



On a Cahn–Hilliard–Keller–Segel model with generalized logistic source describing tumor growth

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Abstract

We propose a new type of diffuse interface model describing the evolution of a tumor mass under the effects of a chemical substance (e.g., a nutrient or a drug). The process is described by utilizing the variables φ , an order parameter representing the local proportion of tumor cells, and σ , representing the concentration of the chemical. The order parameter φ is assumed to satisfy a suitable form of the Cahn–Hilliard equation with mass source and logarithmic potential of Flory–Huggins type (or generalizations of it). The chemical concentration σ satisfies a reaction–diffusion equation where the cross-diffusion term has the same expression as in the celebrated Keller–Segel model. In this respect, the model we propose represents a new coupling between the Cahn–Hilliard equation and a subsystem of the Keller–Segel model. We believe that, compared to other models, this choice is more effective in capturing the chemotactic effects that may occur in tumor growth dynamics (chemically induced tumor evolution and consumption of nutrient/drug by tumor cells). Note that, in order to prevent finite time blowup of σ , we assume a chemical source term of logistic type. Our main mathematical result is devoted to proving existence of weak solutions in a rather general setting that covers both the two- and three- dimensional cases. Under more restrictive assumptions on coefficient and data, and in some cases on the spatial dimension, we prove various regularity results. Finally, in a proper class of smooth solutions we show uniqueness and continuous dependence on the initial data in a number of significant cases.

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1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a smooth and bounded domain, and let $T > 0$ be an assigned final time. In this paper, we consider the following Cahn–Hilliard–Keller–Segel (CHKS) model aimed at describing some classes of tumor growth processes:

$$\varphi_t - \operatorname{div}(\mathfrak{m}(\varphi, \sigma)\nabla\mu) = S(\varphi, \sigma) \quad \text{in } Q := \Omega \times (0, T), \tag{1.1}$$

$$\mu = -\varepsilon\Delta\varphi + \varepsilon^{-1}f(\varphi) - \chi\sigma \quad \text{in } Q, \tag{1.2}$$

$$\sigma_t - \operatorname{div}(\sigma\mathfrak{m}(\varphi, \sigma)\nabla(\ln\sigma + \chi(1 - \varphi))) = b(\varphi, \sigma) \quad \text{in } Q, \tag{1.3}$$

$$\partial_n\varphi = (\mathfrak{m}(\varphi, \sigma)\nabla\mu) \cdot \mathbf{n} = (\sigma\mathfrak{m}(\varphi, \sigma)\nabla(\ln\sigma + \chi(1 - \varphi))) \cdot \mathbf{n} = 0 \quad \text{on } \Sigma := \partial\Omega \times (0, T), \tag{1.4}$$

$$\varphi|_{t=0} = \varphi_0, \quad \sigma|_{t=0} = \sigma_0 \quad \text{in } \Omega. \tag{1.5}$$

Equations (1.1)–(1.2) correspond to a generalized version of the Cahn–Hilliard (CH) system with mass source for the two unknown variables φ and μ . Here, φ denotes an order parameter, or phase-field, representing the difference between the tumor cells and healthy cells volume fractions, and is normalized in such a way that, at least in principle, the level sets $\{\varphi = 1\} := \{x \in \Omega : \varphi(x) = 1\}$ and $\{\varphi = -1\}$ describe the regions occupied by the pure (“tumor” and “healthy”) phases, respectively. These regions are separated by a narrow transition layer of thickness scaling as $\varepsilon \in (0, 1)$, in which $\{-1 < \varphi < 1\}$. As we will specify below, the fact that φ takes value in the reference interval $[-1, 1]$ is enforced by the occurrence of the function f in (1.2), which represents the derivative of what, in the Cahn–Hilliard terminology, is generally noted as a “singular (configuration) potential”. The variable μ is an auxiliary quantity denoting the chemical potential of the phase separation process. Since in tumor growth processes the total mass of the tumor is not conserved, we also assume the occurrence of a volumic source term S on the right-hand side of (1.1). We shall comment on the precise expression of S later on.

The Cahn–Hilliard system (1.1)–(1.2) (cf. [6]) is coupled with the reaction-diffusion equation (1.3) describing the effects of a chemical substance on the evolution of the tumor. This may be a nutrient like oxygen or glucose which constitutes the primary source of nourishment for the tumor cells, as well as a drug or a medicine preventing the tumor to grow. In either case, the concentration of such a substance is represented by the variable σ . We shall extensively comment below on the expression of equation (1.3). The functions $\mathfrak{m}(\varphi, \sigma)$ and $\mathfrak{m}(\varphi, \sigma)$ in (1.1) and (1.3) are nonnegative mobility functions related to the phase-field and the nutrient concentration, respectively. The system is complemented with the Cauchy conditions (1.5) and with the no-flux (i.e., homogeneous Neumann) boundary conditions (1.4), where \mathbf{n} is the outer unit normal vector to $\partial\Omega$.

Diffuse interface models for tumor growth are now receiving a notable attention among the scientific community and the recent mathematical literature is very vast (we may quote, with no claim of completeness [7–9,1,10,12,13,18,19,21,27–31,23,22,46], see also the references

therein). Actually, most of the models considered in these papers turn out to couple a Cahn–Hilliard relation for the tumor cell proportion (which may be of multi-phase type if more than two types of cells are considered, cf., e.g., [17,32,20]) with other equations describing the behavior of further significant quantities, like nutrient concentration (as in our case), macroscopic velocity, or even temperature [33].

Compared to previous tumor growth models of the same type (i.e., based on the coupling of the Cahn–Hilliard system with a reaction-diffusion equation), the main novelty in our system (1.1)–(1.5) is represented by the expression of the reaction-diffusion equation (1.3), which is also what led us to use the terminology “Cahn–Hilliard–Keller–Segel model”. In this direction, we are aware of the recent contribution [15], where a connection between a generalized form of the Keller–Segel system and a relaxed version of the Cahn–Hilliard system is rigorously shown through a suitable limiting procedure). In a sense, the biological effect we would like to represent is *chemotaxis*, basically corresponding to the active movement, in a biological sense, of the tumor cells towards regions of high nutrient concentration. Considering for simplicity the case of a constant mobility $\mathfrak{m} \equiv 1$, in previous models (see, e.g., [21]), this “active transport” effect was described utilizing a relation of the form

$$\sigma_t - \Delta\sigma + \chi \Delta\varphi = b(\varphi, \sigma), \tag{1.6}$$

where b is, as in our case, a volumic nutrient source. However, relation (1.6), which is mathematically simpler compared to (1.3), in our view seems to present several drawbacks from a modeling perspective. First of all, in view of the fact that the term $\chi \Delta\varphi$ has no sign properties, (1.6) does not obey the minimum principle; hence, one cannot exclude, at least in principle, that the variable σ might somewhere assume strictly negative values conflicting with the physical interpretation of σ as a concentration. A further issue can be observed if one integrates (1.6) on a reference volume $V \subset \Omega$. Indeed, applying the Gauss–Green formula, one then obtains

$$\frac{d}{dt} \int_V \sigma = \int_{\partial V} \partial_n \sigma + \int_V b(\varphi, \sigma) - \chi \int_{\partial V} \partial_n \varphi, \tag{1.7}$$

and we may notice that the last integral prescribes that the variation of σ in V depends on the flux of tumor cells across ∂V , *independently* of the value of σ . For instance, if many tumor cells ($\varphi \sim 1$) are present outside V and fewer ones ($\varphi \sim -1$) occur inside V (so that $\partial_n \varphi$ is positive), then there is a nutrient flux from the inside to the outside of V , but this flux is in fact independent of the actual nutrient concentration.

On the other hand, if (1.6) is replaced by our (1.3), then (still in the case $\mathfrak{m} \equiv 1$), (1.7) assumes the different form

$$\frac{d}{dt} \int_V \sigma = \int_{\partial V} \partial_n \sigma + \int_V b(\varphi, \sigma) - \chi \int_{\partial V} \sigma \partial_n \varphi, \tag{1.8}$$

where, as physically expected, the nutrient flux across ∂V driven by consumption by tumor cells is proportional to the actual value of σ : the more nutrient is present, the more it flows away. This is, indeed, the main reason that led us to consider the present expression for the equation (1.3).

It is clear that the above choice, corresponding in the constant mobility case to the equation

$$\sigma_t - \Delta\sigma + \chi \operatorname{div}(\sigma \nabla \varphi) = b(\varphi, \sigma), \tag{1.9}$$

gives rise to a number of mathematical complications mainly due to the quadratic behavior of the cross-diffusion term. This is, indeed, one of the main sources of difficulty in the mathematical analysis of the Keller–Segel (KS) model [35]. Despite the vastness of the mathematical literature dealing with the KS model (cf., e.g., [5,11,34,48–50]), it is worth noting that, up to our knowledge, this is the first paper where the coupling between a “Keller–Segel-like” expression of the form (1.3) (or (1.9)) with the Cahn–Hilliard system is considered. From a modeling perspective, while in true Keller–Segel models, a relation like (1.3) is combined with a *second order* reaction-diffusion equation describing the evolution of a further *concentration*, in the present coupling, relation (1.3) is coupled with a *fourth order* equation describing the evolution of a *proportion*, i.e., of a *normalized* variable, the order parameter φ . This new type of coupling has some implications both on the regularity of solutions and on the mathematical techniques we use to address the system. For instance, we may notice that, compared to the case when the coupling variable φ satisfies a second order relation (like in the true KS model), here φ enjoys *more regularity* in space, but *less regularity* in time. This leads to some modifications of the regularity scenario and of the expected properties of solutions compared to the standard KS case.

It is worth noting that, as also happens in the KS model, the regularity obtained by the a-priori estimate corresponding to the energy balance principle (the variational formulation of the model starting from the free energy balance is presented below) seems not sufficient to prevent finite time blowup of the solution, unless the mass source term b in (1.3) is suitably designed. In particular, as is habitual in the Keller–Segel context, we have to assume b to present a “generalized logistic growth” property (see the next section for the precise assumption); namely, it goes like σ for $\sigma \sim 0$ (so to preserve the minimum principle), while it behaves as $-\sigma^p$ (for suitable $p > 1$, with the reference case given by $p = 2$ corresponding to a “true” logistic growth) for large σ (see [26,48,49] for examples of Keller–Segel models with logistic growth). With this choice, relation (1.8) prescribes that, if the nutrient concentration is high, then there occurs a volumic effect leading it to decrease. We believe this property be biologically reasonable, in addition to being probably unavoidable mathematically.

As anticipated above, system (1.1)–(1.5) could be variationally derived from the free energy functional

$$\mathcal{F}(\varphi, \sigma) = \underbrace{\frac{\varepsilon}{2} \int_{\Omega} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \int_{\Omega} F(\varphi)}_{=: \mathcal{E}(\varphi)} + \underbrace{\int_{\Omega} (\sigma (\ln \sigma - 1) + \chi \sigma (1 - \varphi))}_{=: \mathcal{M}(\varphi, \sigma)}, \tag{1.10}$$

where F is an antiderivative of f . In particular, equation (1.1) is obtained as a balance law by setting

$$\varphi_t + \operatorname{div} \mathbf{J}_\varphi = S(\varphi, \sigma),$$

where, as is typical for the Cahn–Hilliard equation, the flux \mathbf{J}_φ is prescribed as $\mathbf{J}_\varphi = -\mathfrak{m}(\varphi, \sigma) \nabla \mu$ for a mobility function $\mathfrak{m}(\varphi, \sigma)$, and where the chemical potential μ is defined as the variational derivative of the free energy with respect to the order parameter, namely $\mu := \delta \mathcal{F} / \delta \varphi$. Note that also equation (1.3) can be obtained as a balance law for the nutrient flux \mathbf{J}_σ , i.e.,

$$\sigma_t + \operatorname{div} \mathbf{J}_\sigma = b(\varphi, \sigma), \quad \text{with } \mathbf{J}_\sigma := -\sigma \mathfrak{m}(\varphi, \sigma) \nabla \mu_\sigma, \quad \mu_\sigma := \frac{\delta \mathcal{F}}{\delta \sigma} = \frac{\delta \mathcal{M}}{\delta \sigma} = \ln \sigma + \chi(1 - \varphi),$$

where the mobility function has the expression $\sigma_{\mathfrak{m}}(\varphi, \sigma)$, hence, in particular, degenerates (in fact linearly) as $\sigma \searrow 0$ (so guaranteeing the minimum principle).

The above expression (1.10) for the free energy permits us to remark a further peculiarity of the present model. This is related to the coercivity of \mathcal{F} , which is linked to the choice of a “singular potential” F , with the most usual choice in the Cahn–Hilliard literature being given by the Flory–Huggins “logarithmic potential” given by

$$F(r) = (1 + r) \log(1 + r) + (1 - r) \log(1 - r) - \frac{\lambda}{2} r^2, \quad r \in [-1, 1], \quad \lambda \geq 0. \tag{1.11}$$

For the standard Cahn–Hilliard model the expression (1.11) represents a source of mathematical difficulties (cf., e.g., [41]), due to its singular character, and, for this reason, it is often replaced by a double well potential of controlled growth like, e.g., $F_{\text{reg}}(r) = (r^2 - 1)^2$. Here, instead, the singular character of F helps us to get coercivity of the energy functional, and in particular to control the coupling term (i.e., the last summand in (1.10)).

Notice also that such a difficulty does not occur when the nutrient equation has the form (1.6). Indeed, in that case the free energy takes the expression

$$\mathcal{F}_2(\varphi, \sigma) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \int_{\Omega} F(\varphi) + \int_{\Omega} \left(\frac{1}{2} \sigma^2 + \chi \sigma (1 - \varphi) \right),$$

which keeps its coercivity because of the contribution of σ^2 (note also that, in this case, the variational derivation of the model is similar, but one has to consider a mobility of the form $\mathfrak{m}(\sigma, \varphi)$ rather than $\sigma_{\mathfrak{m}}(\sigma, \varphi)$).

We also have to observe a further difficulty occurring in Cahn–Hilliard models with mass source and singular potentials like (1.11). Namely, the forcing term S in (1.1) has to be designed in such a way to prevent the spatial average of φ to become larger than 1 or smaller than -1 , which would be inconsistent with (1.2). Indeed, the mass balance (i.e., the evolutionary law ruling the spatial average of φ) only depends on (1.1), but at the same time its outcome must be consistent with (1.2). Following the lines of [17], we actually assume $S(\varphi, \sigma) = -m\varphi + h(\varphi, \sigma)$, where $m > 0$ is “large” compared to the L^∞ -norm of the (bounded) function h , which is readily seen to be an appropriate choice (see Subsec. 3.1 below for details). Note also that, for constant h , (1.1)-(1.2) reduces to the well-known Cahn–Hilliard–Oono system (see, e.g., [25,40,42,43]).

Our main mathematical results are devoted to proving existence of weak solutions under mild conditions on parameters and data as well as regularity and uniqueness results holding in more restrictive settings. In particular, under the sole “energy regularity” conditions on the initial data (basically corresponding to the finiteness of the functional \mathcal{F} at the initial time), we can prove existence of weak solutions for nonconstant, bounded and nondegenerate mobilities \mathfrak{m} , \mathfrak{m} , and for a wide class of logistic terms. In particular, we provide, depending on the space dimension d , sufficient conditions on the growth of b at infinity in order to exclude the occurrence of blowup. This result is proved by a-priori estimates and weak compactness methods. A possible approximation scheme compatible with the a-priori estimates is also sketched.

In the case of true logistic growth, i.e., for b behaving like $-\sigma^2$ at infinity, we can also present a number of regularity results holding under additional hypotheses on the mobilities and on the other coefficients and data. As is customary for the CH system, some regularity results are only valid in spatial dimension $d = 2$, for reasons depending both on the structure of equation (1.3) (and, in particular, on the quadratic behavior of the cross-diffusion term), and on the occurrence

of the singular potential, which gives rise, in the three-dimensional case, to an upper regularity threshold (see, e.g., [37]). In some cases we can also prove uniqueness; in fact, this is presented as a conditional result stating that two weak solutions starting from the same initial data and obeying some additional regularity properties must coincide. Then, it is observed that these regularity conditions are fulfilled for proper classes of strong solutions, also depending on the regularity of data and on the space dimension.

The plan of the paper is as follows: in the next section, we introduce our precise assumptions and present the statements of all our mathematical results. Then, in Section 3, we prove existence of weak solutions, while in Section 4 we move to the regularity results. Finally, Section 5 is devoted to uniqueness of “strong” solutions.

2. Mathematical preliminaries and main results

2.1. Notation

Before diving into the mathematical details, let us introduce the notation employed in the paper. Letting X be a Banach space, we denote by $\|\cdot\|_X$ the corresponding norm, by X^* the topological dual of X , and by $\langle \cdot, \cdot \rangle_X$ the related duality pairing between X^* and X . Standard Lebesgue and Sobolev spaces defined on Ω , for every $1 \leq p \leq \infty$ and $k \geq 0$, are indicated by $L^p(\Omega)$ and $W^{k,p}(\Omega)$, with associated norms $\|\cdot\|_{L^p(\Omega)} = \|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}(\Omega)}$, respectively. When $p = 2$, these become Hilbert spaces and we use $\|\cdot\| = \|\cdot\|_2$ for the norm of $L^2(\Omega)$ and set $H^k(\Omega) := W^{k,2}(\Omega)$. Moreover, for brevity we introduce the following notation:

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad H_n^2(\Omega) := \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \Gamma\},$$

where we denote by Γ the boundary of Ω , that is $\Gamma = \partial\Omega$.

For every $v \in V^*$, we use $v_\Omega := \frac{1}{|\Omega|} \langle v, 1 \rangle_V$ for the generalized mean value of v . Let us also point out a version of the celebrated Poincaré–Wirtinger inequality:

$$\|v - v_\Omega\| \leq c_\Omega \|\nabla v\|, \quad v \in V, \tag{2.1}$$

where the constant $c_\Omega > 0$ depends only on Ω and the spatial dimension d . The norm in V^* will be simply denoted by $\|\cdot\|_*$. Identifying H with H^* by employing the scalar product of H , we obtain the chain of continuous and dense embeddings $V \subset H \subset V^*$. Moreover, we may denote as V_0, H_0, V_0^* the (closed) subspaces respectively of V, H , and V^* , consisting of functions (or functionals) with zero spatial mean. Then, we observe that the weak version of the operator $-\Delta$ with homogeneous Neumann boundary conditions, i.e.,

$$(-\Delta) : V \rightarrow V^*, \quad \langle (-\Delta)v, z \rangle := \int_\Omega \nabla v \cdot \nabla z, \tag{2.2}$$

for $v, z \in V$, is invertible when it is restricted to the functions with zero spatial mean (i.e., when it operates from V_0 to V_0^*). Its inverse operator will be denoted by $\mathcal{N} : V_0^* \rightarrow V_0$.

Finally, we remark that, for any $v \in V^*$ there exists a positive constant c such that

$$|v_\Omega| = \left| \frac{1}{|\Omega|} \langle v, 1 \rangle_V \right| \leq c \|v\|_*,$$

whence the Poincaré–Wirtinger inequality (2.1) yields

$$\|v\|_V \leq c(\|\nabla v\| + |v_\Omega|) \leq c(\|\nabla v\| + \|v\|_*), \quad v \in V.$$

From now onward, we convey that the small-case symbol c denotes every constant that only relates to structural data of the problem and the norms of the involved functions; thus, its meaning may vary from line to line. When an additional positive constant δ also enters the computation, we use c_δ to stress the dependency of c on δ .

2.2. Main results

We describe here our basic assumptions on coefficients and data, which will be kept for the remainder of the paper. Each assumption will be presented with a number of comments aimed at outlining its meaningfulness in the light of our specific application to tumor growth processes.

Moreover, we observe that more restrictive conditions, needed for the regularity and uniqueness results, will be specified on occurrence.

(A1) - Assumptions on the potential. We assume F to be decomposed as $F = F_1 + F_2$, with F_1 denoting the “singular” convex part and F_2 the “smooth” nonconvex part. The latter is simply given by $F_2(r) := -\lambda r^2/2$, $r \in \mathbb{R}$, with $\lambda \geq 0$ (so including the case $F_2 \equiv 0$ corresponding to a convex potential F). The properties of F_1 are better described by using some basic notions from the theory of subdifferential operators. Namely, we assume $F_1 : \mathbb{R} \rightarrow (-\infty, +\infty]$ be convex and lower semicontinuous with the set $\{r \in \mathbb{R} : F_1(r) < +\infty\}$ (usually indicated as *domain* of F_1 in the convex analysis terminology) coinciding either with $[-1, 1]$ or with $(-1, 1)$. In such a situation it is well-known that the *subdifferential* $f_1 = \partial F_1$ is a maximal monotone, possibly multivalued, operator in \mathbb{R} such that $\{f_1(r)\}$ is nonempty at least for $r \in (-1, 1)$ and at most for $r \in [-1, 1]$. Here, we are not interested in considering nonsmooth operators; for this reason we will also assume $F_1 \in C^2(-1, 1)$ so that $f(r) = f_1(r) + f_2(r) = F'_1(r) + F'_2(r)$ for $r \in (-1, 1)$. Moreover, just for the sake of simplicity, we assume F_1 so normalized that $F'_1(0) = 0$, which implies in particular that $F'_1(r) \geq 0$ for $r \geq 0$ and $F'_1(r) \leq 0$ for $r \leq 0$. Notice that this includes both the case of the Flory–Huggins potential (1.11) (whose *domain* is $[-1, 1]$) as well as the case of “more singular” potentials like that considered in [45], i.e.,

$$F_1(r) = -\log(1 - r^2), \quad r \in (-1, 1). \tag{2.3}$$

Notice however that nonsmooth potentials, like the so-called *double obstacle potential* $F_1(r) = I_{[-1,1]}(r)$, with $I_{[-1,1]}$ denoting the *indicator function* of the interval $[-1, 1]$ (cf., e.g., [4]) may be considered as well, at least for what concerns existence of weak solutions.

(A2) - Assumptions on the mass source term. We assume S to be given by

$$S(\varphi, \sigma) = -m\varphi + h(\varphi, \sigma), \quad (\varphi, \sigma) \in \mathbb{R}^2, \tag{2.4}$$

where $m > 0$ is a constant. Moreover, we assume h to be uniformly bounded and Lipschitz continuous with respect to the complex of its variables. Finally, the following compatibility condition is assumed to hold

$$\frac{K}{m} < 1, \quad \text{where } K := \|h\|_{L^\infty(\mathbb{R} \times \mathbb{R})}. \tag{2.5}$$

Notice that, in principle, only the behavior of h over the physical reference set $\mathcal{H} = [-1, 1] \times [0, +\infty)$ is significant. On the other hand, it is worth assuming h be defined for every value of its arguments because, for instance, in an approximation, it may happen φ to take values outside $[-1, 1]$ (cf. Subsec. 3.3).

It is worth observing that, if h is a constant function (still indicated as h for notational simplicity), the expression of S corresponds to that occurring in the so-called Cahn–Hilliard–Oono equation (see, e.g., [25,40,42,43] and the references therein), i.e.

$$S(\varphi, \sigma) = -m\varphi + h, \quad h \in (-m, m).$$

Moreover, we observe that the case $m \equiv h \equiv 0$, corresponding to the conservation of total tumor mass, is admissible too, and in fact simpler to deal with. The variations needed to consider the situation with no mass source will be outlined on occurrence.

(A3) - Assumptions on the chemical source term. We assume b has a generalized logistic expression of the form

$$b(\varphi, \sigma) = \beta(\varphi)(\kappa_0\sigma - \kappa_\infty\sigma^p), \quad \varphi \in \mathbb{R}, \quad \sigma \geq 0, \tag{2.6}$$

where $p \in (1, 2]$ is a given exponent, and $\kappa_0 > 0, \kappa_\infty > 0$ are positive constants. Note that, in view of the minimum principle holding for equation (1.3) (and preserved in the approximation) it is sufficient to specify the above expression for $\sigma \geq 0$. Here, the function $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be Lipschitz continuous and to satisfy

$$0 \leq \beta(r) \leq B < +\infty \quad \text{for every } r \in \mathbb{R}, \tag{2.7}$$

$$0 < b_0 \leq \beta(r) \quad \text{for every } r \in [-3/2, 3/2], \tag{2.8}$$

$$\beta(r) \equiv 0 \quad \text{for every } r \notin (-2, 2), \tag{2.9}$$

where $b_0, B > 0$ are given constants. In fact, in the limit, the only significant values of $\beta(r)$ will be those assumed as $r \in [-1, 1]$. However, as in the case of h , it is necessary to extend β also outside that interval in view of an approximation. We finally observe that the motivations underlying the choice of a logistic behavior for the chemical source have been extensively detailed in the introduction.

(A4) - Assumptions on the mobility functions. We assume $\mathfrak{m} \in C^0(\mathbb{R} \times [0, +\infty))$ and $\mathfrak{n} \in C^1(\mathbb{R} \times [0, +\infty))$ to be globally Lipschitz continuous in the complex of their arguments, and to satisfy

$$0 < m_0 \leq \mathfrak{m}(\varphi, \sigma), \mathfrak{n}(\varphi, \sigma) \leq M < +\infty, \quad \text{for every } \varphi \in \mathbb{R}, \quad \sigma \geq 0, \tag{2.10}$$

$$|\partial_\varphi \mathfrak{n}(\varphi, \sigma)| \leq M < +\infty, \quad \text{for every } \varphi \in \mathbb{R}, \quad \sigma \geq 0, \tag{2.11}$$

where, again, $m_0, M > 0$ are given constants. In order to properly state a weak formulation of the system, we also set

$$N(\varphi, \sigma) := \int_0^\sigma \mathfrak{n}(\varphi, s) \, ds,$$

and we notice that, thanks to (2.10), N satisfies

$$m_0\sigma \leq N(\varphi, \sigma) \leq M\sigma \quad \text{for every } \varphi \in \mathbb{R}, \sigma \geq 0. \tag{2.12}$$

Moreover, it is not difficult to prove that

$$\begin{aligned} |N(\varphi_1, \sigma_1) - N(\varphi_2, \sigma_2)| &\leq |N(\varphi_1, \sigma_1) - N(\varphi_1, \sigma_2)| + |N(\varphi_1, \sigma_2) - N(\varphi_2, \sigma_2)| \\ &\leq L|\sigma_1 - \sigma_2| + L\sigma_2|\varphi_1 - \varphi_2| \end{aligned} \tag{2.13}$$

where $L > 0$ is a Lipschitz constant. We also need to define

$$\mathfrak{n}_1(\varphi, \sigma) := \partial_\varphi N(\varphi, \sigma) = \int_0^\sigma \partial_\varphi \mathfrak{n}(\varphi, s) \, ds,$$

whence there holds the identity

$$\nabla N(\varphi, \sigma) = \mathfrak{n}(\varphi, \sigma)\nabla\sigma + \mathfrak{n}_1(\varphi, \sigma)\nabla\varphi. \tag{2.14}$$

Moreover, since \mathfrak{n} is assumed to be C^1 , \mathfrak{n}_1 turns out to be continuous and to satisfy

$$|\mathfrak{n}_1(\varphi, \sigma)| \leq M\sigma \quad \text{for every } \varphi \in \mathbb{R}, \sigma \geq 0, \tag{2.15}$$

as a direct check shows.

In addition to the above assumptions, we take the chemotaxis sensitivity χ appearing in (1.2)-(1.3) to be a strictly positive constant. We keep its value explicit because its magnitude will play a role in part of the results. On the other hand, the magnitude of the interfacial energy coefficient $\varepsilon > 0$ has no importance for the mathematical analysis. Hence, for the sake of simplicity, we will directly take $\varepsilon = 1$ from now onward, without further reference.

The above choices lead us to rewrite system (1.1)-(1.3) in the following form, where, for the sake of clarity, some expressions of the source terms have been expanded:

$$\varphi_t - \operatorname{div}(\mathfrak{n}(\varphi, \sigma)\nabla\mu) = -m\varphi + h(\varphi, \sigma) \quad \text{in } Q, \tag{2.16}$$

$$\mu = -\Delta\varphi + F'_1(\varphi) - \lambda\varphi - \chi\sigma \quad \text{in } Q, \tag{2.17}$$

$$\sigma_t - \operatorname{div}(\mathfrak{n}(\varphi, \sigma)\nabla\sigma) - \chi \operatorname{div}(\sigma \mathfrak{n}(\varphi, \sigma)\nabla(1 - \varphi)) = \beta(\varphi)(\kappa_0\sigma - \kappa_\infty\sigma^p) \quad \text{in } Q. \tag{2.18}$$

In particular, we have written the equation for σ in the “decoupled” form (2.18) where the cross-diffusion term is split between two distinct components. Indeed, this is a necessary step in order to deal with a mathematically tractable weak formulation. On the other hand, it is also worth recalling the following “coupled” version of the equation for σ which is more suitable for the derivation of the a-priori estimates:

$$\sigma_t - \operatorname{div}(\sigma \mathfrak{n}(\varphi, \sigma)\nabla(\ln \sigma + \chi(1 - \varphi))) = \beta(\varphi)(\kappa_0\sigma - \kappa_\infty\sigma^p) \quad \text{in } Q. \tag{2.19}$$

It is clear that, as far as “smooth” solutions are considered, relations (2.18) and (2.19) may be interpreted as equivalent. In particular, this may happen in the approximation thanks to additional regularity available at that level.

We can now present our first result for the chemotaxis system (1.1)-(1.5) concerning the existence of weak solutions in dimensions two and three holding under the assumptions detailed above. We observe in particular that, in order to pass to the limit in the cross-diffusion term (and in particular to decouple its components as expressed by equation (2.18)), we will be forced to restrict the admissible range of the exponents p in (2.6) in a way depending on the space dimension d . In the sequel, functions of the form $g(r) = r \ln r$, or similar, are implicitly intended to be extended, by continuity, to $r = 0$ by setting $g(0) = 0$.

Theorem 2.1 (Existence of weak solutions, $d \in \{2, 3\}$). *Suppose that Assumptions (A1)-(A4) are satisfied, let $\chi > 0$ and let $d \in \{2, 3\}$. Moreover, assume that the initial data satisfy*

$$\varphi_0 \in V, \quad F(\varphi_0) \in L^1(\Omega), \quad (\varphi_0)_\Omega \in (-1, 1), \tag{2.20}$$

$$\sigma_0 \geq 0 \text{ a.e. in } \Omega, \quad \sigma_0 \ln \sigma_0 \in L^1(\Omega). \tag{2.21}$$

Moreover, assume that the exponent p in (2.6) satisfies $p \in [3/2, 2]$ for $d = 2$ and $p \in [8/5, 2]$ for $d = 3$. Then, system (1.1)-(1.5) admits at least one weak solution; namely there exists a triplet (φ, μ, σ) satisfying the regularity properties

$$\varphi \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^p(0, T; W^{2,p}(\Omega)), \tag{2.22}$$

$$\varphi \in L^\infty(Q) : -1 \leq \varphi(x, t) \leq 1 \text{ for a.e. } (x, t) \in Q, \tag{2.23}$$

$$\sigma(x, t) \geq 0 \text{ for a.e. } (x, t) \in Q, \tag{2.24}$$

$$\sigma \in C^0([0, T]; \mathcal{W}_n^*) \cap L^\infty(0, T; L^1(\Omega)), \tag{2.25}$$

$$\sigma^p \ln \sigma \in L^1(0, T; L^1(\Omega)), \quad \sigma \ln \sigma \in L^\infty(0, T; L^1(\Omega)), \tag{2.26}$$

$$\sigma^{1/2} \nabla(\ln \sigma + \chi(1 - \varphi)) \in L^2(0, T; H), \tag{2.27}$$

$$\mu \in L^2(0, T; V), \tag{2.28}$$

$$F(\varphi) \in L^\infty(0, T; L^1(\Omega)), \quad f(\varphi) \in L^p(0, T; L^p(\Omega)), \tag{2.29}$$

together with the “pointwise” formulation

$$\mu = -\Delta\varphi + f(\varphi) - \chi\sigma \text{ a.e. in } Q, \tag{2.30}$$

the boundary condition

$$\partial_n \varphi = 0 \text{ in the sense of traces on } \Gamma \times (0, T), \tag{2.31}$$

and the weak variational formulations

$$\langle \varphi_t, v \rangle_V + \int_{\Omega} \mathfrak{m}(\varphi, \sigma) \nabla \mu \cdot \nabla v = \int_{\Omega} S(\varphi, \sigma) v, \quad \text{a.e. in } (0, T), \tag{2.32}$$

$$\begin{aligned} \langle \sigma(t), w(t) \rangle_{\mathcal{W}_n} - \int_0^t \int_{\Omega} N(\sigma, \varphi) \Delta w - \int_0^t \int_{\Omega} \mathfrak{m}_1(\varphi, \sigma) \nabla \varphi \cdot \nabla w - \chi \int_0^t \int_{\Omega} \sigma \mathfrak{m}(\varphi, \sigma) \nabla \varphi \cdot \nabla w \\ = \langle \sigma_0, w(0) \rangle_{\mathcal{W}_n} + \int_0^t \langle \sigma, w_t \rangle_{\mathcal{W}_n} + \int_0^t \int_{\Omega} b(\varphi, \sigma) w, \quad \text{for every } t \in [0, T], \end{aligned} \tag{2.33}$$

for all test functions $v \in V, w \in C^1([0, T]; \mathcal{W}_n)$, where we have set

$$\mathcal{W}_n := \{w \in W^{1,\infty}(\Omega) \cap W^{2,p'}(\Omega) : \partial_n w = 0 \text{ on } \Gamma\},$$

with p' being the conjugate exponent of p , i.e., the exponent such that $1/p + 1/p' = 1$. The space \mathcal{W}_n is naturally endowed with the graph norm, which turns it into a Banach space. Besides, the initial conditions are satisfied in the sense that

$$\varphi|_{t=0} = \varphi_0 \quad \text{a.e. in } \Omega, \tag{2.34}$$

$$\sigma|_{t=0} = \sigma_0 \quad \text{in } \mathcal{W}_n^*. \tag{2.35}$$

Furthermore, if the source term b has a standard logistic growth, i.e., b fulfills (2.6) with $p = 2$, then the solution (φ, μ, σ) obtained before satisfies the additional regularity property

$$\sigma^{1/2} \in L^2(0, T; V). \tag{2.36}$$

It is worth providing some further comments on the above statement. First of all, we notice that, due to (A1), the second condition in (2.20) implies in particular that $\varphi_0 \in L^\infty(\Omega)$ with $-1 \leq \varphi_0 \leq 1$ almost everywhere in Ω . We also observe that relations (2.32)-(2.33) conveniently incorporate the boundary conditions. Finally, we observe that there may be proved the additional regularity property $\sigma \in BV(0, T; \mathcal{W}_n^*)$.

The above result may be improved as soon as the source term is pure logistic, i.e., b verifies (2.6) with $p = 2$. Specifically, that additional assumption, along with natural conditions on the initial data, suffices to improve the regularity of the weak solutions for $d = 2$ without any further restriction. In the three-dimensional case, a similar property holds provided that the chemotactic coefficient χ is assumed small enough and the mobility \mathfrak{m} is taken as a constant function (however this condition may be partially relaxed, see Remark 4.2 below).

Theorem 2.2 (Regularity properties of weak solutions). *Suppose that Assumptions (A1)-(A4) and (2.20)-(2.21) hold with $p = 2$ in (2.6), and assume that the initial datum σ_0 additionally satisfies*

$$\sigma_0 \in H. \tag{2.37}$$

Moreover, if $d = 3$, suppose also that

$$\chi < \sqrt{2\kappa_\infty b_0}, \tag{2.38}$$

$$\mathfrak{m}(\varphi, \sigma) \equiv 1. \tag{2.39}$$

Then, the weak solution (φ, μ, σ) provided by Theorem 2.1 satisfies the following additional regularity properties:

$$\varphi \in H^1(0, T; V^*) \cap L^4(0, T; H_n^2(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)), \tag{2.40}$$

$$\sigma \in H^1(0, T; V^*) \cap C^0([0, T]; H) \cap L^2(0, T; V) \cap L^3(0, T; L^3(\Omega)), \tag{2.41}$$

where $q = 6$ in (2.40) if $d = 3$, whereas one can take any $q \in [1, \infty)$ if $d = 2$. Moreover, the system equations are satisfied in the following sense: (2.30)-(2.31) hold together with the variational equalities

$$\langle \varphi_t, v \rangle_V + \int_\Omega \mathfrak{m}(\varphi, \sigma) \nabla \mu \cdot \nabla v = \int_\Omega S(\varphi, \sigma) v, \tag{2.42}$$

$$\langle \sigma_t, v \rangle_V + \int_\Omega \mathfrak{m}(\varphi, \sigma) \nabla \sigma \cdot \nabla v - \chi \int_\Omega \sigma \mathfrak{m}(\varphi, \sigma) \nabla \varphi \cdot \nabla v = \int_\Omega b(\varphi, \sigma) v, \tag{2.43}$$

for every test function $v \in V$ and almost everywhere in $(0, T)$. Finally, the initial conditions (2.34)-(2.35) are now both satisfied almost everywhere in Ω .

Under the assumptions of the previous theorem (including in particular (2.38) in the three-dimensional case), we can prove additional regularity of solutions for constant mobilities provided that also the initial data are smoother. This is stated in the following theorem.

Theorem 2.3. *Suppose that Assumptions (A1)-(A4) hold, with $p = 2$ in (2.6), together with (2.20)-(2.21). Moreover, assume*

$$\mathfrak{m}(\varphi, \sigma) \equiv \mathfrak{m}(\varphi, \sigma) \equiv 1, \tag{2.44}$$

and, if $d = 3$, assume also (2.38). If the initial data satisfy the additional conditions

$$\varphi_0 \in H_n^2(\Omega), \quad \mu_0 := -\Delta \varphi_0 + f(\varphi_0) - \chi \sigma_0 \in V, \quad \sigma_0 \in V, \tag{2.45}$$

then, the weak solution (φ, μ, σ) provided by Theorem 2.1 satisfies the following additional regularity properties:

$$\varphi \in W^{1,\infty}(0, T; V^*) \cap H^1(0, T; V) \cap L^\infty(0, T; W^{2,q}(\Omega)), \tag{2.46}$$

$$F'_1(\varphi) \in L^\infty(0, T; L^q(\Omega)), \tag{2.47}$$

$$\mu \in L^\infty(0, T; V), \tag{2.48}$$

$$\sigma \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; H^2(\Omega)), \tag{2.49}$$

where $q = 6$ if $d = 3$ and $q \in [1, \infty)$ if $d = 2$. Moreover, (2.42)-(2.43) can be interpreted as equations holding a.e. in Ω with the boundary conditions holding in the sense of traces.

The next result, valid only in the two-dimensional case, extends to the present system a regularity property holding for the Cahn–Hilliard equation for those singular potentials whose convex part fulfills the growth condition

$$|F_1''(r)| \leq e^{C_F(|F_1'(r)|+1)}, \quad \text{for every } r \in (-1, 1), \tag{2.50}$$

for some positive constant C_F . It is well-known that (2.50) is satisfied by the logarithmic potential in (1.11); as one can directly check, it also holds for “more singular” potentials, like (2.3), such that $|F_1'(r)|$ behaves like a negative power of $1 - |r|$ as $|r| \nearrow 1$. It does not hold, instead, in the case of the *double obstacle* potential. Nevertheless, whenever (2.50) holds, we can prove that, for smoother initial data, the solution φ is “separated” from the singular values ± 1 in the uniform norm. This is stated in the following theorem.

Theorem 2.4. *Suppose that Assumptions (A1)-(A4) hold with $p = 2$ in (2.6), together with (2.20)-(2.21). Moreover, assume that $d = 2$, the potential fulfills (2.50), and (2.44) holds. If the initial data satisfy the additional conditions*

$$\varphi_0 \in H_{\mathbf{n}}^2(\Omega), \quad \mu_0 := -\Delta\varphi_0 + f(\varphi_0) - \chi\sigma_0 \in H_{\mathbf{n}}^2(\Omega), \quad \sigma_0 \in V, \tag{2.51}$$

then the weak solution (φ, μ, σ) provided by Theorem 2.1, in addition to the regularity stated in Theorem 2.3, satisfies the following additional properties:

$$\varphi \in W^{1,\infty}(0, T; H) \cap H^1(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^4(\Omega) \cap W^{2,q}(\Omega)), \quad q \in [2, \infty), \tag{2.52}$$

$$\mu \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; H^3(\Omega)), \tag{2.53}$$

$$F_1''(\varphi) \in L^\infty(0, T; L^q(\Omega)). \tag{2.54}$$

Moreover, if the initial data also satisfy

$$\sigma_0 \in L^\infty(\Omega), \tag{2.55}$$

then one also has

$$\sigma \in L^\infty(Q) \tag{2.56}$$

and there exists a computable constant $\delta \in (0, 1)$ only depending on the problem data such that the following “separation property” holds:

$$-1 + \delta \leq \varphi \leq 1 - \delta \quad \text{a.e. in } Q. \tag{2.57}$$

Remark 2.5. A direct check shows that, if (2.51) and (2.56) hold, then the separation property (2.57) holds at the initial time (i.e., its analogue is satisfied by φ_0). Hence, (2.57) is fully compatible with (2.51).

Remark 2.6. The separation property (2.57) is extremely important for singular potentials like (1.11). Indeed, if (2.57) holds, then, the singularity of F is no longer an obstacle for the analysis as, actually, φ is limited to range in a closed subinterval of $(-1, 1)$ where F has controlled growth.

Remark 2.7. Given the parabolic nature of system (1.1)-(1.5), most of our regularity results could be seen as smoothing properties of weak solutions (i.e., of solutions starting from “energy regular” initial data as those constructed in Theorem 2.1), holding for strictly positive times (provided that the required additional assumptions on coefficients, like for instance constant mobilities, hold). Of course, since uniqueness is not known to hold for weak solutions, we can assert that from energy regular initial data starts at least one weak solution that enjoys parabolic smoothing properties. However, we cannot exclude that there might exist other weak solutions which *do not* regularize in time.

Our last result is devoted to establishing uniqueness of solutions in the case of constant mobility functions. We prefer to formulate the result in a general version holding both for $d = 2$ and for $d = 3$ though in a conditional way.

Theorem 2.8 (Uniqueness). *Suppose that assumptions (A1)-(A4) hold. Moreover, let $m, n, \beta \equiv 1$ and $p = 2$ in (2.6). Let us consider a couple of weak solutions $\{(\varphi_i, \mu_i, \sigma_i)\}_{i=1,2}$ additionally satisfying*

$$\varphi_1 \in L^2(0, T; W^{2,6}(\Omega)), \tag{2.58}$$

$$\sigma_1 \in L^4(0, T; H), \tag{2.59}$$

$$\sigma_2 \in L^4(0, T; L^6(\Omega)), \tag{2.60}$$

associated to initial data $\{(\varphi_{0,i}, \sigma_{0,i})\}_{i=1,2}$ fulfilling (2.20)-(2.21) and (2.45). Let us also assume that either h is a constant function, or $F \in C^2(-1, 1)$ and there hold the additional conditions

$$F''(\varphi_1), F''(\varphi_2) \in L^2(0, T; H) \tag{2.61}$$

as well as $\{(\varphi_{0,i}, \sigma_{0,i})\}_{i=1,2}$ also fulfill (2.51). Then, $(\varphi_1, \mu_1, \sigma_1) \equiv (\varphi_2, \mu_2, \sigma_2)$ almost everywhere in Q .

Remark 2.9. We notice that conditions (2.58)-(2.59) are verified under the regularity setting of Theorem 2.2. The main obstacle is represented by (2.60), which holds only under the more restrictive conditions in Theorem 2.3. Finally, the validity of (2.61) is limited to the two-dimensional case under assumption (2.50) and in the regularity setting of Theorem 2.4.

Remark 2.10. It is worth noticing that, under the assumptions of the above theorem, a continuous dependence estimate also holds. For instance, in the case of h constant, one has

$$\begin{aligned} & \|\varphi_1 - \varphi_2 - ((\varphi_1)_\Omega - (\varphi_2)_\Omega)\|_{L^\infty(0,T;V^*)}^2 + \|(\varphi_1)_\Omega - (\varphi_2)_\Omega\|_{L^\infty(0,T)}^2 + \|(\varphi_1)_\Omega - (\varphi_2)_\Omega\|_{L^\infty(0,T)} \\ & + \|\sigma_1 - \sigma_2 - ((\sigma_1)_\Omega - (\sigma_2)_\Omega)\|_{L^\infty(0,T;V^*)}^2 + \|(\sigma_1)_\Omega - (\sigma_2)_\Omega\|_{L^\infty(0,T)}^2 \end{aligned}$$

$$\begin{aligned}
 & + \|\varphi_1 - \varphi_2\|_{L^2(0,T;V)}^2 + \|\sigma_1 - \sigma_2\|_{L^2(0,T;H)}^2 \\
 \leq & K \left(\|\varphi_{0,1} - \varphi_{0,2} - ((\varphi_{0,1})_\Omega - (\varphi_{0,2})_\Omega)\|_{V^*}^2 + |(\varphi_{0,1})_\Omega - (\varphi_{0,2})_\Omega|^2 + |(\varphi_{0,1})_\Omega - (\varphi_{0,2})_\Omega| \right. \\
 & \left. + \|\sigma_{0,1} - \sigma_{0,2} - ((\sigma_{0,1})_\Omega - (\sigma_{0,2})_\Omega)\|_{V^*}^2 + |(\sigma_{0,1})_\Omega - (\sigma_{0,2})_\Omega|^2 \right), \tag{2.62}
 \end{aligned}$$

for some K depending only on the known data, including the norms in (2.58)-(2.59).

Remark 2.11. Aiming at reducing the technical burden, Theorem 2.8 is proved by considering the two- and three-dimensional cases together. As a consequence, it is worth noticing that conditions (2.58)-(2.60) are unlikely to be optimal, especially in dimension two where better inequalities hold, and may be in fact replaced by other similar assumptions. For instance, it will be noted in the proof that (2.60) might be replaced by

$$\sigma_2 \in L^\infty(0, T; L^{3+\delta}(\Omega)) \quad \text{for some } \delta > 0.$$

3. Well-posedness

This section is devoted to the proof of Theorem 2.1, which will be split into several parts presented in separate subsections.

3.1. Mass dynamics

The main tool in the existence proof consists in the derivation of suitable a-priori estimates. For the sake of simplicity, these will be presented by working on a triplet (φ, μ, σ) solving the original system (2.16)-(2.18) plus the initial and boundary conditions, without referring to any explicit approximation or regularization of it. In Subsection 3.3 below we will propose a regularization of the system and explain how the formal estimates derived here may be adapted to the rigorous framework.

In this respect, it is worth observing from the very beginning that a crucial point stands in the fact that the coercivity of the energy functional \mathcal{F} (cf. (1.10)) is tied to the choice of a “singular” potential F . Hence, dealing with the original (i.e., non-regularized system) and assuming in particular that the component φ of the solution satisfies the a-priori information

$$-1 \leq \varphi(x, t) \leq 1 \quad \text{for a.e. } (x, t) \in Q \tag{3.1}$$

represents a real simplification at this level. Indeed, let us recall the expression of the energy functional \mathcal{F} , namely (recall that $\varepsilon = 1$)

$$\mathcal{F}(\varphi, \sigma) = \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 + \int_{\Omega} F(\varphi) + \int_{\Omega} \sigma (\ln \sigma - 1) + \chi \int_{\Omega} \sigma (1 - \varphi).$$

Then, it is clear that, as far as (3.1) holds, the last term is nonnegative due to the expected positive sign of the variable σ . As a consequence, \mathcal{F} turns out to be coercive: it is easy to check that there exists a computable constant $C > 0$ depending only on the known data such that

$$(C + \mathcal{F}(\varphi, \sigma)) \geq \frac{1}{2} \|\varphi\|_V^2 + \frac{1}{2} \|\sigma \ln \sigma\|_1. \tag{3.2}$$

Notice that the above still holds when $\varphi \in L^\infty$, even if (3.1) is not known to hold. Indeed, the coupling term can then be controlled as follows

$$\chi \left| \int_{\Omega} \sigma(1 - \varphi) \right| \leq \chi \|\sigma\|_1 (1 + \|\varphi\|_\infty) \leq \frac{1}{2} \|\sigma \ln \sigma\|_1 - c, \tag{3.3}$$

where the last c also depends on the L^∞ -norm of φ .

On the other hand, if the singular potential F is replaced by a function of controlled growth, then the boundedness of φ is lost, (3.3) cannot be used, and consequently the energy functional loses its coercivity. This is the main issue we will need to fix when detailing the approximation in Subsection 3.3.

That said, we first show that, under assumptions (2.4)-(2.5) the spatial mean of φ is constrained to take values in the physical interval $(-1, 1)$ for every $t \geq 0$. Actually, testing (1.1) by $|\Omega|^{-1}$ and setting for simplicity $y = \varphi_\Omega$, we deduce the ODE-like relation

$$y' + my = \frac{1}{|\Omega|} \int_{\Omega} h(\varphi, \sigma),$$

whence, using (2.5), we obtain the differential inequalities

$$-K \leq y' + my \leq K.$$

Consequently, it holds that, for every $t \in [0, T]$,

$$y(0)e^{-mt} + (1 - e^{-mt}) \left(-\frac{K}{m} \right) \leq y(t) \leq y(0)e^{-mt} + (1 - e^{-mt}) \frac{K}{m}.$$

Using again (2.5) and recalling the last assumption in (2.20), we then deduce that, for some $\delta > 0$ depending only on φ_0, K and m , there holds

$$|(\varphi(t))_\Omega| \leq 1 - \delta \quad \text{for every } t \in [0, T], \tag{3.4}$$

which entails that the total mass of φ is prevented to reach the critical values ± 1 . Of course the same property (3.4) holds also when one takes $S \equiv 0$ because the spatial mean of φ is conserved in that case.

3.2. Formal energy estimate

First of all, we observe that, using assumption (2.21) and applying a standard minimum principle argument, there follows that $\sigma \geq 0$ almost everywhere in Q . Then, testing (2.16) by μ , (2.17) by φ_t , and taking the difference, we infer

$$\frac{d}{dt} \mathcal{E}(\varphi) - \chi \int_{\Omega} \sigma \varphi_t + \int_{\Omega} \text{In}(\varphi, \sigma) |\nabla \mu|^2 = \int_{\Omega} S(\varphi, \sigma) \mu, \tag{3.5}$$

where we recall that \mathcal{E} denotes the standard Ginzburg–Landau energy introduced in (1.10) (with $\varepsilon = 1$). Next, testing (2.19) by $\ln \sigma + \chi(1 - \varphi)$, and setting for the sake of simplicity $L(\sigma) := \sigma(\ln \sigma - 1)$, we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} L(\sigma) + \chi \int_{\Omega} \sigma_t(1 - \varphi) + \int_{\Omega} \sigma \mathfrak{n}(\varphi, \sigma) |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 \\ &= \int_{\Omega} \beta(\varphi)(\kappa_0 \sigma - \kappa_{\infty} \sigma^P)(\ln \sigma + \chi(1 - \varphi)). \end{aligned} \tag{3.6}$$

Adding (3.5) with (3.6) and rearranging, using also (2.10), we deduce

$$\begin{aligned} & \frac{d}{dt} \left[\mathcal{E}(\varphi) + \int_{\Omega} (L(\sigma) + \chi \sigma(1 - \varphi)) \right] \\ &+ m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + m_0 \|\nabla \mu\|^2 + \kappa_{\infty} \int_{\Omega} \beta(\varphi) \sigma^P \ln \sigma \\ &\leq \int_{\Omega} S(\varphi, \sigma) \mu + \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma + \chi \int_{\Omega} \beta(\varphi)(\kappa_0 \sigma - \kappa_{\infty} \sigma^P)(1 - \varphi). \end{aligned} \tag{3.7}$$

To control the first term on the right-hand side, we observe that, replacing the expression of μ given by equation (2.17) and using (A2) along with (3.1) and the Poincaré–Wirtinger inequality (2.1), there follows

$$\int_{\Omega} S(\varphi, \sigma) \mu = \int_{\Omega} S(\varphi, \sigma)(\mu - \mu_{\Omega}) + \mu_{\Omega} \int_{\Omega} S(\varphi, \sigma) \leq (m + K)c_{\Omega} \|\nabla \mu\| + |\Omega|(m + K)|\mu_{\Omega}|, \tag{3.8}$$

where $c_{\Omega} > 0$ is a Poincaré constant. Now, integrating (2.17) over Ω and recalling that $\sigma \geq 0$ almost everywhere in Q and (1.4), we have

$$|\Omega||\mu_{\Omega}| \leq \|f(\varphi)\|_1 + \chi \int_{\Omega} \sigma. \tag{3.9}$$

Replacing (3.8) and (3.9) into (3.7), using (2.7) and (2.8) with the fact $1 - \varphi \geq 0$ (which is, in turn, a consequence of (3.1)), it is not difficult to obtain

$$\begin{aligned} & \frac{d}{dt} \mathcal{F}(\varphi, \sigma) + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + m_0 \|\nabla \mu\|^2 + \kappa_{\infty} b_0 \int_{\Omega} (\sigma^P \ln \sigma + 2) \\ &\leq c + c \int_{\Omega} \sigma + c \|\nabla \mu\| + (m + K) \|f(\varphi)\|_1 \\ &+ \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma + \chi \int_{\Omega} \beta(\varphi)(\kappa_0 \sigma - \kappa_{\infty} \sigma^P)(1 - \varphi) \end{aligned}$$

$$\leq c + c \int_{\Omega} \sigma + \frac{m_0}{4} \|\nabla \mu\|^2 + (m + K) \|f(\varphi)\|_1 + \kappa_0 B \int_{\Omega} |\sigma \ln \sigma| + \chi \kappa_0 B \int_{\Omega} \sigma(1 - \varphi), \tag{3.10}$$

where we recall that \mathcal{F} was defined in (1.10) (with $\varepsilon = 1$). In order to control the norm of $f(\varphi)$ on the right-hand side, we test (2.17) by $\varphi - \varphi_{\Omega}$ to get

$$\int_{\Omega} f(\varphi)(\varphi - \varphi_{\Omega}) + \|\nabla \varphi\|^2 = \int_{\Omega} \mu(\varphi - \varphi_{\Omega}) + \chi \int_{\Omega} \sigma(\varphi - \varphi_{\Omega}). \tag{3.11}$$

Using the mass property (3.4) and proceeding similarly as in [37] (see also, e.g., [16,39]) we then deduce that there exist two constants $\alpha > 0$ (small) and $c > 0$ (large), both depending on the constant δ in (3.4), such that

$$\int_{\Omega} f(\varphi)(\varphi - \varphi_{\Omega}) \geq \alpha \|f(\varphi)\|_1 - c. \tag{3.12}$$

Hence, recalling (3.1) and operating straightforward manipulations in (3.11) leads to

$$\begin{aligned} \alpha \|f(\varphi)\|_1 &\leq c \|\nabla \mu\| \|\nabla \varphi\| + 2\chi \int_{\Omega} \sigma + c \\ &\leq c_{\delta} \|\nabla \varphi\|^2 + \delta \|\nabla \mu\|^2 + 2\chi \int_{\Omega} \sigma + c, \end{aligned} \tag{3.13}$$

where $\delta > 0$ can be taken arbitrarily small, in a way that will be specified later on, and $c_{\delta} > 0$ depends on the choice of δ .

Next, we multiply (3.13) by $P > 0$ large enough such that $P\alpha \geq m + K + 1$; then, we choose δ small enough such that $P\delta \leq m_0/4$. Finally, we add the result of this operation to (3.10) deducing that

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(\varphi, \sigma) + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + \frac{m_0}{2} \|\nabla \mu\|^2 + \alpha P \|f(\varphi)\|_1 + \kappa_{\infty} b_0 \int_{\Omega} (\sigma^p \ln \sigma + 2) \\ \leq c \|\nabla \varphi\|^2 + c + c \int_{\Omega} \sigma + \kappa_0 B \int_{\Omega} |\sigma \ln \sigma| + \chi \kappa_0 B \int_{\Omega} \sigma(1 - \varphi). \end{aligned}$$

In order to get an estimate from the above relation, we observe that, since $p > 1$, we have

$$c \int_{\Omega} \sigma + \kappa_0 B \int_{\Omega} |\sigma \ln \sigma| + \chi \kappa_0 B \int_{\Omega} \sigma(1 - \varphi) \leq c + \frac{\kappa_{\infty} b_0}{2} \int_{\Omega} (\sigma^p \ln \sigma + 2).$$

Then, we may set

$$\mathcal{V} := C + \mathcal{F}(\varphi, \sigma),$$

where $C > 0$ is chosen such that the coercivity property (3.2) holds. As noted above, here we are using the constraint (3.1) in an essential way. With these choices, we arrive at the differential inequality

$$\frac{d}{dt} \mathcal{V} + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + \frac{m_0}{2} \|\nabla \mu\|^2 + \alpha P \|f(\varphi)\|_1 + \frac{\kappa_{\infty} b_0}{2} \int_{\Omega} (\sigma^p \ln \sigma + 2) \leq c \mathcal{V}. \tag{3.14}$$

Noting that $\mathcal{V}(0) < \infty$ thanks to (2.20)-(2.21), we can then apply Grönwall’s lemma to deduce the following a-priori bounds:

$$\begin{aligned} & \|\varphi\|_{L^{\infty}(0,T;V)} + \|\sigma \ln \sigma\|_{L^{\infty}(0,T;L^1(\Omega))} + \|F(\varphi)\|_{L^{\infty}(0,T;L^1(\Omega))} \\ & + \|\sigma^{1/2} \nabla(\ln \sigma + \chi(1 - \varphi))\|_{L^2(0,T;H)} + \|\nabla \mu\|_{L^2(0,T;H)} \\ & + \|f(\varphi)\|_{L^1(0,T;L^1(\Omega))} + \|\sigma^p \ln \sigma\|_{L^1(0,T;L^1(\Omega))} \leq c. \end{aligned} \tag{3.15}$$

Here and below, it is intended that the constant $c > 0$ on the right-hand side, whose explicit value may vary on occurrence, depends only on the known data of the problem, including the initial data, but is independent of any hypothetical approximation parameter.

Next, going back to (3.13), squaring its first row, and integrating in time, using also the information resulting from (3.15), we infer

$$\|f(\varphi)\|_{L^2(0,T;L^1(\Omega))} \leq c. \tag{3.16}$$

Then, we go back to (3.9): squaring and integrating in time, using (3.15) and (3.16) lead us to $\|\mu_{\Omega}\|_{L^2(0,T)} \leq c$, and applying once more the Poincaré–Wirtinger inequality, we arrive at

$$\|\mu\|_{L^2(0,T;V)} \leq c.$$

Next, comparing terms in (2.16) and using in particular assumptions (A2) and (2.10), it is not difficult to deduce

$$\|\varphi_t\|_{L^2(0,T;V^*)} \leq c. \tag{3.17}$$

Finally, we observe that (2.17), complemented with the no-flux boundary condition, can be interpreted as a family of time-dependent elliptic problems with maximal monotone perturbations of the form

$$-\Delta \varphi + F'_1(\varphi) = \mu + \lambda \varphi + \chi \sigma. \tag{3.18}$$

Here, we observe that, since we assumed $p \leq 2$, the maximal summability available for the right-hand side is exactly the L^p -one. Hence, applying standard tools (which basically correspond to testing (3.18) by $|F'_1(\varphi)|^{p-1} \text{sign}(F'_1(\varphi))$ and exploiting the monotonicity of F'_1), we deduce the additional estimates

$$\|F'_1(\varphi)\|_{L^p(0,T;L^p(\Omega))} + \|\Delta \varphi\|_{L^p(0,T;L^p(\Omega))} \leq c. \tag{3.19}$$

Suppose now that b has a true logistic growth, that is (2.6) is fulfilled with $p = 2$. Then, testing (2.18) by $\ln \sigma$ and using (2.12), (2.14), and (2.15), it is not difficult to arrive at

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \sigma (\ln \sigma - 1) + 4m_0 \|\nabla \sigma^{1/2}\|^2 + b_0 \kappa_{\infty} \int_{\Omega} \sigma^2 \ln \sigma \\ & \leq \chi \int_{\Omega} \mathfrak{m}(\varphi, \sigma) \nabla \varphi \cdot \nabla \sigma + \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma \\ & = \chi \int_{\Omega} \nabla N(\varphi, \sigma) \cdot \nabla \varphi - \chi \int_{\Omega} \mathfrak{m}_1(\varphi, \sigma) |\nabla \varphi|^2 + \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma \\ & \leq -\chi \int_{\Omega} N(\varphi, \sigma) \Delta \varphi + c \int_{\Omega} \sigma |\nabla \varphi|^2 + \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma \\ & \leq c \int_{\Omega} \sigma |\Delta \varphi| + c \|\sigma\| \|\varphi\|_{\infty} \|\varphi\|_{H^2(\Omega)} + c \|\sigma\|^2 \\ & \leq c \|\Delta \varphi\|^2 + \frac{b_0 \kappa_{\infty}}{2} \int_{\Omega} \sigma^2 \ln \sigma + c, \end{aligned}$$

where we also used the well-known interpolation inequality

$$\|v\|_{W^{1,4}(\Omega)} \leq c \|v\|_{H^2(\Omega)}^{1/2} \|v\|_{\infty}^{1/2} \quad \text{for every } v \in H^2(\Omega) \cap L^{\infty}(\Omega). \tag{3.20}$$

Hence, integrating the above in time and using (3.19) with $p = 2$, we obtain the additional estimate

$$\|\sigma^{1/2}\|_{L^2(0,T;V)} \leq c. \tag{3.21}$$

3.3. Approximation scheme

In this part, we outline a possible regularization scheme for system (1.1)-(1.5). Usually, in Cahn–Hilliard-based models, approximation is provided by smoothing out the singular term (here represented by F'_1) and replacing it with a Lipschitz continuous function. In this way, at least locally in time, existence of approximate solutions may be proved for instance by using a Faedo–Galerkin or time discretization scheme. Here, however, a further difficulty arises because the coercivity of the free energy (1.10) is tied to the presence of the singular potential F_1 . In other words, if F_1 is smoothed out, it is also necessary to intervene on the coupling term $\chi \sigma(1 - \varphi)$ by suitably truncating it; indeed, approximating F_1 , the property $|\varphi| \leq 1$ is lost and the coupling term becomes supercritical if it is not smoothed out. This is why, similarly, e.g., to [26], we propose a regularized scheme where also σ is properly truncated. Namely, we consider the system

$$\varphi_t - \operatorname{div}(\mathfrak{m}(\varphi, \sigma) \nabla \mu) = S(\varphi, \sigma) \tag{3.22}$$

$$\mu = -\Delta \varphi + F'_n(\varphi) - \lambda \varphi - \chi T_n(\sigma) \tag{3.23}$$

$$T_n(\sigma)_t - \operatorname{div}(\mathfrak{m}(\varphi, \sigma)\nabla\sigma) - \chi \operatorname{div}(\sigma\mathfrak{m}(\varphi, \sigma)\nabla(1 - \varphi)) = \beta(\varphi)(\kappa_0\sigma - \kappa_\infty\sigma^p) \quad \text{in } Q, \tag{3.24}$$

with $n \in \mathbb{N}$ denoting the regularization parameter, intended to go to infinity in the limit. As for the boundary and initial conditions, we consider the same as in (1.4) and (1.5). In this part we will avoid employing the subscript n to denote the approximate solution for the sake of notational clarity. Here, we assume the following properties. First of all, $\{F_n\}$, with $F_n : \mathbb{R} \rightarrow \mathbb{R}$, is a family of convex and regular functions such that, as $n \rightarrow \infty$, F_n tends to F_1 in the sense of Mosco. We refer to [2, Chap. 3] for the necessary background in convex analysis; we just observe that a simple condition ensuring this property holds when, for every fixed $r \in \mathbb{R}$, $F_n(r)$ is increasingly monotone with respect to n and converges to the limit value $F_1(r)$ (which is intended to be $+\infty$ as far $|r|$ is larger than 1, cf. **(A1)**). We also assume the normalization property

$$F'_n(r) \operatorname{sign} r \geq n^3(|r| - 1) \quad \text{for every } |r| \geq 1. \tag{3.25}$$

Indeed, it is apparent that, for every potential F_1 compatible with assumption **(A1)**, an approximation F_n satisfying the above conditions can be constructed by standard methods. For instance one could take the Yosida approximation (see [3,4]) of F_1 of order n^{-1} and add to it $n^3(|r| - 1)_+ \operatorname{sign} r$.

Concerning the truncation operator T_n we assume the following properties:

$$T_n \in C^{1,1}(\mathbb{R}); \quad T_n(r) = r \quad \text{for every } r \leq n, \tag{3.26}$$

$$T_n \text{ is strictly monotone and concave,} \tag{3.27}$$

$$T_n(r) < n + 1 \quad \text{for every } r \in \mathbb{R}, \quad \lim_{r \rightarrow \infty} T_n(r) = n + 1. \tag{3.28}$$

Explicit forms of T_n can be also constructed very easily.

We also need to introduce its inverse function, that is, $\gamma_n := T_n^{-1}$. Then, $\gamma_n \in C^1((-\infty, n + 1); \mathbb{R})$ and γ_n can also be interpreted as a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ so to apply the usual machinery of maximal monotone operator theory. Then, we also set

$$s := T_n(\sigma) \quad \text{so that } \sigma = \gamma_n(s)$$

and equation (3.24) can be consequently restated in terms of the new variable s in the following equivalent way:

$$s_t - \operatorname{div}(\mathfrak{m}(\varphi, \gamma_n(s))\nabla\gamma_n(s)) - \chi \operatorname{div}(\gamma_n(s)\mathfrak{m}(\varphi, \gamma_n(s))\nabla(1 - \varphi)) = \beta(\varphi)(\kappa_0\gamma_n(s) - \kappa_\infty\gamma_n(s)^p). \tag{3.29}$$

Of course, also relations (3.22)-(3.23) could be equivalently reformulated in terms of s . Besides, by using the variable s , the initial conditions may be expressed as follows:

$$s|_{t=0} = s_0 = T_n(\sigma_0), \quad \varphi|_{t=0} = \varphi_0. \tag{3.30}$$

Dealing with the regularized system (3.22)-(3.24), complemented with the initial conditions (3.30) and with the no-flux boundary conditions, may still be nontrivial. Indeed, equation (3.29) contains the *singular* function γ_n . A strategy that could be used in order to obtain at least local in time existence can be sketched as follows:

- (A) Smoothing out the function γ_n (for instance by replacing in with its Yosida regularization $\gamma_{n,\epsilon}$ for a regularization parameter ϵ intended to go to zero in the limit); one may also need to add further regularizing terms to get better properties of approximating solutions;
- (B) Proving local in time existence to the obtained system through the Faedo–Galerkin method;
- (C) Getting a-priori estimates uniform with respect to ϵ and, exploiting these, passing to the limit with respect to the approximation parameter ϵ so to obtain a solution to (3.22)–(3.24) with the initial and boundary conditions.

As said, even if the above strategy (A)–(C) may be not trivial, we believe that the main difficulties are just of technical nature. Indeed, equations of the form

$$s_t - \Delta(\gamma(s)) = f, \tag{3.31}$$

with γ maximal monotone graph (possibly, as here, of singular nature), have been extensively studied in the literature and the proposed strategy in order to get local existence (i.e., smoothing γ , discretizing by Faedo–Galerkin, then going back to the original γ) is very well established. Of course, in our setting, equation (3.29) is more complicated than (3.31) and we also have the additional difficulties resulting from the coupling with the CH system. Nevertheless, to reduce technical details, we assume to have accomplished the above strategy and we just focus on what we believe to be the main difficulty to get an existence theorem, i.e., the passage to the limit $n \rightarrow \infty$.

For this purpose, we first have to reproduce the energy estimate by working on the regularized system (3.22)–(3.24). Of course, we can use either the original nutrient variable σ or the transformed (truncated) variable $s = T_n(\sigma)$ since the formulations in terms of σ and s are equivalent at this level. Notice also that the proposed approximation is devised as to preserve the minimum principle property; hence we can freely assume s and σ to be nonnegative.

That said, using the variable σ so to get an estimate more similar to that obtained in the previous section, we have to test (3.22) by μ , (3.23) by φ_t , and (3.24) by $\ln \sigma + \chi(1 - \varphi)$. Then, by proceeding as before, we arrive at the analogue of (3.7), which takes now the form

$$\begin{aligned} & \frac{d}{dt} \mathcal{F}_n(\varphi, \sigma) + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + m_0 \|\nabla \mu\|^2 + \kappa_{\infty} \int_{\Omega} \beta(\varphi) \sigma^p \ln \sigma \\ & \leq \int_{\Omega} S(\varphi, \sigma) \mu + \kappa_0 \int_{\Omega} \beta(\varphi) \sigma \ln \sigma + \kappa_0 \chi \int_{\Omega} \beta(\varphi) \sigma (1 - \varphi) + \kappa_{\infty} \chi \int_{\Omega} \beta(\varphi) \sigma^p (\varphi - 1), \end{aligned} \tag{3.32}$$

and where the approximated energy takes the expression

$$\mathcal{F}_n(\varphi, \sigma) = \int_{\Omega} \left(\frac{1}{2} |\nabla \varphi|^2 + F_n(\varphi) - \frac{\lambda}{2} \varphi^2 + L_n(\sigma) + \chi T_n(\sigma)(1 - \varphi) \right).$$

The function L_n is defined as follows:

$$L_n(\sigma) = \int_0^{\sigma} T'_n(r) \ln r \, dr,$$

so that we have in particular

$$L_n(\sigma) = \sigma(\ln \sigma - 1) \text{ for } \sigma \leq n \quad \text{and} \quad L_n(\sigma) \geq n(\ln n - 1) \text{ for } \sigma > n. \tag{3.33}$$

Let us now verify that the coupling term $\chi T_n(\sigma)(1 - \varphi)$ can be controlled, uniformly in n , by using the other integrands. We actually notice that

$$\begin{aligned} \chi \int_{\Omega} T_n(\sigma)(1 - \varphi) &= \chi \int_{\Omega} T_n(\sigma) - \chi \int_{\Omega} T_n(\sigma)\varphi \\ &= \chi \int_{\Omega} T_n(\sigma) - \chi \int_{\{|\varphi| \leq 2\}} T_n(\sigma)\varphi - \chi \int_{\{|\varphi| > 2\}} T_n(\sigma)\varphi \\ &\geq \chi \int_{\Omega} T_n(\sigma) - 2\chi \int_{\{|\varphi| \leq 2\}} T_n(\sigma) - (n + 1)\chi \int_{\{|\varphi| > 2\}} \varphi \\ &\geq -\chi \int_{\Omega} T_n(\sigma) - 2(n + 1)\chi \int_{\{|\varphi| > 2\}} (\varphi - 1). \end{aligned}$$

To control the right-hand side, one can first verify that

$$\chi T_n(\sigma) \leq \frac{1}{2}L_n(\sigma) + c, \tag{3.34}$$

for every $\sigma > 0$ and a suitable constant $c \geq 0$ independent of n . Analogously, owing to (3.25), it is clear that, for $c \geq 0$ as above, we have

$$F_n(r) \geq \frac{n^3}{2}(|r| - 1)^2 - c \quad \text{for every } |r| \geq 1. \tag{3.35}$$

Consequently, thanks also to Young’s inequality, we have in particular

$$2(n + 1)\chi(\varphi - 1) \leq \frac{1}{4}F_n(\varphi) + c \quad \text{for every } \varphi \geq 2.$$

Based on the above considerations, and noting also that, by (3.25),

$$\frac{\lambda}{2}\varphi^2 \leq \frac{1}{4}F_n(\varphi) + c,$$

again for $c \geq 0$ independent of n , we conclude that there exists a constant C independent of n such that

$$\mathcal{F}_n(\varphi, \sigma) \geq \int_{\Omega} \left(\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}F_n(\varphi) + \frac{1}{2}L_n(\sigma) - C \right). \tag{3.36}$$

Namely, coercivity of the energy is preserved at the approximate level.

Then, in order to deduce from relation (3.32) an analogue of the energy estimate (3.14), we need to check that we can still control the right-hand side. To this aim, we start observing that the first integral can be managed similarly with (3.8)-(3.12). We only notice that, when the analogue of (3.11) is performed, we can no longer use the uniform boundedness of φ . On the other hand, the mean property (3.4) is preserved also in the approximation. Then, the contribution corresponding to the last integral in (3.11) is now managed as follows:

$$\begin{aligned} \chi \int_{\Omega} T_n(\sigma)(\varphi - \varphi_{\Omega}) &= \chi \int_{\{|\varphi| \leq 3/2\}} T_n(\sigma)(\varphi - \varphi_{\Omega}) + \chi \int_{\{|\varphi| > 3/2\}} T_n(\sigma)(\varphi - \varphi_{\Omega}) \\ &\leq c \int_{\{|\varphi| \leq 3/2\}} |T_n(\sigma)| + c(n + 1) \int_{\{|\varphi| > 3/2\}} (|\varphi| - 1) \\ &\leq c \int_{\Omega} |L_n(\sigma)| + c + c \int_{\Omega} F_n(\varphi), \end{aligned}$$

where we used (3.34), (3.35) and the control (3.4) on the spatial average of φ , which is not affected by the approximation. Then we notice that the last integral on the right-hand side can be controlled by Grönwall’s lemma. In order to estimate the remaining terms in (3.32), we first observe that, using (A3) and in particular (2.9),

$$\begin{aligned} \kappa_{\infty} \chi \int_{\Omega} \beta(\varphi) \sigma^p (\varphi - 1) &= \kappa_{\infty} \chi \int_{\{|\varphi| \leq 2\}} \beta(\varphi) \sigma^p (\varphi - 1) \\ &= \kappa_{\infty} \chi \int_{\{|\varphi| \leq 3/2\}} \beta(\varphi) \sigma^p (\varphi - 1) + \kappa_{\infty} \chi \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) \sigma^p (\varphi - 1) =: \mathbb{I}_1 + \mathbb{I}_2. \end{aligned}$$

Now, using (2.7) with a generalized form of Young’s inequality, we have

$$\mathbb{I}_1 \leq c \int_{\{|\varphi| \leq 3/2\}} \sigma^p \leq c + \frac{\kappa_{\infty} b_0}{4} \int_{\{|\varphi| \leq 3/2\}} |\sigma^p \ln \sigma|. \tag{3.37}$$

Similarly, we have

$$\mathbb{I}_2 \leq c \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) \sigma^p \leq c + \frac{\kappa_{\infty}}{4} \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) |\sigma^p \ln \sigma|. \tag{3.38}$$

The integrals on the right-hand sides of (3.37) and (3.38) can be estimated by noting that the last term on the left-hand side of (3.32) gives

$$\kappa_{\infty} \int_{\Omega} \beta(\varphi) \sigma^p \ln \sigma \geq \kappa_{\infty} b_0 \int_{\{|\varphi| \leq 3/2\}} |\sigma^p \ln \sigma| + \kappa_{\infty} \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) |\sigma^p \ln \sigma| - c, \tag{3.39}$$

as a straightforward check shows. The remaining two integrals on the right-hand side of (3.32) have a slower growth with respect to σ ; hence they can be controlled in a similar but in fact easier way.

As a consequence, it is not difficult to obtain from (3.32) the following inequality:

$$\begin{aligned} \frac{d}{dt} \mathcal{F}_n(\varphi, \sigma) + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + \frac{m_0}{2} \|\nabla \mu\|^2 + \frac{\kappa_{\infty} b_0}{2} \int_{\{|\varphi| \leq 3/2\}} |\sigma^p \ln \sigma| \\ + \frac{\kappa_{\infty}}{2} \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) |\sigma^p \ln \sigma| \leq c \|\nabla \varphi\|^2 + c + c \int_{\Omega} F_n(\varphi) + c \int_{\Omega} |L_n(\sigma)|. \end{aligned}$$

Estimating the last term similarly with (3.37) we end up with

$$\begin{aligned} \frac{d}{dt} (C_0 + \mathcal{F}_n(\varphi, \sigma)) + m_0 \int_{\Omega} \sigma |\nabla(\ln \sigma + \chi(1 - \varphi))|^2 + \frac{m_0}{2} \|\nabla \mu\|^2 + \frac{\kappa_{\infty} b_0}{4} \int_{\{|\varphi| \leq 3/2\}} |\sigma^p \ln \sigma| \\ + \frac{\kappa_{\infty}}{2} \int_{\{3/2 < |\varphi| \leq 2\}} \beta(\varphi) |\sigma^p \ln \sigma| \leq c \|\nabla \varphi\|^2 + c + c \int_{\Omega} F_n(\varphi) + c \int_{\Omega} |L_n(\sigma)|, \end{aligned} \tag{3.40}$$

where $C_0 > 0$ is large enough so that $C_0 + \mathcal{F}_n$ is coercive uniformly with respect to n (cf. (3.36)). Then, we can apply Grönwall’s lemma to the above relation so to deduce the following bounds which are independent of the approximation parameter n :

$$\|\varphi\|_{L^{\infty}(0, T; V)} \leq c, \tag{3.41}$$

$$\|\nabla \mu\|_{L^2(0, T; H)} \leq c, \tag{3.42}$$

$$\|F_n(\varphi)\|_{L^{\infty}(0, T; L^1(\Omega))} + \|L_n(\sigma)\|_{L^{\infty}(0, T; L^1(\Omega))} \leq c, \tag{3.43}$$

$$\|\sigma^{1/2} \nabla(\ln \sigma + \chi(1 - \varphi))\|_{L^2(0, T; H)} \leq c. \tag{3.44}$$

As in the previous part, the uniform control (3.4) of the spatial average of φ permits us to improve (3.42) leading to

$$\|\mu\|_{L^2(0, T; V)} \leq c. \tag{3.45}$$

Now, from the first of (3.43) and (3.35) it is not difficult to deduce that

$$|\{|\varphi(\cdot, t)| \geq 3/2\}| \leq cn^{-3} \quad \text{for a.e. } t \in (0, T).$$

As a consequence, we have

$$\begin{aligned} \int_0^t \int_{\Omega} |\sigma^p \ln \sigma| &= \int_0^t \int_{\{|\varphi(\cdot, t)| \leq 3/2\}} |\sigma^p \ln \sigma| + \int_0^t \int_{\{|\varphi(\cdot, t)| > 3/2\}} |\sigma^p \ln \sigma| \\ &\leq c + cn^p \ln n |\{|\varphi(\cdot, t)| > 3/2\}| \leq c + cn^{p-3} \ln n \leq c, \end{aligned}$$

where we used the control (3.40) and the fact $p \leq 2$. Hence, we end up with

$$\|\sigma^p \ln \sigma\|_{L^1(0,T;L^1(\Omega))} \leq c. \tag{3.46}$$

A similar procedure, combined with the second bound in (3.43), permits us to deduce

$$\|\sigma \ln \sigma\|_{L^\infty(0,T;L^1(\Omega))} \leq c. \tag{3.47}$$

Finally, we notice that the analogues of (3.17) and (3.19) can be obtained reasoning as in the previous section. On the other hand, concerning (3.21), since the property $|\varphi| \leq 1$ is not known to hold at the approximate level, we cannot apply the interpolation inequality (3.20) and for this reason the procedure should be modified by operating a proper truncation of the mobility function \mathfrak{m} . We omit the details since property (3.21) is not essential for the sequel. In addition to that, we notice that this difficulty does not arise when \mathfrak{m} is a constant function.

3.4. Passage to the limit

In this part, we assume to have a sequence $\{(\varphi_n, \mu_n, \sigma_n)\}_n$ of solutions complying with the a-priori estimates uniformly with respect to the parameter n . Such a sequence may be an outcome of the “strategy” (A)-(C) outlined before. In particular, we will assume $(\varphi_n, \mu_n, \sigma_n)$ to solve, at least locally in time, system (3.22)-(3.24) complemented with homogeneous Neumann boundary conditions and suitable initial conditions. Moreover, from now on, the dependence of the approximate solution on the parameter n is stressed.

Moreover, since the estimates derived in the previous part are uniform in time, by standard extension arguments the solution obtained in the limit will acquire a global in time character. For this reason, and for the sake of simplicity too, we will directly assume to have a global solution at the approximate level.

That said, we observe that the approximated version of estimates (3.17) and (3.19), (3.41)-(3.45), (3.46), with standard weak and weak star compactness results, imply that there exist limit functions $\varphi, \sigma, \mu,$ and ξ such that, as $n \rightarrow \infty$,

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly-star in } H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^p(0, T; W^{2,p}(\Omega)), \tag{3.48}$$

$$\sigma_n \rightharpoonup \sigma \quad \text{weakly in } L^p(0, T; L^p(\Omega)), \tag{3.49}$$

$$\mu_n \rightharpoonup \mu \quad \text{weakly in } L^2(0, T; V), \tag{3.50}$$

$$F'_n(\varphi_n) \rightharpoonup \xi \quad \text{weakly in } L^p(0, T; L^p(\Omega)). \tag{3.51}$$

The above convergence relations, as well as the ones that follow, are intended to hold up to the extraction of non-relabelled subsequences of $n \rightarrow \infty$. Since (3.26)-(3.28) imply in particular $|T'_n(r)| \leq 1$ for every $n \in \mathbb{N}$ and $r \in \mathbb{R}$, we also have

$$s_n \rightharpoonup s \quad \text{weakly in } L^p(0, T; L^p(\Omega)),$$

where, at this level, the functions s and σ need not be related to each other.

Next, note that (3.48), applying the Aubin–Lions lemma, also gives

$$\varphi_n \rightarrow \varphi \quad \text{strongly in } C^0([0, T]; H^{1-\delta}(\Omega)) \text{ for every } \delta > 0. \tag{3.52}$$

The above implies, in particular, pointwise (almost everywhere) convergence. As we will see below, properties (3.48)–(3.51) are sufficient to pass to the limit in the Cahn–Hilliard system (2.16)–(2.17). On the other hand, as is common in Keller–Segel-type models, the main difficulties arise when one considers the equation (2.18) for the chemical concentration. In particular, the key step stands in providing a suitable control of the cross-diffusion term, which has a quadratic growth. To this aim, the choice of a logistic source term is crucial and a suitable refined argument has to be devised.

Before detailing the procedure to pass to the limit, we need some preparation. In this direction, we set $Z := W^{1,2p/(p-1)}(\Omega)$ and we first notice that

$$\begin{aligned} & \|\sigma_n \mathfrak{m}(\varphi_n, \sigma_n) \nabla(\ln \sigma_n + \chi(1 - \varphi_n))\|_{L^{2p/(p+1)}(Q)} \\ & \leq M \|\sigma_n^{1/2}\|_{L^{2p}(Q)} \|\sigma_n^{1/2} \nabla(\ln \sigma_n + \chi(1 - \varphi_n))\|_{L^2(Q)} \leq c, \end{aligned} \tag{3.53}$$

the last inequality following from (3.44) and (3.46).

Next, we consider the second term in (3.24) multiplied by $z \in Z$. Integrating by parts and using the analogue on Ω of (3.53), we obtain

$$\begin{aligned} & \int_{\Omega} \sigma_n \mathfrak{m}(\varphi_n, \sigma_n) \nabla(\ln \sigma_n + \chi(1 - \varphi_n)) \cdot \nabla z \\ & \leq \|\sigma_n \mathfrak{m}(\varphi_n, \sigma_n) \nabla(\ln \sigma_n + \chi(1 - \varphi_n))\|_{2p/(p+1)} \|\nabla z\|_{2p/(p-1)} \\ & \leq M \|\sigma_n\|_p^{1/2} \|\sigma_n^{1/2} \nabla(\ln \sigma_n + \chi(1 - \varphi_n))\| \|z\|_Z. \end{aligned} \tag{3.54}$$

Moreover, we observe that, if p is as in the statement of Theorem 2.1, i.e., $p \in [3/2, 2]$ if $d = 2$ and $p \in [8/5, 2]$ if $d = 3$, then we also have $Z \subset L^\infty(\Omega)$ by Sobolev’s embeddings. As a consequence, we have

$$\int_{\Omega} \beta(\varphi_n) (\kappa_0 \sigma_n - \kappa_\infty \sigma_n^p) z \leq c(1 + \|\sigma_n\|_p^p) \|z\|_\infty \leq c(1 + \|\sigma_n\|_p^p) \|z\|_Z. \tag{3.55}$$

Hence, recalling (3.46) and (3.44), it is not difficult to deduce from (3.54) and (3.55) that

$$\|s_{n,t}\|_{L^1(0,T;Z^*)} \leq c. \tag{3.56}$$

Now, to apply once again the Aubin–Lions lemma, we also need an estimate of the gradient of s_n . To this aim, we need to decouple the information on the cross-diffusion term resulting from the energy estimate. In order to achieve this goal, the constraints on the exponent p in (2.6) are essential.

Indeed, we start noticing that, by the second in (3.19) (or the corresponding convergence (3.48)), there holds

$$\|\nabla \varphi_n\|_{L^p(0,T;L^{dp/(d-p)}(\Omega))} \leq c,$$

where, in the critical case $p = d = 2$, $dp/(d - p)$ is intended to be replaced by any $p \in [1, \infty)$.

Interpolating the above information with the $L^\infty(0, T; H)$ -bound resulting from the second in (3.48), it is then not difficult to obtain

$$\|\nabla\varphi_n\|_{L^{p(d+2)/d}(Q)} \leq c, \tag{3.57}$$

which holds also in the critical case $p = d = 2$ thanks to Ladyženskaja’s inequality and the last of (3.48). Combining this fact with the uniform L^p -bound for σ_n , we also deduce that

$$\|\sigma_n \nabla\varphi_n\|_{L^1(Q)} \leq c \quad \text{provided that} \quad \frac{1}{p} + \frac{d}{p(d+2)} \leq 1, \tag{3.58}$$

where the condition on the exponents corresponds exactly to $p \geq 8/5$ for $d = 3$ and $p \geq 3/2$ for $d = 2$, as stated in Theorem 2.1. Next, comparing (3.58) with (3.53) and using also that $|T'_n| \leq 1$, it is not difficult to obtain

$$\|\nabla\sigma_n\|_{L^1(Q)} + \|\nabla s_n\|_{L^1(Q)} \leq c. \tag{3.59}$$

Properties (3.56) and (3.59) allow us to apply to s_n the generalized Aubin–Lions in the form [44, Cor. 4, Sec. 8], which implies, in particular, the pointwise (a.e.) convergence $s_n \rightarrow s$.

More precisely, the control of the last summand in (3.46) provides the following uniform integrability estimate:

$$\|\sigma_n \ln^{1/p}(1 + \sigma_n)\|_{L^p(Q)} \leq c \tag{3.60}$$

and, as before, the same bound holds for s_n . Combining this fact with the pointwise convergence shown above and applying Vitali’s theorem [47], we then infer

$$s_n \rightarrow s \quad \text{strongly in } L^p(Q). \tag{3.61}$$

We now show that, in fact, the functions s and σ do coincide. To this aim, setting $\Omega_n^+ = \Omega_n^+(t) := \{x \in \Omega : \sigma_n(x, t) \geq n\}$, thanks to (3.33) there follows $|\Omega_n^+(t)| \leq c/(n \ln n)$ for almost every $t \in (0, T)$. As a consequence,

$$\begin{aligned} \|\sigma_n - s\|_{L^1(Q)} &\leq \|\sigma_n - s_n\|_{L^1(Q)} + \|s_n - s\|_{L^1(Q)} \\ &= \|\sigma_n - T_n(\sigma_n)\|_{L^1(Q)} + \|s_n - s\|_{L^1(Q)} \\ &\leq \int_0^T \int_{\Omega_n^+(t)} \sigma_n(\cdot, t) \, dt + \|s_n - s\|_{L^1(Q)} \\ &\leq \int_0^T \left(|\Omega_n^+(t)|^{\frac{p-1}{p}} \|\sigma_n\|_{L^p(\Omega_n^+(t))} \right) dt + \|s_n - s\|_{L^1(Q)} \\ &\leq \frac{c}{n^{\frac{p-1}{p}}} \|\sigma_n\|_{L^1(0, T; L^p(\Omega))} + \|s_n - s\|_{L^1(Q)}, \end{aligned} \tag{3.62}$$

and it is readily seen that the right-hand side tends to zero in view of (3.60) and (3.61) as $n \rightarrow \infty$. Comparing with (3.49), we then obtain the identification $s \equiv \sigma$. In particular, the truncation operator T_n disappears in the limit. For this reason, we can drop the use of the letter s in the limit and go back to the original variable σ . Notice also that, applying Vitali’s theorem again, we have

$$\sigma_n \rightarrow \sigma \quad \text{strongly in } L^p(Q). \tag{3.63}$$

We are now ready to take the limit $n \rightarrow \infty$ in the Cahn–Hilliard system. To this aim, we first notice that, by the facts that $\varphi_n \rightarrow \varphi$ and $\sigma_n \rightarrow \sigma$ almost everywhere, combined with the boundedness and Lipschitz continuity of $h, \mathfrak{m}, \mathfrak{n}$, and β , we may deduce, as $n \rightarrow \infty$,

$$h(\varphi_n, \sigma_n), \mathfrak{m}(\varphi_n, \sigma_n), \mathfrak{n}(\varphi_n, \sigma_n), \beta(\varphi_n) \rightarrow h(\varphi, \sigma), \mathfrak{m}(\varphi, \sigma), \mathfrak{n}(\varphi, \sigma), \beta(\varphi) \quad \text{strongly in } L^p(Q), \tag{3.64}$$

for every $p \in [1, \infty)$, thanks also to a generalized version of Lebesgue’s dominated convergence theorem. Combining this with (3.50) we have, by virtue of the weak-strong convergence principle,

$$\mathfrak{m}(\varphi_n, \sigma_n) \nabla \mu_n \rightarrow \mathfrak{m}(\varphi, \sigma) \nabla \mu \quad \text{weakly in } L^2(0, T; H). \tag{3.65}$$

Hence, in view of the above relations, testing (3.22) by a generic test function $v \in V$ and integrating by parts, it is apparent that all terms pass to $n \rightarrow \infty$ so to obtain (2.32) in the limit.

Concerning (2.30), this is obtained by testing (3.23) by $v \in V$ and then letting $n \rightarrow \infty$. To check that this procedure work we just need to take care of the nonlinear term depending on the configuration potential. In other words, going back to the weak convergence in (3.51), we need to identify the limit function ξ . To this aim, we first notice that from (3.52) and Sobolev’s embeddings there follows in particular

$$\varphi_n \rightarrow \varphi \quad \text{strongly in } L^q(0, T; L^q(\Omega)), \tag{3.66}$$

for any $q \in [1, 6)$. Hence, under our assumptions on p , we have in particular

$$\varphi_n \rightarrow \varphi \quad \text{strongly in } L^{p'}(0, T; L^{p'}(\Omega)), \tag{3.67}$$

where p' is the conjugate exponent to p .

Actually, the strong convergence (3.67) combined with the weak convergence (3.51) guarantees the identification $\xi = F'_1(\varphi)$ by means of a suitable version of the standard strong-weak compactness argument for maximal monotone operators. Indeed, we recall that the assumed Mosco-convergence $F_n \rightarrow F_1$ implies a convergence property on the maximal monotone operators induced by the derivatives F'_n . Referring once more to [2, Chap. 3] for the background, what holds is the *graph convergence*

$$F'_n \rightarrow F'_1 \quad \text{in } L^{p'}(Q) \times L^p(Q).$$

This corresponds to saying that, for every couple $[w, \eta] \in L^{p'}(Q) \times L^p(Q)$ such that $\eta = F'_1(w)$ a.e. in Q there exists a sequence $\{[w_n, \eta_n]\} \subset L^{p'}(Q) \times L^p(Q)$, with $\eta_n = F'_n(w_n)$ a.e. in Q and such that

$$[w_n, \eta_n] \rightarrow [w, \eta] \quad \text{strongly in } L^{p'}(Q) \times L^p(Q).$$

Thanks to this property, an appropriate version of the usual monotonicity argument in (reflexive) Banach spaces (cf., e.g., [3]) permits us to achieve that, as $n \rightarrow \infty$,

$$F'_n(\varphi_n) \rightarrow F'_1(\varphi) \quad \text{weakly in } L^p(Q).$$

Hence, we can pass to the limit $n \rightarrow \infty$ in (3.23) so to obtain (2.30).

Finally, we need to take the limit in the equation for σ , which is a bit trickier. First of all, we go back to (3.24), test it by $w \in \mathcal{W}_n$ and integrate by parts. Using the condition $\partial_n w = 0$ on the boundary and (2.14), we then get

$$\begin{aligned} \int_{\Omega} \mathfrak{m}(\varphi_n, \sigma_n) \nabla \sigma_n \cdot \nabla w &= \int_{\Omega} (\nabla N(\varphi_n, \sigma_n) - \mathfrak{m}_1(\varphi_n, \sigma_n) \nabla \varphi_n) \cdot \nabla w \\ &= - \int_{\Omega} N(\varphi_n, \sigma_n) \Delta w - \int_{\Omega} \mathfrak{m}_1(\varphi_n, \sigma_n) \nabla \varphi_n \cdot \nabla w, \end{aligned}$$

which leads us to the n -analogue of (2.33), namely

$$\begin{aligned} \langle s_{n,t}, w \rangle_{\mathcal{W}_n} - \int_{\Omega} N(\sigma_n, \varphi_n) \Delta w - \int_{\Omega} \mathfrak{m}_1(\varphi_n, \sigma_n) \nabla \varphi_n \cdot \nabla w - \chi \int_{\Omega} \sigma_n \mathfrak{m}(\varphi_n, \sigma_n) \nabla \varphi_n \cdot \nabla w \\ = \int_{\Omega} \beta(\varphi_n) (\kappa_0 \sigma_n - \kappa_{\infty} \sigma_n^p) w. \end{aligned} \tag{3.68}$$

To take the limit in this relation, we first observe that, by (3.63) and (3.64),

$$\beta(\varphi_n) (\kappa_0 \sigma_n - \kappa_{\infty} \sigma_n^p) \rightarrow \beta(\varphi) (\kappa_0 \sigma - \kappa_{\infty} \sigma^p) \quad \text{weakly in } L^1(Q). \tag{3.69}$$

Next, by (3.63), (3.67) and (2.13), it turns out that $N(\varphi_n, \sigma_n) \rightarrow N(\varphi, \sigma)$ almost everywhere. As a consequence of the generalized Lebesgue theorem we then deduce

$$N(\varphi_n, \sigma_n) \rightarrow N(\varphi, \sigma) \quad \text{strongly in } L^p(Q). \tag{3.70}$$

Analogously, recalling (2.15), we infer

$$\mathfrak{m}_1(\varphi_n, \sigma_n) \rightarrow \mathfrak{m}_1(\varphi, \sigma) \quad \text{strongly in } L^p(Q). \tag{3.71}$$

To deal with the cross-diffusion terms, for clarity we just consider the worst case, corresponding, as said, to $d = 3$ and $p = 8/5$. In that case, the exponent of the space in (3.57) reduces to $8/3$. Then, using (3.64) with the uniform boundedness of \mathfrak{m} and (3.57), we deduce

$$\mathfrak{m}(\varphi_n, \sigma_n) \nabla \varphi_n \rightarrow \mathfrak{m}(\varphi, \sigma) \nabla \varphi \quad \text{weakly in } L^{8/3}(Q),$$

whence, by virtue of (3.63), as $n \rightarrow \infty$,

$$\sigma_n \mathfrak{m}(\varphi_n, \sigma_n) \nabla \varphi_n \rightharpoonup \sigma \mathfrak{m}(\varphi, \sigma) \nabla \varphi \quad \text{weakly in } L^1(Q). \tag{3.72}$$

Analogously, owing to (3.71), we get, as $n \rightarrow \infty$,

$$\mathfrak{m}_1(\varphi_n, \sigma_n) \nabla \varphi_n \rightharpoonup \mathfrak{m}_1(\varphi, \sigma) \nabla \varphi \quad \text{weakly in } L^1(Q). \tag{3.73}$$

The above relations serve as a starting point to pass to the limit in (3.68). Indeed, the diffusion terms are managed by means of (3.70) and (3.72), whereas the right-hand side goes to the desired limit thanks to (3.69). On the other hand, the best estimate we have on $s_{n,t}$ is given by (3.56). Hence, in order to take the limit $n \rightarrow \infty$, we have to consider, as specified in the statement, $w \in C^1([0, T]; \mathcal{W}_n)$ and integrate (3.68) with respect to time between 0 and $t \leq T$ integrating by parts the first term. In this way, the time derivative of s_n disappears; nevertheless, (3.56) still does not suffice to take the desired limit, unless one uses a generalized tool like Helly’s selection principle. In particular, the limit function s is expected to be only BV with respect to time, which would allow it to have jumps with respect to the time variable.

In order to exclude this fact, we need to refine a bit the information on $s_{n,t}$ by exploiting in a suitable way the uniform integrability property (3.60). This procedure will allow us to recover also the initial datum in the sense (2.35). To this aim, we go back to (3.24) and test it by $w \in \mathcal{W}_n$. Using, in particular, the fact $\mathcal{W}_n \subset W^{1,\infty}(\Omega)$, it is then not difficult to obtain

$$\begin{aligned} \|s_{n,t}\|_{\mathcal{W}_n^*} &\leq c \|\sigma_n^{1/2}\| \|\sigma_n^{1/2} \nabla (\ln \sigma_n + \chi(1 - \varphi_n))\| + c(1 + \|\sigma_n\|_p^p) \\ &\leq c \|\sigma_n^{1/2}\| \|\sigma_n^{1/2} \nabla (\ln \sigma_n + \chi(1 - \varphi_n))\| + c + c\|\sigma_n\|_p^p =: M_{1,n} + c + M_{2,n}. \end{aligned} \tag{3.74}$$

Now, it is clear that $t \mapsto \|\sigma_n^{1/2}(t)\|$ is bounded in $L^{2p}(0, T)$ as a consequence of (3.63) and that $t \mapsto \|\sigma_n^{1/2}(t) \nabla (\ln \sigma_n(t) + \chi(1 - \varphi_n(t)))\|$ is bounded in $L^2(0, T)$ as a consequence of (3.44). Combining these facts, we readily obtain that

$$\|M_{1,n}\|_{L^{2p/(p+1)}(0,T)} \leq c, \tag{3.75}$$

with c independent of n . Let us now set, for $r > 0$, $\Phi(r) := r \ln(e + r)$ and let us notice that Φ is convex and increasing. Then, applying Φ to inequality (3.74) and integrating in time, it is not difficult to check that

$$\begin{aligned} \int_0^T \Phi(\|s_{n,t}\|_{\mathcal{W}_n^*}) &\leq \int_0^T \Phi(M_{1,n} + c + M_{2,n}) \\ &\leq c + c \int_0^T \Phi(M_{1,n}) + c \int_0^T \Phi(M_{2,n}) \\ &\leq c + c \int_0^T \Phi(c\|\sigma_n\|_p^p) \end{aligned}$$

$$\leq c + c \int_{\Omega} \sigma_n^p \ln(e + \sigma_n) \leq C, \tag{3.76}$$

where $C > 0$ is a computable constant depending only on the known data of the problem. Notice also that, to estimate the first integral on the second row, we used (3.75) with the fact $2p/(p + 1) > 1$.

Next, let $0 \leq \tau < t \leq T$. Then, we have

$$\frac{\|s_n(t) - s_n(\tau)\|_{\mathcal{W}_n^*}}{|t - \tau|} \leq \int_{\tau}^t \frac{1}{|t - \tau|} \|s_{n,t}(r)\|_{\mathcal{W}_n^*} \, dr.$$

Using that Φ is nondecreasing and convex, and applying Jensen’s inequality, we then deduce

$$\begin{aligned} \Phi\left(\frac{\|s_n(t) - s_n(\tau)\|_{\mathcal{W}_n^*}}{|t - \tau|}\right) &\leq \Phi\left(\int_{\tau}^t \frac{1}{|t - \tau|} \|s_{n,t}(r)\|_{\mathcal{W}_n^*} \, dr\right) \\ &\leq \int_{\tau}^t \frac{1}{|t - \tau|} \Phi(\|s_{n,t}(r)\|_{\mathcal{W}_n^*}) \, dr \\ &\leq \frac{1}{|t - \tau|} \int_0^T \Phi(\|s_{n,t}(r)\|_{\mathcal{W}_n^*}) \, dr \leq \frac{C}{|t - \tau|}, \end{aligned}$$

where $C > 0$ is the constant introduced in (3.76).

Then, using again the strict monotonicity of Φ , we deduce

$$\frac{\|s_n(t) - s_n(\tau)\|_{\mathcal{W}_n^*}}{|t - \tau|} \leq \Phi^{-1}\left(\frac{C}{|t - \tau|}\right),$$

or, in other words,

$$\|s_n(t) - s_n(\tau)\|_{\mathcal{W}_n^*} \leq |t - \tau| \Phi^{-1}\left(\frac{C}{|t - \tau|}\right).$$

Then, noting that Φ^{-1} is strictly sublinear at infinity, as a direct check shows, we may observe that there holds the following equicontinuity property: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ and every $0 \leq \tau < t \leq T$ with $|t - \tau| < \delta$ there holds

$$\|s_n(t) - s_n(\tau)\|_{\mathcal{W}_n^*} < \varepsilon.$$

Now, using (3.47) with (3.26)-(3.28), it is easy to deduce

$$\|s_n\|_{L^\infty(0,T;L^1(\Omega))} \leq c.$$

Hence, observing that $L^1(\Omega) \subset \mathcal{W}_n^*$ with compact embedding, if we take as \mathcal{Z} a generic (reflexive) Banach space (which, of course, will have a negative order as a Sobolev space) such that

$$L^1(\Omega) \subset\subset \mathcal{Z} \subset \mathcal{W}_n^*,$$

using some interpolation it is not difficult to check that Ascoli’s theorem can be applied to the sequence $\{s_n\}$ in the space $C^0([0, T]; \mathcal{Z})$ so to obtain

$$s_n \rightarrow s = \sigma \quad \text{strongly in } C^0([0, T]; \mathcal{Z}) \tag{3.77}$$

and, a fortiori, in $C^0([0, T]; \mathcal{W}_n^*)$. In particular, since $s_n|_{t=0} = T_n(\sigma_0)$ and $T_n(\sigma_0)$ tends to σ_0 in $L^1(\Omega)$ thanks to Lebesgue’s dominated convergence theorem, we obtain that the initial condition $\sigma|_{t=0} = \sigma_0$ is satisfied in a standard sense, which excludes the occurrence of jumps of σ with respect to the time variable. Moreover, (3.77) allows us to pass to the limit in the time-integrated version of (3.68) so to obtain (2.33) (which, we remark, also incorporates the boundary conditions). We incidentally notice that (2.25) also follows from the above procedure. In particular, the second regularity in (2.26) is a consequence of (3.47) and an equiintegrability argument. Finally, (2.36) follows from the analogue of (3.21). This concludes the proof of Theorem 2.1.

4. Proof of the regularity results

The proofs of the regularity results are mainly based on the derivation of higher-order additional sets of a-priori estimates. It is worth observing from the very beginning that these estimates will be derived in a formal way by working on the “original” system (1.1)–(1.5). We believe that, at this regularity level, obtaining the estimates in a fully rigorous way would require a very lengthy and technical adaptation of the approximation scheme. Since such a procedure would, however, present a very limited mathematical interest, we prefer to proceed formally.

Proof of Theorem 2.2. We distinguish between the 2D and the 3D cases, which have to be managed by different methods.

Two dimensional case. For convenience, let us start with the two dimensional case recalling that now $p = 2$. In that setting, we test (2.18) by σ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + m_0 \|\nabla \sigma\|^2 + \kappa_\infty b_0 \|\sigma\|_3^3 \leq c \|\sigma\|^2 + \chi \int_\Omega \sigma \mathfrak{n}(\varphi, \sigma) \nabla \varphi \cdot \nabla \sigma. \tag{4.1}$$

Then, we test (2.17) by $-\Delta \varphi$ to infer that

$$\int_\Omega F_1''(\varphi) |\nabla \varphi|^2 + \|\Delta \varphi\|^2 \leq -\chi \int_\Omega \sigma \Delta \varphi + \lambda \|\nabla \varphi\|^2 + \|\nabla \varphi\| \|\nabla \mu\|, \tag{4.2}$$

whence, squaring and using the previous estimates with the monotonicity of F_1' , standard manipulations lead us to

$$\|\Delta \varphi\|^4 \leq c(1 + \|\sigma\|^4 + \|\nabla \mu\|^2). \tag{4.3}$$

Then, to control the last term on the right-hand side of (4.1), we observe that, by Hölder’s and standard interpolation inequalities holding for $d = 2$,

$$\begin{aligned}
 \chi \int_{\Omega} \mathfrak{n}(\varphi, \sigma) \sigma \nabla \varphi \cdot \nabla \sigma &\leq c \|\sigma\|_4 \|\nabla \varphi\|_4 \|\nabla \sigma\| \\
 &\leq c \|\sigma\|^{1/2} \|\sigma\|_V^{1/2} \|\varphi\|_V^{1/2} \|\varphi\|_{H^2(\Omega)}^{1/2} \|\nabla \sigma\| \\
 &\leq c \|\sigma\|^{1/2} (\|\sigma\|^{1/2} + \|\nabla \sigma\|^{1/2}) (1 + \|\Delta \varphi\|^{1/2}) \|\nabla \sigma\| \\
 &\leq \frac{m_0}{2} \|\nabla \sigma\|^2 + \frac{1}{2} \|\Delta \varphi\|^4 + c(\|\sigma\|^4 + 1),
 \end{aligned} \tag{4.4}$$

where we used in particular that $t \mapsto \|\varphi(t)\|_V$ is $L^\infty(0, T)$ and Young’s inequality. Summing (4.1) with (4.3) and using (4.4), we then arrive at

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + \frac{m_0}{2} \|\nabla \sigma\|^2 + \kappa_\infty b_0 \|\sigma\|_3^3 + \frac{1}{2} \|\Delta \varphi\|^4 \leq c(1 + \|\sigma\|^4 + \|\nabla \mu\|^2).$$

Next, recalling that (2.26) holds with $p = 2$, an application of Grönwall’s lemma, along with elliptic regularity results, yields the additional regularity bounds

$$\|\varphi\|_{L^4(0,T;H^2(\Omega))} \leq c, \tag{4.5}$$

$$\|\sigma\|_{L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^3(0,T;L^3(\Omega))} \leq c, \tag{4.6}$$

where (2.37) has also been used.

Using (4.5)-(4.6) and comparing terms in (2.18), it is then a standard matter to derive that

$$\|\sigma_t\|_{L^2(0,T;V^*)} \leq c.$$

This permits us to write the nutrient equation in the standard form (2.43) rather than in the integrated form (2.33); moreover, by classical results for second-order parabolic equations, this also gives the continuity property in (2.41). By the above relations we also recover the usual regularity scenario for the Cahn–Hilliard equation with singular potential under the “energy regularity” of the initial data in two space dimensions, i.e.,

$$\varphi \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^4(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)),$$

for any $q \in [1, \infty)$, where the latter regularity property is obtained by considering once more (2.17) as an elliptic equation with maximal monotone nonlinearity, namely

$$-\Delta \varphi + F'_1(\varphi) = \lambda \varphi + \chi \sigma + \mu, \tag{4.7}$$

and noting that the right-hand side lies (or, in a suitable approximation, is uniformly bounded in) $L^2(0, T; L^q(\Omega))$ thanks to continuity of the two-dimensional embedding $V \hookrightarrow L^q(\Omega)$ for any $q \in [1, \infty)$.

Three dimensional case. We now move to the three dimensional case. As said before, we proceed formally and, to begin, we provide an auxiliary estimate which will play an important role in the sequel. To this aim, we set $\gamma(\varphi) := -(F'_1)_-^5(\varphi)$ (where $(\cdot)_-$ denotes the negative part of a quantity). Then, noting that γ is monotone and nonpositive, we test (2.17) by $\gamma(\varphi)$ to obtain

$$\int_{\Omega} F'_1(\varphi)\gamma(\varphi) + \int_{\Omega} \gamma'(\varphi)|\nabla\varphi|^2 = \lambda \int_{\Omega} \varphi\gamma(\varphi) + \chi \int_{\Omega} \sigma\gamma(\varphi) + \int_{\Omega} \mu\gamma(\varphi). \tag{4.8}$$

Now, as $\gamma(\varphi) = -(F'_1)_-^5(\varphi) = F'_1(\varphi)^5\chi_{\{\varphi < 0\}}$ (recall the normalization property $F'_1(0) = 0$, cf. (A1)), it is clear that

$$\int_{\Omega} F'_1(\varphi)\gamma(\varphi) = \|(F'_1)_-(\varphi)\|_6^6. \tag{4.9}$$

Moreover, the second term on the left-hand side of (4.8) is clearly nonnegative, while the second term on the right-hand side is nonpositive due to (2.24). By Hölder’s and Young’s inequalities we also have

$$\begin{aligned} \lambda \int_{\Omega} \varphi\gamma(\varphi) + \int_{\Omega} \mu\gamma(\varphi) &\leq c(\|\varphi\|_6 + \|\mu\|_6)\|\gamma(\varphi)\|_{6/5} \leq c(1 + \|\mu\|_6)\|(F'_1)_-(\varphi)\|_6^5 \\ &\leq c(1 + \|\mu\|_6^6) + \frac{1}{2}\|(F'_1)_-(\varphi)\|_6^6. \end{aligned} \tag{4.10}$$

Hence, replacing (4.9) and (4.10) into (4.8), it is not difficult to deduce

$$\frac{1}{2}\|(F'_1)_-(\varphi)\|_6^6 \leq c(1 + \|\mu\|_6^6).$$

Taking the cubic root, using Sobolev’s embeddings and recalling (2.28) this implies

$$(F'_1)_-(\varphi) \in L^2(0, T; L^6(\Omega)). \tag{4.11}$$

Next, recalling assumptions (2.38)-(2.39), we may test (2.18) (with $n \equiv 1$ and $p = 2$) by σ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + \|\nabla\sigma\|^2 + \kappa_{\infty} b_0 \|\sigma\|_3^3 \leq c\|\sigma\|^2 + \chi \int_{\Omega} \sigma \nabla\varphi \cdot \nabla\sigma \tag{4.12}$$

and we need to properly manipulate the last term on the right-hand side. To this aim, we integrate by parts and exploit the no-flux conditions with relation (1.2) to deduce

$$\begin{aligned} \chi \int_{\Omega} \sigma \nabla\varphi \cdot \nabla\sigma &= \frac{\chi}{2} \int_{\Omega} \nabla\varphi \cdot \nabla(\sigma^2) = -\frac{\chi}{2} \int_{\Omega} \Delta\varphi \sigma^2 \\ &= \frac{\chi}{2} \int_{\Omega} \mu\sigma^2 - \frac{\chi}{2} \int_{\Omega} F'_1(\varphi)\sigma^2 + \frac{\lambda\chi}{2} \int_{\Omega} \varphi\sigma^2 + \frac{\chi^2}{2} \|\sigma\|_3^3 \end{aligned}$$

$$=: \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 + \mathbb{I}_4.$$

We now provide a bound of the various terms on the right-hand side. First of all, for every $\delta > 0$,

$$\mathbb{I}_1 \leq \|\mu\|_6 \|\sigma\|_3 \|\sigma\| \leq \delta \|\sigma\|_3^2 + c_\delta \|\mu\|_V^2 \|\sigma\|^2.$$

The second term is the key one. Using again that $F'_1(\varphi)$ has the same sign as φ , we have

$$\begin{aligned} \mathbb{I}_2 &= -\frac{\chi}{2} \int_{\{\varphi \geq 0\}} F'_1(\varphi) \sigma^2 - \frac{\chi}{2} \int_{\{\varphi < 0\}} F'_1(\varphi) \sigma^2 \leq -\frac{\chi}{2} \int_{\{\varphi < 0\}} F'_1(\varphi) \sigma^2 \\ &= \frac{\chi}{2} \int_{\Omega} |(F'_1)_-(\varphi)| \sigma^2 \leq \frac{\chi}{2} \|(F'_1)_-(\varphi)\|_6 \|\sigma\|_3 \|\sigma\| \\ &\leq \delta \|\sigma\|_3^2 + c_\delta \|(F'_1)_-(\varphi)\|_6^2 \|\sigma\|^2. \end{aligned}$$

The control of \mathbb{I}_3 is immediate, while \mathbb{I}_4 has to be moved to the left-hand side. Collecting the above considerations, then (4.12) gives, for every “small” $\delta > 0$ and correspondingly “large” $c_\delta > 0$,

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + \|\nabla \sigma\|^2 + \left(\kappa_\infty b_0 - \frac{\chi^2}{2} \right) \|\sigma\|_3^3 \leq c_\delta \left(1 + \|(F'_1)_-(\varphi)\|_6^2 + \|\mu\|_V^2 \right) \|\sigma\|^2 + 3\delta \|\sigma\|_3^2.$$

Hence, under the compatibility condition (2.38), recalling (2.28) and the preliminary bound (4.11) we can adjust $\delta \in (0, 1)$ and apply Grönwall’s lemma to deduce

$$\sigma \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^3(0, T; L^3(\Omega)).$$

Finally, with this property at hand, the regularity of φ can be bootstrapped easily by arguing as done above for the two dimensional case. \square

Remark 4.1. It is not difficult to check that, in the three-dimensional case, the “smallness” condition (2.38) might be avoided if one takes a superquadratic logistic term on the right-hand side of (1.3), i.e., $\beta(\varphi)(\kappa_0 \sigma - \kappa_\infty \sigma^{2+\rho})$, where $\rho > 0$ may be arbitrarily small.

Remark 4.2. In the three dimensional case, even when the mobility \mathfrak{m} is not taken as a constant function, something could still be said. Indeed, (4.12) would then be replaced by

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 + m_0 \|\nabla \sigma\|^2 + \kappa_\infty b_0 \|\sigma\|_3^3 \leq c \|\sigma\|^2 + \chi \int_{\Omega} \sigma \mathfrak{m}(\varphi, \sigma) \nabla \varphi \cdot \nabla \sigma$$

and the last term could be integrated by parts as follows:

$$\chi \int_{\Omega} \sigma \mathfrak{m}(\varphi, \sigma) \nabla \varphi \cdot \nabla \sigma = -\chi \int_{\Omega} N_2(\varphi, \sigma) \Delta \varphi - \chi \int_{\Omega} \mathfrak{m}_2(\varphi, \sigma) |\nabla \varphi|^2, \tag{4.13}$$

where we have set

$$N_2(\varphi, \sigma) := \int_0^\sigma \mathfrak{m}(\varphi, s) s \, ds, \quad \mathfrak{m}_2(\varphi, \sigma) := \int_0^\sigma \partial_\varphi \mathfrak{m}(\varphi, s) s \, ds.$$

Then, the procedure performed above can be adapted at least when either \mathfrak{m} depends only on σ (so that the last integral in (4.13) disappears) or \mathfrak{m} satisfies proper structure assumptions ensuring that the last integral in (4.13) is nonnegative (so that it can be moved to the left-hand side and does not need to be controlled). On the other hand, in the general case (i.e., for \mathfrak{m} depending both on φ and σ with no sign conditions), the last integrand in (4.13) behaves like $\sigma^2 |\nabla \varphi|^2$, which appears to have a supercritical behavior in space dimension $d = 3$, as the interested reader can verify.

Proof of Theorem 2.3. Again, we derive additional a-priori estimates in a formal way and without referring to the proposed approximation. We also recall that, at this level, weak solutions are already known to enjoy the regularity in (2.22)-(2.29) and (2.40)-(2.41). That said, we formally differentiate equation (2.17) with respect to time obtaining the identity

$$\mu_t = -\Delta \varphi_t + F_1''(\varphi) \varphi_t - \lambda \varphi_t - \chi \sigma_t. \tag{4.14}$$

Next, we multiply (2.16) (where $\mathfrak{m} \equiv 1$) by μ_t , the above expression (4.14) by φ_t , and add to both sides the term $\|\varphi_t\|^2$. Then, summing the resulting equalities together and integrating by parts we infer

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla \mu\|^2 + \|\varphi_t\|_V^2 + \int_\Omega F_1''(\varphi) |\varphi_t|^2 \\ &= \int_\Omega S(\varphi, \sigma) \mu_t + (1 + \lambda) \|\varphi_t\|^2 + \chi \int_\Omega \sigma_t \varphi_t \\ &= \frac{d}{dt} \int_\Omega S(\varphi, \sigma) \mu - \int_\Omega \partial_t (S(\varphi, \sigma)) \mu + (1 + \lambda) \|\varphi_t\|^2 + \chi \int_\Omega \sigma_t \varphi_t. \end{aligned}$$

Note that here we assumed, just for simplicity, that F_1 is twice differentiable, which is unnecessarily true under our assumption (A1). However, it is easy to see that, using standard convex analysis tools, the argument might be adapted to work for nonsmooth, but convex, F_1 (as in our case). That said, we add to the above relation the result of (2.18) (where $\mathfrak{m} \equiv 1$) tested by σ_t . Using the specific expression of the source term in (2.6) with $p = 2$, after some rearrangements we obtain

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|\nabla \mu\|^2 - \int_\Omega S(\varphi, \sigma) \mu + \frac{1}{2} \|\nabla \sigma\|^2 \right) + \|\varphi_t\|_V^2 + \|\sigma_t\|^2 \\ & \leq (1 + \lambda) \|\varphi_t\|^2 - \int_\Omega \partial_t (S(\varphi, \sigma)) \mu + \chi \int_\Omega \sigma_t \varphi_t - \chi \int_\Omega (\nabla \sigma \cdot \nabla \varphi + \sigma \Delta \varphi) \sigma_t \end{aligned}$$

$$+ \int_{\Omega} \beta(\varphi)(\kappa_0\sigma - \kappa_{\infty}\sigma^2)\sigma_t, \tag{4.15}$$

and we need to control the terms on the right-hand side. First of all, by elementary interpolation, it is clear that

$$(1 + \lambda)\|\varphi_t\|^2 \leq \frac{1}{8}\|\varphi_t\|_V^2 + c\|\varphi_t\|_*^2. \tag{4.16}$$

Next, by the global Lipschitz continuity of S (cf. **(A2)**) and a well-known chain rule formula for Lipschitz functions in Sobolev spaces, we obtain

$$- \int_{\Omega} \partial_t(S(\varphi, \sigma))\mu \leq c \int_{\Omega} (|\varphi_t| + |\sigma_t|)|\mu| \leq \frac{1}{8}\|\varphi_t\|^2 + \frac{1}{8}\|\sigma_t\|^2 + c\|\mu\|^2.$$

Analogously, owing to interpolation once again, we deduce

$$\chi \int_{\Omega} \sigma_t\varphi_t = \chi \langle \sigma_t, \varphi_t \rangle_V \leq \frac{1}{8}\|\varphi_t\|_V^2 + c\|\sigma_t\|_*^2.$$

The estimation of the remaining terms is just a bit more involved. Firstly we notice that, by standard Sobolev’s embeddings holding both in the two- and in the three-dimensional case, we have

$$\begin{aligned} -\chi \int_{\Omega} (\nabla\sigma \cdot \nabla\varphi + \sigma\Delta\varphi)\sigma_t &\leq c(\|\nabla\sigma\| \|\nabla\varphi\|_{\infty} \|\sigma_t\| + \|\sigma\|_4 \|\Delta\varphi\|_4 \|\sigma_t\|) \\ &\leq \frac{1}{8}\|\sigma_t\|^2 + c\|\sigma\|_V^2 \|\varphi\|_{W^{2,4}(\Omega)}^2. \end{aligned}$$

Next, using (2.7), we infer

$$\begin{aligned} \int_{\Omega} \beta(\varphi)(\kappa_0\sigma - \kappa_{\infty}\sigma^2)\sigma_t &\leq B(\kappa_0\|\sigma\| + \kappa_{\infty}\|\sigma\|_4^2)\|\sigma_t\| \\ &\leq \frac{1}{8}\|\sigma_t\|^2 + c(1 + \|\sigma\|_V^4). \end{aligned} \tag{4.17}$$

Replacing the outcome of (4.16)-(4.17) into (4.15), we then arrive at

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2}\|\nabla\mu\|^2 - \int_{\Omega} S(\varphi, \sigma)\mu + \frac{1}{2}\|\nabla\sigma\|^2 \right) &+ \frac{5}{8}\|\varphi_t\|_V^2 + \frac{5}{8}\|\sigma_t\|^2 \\ &\leq c + c\|\varphi_t\|_*^2 + c\|\sigma_t\|_*^2 + c\|\mu\|^2 + c(1 + \|\sigma\|_V^2 + \|\varphi\|_{W^{2,4}(\Omega)}^2)\|\sigma\|_V^2. \end{aligned} \tag{4.18}$$

To close the estimate, we go back to (4.12). Neglecting some nonnegative terms on the left-hand side and performing standard manipulations, that relation implies

$$\frac{1}{2} \frac{d}{dt} \|\sigma\|^2 \leq c \|\sigma\|^2 + c \|\sigma\|_6 \|\nabla \sigma\| \|\nabla \varphi\|_3 \leq c(1 + \|\varphi\|_{H^2(\Omega)}^2) \|\sigma\|_V^2. \tag{4.19}$$

Adding this relation to (4.18), we recover the full V -norm of σ on the left-hand side. Next, we prove that the functional we get under time derivative is coercive. In this direction, we employ (1.2), Young’s inequality and the uniform boundedness of S (in particular we can now use that $-1 \leq \varphi \leq 1$ almost everywhere) to infer that

$$\begin{aligned} - \int_{\Omega} S(\varphi, \sigma)\mu &= - \int_{\Omega} S(\mu - \mu_{\Omega}) - \int_{\Omega} S\mu_{\Omega} \geq -c\|\mu - \mu_{\Omega}\| - c|\mu_{\Omega}| \\ &\geq -c\|\nabla \mu\| - c|\mu_{\Omega}| \geq -\frac{1}{8}\|\nabla \mu\|^2 - c|\mu_{\Omega}| - c \\ &\geq -\frac{1}{8}\|\nabla \mu\|^2 - c(\|F'_1(\varphi)\|_1 + \|\varphi\|_1 + \|\sigma\|_1) \\ &\geq -\frac{1}{8}\|\nabla \mu\|^2 - c\|F'_1(\varphi)\|_1 - c. \end{aligned} \tag{4.20}$$

Next, testing once more (2.17) by $\varphi - \varphi_{\Omega}$ and proceeding similarly with (3.11)-(3.13), we arrive at

$$\alpha \|F'_1(\varphi)\|_1 \leq c(1 + \|\nabla \mu\|), \tag{4.21}$$

where $\alpha > 0$ is as in (3.12) and $c > 0$ on the right-hand side also depends on other quantities that have already been controlled uniformly with respect to time. As a consequence, from (4.18)-(4.19), we deduce the differential inequality

$$\begin{aligned} \frac{d}{dt} \left(\underbrace{\frac{1}{2}\|\nabla \mu\|^2 - \int_{\Omega} S(\varphi, \sigma)\mu + \frac{1}{2}\|\sigma\|_V^2}_{=: \mathcal{J}} \right) + \frac{5}{8}\|\sigma_t\|^2 &\leq c + c\|\varphi_t\|_*^2 + c\|\sigma_t\|_*^2 + c\|\mu\|^2 \\ &+ c(1 + \|\sigma\|_V^2 + \|\varphi\|_{W^{2,4}(\Omega)}^2 + \|\varphi\|_{H^2(\Omega)}^2) \|\sigma\|_V^2, \end{aligned} \tag{4.22}$$

where the functional \mathcal{J} , thanks to (4.20)-(4.21), satisfies

$$\mathcal{J} \geq \frac{3}{8}\|\nabla \mu\|^2 - c\|\nabla \mu\| - c + \frac{1}{2}\|\sigma\|_V^2 \geq \frac{1}{4}\|\nabla \mu\|^2 + \frac{1}{2}\|\sigma\|_V^2 - C,$$

and $C > 0$ depends only on quantities that have already been controlled uniformly in time. Hence, for $C > 0$ as above, (4.22) can be rewritten in the form

$$\begin{aligned} \frac{d}{dt}(\mathcal{J} + C) + \frac{5}{8}\|\varphi_t\|_V^2 + \frac{5}{8}\|\sigma_t\|^2 \\ \leq c + c\|\varphi_t\|_*^2 + c\|\sigma_t\|_*^2 + c\|\mu\|^2 + c(1 + \|\sigma\|_V^2 + \|\varphi\|_{W^{2,4}(\Omega)}^2 + \|\varphi\|_{H^2(\Omega)}^2)(\mathcal{J} + C). \end{aligned}$$

Then, recalling (2.28) and (2.40)-(2.41), an application of Grönwall’s lemma yields the estimate

$$\|\varphi\|_{H^1(0,T;V)} + \|\nabla\mu\|_{L^\infty(0,T;H)} + \|\sigma\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c,$$

provided that the functional \mathcal{J} is finite at the initial time, and we actually note that this follows, at least formally, from (2.45).

Next, using the control on the mean value of μ resulting from (4.21), it is a standard matter to infer that

$$\|\mu\|_{L^\infty(0,T;V)} \leq c.$$

In turn, this also allows us to improve the regularity of φ . Indeed, we may go back to relation (4.7) and notice that, now, the right-hand side lies in the space $L^\infty(0, T; V)$. Then, arguing once more as in [25, Lemmas 7.3 and 7.4], we deduce that

$$\|F'_1(\varphi)\|_{L^\infty(0,T;L^q(\Omega))} + \|\varphi\|_{L^\infty(0,T;W^{2,q}(\Omega))} \leq c, \tag{4.23}$$

where $q = 6$ if $d = 3$ and $q \in [1, \infty)$ if $d = 2$. Finally, by a comparison of terms in (2.16), it is easy to check that

$$\|\varphi_t\|_{L^\infty(0,T;V^*)} \leq c,$$

whereas, applying elliptic regularity in (2.18), one can easily deduce

$$\|\sigma\|_{L^2(0,T;H^2(\Omega))} \leq c.$$

Noting that the continuity property in (2.49) is, once more, a consequence of standard regularity results, this concludes the proof of the theorem. \square

Proof of Theorem 2.4. First of all, proceeding as in [25, Lemmas 7.3 and 7.4] and using the growth condition (2.50) with the Trudinger–Moser inequality (see also [38]), we deduce

$$\|F''_1(\varphi)\|_{L^\infty(0,T;L^q(\Omega))} \leq c, \tag{4.24}$$

for any $q \in [1, \infty)$. This acts as a starting point to prove the additional regularity in the statement. As before, we proceed formally to avoid unnecessary technicalities, noting that rigorous estimates could be performed, e.g., by working on a time discrete level as done in [25]. In this direction, we differentiate in time (2.16) (where, we recall, $\mathfrak{m} \equiv 1$), to find

$$\varphi_{tt} = \Delta\mu_t + (S(\varphi, \sigma))_t = \Delta\mu_t - m\varphi_t + \partial_\varphi h(\varphi, \sigma)\varphi_t + \partial_\sigma h(\varphi, \sigma)\sigma_t.$$

Then, we test the above equation by φ_t . Integrating by parts and using the Lipschitz continuity of h , we deduce

$$\frac{1}{2} \frac{d}{dt} \|\varphi_t\|^2 + \int_{\Omega} \nabla\mu_t \cdot \nabla\varphi_t \leq c(\|\varphi_t\|^2 + \|\sigma_t\|^2). \tag{4.25}$$

Next, differentiating (2.17) in time and testing the result by $-\Delta\varphi_t$, we get

$$\int_{\Omega} \nabla \mu_t \cdot \nabla \varphi_t = \|\Delta \varphi_t\|^2 - \int_{\Omega} F_1''(\varphi) \varphi_t \Delta \varphi_t - \lambda \|\nabla \varphi_t\|^2 + \chi \int_{\Omega} \sigma_t \Delta \varphi_t. \tag{4.26}$$

Combining (4.25) with (4.26) and performing standard manipulations, it is easy to get

$$\frac{1}{2} \frac{d}{dt} \|\varphi_t\|^2 + \frac{1}{2} \|\Delta \varphi_t\|^2 \leq c(\|\varphi_t\|_V^2 + \|\sigma_t\|^2) + \int_{\Omega} F_1''(\varphi) \varphi_t \Delta \varphi_t \tag{4.27}$$

and the last term can be controlled as follows:

$$\begin{aligned} \int_{\Omega} F_1''(\varphi) \varphi_t \Delta \varphi_t &\leq \|F_1''(\varphi)\|_4 \|\varphi_t\|_4 \|\Delta \varphi_t\| \\ &\leq c \|F_1''(\varphi)\|_4^2 \|\varphi_t\|_V^2 + \frac{1}{4} \|\Delta \varphi_t\|^2 \leq c \|\varphi_t\|_V^2 + \frac{1}{4} \|\Delta \varphi_t\|^2, \end{aligned} \tag{4.28}$$

the last inequality following from (4.24). Hence, replacing (4.28) into (4.27), using the known regularity properties (2.46) and (2.49), we deduce

$$\|\varphi_t\|_{L^\infty(0,T;H)} + \|\varphi_t\|_{L^2(0,T;H^2(\Omega))} \leq c, \tag{4.29}$$

provided φ_t lies in H at the initial time. As before, this property has to be read by formally evaluating (2.16) at the time $t = 0$. Then, by a direct check one can verify that this corresponds exactly to the condition on μ_0 postulated in (2.51).

Then, viewing (2.16) as a family of time-dependent elliptic equations whose right-hand sides lie in $L^\infty(0, T; H) \cap L^2(0, T; V)$ due to (4.29), (2.49) and the Lipschitz continuity of h , we deduce

$$\|\mu\|_{L^\infty(0,T;H^2(\Omega))} + \|\mu\|_{L^2(0,T;H^3(\Omega))} \leq c.$$

Note that the above, by Sobolev’s embeddings, also gives

$$\|\mu\|_{L^\infty(Q)} \leq c. \tag{4.30}$$

Next, to improve the regularity of φ , we rewrite (2.17) as

$$-\Delta \varphi = \mu - F_1'(\varphi) + \lambda \varphi + \chi \sigma.$$

Then, recalling (4.23) and (4.24), a simple check permits us to verify that the above right-hand side lies (at least) in $L^\infty(0, T; V)$. Hence, by elliptic regularity we deduce also

$$\|\varphi\|_{L^\infty(0,T;H^3(\Omega))} \leq c. \tag{4.31}$$

Next, to get the L^∞ -bound of σ , we come back to (2.18), which, rearranging, can be rewritten as

$$\sigma_t - \Delta \sigma = -\chi(\nabla \sigma \cdot \nabla \varphi + \sigma \Delta \varphi) + \beta(\varphi)(\kappa_0 \sigma - \kappa_\infty \sigma^2) =: G.$$

We now claim that $G \in L^\infty(0, T; H)$. To check this, we consider only the two cross-diffusion terms, the other ones being simpler to deal with. Indeed, we first observe that

$$\|\nabla\sigma \cdot \nabla\varphi\|_{L^\infty(0,T;H)} \leq c\|\nabla\sigma\|_{L^\infty(0,T;H)}\|\nabla\varphi\|_{L^\infty(Q)} \leq c,$$

thanks to (4.31), (2.49) and Sobolev’s embeddings. Analogously,

$$\|\sigma \Delta\varphi\|_{L^\infty(0,T;H)} \leq c\|\sigma\|_{L^\infty(0,T;L^4(\Omega))}\|\Delta\varphi\|_{L^\infty(0,T;L^4(\Omega))} \leq c.$$

Then, recalling the assumption (2.55) on the initial datum, by an application of [36, Thm. 7.1, p. 181] we readily obtain (2.56). Finally, the above regularity allows us to obtain the separation property (2.57). To this aim, we go back to the expression (3.18) and notice that the right-hand side, thanks to (4.30), (4.31) and (2.56), is now bounded in the $L^\infty(Q)$ -norm. Hence, (2.57) can be obtained by reasoning exactly as in the proof of [14, Thm. 2.2]. This concludes the proof. \square

Remark 4.3. With the separation property (2.57) at disposal, the singular character of F'_1 at ± 1 is essentially lost and the term $F'_1(\varphi)$ in (2.17) behaves like a smooth function of φ with controlled growth. Thanks to this fact, the regularity of solutions may be further improved at least as far as the nonlinear terms (like h , or F_1 itself) satisfy additional regularity properties (e.g., C^k for large k).

5. Uniqueness of strong solutions

This section is devoted to the proof of Theorem 2.8. We first recall that in Subsection 2.1 was introduced the operator $\mathcal{N} : V'_0 \rightarrow V_0$ representing, in a suitable weak sense, the inverse of (minus) the Neumann Laplacian acting on the functions with zero spatial average. Moreover, as anticipated in Remark 2.11, we just consider the case $d = 3$, noting that the conditions may be relaxed in the two-dimensional setting.

Proof of Theorem 2.8. Let us assume to have a couple of solutions $(\varphi_1, \mu_1, \sigma_1)$ and $(\varphi_2, \mu_2, \sigma_2)$ fulfilling the assumptions of the theorem and let us correspondingly set

$$\begin{aligned} \varphi &:= \varphi_1 - \varphi_2, & \mu &:= \mu_1 - \mu_2, & \sigma &:= \sigma_1 - \sigma_2, \\ S_i &:= S(\varphi_i, \sigma_i) \text{ for } i = 1, 2, & \varphi_0 &:= \varphi_{0,1} - \varphi_{0,2}, & \sigma_0 &:= \sigma_{0,1} - \sigma_{0,2}. \end{aligned} \tag{5.1}$$

Then, under the assumptions of the theorem the triplet (φ, μ, σ) turns out to solve the system

$$\varphi_t = \Delta\mu + (S_1 - S_2) \quad \text{in } Q, \tag{5.2}$$

$$\mu = -\Delta\varphi + (f(\varphi_1) - f(\varphi_2)) - \chi\sigma \quad \text{in } Q, \tag{5.3}$$

$$\sigma_t - \Delta\sigma + \chi \operatorname{div}(\sigma\nabla\varphi_1 + \sigma_2\nabla\varphi) = \kappa_0\sigma - \kappa_\infty\sigma(\sigma_1 + \sigma_2) \quad \text{in } Q, \tag{5.4}$$

$$\partial_n\varphi = \partial_n\mu = \partial_n\sigma = 0 \quad \text{on } \Sigma, \tag{5.5}$$

$$\varphi|_{t=0} = \varphi_0, \quad \sigma|_{t=0} = \sigma_0 \quad \text{in } \Omega. \tag{5.6}$$

We then start by integrating (5.2) over Ω to find that

$$\varphi'_\Omega = (S_1 - S_2)_\Omega = \frac{1}{|\Omega|} \int_\Omega (S(\varphi_1, \sigma_1) - S(\varphi_2, \sigma_2)). \tag{5.7}$$

Testing the above by φ_Ω and using Young’s inequality with the Lipschitz continuity of S , we easily deduce

$$\frac{1}{2} \frac{d}{dt} |\varphi_\Omega|^2 \leq |\varphi_\Omega|^2 + c(\|\varphi\|^2 + \|\sigma\|^2). \tag{5.8}$$

Next, we subtract (5.7) from (5.2) and test the resulting equality by $\mathcal{N}(\varphi - \varphi_\Omega)$ obtaining that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi - \varphi_\Omega\|_*^2 + \int_\Omega (\varphi - \varphi_\Omega)(\mu - \mu_\Omega) &= \int_\Omega ((S_1 - S_2) - (S_1 - S_2)_\Omega) \mathcal{N}(\varphi - \varphi_\Omega) \\ &\leq c\|\varphi\|^2 + c\|\sigma\|^2 + c\|\varphi - \varphi_\Omega\|_*^2. \end{aligned} \tag{5.9}$$

Let us now point out that, by the Poincaré–Wirtinger inequality and some elementary interpolation,

$$\begin{aligned} c\|\varphi\|^2 &\leq c(\|\varphi - \varphi_\Omega\|^2 + |\varphi_\Omega|^2) \leq c\|\varphi - \varphi_\Omega\|_V \|\varphi - \varphi_\Omega\|_* + c|\varphi_\Omega|^2 \\ &\leq \delta \|\nabla \varphi\|^2 + c_\delta \|\varphi - \varphi_\Omega\|_*^2 + c|\varphi_\Omega|^2 \end{aligned} \tag{5.10}$$

for “small” $\delta > 0$ and correspondingly “large” $c_\delta > 0$.

Next, noting that $\int_\Omega \mu_\Omega(\varphi - \varphi_\Omega) = 0$, we may use (5.3) to obtain that

$$\int_\Omega (\varphi - \varphi_\Omega)(\mu - \mu_\Omega) = \|\nabla \varphi\|^2 + \int_\Omega (\varphi - \varphi_\Omega)(f(\varphi_1) - f(\varphi_2)) - \chi \int_\Omega (\varphi - \varphi_\Omega)\sigma. \tag{5.11}$$

Using also (5.10), we deduce

$$\chi \left| \int_\Omega (\varphi - \varphi_\Omega)\sigma \right| \leq c\|\sigma\|^2 + c\|\varphi - \varphi_\Omega\|^2 \leq \delta \|\nabla \varphi\|^2 + c_\delta \|\varphi - \varphi_\Omega\|_*^2 + c\|\sigma\|^2, \tag{5.12}$$

for $\delta > 0$ and $c_\delta > 0$ as above.

Summing (5.8) with (5.9) and using (5.11), (5.12) and (5.10) again, we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\varphi - \varphi_\Omega\|_*^2 + |\varphi_\Omega|^2) + \int_\Omega (\varphi - \varphi_\Omega)(f(\varphi_1) - f(\varphi_2)) + (1 - 2\delta) \|\nabla \varphi\|^2 \\ \leq c_\delta (\|\varphi - \varphi_\Omega\|_*^2 + \|\sigma\|^2 + |\varphi_\Omega|^2). \end{aligned}$$

Next, decomposing f into its monotone and remainder parts, it is not difficult to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\varphi - \varphi_\Omega\|_*^2 + |\varphi_\Omega|^2) + (1 - 3\delta) \|\nabla\varphi\|^2 \\ & \leq c_\delta (\|\varphi - \varphi_\Omega\|_*^2 + \|\sigma\|^2 + |\varphi_\Omega|^2) + \int_\Omega (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_\Omega. \end{aligned} \tag{5.13}$$

We now move to the estimation of σ . Integrating (5.4) over Ω we obtain

$$(\sigma_\Omega)_t = \kappa_0 \sigma_\Omega - \kappa_\infty (\sigma_1^2 - \sigma_2^2)_\Omega. \tag{5.14}$$

Subtracting the above from (5.4), we then get

$$(\sigma - \sigma_\Omega)_t - \Delta\sigma + \chi \operatorname{div}(\sigma \nabla\varphi_1 + \sigma_2 \nabla\varphi) = \kappa_0(\sigma - \sigma_\Omega) - \kappa_\infty(\sigma_1^2 - \sigma_2^2 - (\sigma_1^2)_\Omega + (\sigma_2^2)_\Omega). \tag{5.15}$$

Testing (5.14) by σ_Ω , it is not difficult to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\sigma_\Omega|^2 & \leq \kappa_0 |\sigma_\Omega|^2 + \frac{\kappa_\infty}{|\Omega|} |\sigma_\Omega| \int_\Omega |\sigma| |\sigma_1 + \sigma_2| \\ & \leq \kappa_0 |\sigma_\Omega|^2 + c |\sigma_\Omega| (\|\sigma - \sigma_\Omega\| + |\sigma_\Omega|) \|\sigma_1 + \sigma_2\| \\ & \leq \eta \|\sigma - \sigma_\Omega\|^2 + c_\eta (1 + \|\sigma_1\|^2 + \|\sigma_2\|^2) |\sigma_\Omega|^2, \end{aligned} \tag{5.16}$$

where $\eta > 0$ denotes a positive constant whose value will be fixed at the end. Next, we test (5.15) by $\mathcal{N}(\sigma - \sigma_\Omega)$ to deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + \|\sigma - \sigma_\Omega\|^2 & \leq \chi \int_\Omega \sigma \nabla\varphi_1 \cdot \nabla\mathcal{N}(\sigma - \sigma_\Omega) + \chi \int_\Omega \sigma_2 \nabla\varphi \cdot \nabla\mathcal{N}(\sigma - \sigma_\Omega) \\ & + \kappa_0 \|\sigma - \sigma_\Omega\|_*^2 - \kappa_\infty \int_\Omega (\sigma_1^2 - \sigma_2^2 - (\sigma_1^2)_\Omega + (\sigma_2^2)_\Omega) \mathcal{N}(\sigma - \sigma_\Omega). \end{aligned} \tag{5.17}$$

As for the right-hand side, we first notice that

$$\begin{aligned} \chi \int_\Omega \sigma \nabla\varphi_1 \cdot \nabla\mathcal{N}(\sigma - \sigma_\Omega) & \leq c \|\sigma\| \|\nabla\varphi_1\|_\infty \|\nabla\mathcal{N}(\sigma - \sigma_\Omega)\| \\ & \leq c \|\sigma\| \|\nabla\varphi_1\|_\infty \|\sigma - \sigma_\Omega\|_* \leq \eta \|\sigma\|^2 + c_\eta \|\nabla\varphi_1\|_\infty^2 \|\sigma - \sigma_\Omega\|_*^2 \\ & \leq 2\eta \|\sigma - \sigma_\Omega\|^2 + c_\eta |\sigma_\Omega|^2 + c_\eta \|\varphi_1\|_{W^{2,6}(\Omega)}^2 \|\sigma - \sigma_\Omega\|_*^2. \end{aligned} \tag{5.18}$$

To control the second integral, several strategies are possible, leading to different assumptions on σ_2 . Under the conditions in the statement, we may proceed by noting that

$$\begin{aligned} \chi \int_\Omega \sigma_2 \nabla\varphi \cdot \nabla\mathcal{N}(\sigma - \sigma_\Omega) & \leq c \|\sigma_2\|_6 \|\nabla\varphi\| \|\nabla\mathcal{N}(\sigma - \sigma_\Omega)\|_3 \\ & \leq c \|\sigma_2\|_6 \|\nabla\varphi\| \|\mathcal{N}(\sigma - \sigma_\Omega)\|_V^{1/2} \|\mathcal{N}(\sigma - \sigma_\Omega)\|_{H^2(\Omega)}^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq c\|\sigma_2\|_6\|\nabla\varphi\|\|\sigma - \sigma_\Omega\|_*^{1/2}\|\sigma - \sigma_\Omega\|^{1/2} \\ &\leq \eta\|\nabla\varphi\|^2 + \eta\|\sigma - \sigma_\Omega\|^2 + c_\eta\|\sigma_2\|_6^4\|\sigma - \sigma_\Omega\|_*^2. \end{aligned} \tag{5.19}$$

Next, we move to the last term in (5.17), which can be treated as follows:

$$\begin{aligned} &-\kappa_\infty \int_\Omega (\sigma_1^2 - \sigma_2^2 - (\sigma_1^2)_\Omega + (\sigma_2^2)_\Omega)\mathcal{N}(\sigma - \sigma_\Omega) \\ &\leq c\|\sigma_1^2 - \sigma_2^2 - (\sigma_1^2)_\Omega + (\sigma_2^2)_\Omega\|_1\|\mathcal{N}(\sigma - \sigma_\Omega)\|_\infty \\ &\leq c\|\sigma\|(\|\sigma_1\| + \|\sigma_2\|)\|\mathcal{N}(\sigma - \sigma_\Omega)\|_V^{1/2}\|\mathcal{N}(\sigma - \sigma_\Omega)\|_{H^2(\Omega)}^{1/2} \\ &\leq c(\|\sigma - \sigma_\Omega\| + |\sigma_\Omega|)(\|\sigma_1\| + \|\sigma_2\|)\|\sigma - \sigma_\Omega\|_*^{1/2}\|\sigma - \sigma_\Omega\|^{1/2} \\ &\leq c_\eta(1 + \|\sigma_1\|^4 + \|\sigma_2\|^4)\|\sigma - \sigma_\Omega\|_*^2 + \eta\|\sigma - \sigma_\Omega\|^2 + c|\sigma_\Omega|^2. \end{aligned} \tag{5.20}$$

Next, we replace (5.18), (5.19) and (5.20) into (5.17) to deduce that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|\sigma - \sigma_\Omega\|_*^2 + (1 - 4\eta)\|\sigma - \sigma_\Omega\|^2 \\ &\leq c_\eta(1 + \|\sigma_1\|^4 + \|\sigma_2\|_6^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 + \eta\|\nabla\varphi\|^2 + c|\sigma_\Omega|^2. \end{aligned}$$

Adding (5.16) to the above relation gives

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_\Omega\|_*^2 + |\sigma_\Omega|^2) + (1 - 5\eta)\|\sigma - \sigma_\Omega\|^2 \leq c_\eta(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2)|\sigma_\Omega|^2 \\ &\quad + c_\eta(1 + \|\sigma_1\|^4 + \|\sigma_2\|_6^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 + \eta\|\nabla\varphi\|^2. \end{aligned} \tag{5.21}$$

We then take $\delta = 1/6$ in (5.13) and multiply that relation by $\zeta > 0$ to be chosen below. Finally, we add the result to (5.21). This yields

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_\Omega\|_*^2 + |\sigma_\Omega|^2 + \zeta\|\varphi - \varphi_\Omega\|_*^2 + \zeta|\varphi_\Omega|^2) + \frac{\zeta}{2}\|\nabla\varphi\|^2 + (1 - 5\eta)\|\sigma - \sigma_\Omega\|^2 \\ &\leq c_\eta(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2)|\sigma_\Omega|^2 \\ &\quad + c_\eta(1 + \|\sigma_1\|^4 + \|\sigma_2\|_6^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 + \eta\|\nabla\varphi\|^2 \\ &\quad + c\zeta\|\varphi - \varphi_\Omega\|_*^2 + c_1\zeta\|\sigma - \sigma_\Omega\|^2 + c\zeta|\sigma_\Omega|^2 + c\zeta|\varphi_\Omega|^2 + \zeta \left| \int_\Omega (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_\Omega \right|, \end{aligned}$$

where $c_1 > 0$ is a computable constant independent of ζ and η . Then, choosing first $\zeta \leq \min\{1, 1/2c_1\}$, we arrive at

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_\Omega\|_*^2 + |\sigma_\Omega|^2 + \zeta\|\varphi - \varphi_\Omega\|_*^2 + \zeta|\varphi_\Omega|^2) + \frac{\zeta}{2}\|\nabla\varphi\|^2 + \left(\frac{1}{2} - 5\eta\right)\|\sigma - \sigma_\Omega\|^2 \\ &\leq c_\eta(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2)|\sigma_\Omega|^2 \end{aligned}$$

$$\begin{aligned}
 &+ c_\eta(1 + \|\sigma_1\|^4 + \|\sigma_2\|^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 + \eta\|\nabla\varphi\|^2 \\
 &+ c\|\varphi - \varphi_\Omega\|_*^2 + c|\sigma_\Omega|^2 + c|\varphi_\Omega|^2 + \left| \int_\Omega (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_\Omega \right|.
 \end{aligned}$$

Next, choosing $\eta \leq \min\{1/20, \zeta/4\}$, we get

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_\Omega\|_*^2 + |\sigma_\Omega|^2 + \zeta\|\varphi - \varphi_\Omega\|_*^2 + \zeta|\varphi_\Omega|^2) + \frac{\zeta}{4}\|\nabla\varphi\|^2 + \frac{1}{4}\|\sigma - \sigma_\Omega\|^2 \\
 &\leq c(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2)|\sigma_\Omega|^2 + c(1 + \|\sigma_1\|^4 + \|\sigma_2\|^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 \\
 &\quad + c\|\varphi - \varphi_\Omega\|_*^2 + c|\sigma_\Omega|^2 + c|\varphi_\Omega|^2 + \left| \int_\Omega (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_\Omega \right|. \tag{5.22}
 \end{aligned}$$

To obtain a contractive estimate, we need to manage the last term. This is treated in two different ways depending on the assumption on h . Indeed, if h is a constant, we may proceed as in [24] since in that case the ODE relation (5.7) reduces to

$$\varphi'_\Omega + m\varphi_\Omega = 0. \tag{5.23}$$

For constant h we may then proceed by noting that

$$\left| \int_\Omega (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_\Omega \right| \leq (\|F'_1(\varphi_1)\|_1 + \|F'_1(\varphi_2)\|_1)|\varphi_\Omega|. \tag{5.24}$$

Then, testing (5.23) by the sign of φ_Ω (this standard procedure may be also justified by approximation), summing the result to (5.22), and using (5.24), we deduce

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_\Omega\|_*^2 + |\sigma_\Omega|^2 + \zeta\|\varphi - \varphi_\Omega\|_*^2 + \zeta|\varphi_\Omega|^2 + |\varphi_\Omega|) + \frac{\zeta}{4}\|\nabla\varphi\|^2 + \frac{1}{4}\|\sigma - \sigma_\Omega\|^2 \\
 &\leq c(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2)|\sigma_\Omega|^2 + c(1 + \|\sigma_1\|^4 + \|\sigma_2\|^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2)\|\sigma - \sigma_\Omega\|_*^2 \\
 &\quad + c\|\varphi - \varphi_\Omega\|_*^2 + c|\varphi_\Omega|^2 + (\|F'_1(\varphi_1)\|_1 + \|F'_1(\varphi_1)\|_1)|\varphi_\Omega|,
 \end{aligned}$$

where we recall that ζ is a positive constant whose value has already been fixed. Next, we observe that, in view of (2.29) and (2.58)-(2.60), we may apply Grönwall’s lemma, which gives the statement (and, more generally, the continuous dependence estimate (2.62)). This concludes the analysis of the first case.

On the other hand, when h is nonlinear, it does not seem to be possible to proceed as above. For this reason, we need to provide a different control of the last integral term in (5.22). Namely, we may first notice that a simple computation shows, as we are assuming $F \in C^2(-1, 1)$, that

$$F'_1(\varphi_1) - F'_1(\varphi_2) = \ell\varphi, \quad \text{with } \ell = \int_0^1 F''_1(s\varphi_1 + (1-s)\varphi_2) ds.$$

Consequently, by the Young, Hölder and Poincaré–Wirtinger inequalities we infer

$$\begin{aligned} \left| \int_{\Omega} (F'_1(\varphi_1) - F'_1(\varphi_2))\varphi_{\Omega} \right| &= \left| \varphi_{\Omega} \int_{\Omega} \ell \varphi \right| \leq |\varphi_{\Omega}| \|\varphi\| \|\ell\| \\ &\leq |\varphi_{\Omega}| (\|\varphi - \varphi_{\Omega}\| + |\varphi_{\Omega}|) \|\ell\| \leq c |\varphi_{\Omega}| (\|\nabla \varphi\| + |\varphi_{\Omega}|) \|\ell\| \\ &\leq \frac{\zeta}{8} \|\nabla \varphi\|^2 + c(1 + \|\ell\|^2) |\varphi_{\Omega}|^2 \leq \frac{\zeta}{8} \|\nabla \varphi\|^2 + c |\varphi_{\Omega}|^2 (1 + \|F''_1(\varphi_1)\|^2 + \|F''_1(\varphi_2)\|^2), \end{aligned}$$

where the value of ζ was assigned before (and the last constants $c > 0$ also depend on it). Replacing this into (5.22), we then get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\sigma - \sigma_{\Omega}\|_*^2 + |\sigma_{\Omega}|^2 + \zeta \|\varphi - \varphi_{\Omega}\|_*^2 + \zeta |\varphi_{\Omega}|^2) &+ \frac{\zeta}{8} \|\nabla \varphi\|^2 + \frac{1}{4} \|\sigma - \sigma_{\Omega}\|^2 \\ &\leq c(1 + \|\sigma_1\|^2 + \|\sigma_2\|^2) |\sigma_{\Omega}|^2 + c(1 + \|\sigma_1\|^4 + \|\sigma_2\|_6^4 + \|\varphi_1\|_{W^{2,6}(\Omega)}^2) \|\sigma - \sigma_{\Omega}\|_*^2 \\ &\quad + c \|\varphi - \varphi_{\Omega}\|_*^2 + c |\varphi_{\Omega}|^2 (1 + \|F''_1(\varphi_1)\|^2 + \|F''_1(\varphi_2)\|^2). \end{aligned}$$

Once again, using also the additional assumption (2.61), Grönwall’s lemma gives the thesis.

Finally, with reference to Remark 2.11, we notice that the second integral on the right-hand side of (5.17) can alternatively managed in the following way employing Hölder’s inequality:

$$\begin{aligned} \chi \int_{\Omega} \sigma_2 \nabla \varphi \cdot \nabla \mathcal{N}(\sigma - \sigma_{\Omega}) &\leq c \|\sigma_2\|_{3+\delta} \|\nabla \varphi\| \|\nabla \mathcal{N}(\sigma - \sigma_{\Omega})\|_{\frac{6+2\delta}{1+\delta}} \\ &\leq \delta \|\nabla \varphi\|^2 + c_{\delta} \|\sigma_2\|_{L^{\infty}(0,T;L^{3+\delta}(\Omega))}^2 \|\nabla \mathcal{N}(\sigma - \sigma_{\Omega})\|_{\frac{6+2\delta}{1+\delta}}^2, \end{aligned}$$

where $\delta > 0$ is arbitrarily small (but fixed) and the right-hand side can be managed by using interpolation and accordingly adjusting the magnitude of the occurring constants as the interested reader may verify. □

Data availability

No data was used for the research described in the article.

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