 0$. If the energy $E > 0$ lies in a spectral gap of \mathcal{H}_Q then \mathcal{P}_E define a vector bundle $\mathcal{E}_E \rightarrow \mathbb{B}_B$ according to the prescription (3.28). In this case the rank and the Chern class of \mathcal{E}_E can be computed again by formulas of the type (1.6) for the projection \mathcal{P}_E . Even though, we are not computing exactly these numbers, we can have access to some informations by examining the symmetries of \mathcal{H}_Q .

Let us introduce the twisted flip operator $\mathcal{F}' := F \otimes \sigma_2$. A direct check shows that

$$\begin{aligned} \mathcal{F}\mathcal{K}_1\mathcal{F} &= F\mathcal{K}_1F \otimes \mathbf{1}_2 - c_b \mathbf{1}_{L^2} \otimes \sigma_2\gamma_1\sigma_2 = G_1 \otimes \mathbf{1}_2 + c_b \mathbf{1}_{L^2} \otimes \gamma_2, \\ \mathcal{F}\mathcal{K}_2\mathcal{F} &= F\mathcal{K}_2F \otimes \mathbf{1}_2 - c_b \mathbf{1}_{L^2} \otimes \sigma_2\gamma_2\sigma_2 = G_2 \otimes \mathbf{1}_2 + c_b \mathbf{1}_{L^2} \otimes \gamma_1. \end{aligned}$$

Let $\Xi' := \mathcal{F}'C$ be the composition of the twisted flip operator \mathcal{F}' with the complex conjugation C on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. By combining the last relations with (3.17) one gets

$$\Xi'\mathcal{K}_1\Xi'^{-1} = -\mathcal{K}_2, \quad \Xi'\mathcal{K}_2\Xi'^{-1} = -\mathcal{K}_1, \quad (4.18)$$

and in turn

$$\Xi'\mathcal{H}_Q\Xi'^{-1} = \mathcal{H}_Q \quad (4.19)$$

in view of the general structure (4.5) for non-Abelian Landau Hamiltonians. Moreover, a direct computation shows that $\Xi'^2 = \mathcal{F}'C\mathcal{F}'C = -\mathcal{F}'^2 = -\mathbf{1}$, namely Ξ' is an anti-linear anti-involution on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$. Therefore, Ξ' provides \mathcal{H}_Q with a generalized TRS of ‘‘Quaternionic’’ type according to Definition 1.1. The latter fact justifies the name of ‘‘Quaternionic’’ model for \mathcal{H}_Q .

The generalized TRS Ξ' endows the vector bundle \mathcal{E}_E associated with \mathcal{P}_E with a ‘‘Quaternionic’’ structure over the involutive torus $(\mathbb{T}^2, \mathfrak{f})$. The resulting ‘‘Quaternionic’’ vector bundle will be denoted with (\mathcal{E}_E, Ξ') . For the theory and the classification of ‘‘Quaternionic’’ vector bundles we refer to [DG2, DG4, DG5]. Since the fixed point set of $(\mathbb{T}^2, \mathfrak{f})$ is not empty we can deduce from [DG2, Proposition 2.1] that \mathcal{E}_E has even rank. This translates into

$$\text{rk}(\mathcal{E}_E) = \text{Tr}_{\text{Dix}} \left((Q_{B,\xi}^{-1} \otimes \mathbf{1}_2) \mathcal{P}_E \right) \in 2\mathbb{N}$$

by using the same arguments as that in Section 4.2. The second topological information comes from the isomorphism $\text{Vec}_{\Omega}^{2m}(\mathbb{T}^2, \mathfrak{f}) \simeq 2\mathbb{Z}$ (cf. Corollary A.1) which says that the topology of (\mathcal{E}_E, Ξ') is completely determined by the first Chern class and that

$$c_1(\mathcal{E}_E) = \frac{i}{\ell_B^2} \text{Tr}_{\text{Dix}} \left((Q_{B,\xi}^{-1} \otimes \mathbf{1}_2) \mathcal{P}_E [\partial_1(\mathcal{P}_E), \partial_2(\mathcal{P}_E)] \right) \in 2\mathbb{Z}. \quad (4.20)$$

Appendix A. Equivariant cohomology for the flip involution

This section provides the computation of the twisted equivariant cohomology of the two-dimensional $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ endowed with the flip involution $f : (k_1, k_2) \mapsto (-k_2, -k_1)$. Sometimes, it is also useful to use complex coordinates for $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$ through the identification $\mathbb{S}^1 \simeq \{z \in \mathbb{C} \mid |z| = 1\}$. With this parametrization $(z_1, z_2) \in \mathbb{T}^2$ the flip involution reads $f : (z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$.

The next result is needed to complete the proof of Proposition 3.2

Lemma A.1. *The map ι which forgets the \mathbb{Z}_2 action induces the isomorphism*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \xrightarrow{\simeq} H^2(\mathbb{T}^2, \mathbb{Z}).$$

Proof. Let f' be the involution on \mathbb{T}^2 given by $f' : (k_1, k_2) \mapsto (k_2, k_1)$. Then, there is a \mathbb{Z}_2 -equivariant homeomorphism $\varphi : (\mathbb{T}^2, f) \rightarrow (\mathbb{T}^2, f')$ given by $\varphi : (k_1, k_2) \mapsto (k_1, -k_2)$. This means that we can compute the equivariant cohomology groups for the involutive space (\mathbb{T}^2, f') instead of the involutive space (\mathbb{T}^2, f) . The involution f' on \mathbb{T}^2 agrees with that induced from a natural action of the *wallpaper group* cm on \mathbb{R}^2 (cf [Gom2, Section 2.4]). The low degree equivariant cohomology groups $H_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(k))$ for the involutive space (\mathbb{T}^2, f') have been computed in [Gom2, Theorem 1.3 & Theorem 1.6] (by using stable splittings) or in [GT] (by the Gysin exact sequences) and summarized in the following table:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$H_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(0))$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2
$H_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(1))$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}$	\mathbb{Z}	\mathbb{Z}_2

The equivariant cohomology and the ordinary cohomology $H^n(\mathbb{T}^2, \mathbb{Z})$ fit into the long exact sequence [Gom1, Proposition 2.3]:

$$\begin{aligned} \cdots \rightarrow H_{\mathbb{Z}_2}^{n-1}(\mathbb{T}^2, \mathbb{Z}(0)) \rightarrow H_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(1)) \xrightarrow{\iota} H^n(\mathbb{T}^2, \mathbb{Z}) \\ \rightarrow H_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(0)) \rightarrow \cdots \end{aligned}$$

where ι is the homomorphism which forgets the \mathbb{Z}_2 -action. Knowing $H^n(\mathbb{T}^2, \mathbb{Z})$, one concludes that ι provides an isomorphism on the degree $n = 2$. \square

The next step is to compute the equivariant relative cohomology group $H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1))$ where $(\mathbb{T}^2)^f \simeq \mathbb{S}^1$ is the fixed point set of the torus \mathbb{T}^2 with respect to the flip involution.

Lemma A.2. *There is an isomorphism of groups*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1)) \simeq \mathbb{Z}.$$

Proof. In general, if X is a \mathbb{Z}_2 -CW complex and $Z \subseteq Y \subseteq X$ are \mathbb{Z}_2 -sub-complexes, then there is an exact sequence of groups

$$\begin{aligned} \cdots \rightarrow H_{\mathbb{Z}_2}^{n-1}(Y|Z, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^n(X|Y, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^n(X|Z, \mathbb{Z}(1)) \\ \rightarrow H_{\mathbb{Z}_2}^n(Y|Z, \mathbb{Z}(1)) \rightarrow \cdots \end{aligned}$$

This is just a consequence of the excision axiom and the exactness axiom. We apply this exact sequence to $X := \mathbb{T}^2$, $Y := (\mathbb{T}^2)^f$ and a fixed point

$Z := \{*\}$:

$$\begin{aligned} \cdots \rightarrow \tilde{H}_{\mathbb{Z}_2}^{n-1}((\mathbb{T}^2)^f, \mathbb{Z}(1)) &\rightarrow H_{\mathbb{Z}_2}^n(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1)) \\ &\rightarrow \tilde{H}_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(1)) \rightarrow \tilde{H}_{\mathbb{Z}_2}^n((\mathbb{T}^2)^f, \mathbb{Z}(1)) \rightarrow \cdots, \end{aligned}$$

where $\tilde{H}_{\mathbb{Z}_2}^n(M, \mathbb{Z}(1)) := H_{\mathbb{Z}_2}^n(M | \{*\}, \mathbb{Z}(1))$ is the reduced cohomology of the \mathbb{Z}_2 -space M with a fixed point $\{*\}$. By using the direct sum decomposition

$$\begin{aligned} H_{\mathbb{Z}_2}^n(M, \mathbb{Z}(1)) &\simeq \tilde{H}_{\mathbb{Z}_2}^n(M, \mathbb{Z}(1)) \oplus H_{\mathbb{Z}_2}^n(\{*\}, \mathbb{Z}(1)) \\ &= \begin{cases} \tilde{H}_{\mathbb{Z}_2}^n(M, \mathbb{Z}(1)) & n \text{ even} \\ \tilde{H}_{\mathbb{Z}_2}^n(M, \mathbb{Z}(1)) \oplus \mathbb{Z}_2 & n \text{ odd} \end{cases} \end{aligned}$$

(cf. [DG1, Section 5]) and the computation of the equivariant cohomology groups of (\mathbb{T}^2, f) provided in the proof of Proposition 3.2, one can compute $\tilde{H}_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(1))$. The computation of $\tilde{H}_{\mathbb{Z}_2}^n((\mathbb{T}^2)^f, \mathbb{Z}(1))$ follows similarly by observing that $(\mathbb{T}^2)^f$ coincides with \mathbb{S}^1 with the trivial involution (the computation of the related equivariant cohomology groups is in [Gom1]). The values of the various cohomology groups are displayed in the following table:

	$n = 0$	$n = 1$	$n = 2$	$n = 3$
$\tilde{H}_{\mathbb{Z}_2}^n(\mathbb{T}^2, \mathbb{Z}(1))$	0	\mathbb{Z}	\mathbb{Z}	0
$\tilde{H}_{\mathbb{Z}_2}^n((\mathbb{T}^2)^f, \mathbb{Z}(1))$	0	0	\mathbb{Z}_2	0

From the exact sequence above, one immediately gets the isomorphism $H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1)) \simeq \mathbb{Z}$. \square

Lemma A.2 provides the computation of the equivariant relative cohomology group $H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1))$. However, in the proof it is not specified whether the injective homomorphism

$$j : H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \quad (\text{A.1})$$

is the bijection $j : n \mapsto n$ or the multiplication by two $j : n \mapsto 2n$.

Lemma A.3. *The homomorphism (A.1) coincides with the multiplication by two $j : n \mapsto 2n$, $n \in \mathbb{Z}$. As a consequence*

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^2 | (\mathbb{T}^2)^f, \mathbb{Z}(1)) \simeq 2\mathbb{Z}.$$

Proof. By inspecting the exact sequence used in the proof of Lemma A.2, it suffices to show that the restriction homomorphism

$$\mathbb{Z} \simeq H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \rightarrow H_{\mathbb{Z}_2}^2((\mathbb{T}^2)^f, \mathbb{Z}(1)) \simeq \mathbb{Z}_2$$

is surjective. We will show this in a geometric manner. Since $H_{\mathbb{Z}_2}^2(X, \mathbb{Z}(1))$ classifies ‘‘Real’’ line bundles over X , we shall construct a ‘‘Real’’ line bundle over \mathbb{T}^2 whose restriction to $(\mathbb{T}^2)^f$ is a non-trivial real line bundle. Let $\tilde{L}_{\mathbb{T}^2} := \mathbb{R}^2 \times \mathbb{C}$ be the product line bundle over \mathbb{R}^2 . This is equipped with two actions

$$\begin{aligned} T_1 : \tilde{L}_{\mathbb{T}^2} &\rightarrow \tilde{L}_{\mathbb{T}^2}, & T_1(x_1, x_2, z) &:= (x_1 + 1, x_2, e^{i2\pi x_1} z), \\ T_2 : \tilde{L}_{\mathbb{T}^2} &\rightarrow \tilde{L}_{\mathbb{T}^2}, & T_2(x_1, x_2, z) &:= (x_1, x_2 + 1, e^{i2\pi x_1} z). \end{aligned}$$

These two actions commute with each other, so that they make the line bundle $\tilde{L}_{\mathbb{T}^2} \rightarrow \mathbb{R}^2$ into a \mathbb{Z}^2 -equivariant line bundle, where \mathbb{Z}^2 acts on \mathbb{R}^2 by translations. Furthermore, this \mathbb{Z}^2 -action is free. Thus, taking the quotient,

we get a complex line bundle $L_{\mathbb{T}^2} := \tilde{L}_{\mathbb{T}^2}/\mathbb{Z}^2$ over the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. It can be proven that $L_{\mathbb{T}^2}$ is non-trivial and has Chern class $c_1(L_{\mathbb{T}^2}) = 1$. We can define a “Real” structure on $\tilde{L}_{\mathbb{T}^2}$ by

$$\tilde{\Theta} : \tilde{L}_{\mathbb{T}^2} \rightarrow \tilde{L}_{\mathbb{T}^2}, \quad \tilde{\Theta}(x_1, x_2, z) := \left(x_2, x_1, e^{i\pi((x_1+x_2)^2 - (x_1+x_2))} \bar{z} \right).$$

Because of the relations

$$\tilde{\Theta} \circ T_1 = T_2 \circ \tilde{\Theta}, \quad \tilde{\Theta}^2 = \text{Id}_{\tilde{L}_{\mathbb{T}^2}}.$$

$\tilde{\Theta}$ descends to a “Real” structure on $L_{\mathbb{T}^2}$. The restriction $L_{\mathbb{T}^2}|_{(\mathbb{T}^2)^f}$ is isomorphic to the quotient $L_{\mathbb{S}^1} = \tilde{L}_{\mathbb{S}^1}/\mathbb{Z}$ over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ of the product bundle $\tilde{L}_{\mathbb{S}^1} = \mathbb{R} \times \mathbb{C}$ over \mathbb{R} under the free action of \mathbb{Z} generated by

$$T_0 : \tilde{L}_{\mathbb{S}^1} \rightarrow \tilde{L}_{\mathbb{S}^1}, \quad T(x, z) = (x + 1, e^{i4\pi x} z).$$

The “Real” structure Θ_0 on $L_{\mathbb{S}^1} \simeq L_{\mathbb{T}^2}|_{(\mathbb{T}^2)^f}$ is induced from the following “Real” structure on $\tilde{L}_{\mathbb{S}^1}$

$$\tilde{\Theta}_0 : \tilde{L}_{\mathbb{S}^1} \rightarrow \tilde{L}_{\mathbb{S}^1}, \quad \tilde{\Theta}_0(x, z) := (x, e^{i2\pi(2x^2 - x)} \bar{z}).$$

As a complex line bundle, $L_{\mathbb{S}^1} \rightarrow \mathbb{S}^1$ admits a nowhere vanishing section $\sigma : \mathbb{S}^1 \rightarrow L_{\mathbb{S}^1}$ given by $\sigma([x]) = [x, e^{i2\pi x^2}]$. Under the “Real” action Θ_0 , this section behaves as $\Theta_0(\sigma([x])) = e^{-i2\pi x} \sigma([x])$. Then, by using this section, we can construct an isomorphism between $L_{\mathbb{S}^1}$ and the product bundle $\mathbb{S}^1 \times \mathbb{C} \rightarrow \mathbb{C}$ with the “Real” structure $(x, z) \mapsto (x, e^{-i2\pi x} \bar{z})$. The (dual of the) latter “Real” line bundle is shown to be non-trivial in [DG3, Example 3.10]. \square

Corollary A.1. *The first Chern class provides the isomorphism*

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^2, \mathfrak{f}) \stackrel{c_1}{\simeq} 2\mathbb{Z}.$$

Proof. The FKMM invariant κ [DG2, DG4, DG5] and Lemma A.3 provide the isomorphisms

$$\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^2, \mathfrak{f}) \stackrel{\kappa}{\simeq} H_{\mathbb{Z}_2}^2(\mathbb{T}^2|_{(\mathbb{T}^2)^f}, \mathbb{Z}(1)) \simeq 2\mathbb{Z}$$

The nature of the $2\mathbb{Z}$ -valued invariant can be described by observing that

$$H_{\mathbb{Z}_2}^2(\mathbb{T}^2|_{(\mathbb{T}^2)^f}, \mathbb{Z}(1)) \xrightarrow{j} H_{\mathbb{Z}_2}^2(\mathbb{T}^2, \mathbb{Z}(1)) \stackrel{\iota}{\simeq} H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$$

where j is the injection described in Lemma A.3 and ι is the isomorphism which forgets the \mathbb{Z}_2 -action. The resulting map $\text{Vec}_{\mathbb{Q}}^{2m}(\mathbb{T}^2, \mathfrak{f}) \rightarrow H^2(\mathbb{T}^2, \mathbb{Z})$ given by the composition of these homomorphisms is nothing but the first Chern class of the complex vector bundle underlying the “Quaternionic” vector bundle. \square

Appendix B. A primer on Dixmier trace

This appendix is devoted to the construction of the *Mačæev ideals* and the *Dixmier trace*. Useful references for these subjects are [CM, Appendix A], [GBVF, Sect. 7.5 and App. 7.C], [AM] and [Sim].

B.1. Trace, Schatten ideals and Mač̆ev ideals. We will assume the familiarity of the reader with the theory of compact operators (see [RS1, VI.5]). Let \mathcal{H} be a separable Hilbert, $\mathcal{B}(\mathcal{H})$ the C^* -algebra of the bounded operators acting on \mathcal{H} and $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ the two-sided ideal of compact operators. We just recall three important facts about compact operators: (i) $T \in \mathcal{K}(\mathcal{H})$ if and only if T is the norm limit of a sequence of finite rank operators; (ii) the spectrum $\sigma(T)$ of a compact operator is a discrete set having no limit point, except perhaps the zero and every non-zero eigenvalue has finite multiplicity (*Riesz-Schauder Theorem*); (iii) $\mathcal{K}(\mathcal{H})$ is the only non-trivial norm-closed two-sided ideal in $\mathcal{B}(\mathcal{H})$. The latter property implies that $T \in \mathcal{K}(\mathcal{H})$ if and only if $|T| := \sqrt{T^*T} \in \mathcal{K}(\mathcal{H})$.

The *singular values* $\mu_n(T)$ of a compact operator T are the eigenvalues of $|T|$. By convention the singular values will be listed in decreasing order, repeated according to the multiplicity, *i. e.*

$$\mu_0(T) \geq \mu_1(T) \geq \dots \geq \mu_n(T) \geq \mu_{n+1}(T) \geq \dots \geq 0$$

with $\mu_0(T) = \| |T| \|_{\mathcal{B}(\mathcal{H})} = \| T \|_{\mathcal{B}(\mathcal{H})}$.

From an analytic point of view compact operators are in a sense “small” or better *infinitesimal*. Indeed, if T is a compact operator then for all $\varepsilon > 0$ there exists a finite-dimensional subspace $V_\varepsilon \subset \mathcal{H}$ such that $\| T|_{V_\varepsilon^\perp} \|_{\mathcal{B}(\mathcal{H})} < \varepsilon$. Moreover one has that

$$\mu_n(T) = \inf \{ \| T|_{V^\perp} \|_{\mathcal{B}(\mathcal{H})} \mid \dim V = n \}.$$

Compact operators can be classified according to their order of infinitesimal. One says that $T \in \mathcal{K}(\mathcal{H})$ is of *order* $\alpha \in (0, +\infty)$ if there exist a $C > 0$ and a $N_0 \in \mathbb{N}$ such that

$$\text{Mult}[\mu_n(T)] \mu_n(T) \leq C n^{-\alpha} \quad \forall n \geq N_0$$

where $\text{Mult}[\mu_n(T)]$ is the multiplicity of $\mu_n(T)$. This definition is consistent: Indeed if T_1 is an infinitesimal of order α and T_2 an infinitesimal of order β then $T_1 T_2$ is an infinitesimal of order at most $\alpha + \beta$ (as a consequence of the submultiplicative property for singular values). Therefore, the set of the infinitesimals of order α is a (non-closed) two-sided ideal in $\mathcal{B}(\mathcal{H})$.

Let us recall few standard facts about the notion of *trace* on \mathcal{H} (*cf.* [RS1, Sect VI.6]). Every orthonormal basis $\{\phi_n\}_{n \in \mathbb{N}}$ of \mathcal{H} defines a linear functional on the cone of positive operators by

$$\text{Tr}_{\mathcal{H}}(T) := \sum_{n=0}^{+\infty} \langle \phi_n, T \phi_n \rangle_{\mathcal{H}}. \quad (\text{B.1})$$

The linear functional is monotone (with respect to the ordering of the positive operators) and its range is $[0, +\infty]$. Moreover, $\text{Tr}_{\mathcal{H}}$ is independent of the particular choice of the orthonormal basis. A bounded operator T is called *trace class* if and only if $\text{Tr}_{\mathcal{H}}(|T|) < +\infty$. The family of trace class operators is denoted by $\mathcal{L}^1(\mathcal{H})$. One has that $\mathcal{L}^1(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$ and T is trace class if and only if

$$\text{Tr}_{\mathcal{H}}(|T|) = \sum_{n=0}^{+\infty} \mu_n(T) < +\infty.$$

The set $\mathcal{L}^1(\mathcal{H})$ is a two-sided self-adjoint ideal of $\mathcal{B}(\mathcal{H})$ which is not closed with respect to the operator norm but which is closed with respect to the *trace-norm* $\|T\|_1 := \text{Tr}_{\mathcal{H}}(|T|)$. The finite rank operators are $\|\cdot\|_1$ -dense in $\mathcal{L}^1(\mathcal{H})$. The ideal $\mathcal{L}^1(\mathcal{H})$ is the natural domain of definition for the trace functional. Indeed if $T \in \mathcal{L}^1(\mathcal{H})$ (not necessary positive) then the sum (B.1) converges absolutely and the limit is independent of the choice of a particular orthonormal basis. Therefore, $\text{Tr}_{\mathcal{H}} : \mathcal{L}^1(\mathcal{H}) \rightarrow \mathbb{C}$ defines a linear $*$ -functional bounded by the norm $\|\cdot\|_1$. Finally, the *tracial property* $\text{Tr}_{\mathcal{H}}(TS) = \text{Tr}_{\mathcal{H}}(ST)$ holds for all $T \in \mathcal{L}^1(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$.

The definition of the trace class ideal $\mathcal{L}^1(\mathcal{H})$ can be generalized. For each $p \in [1, +\infty)$ one defines the *p-th Schatten class* as the set

$$\mathcal{L}^p(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) \mid \text{Tr}_{\mathcal{H}}(|T|^p) = \sum_{n=0}^{+\infty} \mu_n(T)^p < +\infty \right\}.$$

Every $\mathcal{L}^p(\mathcal{H})$ is a two-sided self-adjoint ideal contained in $\mathcal{K}(\mathcal{H})$ which is not closed with respect to the operator norm but which is closed with respect to the *p-norm* $\|T\|_p := \text{Tr}_{\mathcal{H}}(|T|^p)^{\frac{1}{p}}$. Moreover, $\mathcal{L}^p(\mathcal{H}) \subset \mathcal{L}^q(\mathcal{H})$ for every $p \leq q$ and the finite rank operators are $\|\cdot\|_p$ -dense in $\mathcal{L}^p(\mathcal{H})$. The 2-nd Schatten class $\mathcal{L}^2(\mathcal{H})$ is usually called *Hilbert-Schmidt ideal* and it can be endowed with the structure of a Hilbert space by means of the inner product $\langle T_1, T_2 \rangle_{\text{H.S.}} := \text{Tr}_{\mathcal{H}}(T_1^* T_2)$.

Operator in $\mathcal{L}^p(\mathcal{H})$, for $p \in [1, +\infty)$, are infinitesimal of order strictly greater than $1/p$. To see this, let us introduce the partial sums

$$\sigma_N^p(T) := \sum_{n=0}^{N-1} \mu_n(T)^p. \quad (\text{B.2})$$

Then, $T \in \mathcal{L}^p(\mathcal{H})$ if and only if $\|T\|_p^p = \lim_{N \rightarrow \infty} \sigma_N^p(T) < +\infty$ which is the same of $\text{Mult}[\mu_n(T)] \mu_n(T)^p \leq C n^{-(1+\epsilon)}$ for some positive constants C and ϵ and $n \geq N_0$. Then, the ideal of infinitesimal operator of order $1/p$ is strictly larger than $\mathcal{L}^p(\mathcal{H})$. However, from $\sum_{n=1}^N n^{-1} \sim \log(N)$ one infers that infinitesimals of order 1 have partial sums σ_N^1 which are at most logarithmically divergent. This observation suggests to consider the “regularized” partial sums

$$\gamma_N^p(T) := \frac{\sigma_N^p(T)}{\log(N)} = \frac{1}{\log(N)} \sum_{n=0}^{N-1} \mu_n(T)^p, \quad N > 1. \quad (\text{B.3})$$

The positive sequence $\gamma_N^p(T)$, although bounded if T is an infinitesimal of order $1/p$, does not converge in general when $N \rightarrow \infty$. This suggests to consider the *supremum limit*. The (1^+) -*Mačaev class* is the set

$$\mathcal{L}^{1^+}(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_{1^+} := \sup_{N>1} \gamma_N^1(T) < +\infty \right\}. \quad (\text{B.4})$$

In particular $T \in \mathcal{L}^{1^+}(\mathcal{H})$ when (B.3) converges. The above definitions generalizes for $p > 1$ as follows

$$\mathcal{L}^{p^+}(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_{p^+} := \sup_{N>1} N^{\frac{1-p}{p}} \sigma_N^1(T) < +\infty \right\}. \quad (\text{B.5})$$

These definitions encode the fact that $\mathcal{L}^{p^+}(\mathcal{H})$ is the ideal of infinitesimal operator of order $1/p$. More precisely, one can prove that $T \in \mathcal{L}^{p^+}(\mathcal{H})$ if and only if $\text{Mult}[\mu_n(T)]\mu_n(T)^p \sim O(n^{-1})$ if $p > 1$. For $p = 1$ this characterization fails as showed in [AM, Remark 1.1] but the infinitesimals of order 1 are anyway in $\mathcal{L}^{1^+}(\mathcal{H})$ (cf. Lemma B.1 below). For every $p \in [1, +\infty)$ the set $\mathcal{L}^{p^+}(\mathcal{H})$ is a two-sided self-adjoint ideals which is not closed with respect to the operator norm but which is closed with respect to the (Calderón) norm $\|\cdot\|_{p^+}$. The $\|\cdot\|_{p^+}$ -closure of the finite rank operators does not coincide with $\mathcal{L}^{p^+}(\mathcal{H})$ but defines the (p_0^+) -Mač̆aev (sub-)class which is characterized by

$$\mathcal{L}_0^{p^+}(\mathcal{H}) := \{T \in \mathcal{K}(\mathcal{H}) \mid \lim_{N \rightarrow \infty} \gamma_N^p(T) = 0\}. \quad (\text{B.6})$$

As a consequence the ideals $\mathcal{L}^{p^+}(\mathcal{H})$ are not separable. The spaces $\mathcal{L}_0^{p^+}(\mathcal{H})$ are two-sided self-adjoint ideals and the inclusions $\mathcal{L}^p(\mathcal{H}) \subset \mathcal{L}_0^{p^+}(\mathcal{H}) \subset \mathcal{L}^{p^+}(\mathcal{H}) \subset \mathcal{L}^{p'}$ hold for every $p' > p \geq 1$.

The ideal $\mathcal{L}^{1^+}(\mathcal{H})$, usually called the *Dixmier ideal*, is of particular importance for our aims. Operators $T \in \mathcal{L}^{1^+}(\mathcal{H})$ with a convergent sequence $\gamma_N^1(T)$ are called *measurable*. Evidently, every operator in $\mathcal{L}_0^{1^+}(\mathcal{H})$ is measurable since in this case the sequence $\gamma_N^1(T)$ converges to zero. The set of measurable operators is a closed subspace of $\mathcal{L}^{1^+}(\mathcal{H})$ which is invariant under conjugation by bounded invertible operators [GBVF, Proposition 7.15]. The following result turns out to be a very useful criterion for the measurability of an operator:

Lemma B.1 ([AM, Lemma 1.6]). *Let $T \in \mathcal{K}(\mathcal{H})$ be a compact operator such that $\text{Mult}[\mu_n(T)]\mu_n(T) \sim Cn^{-1}$. Then T is a measurable element in $\mathcal{L}^{1^+}(\mathcal{H})$ and $\lim_{N \rightarrow \infty} \gamma_N^1(T) = \alpha C$ where*

$$\alpha := \lim_{N \rightarrow \infty} \frac{\log(N)}{\log\left(\sum_{n=0}^{N-1} \text{Mult}[\mu_n(T)]\right)}.$$

We point out that in the original statement of [AM, Lemma 1.6] the terms $[\mu_n(T)]$ and α are both omitted. Of course this is possible when $\text{Mult}[\mu_n(T)]$ is a bounded sequence with $\lim_{n \rightarrow \infty} \text{Mult}[\mu_n(T)] = 1$.

B.2. The Dixmier trace. The aim of this section is to define an “integral” which neglects infinitesimal operators of order greater than 1. More precisely, we want a trace functional such that $\mathcal{L}^{1^+}(\mathcal{H})$ is in the domain of such a trace and infinitesimal operator of order higher than 1 have vanishing trace. The usual trace is not appropriate since its domain $\mathcal{L}^1(\mathcal{H})$ is smaller than $\mathcal{L}^{1^+}(\mathcal{H})$. Dixmier [Dix1] has shown that such a trace exists and corresponds “morally” to the operation of the extraction of the limit $\lim_{N \rightarrow \infty} \gamma_N^1(T)$ defined by (B.3). However, this procedure does not define a trace since linearity and convergence are not guaranteed. Then, one needs a more sophisticated object called the *Dixmier trace*.

Given a $T \in \mathcal{K}(\mathcal{H})$ let us define the following family of functions

$$\sigma_\lambda(T) := \inf \{ \|R\|_1 + \lambda \|S\|_{\mathcal{B}(\mathcal{H})} \mid R, S \in \mathcal{K}(\mathcal{H}), R + S = T \} \quad (\text{B.7})$$

indexed by the scale parameter $\lambda \in [1, \infty)$. The functions σ_λ are norms defined on $\mathcal{K}(\mathcal{H})$ and for integer values of the scale parameter $\sigma_{\lambda=N}(T)$ coincides with (B.2). The function $\lambda \mapsto \sigma_\lambda(T)$ is piecewise linear and concave and the inequalities

$$\sigma_\lambda(T_1 + T_2) \leq \sigma_\lambda(T_1) + \sigma_\lambda(T_2) \leq \sigma_{2\lambda}(T_1 + T_2) \quad (\text{B.8})$$

hold true for positive $T_1, T_2 \in \mathcal{K}(\mathcal{H})$ and $\lambda \geq 1$.

The function $\sigma_\lambda(T)$ can be interpreted as the trace of T ‘‘cutoff’’ at the inverse scale λ . Equation (B.8) suggests that for large λ , $\log(\lambda)^{-1}\sigma_\lambda$ is an ‘‘almost additive’’ functional on the cone of the positive compact operators. If it were actually additive, it would be extended by linearity to a trace since the invariance under unitary operator due to $\sigma_\lambda(T) = \sigma_\lambda(UTU^{-1})$. However, a genuine trace can be obtained by suitably averaging the norms induced by $\log(\lambda)^{-1}\sigma_\lambda$. Observe that the norm $\|\cdot\|_{1+}$ defined in (B.4) can be replaced by the equivalent norm (still denoted with the same symbol) $\|T\|_{1+} := \sup_{\lambda \geq e} \log(\lambda)^{-1} \sigma_\lambda(T)$. Consider the following *Cesàro mean*

$$\tau_\lambda(T) := \frac{1}{\log(\lambda)} \int_{\lambda_0}^{\lambda} \frac{\sigma_s(T)}{\log(s)} \frac{ds}{s}, \quad \text{for } \lambda \geq \lambda_0 > e. \quad (\text{B.9})$$

This is still not an additive functional, but it has an ‘‘asymptotic additivity’’ property as shown by the following inequality [CM, Lemma A.4]

$$|\tau_\lambda(T_1 + T_2) - \tau_\lambda(T_1) - \tau_\lambda(T_2)| \leq C_\lambda (\|T_1\|_{1+} + \|T_2\|_{1+}), \quad (\text{B.10})$$

where $C_\lambda = \log(2) \log(\lambda)^{-1} (2 + \log(\log(\lambda)))$, valid for any pair of positive operators $T_1, T_2 \in \mathcal{L}^{1+}(\mathcal{H})$. The function $\log(\lambda)^{-1} \log(\log(\lambda))$ is bounded on the interval $[\lambda_0, \infty)$ and falls to zero at infinity. Thus the function $\lambda \mapsto \tau_\lambda(T)$ lies in $\mathcal{C}_b([\lambda_0, \infty))$ and the left hand side of (B.10) lies in $\mathcal{C}_0([\lambda_0, +\infty))$. Let $\mathfrak{B}_\infty := \mathcal{C}_b([\lambda_0, \infty)) / \mathcal{C}_0([\lambda_0, \infty))$ be the quotient C^* -algebra and denote by $\tau : \mathcal{L}^{1+}(\mathcal{H}) \rightarrow \mathfrak{B}_\infty$ the map which associates to any positive $T \in \mathcal{L}^{1+}(\mathcal{H})$ the equivalence class $\tau(T) := [\lambda \mapsto \tau_\lambda(T)] \in \mathfrak{B}_\infty$. One can prove that τ is additive and positive-homogeneous on the positive cone of $\mathcal{L}^{1+}(\mathcal{H})$. Moreover, τ extends by linearity to a linear map $\tau : \mathcal{L}^{1+}(\mathcal{H}) \rightarrow \mathfrak{B}_\infty$ defined on the full ideal $\mathcal{L}^{1+}(\mathcal{H})$ and which verifies the trace property $\tau(ST) = \tau(TS)$ for all $T \in \mathcal{L}^{1+}(\mathcal{H})$ and bounded S . Finally, $\tau : \mathcal{L}_0^{1+}(\mathcal{H}) \rightarrow \{0\}$.

To define a trace functional with domain the Dixmier ideal $\mathcal{L}^{1+}(\mathcal{H})$, all we have to do is to follow the map τ with a state $\omega : \mathfrak{B}_\infty \rightarrow \mathbb{C}$. The latter is a positive linear form on $\mathcal{C}_b([\lambda_0, +\infty))$ which vanishes on $\mathcal{C}_0([\lambda_0, +\infty))$, normalized by $\omega(1) = 1$. Let $f \in \mathcal{C}_b([\lambda_0, +\infty))$ such that f has limit L when $\lambda \rightarrow \infty$. Then $f - L \in \mathcal{C}_0([\lambda_0, +\infty))$ and $\omega(f - L) = 0$ which implies $\omega(f) = L$ independently of the choice of the state ω . On the other hand, if f has two distinct limit points, one gets two states ω_1 and ω_2 whose values on f are different. Then the states of \mathfrak{B}_∞ correspond to ‘‘generalized limits’’ as $\lambda \rightarrow \infty$, of bounded but not necessarily convergent functions.

Definition B.1 (Dixmier traces). To each state ω on the commutative C^* -algebra \mathfrak{B}_∞ there corresponds a *Dixmier trace* on $\mathcal{L}^{1+}(\mathcal{H})$ define by $\text{Tr}_{\text{Dix}, \omega} := \omega \circ \tau$.

The Dixmier trace has many of the properties of a usual trace. Indeed, each $\text{Tr}_{\text{Dix},\omega}$ is a positive linear functional on $\mathcal{L}^{1+}(\mathcal{H})$ such that $\text{Tr}_{\text{Dix},\omega}(ST) = \text{Tr}_{\text{Dix},\omega}(TS)$ for all $T \in \mathcal{L}^{1+}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. Moreover,

$$|\text{Tr}_{\text{Dix},\omega}(TS)| \leq \text{Tr}_{\text{Dix},\omega}(|TS|) \leq \|S\|_{\mathcal{B}(\mathcal{H})} \text{Tr}_{\text{Dix},\omega}(|T|) \leq \|S\|_{\mathcal{B}(\mathcal{H})} \|T\|_{1+}$$

and the kernel of $\text{Tr}_{\text{Dix},\omega}$ coincide with $\mathcal{L}_0^{1+}(\mathcal{H})$. The (*abstract*) Hölder inequality holds true, namely if $T_1, T_2 \in \mathcal{K}(\mathcal{H})$ such that $T_1^p, T_2^q \in \mathcal{L}^{1+}(\mathcal{H})$ with $p, q \in [0, \infty]$ and $1 = 1/p + 1/q$, then

$$\text{Tr}_{\text{Dix},\omega}(|T_1 T_2|) \leq (\text{Tr}_{\text{Dix},\omega}(|T_1|^p))^{\frac{1}{p}} (\text{Tr}_{\text{Dix},\omega}(|T_2|^q))^{\frac{1}{q}}.$$

As a consequence, one has that if $T \in \mathcal{L}^{1+}(\mathcal{H})$ is positive then also $T^{\frac{1}{2}} S T^{\frac{1}{2}} \in \mathcal{L}^{1+}(\mathcal{H})$ for every $S \in \mathcal{B}(\mathcal{H})$ and

$$\text{Tr}_{\text{Dix},\omega}(TS) = \text{Tr}_{\text{Dix},\omega}\left(T^{\frac{1}{2}} S T^{\frac{1}{2}}\right). \quad (\text{B.11})$$

In general no explicit general formula for $\text{Tr}_{\text{Dix},\omega}(T)$ can be given without specifying the state ω . However, we can at least rewrite it as a generalized limit of a sequence. If $\{a_N\} \in \ell^\infty(\mathbb{N})$ is a bounded sequence we can extend it piecewise-linearly to a function in $\mathcal{C}_b([\lambda_0, \infty))$. Let a_∞ be the image of this function in \mathfrak{B}_∞ . We write $\lim_\omega(a_N) := \omega(a_\infty)$. Clearly \lim_ω defines a positive linear functional on the space $\ell^\infty(\mathbb{N})$, coinciding with the ordinary limit on the subspace of convergent sequences. Moreover, \lim_ω has the *scale invariance property*, i. e. $\lim_\omega(\{a_1, a_2, a_3, a_4 \dots\}) = \lim_\omega(\{a_1, a_1, a_2, a_2 \dots\})$. With this identification in mind one has that

$$\text{Tr}_{\text{Dix},\omega}(T) = \lim_\omega(\gamma_N^1(T)), \quad T \in \mathcal{L}^{1+}(\mathcal{H}), \quad T \geq 0.$$

Then, the value of $\text{Tr}_{\text{Dix},\omega}(T)$ is independent of the choice of the particular state ω if and only if the operator T is measurable. Indeed, in this case the function $\lambda \mapsto \tau_\lambda(T)$ converges when $\lambda \rightarrow \infty$ and the common value of all the Dixmier traces is given by

$$\text{Tr}_{\text{Dix}}(T) = \lim_{\lambda \rightarrow \infty} \tau_\lambda(T) = \lim_{\lambda \rightarrow \infty} \frac{\sigma_\lambda(T)}{\log(\lambda)} = \lim_{N \rightarrow \infty} \gamma_N^1(T)$$

if T is positive, or by a linear combination of the above formula in the generic case. When $T \in \mathcal{L}^{1+}(\mathcal{H})$ is a measurable operator we will use the short notation $\text{Tr}_{\text{Dix}}(T)$ instead of $\text{Tr}_{\text{Dix},\omega}(T)$ in order to emphasize the independence of ω .

There are no general criteria for the calculation of the Dixmier trace. Nevertheless, the following result turns out to be very useful:

Lemma B.2 ([GBVF, Lemma 7.17]). *Let $T \in \mathcal{L}^{1+}(\mathcal{H})$ be a positive operator and let $S \in \mathcal{B}(\mathcal{H})$ be some bounded operator. Let $\{\psi_n \mid n \in \mathbb{N}\}$ be an orthonormal basis of eigenvectors of T (ordered according to the decreasing sequence of eigenvalues). Then*

$$\text{Tr}_{\text{Dix},\omega}(TS) = \lim_\omega \left(\frac{1}{\log(N)} \sum_{n=0}^{N-1} \langle \psi_n, TS \psi_n \rangle_{\mathcal{H}} \right).$$

The following result provides a useful criterion to determine whether $\text{Tr}_{\text{Dix}}(T)$ is independent of ω and to compute its value.

Theorem B.1 ([CM, Appendix A]). *Let $T \in \mathcal{L}^{1+}(\mathcal{H})$ be a positive operator and define the zeta function $\zeta_T(s) := \text{Tr}_{\mathcal{H}}(T^s)$. Then the following convergence conditions are equivalent*

$$\lim_{s \rightarrow 1^+} (s-1)\zeta_T(s) = L \quad \Leftrightarrow \quad \lim_{N \rightarrow \infty} \frac{1}{\log(N)} \sum_{n=0}^{N-1} \mu_n(T) = L$$

and $\text{Tr}_{\text{Dix}}(T) = L$ independently of the choice of the state ω .

For a detailed demonstration the reader can refer to [GBVF, Lemmas 7.19, 7.20] and references therein.

As a corollary of the Lemma B.2 we prove a technical result which will be useful in the following.

Lemma B.3. *Let $T \in \mathcal{L}^{p+}(\mathcal{H})$ and $M \in \text{Mat}_{\ell}(\mathbb{C})$. Then $T \otimes M \in \mathcal{L}^{p+}(\mathcal{H} \otimes \mathbb{C}^{\ell})$. Moreover, if $T \in \mathcal{L}^{1+}(\mathcal{H})$ then*

$$\text{Tr}_{\text{Dix},\omega}(T \otimes M) = \text{Tr}_{\text{Dix},\omega}(T) \text{Tr}_{\mathbb{C}^{\ell}}(M).$$

Proof. Since both $\mathcal{L}^{p+}(\mathcal{H})$ and $\text{Mat}_{\ell}(\mathbb{C})$ are generated by its positive elements, there is no loss of generality in assuming that $T \geq 0$ and $M \geq 0$. In this case we can use the characterization

$$\sigma_N^1(T \otimes M) = \sup \{ \text{Tr}_{\mathcal{H}}(P_V T P_V) \text{Tr}_{\mathbb{C}^{\ell}}(P_U M P_U) \mid \dim V + \dim U = N \}$$

where $V \subset \mathcal{H}$ and $U \subset \mathbb{C}^{\ell}$ are finite dimensional subspaces and P_V and P_U are the related orthogonal projections [GBVF, Lemma 7.32]. This leads immediately to the inequality $\sigma_N^1(T \otimes M) \leq \sigma_N^1(T) \text{Tr}_{\mathbb{C}^{\ell}}(M)$ and consequently $\|T \otimes M\|_{p+} \leq \|T\|_{p+} \text{Tr}_{\mathbb{C}^{\ell}}(M)$. Now, let us assume that $T \in \mathcal{L}^{1+}(\mathcal{H})$ and let $\{\varphi_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ be the orthonormal basis of eigenvectors of T (ordered according to the decreasing series of eigenvalues) and $\{e_j\}_{j=1, \dots, \ell}$ the canonical basis of \mathbb{C}^{ℓ} . Since $T \otimes \mathbf{1}_{\ell} \in \mathcal{L}^{1+}(\mathcal{H} \otimes \mathbb{C}^{\ell})$ for the above argument we can apply Lemma B.2 in order to prove that

$$\text{Tr}_{\text{Dix},\omega}(T \otimes M) = \lim_{\omega} \left(\frac{\log(N)}{\log(\ell N)} \frac{1}{\log(N)} \sum_{n=0}^{N-1} \sum_{j=1}^{\ell} \langle \psi_n, T \psi_n \rangle_{\mathcal{H}} \langle e_j, M e_j \rangle_{\mathbb{C}^{\ell}} \right).$$

This equality concludes the proof. \square

B.3. Dixmier trace and trace per unit volume. Let \mathcal{M}_B be the von Neumann algebra generated by the (spectral) Landau projections Π_j of the Landau Hamiltonian H_B . The algebra \mathcal{M}_B is abelian and its Gelfand spectrum is given by the pure states $\{\delta_k\}_{k \in \mathbb{N}_0}$ defined by $\delta_k(\Pi_j) = \delta_{k,j}$. As a consequence one has the Gelfand isomorphism $\mathcal{M}_B \simeq \ell^{\infty}(\mathbb{N}_0)$. Inside \mathcal{M}_B there is the ideal $\mathcal{M}_B^1 \subset \mathcal{M}_B$ defined by sequence in $\ell^1(\mathbb{N}_0)$, namely $T \in \mathcal{M}_B^1$ if and only if $\{t_k\}_{k \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ where $t_j := \delta_j(T)$. It turns out that \mathcal{M}_B^1 has the *Schur's property* [Meg, Example 2.5.24], namely weakly convergent sequences automatically converge in norm. The map

$$\int(T) := \sum_{k=0}^{\infty} \delta_k(T) = \sum_{k=1}^{\infty} t_k, \quad T \in \mathcal{M}_B^1 \quad (\text{B.12})$$

defines a faithful, semi-finite normal (FSN) trace on \mathcal{M}_B with domain \mathcal{M}_B^1 which coincides with the usual (discrete) integral on $\ell^1(\mathbb{N}_0)$. In this section we will provide two different formulas to compute this trace.

To construct the first formula we need some preliminary results. Let Q_B be the unbounded operator defined by (3.32) and $Q_{B,\xi}^{-s}$ be the compact operator defined by (3.33) for all $s > 0$ and $\xi \geq 0$. The next result concerns with the measurability properties of $Q_{B,\xi}^{-s}$.

Lemma B.4. *Let $Q_{B,\xi}^{-s}$ be the compact operator defined by (3.32) and (3.33). Then:*

(i) $Q_{B,\xi}^{-s}$ is trace class for every $s > 2$, $\xi \geq 0$ and

$$\mathrm{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s}) = \mathfrak{z}(s-1, 1+2\xi) - (1+2\xi) \mathfrak{z}(s, 1+2\xi) \quad (\text{B.13})$$

where \mathfrak{z} is the Hurwitz zeta function⁵;

(ii) $Q_{B,\xi}^{-s}$ is a measurable element of the Dixmier ideal and

$$\mathrm{Tr}_{\mathrm{Dix}}(Q_{B,\xi}^{-2}) = \frac{1}{2} \quad (\text{B.14})$$

independently of $\xi \geq 0$.

Proof. (i) The trace of $Q_{B,\xi}^{-s}$, when it exists, is given by the limit of the increasing sequence $\sigma_N^1(Q_{B,\xi}^{-s})$ of the first N eigenvalues of $Q_{B,\xi}^{-s}$ counted with their multiplicity. This sequence can only converge (whenever it is bounded) or diverge. Both situations can be controlled by the subsequence $\sigma_{j_N}^1(Q_{B,\xi}^{-s})$ of the first $j_N := \frac{1}{2}N(N+1)$ eigenvalues of $Q_{B,\xi}^{-s}$. An explicit computation provides

$$\begin{aligned} \sigma_{j_N}^1(Q_{B,\xi}^{-s}) &= \sum_{j=0}^{N-1} \frac{\mathrm{Mult}[\lambda_j]}{(\lambda_j + 2\xi)^s} = \sum_{j=1}^N \frac{j}{(j+1+2\xi)^s} \\ &= \sum_{j=1}^N \frac{1}{(j+1+2\xi)^{s-1}} - (1+2\xi) \sum_{j=1}^N \frac{1}{(j+1+2\xi)^s}. \end{aligned}$$

This series is absolutely convergent whenever $s > 2$ and $\xi \geq 0$ and in this case $\mathrm{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s}) = \lim_{N \rightarrow \infty} \sigma_{j_N}^1(Q_{B,\xi}^{-s})$ is given by (B.13).

(ii) To prove that $Q_{B,\xi}^{-2}$ is in the Dixmier ideal we need to study the sequence $\gamma_{N'}^1(Q_{B,\xi}^{-2}) := \log(N')^{-1} \sigma_{N'}^1(Q_{B,\xi}^{-2})$ with $N' \in \mathbb{N}$. By observing that for every $N' \in \mathbb{N}$ there is a $N := N(N')$ such that $j_N \leq N' \leq j_{N+1}$ one has that

$$\frac{\log(j_N)}{\log(j_{N+1})} \gamma_{j_N}^1(Q_{B,\xi}^{-2}) \leq \gamma_{N'}^1(Q_{B,\xi}^{-2}) \leq \frac{\log(j_{N+1})}{\log(j_N)} \gamma_{j_{N+1}}^1(Q_{B,\xi}^{-2}).$$

Since $\lim_{N \rightarrow \infty} \frac{\log(j_N)}{\log(j_{N+1})} = 1$ one infers that the convergence of the sequence $\gamma_{N'}^1(Q_{B,\xi}^{-2})$ is equivalent to the convergence of the subsequence

$$\gamma_{j_N}^1(Q_{B,\xi}^{-2}) = \frac{1}{\log(j_N)} \left(\sum_{j=1}^N \frac{1}{j+1+2\xi} - (1+2\xi) \sum_{j=1}^N \frac{1}{(j+1+2\xi)^2} \right).$$

⁵The Hurwitz zeta function is defined by the absolutely convergent series $\mathfrak{z}(s, \xi) := \sum_{j=0}^{\infty} (j+\xi)^{-s}$ for every $s > 0$ and $\xi > 0$. The Riemann zeta function is $\mathfrak{z}_0(s) := \mathfrak{z}(s, 1)$.

From the absolute convergence of the second series inside the brackets and by observing that $\lim_{N \rightarrow \infty} \frac{\log(N)}{\log(jN)} = \frac{1}{2}$ one gets that

$$\lim_{N \rightarrow \infty} \gamma_{jN}^1(Q_{B,\xi}^{-2}) = \frac{1}{2} \lim_{N \rightarrow \infty} \frac{1}{\log(N)} \sum_{j=1}^N \frac{1}{j+1+2\xi} = \frac{1}{2}.$$

The equality $\text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-2}) = \lim_{N \rightarrow \infty} \gamma_N^1(Q_{B,\xi}^{-2}) = \lim_{N \rightarrow \infty} \gamma_{jN}^1(Q_{B,\xi}^{-2})$ concludes the proof. \square

Item (i) of Lemma B.4 can be equivalently stated by saying that $Q_{B,\xi}^{-1}$ is an element of the Schatten ideal $\mathcal{L}^s(L^2(\mathbb{R}^2))$ for all $s > 2$. The measurability of $Q_{B,\xi}^{-2}$ implies that the value of the Dixmier trace in Lemma B.4 (ii) is defined unambiguously. Moreover, from (B.13) one gets

$$\lim_{s \rightarrow 2^+} (s-2) \text{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s}) = \lim_{s \rightarrow 1^+} (s-1) \mathfrak{z}(s, 1+2\xi) = 1$$

which, along with (B.14), implies

$$\text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-2}) = \frac{1}{2} \lim_{s \rightarrow 2^+} (s-2) \text{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s})$$

in accordance with the Connes-Moscovici residue formula described in Theorem B.1.

The measurability properties of $Q_{B,\xi}^{-s}$ change when $Q_{B,\xi}^{-s}$ is multiplied by a Landau projection Π_j .

Lemma B.5. *Let $Q_{B,\xi}^{-s}$ be the compact operator defined by (3.32) and (3.33) and Π_j the j -th Landau projection. Then:*

(i) $Q_{B,\xi}^{-s} \Pi_j$ is trace class for every $s > 1$, $\xi \geq 0$ and

$$\text{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s} \Pi_j) = \mathfrak{z}(s, j+2(1+\xi)) \quad (\text{B.15})$$

where \mathfrak{z} is the Hurwitz zeta function;

(ii) $Q_{B,\xi}^{-1} \Pi_j$ is a measurable element of the Dixmier ideal and

$$\text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} \Pi_j) = 1 \quad (\text{B.16})$$

independently of $\xi \geq 0$.

Proof. (i) The spectrum of $Q_{B,\xi}^{-s} \Pi_j$ is given by

$$\sigma(Q_{B,\xi}^{-s} \Pi_j) = \{(k+j+2(1+\xi))^{-s} \mid k \in \mathbb{N}_0\}$$

and all the eigenvalues are simple. As a consequence one has that

$$\sigma_N^1(Q_{B,\xi}^{-s} \Pi_j) = \sum_{k=0}^{N-1} \frac{1}{[k+j+2(1+\xi)]^s}.$$

This series is absolutely convergent whenever $s > 1$ and in this case $\text{Tr}_{L^2(\mathbb{R}^2)}(Q_{B,\xi}^{-s} \Pi_j) = \lim_{N \rightarrow \infty} \sigma_N^1(Q_{B,\xi}^{-s} \Pi_j)$ is given by (B.15).

(ii) To compute the Dixmier trace we need to analyze the sequence

$$\gamma_N^1(Q_{B,\xi}^{-1} \Pi_j) = \frac{1}{\log(N)} \sum_{k=1}^N \frac{1}{k+(j+1+2\xi)}. \quad (\text{B.17})$$

This series converges to 1 proving the formula (B.16). \square

We are now in position to provide the first formula to compute the integral (B.12). This involves the Dixmier trace and the operator $Q_{B,\xi}^{-1}$.

Lemma B.6. *The equality*

$$\int(T) = \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1}T)$$

holds true for all $T \in \mathcal{M}_B^1$, independently of $\xi \geq 0$.

Proof. In the proof we will use that the Dixmier ideal $\mathcal{L}^{1+}(\mathcal{H})$ endowed with the Calderón norm $\|\cdot\|_{1+}$ is a Banach (hence closed) space. Moreover, from definition (B.4) it follows that

$$\|A\|_{\mathcal{B}(\mathcal{H})} = \mu_0(A) \leq \log(2) \frac{\mu_0(A) + \mu_1(A)}{\log(2)} \leq \|A\|_{1+} \quad (\text{B.18})$$

for every $A \in \mathcal{L}^{1+}(\mathcal{H})$. From Lemma B.5 (ii) one gets that $Q_{B,\xi}^{-1}\Pi_j$ is a measurable element of $\mathcal{L}^{1+}(L^2(\mathbb{R}^2))$. Moreover, from (B.17) it follows that

$$\gamma_N^1(Q_{B,\xi}^{-1}\Pi_j) \leq \frac{1}{\log(N)} \sum_{k=1}^N \frac{1}{k + (1 + 2\xi)} = \gamma_N^1(Q_{B,\xi}^{-1}\Pi_0)$$

and in turn

$$\|Q_{B,\xi}^{-1}\Pi_j\|_{1+} \leq \|Q_{B,\xi}^{-1}\Pi_0\|_{1+}, \quad \forall j \in \mathbb{N}_0. \quad (\text{B.19})$$

Now, let $T \in \mathcal{M}_B^1$. Then, there is a $\{t_k\}_{k \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ such that $T = \sum_{k=0}^{\infty} t_k \Pi_k$. The Schur's property assures that the operator $Q_{B,\xi}^{-1}T$ is the uniform norm limit of the partial sums $Q_{B,\xi}^{-1}T_N = \sum_{k=0}^N t_k Q_{B,\xi}^{-1}\Pi_k$. From (B.19) one can prove that the sequence $\{Q_{B,\xi}^{-1}T_N\}_{N \in \mathbb{N}_0}$ is indeed Cauchy in $\mathcal{L}^{1+}(L^2(\mathbb{R}^2))$, hence it converges to an element in the Dixmier ideal. From (B.18) and the uniqueness of the limit, one gets that this limit coincide with $Q_{B,\xi}^{-1}T$ proving that $Q_{B,\xi}^{-1}T \in \mathcal{L}^{1+}(L^2(\mathbb{R}^2))$ for all $T \in \mathcal{M}_B^1$. Finally, from the linearity of the Dixmier trace and the continuity of the Dixmier trace with respect to the $\|\cdot\|_{1+}$ -norm, one easily gets that

$$\text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1}T) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \text{Tr}_{\text{Dix}}(t_k Q_{B,\xi}^{-1}\Pi_k) = \sum_{k=0}^{\infty} t_k$$

and this concludes the proof. \square

The second formula for (B.12) needs the concept of *trace per unit volume* \mathcal{T}_B . Let us start with a preliminary result. Let $\Lambda \subset \mathbb{R}^2$ be any compact subset and χ_Λ the characteristic function of the set Λ . The function χ_Λ acts on $L^2(\mathbb{R}^2)$ as a (multiplication) self-adjoint projection.

Lemma B.7. *Let $\Lambda \subset \mathbb{R}^2$ be a compact subset and Π_j the j -th Landau projection. Then $\Pi_j \chi_\Lambda$ and $\chi_\Lambda \Pi_j$ are trace class and*

$$\text{Tr}_{L^2(\mathbb{R}^2)}(\Pi_j \chi_\Lambda) = \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_\Lambda \Pi_j) = \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_\Lambda \Pi_j \chi_\Lambda) = \frac{|\Lambda|}{2\pi \ell_B^2}$$

with $|\Lambda|$ the volume of the set Λ .

Proof. From (3.16) one gets

$$|\Pi_j(x, y)| = \frac{1}{2\pi\ell_B^2} e^{-\frac{|x-y|^2}{4\ell_B^2}} \left| L_j^{(0)} \left(\frac{|x-y|^2}{2\ell_B^2} \right) \right| \leq C_j e^{-\frac{|x-y|^2}{8\ell_B^2}}$$

where $C_j := \frac{\alpha_j}{2\pi\ell_B^2}$ and $\alpha_j := \max_{\zeta \geq 0} \left| e^{-\frac{\zeta}{4}} L_j^{(0)}(\zeta) \right|$. Let M_g be the operator of multiplication by the function $g(x) := e^{-\frac{|x|^2}{16\ell_B^2}}$. M_g^{-1} is the multiplication operator by the function $1/g(x)$. One has the identity

$$\chi_\Lambda \Pi_j = (\chi_\Lambda \Pi_j M_g^{-1})(M_g \Pi_j).$$

A direct inspection shows that both the operators $\chi_\Lambda \Pi_j M_g^{-1}$ and $M_g \Pi_j$ have an integral kernel which is in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$. Therefore $\chi_\Lambda \Pi_j M_g^{-1}$ and $M_g \Pi_j$ are Hilbert-Schmidt and $\chi_\Lambda \Pi_j$ is trace class. The trace of $\chi_\Lambda \Pi_j$ can be computed by integrating along the diagonal the integral kernel (3.16). The result is exactly $\frac{|\Lambda|}{2\pi\ell_B^2}$. Since trace class operators form an ideal it follows that $\chi_\Lambda \Pi_j \chi_\Lambda$ is also trace class. The equality $\text{Tr}_{L^2(\mathbb{R}^2)}(\chi_\Lambda \Pi_j) = \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_\Lambda \Pi_j \chi_\Lambda)$ follows from the cyclicity of the trace. The claim for $\Pi_j \chi_\Lambda$ can be proved with a similar argument. \square

Let $\Lambda_n \subseteq \mathbb{R}^2$ be any increasing sequence of compact subsets such that $\Lambda_n \nearrow \mathbb{R}^2$ which meet the *Følner* condition (see e. g. [Gre] for more details). Typical examples for the Λ_n are an increasing sequence of concentric cubes or disks. From Lemma B.7 it follows that

$$\mathcal{T}_B(\Pi_j) := \lim_{n \rightarrow \infty} \frac{1}{|\Lambda_n|} \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_{\Lambda_n} \Pi_j \chi_{\Lambda_n}) = \frac{1}{2\pi\ell_B^2}. \quad (\text{B.20})$$

The quantity \mathcal{T}_B is by definition the trace per unit volume.

Remark B.1 (IDOS of H_B). The trace per unit volume is usually used to define the *integrate density of states* (IDOS) [Ves]. In the case of the Hamiltonian H_B the IDOS is given by [Nak, Appendix B]

$$N_B(E) := \mathcal{T}_B(\chi_E(H_B)) = \frac{1}{2\pi\ell_B^2} \sum_{j=0}^{+\infty} \Theta(E - E_j)$$

where χ_E is the characteristic function of the interval $[0, E]$, Θ is the Heaviside step function and the E_j are the energy levels (3.13). This result follows immediately from (B.20). \blacktriangleleft

All the elements of \mathcal{M}_B^1 admit a trace per unit volume. This is shown in the following result which provides the second formula for the computation fo the integral B.12.

Lemma B.8. *The equality*

$$\int(T) = 2\pi\ell_B^2 \mathcal{T}_B(T)$$

holds true for all $T \in \mathcal{M}_B^1$.

Proof. The strategy of the proof is the same as in Lemma B.6. Let $T \in \mathcal{M}_B^1$. There exists a $\{t_k\}_{k \in \mathbb{N}_0} \in \ell^1(\mathbb{N}_0)$ such that $T = \sum_{k=0}^{\infty} t_k \Pi_k$. Moreover, the Schur's property assures that T can be obtained as the uniform norm limit of the partial sums $T_N := \sum_{k=0}^N t_k \Pi_k$. Let $\Lambda \subset \mathbb{R}^2$ be any compact subset. From Lemma B.7 we know that

$$\|\chi_\Lambda \Pi_k \chi_\Lambda\|_1 = \frac{|\Lambda|}{2\pi \ell_B^2}$$

independently of k . This fact can be used to conclude that $\chi_\Lambda T \chi_\Lambda$ is a trace class operator. Indeed by linearity $\chi_\Lambda T_N \chi_\Lambda$ is trace class for all $N \in \mathbb{N}_0$ and for every $M > N$

$$\|\chi_\Lambda T_M \chi_\Lambda - \chi_\Lambda T_N \chi_\Lambda\|_1 \leq \frac{|\Lambda|}{2\pi \ell_B^2} \sum_{N+1}^M |t_k|$$

showing that $\{\chi_\Lambda T_N \chi_\Lambda\}_{N \in \mathbb{N}_0}$ is a Cauchy sequence with respect to the trace-norm $\|\cdot\|_1$. Since the space of trace class operators is a Banach (hence closed) space with respect to the trace-norm it follows that the $\|\cdot\|_1$ -limit of $\chi_\Lambda T_N \chi_\Lambda$ defines a trace class operator. From the uniqueness of the limit and the fact that trace-norm dominates the operator norm one gets that $\chi_\Lambda T \chi_\Lambda$ is indeed a trace class operator. Moreover since the trace is $\|\cdot\|_1$ -continuous one obtains

$$\begin{aligned} \text{Tr}_{L^2(\mathbb{R}^2)}(\chi_\Lambda T \chi_\Lambda) &= \lim_{N \rightarrow \infty} \text{Tr}_{L^2(\mathbb{R}^2)} \left(\sum_{k=0}^N t_k \chi_\Lambda \Pi_k \chi_\Lambda \right) \\ &= \lim_{N \rightarrow \infty} \frac{|\Lambda|}{2\pi \ell_B^2} \sum_{k=0}^N t_k = \frac{|\Lambda|}{2\pi \ell_B^2} \int(T). \end{aligned}$$

The last equality, along with the arbitrariness of Λ , implies the claim.

Let $\Omega_B := \pi \ell_B^2$ be the area of the magnetic disk of radius ℓ_B . As a consequence of Lemma B.6 and Lemma B.8 one obtains the following result:

Theorem B.2. *The equality*

$$\mathcal{T}_B(T) = \frac{1}{2\Omega_B} \text{Tr}_{\text{Dix}}(Q_{B,\xi}^{-1} T)$$

holds true for all $T \in \mathcal{M}_B^1$, independently of $\xi \geq 0$.

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