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CONVERGENCE ANALYSIS OF PADÉ APPROXIMATIONS FOR HELMHOLTZ FREQUENCY RESPONSE PROBLEMS *

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Abstract. The present work concerns the approximation of the solution map $\mathcal{S}$ associated to the parametric Helmholtz boundary value problem, *i.e.*, the map which associates to each (real) wavenumber belonging to a given interval of interest the corresponding solution of the Helmholtz equation. We introduce a least squares rational Padé-type approximation technique applicable to any meromorphic Hilbert space-valued univariate map, and we prove the uniform convergence of the Padé approximation error on any compact subset of the interval of interest that excludes any pole. This general result is then applied to the Helmholtz solution map $\mathcal{S}$, which is proven to be meromorphic in $\mathbb{C}$, with a pole of order one in every (single or multiple) eigenvalue of the Laplace operator with the considered boundary conditions. Numerical tests are provided that confirm the theoretical upper bound on the Padé approximation error for the Helmholtz solution map.

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1. Introduction

Due to the oscillatory behavior of the solutions, the finite element approximation of time-harmonic wave problems in mid- and high-frequency regimes is challenging: accurate approximations are possible only on very fine meshes or with high polynomial approximation degrees. Moreover, for increasing wave numbers, there is an increasing discrepancy between the best approximation error and the Galerkin discretization error (pollution effect) [2]. For this reason, the direct numerical evaluation of the frequency response functions for a whole range of frequencies is often out of reach.

Model order reduction methods aim at reducing the computational cost by approximating the frequency response function starting from evaluations only at few frequencies. For a survey of model order reduction methods for parametric systems we refer to [4], and for reduced order models for non-coercive and time-harmonic

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problems we refer to [8, 22, 29, 30] and references therein. Some of the model reduction methods for frequency domain wave problems are based on componentwise Padé or Padé-type approximations, on Petrov−Galerkin schemes, or on projections onto Krylov subspaces; see [15, 16] and the references therein, where the authors review interpolatory model order reduction methods, and compare them when applied to structural dynamic, acoustic and vibro-acoustic problems. In this work, we focus on the design and analysis of a numerical method based on the *rational* (*Padé-type*) *approximation* of the solution map to time-harmonic wave problems over a given range of angular frequencies.

Padé-type approximations have been firstly introduced for univariate complex-valued functions. Suppose that a complex function $f:\mathbb{C}\to\mathbb{C}$ holomorphic in a point $z_0\in\mathbb{C}$ (we take for simplicity $z_0=0$) is expressed in power series as $f(z)=\sum_{j=0}^{\infty} f_j z^j$ locally around 0. The Padé approximant of f, denoted as $f_{[M/N]}$, is the ratio between two polynomials $f_{[M/N]}(z)=\frac{p(z)}{q(z)}$, with $p(z)=\sum_{m=0}^{M} p_m z^m\in\mathbb{P}_M(\mathbb{C})$ and $q(z)=\sum_{n=0}^{N} q_n z^n\in\mathbb{P}_N(\mathbb{C})$, such that its Taylor series agrees with the power series of f for as many terms as possible. More precisely, $p\in\mathbb{P}_M(\mathbb{C})$ and $q\in\mathbb{P}_N(\mathbb{C})$ can be found such that

$$\sum_{j=0}^{\infty} f_j z^j=\frac{p(z)}{q(z)}+O(z^{M+N+1}). \tag{1.1}$$

Equation (1.1) is non-linear. In order to compute the coefficients of $p(z)$ and $q(z)$, one can multiply both sides of the equation by $q(z)$, and then identify the coefficients of the monomials of the same order. This procedure leads to the solution of a linear system in the unknowns $p_0,\ \dots,\ p_M,\ q_0,\ \dots,\ q_N$. The trivial solution $p(z),q(z)\equiv 0$ is usually avoided by imposing $q_0=1$. For more details on the classical Padé construction we refer to ([3], Chap. 1).

The convergence theory of the Padé approximant for meromorphic maps has been deeply studied. Suppose that $f(z)$ is meromorphic in the circle $\mathcal{B}(0,R)$, and define ν as the sum of the multiplicities of all the isolated poles of $f(z)$ inside $\mathcal{B}(0,R)$. Denote with $G\subset\mathcal{B}(0,R)$ the set of isolated poles of $f(z)$. When the degree of the denominator N of the Padé approximant is fixed and exactly equal to ν, and the degree of the numerator M is let to infinity, the Montessus de Ballore Theorem ([3, 26], Chap. 6) states the uniform convergence of the approximation error $\left|f(z)-f_{[M/N]}(z)\right|$ on compact subsets of $\mathcal{B}(0,R)\setminus G$. Weaker convergence results can be proved either when N (fixed) is larger or equal to ν, or when N is equal to the degree of the numerator M, and they both go to infinity (see *e.g.* [3], Chap. 6, [11, 31]).

The classical Padé approximation technique described above is also known as *single-point* Padé approximation. Indeed, the construction of the rational Padé approximant is based on the power series of the function $f(z)$ around one single point. A natural generalization is the *multi-point* Padé approximant (see [3], Chap. 7, [9, 27]), which exploits the power series expansions of $f(z)$ at several points $z_0,\dots,z_n$, which may possibly coincide.

Several generalizations of the Padé approximation to the case of a multivariate function $f:\mathbb{C}^d\to\mathbb{C}$, $d\geq 2$, have been proposed and analyzed in literature. We mention, for instance, [12–14, 19]. In this work, we follow the approach proposed in [14]. There, the authors present a *least squares* (LS) Padé approximation technique, which generalizes the classical approach. The Padé approximant, in fact, does not rely on the exact solution of the linear system in the unknowns $p_0,\ \dots,\ p_M,\ q_0,\ \dots,\ q_N$; it is rather defined as the (in general not unique) solution of a related minimization problem. The condition $q_0=1$ on the coefficients of the denominator $q_0,\dots,q_N$ is replaced by the condition $\sum_{n=0}^{N}|q_i|=1$. A convergence result for the LS Padé approximation error similar to the Montessus de Ballore Theorem is stated in [14].

The novelty of our paper consists in the definition of a LS Padé approximant for univariate Hilbert space-valued meromorphic maps $\mathcal{T}:\mathbb{C}\to V$, V being a Hilbert space. The Padé approximant of $\mathcal{T}$, denoted as $\mathcal{T}_{[M/N]}$, is the rational V-valued map $\mathcal{T}_{[M/N]}(z)=\frac{\mathcal{P}(z)}{\mathcal{Q}(z)}$, where $\mathcal{P}(z)=\sum_{i=0}^{M} p_i(z)z^i$, with coefficients $p_i(z)\in V$ for all i, and $\mathcal{Q}\in\mathbb{P}_N(\mathbb{C})$ is a $\mathbb{C}$-valued polynomial of degree N. The main result is Theorem 6.2, where the convergence of the Padé approximant is proved. Suppose $\mathcal{T}(z)$ to be meromorphic in the circle $\mathcal{B}(0,R)$, and define ν as the sum of the multiplicities of the isolated poles of $\mathcal{T}$ contained in the circle $\mathcal{B}(0,R)$. Moreover, denote with

$G \subset \mathcal{B}(0,R)$ the set of the isolated poles of $\mathcal{T}$. Letting the degree of the Padé denominator N be fixed and exactly equal to ν, we prove exponential convergence of the approximation error $\left\|\mathcal{T}(z) - T_{[M/N]}(z)\right\|_V$, as M goes to infinity, on compact subsets of $\mathcal{B}(0,R) \setminus G$. In Section 7, we apply the LS Padé construction and the convergence estimate to the meromorphic solution map $\mathcal{S}$ associated with the parametric Helmholtz problem, namely the map that associates with each (real) wavenumber inside an interval of interest $K \subset \mathbb{R}^+$ the solution of the corresponding Helmholtz problem.

The outline of the paper is the following. In Section 2, we introduce the parametric Helmholtz problem with homogeneous either Dirichlet or Neumann boundary conditions; the problem is set in the Hilbert space $V = H^1_0(D)$ or $V = H^1(D)$, respectively. The wavenumber varies inside an interval of interest $K \subset \mathbb{R}^+$, and we define the solution map $\mathcal{S} : K \to V$. The solution map is then extended to the entire complex plane $\mathcal{S} : \mathbb{C} \to V$, and well-posedness and stability bounds of the corresponding (damped) Helmholtz problem are proved in Section 3. In Section 4, we study the regularity of the solution map $\mathcal{S}$, which is proved to be meromorphic, with a pole of order one in each (single or multiple) eigenvalue of the Laplace problem with the considered boundary condition. The construction of the LS Padé approximant for any meromorphic Hilbert space-valued map $\mathcal{T}$ is described in Section 5, and a convergence result of the approximation error is stated in Section 6. In Section 7, we apply the results obtained in Sections 5 and 6 to the solution map $\mathcal{S}$. Numerical results for the Helmholtz problem in a 2D spatial domain are shown in Section 8, and conclusions are drawn in Section 9.

2. Problem setting

Let D be an open bounded Lipschitz domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$. We consider the Helmholtz problem with parametric wavenumber $k^2 \in K := [k^2_{\min}, k^2_{\max}] \subset \mathbb{R}^+$:

$$-\Delta u - k^2 u = f \qquad \text{in} \quad D, \tag{2.1}$$

endowed with homogeneous either Dirichlet or Neumann boundary conditions. Let us denote with V either the Hilbert space $H^1_0(D)$ or $H^1(D)$, in case problem (2.1) is endowed with Dirichlet or Neumann homogeneous boundary conditions on ∂D, respectively. Moreover we assume the functions in V to be complex-valued.

Given a real positive weight $w > 0$, such that w^{-1} is dimensionally homogeneous to a length, we denote by $\|\cdot\|_{V,w}$ the (weighted) $H^1(D)$-norm:

$$\|v\|^2_{V,w} = \|\nabla v\|^2_{L^2(D)} + w^2 \|v\|^2_{L^2(D)}. \tag{2.2}$$

Note that the (weighted) $H^1(D)$-norm is equivalent to the standard $H^1(D)$-norm, indeed:

$$\sqrt{\min\{1, w^2\}}\, \|u\|_{H^1(D)} \leq \|u\|_{V,w} \leq \sqrt{\max\{1, w^2\}}\, \|u\|_{H^1(D)}. \tag{2.3}$$

Moreover, we notice that standard analysis for the Helmholtz problem for a fixed wavenumber k is done in the weighted $H^1(D)$-norm $\|\cdot\|_{V,w}$ with $w = k$ (see, *e.g.* [25]).

The weak formulation of the parametric problem (2.1) is: given $f \in L^2(D)$, find $u(k^2,\cdot) \in V$ such that

$$\int_D \nabla u(k^2, \mathbf{x}) \cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - k^2 \int_D u(k^2, \mathbf{x}) \overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_D f(\mathbf{x}) \overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \quad \forall v \in V, \tag{2.4}$$

with $k^2 \in K$. We introduce the solution map

$$\begin{aligned} \mathcal{S} : K &\to V, \\ k^2 &\mapsto u(k^2, \cdot). \end{aligned} \tag{2.5}$$

The solution map $\mathcal{S}$ is well-defined provided that $k^2 \notin \Lambda$, $\Lambda := \{\lambda_i\}$ being the set of (real, non negative) eigenvalues of the Laplace operator with the considered boundary conditions.

3. The Helmholtz problem with complex-valued wavenumber

We extend the solution map defined in (2.5) to the complex plane:

$$\begin{aligned} \mathcal{S} : \mathbb{C} &\to V \\ z &\mapsto u(z,\cdot), \end{aligned} \tag{3.1}$$

where $u(z,\cdot)$ solves

$$\int_D \nabla u(z,\mathbf{x}) \cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - z\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_D f(\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \quad \forall v \in V. \tag{3.2}$$

Whenever $\mathrm{Im}\,(z) \neq 0$, problem (3.2) contains the damping term

$$i\,\mathrm{Im}\,(z)\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}.$$

The next theorem states the well-posedness of (3.2) for all $z \in \mathbb{C}\setminus\Lambda$. The reason why we consider also wavenumbers z with negative real part will be clarified in Section 7 (see Rem. 7.2).

Theorem 3.1. *Let $z \in \mathbb{C}\setminus\Lambda$. Then problem (3.2) admits a unique solution. Moreover, if*

$$\min_{j:\lambda_j\in\Lambda} |\lambda_j - z| > \alpha > 0, \tag{3.3}$$

then the unique solution $u(z,\mathbf{x})$ satisfies the a priori bound

$$\|u(z,\cdot)\|_{V,w} \leq \frac{\sqrt{|z-\lambda_{\min}| + |\mathrm{Re}\,(z)| + w^2}}{\alpha}\,\|f\|_{L^2(D)}\,, \tag{3.4}$$

where

$$\lambda_{\min} := \min\{\lambda \in \Lambda\} \tag{3.5}$$

is the smallest eigenvalue of the Laplace operator with the considered boundary conditions.

Proof. We start by proving that problem (3.2) admits a unique solution.

We consider first the case $z \in \mathbb{C}\setminus\mathbb{R}^+$. We prove that the bilinear form

$$B_z(u,v) := \int_D \nabla u(z,\mathbf{x}) \cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - z\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}$$

which appears in problem (3.2) is coercive and continuous.

We distinguish two cases.

(a) Let $z \in \mathbb{C}^- := \mathbb{R}^- + i\mathbb{R}$. In this case, we have

$$|B_z(u,u)| \geq |\mathrm{Re}\,(B_z(u,u))| \geq \min\left\{1, -\frac{\mathrm{Re}\,(z)}{w^2}\right\}\|u\|^2_{V,w}\,.$$

The coercivity of the bilinear form $B_z(\cdot,\cdot)$ is then proved with constant $\min\{1, -\frac{\mathrm{Re}(z)}{w^2}\}$. The continuity of $B_z(\cdot,\cdot)$ holds with continuity constant $\max\{1, \frac{|z|}{w^2}\}$, so that, thanks to the Lax-Milgram Lemma, we conclude the existence and uniqueness of the solution of problem (3.2) for every $z \in \mathbb{C}^-$.

(b) Let $z \in \mathbb{C}^+$ and $\mathrm{Im}\,(z) \neq 0$, with $\mathbb{C}^+ := \mathbb{R}^+ + i\mathbb{R}$. In this case, we prove the coercivity of $B_z(\cdot,\cdot)$ following ([23], Chap. 2). Since for every $\nu \in \mathbb{C}$, $\sqrt{2}\,|\nu| \geq |\mathrm{Re}\,(\nu)| + |\mathrm{Im}\,(\nu)|$, we have

$$\sqrt{2}\,|B_z(u,u)| \geq |\mathrm{Re}\,(B_z(u,u))| + |\mathrm{Im}\,(B_z(u,u))|.$$

Moreover, for every $0 < \varepsilon < 1$, $|\mu| \geq \varepsilon\mu$ with $\mu \in \mathbb{R}$. Hence,

$$\begin{aligned}\sqrt{2}\,|B_z(u,u)| &\geq \varepsilon \mathrm{Re}\,(B_z(u,u)) + |\mathrm{Im}\,(B_z(u,u))| \\ &= \varepsilon \left(\|\nabla u(z,\cdot)\|^2_{L^2(D)} - \mathrm{Re}\,(z)\,\|u(z,\cdot)\|^2_{L^2(D)} \right) + |\mathrm{Im}\,(z)|\,\|u(z,\cdot)\|^2_{L^2(D)} \\ &= \varepsilon\,\|\nabla u(z,\cdot)\|^2_{L^2(D)} + (-\varepsilon\mathrm{Re}\,(z) + |\mathrm{Im}\,(z)|)\,\|u(z,\cdot)\|^2_{L^2(D)} \\ &\geq \min\left\{\varepsilon, \frac{|\mathrm{Im}\,(z)| - \varepsilon\mathrm{Re}\,(z)}{w^2}\right\}\|u(z,\cdot)\|^2_{V,w}.\end{aligned}$$

Provided that $0 < \varepsilon < \min\{1, \frac{|\mathrm{Im}(z)|}{\mathrm{Re}(z)}\}$, the coercivity of the bilinear form $B_z(\cdot,\cdot)$ is then proved with coercivity constant $\frac{1}{\sqrt{2}}\min\{\varepsilon, \frac{|\mathrm{Im}(z)|-\varepsilon\mathrm{Re}(z)}{w^2}\}$. As in the previous case, the continuity of $B_z(\cdot,\cdot)$ holds with constant $\max\{1, \frac{|z|}{w^2}\}$, and the existence and uniqueness of the solution of problem (3.2) follows by the Lax-Milgram Lemma.

For the case $z \in \mathbb{R}^+ \setminus \varLambda$, problem (3.2) admits a unique solution by the Fredholm alternative.

It remains to show the *a priori* bound (3.4).

Let $\{\varphi_l\}$ be the $L^2(D)$-orthogonal set of eigenfunctions of the Laplacian (with the considered boundary conditions) corresponding to the eigenvalues $\{\lambda_l\}$. Observe that the set $\{\varphi_l\}$ is orthogonal also with respect to the (weighted) $H^1(D)$-norm, and $\|\nabla\varphi_l\|^2_{L^2(D)} = \lambda_l\,\|\varphi_l\|^2_{L^2(D)}$.

Replacing the eigenfunction expansions $u(z,\mathbf{x}) = \sum_l u_l(z)\varphi_l(\mathbf{x})$ and $f(\mathbf{x}) = \sum_l f_l\varphi_l(\mathbf{x})$ into the Helmholtz problem, we derive

$$u_j(z) = \frac{f_j}{\lambda_j - z}\cdot \tag{3.6}$$

We express the norm of u as follows:

$$\begin{aligned}\|u(z,\cdot)\|^2_{V,w} &= \|\nabla u(z,\cdot)\|^2_{L^2(D)} + w^2\,\|u(z,\cdot)\|^2_{L^2(D)} \\ &= \int_D \left|\sum_l u_l(z)\nabla\varphi_l(\mathbf{x})\right|^2\,\mathrm{d}\mathbf{x} + w^2\int_D\left|\sum_l u_l(z)\varphi_l(\mathbf{x})\right|^2\,\mathrm{d}\mathbf{x} \\ &= \sum_l |u_l(z)|^2\,\|\nabla\varphi_l\|^2_{L^2(D)} + w^2\sum_l |u_l(z)|^2\,\|\varphi_l\|^2_{L^2(D)} \\ &= \sum_l (\lambda_l + w^2)\,|u_l(z)|^2\,\|\varphi_l\|^2_{L^2(D)},\end{aligned}$$

where the third identity follows from the H^1-orthogonality of the set $\{\varphi_l\}_l$. Using (3.6) we obtain

$$\|u(z,\cdot)\|^2_{V,w} = \sum_l \frac{\lambda_l + w^2}{|\lambda_l - z|^2}\,|f_l|^2\,\|\varphi_l\|^2_{L^2(D)}.$$

The coefficient $\dfrac{\lambda_l + w^2}{|\lambda_l - z|^2}$ can be bounded uniformly with respect to l. In order to do this, define the function $g : \mathbb{R}^+ \to \mathbb{R}$ as $g(x) := \dfrac{x + w^2}{|x - z|^2}$ for $x \neq z$, and observe that

$$g(x) = \frac{x - \mathrm{Re}\,(z) + \mathrm{Re}\,(z) + w^2}{|x - z|^2} \leq \frac{|x - \mathrm{Re}\,(z)| + |\mathrm{Re}\,(z)| + w^2}{|x - z|^2} \leq \frac{|x - z| + |\mathrm{Re}\,(z)| + w^2}{|x - z|^2}.$$

Provided that $x \neq z$, let $y := |x - z|$ and $h(y) := \dfrac{y + |\mathrm{Re}\,(z)| + w^2}{y^2}$, so that $g(x) \leq h(y)$.

Since $h'(y) = -\dfrac{y + 2\,|\mathrm{Re}\,(z)| + 2w^2}{y^3} < 0$, the function h is decreasing and its maximum is achieved when y is the smallest possible. Let $\lambda_z^\star := \mathrm{argmin}_{\lambda \in \Lambda} \{|\lambda - z|\}$ be the closest eigenvalue to z. Then, for any $\lambda_l \in \Lambda$,

$$g(\lambda_l) \leq h(|\lambda_z^\star - z|) = \frac{|\lambda_z^\star - z| + |\mathrm{Re}\,(z)| + w^2}{|\lambda_z^\star - z|^2}.$$

As in formula (3.5), we denote with $\lambda_{\min}$ the smallest eigenvalue of the Laplace operator with the considered boundary conditions. Then,

$$|\lambda_z^\star - z| + |\mathrm{Re}\,(z)| + w^2 \leq |\lambda_{\min} - z| + |\mathrm{Re}\,(z)| + w^2.$$

Using assumption (3.3) we deduce that

$$\frac{\lambda_l + w^2}{|\lambda_l - z|^2} = g(\lambda_l) \leq \frac{|\lambda_{\min} - z| + |\mathrm{Re}\,(z)| + w^2}{\alpha^2}.$$

Finally, we conclude:

$$\begin{aligned} \|u(z,\cdot)\|_{V,w}^2 &\leq \frac{|\lambda_{\min} - z| + |\mathrm{Re}\,(z)| + w^2}{\alpha^2} \sum_l |f_l|^2 \, \|\varphi_l\|_{L^2(D)}^2 \\ &= \frac{|\lambda_{\min} - z| + |\mathrm{Re}\,(z)| + w^2}{\alpha^2} \, \|f\|_{L^2(D)}^2 . \end{aligned}$$ □

A variant of Theorem 3.1 for real values of z previously appeared in [24] (Lem. 2.1), where stability in $H^{1+\gamma}(D)$, $\gamma \in (1/2, 1]$ was proved.

Remark 3.2. For a Helmholtz problem with fixed wavenumber z with $\mathrm{Re}\,(z) > 0$, it is reasonable to take the weighted $H^1(D)$-norm (2.2) with $w^2 = \mathrm{Re}\,(z)$. With this choice, the estimate (3.4) becomes:

$$\|u(z,\cdot)\|_{V,w} \leq \frac{\sqrt{|z - \lambda_{\min}| + 2\mathrm{Re}\,(z)}}{\alpha} \, \|f\|_{L^2(D)} .$$

Figure 1 refers to the Helmholtz problem (3.2) coupled with homogeneous Dirichlet boundary conditions on ∂D, where $D = [0,\pi] \times [0,\pi]$. Let $\nu^2 = 12 \in \mathbb{R}^+ \setminus \Lambda$ and $\mathbf{d} = (\cos(\pi/6), \sin(\pi/6)) \in \mathbb{R}^2$. The loading term $f(\mathbf{x})$ is such that the unique solution $u(\mathbf{x})$ of the considered Helmholtz problem with wavenumber ν^2 is the product between the plane wave traveling along the direction $\mathbf{d}$, $v(\mathbf{x}) = \mathrm{e}^{-i\nu\mathbf{d}\cdot\mathbf{x}}$, and the bubble $\phi(\mathbf{x}) = \frac{16}{\pi^4} x_1 x_2 (x_1 - \pi)(x_2 - \pi)$.

We choose the interval of interest $K = [k_{\min}^2, k_{\max}^2] = [4, 15]$, which contains four eigenvalues of the Laplace problem with the considered boundary conditions: $\lambda = 5$ (double), 8 (single), 10 (double), 13 (double). We partition the interval of interest K in 150 intervals with all the same length. At each point z of the mesh we firstly compute the solution of the Helmholtz problem $u(z,\cdot) \in H_0^1(D)$ *via* the $\mathbb{P}^3$ continuous Finite Element (FE) method. The weighted $H^1(D)$-norm $\|u(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ is calculated, with weight equal to the square root of the real part of $z_0 = 10 + \frac{i}{2}$. We observe that the upper bound (3.4) (dashed line) behaves as the norm of the solution (solid line).

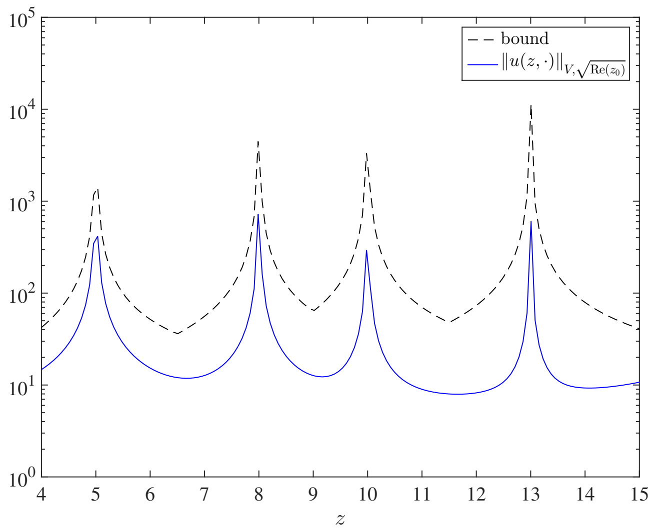


FIGURE 1. Numerical testing of the bound (3.4). The Helmholtz problem (3.2) with homogeneous Dirichlet boundary conditions on ∂D, $D = [0, \pi] \times [0, \pi]$, is considered. The interval of interest $K = [k^2_{\min}, k^2_{\max}] = [4, 15]$ is partitioned into 150 intervals all with the same length. At each point z of the grid, the norm $\|u(z,\cdot)\|_{V,w}$ of the Helmholtz solution, with $w^2 = \mathrm{Re}\,(z_0)$ and $z_0 = 10 + \frac{i}{2}$, is computed (solid line) and compared with the right-hand side of the bound (3.4) (dashed line).

4. Regularity of the solution map

In Section 3, we have introduced the solution map $\mathcal{S}$ (see (3.1)) which associates to each $z \in \mathbb{C} \setminus \varLambda$ the solution $u(z,\cdot)$ of the damped Helmholtz problem (3.2). In the following, we prove the regularity properties of this mapping.

Proposition 4.1. *The solution map $\mathcal{S} : \mathbb{C} \setminus \varLambda \to V$ defined in (3.1), with V endowed with the norm $\|\cdot\|_{V,w}$, is continuous.*

Proof. We have to verify that, for every $z \in \mathbb{C} \setminus \varLambda$,

$$\lim_{h\to 0} \|\mathcal{S}(z+h) - \mathcal{S}(z)\|_{V,w} = 0, \qquad \text{with } h \in \mathbb{C} \setminus \{0\}, \tag{4.1}$$

where $\mathcal{S}(z+h) := u(z+h,\cdot)$ is the unique solution of

$$\int_D \nabla u(z+h,\mathbf{x}) \cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - (z+h)\int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_D f(\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}. \tag{4.2}$$

Recall that $\mathcal{S}(z) := u(z,\cdot)$ is the unique solution of (3.2).

$$\int_D \nabla u(z,\mathbf{x}) \cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - z\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_D f(\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}. \tag{4.3}$$

Taking the difference of the weak formulations (4.2) and (3.2), we find

$$
\begin{aligned}
0 &= \int_D \nabla\left(u(z+h,\mathbf{x}) - u(z,\mathbf{x})\right)\cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - (z+h)\int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \\
&\quad + z\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \\
&= \int_D \nabla\left(u(z+h,\mathbf{x}) - u(z,\mathbf{x})\right)\cdot \overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - (z+h)\int_D \left(u(z+h,\mathbf{x}) - u(z,\mathbf{x})\right)\overline{v}(\mathbf{x})\ dx \\
&\quad - h\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}.
\end{aligned}
$$

The function $w_h(\mathbf{x}) := u(z+h,\mathbf{x}) - u(z,\mathbf{x})$ solves

$$
\int_D \nabla w_h(\mathbf{x})\cdot\overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - (z+h)\int_D w(\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = h\int_D u(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}. \tag{4.4}
$$

Theorem 3.1 states that Problem (4.4) is well-posed and its unique solution satisfies the upper bound (3.4):

$$
\|w_h(z+h,\cdot)\|_{V,w} \le \frac{\sqrt{|z+h-\lambda_{\min}| + |\mathrm{Re}\,(z+h)| + w^2}}{\left|\lambda^\star_{z+h} - z\right|}\,|h|\,\|u(z,\cdot)\|_{L^2(D)}
$$

where $\lambda^\star_{z+h} := \mathrm{argmin}_{\lambda\in\Lambda}\left\{|\lambda - (z+h)|\right\}$ and $\lambda^\star_{z+h} \xrightarrow{h\to 0} \lambda^\star_z$. Hence,

$$
\lim_{h\to 0}\|u(z+h,\cdot) - u(z,\cdot)\|_{V,w} = \lim_{h\to 0}\|w_h(z+h,\cdot)\|_{V,w} = 0,
$$

so that (4.1) is verified. □

Proceeding as in [10, 28], we prove now that the solution map $\mathcal{S}$ admits complex derivative.

Proposition 4.2. *For any $z\in\mathbb{C}\setminus\Lambda$, the solution map $\mathcal{S}$ admits a complex derivative $\dfrac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}$, which is the unique solution of*

$$
\int_D \nabla\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}\cdot\overline{\nabla v}\,\mathrm{d}\mathbf{x} - z\int_D \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}\overline{v}\,\mathrm{d}\mathbf{x} = \int_D \mathcal{S}(z)\overline{v}\,\mathrm{d}\mathbf{x}, \quad \forall v\in V. \tag{4.5}
$$

Proof. The complex derivative $\dfrac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}$ is defined as

$$
\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}(z) := \lim_{h\to 0}\frac{\mathcal{S}(z+h) - \mathcal{S}(z)}{h} = \lim_{h\to 0}\frac{u(z+h,\cdot) - u(z,\cdot)}{h}, \quad h\in\mathbb{C}\setminus\{0\}.
$$

Define the difference quotient

$$
w_h(z,\cdot) := \frac{u(z+h,\cdot) - u(z,\cdot)}{h}. \tag{4.6}
$$

As in the proof of Proposition 4.1, we take the difference between the weak formulations (4.2) and (4.3) solved by $u(z+h)$ and $u(z)$, respectively:

$$
\begin{aligned}
0 &= \int_D \nabla(u(z+h,\mathbf{x}) - u(z,\mathbf{x}))\cdot\overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - z\int_D (u(z+h,\mathbf{x}) - u(z,\mathbf{x}))\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \\
&\quad - (z+h)\int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} + z\int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} \\
&= h\int_D \nabla w_h(z,\mathbf{x})\cdot\overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - zh\int_D w_h(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - h\int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x}.
\end{aligned}
$$

Then, $w_h(z,\cdot)$ is the unique solution of

$$\int_D \nabla w_h(z,\mathbf{x})\cdot\overline{\nabla v}(\mathbf{x})\ \mathrm{d}\mathbf{x} - z\int_D w_h(z,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int_D u(z+h,\mathbf{x})\overline{v}(\mathbf{x})\ \mathrm{d}\mathbf{x},\quad \forall v\in V. \tag{4.7}$$

Taking the limit as $h\to 0$ in (4.7) and using the continuity of $\mathcal{S}$ (see Prop. 4.1), we derive problem (4.5), which is well-posed provided that $z\notin\Lambda$. Hence, for any $z\in\mathbb{C}\setminus\Lambda$, the complex derivative $\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}(z)$ exists and is the unique solution of (4.5). □

Proposition 4.2 states that the solution map $\mathcal{S}$ is holomorphic in $\mathbb{C}$ except in the set of isolated points $\Lambda=\{\lambda_j\}$, *i.e.*, $\mathcal{S}\in\mathcal{H}(\mathbb{C}\setminus\Lambda;V)$, where $\mathcal{H}(U;V)$ is the space of holomorphic mappings from $U\subset\mathbb{C}$ with values in V. Since the multiplicity μ_j of every eigenvalue λ_j is finite for every j (see *e.g.* [20], Chapter 6), to each eigenvalue λ_j there correspond μ_j eigenfunctions $\{\varphi_i\}_{i=1,\ldots,\mu_j}$. The eigenfunction expansion of the solution map $\mathcal{S}$ is then

$$\mathcal{S}(z)=u(z,\mathbf{x})=\sum_{j=1}^{\infty}\sum_{i=1}^{\mu_j}u_{j,i}(z)\varphi_{j,i}(\mathbf{x})\overset{(3.6)}{=}\sum_{j=1}^{\infty}\sum_{i=1}^{\mu_j}\frac{f_{j,i}}{\lambda_j-z}\varphi_{j,i}(\mathbf{x}).$$

We deduce that every eigenvalue λ_j is a pole with multiplicity one, and that the solution map $\mathcal{S}$ is meromorphic, according to the following definition (see *e.g.* [18], p. 7, [6], p. 356).

Definition 4.3. A function $\mathcal{T}:U\subset\mathbb{C}\to V$ is called meromorphic if there exists a discrete subset W of U such that $\mathcal{T}\in\mathcal{H}(U\setminus W;V)$ and, for each $\tau\in W$, there exists $k\in\mathbb{N}$ such that $(z-\tau)^k\mathcal{T}(z)$ admits holomorphic extension in τ. We write $\mathcal{T}\in\mathcal{M}(U;V)$.

5. Construction of the Padé approximant

In this section, we construct a Padé approximant of a holomorphic mapping $\mathcal{T}:\mathbb{C}\to V$. We follow the procedure illustrated in [14], with the difference that we are interested in the case where the mapping $\mathcal{T}$ is univariate, instead of multivariate, but with values in a Hilbert space, instead of in $\mathbb{C}$. We denote the Padé approximation of $\mathcal{T}$ as $\mathcal{T}_{[M/N]}:\mathbb{C}\to V$. It is defined as the ratio of two polynomials of degree M and N respectively:

$$\mathcal{T}_{[M/N]}(z):=\frac{\mathcal{P}_{[M]}(z)}{\mathcal{Q}_{[N]}(z)}. \tag{5.1}$$

The denominator $\mathcal{Q}_{[N]}\in\mathbb{P}_N(\mathbb{C})$, where $\mathbb{P}_N(\mathbb{C})$ denotes the space of polynomials of degree at most N, is a function of z only. The numerator $\mathcal{P}_{[M]}:\mathbb{C}\to V$ is a function of both the complex variable z and the space variable $\mathbf{x}\in D$. More precisely, $\mathcal{P}_{[M]}(z)=\sum_{i=0}^{M}p_i(z)z^i$, with coefficients $p_i(z)\in V$. In the following we denote by $\mathbb{P}_M(\mathbb{C};V)$ the space of polynomials of degree at most M in $z\in\mathbb{C}$ with coefficients in V.

We start the construction of (5.1) by introducing the following notation. Let $\mathcal{T}:\mathbb{C}\to V$ be a mapping which is holomorphic around $z_0\in\mathbb{C}$. Then $\mathcal{T}(z)$ can be expressed as the Taylor series

$$\mathcal{T}(z)=\sum_{\alpha=0}^{\infty}\big(\mathcal{T}(z)\big)_{z_0,\alpha}(z-z_0)^{\alpha},$$

with z in a neighborhood of z_0, where $\big(\mathcal{T}(z)\big)_{z_0,\alpha}\in V$ is the α-th order Fréchet derivative of $\mathcal{T}$ in z_0 divided by $\alpha\,!$ (see *e.g.* [1], Chap. 1). Recall the Cauchy formula:

$$\big(\mathcal{T}(z)\big)_{z_0,\alpha}=\frac{1}{2\pi i}\int_{\gamma}\frac{\mathcal{T}(z)}{(z-z_0)^{\alpha+1}}\mathrm{d}z, \tag{5.2}$$

where γ is a circle centered at z_0 and contained in the region of holomorphy of $\mathcal{T}$. Moreover, given $E \in \mathbb{N}$, denote with $\big(\mathcal{T}(z)\big)^E_{z_0}$ the Taylor polynomial of $\mathcal{T}$ of degree E,

$$\big(\mathcal{T}(z)\big)^E_{z_0} = \sum_{\alpha=0}^{E} \big(\mathcal{T}(z)\big)_{z_0,\alpha}(z-z_0)^\alpha. \tag{5.3}$$

To lighten the notation, in the following we will take $z_0 = 0$, and we will denote the Taylor coefficients $\big(\mathcal{T}(z)\big)_{z_0,\alpha}$ simply as $\big(\mathcal{T}(z)\big)_\alpha$, and $\big(\mathcal{T}(z)\big)^E_{z_0}$, the Taylor polynomial of degree E, simply as $\big(\mathcal{T}(z)\big)^E$. It is understood that all results generalize straightforwardly to the case $0 \neq z_0 \in \mathbb{C}$. The construction of the Padé approximation (5.1) relies on the minimization problem involving the following functional.

Definition 5.1. Let V be a Hilbert space, $\mathcal{T} : \mathbb{C} \to V$ a mapping which is holomorphic around the origin, and $\rho \in \mathbb{R}^+$. Given $P \in \mathbb{P}_M(\mathbb{C}; V)$, $Q \in \mathbb{P}_N(\mathbb{C})$ and $E \in \mathbb{N}$, we define

$$j_{E,\rho}(P,Q) := \left(\sum_{\alpha=0}^{E} \left\| \big(Q(z)\mathcal{T}(z) - P(z)\big)_\alpha \right\|^2_{V,w} \rho^{2\alpha} \right)^{1/2}. \tag{5.4}$$

The functional (5.4) can be defined equivalently using the following characterization.

Lemma 5.2. *Set $\gamma := \partial\mathcal{B}(0,\rho)$, where $\mathcal{B}(0,\rho)$ is the open disk centered at the origin and with radius $\rho > 0$. Then it holds*

$$\begin{aligned} j_{E,\rho}(P,Q) &= \left(\frac{1}{2\pi i} \int_\gamma \left\| \big(Q(z)\mathcal{T}(z) - P(z)\big)^E \right\|^2_{V,w} \frac{1}{z} \mathrm{d}z \right)^{1/2} \\ &= \left(\int_0^1 \left\| \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)^E \right\|^2_{V,w} \mathrm{d}\theta \right)^{1/2}, \end{aligned} \tag{5.5}$$

where the notation with the index E is defined in (5.3) and in the text thereafter.

Proof. The second identity in (5.5) simply follows from the change of variable $z = \rho \mathrm{e}^{2\pi i\theta}$.

Denoting with $\langle \cdot, \cdot \rangle_{V,w}$ the weighted scalar product in V which induces the norm $\|\cdot\|_{V,w}$, we have

$$\begin{aligned} &\int_0^1 \left\| \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)^E \right\|^2_{V,w} \mathrm{d}\theta \\ &= \int_0^1 \left\langle \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)^E, \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)^E \right\rangle_{V,w} \mathrm{d}\theta \\ &\overset{(5.3)}{=} \int_0^1 \left\langle \sum_{\alpha=0}^{E} \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)_\alpha \rho^\alpha \mathrm{e}^{2\pi i\theta\alpha}, \right. \\ &\qquad \left. \sum_{\beta=0}^{E} \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)_\beta \rho^\beta \mathrm{e}^{2\pi i\theta\beta} \right\rangle_{V,w} \mathrm{d}\theta \\ &= \sum_{\alpha=0}^{E}\sum_{\beta=0}^{E} \left\langle \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)_\alpha, \big(Q(\rho \mathrm{e}^{2\pi i\theta})\mathcal{T}(\rho \mathrm{e}^{2\pi i\theta}) - P(\rho \mathrm{e}^{2\pi i\theta})\big)_\beta \right\rangle_{V,w} \\ &\qquad \cdot \int_0^1 \rho^\alpha \mathrm{e}^{2\pi i\theta\alpha} \rho^\beta \overline{\mathrm{e}^{2\pi i\theta\beta}} \mathrm{d}\theta. \end{aligned}$$

Since

$$\int_0^1 \rho^\alpha e^{2\pi i\theta\alpha} \rho^\beta \overline{e^{2\pi i\theta\beta}} d\theta = \begin{cases} \rho^{2\alpha}, & \text{if} \quad \alpha = \beta \\ 0, & \text{if} \quad \alpha \neq \beta, \end{cases}$$

taking into account the formula (5.4), we obtain the first identity in (5.5). ◻

We can now define the Padé approximant of $\mathcal{T}$.

Definition 5.3. Let $M, N \in \mathbb{N}$, $E \geq M + N$, and $\rho \in \mathbb{R}^+$. Let $\mathcal{T} : \mathbb{C} \to V$ be as in Definition 5.1. A Padé approximant of $\mathcal{T}$ is a quotient $\dfrac{P}{Q}$ with $P \in \mathbb{P}_M(\mathbb{C}; V)$, $Q \in \mathbb{P}_N(\mathbb{C})$, $\sum_{\alpha=0}^N \left|(Q)_\alpha\right|^2 = 1$ such that

$$j_{E,\rho}(P, Q) \leq j_{E,\rho}(R, S) \quad \forall R \in \mathbb{P}_M(\mathbb{C}; V), \ \forall S \in \mathbb{P}_N(\mathbb{C}) \text{ with } \sum_{\alpha=0}^N \left|(S)_\alpha\right|^2 = 1. \tag{5.6}$$

A solution of this problem will be denoted as $\mathcal{T}_{[M/N]} = \dfrac{\mathcal{P}_{[M]}}{\mathcal{Q}_{[N]}}$.

Comments on the choice of ρ will be given at end of Section 6.

Proposition 5.4 (Existence of the Padé approximant)**.** *The minimization problem* (5.6) *admits at least one solution.*

Proof. Note that

$$\begin{aligned} j_{E,\rho}(P, Q)^2 &= \sum_{\alpha=0}^M \left\|\left(Q(z)\mathcal{T}(z) - P(z)\right)_\alpha\right\|^2_{V,w} \rho^{2\alpha} + \sum_{\alpha=M+1}^E \left\|\left(Q(z)\mathcal{T}(z) - P(z)\right)_\alpha\right\|^2_{V,w} \rho^{2\alpha} \\ &= \sum_{\alpha=0}^M \left\|\left(Q(z)\mathcal{T}(z) - P(z)\right)_\alpha\right\|^2_{V,w} \rho^{2\alpha} + \sum_{\alpha=M+1}^E \left\|\left(Q(z)\mathcal{T}(z)\right)_\alpha\right\|^2_{V,w} \rho^{2\alpha}. \end{aligned}$$

Taking $P = \bar{P}(Q)$, where $\bar{P}(Q)$ satisfies $\left(\bar{P}(z)\right)_\alpha = \left(Q(z)\mathcal{T}(z)\right)_\alpha$ for $0 \leq \alpha \leq M$, problem (5.6) can be formulated as a minimization problem in Q only (see [17], Rem. 2.3): find $Q \in \mathbb{P}_N(\mathbb{C})$ s.t. $\sum_{\alpha=0}^N \left|(Q)_\alpha\right|^2 = 1$ and

$$j_{E,\rho}(\bar{P}(Q), Q) \leq j_{E,\rho}(\bar{P}(S), S) \quad \forall S \in \mathbb{P}_N(\mathbb{C}) \qquad \text{with} \qquad \sum_{\alpha=0}^N \left|(S)_\alpha\right|^2 = 1. \tag{5.7}$$

Since $j_{E,\rho}(\bar{P}, \cdot)$ is continuous and the unit sphere in $\mathbb{C}^{N+1}$ is compact, the minimization problem (5.7) admits at least one solution. ◻

Note that Definition 5.1 and Definition 5.3 generalize without difficulty to the case $\mathcal{T} : \mathbb{C} \to V$ holomorphic in the open disk $\mathcal{B}(z_0, \rho)$, $z_0 \in \mathbb{C}$, expanded around z_0.

6. Padé approximation properties

The main result of this section is Theorem 6.2, which adapts the result of [14] to the case of $\mathcal{T} : \mathbb{C} \to V$. We make the following assumptions on $\mathcal{T}$.

- $\mathcal{T}$ is meromorphic in the closed disk $\overline{\mathcal{B}(0, R)}$, with $R > 0$.

- $\mathcal{T}(z) = \dfrac{h(z)}{g(z)}$ an irreducible fraction, with $h : \mathbb{C} \to V$ holomorphic in $\overline{\mathcal{B}(0,R)}$ and $g \in \mathbb{P}_N(\mathbb{C})$ such that $g(0) \neq 0$ and g is N-maximal, *i.e.*, for every polynomial f, the condition $gf \in \mathbb{P}_N(\mathbb{C})$ implies $f \in \mathbb{C}$. We assume g to be such that $\sum_{\alpha=0}^{N} \left|(g)_\alpha\right|^2 = 1$.

Lemma 6.1. *Let $(P^M, Q^N) \in (\mathbb{P}_M(\mathbb{C};V), \mathbb{P}_N(\mathbb{C}))$ be a solution of problem (5.6), and set $E = M + N + \delta$, with $\delta \geq 0$ independent of M. Define the mapping $H^M : \mathbb{C} \to V$ as*

$$H^M(z) := \left(Q^N(z)\mathcal{T}(z) - P^M(z)\right)g(z), \tag{6.1}$$

which is holomorphic in the closed disk $\overline{\mathcal{B}(0,R)}$. Then, for any z such that $|z| < \rho < R$, it holds

$$\left\|H^M(z)\right\|_{V,w} \leq C_H \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w} \left(\frac{\rho}{R}\right)^{M+1}, \tag{6.2}$$

where the constant $C_H > 0$ depends on N, $|z|$, ρ and R. Moreover, for any $z \in \mathcal{B}(0,R)$, it holds

$$\lim_{M\to\infty} \left\|H^M(z)\right\|_{V,w} = 0 \tag{6.3}$$

uniformly on all compact subsets of $\mathcal{B}(0,R)$.

Proof. The proof of Lemma 6.1 follows the same steps as the proof of Lemma 3.2 in [14]. We first prove the upper bound (6.2), and then we derive the limit (6.3).

Proof of the upper bound (6.2).

Let us fix $z \in \mathcal{B}(0,\rho)$ with $\rho < R$. Since $H^M \in \mathcal{H}(\mathcal{B}(0,R);V)$, it coincides with its Taylor series

$$H^M(z) = \sum_{\alpha=0}^{\infty} \left(H^M(z)\right)_\alpha z^\alpha = \sum_{\alpha=0}^{E} \left(H^M(z)\right)_\alpha z^\alpha + \sum_{\alpha>E} \left(H^M(z)\right)_\alpha z^\alpha.$$

In the rest of this proof, we omit the argument z (or $R, \mathrm{e}^{2\pi i\theta}$) of the functions whenever this does not generate confusion.

In order to prove (6.2), we bound the norm $\|\cdot\|_{V,w}$ of the coefficients $\left(H^M(z)\right)_\alpha$. We distinguish the two cases $0 \leq \alpha \leq E$ and $\alpha > E$.

Case $\boldsymbol{\alpha > E}$. Observe that $gP^M \in \mathbb{P}_{M+N}(\mathbb{C};V)$. Since $E \geq M+N$, then $\left(H^M\right)_\alpha = \left(g\,Q^N\mathcal{T}\right)_\alpha = \left(Q^N h\right)_\alpha$ and, using the Cauchy formula, $\left(H^M\right)_\alpha = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\mathcal{B}(0,R)} \frac{Q^N(z)h(z)}{z^{\alpha+1}}\mathrm{d}z$. Hence,

$$\begin{aligned}\left\|\left(H^M\right)_\alpha\right\|_{V,w} &= \left\|\frac{1}{2\pi i}\int_{\partial\mathcal{B}(0,R)} \frac{Q^N(z)h(z)}{z^{\alpha+1}}\mathrm{d}z\right\|_{V,w} = \left\|\int_0^1 \frac{Q^N(R\,\mathrm{e}^{2\pi i\theta})h(R\,\mathrm{e}^{2\pi i\theta})}{(R\,\mathrm{e}^{2\pi i\theta})^\alpha}\mathrm{d}\theta\right\|_{V,w}\\ &\leq \sup_{z\in\partial\mathcal{B}(0,R)} \left\|Q^N(z)h(z)\right\|_{V,w} \frac{1}{R^\alpha}.\end{aligned} \tag{6.4}$$

Case $\mathbf{0 \leq \alpha \leq E}$. Due to the formula (6.1) of H^M, $\left(H^M\right)_\alpha = \left(\left(Q^N \mathcal{T} - P^M\right)^E g\right)_\alpha$. Thanks to the Cauchy formula, $\left(H^M\right)_\alpha = \frac{1}{2\pi i} \int_{\partial\mathcal{B}(0,\rho)} \frac{\left(Q^N \mathcal{T} - P^M\right)^E g}{z^{\alpha+1}} \mathrm{d}z$. Hence,

$$
\begin{aligned}
\left\|\left(H^M\right)_\alpha\right\|_{V,w} &= \left\| \frac{1}{2\pi i} \int_{\partial\mathcal{B}(0,\rho)} \frac{\left(Q^N \mathcal{T} - P^M\right)^E g}{z^{\alpha+1}} \mathrm{d}z \right\|_{V,w} = \left\| \int_0^1 \frac{\left(Q^N \mathcal{T} - P^M\right)^E g}{(\rho\,\mathrm{e}^{2\pi i\theta})^\alpha} \mathrm{d}\theta \right\|_{V,w} \\
&\leq \int_0^1 \left\| \frac{\left(Q^N \mathcal{T} - P^M\right)^E g}{(\rho\,\mathrm{e}^{2\pi i\theta})^\alpha} \right\|_{V,w} \mathrm{d}\theta = \int_0^1 \frac{|g|}{\rho^\alpha} \left\| \left(Q^N \mathcal{T} - P^M\right)^E \right\|_{V,w} \mathrm{d}\theta \\
&\leq \left(\int_0^1 \left(\frac{|g|}{\rho^\alpha}\right)^2 \mathrm{d}\theta \right)^{1/2} \left(\int_0^1 \left\| \left(Q^N \mathcal{T} - P^M\right)^E \right\|_{V,w}^2 \mathrm{d}\theta \right)^{1/2} \\
&\overset{(5.5)}{=} \left(\int_0^1 \left(\frac{|g|}{\rho^\alpha}\right)^2 \mathrm{d}\theta \right)^{1/2} j_{E,\rho}(P^M, Q^N) \\
&\leq \sup_{z \in \partial\mathcal{B}(0,\rho)} |g(z)| \, \frac{1}{\rho^\alpha} \, j_{E,\rho}(P^M, Q^N),
\end{aligned}
\tag{6.5}
$$

with $j_{E,\rho}(P^M, Q^N)$ as in Definition 5.1. By assumption, (P^M, Q^N) is a solution of (5.6), so that $j_{E,\rho}(P^M, Q^N) \leq j_{E,\rho}(h_M, g)$, where $h_M = \sum_{\alpha=0}^{M} \left(h\right)_\alpha z^\alpha \in \mathbb{P}_M(\mathbb{C}; V)$ and $g \in \mathbb{P}_N(\mathbb{C})$ with $\sum_{\alpha=0}^{N} \left|\left(g\right)_\alpha\right|^2 = 1$. Using formula (5.4), we find

$$
\begin{aligned}
j_{E,\rho}(h_M, g) &= \left(\sum_{\alpha=0}^{E} \left\| \left(g\mathcal{T} - h_M\right)_\alpha \right\|_{V,w}^2 \rho^{2\alpha} \right)^{1/2} = \left(\sum_{\alpha=0}^{E} \left\| \left(h - h_M\right)_\alpha \right\|_{V,w}^2 \rho^{2\alpha} \right)^{1/2} \\
&= \left(\sum_{\alpha=M+1}^{E} \left\| \left(h\right)_\alpha \right\|_{V,w}^2 \rho^{2\alpha} \right)^{1/2} \overset{(5.5)}{=} \left(\frac{1}{2\pi i} \int_{\partial\mathcal{B}(0,\rho)} \left\| \sum_{\alpha=M+1}^{E} \left(h\right)_\alpha z^\alpha \right\|_{V,w}^2 \mathrm{d}z \right)^{1/2} \\
&= \left(\int_0^1 \left\| \sum_{\alpha=M+1}^{E} \left(h\right)_\alpha (\rho\,\mathrm{e}^{2\pi i\theta})^\alpha \right\|_{V,w}^2 \mathrm{d}\theta \right)^{1/2}.
\end{aligned}
\tag{6.6}
$$

We bound the term $\left\| \sum_{\alpha=M+1}^{E} \left(h\right)_\alpha (\rho\,\mathrm{e}^{2\pi i\theta})^\alpha \right\|_{V,w}$ as follows:

$$
\begin{aligned}
&\left\| \sum_{\alpha=M+1}^{E} \left(h\right)_\alpha (\rho\,\mathrm{e}^{2\pi i\theta})^\alpha \right\|_{V,w} = \left\| \sum_{\alpha=M+1}^{E} \frac{1}{2\pi i} \int_{\partial\mathcal{B}(0,R)} \frac{h(\xi)}{\xi^{\alpha+1}} (\rho\,\mathrm{e}^{2\pi i\theta})^\alpha d\xi \right\|_{V,w} \\
&= \left\| \sum_{\alpha=M+1}^{E} \int_0^1 \frac{h(R\,\mathrm{e}^{2\pi i\omega})}{(R\,\mathrm{e}^{2\pi i\omega})^\alpha} (\rho\,\mathrm{e}^{2\pi i\theta})^\alpha \mathrm{d}\omega \right\|_{V,w} \leq \sum_{\alpha=M+1}^{E} \left(\frac{\rho}{R}\right)^\alpha \int_0^1 \left\| h(R\,\mathrm{e}^{2\pi i\omega}) \right\|_{V,w} \mathrm{d}\omega \\
&= \frac{1 - \left(\frac{\rho}{R}\right)^{E-M}}{1 - \frac{\rho}{R}} \left(\frac{\rho}{R}\right)^{M+1} \sup_{z \in \partial\mathcal{B}(0,R)} \left\| h(z) \right\|_{V,w}.
\end{aligned}
\tag{6.7}
$$

Using the estimate (6.7) inside (6.6), we have

$$
j_{E,\rho}(h_M, g) \leq \frac{1 - \left(\frac{\rho}{R}\right)^{E-M}}{1 - \frac{\rho}{R}} \left(\frac{\rho}{R}\right)^{M+1} \sup_{z \in \partial\mathcal{B}(0,R)} \left\| h(z) \right\|_{V,w}.
$$

Hence, thanks to (6.5), we conclude

$$\left\|\left(H^M\right)_\alpha\right\|_{V,w} \le \frac{1-\left(\frac{\rho}{R}\right)^{E-M}}{1-\frac{\rho}{R}} \sup_{z\in\partial\mathcal{B}(0,R)} \|h(z)\|_{V,w} \sup_{z\in\partial\mathcal{B}(0,\rho)} |g(z)| \frac{1}{\rho^\alpha}\left(\frac{\rho}{R}\right)^{M+1}. \tag{6.8}$$

Putting together the bounds (6.4) and (6.8) we have obtained for $\left\|\left(H^M(z)\right)_\alpha\right\|_{V,w}$ in the cases $0\le\alpha\le E$ and $\alpha > E$, respectively, we get

$$\begin{aligned}\left\|H^M(z)\right\|_{V,w} \le & c_1 \sup_{z\in\partial\mathcal{B}(0,R)} \|h(z)\|_{V,w} \sup_{z\in\partial\mathcal{B}(0,\rho)} |g(z)| \left(\frac{\rho}{R}\right)^{M+1} \sum_{\alpha=0}^{E}\left(\frac{|z|}{\rho}\right)^\alpha \\ & + \sup_{z\in\partial\mathcal{B}(0,R)} \left\|Q^N(z)h(z)\right\|_{V,w} \sum_{\alpha>E}\left(\frac{|z|}{R}\right)^\alpha,\end{aligned} \tag{6.9}$$

with $c_1 = \frac{1-\left(\frac{\rho}{R}\right)^{E-M}}{1-\frac{\rho}{R}}$. Observe that

$$\sum_{\alpha=0}^{E}\left(\frac{|z|}{\rho}\right)^\alpha = \frac{1-\left(\frac{|z|}{\rho}\right)^{E+1}}{1-\frac{|z|}{\rho}},$$

and, using $|z|<\rho$,

$$\sum_{\alpha>E}\left(\frac{|z|}{R}\right)^\alpha = \sum_{\alpha=0}^{\infty}\left(\frac{|z|}{R}\right)^\alpha - \sum_{\alpha=0}^{E}\left(\frac{|z|}{R}\right)^\alpha = \frac{1}{1-\frac{|z|}{R}}\left(\frac{|z|}{R}\right)^{E+1} \le \frac{1}{1-\frac{|z|}{R}}\left(\frac{\rho}{R}\right)^{E+1}.$$

Therefore, the bound (6.9) gives

$$\begin{aligned}\left\|H^M(z)\right\|_{V,w} \le & c_1 \frac{1-\left(\frac{|z|}{\rho}\right)^{E+1}}{1-\frac{|z|}{\rho}} \sup_{z\in\partial\mathcal{B}(0,R)} \|h(z)\|_{V,w} \sup_{z\in\partial\mathcal{B}(0,\rho)} |g(z)| \left(\frac{\rho}{R}\right)^{M+1} \\ & + \frac{1}{1-\frac{|z|}{R}} \sup_{z\in\partial\mathcal{B}(0,R)} \left\|Q^N(z)h(z)\right\|_{V,w} \left(\frac{\rho}{R}\right)^{E+1}.\end{aligned} \tag{6.10}$$

We bound now the three quantities $\sup_{z\in\partial\mathcal{B}(0,\rho)}|g(z)|$, $\sup_{z\in\partial\mathcal{B}(0,R)}\|h(z)\|_{V,w}$ and $\sup_{z\in\partial\mathcal{B}(0,R)}$ $\left\|Q^N(z)h(z)\right\|_{V,w}$. Since

$$|g(z)| = \left|\sum_{\alpha=0}^{N}(g)_\alpha z^\alpha\right| \le \left(\sum_{\alpha=0}^{N}\left|(g)_\alpha\right|^2\right)^{1/2}\left(\sum_{\alpha=0}^{N}|z|^{2\alpha}\right)^{1/2} = \left(\sum_{\alpha=0}^{N}|z|^{2\alpha}\right)^{1/2},$$

then

$$\sup_{z\in\partial\mathcal{B}(0,\rho)} |g(z)| \le \sup_{z\in\partial\mathcal{B}(0,\rho)} \left(\sum_{\alpha=0}^{N}|z|^{2\alpha}\right)^{1/2} = \left(\sum_{\alpha=0}^{N}\rho^{2\alpha}\right)^{1/2} = \left(\frac{1-\rho^{2(N+1)}}{1-\rho^2}\right)^{1/2} := c_1'. \tag{6.11}$$

In the same way,

$$\sup_{z\in\partial\mathcal{B}(0,R)} |g(z)| \le \left(\frac{1-R^{2(N+1)}}{1-R^2}\right)^{1/2} := c_1''. \tag{6.12}$$

In order to bound $\sup_{z\in\partial\mathcal{B}(0,R)}\|h(z)\|_{V,w}$ we recall that $\mathcal{T}(z) = \dfrac{h(z)}{g(z)}$ and thus

$$\sup_{z\in\partial\mathcal{B}(0,R)} \|h(z)\|_{V,w} \le \sup_{z\in\partial\mathcal{B}(0,R)} |g(z)| \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w} \le c_1'' \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w}. \tag{6.13}$$

Proceeding as in (6.11), we obtain that

$$\sup_{z\in\partial\mathcal{B}(0,R)} \left|Q^N(z)\right| \leq \left(\frac{1-R^{2(N+1)}}{1-R^2}\right)^{1/2} := c_2. \tag{6.14}$$

Finally,

$$\sup_{z\in\partial\mathcal{B}(0,R)} \left\|Q^N(z)h(z)\right\|_{V,w} \leq \sup_{z\in\partial\mathcal{B}(0,R)} \left|Q^N(z)\right| \sup_{z\in\partial\mathcal{B}(0,R)} \|h(z)\|_V \leq c_2\, c_1'' \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w}\,. \tag{6.15}$$

Thus, using (6.13), (6.11) and (6.15) inside (6.10), we conclude

$$\left\|H^M(z)\right\|_{V,w} \leq \left[C_1 \frac{1-\left(\frac{|z|}{\rho}\right)^{E+1}}{1-\frac{|z|}{\rho}} + C_2 \frac{1}{1-\frac{|z|}{R}}\left(\frac{\rho}{R}\right)^{E-M}\right] \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w} \left(\frac{\rho}{R}\right)^{M+1}, \tag{6.16}$$

with $C_1 = c_1\, c_1'\, c_1''$ and $C_2 = c_2\, c_1''$. The upper bound (6.2) follows from (6.16).

Proof of the limit (6.3).

Let $A \subset \mathcal{B}(0,R)$ be compact, and let ρ_A be the Hausdorff distance between $\{0\}$ and A, *i.e.*, $\rho_A := \operatorname{dist}(0,A) = \max_{z\in A}|z|$. In the case $\rho_A < \rho$, the limit (6.3) follows by (6.9) observing that

$$\sum_{\alpha=0}^{E}\left(\frac{|z|}{\rho}\right)^{\alpha} \leq \sum_{\alpha=0}^{E}\left(\frac{\rho_A}{\rho}\right)^{\alpha} = \frac{1-\left(\frac{\rho_A}{\rho}\right)^{E+1}}{1-\frac{\rho_A}{\rho}},$$

and

$$\sum_{\alpha>E}\left(\frac{|z|}{R}\right)^{\alpha} \leq \sum_{\alpha>E}\left(\frac{\rho_A}{R}\right)^{\alpha} = \frac{1}{1-\frac{\rho_A}{R}}\left(\frac{\rho_A}{R}\right)^{E+1}. \tag{6.17}$$

On the other hand, if $\rho_A > \rho$, the limit (6.3) follows by (6.9) when using (6.17) and observing that

$$\begin{aligned}
\left(\frac{\rho}{R}\right)^{M+1}\sum_{\alpha=0}^{E}\left(\frac{|z|}{\rho}\right)^{\alpha} &\leq \left(\frac{\rho}{R}\right)^{M+1}\sum_{\alpha=0}^{E}\left(\frac{\rho_A}{\rho}\right)^{\alpha} = \left(\frac{\rho_A}{R}\right)^{M+1}\sum_{\alpha=0}^{E}\left(\frac{\rho}{\rho_A}\right)^{M+1-\alpha}\\
&= \left(\frac{\rho_A}{R}\right)^{M+1}\sum_{n=M+1-E}^{M+1}\left(\frac{\rho}{\rho_A}\right)^{n}\\
&= \left(\frac{\rho_A}{R}\right)^{M+1}\left[\sum_{n=M+1-E}^{-1}\left(\frac{\rho}{\rho_A}\right)^{n} + \sum_{n=0}^{M+1}\left(\frac{\rho}{\rho_A}\right)^{n}\right]\\
&= \left(\frac{\rho_A}{R}\right)^{M+1}\left[\sum_{n=1}^{E-M-1}\left(\frac{\rho_A}{\rho}\right)^{n} + \sum_{n=0}^{M+1}\left(\frac{\rho}{\rho_A}\right)^{n}\right]\\
&\leq \left(\frac{\rho_A}{R}\right)^{M+1}\left[\sum_{n=1}^{E-M-1}\left(\frac{\rho_A}{\rho}\right)^{n} + \frac{1}{1-\frac{\rho}{\rho_A}}\right] \xrightarrow{M\to\infty} 0
\end{aligned}$$

since $E - M = N + \delta$, and thus is independent of M by assumption, and $\frac{\rho_A}{R} < 1$. Finally, the case $\rho = \rho_A$ follows easily:

$$\left(\frac{\rho}{R}\right)^{M+1}\sum_{\alpha=0}^{E}\left(\frac{|z|}{\rho}\right)^{\alpha} \leq \left(\frac{\rho}{R}\right)^{M+1}\sum_{\alpha=0}^{E}\left(\frac{\rho_A}{\rho}\right)^{\alpha} = (E+1)\left(\frac{\rho}{R}\right)^{M+1} \xrightarrow{M\to\infty} 0$$

since, by assumption, $(E+1) = (M+1) + N + \delta$, with $N + \delta$ independent of M. ☐

In the next theorem, we prove the convergence of a Padé approximant uniformly on all compact subsets of $\mathcal{B}(0,R)\setminus G$, where G is the set of all the N roots of the polynomial $g(z)$.

Theorem 6.2. *Let $G := \{z \in \mathbb{C} : g(z) = 0\}$ be the set containing the roots of $g(z)$, and assume that $G \subset \mathcal{B}(0,R)$, and $h(z) \neq 0$ $\forall z \in G$. Let $\mathcal{T}_{[M/N]}$ be as in Definition 5.3. Then,*

$$\lim_{M\to\infty} \left\|\mathcal{T}_{[M/N]}(z) - \mathcal{T}(z)\right\|_{V,w} = 0 \tag{6.18}$$

uniformly on all compact subsets of $\mathcal{B}(0,R)\setminus G$.

In particular, for any compact subset $A \subset \mathcal{B}(0,\rho)\setminus G$ there exists $M^\star$ such that, for all $M \geq M^\star$ it holds

$$\left\|\mathcal{T}_{[M/N]}(z) - \mathcal{T}(z)\right\|_{V,w} \leq C \sup_{z\in\partial\mathcal{B}(0,R)} \|\mathcal{T}(z)\|_{V,w} \left(\frac{\rho}{R}\right)^{M+1} \tag{6.19}$$

where the constant $C > 0$ depends on $\rho_A = dist(0,A)$ ($\rho_A < \rho$ by assumption), ρ, R, N and g_A, with $g_A = \min_{z\in A}|g(z)|$, but is independent of M (if $\rho_A \to \rho$, $C = \mathcal{O}(M)$).

Proof. The proof of Theorem 6.2 is the generalization of the proof of Theorem 3.1 in [14].

Let $(P^M, Q^N) \in (\mathbb{P}_M(\mathbb{C};V), \mathbb{P}_N(\mathbb{C}))$ be as in Lemma 6.1. Observe that both P^M and Q^N depend on M. To emphasize this dependence, along this proof we denote Q^N as Q^N_M. The proof is based on two steps. We first prove that the sequence of Padé denominators $\{Q^N_M\}_M$ converges to the polynomial g uniformly on all compact subsets of $\mathbb{C}$. Then we prove the error bound (6.19).

Convergence of the Padé denominator. The sequence $\{Q^N_M\}_M$ is bounded in the finite dimensional space $\mathbb{P}_N(\mathbb{C})$ endowed with the norm $\|P\| = (\sum_{\alpha=0}^N |P_\alpha|^2)^{1/2}$, since, by construction, $\left\|Q^N_M\right\| = 1$ for all M. Consider now an arbitrary subsequence $\{Q^N_{M_j}\}_{M_j}$ which converges to a polynomial $Q \in \mathbb{P}_N(\mathbb{C})$, *i.e.*, $\left\|Q^N_{M_j} - Q\right\| \xrightarrow{M_j\to\infty} 0$. The convergence in the norm $\|\cdot\|$ implies the uniform convergence on all compact subsets of $\mathbb{C}$. Indeed, for any compact subset $A \subset \mathbb{C}$, it holds

$$\begin{aligned}
\max_{z\in A}\left|(Q^N_{M_j} - Q)(z)\right| &= \max_{z\in A}\left|\sum_{\alpha=0}^N \left(Q^N_{M_j} - Q\right)_\alpha z^\alpha\right| \leq \max_{z\in A}\sum_{\alpha=0}^N \left|\left(Q^N_{M_j} - Q\right)_\alpha\right| |z|^\alpha \\
&\leq \max_{z\in A}\left[\left(\sum_{\alpha=0}^N \left|\left(Q^N_{M_j} - Q\right)_\alpha\right|^2\right)^{1/2}\left(\sum_{\alpha=0}^N |z|^{2\alpha}\right)^{1/2}\right] = \left\|Q^N_{M_j} - Q\right\| \max_{z\in A}\left(\sum_{\alpha=0}^N |z|^{2\alpha}\right)^{1/2}.
\end{aligned}$$

Therefore, $\left\|Q^N_{M_j} - Q\right\| \xrightarrow{M_j\to\infty} 0$ implies $\max_{z\in A}\left|(Q^N_{M_j} - Q)(z)\right| \xrightarrow{M_j\to\infty} 0$.

We prove that $Q = g$. Fix $z_0 \in G$. Using formula (6.1), we have

$$H^{M_j}(z_0) := Q^N_{M_j}(z_0)\mathcal{T}(z_0)g(z_0) - P^{M_j}(z_0)g(z_0) = Q^N_{M_j}(z_0)h(z_0).$$

Thanks to Lemma 6.1, $\lim_{M_j\to\infty}\left\|H^{M_j}(z_0)\right\|_{V,w} = 0$. Hence,

$$\begin{aligned}
0 = \lim_{M_j\to\infty}\left\|H^{M_j}(z_0)\right\|_{V,w} &= \lim_{M_j\to\infty}\left\|Q^N_{M_j}(z_0)h(z_0)\right\|_{V,w} \\
&= \lim_{M_j\to\infty}\left|Q^N_{M_j}(z_0)\right| \|h(z_0)\|_{V,w} = |Q(z_0)|\, \|h(z_0)\|_{V,w}.
\end{aligned}$$

Since, by assumption, $h(z_0) \neq 0$, then $Q(z_0) = 0$. This is true for any $z_0 \in G$; therefore $Q \in \mathbb{P}_N(\mathbb{C})$ has the same N roots as g, and thus $Q = g$.

We have proved that any convergent subsequence of $\{Q^N_M\}_M$ converges to g in the $\|\cdot\|$ norm and thus uniformly in all compact subsets of $\mathbb{C}$. It follows that $\{Q^N_M\}_M$ itself converges to g in the $\|\cdot\|$ norm and thus uniformly in all compact subsets of $\mathbb{C}$.

Error bound. Let $A \subset \mathcal{B}(0,\rho) \setminus G$ be compact, and define

$$g_A := \min_{z \in A} |g(z)| . \tag{6.20}$$

Since the sequence $\{Q_M^N\}_M$ converges to g uniformly on all compact subsets of $\mathbb{C}$, there exists M_{g_A} such that, for all $M \geq M_{g_A}$, $\sup_{z\in A} \left|(Q_M^N - g)(z)\right| \leq \frac{g_A}{2}$. Then, for any $z \in A$, it holds

$$|g(z)| \leq \left|g(z) - Q_M^N(z)\right| + \left|Q_M^N(z)\right| \leq \frac{g_A}{2} + \left|Q_M^N(z)\right| ,$$

which implies

$$\left|Q_M^N(z)\right| \geq |g(z)| - \frac{g_A}{2} \geq \frac{g_A}{2}. \tag{6.21}$$

For any fixed $z \in A$, it holds

$$\begin{aligned}
&\left\|\mathcal{T}(z) - \mathcal{T}_{[M/N]}(z)\right\|_{V,w} \\
&= \left\|\left(Q_M^N(z)\mathcal{T}(z) - P^M(z)\right)\frac{1}{Q_M^N(z)}\right\|_{V,w} = \frac{1}{\left|Q_M^N(z)\right|}\left\|Q_M^N(z)\mathcal{T}(z) - P^M(z)\right\|_{V,w} \\
&= \frac{1}{\left|Q_M^N(z)\right|}\frac{1}{|g(z)|}\left\|\left(Q_M^N(z)\mathcal{T}(z) - P^M(z)\right)g(z)\right\|_{V,w} = \frac{1}{\left|Q_M^N(z)\right|}\frac{1}{|g(z)|}\left\|H^M(z)\right\|_{V,w} \\
&\leq \frac{2}{(g_A)^2}\left\|H^M(z)\right\|_{V,w} \\
&\overset{(6.16)}{\leq} \frac{2}{(g_A)^2}\left[C_1\frac{1-\left(\frac{|z|}{\rho}\right)^{E+1}}{1-\frac{|z|}{\rho}}\left(\frac{\rho}{R}\right)^{M+1} + C_2\frac{1}{1-\frac{|z|}{R}}\left(\frac{|z|}{R}\right)^{E+1}\right]\sup_{z\in\partial\mathcal{B}(0,R)}\|\mathcal{T}(z)\|_{V,w} .
\end{aligned}$$

Let $\rho_A = \operatorname{dist}(0, A)$, so that $|z| < \rho_A$. Since, by assumption, $\rho_A < \rho < R$, we obtain

$$\begin{aligned}
&\left\|\mathcal{T}(z) - \mathcal{T}_{[M/N]}(z)\right\|_{V,w} \\
&\leq \frac{2}{(g_A)^2}\left[C_1\frac{1-\left(\frac{\rho_A}{\rho}\right)^{E+1}}{1-\frac{\rho_A}{\rho}}\left(\frac{\rho}{R}\right)^{M+1} + C_2\frac{1}{1-\frac{\rho_A}{R}}\left(\frac{\rho_A}{R}\right)^{E+1}\right]\sup_{z\in\partial\mathcal{B}(0,R)}\|\mathcal{T}(z)\|_{V,w} \\
&\leq \frac{2}{(g_A)^2}\left[C_1\frac{1-\left(\frac{\rho_A}{\rho}\right)^{E+1}}{1-\frac{\rho_A}{\rho}} + C_2\frac{1}{1-\frac{\rho_A}{R}}\left(\frac{\rho_A}{R}\right)^{E-M}\right]\sup_{z\in\partial\mathcal{B}(0,R)}\|\mathcal{T}(z)\|_{V,w}\left(\frac{\rho}{R}\right)^{M+1}
\end{aligned}$$

Hence, inequality (6.19) follows with $C = \frac{2}{(g_A)^2}[C_1\frac{1-(\frac{\rho_A}{\rho})^{E+1}}{1-\frac{\rho_A}{\rho}} + C_2\frac{1}{1-\frac{\rho_A}{R}}(\frac{\rho_A}{R})^{E-M}]$. ☐

Note that, given a compact subset $A \subset \mathcal{B}(0,\rho)\backslash G$, the rate of convergence of a Padé approximation is $\left(\frac{\rho}{R}\right)^{M+1}$, with $\rho_A < \rho < R$. Therefore, it is convenient to take ρ as small as possible, provided that $A \subset \mathcal{B}(0,\rho) \setminus G$ is satisfied; see Figure 2.

7. Padé approximation of the Helmholtz equation with parametric wavenumber

In this section, we detail the results obtained in Section 6 for the Helmholtz solution map $\mathcal{S}$ defined in (3.1). In Section 4 we have shown that $\mathcal{S}$ is meromorphic and the set Λ of eigenvalues of the Laplace problem with

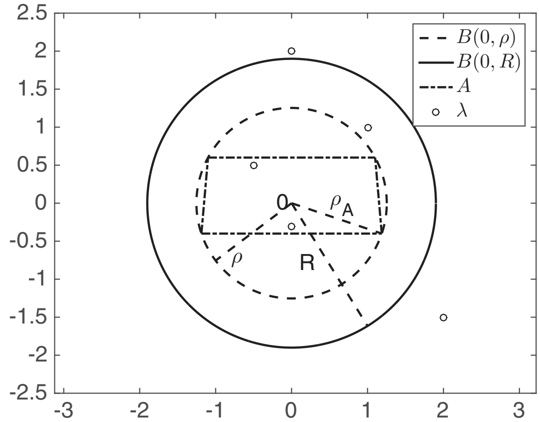


FIGURE 2. Five poles of a meromorphic mapping $\mathcal{T}$ are represented *(small circles)*. The disk $\mathcal{B}(0,R)$ *(solid line)* contains exactly three poles of $\mathcal{T}$. We consider a Padé approximant $\mathcal{T}_{[M/N]}$ with $N=3$. We are interested in approximating the map $\mathcal{T}$ inside the compact subset A. Hence, we take $\rho=\rho_A+\varepsilon$, with $0<\varepsilon\ll 1$, and we construct the disk $\mathcal{B}(0,\rho)$ *(dashed line)*. The rate of approximation of the Padé approximant is $\left(\frac{\rho}{R}\right)^{M+1}$; see Theorem 6.2.

the considered boundary conditions coincide with the set of poles of $\mathcal{S}$. Specifically, each (single or multiple) eigenvalue $\lambda\in\Lambda$ is a pole of order one of $\mathcal{S}$.

Let $K=[k^2_{\min},k^2_{\max}]\subset\mathbb{R}^+$ be the frequency interval of interest, and $z_0\in\mathbb{C}\setminus\Lambda$. To fix the ideas, we take $z_0=\frac{k^2_{\min}+k^2_{\max}}{2}+\delta i$, with $\delta\in\mathbb{R}$ arbitrary. Set $\rho=\rho_K+\varepsilon$, with $0<\varepsilon\ll 1$ and ρ_K the Hausdorff distance between $\{z_0\}$ and K, *i.e.*, $\rho_K=\operatorname{dist}(z_0,K)=\max_{z\in K}|z_0-z|$. Moreover, let $N\in\mathbb{N}$ be fixed, $M\geq N$ and $E\geq M+N$, with $M,E\in\mathbb{N}$. Denote with $\mathcal{S}_{[M/N]}:=\frac{\mathcal{P}_{[M]}}{\mathcal{Q}_{[N]}}$ a Padé approximant of the solution map $\mathcal{S}$ centered in z_0, where $\mathcal{P}_{[M]}\in\mathbb{P}_M(\mathbb{C};V)$ and $\mathcal{Q}_{[N]}\in\mathbb{P}_N(\mathbb{C})$.

Let $R\in\mathbb{R}^+$ be such that $\overline{\mathcal{B}(z_0,R)}$ contains exactly N poles of $\mathcal{S}$, $\lambda_{\ell+1},\ldots,\lambda_{\ell+N}$, and such that

$$\min_{\lambda\in\Lambda\setminus G}\left(\inf_{z\in\mathcal{B}(z_0,R)}|z-\lambda|\right)>\varepsilon,$$

where $G=\{\lambda_{\ell+1},\ldots,\lambda_{\ell+N}\}$. We depict in Figure 3 a particular situation as an example ($N=4$ left and $N=5$ right).

Since $\mathcal{S}$ is meromorphic in $\overline{\mathcal{B}(z_0,R)}$, we can write $\mathcal{S}$ as the ratio between a holomorphic map $h:\overline{\mathcal{B}(z_0,R)}\to V$ and a polynomial g of degree N, with $h(z)\neq 0$ for every z such that $g(z)=0$. Notice that $g(z)=0$ if and only if $z\in G$.

In Theorem 6.2, we have proved that, as M increases, the normalized sequence $\{\mathcal{Q}_{[N]}\}_M$ converges uniformly on all compact sets of $\mathbb{C}$ to a polynomial $Q(z)=\sum_{\alpha=0}^{N}(Q)_\alpha(z-z_0)^\alpha\in\mathbb{P}_N(\mathbb{C})$, with $\sum_{\alpha=0}^{N}\left|(Q)_\alpha\right|^2=1$. Moreover, from the proof of Theorem 6.2, we have that $Q=g$. It follows that the N roots of Q coincide with the set G.

Note that, in general, $K\cap\Lambda\subsetneq G$, *i.e.*, not all the eigenvalues we are approximating belong to the interval of interest.

Theorem 6.2 gives an upper bound on the weighted $H^1(D)$-norm $\|\cdot\|_{V,w}$ of $\mathcal{S}(z)-\mathcal{S}_{[M/N]}(z)$ for any $z\in\mathcal{B}(z_0,R)$. In this section, we choose the weight $w=\sqrt{\operatorname{Re}(z_0)}$ and deduce the following corollary.

Corollary 7.1. *Given $\alpha>0$ small enough, introduce the open subset $K_\alpha\subset K$*

$$K_\alpha:=\bigcup_{\lambda\in\Lambda\cap K}(\lambda-\alpha,\lambda+\alpha). \tag{7.1}$$

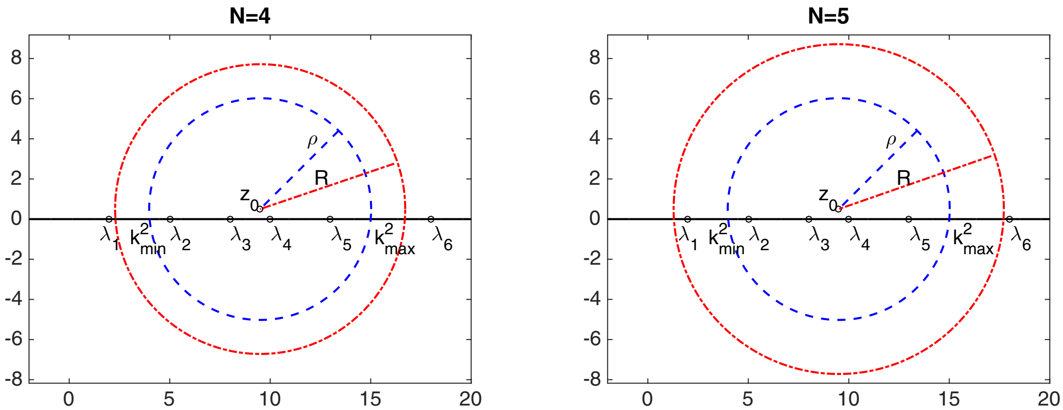


FIGURE 3. Let $K = [k^2_{\min}, k^2_{\max}] = [4, 15]$ be the interval of interest, and $z_0 = \frac{k^2_{\min}+k^2_{\max}}{2} + \frac{i}{2}$ be the center of the Padé approximation. The disk $\mathcal{B}(z_0, \rho)$ *(dashed line)* contains K. Given $N = 4$ ($N = 5$, respectively), the radius R in the picture on the left (on the right, respectively) is chosen such that $\overline{\mathcal{B}(z_0, R)}$ contains exactly four eigenvalues, *i.e.*, λ_2, λ_3, λ_4, λ_5 (five eigenvalues, *i.e.*, λ_1, λ_2, λ_3, λ_4, λ_5, respectively), and the distance to the first neglected one is $> \varepsilon$.

Moreover, let $N \in \mathbb{N}$ be fixed, and let $R \in \mathbb{R}^+$ be such that the disk $\overline{\mathcal{B}(z_0, R)}$ contains exactly N poles of $\mathcal{S}$. Then there exists $M^\star \in \mathbb{N}$ such that, for any $M \geq M^\star$ and for any $z \in K \setminus K_\alpha$, it holds

$$\left\|\mathcal{S}(z) - \mathcal{S}(z)_{[M/N]}\right\|_{V,\sqrt{\mathrm{Re}(z_0)}} \leq C\frac{1}{\alpha}\left(\frac{\rho}{R}\right)^{M+1}, \tag{7.2}$$

where $\rho_K < \rho < R$, and the constant $C > 0$ depends on ρ_K, ρ, R, N, $g_{K,\alpha} = \min_{z\in K\setminus K_\alpha} |g(z)|$, z_0, $\lambda_{\min} = \min\{\lambda \in \Lambda\}$, and $\|f\|_{L^2(D)}$.

Proof. Theorem 6.2 applied to the solution map $\mathcal{S}$ states that there exists $M^\star$ such that, for all $M \geq M^\star$ and any $z \in K \setminus K_\alpha$, it holds

$$\left\|\mathcal{S}_{[M/N]}(z) - \mathcal{S}(z)\right\|_{V,\sqrt{\mathrm{Re}(z_0)}} \leq C' \sup_{z\in\partial\mathcal{B}(z_0,R)} \|\mathcal{S}(z)\|_{V,\sqrt{\mathrm{Re}(z_0)}} \left(\frac{\rho}{R}\right)^{M+1}, \tag{7.3}$$

where $R > \rho > \rho_K$, and C' depends on ρ_K, ρ, R, N, and $g_{K,\alpha}$. Given $\lambda_{\min} = \min\{\lambda \in \Lambda\}$, Theorem 3.1 states that

$$\begin{aligned}
\sup_{z\in\partial\mathcal{B}(z_0,R)} \|\mathcal{S}(z)\|_{V,\sqrt{\mathrm{Re}(z_0)}} &\leq \sup_{z\in\partial\mathcal{B}(z_0,R)} \left(\frac{\sqrt{|z-\lambda_{\min}| + 2\mathrm{Re}\,(z_0)}}{\alpha}\|f\|_{L^2(D)}\right)\\
&= \frac{1}{\alpha}\left(\sup_{z\in\partial\mathcal{B}(z_0,R)} |z-\lambda_{\min}| + 2\mathrm{Re}\,(z_0)\right)^{1/2} \|f\|_{L^2(D)}\\
&\leq \frac{1}{\alpha}\left(\sup_{z\in\partial\mathcal{B}(z_0,R)} (|z-z_0| + |z_0-\lambda_{\min}|) + 2\mathrm{Re}\,(z_0)\right)^{1/2} \|f\|_{L^2(D)}\\
&= \frac{\sqrt{R + |z_0-\lambda_{\min}| + 2\mathrm{Re}\,(z_0)}}{\alpha}\|f\|_{L^2(D)}
\end{aligned}$$

so that we conclude (7.2) with $C = C'\sqrt{R + |z_0-\lambda_{\min}| + 2\mathrm{Re}\,(z_0)}\,\|f\|_{L^2(D)}$. $\square$

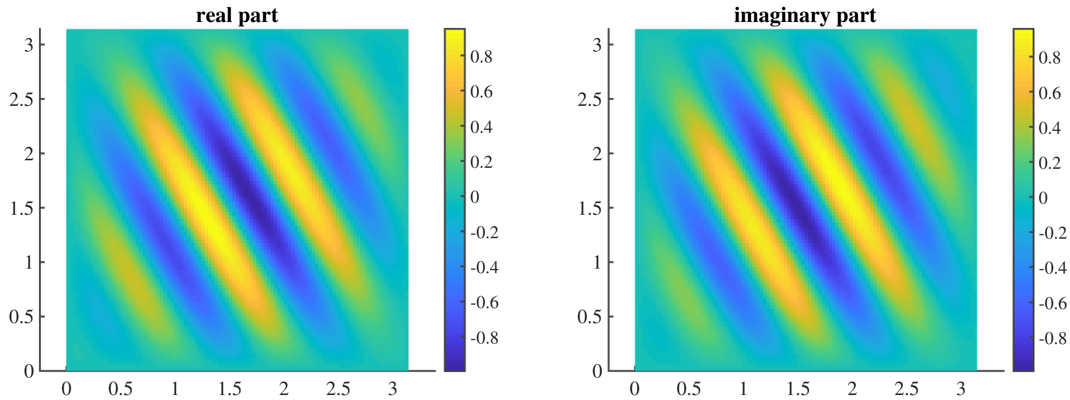


FIGURE 4. Real part *(left)* and imaginary part *(right)* of $u(\mathbf{x}) = \frac{16}{\pi^4} x_1 x_2 (x_1 - \pi)(x_2 - \pi) \mathrm{e}^{-i\nu \mathbf{d}\cdot\mathbf{x}}$ with $\nu = 7$ and $\mathbf{d} = (\cos(\pi/6), \sin(\pi/6))$.

Remark 7.2. Note that, for R large enough, the disk $\mathcal{B}(z_0, R)$ may contain complex numbers z with $\mathrm{Re}\,(z) < 0$. Since the result of Theorem 3.1 is valid also for wavenumbers with negative real part, the bound (7.2) holds true with no additional restrictions.

8. Numerical results

We present here some numerical results aimed at verifying the error estimates of the Padé approximation for the solution map $\mathcal{S}$ proved in Theorem 7.1. As the focus of the present paper is on the approximation properties, we omit the algorithmical details of the Padé approximation construction.

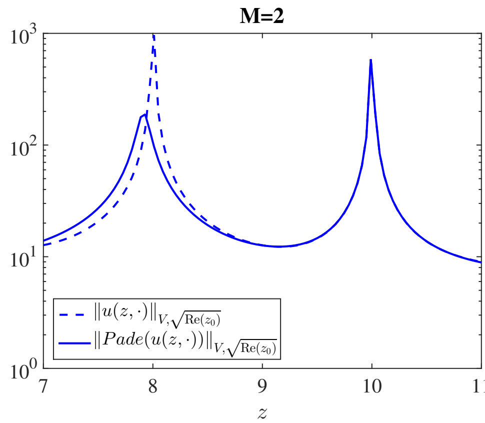
Consider the two-dimensional domain $D = (0, \pi) \times (0, \pi)$. Let $\nu^2 \in \mathbb{R}^+ \setminus \Lambda$ and $\mathbf{d} = (d_1, d_2) \in \mathbb{R}^2$ be fixed. We set $u(\mathbf{x}) = v(\mathbf{x}) w(\mathbf{x})$, where $v(\mathbf{x}) = \mathrm{e}^{-i\nu\mathbf{d}\cdot\mathbf{x}}$, the plane wave traveling along the direction $\mathbf{d}$ with wavenumber ν^2, and $w(\mathbf{x}) = \frac{16}{\pi^4} x_1 x_2 (x_1 - \pi)(x_2 - \pi)$, a bubble function vanishing on ∂D (see Fig. 4). We define $f(\mathbf{x}) = -\Delta u(\mathbf{x}) - \nu^2 u(\mathbf{x})$, *i.e.*,

$$\begin{aligned} f(\mathbf{x}) = f(x_1, x_2) = \frac{16}{\pi^4} \mathrm{e}^{-i\nu\mathbf{d}\cdot\mathbf{x}} \big[& 2i\nu d_1 \left(2x_1 x_2^2 - 2\pi x_1 x_2 - \pi x_2^2 + \pi^2 x_2 \right) \\ & + 2i\nu d_2 \left(2x_1^2 x_2 - \pi x_1^2 - 2\pi x_1 x_2 + \pi^2 x_1 \right) \\ & - \left(2x_2^2 - 2\pi x_1 x_2 + 2x_1^2 - 2\pi x_1 \right) \big].) \end{aligned} \tag{8.1}$$

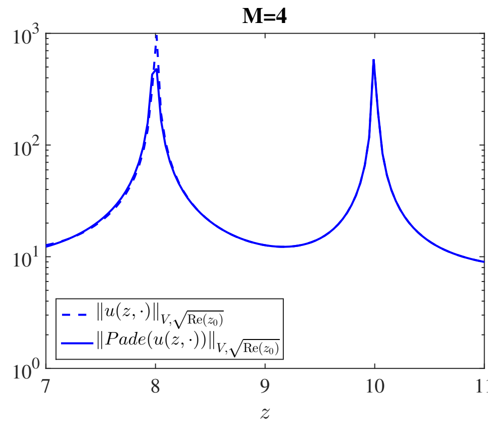
In the following tests, we consider the Helmholtz problem (3.2) in D with homogeneous Dirichlet boundary conditions on ∂D, and loading term (8.1) with $\mathbf{d} = (\cos(\pi/6), \sin(\pi/6))$ and $\nu^2 = 12$.

In the first test, we choose as frequency interval of interest $K = [k^2_{\min}, k^2_{\max}] = [7, 11]$, which contains two eigenvalues of the Laplace problem with the considered boundary conditions: $\lambda = 8$ (multiplicity one), and $\lambda = 10$ (multiplicity two), *i.e.*, two simple poles of the solution map $\mathcal{S}$.

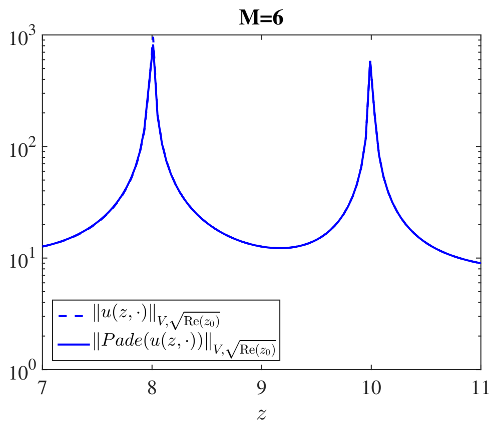
Given N equal to the number of eigenvalues in K, *i.e.*, $N = 2$, we construct a Padé approximation $\mathcal{S}_{[M/N]}(z) =: u_P(z, \cdot)$ centered in $z_0 = 10 + 0.5i$. We partition the interval of interest K uniformly into 100 subintervals. At each point z of the mesh, the numerical solution $u_h(z, \cdot) \in H^1_0(D)$ of the Helmholtz problem is computed *via* the $\mathbb{P}^3$ continuous finite element method (FEM), and its weighted $H^1(D)$-norm $\|u_h(z, \cdot)\|_{V, \sqrt{\mathrm{Re}(z_0)}}$ is calculated. In Figure 5 the norm $\|u_h(z, \cdot)\|_{V, \sqrt{\mathrm{Re}(z_0)}}$ (dashed line) is compared with the norm $\|u_{P,h}(z, \cdot)\|_{V, \sqrt{\mathrm{Re}(z_0)}}$ (solid line), $u_{P,h}(z, \cdot)$ being a Padé approximation with denominator of degree $N = 2$



(a)



(b)



(c)

FIGURE 5. Comparison between $\|u_h(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ *(dashed line)* and $\|u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ *(solid line)*, where $u_h(z,\cdot)$ is the numerical solution to the considered Helmholtz problem computed *via* the $\mathbb{P}^3$ continuous FEM, and $u_{P,h}(z,\cdot)$ is a Padé approximant of u centered in $z_0 = 10 + 0.5i$, evaluated in $z \in K = [7, 11]$, and of degrees $N = 2$ (denominator), and $M = 2$ (a), $M = 4$ (b) and $M = 6$ (c).

and numerator of degree $M = 2$ (Fig. 5a), $M = 4$ (Fig. 5b) and $M = 6$ (Fig. 5c). As M increases, the Padé approximation becomes more accurate.

For the second test, we consider the interval of interest $K = [k^2_{\min}, k^2_{\max}] = [14, 19]$, which contains two eigenvalues of the Laplace problem with the considered boundary conditions: $\lambda = 17$ (multiplicity two), and $\lambda = 18$ (multiplicity one). Again with $N = 2$, we construct the Padé approximation $u_{P,h}(z,\cdot)$ centered in $z_0 = 16.5 + 0.5i$. In Figure 6, we plot the error $\|u_h(z,\cdot) - u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ as a function of the degree of the Padé numerator M, where $u_h(z,\cdot)$ is the solution of the Helmholtz problem computed *via* the $\mathbb{P}^3$ continuous FEM, and $z = 17.5$ ($z = 14$) in Figure 6, *left (right,* respectively). The error (solid line) is compared with the predicted convergence rate $\left(\frac{\rho}{R}\right)^{M+1}$ *(dashed line)* proved in Corollary 7.1. Here, $\rho = |z - z_0|$ and $R > 0$ is such that the disk $\mathcal{B}(z_0, R)$ contains exactly $N = 2$ poles of the solution map $\mathcal{S}$. Specifically, $\rho = |16.5 + 0.5i - 17.5|$ and $R = |16.5 + 0.5i - 20| = |16.5 + 0.5i - 13|$ in Figure 6, *left*, and $\rho = |16.5 + 0.5i - 14|$ and $R = |16.5 + 0.5i - 20| = |16.5 + 0.5i - 13|$ in Figure 6, *right*. Note that $\lambda = 13$

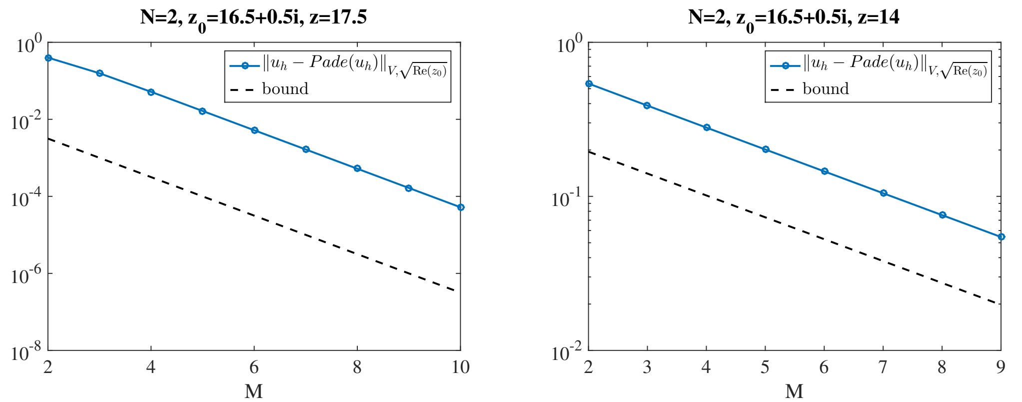


FIGURE 6. Comparison between the computed error $\|u_h(z,\cdot) - u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ *(solid line)* and the predicted slope of convergence $\left(\frac{\rho}{R}\right)^{M+1}$ *(dashed line)* proved in Corollary 7.1. Here $K = [14, 19]$, and $z_0 = 16.5 + 0.5i$. Moreover, $\rho = |16.5 + 0.5i - 17.5|$ ($\rho = |16.5 + 0.5i - 14|$, respectively) and $R = |16.5 + 0.5i - 20| = |16.5 + 0.5i - 13|$ in the *left* (*right*, respectively) picture.

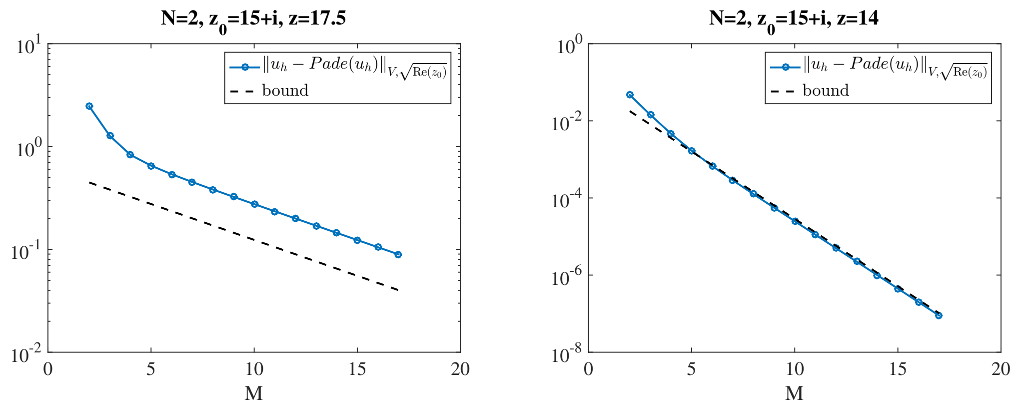


FIGURE 7. Comparison between the computed error $\|u_h(z,\cdot) - u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ *(solid line)* and the predicted slope of convergence $\left(\frac{\rho}{R}\right)^{M+1}$ *(dashed line)* proved in Corollary 7.1. Here $K = [12.5, 17.5]$, and $z_0 = 15 + i$. Moreover, $\rho = |15 + i - 17.5|$ ($\rho = |15 + i - 14|$, respectively) and $R = |15 + i - 18|$ in *left* (*right*, respectively) picture.

and $\lambda = 20$ are the closest eigenvalues of the considered Laplace problem outside the interval of interest. The predicted slope of convergence $(\frac{\rho}{R})^{M+1}$ is then numerically confirmed.

The same quantities are represented in Figure 7, where the interval of interest is $K = [12.5, 17.5]$, the center of the Padé approximation is $z_0 = 15+i$, $\rho = |15 + i - 17.5|$ and $R = |15 + i - 18|$ in Figure 7, *left*, $\rho = |15 + i - 14|$ and $R = |15 + i - 18|$ in Figure 7, *right*.

Remark 8.1. Let $z \in K \setminus K_\alpha$ be a fixed wavenumber. According to the theoretical results in Section 5, the Padé approximant $\mathcal{S}_{[M/N]}$ in z is defined through the minimization of the functional $j_{E,\rho}$ (formula (5.4)) with the choice $\rho = |z_0 - z|$, z_0 being the center of the Padé approximation. Corollary 7.1 states then that the slope of convergence of the Padé approximation error $\left\|\mathcal{S}(z) - \mathcal{S}_{[M/N]}(z)\right\|_{V,\sqrt{\mathrm{Re}(z_0)}}$ is $(\frac{|z_0-z|}{R})^{M+1}$. On the other hand,

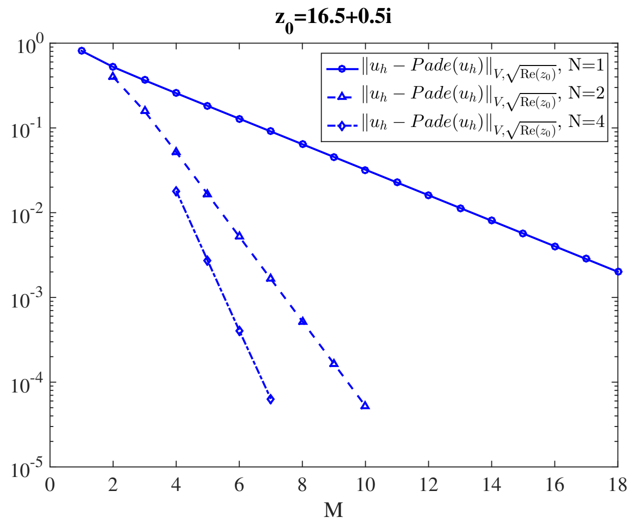


FIGURE 8. Computed error $\|u_h(z,\cdot) - u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$, where $u_{P,h}(z,\cdot)$ is the Padé approximant centered in $z_0 = 16.5 + 0.5i$ with denominator degrees $N = 1, 2, 4$.

the numerical experiments we have run show that, even with the choice $\rho = \rho_K$ in the construction of the functional $j_{E,\rho}$, the same slope $(\frac{|z_0-z|}{R})^{M+1}$ of the Padé error is observed.

In the last test, we take again $K = [14, 19]$ and compare the error $\|u_h(z,\cdot) - u_{P,h}(z,\cdot)\|_{V,\sqrt{\mathrm{Re}(z_0)}}$, where $u_{P,h}(z,\cdot)$ is the Padé approximant centered in $z_0 = 16.5 + 0.5i$ with denominator degrees $N = 1, 2, 4$. As N increases, similar accuracy is reached with a smaller degree of the Padé numerator M; see Figure 8.

9. Conclusions

In the present paper, we have considered Hilbert space-valued rational Padé approximations of the Helmholtz solution map $\mathcal{S}$ which associates with a given wavenumber the corresponding Helmholtz solution. We have focused on Padé expansions in the least squares sense around a single complex frequency (*single-point* Padé expansion) close to a (real) frequency interval of interest.

For meromorphic Hilbert space-valued univariate maps, a uniform convergence result, on any compact subset of the interval of interest that excludes any pole, has been proved for the Padé approximation error. Error estimates have been derived in a funcional space norm. Numerical results for a two-dimensional problem confirm the theoretical upper bound on the Padé approximation error for the Helmholtz solution map.

The description of the algorithmic aspects of the least squares Padé expansion will be carried out in a forthcoming paper, where we will also apply it to the stochastic Helmholtz problem, *i.e.*, the Helmholtz problem where the wavenumber is modeled as a random variable. We are currently investigating the extension of the methodology and of its convergence analysis to the case of *multi-point* Padé expansions, where moments are identified at multiple frequencies.

The proposed least squares Padé approximant delivers an approximation of the solution map $\mathcal{S}(z)$ in the linear space spanned by $\{\mathcal{S}(z_0), \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}z}(z_0), \ldots, \frac{\mathrm{d}^E\mathcal{S}}{\mathrm{d}z^E}(z_0)\}$. As such, the error estimate derived in this paper provides an upper bound of the Kolmogorov n-width for the solution map of the Helmholtz problem and can therefore be useful also to analyze the convergence of a reduced basis method (see *e.g.* [5, 7, 21]).

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