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The Expected Markov Property for Quantum Markov Fields

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Abstract. After a short introduction to the Algebraic formulation of classical stochastic processes and fields, we extend to the quantum case the notion of expected classical Markov field and we prove the equivalence of several formulations of the multi-dimensional Markov property. To this goal we need a Kadison–Schwarz inequality for completely positive maps on $*$-algebras which is proved in the appendix.

Contents

1. Introduction

The notion of markovianity (or conditional independence, or localized dependence) was introduced in probability theory and plays an important role in all scientific disciplines where probability plays a role, from physics to engineering to economics. More recently this notion entered also in disciplines traditionally far from probability theory, such as operator theory [5] where it has been re-named *the commuting square property*, in computer theory and in the theory of Bayesian networks [10].

It is known that the difficulty to construct a quantum theory of Markov processes is rooted in the fact that, given a von Neumann algebra $\mathcal{A}$, a state φ on it

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and a von Neumann sub-algebra $\mathcal{B} \subset \mathcal{A}$ a conditional expectation E, from $\mathcal{A}$ onto $\mathcal{B}$ compatible with φ (i.e. $\varphi \circ E = \varphi$), in general does not exist [11]. If it exists, $\mathcal{B}$ is called **φ-expected**. There exists however a class of quantum Markov processes, important for the applications to the theory of quantum open systems [3], with the property that all the algebras of the past filtration of any processes in this class are φ-expected, where φ is the state defining the expectations of the process. The processes in this class are called expected (in the classical case all processes are expected). In the classical case, the theory of Markov processes has been extended by Nelson [8]. to fields, i.e. processes whose index set is not 1-dimensional (interpreted as *time* in classical probability), but multi-dimensional, for example $\mathbb{R}^d$ (interpreted as *position* in statistical mechanics and quantum field theory) and this extension has found important applications in Euclidean quantum field theory.

It is therefore natural to ask oneself if such a multi-dimensional extension can be done also in the quantum case. In the first part of this paper we prove that such an extension exists in the case of expected Markov fields (see Theorem 6 where it is shown that all equivalent formulations of the classical multi-dimensional Markov property continue to hold in the quantum case). In particular Theorem 6 extends to quantum fields the known equivalence between the forward and the backward classical Markov property.

Since multi-dimensional index sets have no natural total order, for them the terms *forward* and *backward*, used in classical probability, make no sense because in this case one conditions the configurations of the field inside a volume on the configurations outside. In order to use a language that includes both cases, we speak respectively of *out–in* and *in–out* Markov property. Finally in Sect. 4 we prove that 2-positive maps on $*$-algebras satisfy the Kadison–Schwarz inequality. For completely positive maps, the KS inequality is a simple consequence of the Stinespring representation which holds on $*$-algebras (see [7]). Our proof for 2-positive maps does not depend on the Stinespring representation.

2. Algebraic Formulation of Classical Stochastic Processes and Fields

The following is the definition of stochastic process as originally formulated by Kolmogorov [6].

Definition 1. A classical stochastic process is a triple $X \equiv \{(\Omega, \mathcal{F}, \mu), (S_t, \mathcal{B}_t)_{t\in T}, (X_t)_{t\in T}\}$ where T is a set, $(\Omega, \mathcal{F}, \mu)$ is a probability space, for each $t \in T$, $(S_t, \mathcal{B}_t)$ is a measurable space and $X_t \colon (\Omega, \mathcal{F}) \to (S_t, \mathcal{B}_t)$ is a measurable function. The finite dimensional joint expectations of X are the expectation values

$$E\left(f_1(X_{t_1}) \cdots f_n(X_{t_n})\right)$$

where $n \in \mathbb{N}$, $\{t_1, \dots t_n\}$ is a finite sub-set of T and, for each $j \in \{1, \dots n\}$, $f_j \in L^\infty_{\mathbb{C}}(S_t, \mathcal{B}_t)$. Two processes, with the same index set and the same state spaces are called **stochastically equivalent** if their finite dimensional joint expectations coincide.

Let

$$\mathcal{I}_{fin}(T) := \text{ the family of finite sub-sets of } T$$

For $F \in \mathcal{I}_{fin}(T)$, denote $\mathcal{F}_F$ the σ-algebra generated by the random variables $\{X_t \ \ t \in F\}$, i.e.

$$\mathcal{F}_F = \bigvee_{t \in F} X_t^{-1}(\mathcal{B})$$

let μ_F be the restriction of μ on $\mathcal{F}_F$ and denote

$$\mathcal{A}_F = L^\infty(\Omega, \mathcal{F}_F, \mu_F)$$
$$\mathcal{A}_{loc} = \bigcup_{F \in \mathcal{I}_{fin}(T)} \mathcal{A}_F$$
$$\mathcal{A} = \text{ norm closure of } \bigcup_{F \in \mathcal{I}_{fin}(T)} \mathcal{A}_F$$

the norm closure being meant in the sense of the sup-norm on $L^\infty(\Omega, \mathcal{F}, \mu)$. Throughout the present paper we shall adopt, for what concerns C^*- and W^*-algebras, the notations and terminology of S. Sakai's monograph [9]. The measure μ induces a state (i.e. a positive normalized linear functional), still denoted μ, on the C^*-algebra $\mathcal{A}$, defined by:

$$\mu(f) = \int_\Omega f d\mu; \quad f \in \mathcal{A}$$

Thus, every classical stochastic process $X \equiv \{(\Omega, \mathcal{F}, \mu), (S, \mathcal{B}), (X_t)_{t \in T}\}$, it is naturally associated the triple:

$$(\mathcal{A}, (\mathcal{A}_F)_{F \in \mathcal{I}_{fin}(T)}, \mu)$$

Proposition 2. *Two stochastic processes with associated triples* $(\mathcal{A}, (\mathcal{A}_F)_{F \in \mathcal{I}_{fin}(T)}, \mu)$ *and* $(\mathcal{A}', (\mathcal{A}'_F)_{F \in \mathcal{I}_{fin}(T)}, \mu')$ *are* ***stochastically equivalent*** *if there exists an isomorphism of* C^**-algebras*

$$u : \mathcal{A} \to \mathcal{A}'$$

such that

$$u(\mathcal{A}_F) = \mathcal{A}'_F; \qquad F \in \mathcal{F}$$
$$\mu' \cdot u = \mu$$

Proof. See [1].

Remark. An element $\varphi \in \mathcal{A}_F$ $(F \in \mathcal{I}_{fin}(T))$ is a bounded measurable function of the random variables X_t $(t \in F)$. Often one has to deal with bounded measurable functions of all the random variables X_t with index t in an arbitrary sub-set $I \subseteq T$: these are precisely the measurable function for the σ-algebra $\mathcal{F}_I$ generated by $\{X_t \colon t \in I\}$ (see [4]). Typical examples where this situation arises are given by the Feynmann–Kac formulas, when one considers functionals of the type $\int_0^t V(X_s ds)$ where V, interpreted as *potential*, is a measurable function such that the integral exists and $t > 0$. This leads to the consideration of local algebras $\mathcal{A}_I$ where I is no longer a finite subset of T. For any sub-set $I \subseteq T$ one can define the σ-algebra

$$\mathcal{F}_I = \bigvee_{t \in I} X_t^{-1}(\mathcal{B}); \quad \mu_I = \mu\Big|_{\mathcal{F}_I}$$

and the corresponding $*$-algebra (in this case a C^*-algebra)

$$\mathcal{A}_I = L^\infty(\Omega, \mathcal{F}_I, \mu_I)$$

The choice of a family of local algebras associated to a stochastic process is not canonical but depends on the process. For example, if the probability measure has all moments, $L^\infty(\Omega, \mathcal{F}_I, \mu_I)$ can be replaced by the algebra of the algebra of all polynomials in the random variables X_t, with $t \in I$. This gives a natural example of a $*$-algebra which is not a C^*-algebra. Most of these choices are sufficient to determine the class of stochastic equivalence of the process in the sense that their weak closures in the *GNS* representation associated to the state are algebraically isomorphic (cf. [2] for a precise formulation).

3. Algebraic Stochastic Processes

As usual in the transition from commutative to non-commutative, once given a purely algebraic formulation of a classical category, one can extend this category by dropping the requirement that the algebras involved are commutative. In this section we sum up some results of this extension method in the case of the category of **classical stochastic processes**.

It is possible to characterize, up to different notions of stochastic equivalence, those triples $\{\mathcal{A}, (\mathcal{A}_I), \mu\}$ which come from classical stochastic processes in Kolmogorov's sense (cf. [1]). More generally, every triple $\{\mathcal{A}, (\mathcal{A}_I), \varphi\}$ such that

- $\mathcal{A}$ is a commutative $*$-algebra.
- $\mathcal{A}_I$ is a $*$-algebra $\subseteq \mathcal{A}$.
- $\mathcal{I}$ is a family of sub-sets of T. Typically $\mathcal{I}$ is an increasing net, by inclusion, the union of whose elements is T.
- $\mathcal{A} =$ union of $\bigcup\{\mathcal{A}_I : I \in \mathcal{I}\}$.
- $I \subseteq G \Rightarrow \mathcal{A}_I \subseteq \mathcal{A}_G$, $I \in \mathcal{I}$.
- φ is a state on $\mathcal{A}$.

defines a classical stochastic process. This motivates the following definition.

Definition 3. Let T be a set, $\mathcal{I}$ a family of sub-sets of T and $\mathcal{A}$ a $*$-algebra. A family $(\mathcal{A}_I)_{I\in\mathcal{I}}$ of sub-$*$-algebras of $\mathcal{A}$ satisfying

$$I \subseteq J \ \Rightarrow \ \mathcal{A}_I \subseteq \mathcal{A}_J \quad , \quad \forall I, J \in \mathcal{I}$$

is called a **localization** on $\mathcal{A}$ based on T (a W^*-localization (resp. C^*-localization) if the $\mathcal{A}_I$ are W^*-algebras (resp. C^*-algebras)). In this case, we also say that the pair $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{I}}\}$ is **a family of local algebras**.

Definition 4. A triple $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{I}}, \varphi\}$ where $\{\mathcal{A}, (\mathcal{A}_I)\}$ is a family of local algebras and φ is a state on $\mathcal{A}$ will be called a (algebraic) stochastic process localized on $\mathcal{I}$ (a **quantum stochastic process** if $\mathcal{A}$ is not abelian), classical if it is. If the $\mathcal{A}_I$ are W^*-algebras φ is required to be locally normal.

Remark. Any triple $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{F}}, \varphi\}$ with $\mathcal{A}$ abelian determines an **equivalence class** of classical stochastic processes. (cf. [1,2] for a more detailed discussion).

Remark. The representation theory of the CCR and the CAR provides several examples of quantum stochastic process in the sense of Definition 4.

3.1. Conditional Expectations

Let $\{\mathcal{A},(\mathcal{A}_I)_{I\in\mathcal{I}},\varphi\}$ be a stochastic process localized on $\mathcal{I}$ in the sense of Definition 4. For $I\in\mathcal{I}$, a **Umegaki conditional expectation** $\mathcal{A}\to\mathcal{A}_I$ is a linear map $E_I:\mathcal{A}\to\mathcal{A}_I$ such that

$$\begin{aligned} a\geq 0 \quad a\in\mathcal{A} \ \Rightarrow E_I(a)&\geq 0 \\ E_I(1)&=1 \\ E_I(a_Ia)=a_IE_I(a) \quad a_I\in\mathcal{A}_I \quad a&\in\mathcal{A} \\ E_I(a^*)=E_I(a)^* \quad a&\in\mathcal{A} \end{aligned}$$

Such a map is automatically completely positive. For the notion of complete positivity in $*$-algebras, see [7]. E_I is called **faithful** if

$$a\in\mathcal{A} \ \ a\geq 0 \ \ a\neq 0\Rightarrow E_I(a^*a)>0$$

E_I is called **compatible** with a state φ on $\mathcal{A}$ if

$$\varphi\circ E_I=\varphi$$

A family of conditional expectations $(E_I)_{I\in\mathcal{I}}$, $E_I:\mathcal{A}\to\mathcal{A}_I$, is called **projective** if

$$I\subseteq J\Rightarrow E_IE_J=E_I \tag{3.1}$$

In the classical case, the conditional expectations compatible with φ

$$E_I:\mathcal{A}\to\mathcal{A}_I=L^\infty(\Omega,\mathcal{F}_I,\mu_I)$$

are always defined and satisfy (3.1).

3.2. Examples of Umegaki Conditional Expectations on $*$-Algebras

A family of local algebras $\{\mathcal{A},(\mathcal{A}_I)_{I\in\mathcal{I}}\}$ is called **factorizable** if

$$I,J\in\mathcal{I} \ \ I\cap J=\emptyset\Rightarrow\mathcal{A}_{I\cup J}\sim\mathcal{A}_I\otimes\mathcal{A}_J \tag{3.2}$$

where $\sim$ denotes isomorphism of $*$-algebras and $\otimes$ denotes the algebraic tensor product. A state φ on $\mathcal{A}$ is called factorizable if, for any I,J as in 3.2,

$$\varphi(a_I\otimes a_J)=\varphi(a_I)\varphi(a_J) \quad , \quad \forall a_I\in\mathcal{A}_I \ \ a_J\in\mathcal{A}_J$$

If both $\{\mathcal{A},(\mathcal{A}_I)_{I\in\mathcal{I}}\}$ and φ are factorizable, for every $I\in\mathcal{I}$, one can define the map

$$E_I\colon a_I\otimes a_{I^c}\in\mathcal{A}\sim\mathcal{A}_I\otimes\mathcal{A}_{I^c}\to a_I\varphi(a_{I^c})\in\mathcal{A}_I \tag{3.3}$$

and one easily verifies that E_I satisfies all the properties that define a Umegaki conditional expectation. The CCR algebra over $L^2(\mathbb{R}^d)$ ($d\in\mathbb{N}$), i.e. the $*$-algebra with generators $\{A_f,A_g^+\colon f,g\in L^2(\mathbb{R}^d)\}$ and relations

$$[A_f,A_g^+]=\langle f,g\rangle$$

is a typical example of factorizable local algebra. The Fock state on it φ_F, characterized by the property

$$\varphi_F(A_f^+A_f)=0, \quad \forall f\in L^2(\mathbb{R}^d) \tag{3.4}$$

is a typical example of factorizable state. The associated conditional expectations E_I are not faithful because of (3.4). Examples of factorizable state whose associated conditional expectations are faithful are the Gaussian (β, z)-Equilibrium states with pair correlations

$$\varphi_{\beta,z}(A_f A_g^+) := \int_{\mathbb{R}^d} \frac{z}{e^{\beta\omega_k} - z} \bar{f}(k) g(k) dk$$

(where $\beta > 0$ and $z \geq 1$ are constants and $k \in \mathbb{R}^d \mapsto \omega_k$ is an almost everywhere strictly positive function) and the Gaussian $(\beta(\,\cdot\,), z)$-local equilibrium states with pair correlations

$$\varphi_{\beta,z}(A_f A_g^+) := \int_{\mathbb{R}^d} \frac{z}{e^{\beta_k\omega_k} - z} \bar{f}(k) g(k) dk$$

where z and ω_k are as above and $k \in \mathbb{R}^d \mapsto \beta_k$ is an almost everywhere strictly positive function.

Conditional expectations of this kind are routinely used, for example in quantum stochastic calculus and in the theory of open systems.

3.3. The Markov Property for Completely Positive Maps

In this section we consider a family of local algebras $\{\mathcal{A}, (\mathcal{A}_I)_{I\in\mathcal{I}}\}$ and triples $I, \partial I, I'$ of elements of $\mathcal{I}$ such that $\partial I \subseteq I \cap I'$ such that

$$\mathcal{A}_{\partial I} \subseteq \mathcal{A}_I \cap \mathcal{A}_{I'} \tag{3.5}$$

If $T \subseteq \mathbb{R}$ is interpreted as *time* and

$$I = (-\infty, t] \quad ; \quad \partial I = \{t\}; \quad I' = [t, \infty)$$

one calls them respectively **the past, the present and the future algebra**; if $T \subseteq \mathbb{R}^d$ is interpreted as *space*, one often calls them **the interior, the boundary and the exterior algebra** but many other interpretations are possible. We suppose that

$$E_X : \mathcal{A} \to \mathcal{A}_X; \qquad X = I, I', \partial I \tag{3.6}$$

are completely positive maps.

Definition 5. A map $E_I : \mathcal{A} \to \mathcal{A}_I$ is said to enjoy the **Markov property** with respect to the triple

$$\mathcal{A}_{\partial I} \subseteq \mathcal{A}_I \subseteq \mathcal{A} \tag{3.7}$$

if

$$E_I(\mathcal{A}_{I'}) \subseteq \mathcal{A}_{\partial I} \tag{3.8}$$

3.4. The Markov Property for Umegaki Conditional Expectations

Suppose that the maps (3.6) are Umegaki conditional expectations satisfying the projectivity condition

$$E_{\partial I} E_I = E_{\partial I} E_{I'} = E_{\partial I} \tag{3.9}$$

In the formulation (3.7) of the Markov property, the past and the future do not enter in a symmetric way. There is an **essentially** equivalent formulation in which the roles of past and future are symmetric.

Theorem 6. *Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{A}_I, \mathcal{A}_{I'}, \mathcal{A}_{\partial I}$ $*$-sub-algebras of $\mathcal{A}$ satisfying*

$$\mathcal{A}_{\partial I} \subseteq \mathcal{A}_I \cap \mathcal{A}_{I'} \tag{3.10}$$

For a triple of Umegaki conditional expectations

$$E_I : \mathcal{A} \to \mathcal{A}_I; \quad E_{I'} : \mathcal{A} \to \mathcal{A}_{I'}; \quad E_{\partial I} : \mathcal{A} \to \mathcal{A}_{\partial I}$$

consider the following identities (where, here and in the following, $\Big|_{\cdot}$ denotes restriction):

$$E_I\Big|_{\mathcal{A}_{I'}} = E_{\partial I}\Big|_{\mathcal{A}_{I'}} \tag{3.11}$$

$$E_I E_{I'} = E_{\partial I} \tag{3.12}$$

$$E_I(\mathcal{A}_{I'}) \subseteq \mathcal{A}_{\partial I} \tag{3.13}$$

$$E_{\partial I}(a_I a_{I'}) = E_{\partial I}(a_I) \cdot E_{\partial I}(a_{I'}); \qquad \forall a_I \in \mathcal{A}_I \;\; a_{I'} \in \mathcal{A}_{I'} \tag{3.14}$$

$$E_{\partial I}(a_{I'} a_I) = E_{\partial I}(a_{I'}) E_{\partial I}(a_I); \qquad \forall a_I \in \mathcal{A}_I \;\; a_{I'} \in \mathcal{A}_{I'} \tag{3.15}$$

$$E_{I'}\Big|_{\mathcal{A}_I} = E_{\partial I}\Big|_{\mathcal{A}_I} \tag{3.16}$$

$$E_{I'} E_I = E_{\partial I} \tag{3.17}$$

$$E_{I'}(\mathcal{A}_I) \subseteq \mathcal{A}_{\partial I} \tag{3.18}$$

(i) *One has:* (3.11) $\iff$ (3.12) $\iff$ (3.13) $\Rightarrow$ (3.14), (3.15) $\Leftarrow$ (3.18) $\iff$ (3.17) $\iff$ (3.16).

(ii) *If* (3.15) *holds, then for each $a_I \in \mathcal{A}_I$ and $a_{I'} \in \mathcal{A}_{I'}$*

$$E_{\partial I}\left(|E_{\partial I}(a_{I'}) - E_I(a_{I'})|^2 \right) = 0 \tag{3.19}$$

If (3.14) *holds, then for each $a_I \in \mathcal{A}_I$ and $a_{I'} \in \mathcal{A}_{I'}$*

$$E_{\partial I}\left(|E_{\partial I}(a_I) - E_{I'}(a_I)|^2 \right) = 0 \tag{3.20}$$

(iii) *If $E_{\partial I}$ is **faithful** then all the identities* (3.11), . . . , (3.18) *are equivalent.*

(iv) *If both* (3.12) *and* (3.17) *hold, then*

$$E_{I'} E_I = E_I E_{I'} = E_{\mathcal{A}_I \cap \mathcal{A}_{I'}} = E_{\partial I} \quad ; \quad \mathcal{A}_I \cap \mathcal{A}_{I'} = \mathcal{A}_{\partial I} \tag{3.21}$$

Proof. (i) We first prove that (3.11) $\Leftrightarrow$(3.12)$\Rightarrow$ (3.13) $\Rightarrow$ (3.11). Because of surjectivity and projectivity, (3.11) is equivalent to

$$E_I E_{I'} = E_{\partial I} E_{I'} = E_{\partial I}$$

which is (3.12). If (3.12) holds then

$$E_I(\mathcal{A}_{I'}) = E_I E_{I'}(\mathcal{A}) = E_{\partial I}(\mathcal{A}) \subseteq \mathcal{A}_{\partial I}$$

which is (3.13). If (3.13) holds, then for any $a_{I'} \in \mathcal{A}_{I'}$ $E_I(a_{I'}) \in \mathcal{A}_{\partial I}$, by projectivity

$$E_I(a_{I'}) = E_{\partial I} E_I(a_{I'}) = E_{\partial I}(a_{I'})$$

which is (3.11). Thus (3.11), (3.12), (3.13) are equivalent. Suppose that one of these 3 properties holds. Then for any $a_I \in \mathcal{A}_I$ $a_{I'} \in \mathcal{A}_{I'}$, one has

$$E_{\partial I}(a_I a_{I'}) = E_{\partial I}E_I(a_I a_{I'}) = E_{\partial I}(a_I E_I(a_{I'})) = E_{\partial I}(a_I E_{\partial I}(a_{I'})) = E_{\partial I}(a_I)E_{\partial I}(a_{I'})$$

which is (3.14). Similarly

$$E_{\partial I}(a_{I'} a_I) = E_{\partial I}E_I(a_{I'} a_I) = E_{\partial I}(E_I(a_{I'})a_I) = E_{\partial I}(E_{\partial I}(a_{I'})a_I) = E_{\partial I}(a_{I'})E_{\partial I}(a_I)$$

which is (3.15). The equivalences (3.16) $\iff$ (3.17) $\iff$ (3.18) are obtained exchanging the role of I and I' in the corresponding equivalences (3.11) $\iff$ (3.12) $\iff$ (3.13). The implication (3.18) $\longrightarrow$ (3.14), (3.15) follows replacing $a_{I'}$ by a_I and E_I by $E_{I'}$ in (3.19) and using (3.16) which is equivalent to (3.18). This proves (i).

(ii) Suppose that (3.15) holds. Then

$$\begin{aligned}
&E_{\partial I}\Big(|E_{\partial I}(a_I) - E_{I'}(a_I)|^2\Big)\\
&= E_{\partial I}\Big(E_{\partial I}(a_I^*)E_{\partial I}(a_I) - E_{\partial I}(a_I^*)E_{I'}(a_I) - E_{I'}(a_I^*)E_{\partial I}(a_I) + E_{I'}(a_I^*)E_{I'}(a_I)\Big)\\
&= E_{\partial I}\Big(E_{\partial I}(a_I^*)E_{\partial I}(a_I) - E_{\partial I}(a_I^*)E_{\partial I}E_{I'}(a_I)\\
&\qquad -E_{\partial I}(E_{I'}(a_I^*))E_{\partial I}(a_I) + E_{I'}(a_I^*)E_{I'}(a_I)\Big)\\
&= E_{\partial I}\Big(E_{\partial I}(a_I^*)E_{\partial I}(a_I) - E_{\partial I}(a_I^*)E_{\partial I}(a_I) - E_{\partial I}(a_I^*)E_{\partial I}(a_I) + E_{I'}(a_I^*)E_{I'}(a_I)\Big)\\
&= E_{\partial I}\Big(E_{I'}(a_I^*)E_{I'}(a_I) - E_{\partial I}(a_I^*)E_{\partial I}(a_I)\Big)\\
&= E_{\partial I}\Big(E_{I'}(a_I^*)E_{I'}(a_I)\Big) - E_{\partial I}(a_I^*)E_{\partial I}(a_I)\\
&= E_{\partial I}\Big(|E_{I'}(a_I)|^2\Big) - |E_{\partial I}(a_I)|^2\\
&\le E_{\partial I}\Big(E_{I'}(|a_I|^2)\Big) - |E_{\partial I}(a_I)|^2 = E_{\partial I}(|a_I|^2) - |E_{\partial I}(a_I)|^2 = 0
\end{aligned}$$

where in the last step we have used the Kadison–Schwarz inequality for $*$-algebras (see Sect. 4). Since the left hand side is ≥ 0, this is possible if and only if it is zero. Thus (3.15) implies (3.19). The same argument with a_I replacing $a_{I'}$ and E_I, $E_{I'}$ in (3.19) shows that (3.14) implies (3.20). This proves (ii).

(iii) We know that both (3.11) and (3.16) imply (3.14) and (3.15). But, if $E_{\partial I}$ is faithful, (3.14) implies (3.11) and (3.15) implies (3.16). This proves (iii).

(iv) If both (3.12) and (3.17) hold, then $E_I E_{I'} = E_{I'} E_I$ because they are both equal to $E_{\partial I}$, so they commute. Therefore $E_I E_{I'}$ is a norm-1 projector satisfying

$$E_I E_{I'}\big|_{\mathcal{A}_I \cap \mathcal{A}_{I'}} = id$$

i.e. $\mathrm{Range}(E_I E_{I'}) \supseteq \mathcal{A}_I \cap \mathcal{A}_{I'}$. Conversely, suppose that x is invariant under $E_I E_{I'}$. Then $x = E_I E_{I'}(x) \in \mathcal{A}_I$ and $x = E_{I'} E_I(x) \in \mathcal{A}_{I'}$. Thus if (3.21) holds, $\mathrm{Range}(E_I E_{I'}) \subseteq \mathcal{A}_I \cap \mathcal{A}_{I'}$ therefore $\mathrm{Range}(E_I E_{I'}) = \mathcal{A}_I \cap \mathcal{A}_{I'}$. In conclusion

$$\mathcal{A}_I \cap \mathcal{A}_{I'} = \mathrm{Range}(E_I E_{I'}) = \mathrm{Range}(E_{\partial I}) = \mathcal{A}_{\partial I}$$

i.e. (3.21) holds. This proves (iv).

□

Remark 7. $E_{\partial I}$ is always faithful if it is compatible with a faithful state. In classical probability this is always the case for algebras such as $L^\infty(\Omega, \mathcal{F}, P)$ or $\bigcap_{p\geq 1} L^p(\Omega, \mathcal{F}, P)$.

4. Appendix: Kadison–Schwarz Inequality for ∗-Algebras

Definition 8. Let $n \in \mathbb{N}^*$ and $\mathcal{A}$, $\mathcal{B}$ be ∗-algebras. A linear map P of $\mathcal{A}$ into $\mathcal{B}$ is said to be n-**positive** if, $\forall\, n \in \mathbb{N}$, $\forall\, b_1, \dots, b_n \in \mathcal{B}$, $\forall\, a_1, \dots, a_n \in \mathcal{A}$

$$\sum_{j,k=1}^{n} b_j^* \, P(a_j^* \, a_k) b_k \geq 0 \tag{4.1}$$

If P is n-positive for all $n \in \mathbb{N}$, then it is called **completely positive**.

Proposition 9. *Let $\mathcal{A}$ be a ∗-algebra with unit and let $P : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be a 2-positive map such that $P(1) = 1$. Then P satisfies the inequality*

$$P(a^*)P(a) \leq P(a^* a) \qquad ; \quad \forall a \in \mathcal{A} \tag{4.2}$$

In particular the sesqui-linear, $\mathcal{B}$-valued kernel:

$$K(a, b) := P(a^* b) - P(a^*)P(b) \tag{4.3}$$

is Hermitean positive definite.

Proof. Let $e \equiv (e_{ij})$ be a system of matrix units in M_2. By construction the operator

$$(a \otimes e_{21} + 1)^*(a \otimes e_{21} + 1) = a^* a \otimes e_{11} + a^* \otimes e_{12} + a \otimes e_{21} + 1$$
$$= \begin{pmatrix} a^* a & a^* \\ a & 1 \end{pmatrix} = \begin{pmatrix} a^* a & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}^* \in \mathcal{B} \otimes M_2 \equiv \mathcal{B}(\mathcal{K} \otimes \mathbb{C}^2) \equiv \mathcal{B}(\mathcal{K} \oplus \mathcal{K})$$

is positive. Therefore the 2-positivity of P and $P(1) = 1$ imply that the operator

$$\begin{pmatrix} a^* a & a^* \\ a & 1 \end{pmatrix} \mapsto \begin{pmatrix} P(a^* a) & P(a^*) \\ P(a) & 1 \end{pmatrix}$$

is positive, i.e. for every $u, v \in \mathcal{K}$:

$$\langle u, P(a^* a) u \rangle + \langle u, P(a^*) v \rangle + \langle v, P(a) u \rangle + \langle v, v \rangle \geq 0$$

Taking $v = -P(a)u$ one has the inequality

$$\begin{aligned}&\langle u, P(a^*a)u\rangle - \langle u, P(a^*)P(a)u\rangle - \langle u, P(a^*)P(a)u\rangle + \langle P(a)u, P(a)u\rangle \geq 0\\ &\quad \iff \langle u, P(a^*a)u\rangle - \langle u, P(a^*)P(a)u\rangle - \langle u, P(a^*)P(a)u\rangle + \langle u, P(a^*)P(a)u\rangle\\ &\quad = \langle u, P(a^*a)u\rangle - \langle u, P(a^*)P(a)u\rangle \geq 0\\ &\quad \iff \langle u, P(a^*)P(a)u\rangle \leq \langle u, P(a^*a)u\rangle\,; \quad \forall u \in \mathcal{K}\end{aligned}$$

which is (4.2). Finally the kernel (4.3) is Hermitean because P is linear, Hermitean because

$$K(a,b)^* := P(a^*b)^* - P(b)^*P(a^*)^* = P(b^*a) - P(b^*)P(a) = K(b,a)$$

and positive definite because, for every integer $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $a_1, \ldots, a_n \in \mathcal{A}$, one has with summation over repeated indexes and using (4.2),

$$\begin{aligned}\overline{\lambda}_j(P(a_j^*a_k) - P(a_j^*)P(a_k))\lambda_k &= P(\overline{\lambda}_j a_j^* a_k \lambda_k) - P(\overline{\lambda}_j a_j^*)P(a_k\lambda_k)\\ &= P(|a_k\lambda_k|^2) - |P(a_k\lambda_k)|^2 \geq 0\end{aligned}$$

□

Definition 10. The inequality (4.2) is called the **Kadison–Schwarz inequality**. A map P satisfying (4.2) is called a **Schwarz map**.

Remark. A theory of completely positive maps on $*$-algebras has been developed in [7] where a Stinespring theorem was developed for such maps. However in [7] Schwarz maps are not considered.

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