

Second-gradient continua: From Lagrangian to Eulerian and back

Mathematics and Mechanics of Solids
2022, Vol. 27(12) 2715–2750

© The Author(s) 2022



Article reuse guidelines:

sagepub.com/journals-permissions

DOI: 10.1177/10812865221078822

journals.sagepub.com/home/mms



Francesco dell'Isola

Department of Civil, Construction-Architecture and Environmental Engineering, University of L'Aquila, L'Aquila, Italy

Simon R Eugster 

University of Stuttgart, Stuttgart, Germany

Roberto Fedele 

Politecnico di Milano, Milano, Italy

Pierre Seppecher 

University of Toulon, La Garde, France

Received 15 December 2021; accepted 20 January 2022

Abstract

In this paper, we represent second-gradient internal work functionals in Lagrangian (referential) and Eulerian (spatial) descriptions, and we deduce the corresponding expressions for the Piola transformations of stress and double-stress tensors and of external forces and double-forces. We also derive, in both the Eulerian and Lagrangian description, the expression of surface and edge contact interactions (which include forces and double-forces) for second-gradient continua in terms of the normal and the curvature of contact boundary surfaces and edge shapes.

Keywords

Continuum mechanics, second-gradient continua, Piola transformation, principle of virtual work, Lagrangian formulation, Eulerian formulation

1. Introduction

We use the postulation scheme for continuum mechanics based on the principle of virtual work. In it, the work is the primitive concept. This was done, for instance, already in the previous papers [1–5], which are all written in the spirit of the works by D'Alembert, Lagrange, Piola, and Hellinger [6–9]. For a more detailed discussion about this postulation scheme and related methodological topics, the reader

Corresponding author:

Simon R Eugster, University of Stuttgart, 70569 Stuttgart, Germany.

Email: eugster@inm.uni-stuttgart.de

is referred to dell'Isola et al. [10]. Rephrasing D'Alembert and Lagrange, the principle of virtual work states that the motion of a continuum can be calculated: "By equating to zero the sum of the internal work plus external and inertial work expended for any virtual displacement."

In the context of a D'Alembert–Lagrange postulation scheme, which is founded on the principle of virtual work, the difference between first-gradient and second-gradient continua can be simply specified by referring to the order of internal work functional, when regarding it as a distribution and virtual displacements as the corresponding test functions. On the contrary, in the light of Cauchy's postulation scheme, which is based on balance of forces and moments of forces, to specify such a difference becomes nearly insurmountable.

The first systematic formulation of continuum mechanics based on the principle of virtual work (or virtual velocities, as he preferred to say) is due to Gabrio Piola [8]. In fact, the generalization of the principle of minimum potential energy led him to the definition of an internal work functional, where, for a first-gradient theory, the stress appears as dual quantity to the first gradient of the virtual displacement field. Piola started from the formulation of this principle in the Lagrangian description, transformed the Lagrangian expression of internal and compatible external work functionals into the Eulerian description and finally obtained the Eulerian equilibrium conditions as well as the dependence of contact interactions on the shapes of so-called Cauchy cuts.

In contrast, Cauchy preferred to base continuum mechanics on the balances of forces and moments of forces formulated at first in the Eulerian description. Within Cauchy's postulation scheme forces and moments of forces are the primitives concepts of the theory. Moreover, in Cauchy's framework, one has to assume that subbodies interact exclusively by contact surface forces and that these surface forces depend only on the position in the continuum and on the normal of the Cauchy cut. The latter assumption is generally referred to as Cauchy postulate. With his assumption, Cauchy restricted the possible constitutive equations for the continua to be considered. In fact, it results that Cauchy did limit his theory to first-gradient continua.

Instead, as already envisaged by Piola and fully proved by Mindlin and Toupin (among many other scholars, see dell'Isola et al. [11]), it is possible to formulate continuum theories in which contact interactions on Cauchy cuts depend also on their curvature. This is, for instance, the case for the theory of second-gradient continua, in which Cauchy cuts not only contact surface forces but also contact edge forces and double-forces do appear. While the generalization of Cauchy's theory is almost impossible, following D'Alembert–Lagrange continuum mechanics, it is conceptually straightforward to generalize the first-gradient theory [12] to the second-gradient theory [9]: albeit Piola could not complete his scientific program, he fully understood this.

1.1. Notation

Following the postulation accepted for Galilean Mechanics, the physical space, where the material particles of considered second-gradient continua can be placed, is modeled as a three-dimensional Euclidean vector space \mathbb{E}^3 with the inner product denoted by $\langle \cdot, \cdot \rangle$. We assume the reference configuration of the considered continuous body \mathcal{B} to be a subset $\Omega \subset \mathbb{E}^3$, which is sufficiently regular to perform all the required calculations, see Figure 1 for a particular example of such a configuration. The topological boundary of Ω is denoted by $\partial\Omega$. The boundary $\partial\Omega$ is assumed to be the union of a finite number of two-dimensional orientable surfaces with boundary, called faces of Ω . The faces are oriented in accordance with their corresponding outward-pointing unit normal fields N , and therefore the orthogonal projections M_{\parallel} on each tangent space is well defined. Each of the faces' boundary curves are assumed to have a piecewise continuous tangent T as well as an outward-pointing unit normal V that is tangent to the face. The union of all boundary curves is denoted by $\partial\partial\Omega$. Remark that each curve constituting the boundaries of the faces, which are called edges of Ω , must be regarded as part of the boundary of both concurring faces. Edges of Ω are assumed to concur in a finite number of wedges together with a finite number of other edges. See dell'Isola et al. [13] for more details about the differential-geometric assumptions and notations used here.

A placement of the body \mathcal{B} is defined as a suitably regular map $\Pi : \Omega \rightarrow \mathbb{E}^3, X \mapsto x = \Pi(X)$, which is assumed to be a one-to-one map. The image $\omega = \Pi(\Omega) \subset \mathbb{E}^3$ is called the current configuration of the body \mathcal{B} and represents the spatial positions occupied by the body \mathcal{B} in its deformed state. Due to the

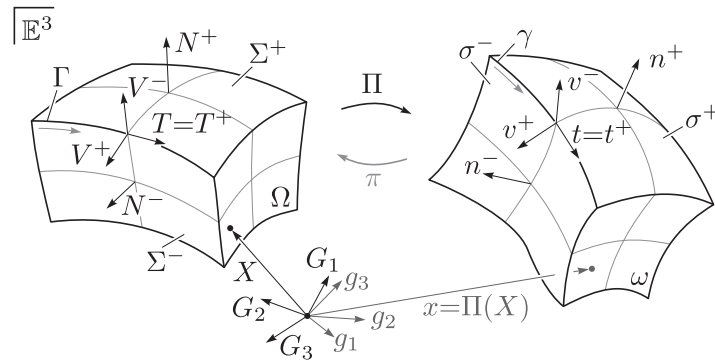


Figure 1. Reference and current configuration of a body with a piecewise continuous boundary surfaces and piecewise continuous boundary curves.

regularity assumption made for the placement, for a current configuration, we have that the faces $\partial\omega = \Pi(\partial\Omega)$ and edges $\partial\partial\omega = \Pi(\partial\partial\Omega)$. The faces are oriented by the outward-pointing unit normal n inducing the orthogonal projection m_{\parallel} . Clearly, the faces' boundary curves have still a piecewise continuous tangent t as well as an outward-pointing unit normal v that is tangent to the face.

We use the upper-case $X \in \Omega$ and the lower-case $x \in \omega$ to denote points in the reference and current configuration, respectively: they label material particles and positions occupied by material particles in the considered configuration. We use an arbitrary right-handed basis (G_1, G_2, G_3) to represent the referential position of material particles as linear combination $X = X^A G_A$. Note that we apply Levi-Civita–Einstein's summation convention, which implies summation over upper contravariant and lower covariant indices that appear twice in a term. The positions in the current configuration are similarly represented with respect to an alternative basis (g_1, g_2, g_3) . Henceforth, we refer to these two bases as referential and current basis and follow the convention for the Lagrangian upper-case and Eulerian lower-case letters. The referential and current metric components are given by $G_{AB} = \langle G_A, G_B \rangle$ and $g_{ij} = \langle g_i, g_j \rangle$. As usually done in index notation G^{AB} and g^{ij} denote the corresponding inverse metrics. Let $u, v \in \mathbb{E}^3$ and A be an endomorphism on \mathbb{E}^3 , the transposed A^T is defined by the relation $\langle u, A \cdot v \rangle = \langle A^T \cdot u, v \rangle$, whose components can then be expressed as $(A^T)^B_i = g_{ij} A^j_C G^{CB} = A^B_i$. The placement map Π is represented in such a way that

$$x = x^i g_i = \Pi(X) = \Pi^i(X) g_i.$$

The components of the first gradient $F = \nabla \Pi$ and the second gradient $\mathbb{F} = \nabla F$ of the placement map are then given as

$$F^i_A := (\nabla \Pi)^i_A = \frac{\partial \Pi^i}{\partial X^A}, \quad \mathbb{F}^i_{AB} := (\nabla F)^i_{AB} = \frac{\partial^2 \Pi^i}{\partial X^A \partial X^B}, \tag{1}$$

both of which are functions of X .

We introduce virtual displacements $\delta \Pi$ as small variations of the placement map Π . Note that we still follow our notational convention to denote Lagrangian fields, which are defined on Ω , with upper-case letters. The first and second gradient of the virtual displacement are given by the relations

$$\delta F^i_A = \frac{\partial \delta \Pi^i}{\partial X^A}, \quad \delta \mathbb{F}^i_{AB} = \frac{\partial^2 \delta \Pi^i}{\partial X^A \partial X^B}.$$

In the Lagrangian description, all vector and tensor valued functions depend on the referential points X . Since the placement $\Pi : \Omega \rightarrow \mathbb{E}^3$ is invertible, we can introduce its inverse for which the position $x \in \Pi(\Omega) = \omega$ is regarded to be the independent variable. The inverse function $\pi = \Pi^{-1} : \omega \rightarrow \Omega$ is written with lower-case letters, as it will be done henceforth for every map with ω as its domain. Therefore, every Lagrangian field can be regarded to be an Eulerian field when using the composition with π . In

particular, we consider $\delta\Pi\circ\pi$, called Eulerian virtual displacement, and its first and second Eulerian gradients:

$$\delta d_j^i(x) := \frac{\partial(\delta\Pi^i\circ\pi)}{\partial x^j}(x), \quad \delta \mathbf{d}_{jk}^i(x) := \frac{\partial^2(\delta\Pi^i\circ\pi)}{\partial x^j\partial x^k}(x).$$

As the Eulerian gradients can only be defined when $\delta\Pi^i$ is composed with π , we abbreviate the above expression as

$$\delta d_j^i = \frac{\partial\delta\Pi^i}{\partial x^j}, \quad \delta \mathbf{d}_{jk}^i := \frac{\partial^2\delta\Pi^i}{\partial x^j\partial x^k}.$$

We will make the same abuse of notation for every field we introduce, be it Lagrangian or Eulerian: we use the same symbol independently of the presence of a composition with π or Π . The indication of the composition may be complemented by the reader if he feels it is necessary.

1.2. Main results of the paper

D'Alembert–Lagrange continuum mechanics is based on the Lagrangian form of the principle of virtual work which demands the equality

$$\delta W_\Omega^{\text{tot}}(\delta\Pi) := \delta W_\Omega^{\text{int}}(\delta\Pi) + \delta W_\Omega^{\text{ext}}(\delta\Pi) = 0 \quad (2)$$

to hold for every virtual displacement field $\delta\Pi$, admissible with respect to the assumed kinematical constraints.

For a second-gradient material, the internal work functional is defined as

$$\delta W_\Omega^{\text{int}}(\delta\Pi) := - \int_\Omega (P_i^A \delta F_A^i + \mathbb{P}_i^{AB} \delta \mathbb{F}_{AB}^i), \quad (3)$$

where P_i^A and \mathbb{P}_i^{AB} , called Piola–Lagrange stress P and Piola–Lagrange double-stress \mathbb{P} , are work conjugate to the first and second gradient of virtual displacements.

As a consequence of the principle of virtual work, the external work functional compatible to the internal work functional must have the form (see [1, 4])

$$\delta W_\Omega^{\text{ext}}(\delta\Pi) := \int_\Omega \mathfrak{F}_i^\Omega \delta\Pi^i + \int_{\partial\Omega} \mathfrak{F}_i^{\partial\Omega} \delta\Pi^i + \int_{\partial\Omega} \mathfrak{D}_i^{\partial\Omega} \frac{\partial\delta\Pi^i}{\partial X^C} N^C + \int_{\partial\partial\Omega} \mathfrak{F}_i^{\partial\partial\Omega} \delta\Pi^i.$$

In this expression, the co-vector fields \mathfrak{F}_i^Ω , $\mathfrak{F}_i^{\partial\Omega}$, and $\mathfrak{F}_i^{\partial\partial\Omega}$ are dual to virtual displacements and are, due to their integration domain, forces per unit reference volume, surface, and line, respectively. Moreover, an additional surface density field $\mathfrak{D}_i^{\partial\Omega}$ appears, which is called surface density of double-forces: these contributions are dual to the normal derivative of the virtual displacement.

After the Lagrangian Eulerian change of variable in the Lagrangian internal work functional (3), it is easy to verify that the Eulerian internal work functional is still a second-order distribution of the form

$$\delta w_\omega^{\text{int}}(\delta\Pi) = - \int_\omega (c_i^j \delta d_j^i + \mathfrak{c}_i^{jk} \delta \mathbf{d}_{jk}^i),$$

where c_i^j and \mathfrak{c}_i^{jk} are the components of the work conjugates to the first and second gradient of the spatial virtual displacement. We call them the Cauchy–Euler stress c and the Cauchy–Euler double-stress \mathfrak{c} .

One of the main results of this paper consists in showing the Piola transformation of Lagrangian stresses into Eulerian stresses

$$c_i^j = J^{-1} (P_i^A F_A^j + \mathbb{P}_i^{AB} \mathbb{F}_{AB}^j), \quad \mathfrak{c}_i^{jk} = J^{-1} \mathbb{P}_i^{AB} F_A^j F_B^k, \quad (4)$$

which can be written, with an obvious meaning of the symbols, in a more synthetic way as

$$c = J^{-1}(P \cdot F^t + \mathbb{P} : \mathbb{F}^t), \quad \mathbf{c} = J^{-1}\mathbb{P} : (F^t \otimes F^t).$$

Note that with our notation, either the left hand sides have to be composed with Π or the right hand sides with π .

The reader will remark that by simply assuming $\mathbb{P} = 0$, we get the Piola transformation formula $c = J^{-1}P \cdot F^t$ for first-gradient continua. Moreover, in this case, the Piola transformation is usually obtained by exploiting the relation between nominal and current surface forces and using so-called Nanson's formula. In contrast, already Piola proved that the easiest way to get the transformation of stress consists in the change variables in the volume expression of the internal work functional. While this procedure naturally generalizes in the case of second-gradient continua, in section 5, it is shown that the generalization of the other one is nearly impossible.

Similarly to what happens for its Lagrangian counterpart, the Eulerian external work functional compatible with a second-gradient Eulerian internal work functional is given by

$$\delta w_\omega^{\text{ext}}(\delta \Pi) = \int_\omega f_i^\omega \delta \Pi^i + \int_{\partial \omega} f_i^{\partial \omega} \delta \Pi^i + \int_{\partial \omega} \mathfrak{d}_i^{\partial \omega} \frac{\partial \delta \Pi^i}{\partial x^c} n^c + \int_{\partial \partial \omega} f_i^{\partial \partial \omega} \delta \Pi^i.$$

The co-vector fields f^ω , $f^{\partial \omega}$, and $f^{\partial \partial \omega}$ are forces per unit current volume, surface, and line, respectively. Also in the Eulerian description, a surface density of double-forces $\mathfrak{d}^{\partial \omega}$ appears, which is a density per unit current surface and which is dual to the normal derivative with respect to the current normal vector.

To formulate the Piola transformation of external interactions, we introduce the inverse of the right Cauchy-Green strain C^{-1} having as components

$$(C^{-1})^{AB} = (F^{-1})_i^A g^{ij} (F^{-1})_j^B, \tag{5}$$

as well as the Lagrangian vector field K defined as

$$K^A := M_{\|C}^A (C^{-1})^{CB} N_B = ((F^{-1})_i^A g^{ij} (F^{-1})_j^B - \|F^{-T} \cdot N\|^2 G^{AB}) N_B.$$

Introducing the Jacobians for volume, area, and length $J = \det(F)$, $J_\Sigma = \|JF^{-T} \cdot N\|$, and $J_\Lambda = \|F \cdot T\|$, respectively, the Piola transformation formulas in second-gradient continua for the external forces are found to be

$$\begin{aligned} \mathfrak{F}_i^\Omega &= J f_i^\omega, \quad \mathfrak{F}_i^{\partial \Omega} = J_\Sigma f_i^{\partial \omega} - M_{\|A}^B \frac{\partial}{\partial X^B} (J \mathfrak{d}_i^{\partial \omega} K^A), \\ \mathfrak{F}_i^{\partial \partial \Omega} &= J_\Lambda f_i^{\partial \partial \omega} + (J(C^{-1})^{AB} V_A N_B \mathfrak{d}_i^{\partial \omega})^+ + (J(C^{-1})^{AB} V_A N_B \mathfrak{d}_i^{\partial \omega})^-, \end{aligned} \tag{6}$$

which are translated to a direct notation reads as

$$\begin{aligned} \mathfrak{F}^\Omega &= J f^\omega, \quad \mathfrak{F}^{\partial \Omega} = J_\Sigma f^{\partial \omega} - \text{DIV}_{\|}^{\partial \Omega} (J \mathfrak{d}^{\partial \omega} \otimes K) \\ \mathfrak{F}^{\partial \partial \Omega} &= J_\Lambda f^{\partial \partial \omega} + (J(V \cdot C^{-1} \cdot N) \mathfrak{d}^{\partial \omega})^+ + (J(V \cdot C^{-1} \cdot N) \mathfrak{d}^{\partial \omega})^-. \end{aligned}$$

The Piola transformation of the surface double-force, once in index and once in direct notation, are given by

$$\mathfrak{D}_i^{\partial \Omega} = J \|F^{-T} \cdot N\|^2 \mathfrak{d}_i^{\partial \omega}, \quad \mathfrak{D}^{\partial \Omega} = J \|F^{-T} \cdot N\|^2 \mathfrak{d}^{\partial \omega}. \tag{7}$$

Finally, we remark that only via a change of variable within the internal work functional, we obtained the Piola transformation of the stresses (4). Since Eulerian normal derivatives to $\partial \omega$ do not transform into Lagrangian normal derivatives, via the placement Π , extra tangent terms to $\partial \Omega$ arise when changing the variable in the Eulerian double-force functional. This is the reason for which Eulerian external double-forces give rise to not only Lagrangian external double-forces (7) but also to extra surface and

edge forces (6). In conclusion, we cannot follow the procedure used for instance by Gurtin [14], where starting from the transformation of external surface forces the Piola transformation of stress is deduced: instead the only possibility we have is to resort to the transformation of the internal and external work functionals via the change of variable given by the placement Π .

1.3. Outline of the paper

In the introduction at hand, we present the used notation and summarize the main results of this paper. In section 2, we postulate the virtual work principle for second-gradient continua in Lagrangian description. In fact, it is the generalization of the principle of minimum of total energy envisaged by Piola [8] and formalized by Paul Germain [3] in Eulerian description. Using an integration by parts procedure, we formulate the strong form of the equilibrium conditions, including the local equilibrium partial differential equations and corresponding natural boundary conditions. In section 3, the Eulerian virtual work principle is deduced from the Lagrangian virtual work principle. An integration by parts procedure similar to the one used in section 2 allows for the deduction of the structure of contact interactions in Eulerian description. The dependence of contact interactions on the shape of Cauchy cuts is explicitly shown so that it becomes clear that Cauchy postulate is not applicable to second-gradient continua. In both sections 2 and 3, we deduce the compatible form of external interactions between second-gradient continua and the external world. In section 4, the transformation formulas between the Lagrangian and Eulerian fields are presented, generalizing the Piola transformations valid for first-gradient continua. In the last section 5, we check the consistency among: (1) Piola transformations of stress and double-stress, (2) Piola transformations of external forces and double-forces, and (3) the obtained expressions of Lagrangian and Eulerian contact interactions.

2. The Lagrangian virtual work principle for second-gradient continua: equilibrium conditions

If one insists to use the postulation scheme put forward by Cauchy for continuum mechanics, in generalizing the theory to the case of second-gradient continua he finds some intrinsic, and nearly insurmountable, difficulties, see for instance dell'Isola and colleagues [15–17]. In fact, Cauchy's approach¹ is based on the primitive concepts of force and moment of forces together with the formulation of corresponding balance laws, and, by means of the tetrahedron argument, on the introduction of the concept of stress. Cauchy's approach requires some major and *ad hoc* modifications to include the case of second-gradient continua, see dell'Isola and Seppecher [18] and dell'Isola et al. [11]. Instead, in D'Alembert–Lagrange continuum mechanics [6,7], it is very easy to define higher gradient continua simply modifying the internal work functional (see e.g. Germain [1,3] and Epstein and Smelser [19]).

In Cauchy's approach, stress is a derived concept, and its existence must be proven, while in the D'Alembert–Lagrange approach it is a primitive concept. Vice versa in Cauchy's approach, the concept of force and the balance law of force are primitive while in D'Alembert–Lagrange's approach they are derived concepts. As discussed in detail by Eugster [9] and Eugster and dell'Isola [20–22], Truesdell (see [23]) interpreted the D'Alembert–Lagrange postulation scheme from his point of view and therefore claims

The derivation given by HELLINGER [...] fails through *petitio principii*[sic!], since the stress components appear in the original variational principle. We do not understand the remark attributed to CARATHÉODORY by MÜLLER and TIMPE [...]. Existence of the stress tensor can be proved from variational principles which assume the existence of an internal energy having a special functional form. (p. 595)

In this section, we show how the principle of virtual work can be postulated to generalize the principle of minimum of total energy, as already discussed by D'Alembert [7] and in particular for continuum mechanics by Piola [8].

2.1. The principle of minimum of total energy generalized into the principle of virtual work

Once the kinematics of a mathematical model is established, it is necessary to find the time sequence of configurations, that is, the motion that predicts the behavior of the modeled system. One can start by looking for the predictions concerning equilibrium configurations. The principle of minimum of total energy characterizes the stable equilibrium configurations and states that,

1. In every model for physical phenomena, the specification of kinematics has to be completed by the choice of a functional defined on the set of admissible configurations, called functional of total (potential) energy.
2. Stable equilibrium configurations are the minima of the total potential energy functional.

The total potential energy functional, for deformable bodies, includes the deformation energy and the potential energy describing the interactions between the considered body with the external world.

D'Alembert, Lagrange, Piola, Hamilton, and Rayleigh (see [6–8,24] and [10,11,16,25] for a historical overview) proposed to generalize the principle of minimum of total potential energy to enlarge the predictive scope of formulated models to include the possibility to describe non-conservative interactions and dissipation phenomena. Their line of thoughts can be reformulated with the following thread of reasoning.

Let us assume that the total potential energy functional \mathcal{E}^{tot} be differentiable. D'Alembert and Lagrange formulated this assumption assuming that one can calculate its first variation, that is, the linear term, in its Taylor expansion, corresponding to an admissible² infinitesimal variation $\delta\Pi$ of the configuration and that this first variation is continuous. Using the modern conceptual frame given by functional analysis, we may state that the total potential energy is continuously differentiable in the sense of Fréchet (see Rudin [26]). Then, in stable equilibrium configurations Π^{equi} , this first variation must vanish because these configurations are minima for the total energy functional. In formulas: every equilibrium configuration Π^{equi} verifies the condition

$$(\forall\delta\Pi) (\delta\mathcal{E}^{\text{tot}}(\Pi^{\text{equi}}; \delta\Pi) = 0). \quad (8)$$

Remark that in the previous condition the dependence on the variable $\delta\Pi$ is linear, while the dependence on Π^{equi} may be nonlinear, if the functional \mathcal{E}^{tot} is not quadratic. In other words, for every admissible configuration Π , the linear continuous first-variation functional $\delta\mathcal{E}^{\text{tot}}$, depending on the infinitesimal variation $\delta\Pi$, is well defined and when $\Pi = \Pi^{\text{equi}}$ such a functional vanishes for every $\delta\Pi$.

One may decompose \mathcal{E}^{tot} , and consequently $\delta\mathcal{E}^{\text{tot}}$, into the sum of an internal part \mathcal{E}^{int} , relative to the body's internal interactions, and an external interactions part \mathcal{E}^{ext} and assume that both these functionals can be differentiated. The condition (8) thus becomes³:

$$(\forall\delta\Pi) (\delta\mathcal{E}^{\text{int}}(\Pi^{\text{equi}}; \delta\Pi) + \delta\mathcal{E}^{\text{ext}}(\Pi^{\text{equi}}; \delta\Pi) = 0).$$

D'Alembert generalized the previous condition in the more general case of non-conservative and dissipative models. He postulates the existence, for every admissible configuration Π , of linear and continuous functionals depending on the admissible infinitesimal variations $\delta\Pi$. These functionals are called by Piola internal (virtual) work functionals and external (virtual) work functionals. In the framework of D'Alembert–Lagrange continuum mechanics, then, the material properties of a specific continuum are fully mathematically described when assigning the internal and external work functional. When the system is conservative, these functionals can be calculated as Fréchet derivative of the deformation energy and potential energy functionals. In fact, in general, internal and external work functionals are not first variations of some energy functionals: in this sense, the subsequent D'Alembert virtual work identity generalizes the total energy stationarity condition. Finally, to include inertial effects, and therefore to find the prediction of the time evolution of considered systems, D'Alembert postulates the existence of an inertial (virtual) work functional, also specified by a constitutive equation.

In conclusion, the basic assumption in D'Alembert–Lagrange continuum mechanics consists in postulating that the motion of every continuum can be characterized by suitably choosing the constitutive

equations for the three work functionals $\delta W_{\Omega}^{\text{int}}$, $\delta W_{\Omega}^{\text{ext}}$, and $\delta W_{\Omega}^{\text{dyn}}$ and by assuming that the D'Alembert virtual work identity

$$\delta W_{\Omega}^{\text{tot}}(\delta\Pi) := \delta W_{\Omega}^{\text{int}}(\delta\Pi) + \delta W_{\Omega}^{\text{ext}}(\delta\Pi) + \delta W_{\Omega}^{\text{dyn}}(\delta\Pi) = 0$$

holds, at every time instant, for every small admissible displacement $\delta\Pi$.

The functional $\delta W_{\Omega}^{\text{int}}$ allows for the calculation of the work expended in the internal interactions among parts of the considered body on every deformation process involving it. The functional $\delta W_{\Omega}^{\text{ext}}$ allows for the calculation of the work expended in the interactions of the body with its external world. Finally, $\delta W_{\Omega}^{\text{dyn}}$ gives the inertial work expended on virtual displacements and deformations. In the following, we will focus on the static case and apply the principle of virtual work without the inertial part.

Albeit, in such a generalization, the original meaning of such small variation of placement (i.e. test displacement used to check minimality of total (potential) energy) is lost, D'Alembert kept calling "virtual displacement" the generic displacement that may be added, at any instant, to the configuration attained by the motion to get another admissible placement.

D'Alembert virtual work identity is intended to hold, at every time instant, for every admissible virtual displacement $\delta\Pi$. Commonly, we will face two kinds of kinematical constraints. Constraints on positions $g = g(\Pi) = 0$ as well as on deformations $h = h(F) = 0$. For these types of kinematical restrictions, a variation of a placement $\hat{\Pi} = \Pi + \delta\Pi$ is admissible if $g(\hat{\Pi}) = 0$ and $h(\nabla\hat{\Pi}) = 0$. Taking the first variation of these expressions leads to the conditions which must be satisfied by admissible virtual displacements and admissible gradients of virtual displacements. These are

$$(\forall\delta\Pi = \delta\Pi_{\text{adm}}) \quad \frac{\partial g}{\partial\Pi^i} \delta\Pi^i = 0, \quad \frac{\partial h}{\partial F_A^i} \frac{\partial\delta\Pi^i}{\partial X^A} = \frac{\partial h}{\partial F_A^i} \delta F_A^i = 0.$$

2.2. Work functionals as distributions

A fundamental part of the principle of virtual work as formulated in a modern language consists in postulating that the functional which associates to every virtual infinitesimal displacement the work expended in any specific interactions among and inside bodies is: (1) linear and (2) continuous.

Of course, one needs to introduce a topology in the set of admissible virtual displacements, if one wants to be able to talk about continuity of work functionals. Therefore, the distribution theory of L. Schwartz [27] seems suitable to give the conceptual frame needed to formulate continuum mechanics (see Germain [3]). The linear continuous functional that associates the virtual work corresponding to every virtual displacement is a *distribution*. We assume to have bounded reference configurations so that $\bar{\Omega}$ is compact. The set of distributions that we consider here is, in fact, the dual space of the set of $C^{\infty}(\bar{\Omega})$ functions having compact support and endowed with the topology induced by the derivatives seminorms (see Schwartz [27] and Reed and Simon [28]).

A very general kinematical assumption that we accept is that the set of admissible virtual displacements include (let us underline: we are not stating that it coincides with) the set $C^{\infty}(\bar{\Omega})$ constituted by the infinitely differentiable functions having compact support in $\bar{\Omega}$.

Note that, the smaller is the space of test functions the larger is its dual space.⁴ Therefore, considering the dual of $C^{\infty}(\bar{\Omega})$ supplies us, under the stated assumption, with the widest possible set of linear and continuous functionals. This set seems suitable for giving a firm mathematical basis for D'Alembert–Lagrange continuum mechanics.

2.3. Second-gradient deformation energy functional

An internal work functional is said to be conservative if it is the derivative of an energy functional, which we call deformation energy functional. We assume that, in general, internal work functionals are the sum of a conservative plus a non-conservative part. For instance, following Hamilton–Rayleigh's postulation scheme (see e.g. [10,29]), the non-conservative part of the internal work functional can be calculated from a so-called Rayleigh dissipation functional. An interesting physical system in which a phase transition occurs (similar to what has been described by Javanbakht et al. [30]) in large deformations is

given by Spagnuolo and Cazzani [31]: Pantographic metamaterials may behave as a second-gradient continua until a certain threshold is reached, beyond which, because of friction phenomena, they behave as standard first-gradient continuum.

Elastic second-gradient continua are continua whose deformation energies depend on F and ∇F . In Lagrangian description, this means that there exists a constitutive function for volume density of deformation energy

$$W^{\text{def}} = W^{\text{def}}(X, F, \nabla F),$$

such that the total deformation energy \mathcal{E}^{def} corresponding to a placement Π is given by the functional

$$\mathcal{E}^{\text{def}}(\Pi) = \int_{\Omega} W^{\text{def}}(X, F(X), \nabla F(X)).$$

When calculating the first variation of the deformation energy, a special role is played by the elastic Piola–Lagrange stress and double-stress tensors defined as

$$(P^e)_i^A := \frac{\partial W^{\text{def}}}{\partial F_A^i}, \quad (\mathbb{P}^e)_i^{AB} := \frac{\partial W^{\text{def}}}{\partial \mathbb{F}_{AB}^i}.$$

The first variation of the second-gradient deformation energy functional has the following form

$$\delta \mathcal{E}^{\text{def}} = \int_{\Omega} (P^e)_i^A \delta F_A^i + \int_{\Omega} (\mathbb{P}^e)_i^{AB} \delta \mathbb{F}_{AB}^i. \tag{9}$$

D’Alembert and, then, Lagrange and Piola, generalized this approach to non-conservative internal interactions. Albeit, in this last case, the internal interactions are not fully determined by a volume density of deformation energy, they assume that it is still possible to introduce a linear (and continuous) functional $\delta W_{\Omega}^{\text{int}}$ depending on the variation $\delta \Pi$, which has the same structure as equation (9). The functional $\delta W_{\Omega}^{\text{int}}$ allows for the calculation of the work expended on the virtual displacements by the continuum’s internal interactions. In general, it is not the first variation of a deformation energy functional but can, for second-gradient continua, always be represented as follows:

$$\delta W_{\Omega}^{\text{int}}(\Pi; \delta \Pi) := - \int_{\Omega} (P_i^A(X) \delta F_A^i(X) + \mathbb{P}_i^{AB}(X) \delta \mathbb{F}_{AB}^i(X)), \tag{10}$$

where we have introduced the Piola–Lagrange stress and double-stress tensors P_i^A and \mathbb{P}_i^{AB} , which are to be assigned by means of suitable constitutive assumptions depending on $(\Pi, F, \nabla F)$.⁵ In case of elastic continua, we have

$$\delta W_{\Omega}^{\text{int}}(\delta \Pi) = - \delta \mathcal{E}^{\text{def}}(\delta \Pi).$$

2.4. External work functionals in second-gradient continua

The external work functional specifies the interactions between the considered continuum and its external world. Once we have defined the internal, inertial, and external work functionals, postulating the D’Alembert identity for every virtual displacement $\delta \Pi$ allows for the determination of the motion. However, this determination is possible only if the external work functional belongs to a specific class, which is compatible with $\delta W_{\Omega}^{\text{int}}$ and $\delta W_{\Omega}^{\text{dyn}}$. In other words, when an internal and inertial work functional are postulated, then, in the corresponding D’Alembert identity, only external work functional of a particular class can be used.

A classical illustration of this fact, already presented by Piola [8,24], is given by perfect fluids. One assumes that perfect fluid’s internal energy depends only on their current mass density $\rho = (\det F)^{-1} \rho_0$. As a consequence of the D’Alembert identity, it is easy to prove that perfect fluids cannot interact with

the external world by shear contact forces on the boundary of the region that they currently occupy (see for instance [32]). Therefore, one should not be surprised when observing that the inclusion of second gradient of placement in the volume density of the deformation energy enlarges the “possibilities” of interactions that are allowed to second-gradient materials in comparison to first-gradient materials.⁶ This section is dedicated to the description of the class of those “compatible” external interactions which are allowed in the case of first- and second-gradient continua.

The internal virtual work equation (10) is, in fact, a representation of a second-order distribution. Using the generalized Schwartz representation theorem for second-order distributions (see Schwartz [27]) as proven in Appendix A by successive application of the divergence theorem, the internal work functional can also be represented⁷ as

$$\begin{aligned} \delta W_{\Omega}^{\text{int}}(\delta\Pi) = & \int_{\Omega} \frac{\partial \bar{P}_i^A}{\partial X^A} \delta\Pi^i - \int_{\partial\Omega} \bar{P}_i^A N_A \delta\Pi^i + \int_{\partial\Omega} M_{\parallel L}^C \frac{\partial}{\partial X^C} (\mathbb{P}_i^{AB} N_B M_{\parallel A}^L) \delta\Pi^i \\ & - \int_{\partial\Omega} (\mathbb{P}_i^{AB} N_A N_B) \frac{\partial \delta\Pi^i}{\partial X^C} N^C - \int_{\partial\partial\Omega} (\mathbb{P}_i^{AB} V_A N_B) \delta\Pi^i, \end{aligned} \quad (11)$$

where

$$\bar{P}_i^A = P_i^A - \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B}.$$

Therefore, the representation (11), together with the virtual work principle (2), implies that the external work functionals must have the form

$$\delta W_{\Omega}^{\text{ext}}(\delta\Pi) = \int_{\Omega} \mathfrak{F}_i^{\Omega} \delta\Pi^i + \int_{\partial\Omega} \mathfrak{F}_i^{\partial\Omega} \delta\Pi^i + \int_{\partial\Omega} \mathfrak{D}_i^{\partial\Omega} \frac{\partial \delta\Pi^i}{\partial X^C} N^C + \int_{\partial\partial\Omega} \mathfrak{F}_i^{\partial\partial\Omega} \delta\Pi^i. \quad (12)$$

As already discussed in the introduction, in this expression, the co-vector fields \mathfrak{F}^{Ω} , $\mathfrak{F}^{\partial\Omega}$, and $\mathfrak{F}^{\partial\partial\Omega}$ are dual to virtual displacements and are, due to their integration domain, forces per unit reference volume, surface, and line, respectively. Moreover, an additional surface density field $\mathfrak{D}^{\partial\Omega}$ appears, which is called surface density of double-forces. This last field is dual to the normal derivatives of the virtual displacement.

2.5. Boundary value problem in second-gradient continua

When the class of external work functionals compatible with the internal work functionals is specified, the essential and natural boundary conditions, supplying well-posed boundary value problems, can be easily determined. We underline that the variational methods introduced in modern mechanics by D'Alembert allow for the simultaneous and logically coherent determination of the strong form of the PDEs that govern the evolution of the considered systems together with the corresponding boundary conditions. Other postulation schemes must, instead, face a difficult problem related to the independent postulations of bulk PDEs and boundary conditions: one has to verify then that the chosen postulates lead to well-posed problems.

The PDEs implied by the D'Alembert identity (2), when using $\delta F = \nabla(\delta\Pi)$ and $\delta\mathbb{F} = \nabla(\delta F)$, in view of equations (11) and (12) are given by

$$\frac{\partial}{\partial X^A} \left(P_i^A - \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) + \mathfrak{F}_i^{\Omega} = 0 \quad \text{in } \Omega. \quad (13)$$

In order to get well-posed problems, to these PDEs suitable boundary conditions must be added. The structure of both equations (11) and (12) obviously indicate that in second-gradient continua, one can assign as essential (kinematical) boundary conditions not only the placements on a subset Σ_{Π} of the

boundary $\partial\Omega$ but also the placements' normal derivatives on another subset Σ_\perp of $\partial\Omega$.⁸ The natural boundary conditions associated to equation (13) can be found by considering, in the D'Alembert identity, all the non-vanishing admissible virtual displacements (outside of Σ_Π) and normal derivatives of virtual displacements (outside of Σ_\perp) on the boundary of Ω , allowed by essential boundary conditions, to get

$$\mathfrak{F}_i^{\partial\Omega} = \bar{P}_i^A N_A - M_{\parallel L}^C \frac{\partial}{\partial X^C} (\mathbb{P}_i^{AB} N_B M_{\parallel A}^L) \quad \text{on } \partial\Omega \setminus \Sigma_\Pi \quad (14)$$

$$\mathfrak{F}_i^{\partial\partial\Omega} = (\mathbb{P}_i^{AB} V_A N_B)^+ + (\mathbb{P}_i^{AB} V_A N_B)^- \quad \text{on } \partial\partial\Omega \setminus \Sigma_\Pi \quad (15)$$

$$\mathfrak{D}_i^{\partial\Omega} = \mathbb{P}_i^{AB} N_A N_B \quad \text{on } \partial\Omega \setminus \Sigma_\perp \quad (16)$$

We recall that (see equation (56) in Appendix 1) the symbols $()^\pm$ denote the limits on the curves constituting $\partial\partial\Omega$ from the faces \pm of the quantities in the brackets.

2.6. Contact interactions in second-gradient continua: dependence on the shape of Cauchy cuts

The concept of contact interactions inside deformable bodies was developed in the third decade of the 19th century by Piola and Cauchy (a detailed discussion about the priority between them deserves further investigations: see e.g. [24,33]). While Piola, following Lagrange, considered contact interactions as derived concepts, Cauchy based his analysis on the laws of balance of forces and moments of forces and therefore treated contact forces as primitive concepts.

2.6.1. Cauchy cuts inside deformable bodies. Cauchy cuts are (suitably regular) surfaces in the Lagrangian or Eulerian configurations that are introduced to divide a continuum into disjoint subbodies. Cauchy, in his foundation of continuum mechanics, assumed that the interaction between two subbodies of a given deformable body, having in common a surface, is localized on such cuts. As shown in Truesdell [34] using a modern formalism, the contact interactions concentrated on Cauchy cuts represent the primitive concept by means of which, assuming as fundamental hypotheses the balance of force and balance of momentum of forces, the existence of the stress tensor can be proven. The key point of this proof is given by the celebrated Cauchy tetrahedron argument. However, Cauchy's argument is based on some assumptions which complicate the generalization to the case of second-gradient continua: for instance the absence of edge contact forces (for a more detailed discussion of this point see [15,18]).

Following an analysis that can already be found in the works by Piola and choosing the principle of virtual work as the most fundamental postulate of continuum mechanics, we show, in this section, that the concept of contact interaction can be formulated also for second-gradient continua, but as a derived concept. The question is rather delicate: in fact, it is true that even for N th gradient continua⁹ the interaction between subbodies is concentrated on Cauchy cuts, see dell'Isola and colleagues [13,16] where a discussion of the original results by Piola and Lagrange can be found.

In this aspect, D'Alembert–Lagrange's approach to continuum mechanics does not differ from Cauchy's. However, as we will show in the following of this subsection, one of the most important among the assumptions accepted by Cauchy, the so-called Cauchy postulate,¹⁰ has not a general validity.

The so-called Cauchy postulate has to be regarded as a property specific to first-gradient continua being valid for a particular class of deformation energy constitutive equations. As a consequence the choice of the word "postulate" seems rather inappropriate.

Indeed, within the variational postulation scheme, where the stresses are defined as duals in work to the gradients of the virtual displacement, the contact interactions between a subbody and its complement, divided by the Cauchy cut, are a derived concept.

2.6.2. *Validity of the principle of virtual work for subbodies and contact interactions in second-gradient continua: reasoning à la Piola.* In the following, we define contact interactions also in second-gradient continua, and we show how they do depend on the shape of the Cauchy cut. We will see that the contact interactions for second-gradient continua can be expressed in terms of

1. The value of the Piola–Lagrange stress and double-stress tensors in the considered point of a Cauchy cut.
2. The local shape of a Cauchy cut.

Let us consider an inner subbody $\hat{\Omega}$ of Ω : that is a connected subset of Ω as regular as Ω and which has no common boundary points neither with the faces nor with the edges of Ω , that is, $\partial\hat{\Omega} \cap \partial\Omega = \emptyset$ and $\partial\partial\hat{\Omega} \cap \partial\partial\Omega = \emptyset$. Following our notation, we denote by \hat{N} and \hat{V} the outward-pointing unit normals to the boundary surface $\partial\hat{\Omega}$ and to the boundary curves $\partial\partial\hat{\Omega}$. For what concerns the complement $\hat{\Omega}^c = \Omega \setminus \hat{\Omega}$, clearly the normals \hat{N}^c and \hat{V}^c are given by $\hat{N}^c = N$ on $\partial\Omega$ and $\hat{V}^c = V$ on $\partial\partial\Omega$ as well as $\hat{N}^c = -\hat{N}$ on the part of $\partial\hat{\Omega}^c$ which is included in $\partial\hat{\Omega}$ and $\hat{V}^c = -\hat{V}$ on the part of $\partial\partial\hat{\Omega}^c$ which is included in $\partial\partial\hat{\Omega}$. The virtual work identity (2), valid for every admissible $\delta\Pi$, can be written as

$$-\int_{\hat{\Omega}} (P_i^A \delta F_A^i + \mathbb{P}_i^{AB} \delta \mathbb{F}_{AB}^i) + \int_{\hat{\Omega}} \mathfrak{F}_i^\Omega \delta \Pi^i + \delta W_\Omega^{\text{tot}}|_{\hat{\Omega}^c}(\delta\Pi) = 0, \quad (17)$$

where

$$\begin{aligned} \delta W_\Omega^{\text{tot}}|_{\hat{\Omega}^c}(\delta\Pi) = & - \int_{\hat{\Omega}^c} (P_i^A \delta F_A^i + \mathbb{P}_i^{AB} \delta \mathbb{F}_{AB}^i) + \int_{\hat{\Omega}^c} \mathfrak{F}_i^\Omega \delta \Pi^i + \int_{\partial\Omega} \mathfrak{F}_i^{\partial\Omega} \delta \Pi^i \\ & + \int_{\partial\hat{\Omega}} \mathfrak{D}_i^{\partial\Omega} \frac{\partial \delta \Pi^i}{\partial X^C} N^C + \int_{\partial\partial\hat{\Omega}} \mathfrak{F}_i^{\partial\partial\Omega} \delta \Pi^i. \end{aligned}$$

Inserting the representation of internal work given by generalized Schwartz theorem (11) in its unique form involving transverse derivatives to the boundary $\partial\hat{\Omega}^c$, we obtain

$$\begin{aligned} \delta W_\Omega^{\text{tot}}|_{\hat{\Omega}^c}(\delta\Pi) = & \int_{\hat{\Omega}^c} \frac{\partial \bar{P}_i^A}{\partial X^A} \delta \Pi^i - \int_{\partial\hat{\Omega}^c} \bar{P}_i^A (\hat{N}^c)_A \delta \Pi^i + \int_{\partial\hat{\Omega}^c} \hat{M}_{\parallel L}^{cC} \frac{\partial}{\partial X^C} (\mathbb{P}_i^{AB} (\hat{N}^c)_B \hat{M}_{\parallel A}^{cL}) \delta \Pi^i \\ & - \int_{\partial\hat{\Omega}^c} (\mathbb{P}_i^{AB} (\hat{N}^c)_A (\hat{N}^c)_B) \frac{\partial \delta \Pi^i}{\partial X^C} (\hat{N}^c)^C - \int_{\partial\partial\hat{\Omega}^c} (\mathbb{P}_i^{AB} (\hat{V}^c)_A (\hat{N}^c)_B) \delta \Pi^i, \\ & + \int_{\hat{\Omega}^c} \mathfrak{F}_i^\Omega \delta \Pi^i + \int_{\partial\Omega} \mathfrak{F}_i^{\partial\Omega} \delta \Pi^i + \int_{\partial\hat{\Omega}} \mathfrak{D}_i^{\partial\Omega} \frac{\partial \delta \Pi^i}{\partial X^C} N^C + \int_{\partial\partial\hat{\Omega}} \mathfrak{F}_i^{\partial\partial\Omega} \delta \Pi^i. \end{aligned}$$

Using $\partial\hat{\Omega}^c = \partial\hat{\Omega} \cup \partial\Omega$ and $\partial\partial\hat{\Omega}^c = \partial\partial\hat{\Omega} \cup \partial\partial\Omega$ together with the boundary conditions (14)–(16) as well as the equilibrium equations (13), the expression can be simplified to

$$\begin{aligned} \delta W_\Omega^{\text{tot}}|_{\hat{\Omega}^c}(\delta\Pi) = & \int_{\partial\hat{\Omega}} \bar{P}_i^A \hat{N}_A \delta \Pi^i - \int_{\partial\hat{\Omega}} \hat{M}_{\parallel L}^C \frac{\partial}{\partial X^C} (\mathbb{P}_i^{AB} \hat{N}_B \hat{M}_{\parallel A}^L) \delta \Pi^i \\ & + \int_{\partial\hat{\Omega}} (\mathbb{P}_i^{AB} \hat{N}_A \hat{N}_B) \frac{\partial \delta \Pi^i}{\partial X^C} \hat{N}^C + \int_{\partial\partial\hat{\Omega}} (\mathbb{P}_i^{AB} \hat{V}_A \hat{N}_B) \delta \Pi^i. \end{aligned}$$

By defining the work functional of contact interaction for $\hat{\Omega}$ as follows

$$\delta W_{\hat{\Omega}}^{\text{ext, con}}(\delta \Pi) := \int_{\partial \hat{\Omega}} \mathfrak{F}_i^{\partial \hat{\Omega}} \delta \Pi^i + \int_{\partial \hat{\Omega}} \mathfrak{D}_i^{\partial \hat{\Omega}} \frac{\partial \delta \Pi^i}{\partial X^C} \hat{N}^C + \int_{\partial \partial \hat{\Omega}} \mathfrak{F}_i^{\partial \partial \hat{\Omega}} \delta \Pi^i,$$

with

$$\begin{aligned} \mathfrak{F}_i^{\partial \hat{\Omega}} &= \bar{P}_i^A \hat{N}_A - \hat{M}_{\parallel L}^C \frac{\partial}{\partial X^C} (\mathbb{P}_i^{AB} \hat{N}_B \hat{M}_{\parallel A}^L) && \text{on } \partial \hat{\Omega} \\ \mathfrak{D}_i^{\partial \hat{\Omega}} &= \mathbb{P}_i^{AB} \hat{N}_A \hat{N}_B && \text{on } \partial \hat{\Omega} \\ \mathfrak{F}_i^{\partial \partial \hat{\Omega}} &= (\mathbb{P}_i^{AB} \hat{V}_A \hat{N}_B)^+ + (\mathbb{P}_i^{AB} \hat{V}_A \hat{N}_B)^- && \text{on } \partial \partial \hat{\Omega}, \end{aligned} \tag{18}$$

we are let to define the external work functional for $\hat{\Omega}$ as

$$\delta W_{\hat{\Omega}}^{\text{ext}}(\delta \Pi) := \int_{\hat{\Omega}} \mathfrak{F}_i^{\Omega} \delta \Pi^i + \delta W_{\hat{\Omega}}^{\text{ext, con}}(\delta \Pi). \tag{19}$$

By defining

$$\delta W_{\hat{\Omega}}^{\text{int}}(\delta \Pi) := - \int_{\hat{\Omega}} (P_i^A \delta F_A^i + \mathbb{P}_i^{AB} \delta \mathbb{F}_{AB}^i),$$

we get that the D'Alembert identity (17) for Ω implies the D'Alembert identity for $\hat{\Omega}$:

$$\delta W_{\hat{\Omega}}^{\text{int}} + \delta W_{\hat{\Omega}}^{\text{ext}} = 0.$$

The presented derivation proves that we can obtain the formulation of the principle of virtual work for any inner subbody $\hat{\Omega}$ from its formulation for the body Ω . The definition (19) can be interpreted saying that the complement $\hat{\Omega}^c$ acts on the subbody $\hat{\Omega}$ via contact interactions which are seen from the subbody as external virtual work functionals.

We recall here that the presented definitions and reasonings parallel closely those used by Piola (see dell'Isola et al. [24]) for introducing contact interactions in the context of first-gradient theory.

Unfortunately, this circumstance was not remarked somewhere in the literature: In Fried and Gurtin [35], the principle of virtual work is postulated for every subbody introducing a so-called non-standard form of the principle of virtual power. However, this non-standard form was already presented by Germain [1,3].

2.6.3. *How contact interactions in second-gradient continua depend on the shape of the Cauchy cut.* Considering equation (18), evidently one sees that

- Contact double-forces $\mathfrak{D}_i^{\partial \hat{\Omega}}$ at X depend: (1) on the shape of the Cauchy cuts only via its unit normal at X , and this dependence is quadratic, (2) on the values at X of the Piola–Lagrange double-stress tensor.
- Edge contact forces $\mathfrak{F}_i^{\partial \partial \hat{\Omega}}$ at X depend: (1) on the shape of the edge of the Cauchy cuts only via the normals \hat{N}^{\pm} and \hat{V}^{\pm} at X and this dependence is bilinear, (2) on the values at X of the Piola–Lagrange double-stress tensor.

Very important, for understanding the true nature of the so-called Cauchy postulate, is the dependence of the surface contact forces $\mathfrak{F}_i^{\partial \hat{\Omega}}$ with respect to the shape of the Cauchy cut $\partial \hat{\Omega}$ at the point

$X \in \partial\hat{\Omega}$. To make more explicit this dependence, we use equation (57) of Appendix A together with $\mathbb{P}_i^{AB} = \mathbb{P}_i^{BA}$, which leads to

$$\mathfrak{F}_i^{\partial\hat{\Omega}} = \left(P_i^A - 2 \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) \hat{N}_A - \mathbb{P}_i^{AB} \frac{\partial \hat{N}_A}{\partial X^B} + \frac{\partial \mathbb{P}_i^{AB}}{\partial X^C} \hat{N}_B \hat{N}^C \hat{N}_A + \mathbb{P}_i^{AB} \frac{\partial \hat{N}^C}{\partial X^C} \hat{N}_B \hat{N}_A.$$

Hence, for second-gradient continua, the Lagrangian expression for surface contact forces depends polynomially on the normal and on the curvature of Cauchy cuts.¹¹ This polynomial includes a linear and a cubic term in the components of the normal vector, a linear term in the curvature and a mixed third-order polynomial quadratic in the normals and linear on the surface mean curvature. Only, when \mathbb{P} vanishes, we recover that contact interactions depend only linearly on the normal of Cauchy cuts.

3. The Eulerian virtual work principle for second-gradient continua: equilibrium conditions

Whether the problem is formulated in Lagrangian or Eulerian description, we still model the same physical phenomena. For this reason, the value of the virtual work for corresponding virtual displacements must be the same in Lagrangian and Eulerian descriptions:

$$\delta w_\omega^{\text{int}}(\delta \Pi) := \delta W_\Omega^{\text{int}}(\delta \Pi), \quad \delta w_\omega^{\text{ext}}(\delta \Pi) := \delta W_\Omega^{\text{ext}}(\delta \Pi). \quad (20)$$

Consequently, the virtual work equality holds also in the Eulerian description for every admissible Eulerian virtual displacement:

$$\delta w_\omega^{\text{tot}}(\delta \Pi) := \delta w_\omega^{\text{int}}(\delta \Pi) + \delta w_\omega^{\text{ext}}(\delta \Pi) = 0. \quad (21)$$

Since, after the simple change of variables given by Π , the Eulerian internal virtual work is still a second-order distribution (see Schwartz [27] and the subsequent section 4) and can be represented as

$$\delta w_\omega^{\text{int}}(\delta \Pi) = - \int_\omega (c_i^j \delta d_j^i + \mathfrak{c}_i^{jk} \delta \mathfrak{d}_{jk}^i), \quad (22)$$

where c_i^j and \mathfrak{c}_i^{jk} are the components of the work conjugates to the first and second gradient of the spatial virtual displacement. We call them the Cauchy–Euler stress c and the Cauchy–Euler double-stress \mathfrak{c} .

The “Axiom of Power of Internal Forces”¹² in Eulerian form (as postulated in Germain [3]), which must hold for any suitably regular subbody $\hat{\omega} \subseteq \omega$, requires the following identity

$$\delta w_\omega^{\text{int}}|_{\hat{\omega}}(\delta \Pi_{\text{rig}}) = 0 \quad \forall \delta \Pi_{\text{rig}},$$

whereas the rigid virtual displacements in Eulerian form are parameterized by

$$\delta \Pi_{\text{rig}}^i = a^i + W_j^i x^j, \quad \text{where } W = -W^T.$$

Since the second gradient of the rigid virtual displacement vanishes, we get the symmetry of the Cauchy–Euler stress c , that is

$$c = c^T.$$

To characterize the compatible external work functional, the same integration by parts procedure as in the Lagrangian formulation can be applied. Defining

$$\bar{c}_i^j = c_i^j - \frac{\partial \mathfrak{c}_i^{jk}}{\partial x^k},$$

and using the results presented in Appendix A, it is proven that the Eulerian internal work functional has the following representation:

$$\begin{aligned} \delta w_\omega^{\text{int}}(\delta\Pi) = & \int_\omega \frac{\partial \bar{c}_i^j}{\partial x^j} \delta\Pi^i - \int_{\partial\omega} \bar{c}_i^j n_j \delta\Pi^i + \int_{\partial\omega} m_{\parallel l}^c \frac{\partial}{\partial x^c} (\mathfrak{c}_i^{jk} n_k m_{\parallel j}^l) \delta\Pi^i \\ & - \int_{\partial\omega} (\mathfrak{c}_i^{jk} n_j n_k) \frac{\partial \delta\Pi^i}{\partial x^c} n^c - \int_{\partial\partial\omega} (\mathfrak{c}_i^{jk} v_j n_k) \delta\Pi^i. \end{aligned} \tag{23}$$

Consequently, the compatible external work functionals must be of the form

$$\delta w_\omega^{\text{ext}}(\delta\Pi) = \int_\omega \mathfrak{f}_i^\omega \delta\Pi^i + \int_{\partial\omega} \mathfrak{f}_i^{\partial\omega} \delta\Pi^i + \int_{\partial\omega} \mathfrak{d}_i^{\partial\omega} \frac{\partial \delta\Pi^i}{\partial x^c} n^c + \int_{\partial\partial\omega} \mathfrak{f}_i^{\partial\partial\omega} \delta\Pi^i, \tag{24}$$

where the co-vector fields \mathfrak{f}^ω , $\mathfrak{f}^{\partial\omega}$, and $\mathfrak{f}^{\partial\partial\omega}$ are forces per unit current volume, surface, and line, respectively. Also in the Eulerian framework, there appears a surface density of double-forces $\mathfrak{d}^{\partial\omega}$, which is a density per unit current surface and which is dual to the derivative of the Eulerian virtual displacement with respect to the current normal vector.

Inserting equations (23) and (24) in equation (21), we obtain the equilibrium equations

$$\frac{\partial}{\partial x^j} \left(c_i^j - \frac{\partial \mathfrak{c}_i^{jk}}{\partial x^k} \right) + \mathfrak{f}_i^\omega = 0 \quad \text{in } \omega, \tag{25}$$

and by considering the dual in work of virtual displacement left free by imposed essential boundary conditions, we get

$$\mathfrak{f}_i^{\partial\omega} = \bar{c}_i^j n_j - m_{\parallel l}^c \frac{\partial}{\partial x^c} (\mathfrak{c}_i^{jk} n_k m_{\parallel j}^l) \quad \text{on } \partial\omega \setminus \Pi(\Sigma_\Pi), \tag{26}$$

$$\mathfrak{f}_i^{\partial\partial\omega} = (\mathfrak{c}_i^{jk} v_j n_k)^+ + (\mathfrak{c}_i^{jk} v_j n_k)^- \quad \text{on } \partial\partial\omega \setminus \Pi(\Sigma_\Pi). \tag{27}$$

We recall that (see equation (56) in Appendix A) the symbols $()^\pm$ denote the limits on the curves constituting $\partial\partial\omega$ from the faces \pm of the quantities in the brackets.

By considering the dual in work of the normal derivative of virtual displacement left free by imposed essential boundary conditions, we get

$$\mathfrak{d}_i^{\partial\omega} = \mathfrak{c}_i^{jk} n_j n_k \quad \text{on } \partial\omega \setminus \Pi(\Sigma_\perp). \tag{28}$$

Using the same procedure as in the Lagrangian framework, the virtual work of the contact interaction between a subbody $\hat{\omega} \subset \omega$ and its complement $\hat{\omega}^c$ can be recognized as

$$\delta w_{\hat{\omega}}^{\text{ext, con}}(\delta\Pi) = \int_{\partial\hat{\omega}} \mathfrak{f}_i^{\partial\hat{\omega}} \delta\Pi^i + \int_{\partial\hat{\omega}} \mathfrak{d}_i^{\partial\hat{\omega}} \frac{\partial \delta\Pi^i}{\partial x^c} \hat{n}^c + \int_{\partial\partial\hat{\omega}} \mathfrak{f}_i^{\partial\partial\hat{\omega}} \delta\Pi^i,$$

where the contact surface forces $\mathfrak{f}^{\partial\hat{\omega}}$, contact surface double-forces $\mathfrak{d}^{\partial\hat{\omega}}$, and contact line forces $\mathfrak{f}^{\partial\partial\hat{\omega}}$ are given as

$$\begin{aligned} \mathfrak{f}_i^{\partial\hat{\omega}} &= \bar{c}_i^j \hat{n}_j - \hat{m}_{\parallel l}^c \frac{\partial}{\partial x^c} (\mathfrak{c}_i^{jk} \hat{n}_k \hat{m}_{\parallel j}^l) && \text{on } \partial\hat{\omega}, \\ \mathfrak{d}_i^{\partial\hat{\omega}} &= \mathfrak{c}_i^{jk} \hat{n}_j \hat{n}_k && \text{on } \partial\hat{\omega}, \\ \mathfrak{f}_i^{\partial\partial\hat{\omega}} &= (\mathfrak{c}_i^{jk} \hat{v}_j \hat{n}_k)^+ - (\mathfrak{c}_i^{jk} \hat{v}_j \hat{n}_k)^- && \text{on } \partial\partial\hat{\omega}. \end{aligned} \tag{29}$$

Here, \hat{n} denotes the outward-pointing unit normal to $\partial\hat{\omega}$ and $\hat{\nu}$ the outward-pointing unit normal to the boundary curves $\partial\partial\hat{\omega}$. Using equation (57) of Appendix A together with $\mathfrak{c}_i^{jk} = \mathfrak{c}_i^{kj}$, the contact surface force can be expressed as

$$\mathfrak{f}_i^{\partial\hat{\omega}} = \left(c_i^j - 2 \frac{\partial \mathfrak{c}_i^{jk}}{\partial x^k} \right) \hat{n}_j - \mathfrak{c}_i^{jk} \frac{\partial \hat{n}_j}{\partial x^k} + \frac{\partial \mathfrak{c}_i^{jk}}{\partial x^l} \hat{n}_k \hat{n}^l \hat{n}_j + \mathfrak{c}_i^{jk} \frac{\partial \hat{n}^l}{\partial x^l} \hat{n}_j \hat{n}_k. \quad (30)$$

The expression for contact forces, which we just obtained, must be compared with the expression obtained following the Cauchy tetrahedron procedure. First of all, we note that when \mathfrak{c} vanishes, we recover Cauchy's representation formula. However, in the case of non-vanishing \mathfrak{c} , we can immediately see in equation (30) that $\mathfrak{f}^{\partial\hat{\omega}}$ depends also on the curvature of the Cauchy cut. Clearly the so-called Cauchy postulate is not verified for second-gradient continua. As the dependence of the deformation energy on the second gradient of placement produces a non-vanishing tensor \mathfrak{c} , we must conclude that the logical status of the Cauchy postulate is different from that of the principle of virtual work, as it holds only for a specific class of continua.

Finally, we remark that, in second-gradient continua, contact interactions must include double-forces. As shown in Germain [3], the tangent part of contact double-forces can be interpreted as contact couples (see also Toupin [4]), whereas the normal part of contact double-forces are a kind of interaction completely independent of forces (i.e. interactions expending work on displacements) and couples (i.e. interactions expending work on rotations). Therefore, it appears evident that postulating the balance of forces and moment of forces is not enough, in second-gradient continua, to get all necessary conditions which follow from the principle of virtual work. This circumstance shows the intrinsic weakness of Cauchy's postulation scheme in producing the theory of generalized continua.

4. Piola transformations in second-gradient continua

In the previous section, we have introduced as Eulerian dual-quantities the Cauchy–Euler stresses together with the Eulerian external forces and double-forces. The Piola transformation problem consists in finding the relationships between the Lagrangian and Eulerian stresses and double-stresses as well as external forces and double-forces implied by the identities (20).

Let Φ be a Lagrangian field with domain Ω related to the corresponding Eulerian field ϕ with domain ω by $\Phi(X) = \phi(\Pi(X))$. Recalling (1), the chain rule implies that the gradients of the Lagrangian and Eulerian fields are connected by

$$\frac{\partial \Phi}{\partial X^A} (X) = \frac{\partial \phi}{\partial x^j} (\Pi(X)) \frac{\partial \Pi^j}{\partial X^A} (X) = \frac{\partial \phi}{\partial x^j} (\Pi(X)) F_A^j (X). \quad (31)$$

As this relation can also be written as

$$\frac{\partial \Phi}{\partial X^A} (\pi(x)) = \frac{\partial \phi}{\partial x^j} (x) F_A^j (\pi(x)), \quad (32)$$

we will drop the arguments in what follows. Using this convention together with equation (1), we obtain, by taking once more the gradient of equation (31), the expression

$$\frac{\partial^2 \Phi}{\partial X^A \partial X^B} = \frac{\partial^2 \phi}{\partial x^j \partial x^k} F_A^j F_B^k + \frac{\partial \phi}{\partial x^j} \mathbb{F}_{AB}^j.$$

Consequently, the gradients of the Lagrangian and Eulerian virtual displacement fields are related by

$$\delta F_A^i = \delta d_j^i F_A^j, \quad \delta \mathbb{F}_{AB}^i = \delta d_{jk}^i F_A^j F_B^k + \delta d_j^i \mathbb{F}_{AB}^j. \quad (33)$$

When the gradient of the Eulerian field ϕ should be expressed in terms of its Lagrangian counterpart, we use the relation $\phi = \Phi \circ \pi$ as well as $(F^{-1})^A_i = \partial \pi^A / \partial x^i$ to end up with

$$\frac{\partial \phi}{\partial x^i} = \frac{\partial \Phi}{\partial X^A} \frac{\partial \pi^A}{\partial x^i} = \frac{\partial \Phi}{\partial X^A} (F^{-1})^A_i. \tag{34}$$

In the following subsections, we will use these relations together with the formulas of the change of variables of volume, surface, and line integrals of Appendix B.4. We obtain the Piola transformations for stresses, double-stresses as well as external forces and external double-forces.

4.1. Piola transformation of stress and double-stress

For a scalar valued function $\Phi: \Omega \rightarrow \mathbb{R}$, the change of variables is of the form

$$\int_{\Omega} \Phi(X) = \int_{\pi(\omega)} \Phi(X) = \int_{\omega} \Phi(\pi(x)) j(x), \tag{35}$$

where $j(x) = J^{-1}(\pi(x)) = \det(F(\pi(x)))^{-1}$ is the volume density change induced by π and $J = \det(F)$. The change of variables of the internal work functional leads to

$$\delta W_{\Omega}^{\text{int}}(\delta \Pi) = - \int_{\omega} J^{-1} (P^A_i \delta F^i_A + \mathbb{P}^{AB}_i \delta \mathbb{F}^i_{AB}),$$

where all functions in the integral are to be composed with π . If we introduce the relation (33), we obtain the functional

$$\delta W_{\Omega}^{\text{int}}(\delta \Pi) = - \int_{\omega} J^{-1} P^A_i \delta d^i_j F^j_A - \int_{\omega} J^{-1} \mathbb{P}^{AB}_i (\delta \mathfrak{d}^i_{jk} F^j_A F^k_B + \delta d^i_j \mathbb{F}^j_{AB}). \tag{36}$$

Because of equation (20), $\delta w_{\omega}^{\text{int}}$ is a second-order distribution, which can be represented as

$$\delta w_{\omega}^{\text{int}}(\delta \Pi) = - \int_{\omega} (c^j_i \delta d^i_j + \mathfrak{c}^{jk}_i \delta \mathfrak{d}^i_{jk}). \tag{37}$$

By using equations (20), (36), and (37), we get the relations between the Piola–Lagrange and the Cauchy–Euler stresses and double-stresses

$$c^j_i = J^{-1} (P^A_i F^j_A + \mathbb{P}^{AB}_i \mathbb{F}^j_{AB}), \quad \mathfrak{c}^{jk}_i = J^{-1} \mathbb{P}^{AB}_i F^j_A F^k_B, \tag{38}$$

which are called the Piola transformation of stress and double-stress.

4.2. Piola transformations of external forces and double-forces

The Eulerian external work functional (24) is the sum of four different terms

$$\int_{\omega} \mathfrak{f}^{\omega}_i \delta \Pi^i, \quad \int_{\partial \omega} \mathfrak{f}^{\partial \omega}_i \delta \Pi^i, \quad \int_{\partial \omega} \mathfrak{d}^{\partial \omega}_i \frac{\partial \delta \Pi^i}{\partial x^c} n^c, \quad \int_{\partial \partial \omega} \mathfrak{f}^{\partial \partial \omega}_i \delta \Pi^i.$$

The Lagrangian work functional (12) admits a similar decomposition in four terms

$$\int_{\Omega} \mathfrak{F}^{\Omega}_i \delta \Pi^i, \quad \int_{\partial \Omega} \mathfrak{F}^{\partial \Omega}_i \delta \Pi^i, \quad \int_{\partial \Omega} \mathfrak{D}^{\partial \Omega}_i \frac{\partial \delta \Pi^i}{\partial X^C} N^C, \quad \int_{\partial \partial \Omega} \mathfrak{F}^{\partial \partial \Omega}_i \delta \Pi^i.$$

The difficulty arises because the change of variable from Eulerian to Lagrangian descriptions does not induce a one-to-one correspondence between the listed terms. In particular, the work of external double-forces in Eulerian description does not produce only a term which can be recognized as work of Lagrangian double-forces. In fact, Eulerian work of double-forces, once transformed into Lagrangian description can be uniquely decomposed into the sum of work functional of double-forces plus work of surface forces plus work of edge forces. This is due to the fact that Eulerian normal derivatives, once transformed into Lagrangian description, are derivatives along a direction not orthogonal to the referential boundary $\partial\Omega$.

4.2.1. Transformations of external forces. The external work functionals due to Eulerian force densities can readily be transformed into Lagrangian description when applying the corresponding change of variables according to equation (64) with the volume Jacobian $J = \det(F)$, the area Jacobian $J_\Sigma = \|JF^{-T} \cdot N\|$ as well as the length Jacobian $J_\Lambda = \|F \cdot T\|$.

The transformations are: for the volume forces

$$\int_{\omega} f_i^\omega \delta\Pi^i = \int_{\Omega} J f_i^\omega \delta\Pi^i, \quad (39)$$

for the surface forces

$$\int_{\partial\omega} f_i^{\partial\omega} \delta\Pi^i = \int_{\partial\Omega} J_\Sigma f_i^{\partial\omega} \delta\Pi^i, \quad (40)$$

and finally for the edge forces

$$\int_{\partial\partial\omega} f_i^{\partial\partial\omega} \delta\Pi^i = \int_{\partial\partial\Omega} J_\Lambda f_i^{\partial\partial\omega} \delta\Pi^i. \quad (41)$$

4.2.2. Transformation of external surface double-forces. The external work functional of Eulerian surface double-forces is $\int_{\partial\omega} \mathfrak{d}_i^{\partial\omega} (\partial\delta\Pi^i / \partial x^r) n^r$. Using the change of variable (64) together with equation (34) for the current gradient of the virtual displacement as well as the expression (62) of the Eulerian unit normal in terms of the Lagrangian unit normal, the work functional takes the form

$$\int_{\partial\Omega} \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^R} (F^{-1})_r^R \left(g^{rs} \frac{(F^{-1})_s^S N_S}{\|F^{-T} \cdot N\|} \right) \|JF^{-T} \cdot N\|.$$

Identifying the inverse of the right Cauchy–Green strain C^{-1} from equation (5), using simple algebra, the work functional becomes

$$\int_{\partial\Omega} J \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^R} (C^{-1})^{RS} N_S. \quad (42)$$

Clearly, this expression involves derivatives of $\delta\Pi$ which are not normal to the boundary $\partial\Omega$. Hence, it cannot coincide with the work expended by the Lagrangian double-forces.

We decompose it into Lagrangian normal and tangential derivatives as follows: using the Kronecker-delta δ_R^E and writing the gradient as

$$\frac{\partial\delta\Pi^i}{\partial X^R} = \frac{\partial\delta\Pi^i}{\partial X^E} \delta_R^E = \frac{\partial\delta\Pi^i}{\partial X^E} (N^E N_R + M_{\parallel R}^E)$$

together with the equality

$$N_R(C^{-1})^{RS}N_S = N_R(F^{-1})_r^R g^{rs}(F^{-1})_s^S N_S = \|F^{-T} \cdot N\|^2$$

the normal part of equation (42) is given by

$$\int_{\partial\Omega} J \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^E} N^E N_R(C^{-1})^{RS} N_S = \int_{\partial\Omega} J \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^E} N^E \|F^{-T} \cdot N\|^2.$$

The residual tangential part is written as

$$\int_{\partial\Omega} J \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^E} M_{\parallel R}^E(C^{-1})^{RS} N_S = \int_{\partial\Omega} S_i^{RS} N_S \frac{\partial\delta\Pi^i}{\partial X^E} M_{\parallel R}^E,$$

with

$$S_i^{RS} = J \mathfrak{d}_i^{\partial\omega}(C^{-1})^{RS}.$$

For a fixed index i , this functional is of the form studied in Appendix A equation (53). Hence, it can be represented by the sum of the two functionals as given by equations (54) and (55), which reads in the present case as the sum of the following terms

$$\int_{\partial\partial\Omega} (J \mathfrak{d}_i^{\partial\omega}(C^{-1})^{RS} N_S V_R) \delta\Pi^i$$

and

$$- \int_{\partial\Omega} M_{\parallel D}^E \frac{\partial}{\partial X^E} (J \mathfrak{d}_i^{\partial\omega} K^D) \delta\Pi^i,$$

where we have introduced the Lagrangian vector field

$$K^D := M_{\parallel R}^D(C^{-1})^{RS} N_S = ((F^{-1})_i^D g^{ij}(F^{-1})_j^S - \|F^{-T} \cdot N\|^2 G^{DS}) N_S.$$

Consequently, the external work functional of Eulerian double-forces can be written as

$$\begin{aligned} \int_{\partial\omega} \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial x^c} n^c &= \int_{\partial\Omega} J \mathfrak{d}_i^{\partial\omega} \frac{\partial\delta\Pi^i}{\partial X^E} N^E \|F^{-T} \cdot N\|^2 \\ &- \int_{\partial\Omega} M_{\parallel D}^E \frac{\partial}{\partial X^E} (J \mathfrak{d}_i^{\partial\omega} K^D) \delta\Pi^i + \int_{\partial\partial\Omega} (J \mathfrak{d}_i^{\partial\omega}(C^{-1})^{RS} N_S V_R) \delta\Pi^i. \end{aligned} \tag{43}$$

Remark that the two last terms will intervene in the Lagrangian expression for external surface and edge work functionals.

4.3. Identification of Piola transformations

Both Lagrangian and Eulerian external work functionals (12) and (24) are unique representations in terms of transverse derivatives. Hence, when transforming the Eulerian work functional into Lagrangian description, the unique relationships between the Lagrangian and Eulerian external forces and double-forces can be identified.

Owing to the previously discussed transformations (39)–(41) and to expression (43) for the external double-force work functional, the external virtual work functional can be written as

$$\begin{aligned}
\delta w_{\omega}^{\text{ext}}(\delta \Pi) &= \int_{\Omega} J \mathfrak{f}_i^{\omega} \delta \Pi^i + \int_{\partial \Omega} \left(J_{\Sigma} \mathfrak{f}_i^{\partial \omega} - M_{\parallel D}^E \frac{\partial}{\partial X^E} (J \mathfrak{d}_i^{\partial \omega} K^D) \right) \delta \Pi^i \\
&+ \int_{\partial \Omega} J \mathfrak{d}_i^{\partial \omega} \| F^{-T} \cdot N \|^2 \frac{\partial \delta \Pi^i}{\partial X^E} N^E \\
&+ \int_{\partial \partial \Omega} J_{\Lambda} \mathfrak{f}_i^{\partial \partial \omega} + \int_{\partial \partial \Omega} (J \mathfrak{d}_i^{\partial \omega} (C^{-1})^{RS} N_S V_R) \delta \Pi^i.
\end{aligned} \tag{44}$$

Note, the last integral expression must be understood in the sense of the convention specified in equation (56) of Appendix A. In agreement with the identity (20), comparison of equation (44) with equation (12) induces the following transformation formulas of the force densities

$$\mathfrak{F}_i^{\Omega} = J \mathfrak{f}_i^{\omega}, \tag{45}$$

$$\mathfrak{F}_i^{\partial \Omega} = J_{\Sigma} \mathfrak{f}_i^{\partial \omega} - M_{\parallel D}^E \frac{\partial}{\partial X^E} (J \mathfrak{d}_i^{\partial \omega} K^D), \tag{46}$$

$$\mathfrak{F}_i^{\partial \partial \Omega} = J_{\Lambda} \mathfrak{f}_i^{\partial \partial \omega} + (J \mathfrak{d}_i^{\partial \omega} (C^{-1})^{RS} V_R N_S)^+ + (J \mathfrak{d}_i^{\partial \omega} (C^{-1})^{RS} V_R N_S)^-, \tag{47}$$

as well as the Piola transformation of the surface double-force

$$\mathfrak{D}_i^{\partial \Omega} = J \| F^{-T} \cdot N \|^2 \mathfrak{d}_i^{\partial \omega}. \tag{48}$$

5. Consistency of Piola transformations

In section 2, we have derived the equilibrium equations and boundary conditions in Lagrangian description. In section 3, we have repeated the same procedure to obtain the corresponding Eulerian boundary value problem. Section 4 was then dedicated to find the Piola transformations relating Piola–Lagrange stress and double-stress with Cauchy–Euler stress and double-stress. Moreover, the transformation formulas for the external force and double-force contributions have been derived. Essentially, we have obtained all the desired results. However, as the transformation formulas (46)–(48) are novel and not very intuitive, a consistency check would be desirable. This is exactly what this section is for. Indeed, Piola transformation for external forces and double-forces can also be deduced from the Piola transformations (38) of stress and double-stress, by making use of the equilibrium conditions both in Eulerian and Lagrangian frameworks.

5.1. Transformation of local equilibrium equations

From the Eulerian principle of virtual work, we obtain the equilibrium equations (25), which are

$$\frac{\partial c_i^a}{\partial x^a} - \left(\frac{\partial^2 \mathfrak{c}_i^{ab}}{\partial x^b \partial x^a} \right) + \mathfrak{f}_i^{\omega} = 0. \tag{49}$$

Inserting the relations from equation (38) into equation (49) leads to

$$\frac{\partial}{\partial x^a} (J^{-1} P_i^A F_A^a + J^{-1} \mathbb{P}_i^{AB} \mathbb{F}_{AB}^a) - \frac{\partial}{\partial x^a} \left(\frac{\partial}{\partial x^b} (J^{-1} \mathbb{P}_i^{AB} F_A^a F_B^b) \right) + \mathfrak{f}_i^{\omega} = 0.$$

For a fixed index i , applying the Piola identity $\text{div}(J^{-1} F \cdot T) = J^{-1} \text{Div}(T)$ from equation (65), Appendix C, we obtain

$$J^{-1} \frac{\partial P_i^A}{\partial X^A} + \frac{\partial}{\partial x^a} \left(J^{-1} \mathbb{P}_i^{AB} \mathbb{F}_{AB}^a - J^{-1} \frac{\partial}{\partial X^B} (\mathbb{P}_i^{AB} F_A^a) \right) + f_i^\omega = 0.$$

Using Leibniz' rule in the last term, two terms cancel and we end up with

$$J^{-1} \frac{\partial P_i^A}{\partial X^A} - \frac{\partial}{\partial x^a} \left(J^{-1} F_A^a \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) + f_i^\omega = 0.$$

Applying once more the Piola identity (65) on the second term, the equality reduces to

$$J^{-1} \frac{\partial P_i^A}{\partial X^A} - J^{-1} \frac{\partial}{\partial X^A} \left(\frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) + f_i^\omega = 0.$$

Using the Lagrangian equilibrium equations (13) in the last expression, we immediately obtain the relation

$$J f_i^\omega = - \frac{\partial P_i^A}{\partial X^A} + \frac{\partial}{\partial X^A} \left(\frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) = \mathfrak{F}_i^\Omega,$$

and we recover the Piola transformation (45) relating Eulerian and Lagrangian external volume force densities.

5.2. Transformation of boundary conditions

The boundary conditions (14)–(16) in Lagrangian or the boundary conditions (26)–(28) in Eulerian form relate the external interactions with the stresses and double-stresses. In the following, we show that inserting the Piola transformation of the stress and double-stress into the boundary conditions, confirms the transformation rules for the external force and double-force densities. The transformations are carried out in the same order as in section 4. Hence, we start with the double-force density followed by the edge forces and close the subsection with the most tedious transformation of the surface force densities.

5.2.1. External surface double-forces. The external double-force densities must satisfy the boundary condition (28), which, after inserting equation (38), can be expressed as

$$\mathfrak{D}_i^{\partial\omega} = \mathfrak{C}_i^{jk} n_j n_k = J^{-1} \mathbb{P}_i^{AB} F_A^j F_B^k n_j n_k.$$

Using (62) to get $F_A^j n_j = N_A \parallel F^{-T} \cdot N \parallel^{-1}$, the previous expression becomes

$$\mathfrak{D}_i^{\partial\omega} = J^{-1} \mathbb{P}_i^{AB} N_A N_B \parallel F^{-T} \cdot N \parallel^{-2} = J^{-1} \mathfrak{D}_i^{\partial\Omega} \parallel F^{-T} \cdot N \parallel^{-2},$$

where the Lagrangian boundary condition (16) has been used. Clearly, this identity is equivalent to the Piola transformation (48) of external double-forces.

5.2.2. External edge forces. When dealing with edge force densities, we use the transformations (62) and (63) for the normal to the faces and for the tangent-normal to the edge (Appendix B.2). We obtain

$$\begin{aligned}
\mathfrak{C}_i^{jk} v_j n_k &= \mathfrak{C}_i^{jk} n_k \left[(F^{-1})_j^R V_R - \frac{(C^{-1})^{RS} V_R N_S (F^{-1})_j^Q N_Q}{\|F^{-T} \cdot N\|^2} \right] J_\Sigma J_\Lambda^{-1} \\
&= \mathfrak{C}_i^{jk} \frac{(F^{-1})_k^L N_L}{\|F^{-T} \cdot N\|} (F^{-1})_j^R V_R \|JF^{-T} \cdot N\| J_\Lambda^{-1} \\
&\quad - \mathfrak{C}_i^{jk} n_k \frac{(C^{-1})^{RS} V_R N_S (F^{-1})_j^Q N_Q}{\|F^{-T} \cdot N\|} J J_\Lambda^{-1}.
\end{aligned}$$

Replacing the Cauchy-Euler double-stress with equation (38) in the first term of the right hand side and using equation (62) in the second term, we obtain the following equation:

$$\mathfrak{C}_i^{jk} v_j n_k = (\mathbb{P}_i^{AB} F_A^j F_B^k (F^{-1})_k^L N_L (F^{-1})_j^R V_R - J c_i^{jk} n_k n_j (C^{-1})^{RS} V_R N_S) J_\Lambda^{-1}.$$

Using the boundary condition (28), we get

$$\mathfrak{C}_i^{jk} v_j n_k = (\mathbb{P}_i^{AB} V_A N_B - J \mathfrak{d}_i^{\partial\omega} (C^{-1})^{RS} V_R N_S) J_\Lambda^{-1}.$$

Using the last expression in the Eulerian boundary condition (27) together with the Lagrangian boundary condition (15), we obtain

$$\begin{aligned}
J_\Lambda \mathfrak{f}_i^{\partial\partial\omega} &= J_\Lambda [(\mathfrak{C}_i^{jk} v_j n_k)^+ + (\mathfrak{C}_i^{jk} v_j n_k)^-] \\
&= (\mathbb{P}_i^{AB} V_A N_B)^+ + (\mathbb{P}_i^{AB} V_A N_B)^- - (J \mathfrak{d}_i^{\partial\omega} (C^{-1})^{RS} V_R N_S)^+ \\
&\quad - (J \mathfrak{d}_i^{\partial\omega} (C^{-1})^{RS} V_R N_S)^- \\
&= \mathfrak{F}_i^{\partial\partial\Omega} - (J \mathfrak{d}_i^{\partial\omega} (C^{-1})^{RS} V_R N_S)^+ - (J \mathfrak{d}_i^{\partial\omega} (C^{-1})^{RS} V_R N_S)^-,
\end{aligned}$$

which corresponds to the Piola transformation of edge forces (47).

5.2.3. External surface forces. For the transformation of the surface force density, we consider the two terms in the boundary conditions (26) separately. Using equations (38) and (62), we get

$$\left(c_i^j - \frac{\partial \mathfrak{C}_i^{jk}}{\partial x^k} \right) n_j = \left(J^{-1} P_i^A F_A^j + J^{-1} \mathbb{P}_i^{AB} \mathbb{F}_{AB}^j - \frac{\partial}{\partial x^k} (J^{-1} \mathbb{P}_i^{AB} F_A^j F_B^k) \right) \frac{(F^{-1})_j^L N_L}{\|F^{-T} \cdot N\|}.$$

Using the Piola identity (65) in the third term, we obtain

$$\left(c_i^j - \frac{\partial \mathfrak{C}_i^{jk}}{\partial x^k} \right) n_j = \left(P_i^A F_A^j + \mathbb{P}_i^{AB} \mathbb{F}_{AB}^j - \frac{\partial}{\partial X^B} (\mathbb{P}_i^{AB} F_A^j) \right) \frac{(F^{-1})_j^L N_L}{\|JF^{-T} \cdot N\|}.$$

Applying Leibniz' rule in the last term, the expression simplifies further to

$$\left(c_i^j - \frac{\partial \mathfrak{C}_i^{jk}}{\partial x^k} \right) n_j = J_\Sigma^{-1} \left(P_i^A - \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) N_A.$$

For treating the second term that appears in the Eulerian boundary condition (26), we must use the surface Piola-type identity in the form equation (69) (Appendix C). We get the relation

$$m_{\parallel l}^c \frac{\partial}{\partial x^c} (\mathfrak{C}_i^{jk} n_k m_{\parallel j}^l) = J_\Sigma^{-1} M_{\parallel S}^A \frac{\partial}{\partial X^A} (J_\Sigma M_{\parallel R}^S (F^{-1})_j^R m_{\parallel j}^l \mathfrak{C}_i^{jk} n_k).$$

Inserting $m_{\parallel j}^l = \delta_j^l - n^l n_j$ and using equations (38) and (28), we get

$$m_{||_I}^c \frac{\partial}{\partial x^c} (\mathfrak{c}_i^{jk} n_k m_{||_j}^l) = J_{\Sigma}^{-1} M_{||_S}^A \frac{\partial}{\partial X^A} (M_{||_R}^S \parallel JF^{-T} \cdot N \parallel (F^{-1})_j^R J^{-1} \mathbb{P}_i^{CD} F_C^j F_D^k n_k - M_{||_R}^S \parallel JF^{-T} \cdot N \parallel (F^{-1})_i^R \mathfrak{d}_i^{\partial\omega} n^l).$$

Since $\parallel F^{-T} \cdot N \parallel F_D^k n_k = N_D$ and

$$M_{||_R}^S (F^{-1})_i^R n^l \parallel F^{-T} \cdot N \parallel = M_{||_R}^S (F^{-1})_i^R g^{lj} (F^{-1})_j^C N_C = M_{||_R}^S (C^{-1})^{RC} N_C = K^S,$$

we can modify the expression further to

$$m_{||_I}^c \frac{\partial}{\partial x^c} (\mathfrak{c}_i^{jk} n_k m_{||_j}^l) = J_{\Sigma}^{-1} M_{||_S}^A \frac{\partial}{\partial X^A} (M_{||_C}^S \mathbb{P}_i^{CD} N_D - J \mathfrak{d}_i^{\partial\omega} K^S)$$

and use it to finally write

$$\begin{aligned} \mathfrak{f}_i^{\partial\omega} &= J_{\Sigma}^{-1} \left[\left(P_i^A - \frac{\partial \mathbb{P}_i^{AB}}{\partial X^B} \right) N_A - M_{||_S}^A \frac{\partial}{\partial X^A} (M_{||_C}^S \mathbb{P}_i^{CD} N_D) + M_{||_S}^A \frac{\partial}{\partial X^A} (J \mathfrak{d}_i^{\partial\omega} K^S) \right] \\ &= J_{\Sigma}^{-1} \left[\mathfrak{F}_i^{\partial\Omega} + M_{||_S}^A \frac{\partial}{\partial X^A} (J \mathfrak{d}_i^{\partial\omega} K^S) \right]. \end{aligned}$$

The last equality confirms the Piola transformation (46) for external surface forces. Hence, we succeeded in showing with an alternative way the Piola transformation formulas for the external forces and double-forces.

In first-gradient theory, where $\mathbb{P} = 0$, it is quite common (see e.g. [14,36]) to show the Piola transformation of the stresses by assuming the transformation rule for the surface forces

$$J_{\Sigma}^{-1} \mathfrak{F}_i^{\partial\Omega} = \mathfrak{f}_i^{\partial\omega},$$

and then representing contact surface forces in terms of stress, by using equations (18) and (29) into the above expression to get

$$\parallel JF^{-T} \cdot N \parallel^{-1} P_i^A N_A = c_i^j n_j = c_i^j (F^{-1})_j^A \parallel F^{-T} \cdot N \parallel^{-1} N_A.$$

This results in the Piola transformation $c_i^j = J^{-1} P_i^A F_A^j$. For second-gradient continua, however, we have seen that this procedure cannot be applied as even the external surface force density transforms in a completely unexpected way.

In fact, Piola transformations can be deduced only by considering the change of variables introduced by the placement Π in the work functionals.

6. Conclusion

In this paper, it has been chosen to base continuum mechanics on the principle of virtual work. Note that this principle, established by D'Alembert and Lagrange, has been first called "principle of virtual velocities" and was applied to fluid mechanics. This is true also for its application to second-gradient continua. Indeed the so-called "capillary fluids" were the first continua of this type to be described ([32,37,38], more historical remarks can be found in previous studies [39–42]). We have argued that D'Alembert–Lagrange postulation scheme is more suitable than Cauchy's postulation scheme for introducing generalized continuum models (see among others [43–49]). In fact, while it is impossible to generalize Cauchy's tetrahedron argument based on the postulation of balance of forces and moments of forces to formulate generalized higher gradient continuum models, instead by using different forms of internal work functionals such a generalization becomes very natural. Specifically, we have discussed how the definition of the internal work functional as a second-order distribution restricts the compatible external work functionals and how it determines the contact interactions which can be exerted in second-gradient continua. Moreover, we deduced the equilibrium conditions from the principle of virtual work in Lagrangian description first and then in Eulerian description. The novel contribution of

this paper is the Piola transformations of all mechanically relevant tensor quantities from Lagrangian to Eulerian description in the case of second-gradient continua.

With the transformations given by equations (46)–(48), we found that, in addition to the geometry of the boundary

1. Lagrangian surface forces are expressed in terms of the Eulerian surface forces and of Eulerian double-forces.
2. Lagrangian edge forces are expressed in terms of the Eulerian edge forces and of the jump of the Eulerian double-forces.
3. Lagrangian double-forces are simply expressed in terms of the Eulerian double-forces.

Sometimes, it has been questioned the importance of second-gradient continuum theories based on a presumed absence of physical systems which are described by such theories. The homogenization methods in previous studies [50,51], or methods based on statistical mechanics [52–55], prove that there exist specific micro-structures that, at macro level, produce a second-gradient behavior [56–63]. The pertinent micro-structures are constituted by lattices of beams [64–71] connected via elastic or perfect pivots [72]. Experimentally, the deformation of such micro-structures can be captured by X-ray micro-tomography in combination with digital image correlation procedures [73,74].

It has to be remarked that already Piola considered N th gradient continua as a *local* approximation of continua in which particles can interact over distance [8,16]. Piola started from non-local energies and based his reasoning on a truncated Taylor expansion: more recently by Silling [75], Piola's results were rediscovered and used for getting predictions in crack generation and many other applications [60,76–79]. However, in peridynamics literature, the generalization of Piola transformations are not yet fully developed.

Piola transformation for second-gradient continua can have a great impact in applications. In fact, a large class of novel metamaterials (those showing a pantographic micro-structure) [72,80–85] produces greater exotic effects in large deformation regimes (e.g. low sensitivity to micro-structure defects [86–88]) and this is exactly the context where the Piola transformations play the most important role. Moreover, in the study of problems in which natural boundary conditions are assigned, deadloads are usually formulated in the Eulerian description. In that case, numerical methods [86,89–94] are generally used to get predictions and Piola transformations are mandatory for formulating effective numerical integration schemes [95–98] in Lagrangian description.


Concerning the modeling of damage and plasticity phenomena [99–102], we remark that second-gradient continua supply an important tool for getting mathematically well-posed problems. The problem of force concentration on crack tips has attracted particular interest: In this context, describing edge force effects is of utmost relevance. Because of the different nature of Lagrangian and Eulerian edge forces, the presented results may clarify some apparent paradoxes.


In perspective, it is interesting to consider the case of second-gradient continua in which new edges can appear in the Eulerian configuration. We mean here, Eulerian edges which are not the image, under the placement mapping, of Lagrangian edges. Moreover, it is challenging to generalize the presented results to the case of N th gradient continua, albeit the related formulas of tensor calculus seem to have a complex recursive structure, see dell'Isola et al. [13]. For what concerns applications, second-gradient continua in large deformations may be useful in describing bone reconstruction [103], and pantographic metamaterial properties may be optimized for being resilient to damage phenomena [104].


Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

ORCID iDs

Simon R Eugster  <https://orcid.org/0000-0002-4562-1287>

Roberto Fedele  <https://orcid.org/0000-0003-1057-9581>

Pierre Seppecher  <https://orcid.org/0000-0002-7225-3598>

Notes

1. Whose inspiring ideas are described by Truesdell [105].
2. An admissible $\delta\Pi$ is such that the placement $\Pi + \delta\Pi$ is admissible when such is Π .
3. Henceforth the dependence on Π will be omitted, if it does not cause any misunderstandings.
4. The dual spaces are decreasing, in the partial order relation given by the inclusion, when the sets of test functions are increasing.
5. Note that as the tensor \mathbb{P}_i^{AB} represents a linear form over the variations $\delta\mathbb{F}_{AB}^i$, which satisfy the symmetry $\delta\mathbb{F}_{AB}^i = \delta\mathbb{F}_{BA}^i$, then the symmetry $\mathbb{P}_i^{AB} = \mathbb{P}_i^{BA}$ is satisfied.
6. This is another consequence of the fact that the dual space is decreasing with increasing set of test functions: considering test functions in H^2 instead of test functions in H^1 enlarges the set of work functionals.
7. This representation in terms of transverse to $\partial\Omega$ derivatives is unique.
8. Note that Σ_Π and Σ_\perp must be non-vanishing subsets of $\partial\Omega$, where the trace of H^2 functions can be defined.
9. An elastic N th gradient continuum is a continuum whose deformation energy is given by $W^{\text{def}} = W^{\text{def}}(X, F, \nabla F, \dots, \nabla^N F)$.
10. It has many equivalent versions including those which are formulated as hypotheses in the Hamel–Noll theorem (see e.g. Truesdell [106]). A careful reading of the hypotheses presented there shows that, in Noll's analysis, it is systematically assumed that contact forces cannot be concentrated on curves. This seems to be, ultimately, one of the most important parts of the basic assumptions in Cauchy's approach.
11. This property holds more generally also for N th gradient continua, see dell'Isola et al. [13].
12. Certainly, in our context, it is a condition on the internal work functional.
13. Note that some authors introduce $\bar{v}^\pm = t^+ \wedge n^\pm$, see [3,4,107]. Then they get a different sign in the second term of (56).

References

- [1] Germain, P. La méthodes des puissances virtuelles en mécanique des milieux continus - 1ère Partie, Théorie du second gradient. *J Mécanique* 1973; 12: 235–274.
- [2] Germain, P. The method of virtual power in continuum mechanics. Part 2: microstructure. *SIAM J Appl Math* 1973; 25: 556–575.
- [3] Germain, P. The method of virtual power in the mechanics of continuous media, I: second-gradient theory. *Math Mech Complex Syst* 2020; 8(2): 153–190.
- [4] Toupin, RA. Elastic materials with couple-stresses. *Arch Ration Mech Anal* 1962; 11: 385–414.
- [5] Mindlin, RD. Second gradient of strain and surface-tension in linear elasticity. *Int J Solid Struct* 1965; 1(4): 417–438.
- [6] Lagrange, J-L. *Mécanique analytique*, vol. 2. Nouvelle ed. Paris: Courcier, 1815.
- [7] D'Alembert, JLR. *Traité de Dynamique*. Paris: Kessinger, 1743.
- [8] dell'Isola, F, Maier, G, Perego, U, et al. (eds). *The complete works of Gabrio Piola: Volume I: Commented English translation. Advanced structured materials*. Cham: Springer, 2014.
- [9] Eugster, SR. Hellinger's 1913 encyclopedia article on the fundamentals of the mechanics of continua. In: dell'Isola, F, Eugster, SR, Spagnuolo, M, et al. (eds) *Evaluation of scientific sources in mechanics*. Cham: Springer, 2022, pp. 99–113.
- [10] dell'Isola, F, Seppecher, P, Placidi, L, et al. Least action and virtual work principles for the formulation of generalized continuum models. In: dell'Isola, F, and Steigmann, DJ (eds) *Discrete and continuum models for complex metamaterials*. Cambridge: Cambridge University Press, 2020, pp. 327–394.
- [11] dell'Isola, F, Della Corte, A, and Giorgio, I. Higher-gradient continua: the legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives. *Math Mech Solid* 2016; 22: 852–872.
- [12] Fedele, R. Piola's approach to the equilibrium problem for bodies with second gradient energies. Part I: first gradient theory and differential geometry. *Contin Mech Thermodyn*. Epub ahead of print 8 January 2022. DOI: 10.1007/s00161-021-01064-6.
- [13] dell'Isola, F, Seppecher, P, and Madeo, A. How contact interactions may depend on the shape of Cauchy cuts in N th gradient continua: approach “à la D'Alembert.” *Z Angew Math Phys* 2012; 63(6): 1119–1141.
- [14] Gurtin, ME. *An introduction to continuum mechanics*. New York: Academic Press, 1981, p. 178.
- [15] dell'Isola, F, Seppecher, P, and Corte, AD. The postulations à la d'Alembert and à la Cauchy for higher gradient continuum theories are equivalent: a review of existing results. *Proc R Soc London A: Math Phys Eng Sci* 2015; 471(2183): 20150415.
- [16] dell'Isola, F, Andreaus, U, and Placidi, L. At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: an underestimated and still topical contribution of Gabrio Piola. *Math Mech Solid* 2014; 20: 887–928.
- [17] dell'Isola, F, Madeo, A, and Seppecher, P. Cauchy tetrahedron argument applied to higher contact interactions. *Arch Ration Mech Anal* 2016; 219(3): 1305–1341.
- [18] dell'Isola, F, and Seppecher, P. Edge contact forces and quasi-balanced power. *Meccanica* 1997; 32(1): 33–52.

- [19] Epstein, M, and Smelser, R. An appreciation and discussion of Paul Germain's "The method of virtual power in the mechanics of continuous media, I: Second-gradient theory." *Math Mech Complex Syst* 2020; 8(2): 191–199.
- [20] Eugster, SR, and dell'Isola, F. Exegesis of the introduction and sect. I from "Fundamentals of the mechanics of continua" by E, Hellinger. *Z Angew Math Mech* 2017; 97(4): 477–506.
- [21] Eugster, SR, and dell'Isola, F. Exegesis of sect. II and III.A from "Fundamentals of the mechanics of continua" by E Hellinger. *Z Angew Math Mech* 2018; 98(1): 31–68.
- [22] Eugster, SR, and dell'Isola, F. Exegesis of sect. III.B from "Fundamentals of the mechanics of continua" by E Hellinger. *Z Angew Math Mech* 2018; 98(1): 69–105.
- [23] Truesdell, C, and Toupin, R. The classical field theories. In: Flügge, S (ed.) *Principles of classical mechanics and field theory, volume III/1 of encyclopedia of physics*. Cham: Springer, 1960, pp. 226–858.
- [24] dell'Isola, F, Andreaus, U, Cazzani, A, et al. (eds) *The complete works of Gabrio Piola: Volume II: Commented English translation*. Cham: Springer, 2019.
- [25] dell'Isola, F, and Placidi, L. Variational principles are a powerful tool also for formulating field theories. In: dell'Isola, F, and Gavriluk, S (eds) *Variational models and methods in solid and fluid mechanics*. Cham: Springer, 2011, pp. 1–15.
- [26] Rudin, W. *Functional analysis*. 2nd ed. New York: McGraw-Hill Inc., 1991.
- [27] Schwartz, L. *Théorie des distributions*. Paris: Hermann, 1966, p. 102.
- [28] Reed, M, and Simon, B. *Methods of modern mathematical physics: functional analysis*. New York: Academic Press, 1980, pp. 145ff.
- [29] Ciallella, A, Pasquali, D, Goaszewski, M, et al. A rate-independent internal friction to describe the hysteretic behavior of pantographic structures under cyclic loads. *Mech Res Commun* 2021; 116: 103761.
- [30] Javanbakht, M, Ghaedi, MS, Barchiesi, E, et al. The effect of a pre-existing nanovoid on martensite formation and interface propagation: a phase field study. *Math Mech Solid* 2021; 26(1): 90–109.
- [31] Spagnuolo, M, and Cazzani, AM. Contact interactions in complex fibrous metamaterials. *Contin Mech Thermodyn* 2021; 33(4): 1873–1889.
- [32] Auffray, N, dell'Isola, F, Eremeyev, VA, et al. Analytical continuum mechanics à la Hamilton–Piola least action principle for second gradient continua and capillary fluids. *Math Mech Solid* 2015; 20(4): 375–417.
- [33] Maugin, GA. What happened on September 30, 1822, and what were its implications for the future of continuum mechanics? In: Maugin, GA (ed.) *Continuum mechanics through the eighteenth and nineteenth centuries: Historical perspectives from John Bernoulli (1727) to Ernst Hellinger (1914), solid mechanics and its applications*. Cham: Springer International Publishing 2014, pp. 33–54.
- [34] Truesdell, C. Cauchy and the modern mechanics of continua. *Rev d'hist Scie* 1992; 45: 5–24.
- [35] Fried, E, and Gurtin, ME. Traction, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales. *Arch Ration Mech Anal* 2006; 182(3): 513–554.
- [36] Holzapfel, GA. *Nonlinear solid mechanics. A continuum approach for engineering*. Chichester: Wiley, 2001.
- [37] Rosi, G, Giorgio, I, and Eremeyev, VA. Propagation of linear compression waves through plane interfacial layers and mass adsorption in second gradient fluids. *Z Angew Math Mech* 2013; 93(12): 914–927.
- [38] Eremeyev, VA, and Altenbach, H. Equilibrium of a second-gradient fluid and an elastic solid with surface stresses. *Meccanica* 2014; 49(11): 2635–2643.
- [39] Spagnuolo, M, Ciallella, A, and Scerrato, D. The loss and recovery of the works by Piola and the Italian tradition of mechanics. In: dell'Isola, F, Eugster, SR, Spagnuolo, M, et al. (eds) *Evaluation of scientific sources in mechanics*. Cham: Springer, 2022, pp. 315–340.
- [40] Barchiesi, E, Ciallella, A, and Scerrato, D. A partial report on the controversies about the principle of virtual work: from Archytas of Tarentum to Lagrange, Piola, Mindlin and Toupin. In: dell'Isola, F, Eugster, SR, Spagnuolo, M, et al. (eds) *Evaluation of scientific sources in mechanics*. Cham: Springer, 2022, pp. 341–377.
- [41] Turco, E. Discrete is it enough? The revival of Piola–Hencky keynotes to analyze three-dimensional elastica. *Contin Mech Thermodyn* 2018; 30(5): 1039–1057.
- [42] Eugster, SR, and Glocker, C. On the notion of stress in classical continuum mechanics. *Math Mech Complex Syst* 2017; 5(3–4): 299–338.
- [43] De Angelo, M, Placidi, L, NejadSadeghi, N, et al. Non-standard Timoshenko beam model for chiral metamaterial: identification of stiffness parameters. *Mech Res Commun* 2020; 103: 103462.
- [44] Giorgio, I, Spagnuolo, M, Andreaus, U, et al. In-depth gaze at the astonishing mechanical behavior of bone: a review for designing bio-inspired hierarchical metamaterials. *Math Mech Solid* 2020; 26(7): 1074–1103.
- [45] Giorgio, I, De Angelo, M, Turco, E, et al. A Biot–Cosserat two-dimensional elastic nonlinear model for a micromorphic medium. *Contin Mech Thermodyn* 2020; 32(5): 1357–1369.
- [46] Eremeyev, VA, and Altenbach, H. On the direct approach in the theory of second gradient plates. In: Altenbach, H, and Mikhasev, GI (eds) *Shell and membrane theories in mechanics and biology: From macro- to nanoscale structures*. Cham: Springer International Publishing, 2015, pp. 147–154.
- [47] Malikan, M, Krashennnikov, M, and Eremeyev, VA. Torsional stability capacity of a nano-composite shell based on a nonlocal strain gradient shell model under a three-dimensional magnetic field. *Int J Eng Sci* 2020; 148: 103210.

- [48] LaValle, G, and Massoumi, S. A new deformation measure for micropolar plates subjected to in-plane loads. *Contin Mech Thermodyn* 2021; 34: 243–257.
- [49] Misra, A, Placidi, L, dell'Isola, F, et al. Identification of a geometrically nonlinear micromorphic continuum via granular micromechanics. *Z Angew Math Phys* 2021; 72(4): 157.
- [50] Boutin, C, dell'Isola, F, Giorgio, I, et al. Linear pantographic sheets: asymptotic micro-macro models identification. *Math Mech Complex Syst* 2017; 5(2): 127–162.
- [51] Turco, E, Misra, A, Pawlikowski, M, et al. Enhanced piola-hencky discrete models for pantographic sheets with pivots without deformation energy: numerics and experiments. *Int J Solid Struct* 2018; 147: 94–109.
- [52] Ciallella, A, Cirillo, EN, Curşeu, PL, et al. Free to move or trapped in your group: mathematical modeling of information overload and coordination in crowded populations. *Math Model Meth Appl Sci* 2018; 28(9): 1831–1856.
- [53] Ciallella, A, Cirillo, EN, and Sohier, J. Residence time of symmetric random walkers in a strip with large reflective obstacles. *Phys Rev E* 2018; 97: 052116.
- [54] Ciallella, A, and Cirillo, EN. Linear Boltzmann dynamics in a strip with large reflective obstacles: stationary state and residence time. *Kinet Relat Model* 2018; 11(6): 1475–1501.
- [55] Ciallella, A, and Cirillo, EN. Conditional expectation of the duration of the classical gambler problem with defects. *Eur Phys J Spec Top* 2019; 228(1): 111–128.
- [56] Spagnuolo, M, Andreaus, U, Misra, A, et al. Mesoscale modeling and experimental analyses for pantographic cells: effect of hinge deformation. *Mech Mater* 2021; 160: 103924.
- [57] Giorgio, I. Lattice shells composed of two families of curved Kirchhoff rods: an archetypal example, topology optimization of a cycloidal metamaterial. *Contin Mech Thermodyn* 2021; 33(4): 1063–1082.
- [58] Battista, A, Rosa, L, dell'Erba, R, et al. Numerical investigation of a particle system compared with first and second gradient continua: deformation and fracture phenomena. *Math Mech Solid* 2016; 22(11): 2120–2134.
- [59] Falsone, G, and La Valle, G. A homogenized theory for functionally graded Euler–Bernoulli and Timoshenko beams. *Acta Mechanica* 2019; 230(10): 3511–3523.
- [60] Yang, H, Abali, BE, Timofeev, D, et al. Determination of metamaterial parameters by means of a homogenization approach based on asymptotic analysis. *Contin Mech Thermodyn* 2020; 32(5): 1251–1270.
- [61] dell'Isola, F, Giorgio, I, Pawlikowski, M, et al. Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proc R Soc London A: Math Phys Eng Sci* 2016; 472(2185): 2015.0790.
- [62] dell'Isola, F, Seppecher, P, Alibert, JJ, et al. Pantographic metamaterials: an example of mathematically driven design and of its technological challenges. *Contin Mech Thermodyn* 2019; 31(4): 851–884.
- [63] dell'Isola, F, Seppecher, P, Spagnuolo, M, et al. Advances in pantographic structures: design, manufacturing, models, experiments and image analyses. *Contin Mech Thermodyn* 2019; 31(4): 1231–1282.
- [64] Placidi, L, and Dhaba, ARE. Semi-inverse method à la Saint-Venant for two-dimensional linear isotropic homogeneous second-gradient elasticity. *Math Mech Solid* 2017; 22(5): 919–937.
- [65] Spagnuolo, M, and Andreaus, U. A targeted review on large deformations of planar elastic beams: extensibility, distributed loads, buckling and post-buckling. *Math Mech Solid* 2018; 24(1): 258–280.
- [66] Eremeyev, VA, and Turco, E. Enriched buckling for beam-lattice metamaterials. *Mech Res Commun* 2020; 103: 103458.
- [67] Turco, E, and Barchiesi, E. Equilibrium paths of Hencky pantographic beams in a three-point bending problem. *Math Mech Complex Syst* 2019; 7(4): 287–310.
- [68] Taig, G, Ranzi, G, and D'Annibale, F. An unconstrained dynamic approach for the generalised beam theory. *Contin Mech Thermodyn* 2015; 27(4): 879–904.
- [69] Turco, E. Stepwise analysis of pantographic beams subjected to impulsive loads. *Math Mech Solid* 2021; 26(1): 62–79.
- [70] Eugster, SR, and Harsch, J. A variational formulation of classical nonlinear beam theories. In: Abali, BE, and Giorgio, I (eds) *Developments and novel approaches in nonlinear solid body mechanics*. Cham: Springer International Publishing, 2020, pp. 95–121.
- [71] Harsch, J, Capobianco, G, and Eugster, SR. Finite element formulations for constrained spatial nonlinear beam theories. *Math Mech Solid* 2021; 26: 1838–1863.
- [72] Spagnuolo, M, Peyre, P, and Dupuy, C. Phenomenological aspects of quasi-perfect pivots in metallic pantographic structures. *Mech Res Commun* 2019; 101: 103415.
- [73] Fedele, R, Ciani, A, Galantucci, L, et al. A regularized, pyramidal multi-grid approach to global 3D-volume digital image correlation based on X-ray micro-tomography. *Fund Inform* 2013; 125(3–4): 361–376.
- [74] Fedele, R. Simultaneous assessment of mechanical properties and boundary conditions based on digital image correlation. *Exp Mech* 2015; 55(1): 139–153.
- [75] Silling, SA. Reformulation of elasticity theory for discontinuities and long-range forces. *J Mech Phys Solid* 2000; 48(1): 175–209.
- [76] Huang, Y, Oterkus, S, Hou, H, et al. Peridynamic model for visco-hyperelastic material deformation in different strain rates. *Contin Mech Thermodyn*. Epub ahead of print 20 November 2019. DOI: 10.1007/s00161-019-00849-0.

- [77] Wang, B, Oterkus, S, and Oterkus, E. Determination of horizon size in state-based peridynamics. *Contin Mech Thermodyn*. Epub ahead of print 16 June 2020. DOI: 10.1007/s00161-020-00896-y.
- [78] Imachi, M, Tanaka, S, Ozdemir, M, et al. Dynamic crack arrest analysis by ordinary state-based peridynamics. *Int J Fract* 2020; 221(2): 155–169.
- [79] Vazic, B, Wang, H, Diyaroglu, C, et al. Dynamic propagation of a macrocrack interacting with parallel small cracks. *AIMS Mater Sci* 2017; 4(1): 118–136.
- [80] Giorgio, I, Ciallella, A, and Scerrato, D. A study about the impact of the topological arrangement of fibers on fiber-reinforced composites: some guidelines aiming at the development of new ultra-stiff and ultra-soft metamaterials. *Int J Solid Struct* 2020; 203: 73–83.
- [81] Ciallella, A. Research perspective on multiphysics and multiscale materials: a paradigmatic case. *Contin Mech Thermodyn* 2020; 32(3): 527–539.
- [82] Laudato, M, and Ciallella, A. Perspectives in generalized continua. In: Abali, BE, and Giorgio, I (eds) *Developments and novel approaches in biomechanics and metamaterials*. Cham: Springer International Publishing, 2020, pp. 1–13.
- [83] Placidi, L, Barchiesi, E, Turco, E, et al. A review on 2D models for the description of pantographic fabrics. *Z Angew Math Phys* 2016; 67(5): 121.
- [84] Spagnuolo, M, Yildizdag, ME, Andreaus, U, et al. Are higher-gradient models also capable of predicting mechanical behavior in the case of wide-knit pantographic structures? *Math Mech Solid* 2021; 26(1): 18–29.
- [85] Placidi, L, Andreaus, U, and Giorgio, I. Identification of two-dimensional pantographic structure via a linear D4 orthotropic second gradient elastic model. *J Eng Math* 2017; 103(1): 1–21.
- [86] De Angelo, M, Barchiesi, E, Giorgio, I, et al. Numerical identification of constitutive parameters in reduced-order bi-dimensional models for pantographic structures: application to out-of-plane buckling. *Arch Appl Mech* 2019; 89(7): 1333–1358.
- [87] Turco, E, and Rizzi, NL. Pantographic structures presenting statistically distributed defects: numerical investigations of the effects on deformation fields. *Mech Res Commun* 2016; 77: 65–69.
- [88] De Angelo, M, Spagnuolo, M, D’Annibale, F, et al. The macroscopic behavior of pantographic sheets depends mainly on their microstructure: experimental evidence and qualitative analysis of damage in metallic specimens. *Contin Mech Thermodyn* 2019; 31(4): 1181–1203.
- [89] Reiher, CJ, Giorgio, I, and Bertram, A. Finite-element analysis of polyhedra under point and line forces in second-strain gradient elasticity. *J Eng Mech* 2017; 143(2): 04016112.
- [90] Greco, L, and Cuomo, M. B-spline interpolation of Kirchhoff–Love space rods. *Comput Meth Appl Mech Eng* 2013; 256: 251–269.
- [91] Cuomo, M, Contrafatto, L, and Greco, L. A variational model based on isogeometric interpolation for the analysis of cracked bodies. *Int J Eng Sci* 2014; 80: 173–188.
- [92] Greco, L. An iso-parametric -conforming finite element for the nonlinear analysis of Kirchhoff rod. Part I: the 2D case. *Contin Mech Thermodyn* 2020; 32(5): 1473–1496.
- [93] Barchiesi, E, Misra, A, Placidi, L, et al. Granular micromechanics-based identification of isotropic strain gradient parameters for elastic geometrically nonlinear deformations. *ZAMM* 2021; 101: e202100059.
- [94] Capobianco, G, Harsch, J, Eugster, SR, et al. A nonsmooth generalized-alpha method for mechanical systems with frictional contact. *Int J Num Meth Eng* 2021; 122(22): 6497–6526.
- [95] Placidi, L, Misra, A, and Barchiesi, E. Simulation results for damage with evolving microstructure and growing strain gradient moduli. *Contin Mech Thermodyn* 2019; 31(4): 1143–1163.
- [96] Greco, L, Cuomo, M, and Contrafatto, L. Two new triangular g1-conforming finite elements with cubic edge rotation for the analysis of kirchhoff plates. *Comput Meth Appl Mech Eng* 2019; 356: 354–386.
- [97] Turco, E, Goaszewski, M, Giorgio, I, et al. Pantographic lattices with non-orthogonal fibres: experiments and their numerical simulations. *Compos Part B: Eng* 2017; 118: 1–14.
- [98] Abali, BE, Müller, WH, and Eremeyev, VA. Strain gradient elasticity with geometric nonlinearities and its computational evaluation. *Mech Adv Mater Mod Process* 2015; 1(1): 4.
- [99] Placidi, L, Barchiesi, E, and Misra, A. A strain gradient variational approach to damage: a comparison with damage gradient models and numerical results. *Math Mech Complex Syst* 2018; 6(2): 77–100.
- [100] Placidi, L, Misra, A, and Barchiesi, E. Two-dimensional strain gradient damage modeling: a variational approach. *Z Angew Math Phys* 2018; 69(3): 56.
- [101] Solyaev, Y, Lurie, S, Barchiesi, E, et al. On the dependence of standard and gradient elastic material constants on a field of defects. *Math Mech Solid* 2019; 25(1): 35–45.
- [102] Placidi, L, and Barchiesi, E. Energy approach to brittle fracture in strain-gradient modelling. *Proc R Soc A: Math Phys Eng Sci* 2018; 474(2210): 20170878.
- [103] Scerrato, D, Bersani, AM, and Giorgio, I. Bio-inspired design of a porous resorbable scaffold for bone reconstruction: a preliminary study. *Biomimetics* 2021; 6(1): 18.
- [104] Desmorat, B, Spagnuolo, M, and Turco, E. Stiffness optimization in nonlinear pantographic structures. *Math Mech Solid* 2020; 25(12): 2252–2262.

- [105] Truesdell, C. The creation and unfolding of the concept of stress. In: Truesdell, C (ed.) *Essays in the history of mechanics*. Cham: Springer, 1968, pp. 184–238.
- [106] Truesdell, C. *A first course in rational continuum mechanics*. New York: Academic Press, 1977, p. 172.
- [107] Eugster, SR, dell'Isola, F, and Steigmann, D. Continuum theory for mechanical metamaterials with a cubic lattice substructure. *Math Mech Complex Syst* 2019; 7(1): 75–98.
- [108] Capobianco, G, and Eugster, SR. On the divergence theorem for submanifolds of Euclidean vector spaces within the theory of second-gradient continua. *Z Angew Math Phys* 2022

A. Equivalent form for second-order distributions

Let us consider a regular manifold \mathcal{V} embedded in n -dimensional Euclidean vector space \mathbb{E}^n and the projectors m_{\parallel} and m_{\perp} on its tangent and normal spaces. When \mathcal{V} has co-dimension one, and if n denotes its unit normal vector, we have

$$m_{\perp\alpha}^{\gamma} = n^{\gamma}n_{\alpha}, \quad m_{\parallel\alpha}^{\gamma} = \delta_{\alpha}^{\gamma} - n^{\gamma}n_{\alpha}. \tag{50}$$

Given a vector field w defined in the neighborhood of \mathcal{V} , the divergence theorem for Riemannian submanifolds with boundaries is stated as

$$\int_{\mathcal{V}} m_{\parallel\alpha}^{\gamma} \frac{\partial}{\partial x^{\gamma}} \left(m_{\parallel\beta}^{\alpha} w^{\beta} \right) = \int_{\partial\mathcal{V}} \left(m_{\parallel\beta}^{\alpha} w^{\beta} \right) m_{\parallel\alpha}^{\gamma} \nu_{\gamma} = \int_{\partial\mathcal{V}} w^{\beta} \nu_{\beta}, \tag{51}$$

where $\partial\mathcal{V}$ denotes the boundary of \mathcal{V} and where the unit vector ν is tangent to \mathcal{V} and normal to $\partial\mathcal{V}$. Defining $\text{div}_{\parallel}^{\mathcal{V}}$ by setting for all smooth fields ϕ

$$\left(\text{div}_{\parallel}^{\mathcal{V}}(\phi) \right)_{\alpha} := m_{\parallel\alpha}^{\gamma} \frac{\partial}{\partial x^{\gamma}}(\phi),$$

the divergence theorem, see Capobianco and Eugster [108], reads

$$\int_{\mathcal{V}} \text{div}_{\parallel}^{\mathcal{V}}(m_{\parallel} \cdot w) = \int_{\partial\mathcal{V}} w \cdot \nu.$$

In accordance with the theory of distributions [27], both the virtual work expressions in Lagrangian and Eulerian descriptions can be considered as distributions \mathcal{D} represented in the form

$$\mathcal{D}(\phi) = \int_{\mathcal{V}} \left(s^{\alpha} \frac{\partial \phi}{\partial y^{\alpha}} + s^{\alpha\beta} \frac{\partial^2 \phi}{\partial y^{\alpha} \partial y^{\beta}} \right), \tag{52}$$

where the derivatives of the test functions ϕ are taken with respect to the coordinates y^{α} of a three-dimensional Euclidean space. Note that the index i appearing in both equations (10) and (22) does not play any role in the present considerations and is therefore omitted. \mathcal{D} is a second-order distribution. The symbol ϖ denotes the generic integration domain which satisfies the same regularity requirements as discussed in section 1.1 for the reference configuration Ω . The faces of the subset \mathcal{V} are denoted by $\partial\varpi$ and come along with the outward-pointing unit normal field n . The symbol $\partial\partial\varpi$ denotes the edges on which the outward-pointing unit normals ν are defined. Moreover, the unit normal ν lies in the tangent plane to the faces constituting $\partial\mathcal{V}$.

Using the product rule in the second integrand of equation (52), we can write

$$\mathcal{D}(\phi) = \int_{\varpi} \left(s^{\alpha} - \frac{\partial s^{\alpha\beta}}{\partial y^{\beta}} \right) \frac{\partial \phi}{\partial y^{\alpha}} + \int_{\varpi} \frac{\partial}{\partial y^{\beta}} \left(s^{\alpha\beta} \frac{\partial \phi}{\partial y^{\alpha}} \right).$$

With the abbreviation $\bar{s}^\alpha = s^\alpha - \partial s^{\alpha\beta} / \partial y^\beta$ and applying Leibniz' rule for the first integrand, we end up with

$$\mathcal{D}(\phi) = \int_{\partial\omega} \frac{\partial}{\partial y^\alpha} (\bar{s}^\alpha \phi) - \int_{\partial\omega} \frac{\partial \bar{s}^\alpha}{\partial y^\alpha} \phi + \int_{\partial\omega} \frac{\partial}{\partial y^\beta} \left(s^{\alpha\beta} \frac{\partial \phi}{\partial y^\alpha} \right).$$

Using the divergence theorem for the first and the third term and introducing the distributions

$$\mathcal{D}_{\partial\omega}^0(\phi) := - \int_{\partial\omega} \frac{\partial \bar{s}^\alpha}{\partial y^\alpha} \phi, \quad \mathcal{D}_{\partial\omega}^0(\phi) := \int_{\partial\omega} \bar{s}^\alpha n_\alpha \phi.$$

Equation (52) can be written in the form

$$\mathcal{D}(\phi) = \mathcal{D}_{\partial\omega}^0(\phi) + \mathcal{D}_{\partial\omega}^0(\phi) + \int_{\partial\omega} s^{\alpha\beta} \frac{\partial \phi}{\partial y^\alpha} n_\beta.$$

The last term here, is the only expression in which derivatives of ϕ yet appear. Therefore, we will manipulate this term further by using the projectors (50) for the faces $\partial\omega$

$$\begin{aligned} \int_{\partial\omega} s^{\alpha\beta} \frac{\partial \phi}{\partial y^\alpha} n_\beta &= \int_{\partial\omega} s^{\alpha\beta} \frac{\partial \phi}{\partial y^\gamma} n_\beta \delta_\alpha^\gamma \\ &= \int_{\partial\omega} s^{\alpha\beta} \frac{\partial \phi}{\partial y^\gamma} n_\beta \left(m_{\parallel\alpha}^\gamma + m_{\perp\alpha}^\gamma \right) \\ &= \int_{\partial\omega} s^{\alpha\beta} \frac{\partial \phi}{\partial y^\gamma} n_\beta m_{\parallel\alpha}^\gamma + \mathcal{D}_{\partial\omega}^I(\phi), \end{aligned} \quad (53)$$

where we have introduced the distribution

$$\mathcal{D}_{\partial\omega}^I(\phi) := \int_{\partial\omega} (s^{\alpha\beta} n_\alpha n_\beta) \frac{\partial \phi}{\partial y^\gamma} n^\gamma.$$

The distribution $\mathcal{D}_{\partial\omega}^I$ involves the normal derivative of the test function $(\partial\phi/\partial y^\gamma)n^\gamma$ and cannot be reduced any further.

Applying once more Leibniz' rule, we can manipulate the first term in the last line of equation (53) in the following way

$$\begin{aligned} \int_{\partial\omega} \left(s^{\alpha\beta} \frac{\partial \phi}{\partial y^\gamma} n_\beta m_{\parallel\alpha}^\lambda \right) m_{\parallel\lambda}^\gamma &= \int_{\partial\omega} \left\{ \frac{\partial}{\partial y^\gamma} (s^{\alpha\beta} n_\beta m_{\parallel\alpha}^\lambda \phi) m_{\parallel\lambda}^\gamma - m_{\parallel\lambda}^\gamma \frac{\partial}{\partial y^\gamma} (s^{\alpha\beta} n_\beta m_{\parallel\alpha}^\lambda \phi) \right\} \\ &= \mathcal{D}_{\partial\omega}^0(\phi) + \tilde{\mathcal{D}}_{\partial\omega}^0(\phi). \end{aligned}$$

In the last step, we have introduced the distributions

$$\mathcal{D}_{\partial\omega}^0(\phi) := \int_{\partial\omega} (s^{\alpha\beta} n_\beta v_\alpha) \phi, \quad (54)$$

$$\tilde{\mathcal{D}}_{\partial\omega}^0(\phi) := - \int_{\partial\omega} m_{\parallel\lambda}^\gamma \frac{\partial}{\partial y^\gamma} (s^{\alpha\beta} n_\beta m_{\parallel\alpha}^\lambda) \phi. \quad (55)$$

To obtain $\mathcal{D}_{\partial\partial\varpi}^0$, the divergence theorem (51) has been applied leading to a line integral along the edges of ϖ . We explicitly remark that in equation (54), we used a notational convention: as depicted in Figure 1, we observe that in an edge γ two faces σ^+ and σ^- concur. Hence, in the performed integration by parts, γ is traversed twice: with the surface normal n^- , edge normal v^- and the limit $(\mathbb{s}^-)^{\alpha\beta}$ approach from the surface σ^- , and similarly from σ^+ with the corresponding n^+ , v^+ , and $(\mathbb{s}^+)^{\alpha\beta}$. Consequently, if we denote each edge curve by γ_i for $i = 1, \dots, n_e$, then the integral expression of equation (54) reads¹³

$$\int_{\partial\partial\varpi} (\mathbb{s}^{\alpha\beta} n_\beta v_\alpha) \phi := \sum_{i=1}^{n_e} \int_{\gamma_i} [(\mathbb{s}^{\alpha\beta} n_\beta v_\alpha)^+ + (\mathbb{s}^{\alpha\beta} n_\beta v_\alpha)^-] \phi. \tag{56}$$

In conclusion, from the point of view of the theory of distributions, the second-order distribution $D(\phi)$ from equation (52) can equivalently be represented as

$$\mathcal{D} = \mathcal{D}_{\varpi}^0 + (\mathcal{D}_{\partial\varpi}^0 + \tilde{\mathcal{D}}_{\partial\varpi}^0) + \mathcal{D}_{\partial\varpi}^I + \mathcal{D}_{\partial\partial\varpi}^0.$$

This equivalence can be applied to the Lagrangian or Eulerian internal work functionals.

As a last thing, we work out the explicit dependence of $\tilde{\mathcal{D}}_{\partial\varpi}^0$ on the normal and the curvature of the faces $\partial\varpi$. For the sake of compact notation, in the following computations, partial derivatives $\partial/\partial y^\alpha$ are written as $(\cdot)_{,\alpha}$ and the abbreviation $\mathbb{s}_n^\alpha = \mathbb{s}^{\alpha\beta} n_\beta$ is used. Inserting the tangent projector (50), we obtain

$$\begin{aligned} (\mathbb{s}_n^\alpha m_{\|\alpha}^\lambda)_{,\gamma} m_{\|\lambda}^\gamma &= (\mathbb{s}_n^\alpha \delta_\alpha^\lambda)_{,\gamma} \delta_\lambda^\gamma - (\mathbb{s}_n^\alpha \delta_\alpha^\lambda)_{,\gamma} n^\gamma n_\lambda \\ &\quad - (\mathbb{s}_n^\alpha n^\lambda n_\alpha)_{,\gamma} \delta_\lambda^\gamma + (\mathbb{s}_n^\alpha n^\lambda n_\alpha)_{,\gamma} n^\gamma n_\lambda, \end{aligned}$$

which is obviously equivalent to

$$(\mathbb{s}_n^\alpha m_{\|\alpha}^\lambda)_{,\gamma} m_{\|\lambda}^\gamma = (\mathbb{s}_n^\alpha)_{,\alpha} - (\mathbb{s}_n^\alpha)_{,\gamma} n^\gamma n_\alpha - (\mathbb{s}_n^\alpha n^\gamma n_\alpha)_{,\gamma} + (\mathbb{s}_n^\alpha n^\lambda n_\alpha)_{,\gamma} n^\gamma n_\lambda.$$

We can further manipulate the expression to

$$\begin{aligned} (\mathbb{s}_n^\alpha m_{\|\alpha}^\lambda)_{,\gamma} m_{\|\lambda}^\gamma &= (\mathbb{s}^{\alpha\beta} n_\beta)_{,\alpha} - (\mathbb{s}^{\alpha\beta} n_\beta)_{,\gamma} n^\gamma n_\alpha - (\mathbb{s}^{\alpha\beta})_{,\gamma} n_\beta n^\gamma n_\alpha - \mathbb{s}^{\alpha\beta} (n_\beta n^\gamma n_\alpha)_{,\gamma} \\ &\quad + (\mathbb{s}^{\alpha\beta})_{,\gamma} n^\gamma n_\beta n^\lambda n_\alpha n_\lambda + \mathbb{s}^{\alpha\beta} (n_\beta n^\lambda n_\alpha)_{,\gamma} n^\gamma n_\lambda. \end{aligned}$$

We assume that the faces are regular enough for extending any field in their vicinity as constant along the normal. Using

$$n^\lambda n_\lambda = 1, \quad \left(\frac{\partial n^\lambda}{\partial y^\gamma}\right) n^\gamma = 0, \quad \left(\frac{\partial n_\delta}{\partial y^\gamma}\right) n^\gamma = 0,$$

the above expression simplifies to

$$(\mathbb{s}_n^\alpha m_{\|\alpha}^\lambda)_{,\gamma} m_{\|\lambda}^\gamma = (\mathbb{s}^{\alpha\beta} n_\beta)_{,\alpha} - (\mathbb{s}^{\alpha\beta})_{,\gamma} n_\beta n^\gamma n_\alpha - \mathbb{s}^{\alpha\beta} n_\beta n_\alpha (n^\gamma)_{,\gamma}. \tag{57}$$

Inserting equation (57) in equation (55), the distribution $\tilde{\mathcal{D}}_{\partial\varpi}^0$ can finally be written as

$$\tilde{\mathcal{D}}_{\partial\varpi}^0(\phi) = - \int_{\partial\varpi} ((\mathbb{s}^{\alpha\beta} n_\beta)_{,\alpha} - (\mathbb{s}^{\alpha\beta})_{,\gamma} n_\beta n^\gamma n_\alpha - \mathbb{s}^{\alpha\beta} n_\beta n_\alpha (n^\gamma)_{,\gamma}) \phi.$$

B. Piola transformations of tangents, surface normals, and edge normals

B.1. Piola transformation of unit tangent vectors

Given any curve in the reference configuration $\Gamma: S \mapsto \Gamma(S) \subset \Omega$, this curve is transported by the placement Π to a curve in the current configuration $\gamma(S) = \Pi(\Gamma(S))$. Assume S to be the arc length parameter

of Γ such that $T = d\Gamma/dS$ is a unit vector. The application of the chain rule readily implies that the referential tangent vector T is mapped to the current tangent vector \tilde{t} by

$$\tilde{t}^i(S) := \frac{d\gamma^i}{dS}(S) = F_A^i(\Gamma(s))T^A(S),$$

which following the convention of omitting the arguments is written as

$$\tilde{t}^i = F_A^i T^A. \quad (58)$$

Because \tilde{t} is generally not a unit vector, the Piola transformation of the unit tangent vector T to the curve Γ to the unit tangent vector t to the curve γ is given by

$$t = \tilde{t} \|\tilde{t}\|^{-1} = F_A^i T^A \|F \cdot T\|^{-1}. \quad (59)$$

B.2. Piola transformation of unit normals

Let $\Sigma \subset \Omega$ be a surface that is transported by the placement Π to the surface $\sigma \subset \omega$. Considering a pair of independent vectors (V, W) both of which are tangent to the surface Σ . Then the referential unit normal to the surface Σ can be constructed by

$$N = \frac{V \wedge W}{\|V \wedge W\|}, \quad N_A = \frac{\sqrt{\det(G_{LM})} \varepsilon_{ABC} V^B W^C}{\|V \wedge W\|},$$

where ε_{ABC} denotes the Levi-Civita permutation symbol and \wedge the vector product in \mathbb{E}^3 . According to equation (58), the tangent vectors V and W are mapped to $\tilde{v} = F \cdot V$ and $\tilde{w} = F \cdot W$, respectively, both of which are tangent to σ . The current unit normal to the surface σ is then given by

$$n = \frac{\tilde{v} \wedge \tilde{w}}{\|\tilde{v} \wedge \tilde{w}\|}, \quad n_i = \frac{\sqrt{\det(g_{lm})} \varepsilon_{ijk} \tilde{v}^j \tilde{w}^k}{\|\tilde{v} \wedge \tilde{w}\|}.$$

As for any $U, V, W \in \mathbb{E}^3$, the determinant $\det(F)$ of the map F is defined by $\langle F \cdot U, (F \cdot V) \wedge (F \cdot W) \rangle = \det(F) \langle U, V \wedge W \rangle$ and consequently

$$\sqrt{\det(g_{ij})} \varepsilon_{ijk} F_A^i U^A F_B^j V^B F_C^k W^C = \det(F) \sqrt{\det(G_{IJ})} \varepsilon_{ABC} U^A V^B W^C,$$

one can carry out the following computations:

$$\begin{aligned} (\tilde{v} \wedge \tilde{w})_l &= \sqrt{\det(g_{ij})} \varepsilon_{ijk} \tilde{v}^j \tilde{w}^k \\ &= \sqrt{\det(g_{ij})} \varepsilon_{ijk} F_B^j V^B F_C^k W^C \\ &= \sqrt{\det(g_{ij})} \varepsilon_{ijk} (F_A^i (F^{-1})_l^A) F_B^j V^B F_C^k W^C \\ &= \det(F) (F^{-1})_l^A \sqrt{\det(G_{IJ})} \varepsilon_{ABC} V^B W^C \\ &= J (F^{-T} \cdot (V \wedge W))_l. \end{aligned} \quad (60)$$

Note, in the last line, we have introduced $J := \det(F)$ and used $(F^{-T} \cdot B)_l = (F^{-1})_l^A B_A$, see Auffray et al. [32] for more details. Using the Piola transformation of the vector product (60), we can relate the referential and current normals in accordance with

$$\begin{aligned} n &= \frac{\tilde{v} \wedge \tilde{w}}{\|\tilde{v} \wedge \tilde{w}\|} = \frac{(F \cdot V) \wedge (F \cdot W)}{\|(F \cdot V) \wedge (F \cdot W)\|} = \frac{J F^{-T} \cdot (V \wedge W)}{\|J F^{-T} \cdot (V \wedge W)\|} \\ &= \frac{J F^{-T} \cdot (V \wedge W)}{\|J F^{-T} \cdot N\| \| (V \wedge W) \|} = \frac{F^{-T} \cdot N}{\|F^{-T} \cdot N\|}. \end{aligned} \quad (61)$$

In components, this transformation reads as

$$n_i = \frac{(F^{-1})^A_i N_A}{\|F^{-T} \cdot N\|}, \quad n^i = \frac{g^{ij}(F^{-1})^A_j N_A}{\|F^{-T} \cdot N\|}. \tag{62}$$

This formula appears already in Piola, see dell'Isola et al. [8].

B.3. Piola transformation of unit edge normals

Consider two surfaces Σ^+ and Σ^- that concur in an edge Γ . As depicted in Figure 1, the referential edge normals V^\pm are the outward-pointing unit normals to the edge Γ that are tangent to the surfaces Σ^\pm , that is, the unit vectors that are normal to T^+ and N^\pm . When the tangent vector T^+ is introduced as in Figure 1, then $V^\pm = \pm T^+ \wedge N^\pm$. In the following, we consider only Σ^+ for which the triple (T^+, N^+, V^+) constitute a right-handed orthonormal system of \mathbb{E}^3 and we drop the superscript $(\cdot)^+$. Through the placement Π , the surface is transported to $\sigma = \Pi(\Sigma)$ and the edge to $\gamma = \Pi(\Gamma)$. The tangent and normal vector T and N are transported to the unit vectors t and n following equations (59) and (61). The question is now how the current edge normal $v = t \wedge n$ is related to the referential edge normal V . To derive this relation, we will essentially apply the Gram-Schmidt process. The transformation rules (59) and (61) were derived such that $\langle t, t \rangle = 1$ and $\langle n, n \rangle = 1$. Moreover, the orthogonality between n and t is preserved. Introducing $J_\Lambda = \|F \cdot T\|$ and $J_\Sigma = \|F^{-T} \cdot N\|$, we remark that

$$J_\Sigma J_\Lambda \langle t, n \rangle = \langle F \cdot T, F^{-T} \cdot N \rangle = \langle T, F^T \cdot F^{-T} \cdot N \rangle = \langle T, N \rangle = 0.$$

The triple $(t, n, F^{-T} \cdot V)$ generates a basis for \mathbb{E}^3 . Indeed,

$$J_\Sigma (n \wedge (F^{-T} \cdot V)) = (F^{-T} \cdot N) \wedge (F^{-T} \cdot V) = J^{-1} F (N \wedge V) = J^{-1} t$$

is not the zero vector, and $(t, n, F^{-T} \cdot V)$ spans the three-dimensional \mathbb{E}^3 . Moreover,

$$J_\Lambda \langle t, F^{-T} \cdot V \rangle = \langle F \cdot T, F^{-T} \cdot V \rangle = \langle T, V \rangle = 0,$$

which shows that $F^{-T} \cdot V$ lies in a plane orthogonal to t . To get a basis vector \tilde{v} which is orthogonal also to n , we must subtract from $F^{-T} \cdot V$ its component in n direction. Setting

$$\tilde{v} := F^{-T} \cdot V - \langle F^{-T} \cdot V, n \rangle n = F^{-T} \cdot V - \frac{\langle F^{-T} \cdot V, F^{-T} \cdot N \rangle}{\langle F^{-T} \cdot N, F^{-T} \cdot N \rangle} F^{-T} \cdot N.$$

Introducing $a = F^{-T} \cdot V$ and $b = F^{-T} \cdot N$, the norm of \tilde{v} can be computed as

$$\begin{aligned} \langle \tilde{v}, \tilde{v} \rangle &= \langle a, a \rangle - 2 \frac{\langle a, b \rangle^2}{\langle b, b \rangle} + \frac{\langle a, b \rangle^2}{\langle b, b \rangle^2} \langle b, b \rangle = \frac{\langle a, a \rangle \langle b, b \rangle - \langle a, b \rangle^2}{\langle b, b \rangle} = \frac{\|a \wedge b\|}{\langle b, b \rangle} \\ &= \frac{\|(F^{-T} \cdot V) \wedge (F^{-T} \cdot N)\|^2}{\|F^{-T} \cdot N\|^2} = \frac{\|J^{-1} F \cdot (V \wedge N)\|^2}{\|F^{-T} \cdot N\|^2} = \frac{\|F \cdot T\|^2}{\|J F^{-T} \cdot N\|^2}. \end{aligned}$$

Consequently, the current unit edge normal v is given by

$$v = J_\Sigma J_\Lambda^{-1} \left[F^{-T} \cdot V - \frac{\langle F^{-T} \cdot V, F^{-T} \cdot N \rangle}{\|F^{-T} \cdot N\|^2} F^{-T} \cdot N \right].$$

In terms of the covariant components of the edge normal, the transformation rule can be written as

$$\begin{aligned} v_i &= J_{\Sigma} J_{\Lambda}^{-1} \left[(F^{-1})_i^R V_R - \frac{(F^{-1})_k^S V_S G^{kl} (F^{-1})_l^P N_P}{\|F^{-T} \cdot N\|^2} (F^{-1})_i^R N_R \right] \\ &= J_{\Sigma} J_{\Lambda}^{-1} \left[(F^{-1})_i^R V_R - \frac{(C^{-1})^{SP} V_S N_P}{\|F^{-T} \cdot N\|^2} (F^{-1})_i^R N_R \right]. \end{aligned} \quad (63)$$

B.4. Change of variables for volume, surface, and line integrals

In this subsection, we give a brief summary of the transformation rules for the change of variables for volumes, surfaces, and line integrals. With the regularity assumptions made in section 1.1, the reference configuration $\Omega \subset \mathbb{E}^3$ with boundary faces $\partial\Omega$ and edges $\partial\partial\Omega$ is mapped to $\omega = \Pi(\Omega)$ with faces $\partial\omega = \Pi(\partial\Omega)$ and edges $\partial\partial\omega = \Pi(\partial\partial\Omega)$. Denoting the outward-pointing unit normal to the boundary surfaces $\partial\Omega$ by N and the tangent vector to $\partial\partial\Omega$ by T , the volume, area, and length Jacobians can be introduced as

$$J := \det F, \quad J_{\Sigma} := \|JF^{-T} \cdot N\|, \quad J_{\Lambda} := \|F \cdot T\|.$$

It can be shown that the following equalities hold when changing the variables within the integral:

$$\int_{\omega} 1 = \int_{\Omega} J, \quad \int_{\partial\omega} 1 = \int_{\partial\Omega} J_{\Sigma}, \quad \int_{\partial\partial\omega} 1 = \int_{\partial\partial\Omega} J_{\Lambda}. \quad (64)$$

The proof of these relation makes use of the incremental version of the following equalities.

Let $U, V, W \in \mathbb{E}^3$ be three independent referential vectors. These are mapped by the deformation gradient F to the current vectors $u = F \cdot U, v = F \cdot V$ and $w = F \cdot W$. Then the current volume $\text{Vol}(u, v, w) = |\langle u, v \wedge w \rangle|$ spanned by the triad (u, v, w) can be related to the referential volume $\text{Vol}(U, V, W) = |\langle U, V \wedge W \rangle|$ by

$$\text{Vol}(u, v, w) = |\langle u, v \wedge w \rangle| = |\det(F)| |\langle U, V \wedge W \rangle| = J \text{Vol}(U, V, W).$$

Using equation (60), the current area spanned by the two vectors u and v can be expressed as

$$\begin{aligned} \text{Area}(u, v) &= \|u \wedge v\| = \|(F \cdot U) \wedge (F \cdot V)\| = \|JF^{-T} \cdot (U \wedge V)\| \\ &= \|JF^{-T} \cdot N\| \|U \wedge V\| = J_{\Sigma} \text{Area}(U, V). \end{aligned}$$

Introducing $T = U \|U\|^{-1}$, the length of the vector u is given as

$$\text{Length}(u) = \|u\| = \|F \cdot U\| = \|F \cdot T\| \|U\| = J_{\Lambda} \text{Length}(U).$$

C. Piola-type identities

The Piola-type identities are essential in getting a direct transformation between the equilibrium equations in Lagrangian and Eulerian form. In fact, they allow for the expression of the Lagrangian divergence operator in terms of the Eulerian divergence operators. As the divergence operator can also be defined on an arbitrary submanifold, Piola identities for such submanifolds can be formulated. For our purposes, we need such an identity for two- and three-dimensional domains.

C.1. Volume Piola identities

It is well known (see e.g. Auffray et al. [32]) that the Lagrangian and Eulerian volume divergence operators can be related by means of celebrated identities that are unanimously attributed to Gabrio Piola (see dell'Isola and colleagues [11,16]). Let $T = T(X)$ be a vector field over the reference configuration. Then the first Piola identity is

$$\frac{\partial}{\partial x^a} ((J^{-1}F_A^a T^A) \circ \pi)(x) = \left(J^{-1} \frac{\partial T^A}{\partial X^A} \right) \circ \pi(x),$$

which, after dropping the arguments, reads as

$$\frac{\partial}{\partial x^a} (J^{-1}F_A^a T^A) = J^{-1} \frac{\partial T^A}{\partial X^A}, \quad \text{div}(J^{-1}F \cdot T) = J^{-1} \text{Div}(T). \tag{65}$$

The proof of this identity can be carried out as follows. Let $\phi = \phi(x)$ be a scalar test function on the current configuration ω with compact support vanishing on the boundary $\partial\omega$, that is, $\phi(x) = 0$ for $x \in \partial\omega$. The integral over ω of the left hand side of equation (65) multiplied with the test function leads after an integration by parts to

$$\int_{\omega} \frac{\partial}{\partial x^a} (J^{-1}F_A^a T^A) \phi = - \int_{\omega} J^{-1}F_A^a T^A \frac{\partial \phi}{\partial x^a}.$$

Introducing the Lagrangian test function $\Phi = \phi \circ \Pi$, together with equation (32), we can write

$$\int_{\omega} \frac{\partial}{\partial x^a} (J^{-1}F_A^a T^A) \phi = - \int_{\omega} J^{-1}T^A \frac{\partial \Phi}{\partial X^A} = - \int_{\Omega} T^A \frac{\partial \Phi}{\partial X^A}$$

where in the last equality, we made use of the change of variables (35). A subsequent integration by parts followed by a change of variables with equation (64) leads to

$$\int_{\omega} \frac{\partial}{\partial x^a} (J^{-1}F_A^a T^A) \phi = \int_{\Omega} \frac{\partial T^A}{\partial X^A} \Phi = \int_{\omega} J^{-1} \frac{\partial T^A}{\partial X^A} \phi.$$

Since the equality must hold for all test functions, equation (65) follows directly.

Note that, setting $t := J^{-1}F \cdot T$, we get the so-called second Piola identity:

$$J \frac{\partial t^a}{\partial x^a} = \frac{\partial}{\partial X^A} (J(F^{-1})_a^A t^a), \quad J \text{div}(t) = \text{Div}(JF^{-1} \cdot t). \tag{66}$$

Note also that, considering in equations (65) and (66) constant tensor fields T or t , the equalities reduce to

$$0 = \frac{\partial}{\partial x^a} (J^{-1}F_A^a), \quad 0 = \frac{\partial}{\partial X^A} (J(F^{-1})_a^A). \tag{67}$$

C.2. Surface Piola-type identity

Similar to the volume Piola-type identity, which relates the current volume divergence operator with the referential one, we can derive an identity for the surface divergence operators. The relation corresponding to equation (66) in terms of the surface operators is expressed in a symbolic way as

$$J_{\Sigma} \text{div}_{\parallel}^{\sigma} (t_{\parallel}) = \text{DIV}_{\parallel}^{\Sigma} (J_{\Sigma} F^{-1} \cdot t_{\parallel})$$

which in components reads as

$$J_{\Sigma} \frac{\partial}{\partial x^c} (m_{\parallel b}^a t^b) m_{\parallel a}^c = \frac{\partial}{\partial X^C} (J_{\Sigma} (F^{-1})_a^A m_{\parallel b}^a t^b) M_{\parallel A}^C. \tag{68}$$

The proof follows the same strategy as for the volume identity. Let Σ be a Lagrangian surface with boundary $\partial\Sigma$. Then, the surface with boundary $\sigma = \Pi(\Sigma)$ is diffeomorphic to Σ . Let us assume that a

sufficiently regular Eulerian vector field t is defined in the neighborhood of σ . Let $\phi = \phi(x)$ be a scalar test function on the current surface σ with compact support vanishing on the boundary $\partial\sigma$. We have

$$\int_{\sigma} \frac{\partial}{\partial x^c} (m_{\parallel b}^a t^b) m_{\parallel a}^c \phi = - \int_{\sigma} m_{\parallel b}^a t^b \frac{\partial \phi}{\partial x^c} m_{\parallel a}^c.$$

Changing integration variable in agreement with equation (64), using equation (34) and the idempotence of m_{\parallel} , one obtains

$$\int_{\sigma} \frac{\partial}{\partial x^c} (m_{\parallel b}^a t^b) m_{\parallel a}^c \phi = - \int_{\Sigma} J_{\Sigma} m_{\parallel b}^c t^b \frac{\partial \Phi}{\partial X^A} (F^{-1})_c^A.$$

By definition of m_{\parallel} , the vector $t = m_{\parallel} \cdot w$ is a vector tangent to σ . According to equation (58), $T = F^{-1} \cdot t$ is tangent to Σ . We have $T = M_{\parallel} \cdot T$ that is $F^{-1} \cdot t = M_{\parallel} \cdot F^{-1} \cdot t$, hence

$$t = F \cdot M_{\parallel} \cdot F^{-1} \cdot t, \quad t^c = F_I^c M_{\parallel J}^I (F^{-1})^J t^a.$$

Using this identity, we can modify our expression further to

$$\begin{aligned} \int_{\sigma} \frac{\partial}{\partial x^c} (m_{\parallel b}^a t^b) m_{\parallel a}^c \phi &= - \int_{\Sigma} J_{\Sigma} (F_I^c M_{\parallel J}^I (F^{-1})_a^J) m_{\parallel b}^a t^b \frac{\partial \Phi}{\partial X^A} (F^{-1})_c^A \\ &= - \int_{\Sigma} J_{\Sigma} (M_{\parallel J}^A (F^{-1})_a^J) m_{\parallel b}^a t^b \frac{\partial \Phi}{\partial X^A} \\ &= - \int_{\Sigma} J_{\Sigma} (M_{\parallel B}^A M_{\parallel J}^B (F^{-1})_a^J) m_{\parallel b}^a t^b \frac{\partial \Phi}{\partial X^A} \\ &= \int_{\Sigma} \Phi M_{\parallel B}^A \frac{\partial}{\partial X^A} (J_{\Sigma} (F^{-1})_a^J m_{\parallel b}^a t^b M_{\parallel J}^B) \\ &= \int_{\sigma} \phi M_{\parallel B}^A \frac{\partial}{\partial X^A} (J_{\Sigma} (F^{-1})_a^J m_{\parallel b}^a t^b M_{\parallel J}^B) J_{\Sigma}^{-1}, \end{aligned}$$

resulting in the identity

$$\frac{\partial}{\partial x^c} (m_{\parallel b}^a t^b) m_{\parallel a}^c = J_{\Sigma}^{-1} M_{\parallel B}^A \frac{\partial}{\partial X^A} (J_{\Sigma} M_{\parallel J}^B (F^{-1})_a^J m_{\parallel b}^a t^b). \quad (69)$$

Since $m_{\parallel} \cdot t$ is tangent to σ , the Lagrangian vector $F^{-1} \cdot m_{\parallel} \cdot t$ is tangent to Σ and

$$M_{\parallel J}^B (F^{-1})_a^J m_{\parallel b}^a t^b = (F^{-1})_a^B m_{\parallel b}^a t^b,$$

with which the Piola-type identity (68) follows.

We conclude this subsection by remarking that the volume Piola identity (67)

$$0 = \frac{\partial}{\partial x^a} (J^{-1} F_A^a)$$

cannot be generalized easily for surfaces. To discuss this point, we should delve into the problem of determining Levi-Civita parallel transport on the submanifolds $\partial\Omega$ and $\partial\omega$.