

Sensitivity of Optimal Retirement Problem to Liquidity Constraints

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Abstract: In this work we analytically solve an optimal retirement problem, in which the agent optimally allocates the risky investment, consumption and leisure rate to maximise a gain function characterised by a power utility function of consumption and leisure, through the duality method. We impose different liquidity constraints over different time spans and conduct a sensitivity analysis to discover the effect of this kind of constraint.

Keywords: Liquidity Constraints, Retirement Stopping Time, Consumption-Portfolio-Leisure Controls, Duality Method, Variational Inequalities

1 Introduction

We study a stochastic control problem involving the consumption-portfolio-leisure policy and the optimal stopping time of retirement. By determining the continuous and stopping regions of the corresponding optimal stopping time problem, we prove that the optimal retirement time is the first hitting time of the wealth process $X(t)$ upward to a critical wealth boundary. We implement different liquidity constraints over different time spans, which are $X(t) \geq R_{pre}$ and $X(t) \geq R_{post}$ separately for pre- and post-retirement periods. The numerical analysis shows that the wealth boundary triggering the retirement is decreasing to R_{pre} but increasing to R_{post} . The additional retirement option impels the agent to consume less and invest more as the wealth approaches the retirement boundary, and this incentive becomes weaker as R_{pre} decreases.

The considered retirement mechanism is directly referred to [1, 2]. More precisely, [1] studied the optimal retirement model regarding the consumption-portfolio-leisure strategy, in which the leisure rate is limited to the binomial choice. [2] investigated a more complex optimization problem that endows the agent the flexibility in labour supply in the context of retirement planning. We extend their research and adopt a different utility function, a power utility function, as in [3], instead of the Constant Elasticity of Substitution (CES) function. Additionally, compared to [2], other extensions are i) the introduction of a continuous debt repayment the agent should face, ii) the different liquidity constraints before and after retirement, which is the main contribution of this work.

2 Problem Formulation

We deal with a financial market in which two kinds of investment are provided: the money market, concerning a fixed risk-free rate $r > 0$, and a risky asset, which dynamics is described by the stochastic differential equation $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$, $S(0) = S_0$, with μ and σ representing the constant drift and diffusion coefficients.

$B(t)$ represents a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t, 0 \leq t < \infty\}$ is the augmented natural filtration on $B(t)$. Moreover, by introducing the market price of risk as $\theta \triangleq \frac{\mu-r}{\sigma}$, we can define the state-price density process as $H(t) \triangleq \xi(t)\tilde{Z}(t)$ following [4], where $\xi(t) \triangleq e^{-rt}$ and $\tilde{Z}(t) \triangleq e^{-\frac{\theta^2}{2}t - \theta B(t)}$ indicate the discount process and an exponential martingale, respectively. Then we define the equivalent martingale measure $\tilde{\mathbb{P}}$ by $\tilde{\mathbb{P}}(A) \triangleq \mathbb{E}[\tilde{Z}(t)\mathbb{I}_A]$, $\forall A \in \mathcal{F}_t$. Based on the Girsanov Theorem, a standard Brownian motion under $\tilde{\mathbb{P}}$ measure can be defined as $\tilde{B}(t) \triangleq B(t) + \theta t$, $\forall t \geq 0$.

We now describe the optimization problem. The agent needs to optimally allocate the consumption $c(t)$, the amount of money for the risky investment $\pi(t)$ and the leisure rate $l(t)$. The sum of labour and leisure rates equals the constant \bar{L} . Furthermore, denoting the retirement time as τ , the retirement mechanism can be elaborated as: $0 \leq l(t) \leq \bar{L} < \bar{L}$ on $0 \leq t \leq \tau$, i.e., the leisure rate, as the complement of labour rate, is upper bounded for keeping the employment state; and $l(t) \equiv \bar{L}$ on $t > \tau$, since the agent enjoys the entire leisure \bar{L} after declaring retirement. Then the dynamics of the wealth process $X(t)$, i.e., the state variable of the optimization, is

$$dX(t) = [rX(t) + \pi(t)(\mu - r) - c(t) - d + w(\bar{L} - l(t))]dt + \sigma\pi(t)dB(t), \quad \forall t \geq 0,$$

d and w are the constant debt repayment and the wage rate, respectively. The initial wealth is $X(0) = x$. The considered optimal retirement problem (P) is

$$V(x) \triangleq \sup_{(\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}(x)} J(x; c, \pi, l, \tau) = \sup_{(\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty e^{-\gamma t} u(c(t), l(t)) dt \right], \quad (P)$$

in which γ is the subjective discount rate, and the utility is characterized by a power function

$$u(c, l) = \frac{(c^\delta l^{1-\delta})^{1-k}}{\delta(1-k)}, \quad 0 < \delta < 1, \quad k > 1.$$

The admissible control set $\mathcal{A}(x)$ follows the standard definition, e.g., [5, Definition 2.1], imposing liquidity constraints: $X(t) \geq R_{pre}$ for $0 \leq t < \tau$, $X(\tau) \geq R_{pre} \vee R_{post}$, and $X(t) \geq R_{post}$ for $t > \tau$ a.s.. Notice that we must impose $R_{pre} \geq \frac{d-w\bar{L}}{r}$ and $R_{post} \geq \frac{d}{r}$ to have the existence of an admissible solution, where $\frac{d-w\bar{L}}{r}$ represents the discounted value of the full debt repayment minus the maximum amount to borrow against the future labour income (in the pre-retirement period).

3 Solution of Optimization Problem

Defining $J_{PR}(X(\tau); c, \pi) \triangleq \mathbb{E} \left[\int_\tau^\infty e^{-\gamma(s-\tau)} u(c(s), \bar{L}) ds \mid \mathcal{F}_\tau \right]$, the gain function of Problem (P) can be rewritten as the expectation of two separated terms representing the pre- and post-retirement part

$$J(x; c, \pi, l, \tau) = \mathbb{E} \left[\int_0^\tau e^{-\gamma t} u(c(t), l(t)) dt + e^{-\gamma \tau} J_{PR}(X(\tau); c, \pi) \right],$$

where the subscript PR indicates that the corresponding variables and functions are related to the post-retirement problem.

The solutions of the pre- and post-retirement part are based on similar techniques, therefore in this letter we only report the solution of the post-retirement part, referring to the Online Appendix, Section A, for details. Depending on the value of R_{post} , the solution of the post-retirement problem is divided into two different cases: one is $R_{post} = \frac{d}{r}$, in which the liquidity constraint has no restriction on the optimization, and the other is $R_{post} > \frac{d}{r}$, with the optimal solution being binded by the liquidity constraint.

Lemma 3.1. *The post-retirement value function*

$$U(x) \triangleq \sup_{\{c(t), \pi(t)\}} J_{PR}(x; c, \pi),$$

for $x \geq R_{post}$, is given by:

$$U(x) = \begin{cases} \left(x - \frac{d}{r}\right)^{\delta(1-k)} K_1^{1-\delta(1-k)} \bar{L}^{(1-k)(1-\delta)} \frac{1}{\delta(1-k)}, & \text{if } R_{post} = \frac{d}{r}, \\ B_{2,PR}(\lambda_{PR}^*)^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (\lambda_{PR}^*)^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \lambda_{PR}^* + \lambda_{PR}^* x, & \text{if } R_{post} > \frac{d}{r}. \end{cases}$$

The Legendre-Fenchel transform of $U(x)$, $\tilde{U}(z) \triangleq \sup_{x \geq R_{post}} [U(x) - zx]$, is:

$$\begin{aligned} \bullet \quad \tilde{U}(z) &= \frac{1-\delta(1-k)}{\delta(1-k)} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - \frac{d}{r} z, \quad z > 0, \text{ if } R_{post} = \frac{d}{r}; \\ \bullet \quad \tilde{U}(z) &= \begin{cases} B_{2,PR} \hat{z}_{PR}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \hat{z}_{PR} - R_{post}(z - \hat{z}_{PR}), & z \geq \hat{z}_{PR}, \\ B_{2,PR} z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z, & 0 < z < \hat{z}_{PR}, \end{cases} \quad \text{if } R_{post} > \frac{d}{r}. \end{aligned}$$

Proof. See the Online Appendix A for the proof and the definition of the constants λ_{PR}^* , K_1 , n_2 , \hat{z}_{PR} and $B_{2,PR}$. \square

3.1 Pre-retirement Part

Based on the dynamic programming principle, we can only consider a subset of the admissible control set of Problem (P), that is $\mathcal{A}_1(x) \subset \mathcal{A}(x)$, in which any policy achieves the maximum of the post-retirement problem's gain function. Hence we have

$$V(x) = \sup_{(\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}_1(x)} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} u(c(t), l(t)) dt + e^{-\gamma \tau} U \left(X^{x, c, \pi, l}(\tau) \right) \right],$$

where $U \left(X^{x, c, \pi, l}(\tau) \right) \triangleq \sup_{\{c(t), \pi(t), l(t)\} \in \mathcal{A}_1(x)} \mathbb{E} \left[\int_\tau^\infty e^{-\gamma(s-\tau)} u(c(s), \bar{L}) ds \mid \mathcal{F}_\tau \right]$ is given in the previous lemma.

We first define an admissible control set corresponding to a fixed stopping time $\tau \in \mathcal{T}$, with \mathcal{T} representing the set of \mathcal{F}_t -stopping times, as $\mathcal{A}_\tau(x) \triangleq \{ \{c(t), \pi(t), l(t)\} : (\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}(x) \}$, and the utility maximization problem

$$V_\tau(x) \triangleq \sup_{\{c(t), \pi(t), l(t)\} \in \mathcal{A}_\tau(x)} J(x; c, \pi, l, \tau). \quad (P_\tau)$$

Then, Problem (P) is converted into an optimal stopping time problem, that is

$$V(x) = \sup_{\tau \in \mathcal{T}} V_\tau(x).$$

Similar to the post-retirement problem, the primal optimization problem's solution depends on the value of R_{pre} , which prompts us to solve it in two different cases. Before the discussion, we follow [5, Proposition 2.1] to provide the pre-retirement budget constraint, that is:

$$\mathbb{E} \left[\int_0^\tau H(t) (c(t) + d + wl(t) - w\bar{L}) dt + H(\tau) X(\tau) \right] \leq x. \quad (3.1)$$

Additionally, we define the Legendre-Fenchel transform of $u(c, l)$ by

$$\tilde{u}(y) \triangleq \sup_{c \geq 0, 0 \leq l \leq L} [u(c, l) - (c + wl)y].$$

3.1.1 Duality Approach with $R_{pre} = \frac{d-w\bar{L}}{r}$

Following the method from [6, Section 6], we first deduce an inequality of $J(x; c, \pi, l, \tau)$ by introducing a Lagrange multiplier $\lambda > 0$ and using the budget constraint (3.1),

$$J(x; c, \pi, l, \tau) \leq \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(\lambda e^{\gamma t} H(t)) - (d - w\bar{L})\lambda e^{\gamma t} H(t)) dt + e^{-\gamma \tau} \tilde{U}(\lambda e^{\gamma \tau} H(\tau)) \right] + \lambda x.$$

The inequality turns to equality if and only if the conditions

$$c(t) + wl(t) = -\tilde{u}'(\lambda e^{\gamma t} H(t)), \quad \forall t \in [0, \tau], \quad X(\tau) = -\tilde{U}'(\lambda e^{\gamma \tau} H(\tau)), \quad \text{a.s.},$$

and $\mathbb{E} \left[\int_0^\tau (c(t) + wl(t) + d - w\bar{L}) H(t) dt + X(\tau) H(\tau) \right] = x$ hold.

Additionally, Lemma 3.1 implies $X(\tau) = -\tilde{U}'(\lambda e^{\gamma \tau} H(\tau)) \geq R_{post} \geq \frac{d-w\bar{L}}{r}$. Then the following lemma shows that under the above conditions, there always exists a portfolio policy to ensure $X^{x,c,\pi,l}(t) \geq \frac{d-w\bar{L}}{r} = R_{pre}$, $\forall t \in [0, \tau]$, which implies the liquidity constraint is satisfied automatically.

Lemma 3.2. *For any given initial wealth $x \geq R_{pre}$, any fixed stopping time $\tau \in \mathcal{T}$, any \mathcal{F}_τ -measurable random variable K with $\mathbb{P}(K \geq \frac{d-w\bar{L}}{r}) = 1$ under the \mathbb{P} measure, and any given progressively measurable consumption and leisure processes $c(t) \geq 0$, $l(t) \geq 0$, $\forall t \geq 0$, satisfying $\mathbb{E} \left[\int_0^\tau H(t)(c(t) + wl(t) + d - w\bar{L}) dt + H(\tau)K \right] = x$, there exists a portfolio process $\pi(t)$ making $X^{x,c,\pi,l}(t) \geq \frac{d-w\bar{L}}{r}$, $\forall t \in [0, \tau]$, and $X^{x,c,\pi,l}(\tau) = K$ hold almost surely.*

Proof. See Online Appendix B. □

Moreover, the Lagrange method indicates that $V_\tau(x) = \inf_{\lambda > 0} [\tilde{J}_\tau(\lambda) + \lambda x]$ with introducing

$$\tilde{J}_\tau(\lambda) \triangleq \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(\lambda e^{\gamma t} H(t)) - (d - w\bar{L})\lambda e^{\gamma t} H(t)) dt + e^{-\gamma \tau} \tilde{U}(\lambda e^{\gamma \tau} H(\tau)) \right],$$

and the value function of Problem (P) can be transformed as

$$V(x) = \sup_{\tau \in \mathcal{T}} V_\tau(x) = \sup_{\tau \in \mathcal{T}} \inf_{\lambda > 0} [\tilde{J}_\tau(\lambda) + \lambda x] \leq \inf_{\lambda > 0} \sup_{\tau \in \mathcal{T}} [\tilde{J}_\tau(\lambda) + \lambda x] = \inf_{\lambda > 0} [\sup_{\tau \in \mathcal{T}} \tilde{J}_\tau(\lambda) + \lambda x].$$

Defining $\tilde{V}(\lambda) \triangleq \sup_{\tau \in \mathcal{T}} \tilde{J}_\tau(\lambda)$, [6, Section 8, Theorem 8.5] shows that $V(x) = \inf_{\lambda > 0} [\tilde{V}(\lambda) + \lambda x]$ holds under the condition that the function $\tilde{V}(\lambda)$ exists and it is differentiable for any $\lambda > 0$. Then, the process to solve Problem (P) is divided into two steps: the first is involved in the pure optimal stopping time problem of $\tilde{V}(\lambda)$, and the second step mainly concerns finding the optimal Lagrange multiplier. We begin with the first step and introduce a new process, $Z(t) \triangleq \lambda e^{\gamma t} H(t)$. Then $\tilde{V}(\lambda)$ can be rewritten as

$$\tilde{V}(\lambda) = \sup_{\tau \in \mathcal{S}} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(Z(t)) - (d - w\bar{L})Z(t)) dt + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \right].$$

We proceed with a generalized optimal stopping time problem

$$\phi(t, z) = \sup_{\tau \geq t} \mathbb{E} \left[\int_t^\tau e^{-\gamma s} (\tilde{u}(Z(s)) - (d - w\bar{L})Z(s)) ds + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \middle| Z(t) = z \right], \quad (3.2)$$

which shows that $\tilde{V}(\lambda) = \phi(0, \lambda)$. The following lemma gives the continuous region and stopping region of the above optimal stopping time problem.

Lemma 3.3. *Considering the optimal stopping time problem (3.2) with the state variable $Z(t)$, the continuous region is $\Omega_1 = \{Z(t) > \bar{z}\}$, the stopping region is $\Omega_2 = \{0 < Z(t) \leq \bar{z}\}$, where \bar{z} denotes the boundary that separates Ω_1 and Ω_2 .*

Proof. See Online Appendix C. □

Straight follows, with the operator $\mathcal{L} = \frac{\partial}{\partial t} + (\gamma - r)z \frac{\partial}{\partial z} + \frac{\theta^2}{2} z^2 \frac{\partial^2}{\partial z^2}$, the optimal stopping time problem (3.2) is equivalent to solving the free boundary problem below.

Variational Inequalities: Find a free boundary $\bar{z} > 0$ (Retirement level), and a function $\phi(t, z) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times (\mathbb{R}^+ \setminus \{\bar{z}\}))$ satisfying

$$\begin{cases} (V1) & \mathcal{L}\phi(t, z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z) = 0, & z > \bar{z}, \\ (V2) & \mathcal{L}\phi(t, z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z) \leq 0, & 0 < z \leq \bar{z}, \\ (V3) & \phi(t, z) \geq e^{-\gamma t} \tilde{U}(z), & z > \bar{z}, \\ (V4) & \phi(t, z) = e^{-\gamma t} \tilde{U}(z), & 0 < z \leq \bar{z}, \end{cases} \quad (3.3)$$

for any $t \geq 0$, with the smooth fit conditions $\phi(t, \bar{z}) = e^{-\gamma t} \tilde{U}(\bar{z})$ and $\frac{\partial \phi}{\partial z}(t, \bar{z}) = e^{-\gamma t} \tilde{U}'(\bar{z})$. The analytical solution of the above inequalities is presented in Online Appendix D.

Once ϕ is computed, we recover $\tilde{V}(\lambda) = \phi(0, \lambda)$, and the value function is given by

$$V(x) = \inf_{\lambda > 0} [\tilde{V}(\lambda) + \lambda x] = \tilde{V}(\lambda^*) + \lambda^* x,$$

x being the initial wealth. The retirement time is the first time the process $Z^*(t) \triangleq \lambda^* e^{\gamma t} H(t)$ touches the barrier \bar{z} from above. The optimal strategies are reported at the end of the Online Appendix D.

Remark 3.1. *The optimal retirement time is the first time the process $Z^*(t)$ touches the lower barrier \bar{z} . The same can be obtained with respect to the wealth level $X(t)$. In fact, the optimal process Z^* is connected to the optimal wealth X by the relation $X(t) = -v'(Z^*(t))$, being $\phi(t, z) = e^{-\gamma t} v(z)$, see the online appendix. The convex property of $v(\cdot)$, see [4, Section 3.4, Lemma 4.3], indicates that $X(t)$ is a decreasing function of $Z^*(t)$, therefore, in this case the optimal retirement time is the first time the process $X(t)$ touches an upper barrier $\bar{x} = -v'(\bar{z})$.*

3.1.2 Duality Approach with $R_{pre} > \frac{d-w\bar{L}}{r}$

Before proceeding to solve the problem, we present the following proposition to construct expectation form of the liquidity constraint related to $X(t) \geq R_{pre}$, $\forall t \in [0, \tau]$.

Proposition 3.1. *The liquidity constraint of the considered problem is*

$$\mathbb{E} \left[\int_t^\tau \frac{H(s)}{H(t)} (c(s) + d + wl(s) - w\bar{L}) ds + \frac{H(\tau)}{H(t)} X(\tau) \middle| \mathcal{F}_t \right] \geq R_{pre}, \quad \forall t \in [0, \tau]. \quad (3.4)$$

Proof. See [5, Proposition 4.1]. □

Considering the budget and liquidity constraints, (3.1) and (3.4), and introducing a Lagrange multiplier $\lambda > 0$ and a non-increasing process $D(t) \geq 0$ [6, 7], the following inequality is obtained:

$$\begin{aligned} J(x; c, \pi, l, \tau) \leq & \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(\lambda D(t) e^{\gamma t} H(t)) - (d - w\bar{L}) \lambda e^{\gamma t} D(t) H(t)) dt + e^{-\gamma \tau} \tilde{U}(\lambda D(\tau) e^{\gamma \tau} H(\tau)) \right] \\ & + \lambda \mathbb{E} \left[\int_0^\tau R_{pre} H(t) dD(t) \right] + \lambda x, \end{aligned}$$

which inspires us to define the dual individual's shadow price problem

$$\begin{aligned} \tilde{V}_\tau(\lambda) \triangleq & \inf_{D(t) \in \mathcal{D}} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(\lambda D(t) e^{\gamma t} H(t)) - (d - w\bar{L}) \lambda e^{\gamma t} D(t) H(t)) dt \right. \\ & \left. + e^{-\gamma \tau} \tilde{U}(\lambda D(\tau) e^{\gamma \tau} H(\tau)) \right] + \lambda \mathbb{E} \left[\int_0^\tau R_{pre} H(t) dD(t) \right], \end{aligned} \quad (S_\tau)$$

where \mathcal{D} is the set of non-negative, non-increasing and progressively measurable processes. Then we establish the duality between Problem (S_τ) and (P_τ) .

Theorem 3.1. (Duality Theorem) Suppose $D^*(t)$ is the optimal solution to Problem (S_τ) , then $c^*(t) + w l^*(t) = -\tilde{u}'(Z^*(t))$ and $X^{x, c^*, \pi^*, l^*}(\tau) = -\tilde{U}'(Z^*(\tau))$ coincide with the optimal solution of Problem (P_τ) , and there exists $V_\tau(x) = \inf_{\lambda > 0} [\tilde{V}_\tau(\lambda) + \lambda x]$, $\forall x \geq R_{pre}$. Here $Z^*(t) = \lambda^* e^{\gamma t} D^*(t) H(t)$, where λ^* and $D^*(t)$ are the parameters λ and $D(t)$ giving the infimum.

Proof. See [5, Theorem 4.1]. □

This duality theorem allows us to link Problem (P) with the shadow price problem through

$$V(x) = \sup_{\tau \in \mathcal{T}} V_\tau(x) = \sup_{\tau \in \mathcal{T}} \inf_{\lambda > 0} [\tilde{V}_\tau(\lambda) + \lambda x] \leq \inf_{\lambda > 0} \sup_{\tau \in \mathcal{T}} [\tilde{V}_\tau(\lambda) + \lambda x] = \inf_{\lambda > 0} [\sup_{\tau \in \mathcal{T}} \tilde{V}_\tau(\lambda) + \lambda x].$$

Defining $\tilde{V}(\lambda) \triangleq \sup_{\tau \in \mathcal{T}} \tilde{V}_\tau(\lambda)$, [6, Section 8, Theorem 8.5] indicates that the last inequality takes the equal sign with the condition that $\tilde{V}(\lambda)$ exists and is differentiable for any $\lambda > 0$. Thereafter, the objective optimization problem can be divided into two parts:

$$\begin{cases} \tilde{V}(\lambda) = \sup_{\tau \in \mathcal{T}} \tilde{V}_\tau(\lambda), \\ V(x) = \inf_{\lambda > 0} [\tilde{V}(\lambda) + \lambda x] \triangleq \tilde{V}(\lambda^*) + \lambda^* x. \end{cases}$$

We now consider the technique of [8] and insert an assumption on the process $D(t)$ for acquiring a closed-form solution.

Assumption 3.1. The non-increasing process $D(t)$ is absolutely continuous with respect to t . Hence, there is a non-negative process $\psi(t)$ such that $dD(t) = -\psi(t)D(t)dt$.

Then, by means of a new defined process $Z(t) \triangleq \lambda D(t) e^{\gamma t} H(t)$, the value function of the individual's shadow price problem can be written as

$$\tilde{V}_\tau(\lambda) = \inf_{\psi(t) \geq 0} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} (\tilde{u}(Z(t)) - (d - w\bar{L}) Z(t) - R_{pre} \psi(t) Z(t)) dt + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \right],$$

where $\psi(t)$ is the control variable, and $Z(t)$ is the state variable. Introducing a generalized problem

$$\phi(t, z) \triangleq \sup_{\tau \geq t} \inf_{\psi(s) \geq 0} \mathbb{E} \left[\int_t^\tau e^{-\gamma s} (\tilde{u}(Z(s)) - (d - w\bar{L}) Z(s) - R_{pre} \psi(s) Z(s)) ds + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \middle| Z(t) = z \right],$$

the solution of $\tilde{V}(\lambda)$ is turned to $\phi(t, z)$ with $\tilde{V}(\lambda) = \phi(0, \lambda)$. We first handle the infimum part by defining

$$\phi_{inf}(t, z) \triangleq \inf_{\psi(s) \geq 0} \mathbb{E} \left[\int_t^\tau e^{-\gamma s} (\tilde{u}(Z(s)) - (d - w\bar{L}) Z(s) - R_{pre} \psi(s) Z(s)) ds + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \middle| Z(t) = z \right].$$

The corresponding Bellman equation is

$$\min_{\psi \geq 0} \left\{ \mathcal{L}\phi_{inf}(t, z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z) - \psi z \left[\frac{\partial \phi_{inf}}{\partial z}(t, z) + R_{pre}e^{-\gamma t} \right] \right\} = 0.$$

The optimum ψ^* has the following characterization,

- $\frac{\partial \phi_{inf}}{\partial z}(t, z) + R_{pre}e^{-\gamma t} = 0 \Rightarrow \psi^* \geq 0$ and $\frac{\partial \phi}{\partial z}(t, z) = \frac{\partial \phi_{inf}}{\partial z}(t, z) = -R_{pre}e^{-\gamma t}$, $z \geq \hat{z}$.
- $\frac{\partial \phi_{inf}}{\partial z}(t, z) + R_{pre}e^{-\gamma t} \leq 0 \Rightarrow \psi^* = 0$, then $\phi(t, z)$ switches to a pure optimal stopping time problem,

$$\phi(t, z) = \sup_{\tau \geq t} \mathbb{E} \left[\int_t^\tau e^{-\gamma s} (\tilde{u}(Z(s)) - (d - w\bar{L})Z(s)) ds + e^{-\gamma \tau} \tilde{U}(Z(\tau)) \mid Z(t) = z \right],$$

which has the same form as (3.2) but applies to the interval $0 < z < \hat{z}$.

Lemma 3.3 can be easily extended also in this case, therefore the optimal retirement time is the first time the process $Z^*(t)$ touches the lower barrier \bar{z} . Therefore, we need to compare the value of \bar{z} and \hat{z} , and split the discussion into two cases: the first one is $\bar{z} < \hat{z}$, which corresponds to the case where the liquidity constraint boundary, R_{pre} , is lower than the retirement threshold.

Variational Inequalities assuming $\bar{z} < \hat{z}$: Find the free boundaries $\bar{z} > 0$ (retirement), $\hat{z} > 0$ (R_{pre} -wealth level), and a function $\phi(\cdot, \cdot) \in C^1((0, \infty) \times \mathbb{R}^+) \cap C^2((0, \infty) \times \mathbb{R}^+ \setminus \{\bar{z}\})$ satisfying

$$\begin{cases} (V1) & \frac{\partial \phi}{\partial z}(t, z) + R_{pre}e^{-\gamma t} = 0, & z \geq \hat{z}, \\ (V2) & \frac{\partial \phi}{\partial z}(t, z) + R_{pre}e^{-\gamma t} \leq 0, & 0 < z < \hat{z}, \\ (V3) & \mathcal{L}\phi(t, z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z) = 0, & \bar{z} < z < \hat{z}, \\ (V4) & \mathcal{L}\phi(t, z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z) \leq 0, & 0 < z \leq \bar{z}, \\ (V5) & \phi(t, z) \geq e^{-\gamma t} \tilde{U}(z), & \bar{z} < z < \hat{z}, \\ (V6) & \phi(t, z) = e^{-\gamma t} \tilde{U}(z), & 0 < z \leq \bar{z}, \end{cases} \quad (3.5)$$

for any $t \geq 0$, with the smooth fit conditions

$$\frac{\partial \phi}{\partial z}(t, \hat{z}) = -R_{pre}e^{-\gamma t}, \quad \frac{\partial^2 \phi}{\partial z^2}(t, \hat{z}) = 0, \quad \phi(t, \bar{z}) = e^{-\gamma t} \tilde{U}(\bar{z}), \quad \text{and} \quad \frac{\partial \phi}{\partial z}(t, \bar{z}) = e^{-\gamma t} \tilde{U}'(\bar{z}).$$

The analytical solution of the variational equation (3.5) is reported in Online Appendix E. Once ϕ and \bar{z} are computed, the value function and the optimal retirement decision can be recovered as in Section 3.1.1. The optimal strategies are reported at the end of the Online Appendix E.

If the first case does not admit a solution, that is, the liquidity constraint boundary R_{pre} is high enough (and larger than R_{post}) to make the agent declare retirement at time 0 for any admissible initial wealth, we deal with an immediate retirement, and therefore $V(x) = U(x)$, and all the optimal strategies are the ones of the post-retirement problem.

4 Numerical Analysis

We now perform the sensitivity analysis to the liquidity constraint boundaries. All the input parameters are reported in Table 4.1. We change the values of R_{pre} , R_{post} and keep all other input parameters consistent with

Table 4.1: Input Parameters

δ	k	r	μ	σ	γ	d	w	R_{pre}	R_{post}	\bar{L}	L
0.6	3	0.02	0.07	0.15	0.1	0.3	1.5	0	15	1	0.8

Figure 4.1: Convergence w.r.t. Liquidity Constraint Boundary of Pre- and Post-Retirement Part

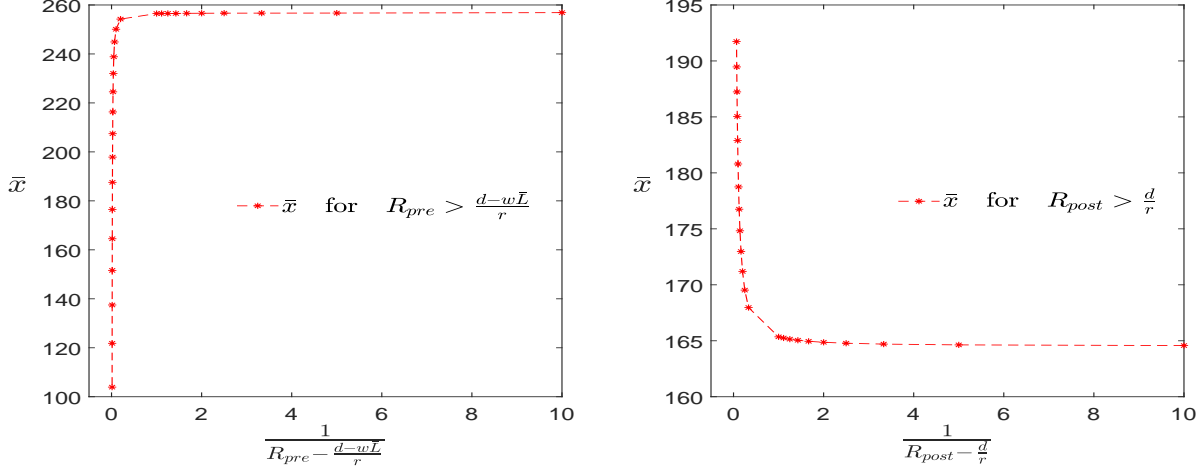
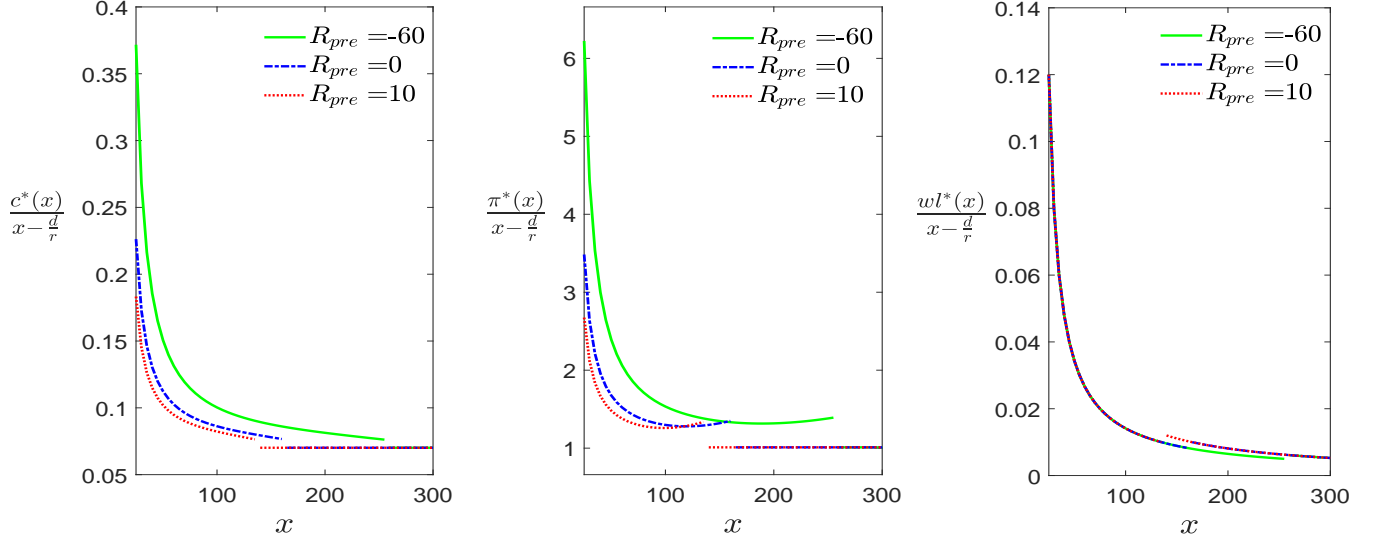


Table 4.1 to discover the different convergence phenomena of retirement wealth threshold concerning the pre- and post-retirement liquidity constraints.

Figure 4.1 shows that the retirement wealth threshold \bar{x} is a decreasing function of R_{pre} due to the fact that the agent with higher R_{pre} values prefers to set a lower wealth threshold to make sure entering in retirement ahead of schedule such that getting rid of the restriction caused by R_{pre} . Whereas, the critical wealth level of retirement is increasing with respect to R_{post} . Since the pre-retirement restriction keeps constant, a higher value of R_{post} , which implies a more rigorous circumstance for the post-retirement period, impels the agent to step into retirement with a higher wealth level.

Moreover, we provide figures to illustrate the sensitivity of optimal consumption, portfolio and leisure fractions in terms of $x - \frac{d}{r}$ with respect to different values of R_{pre} and R_{post} . We begin this kind of analysis fixing the value of R_{post} and arranging three values to R_{pre} . The optimal control strategies for different cases are presented in Proposition A.3, Proposition D.1 and Proposition E.1 in the online appendix. In Figure 4.2, R_{post} is set equal to $\frac{d}{r} = 15$, which implies that the post-retirement part is not restricted by the liquidity constraint. We can observe that the optimal consumption and portfolio fractions suffer a downward jump for various R_{pre} values. This is due to the discontinuity of the leisure rate at the retirement time, which leads to a shrinkage of labour income and reduces the resources allocated to the consumption and investment. In fact, if $x > 256.6913$ ($R_{pre} = -60$), $x > 164.5320$ ($R_{pre} = 0$), $x > 137.4776$ ($R_{pre} = 10$), i.e., the initial wealth is larger than the retirement threshold \bar{x} , the agent is facing the post-retirement region, with $l^*(t) = \bar{L} = 1$ (full leisure). In addition, it should be noted that for different R_{pre} values, the jump happens at different wealth levels. As also shown in the left plot of Figure 4.1, the agent with a higher R_{pre} value experiences the jump at a lower wealth threshold of retirement \bar{x} . Moreover, since the value of R_{post} keeps identical, the optimal consumption and portfolio fractions of different curves are coincident for the post-retirement part and equal to a constant, in line with the Merton classical problem.

Figure 4.2: Optimal Control Fractions w.r.t. Liquidity Constraint Boundary of Pre-Retirement Part

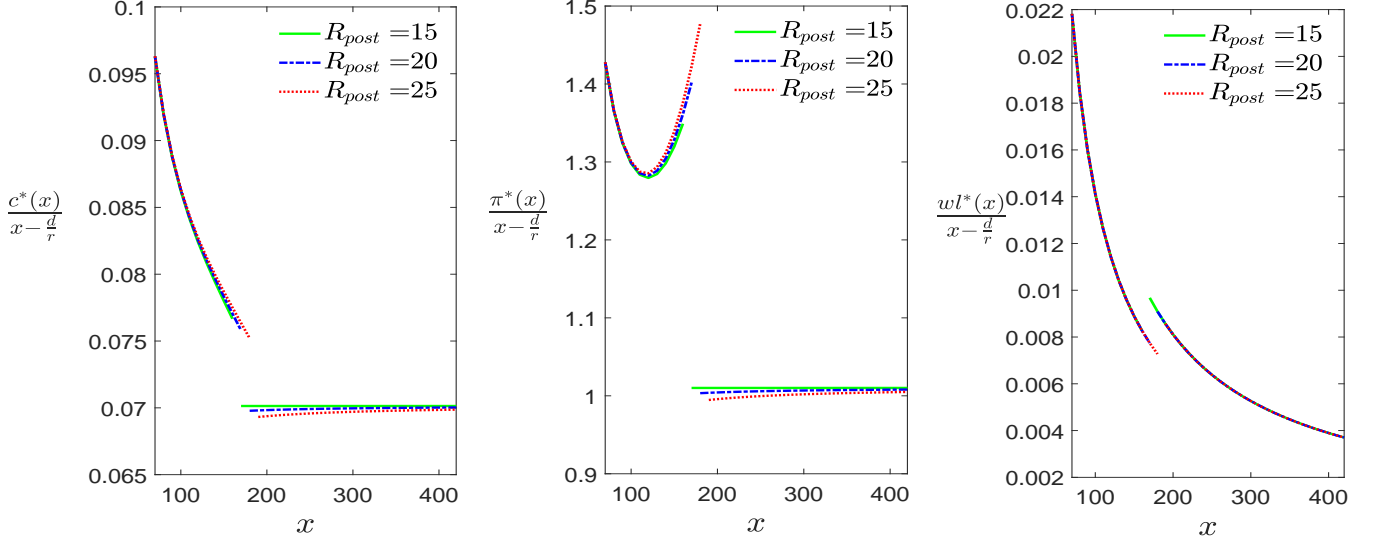


Then we conduct a similar sensitivity analysis with respect to R_{post} . Figure 4.3 shows that the retirement threshold is increasing with the value of R_{post} ($\bar{x} = 164.5320$ for $R_{post} = 15$, $\bar{x} = 171.1993$ for $R_{post} = 20$, $\bar{x} = 180.7943$ for $R_{post} = 25$), in line with the right plot of Figure 4.1, and describes that the optimal control fractions for the post-retirement part of blue dashed and red dotted curves, whose R_{post} values are greater than the boundary $\frac{d}{r}$, i.e., the liquidity constraints impose restrictions on optimal solutions, converge to the ones of the green curve ($R_{post} = \frac{d}{r}$) as x increases. It can be explained by the fact that the liquidity constraint plays a slighter role as the wealth becomes comparably larger and imposes a weaker restriction on the admissible control set. Moreover, we also notice that a high liquidity constraint for the post-retirement part induces the agent to take a large risk (high value of π^*) when the retirement threshold is close.

Finally, we conduct the sensitivity analysis of optimal control strategies to both the liquidity constraint boundary and the retirement option. In Figure 4.4, we fix the value of R_{post} to $\frac{d}{r}$ and plot the curves of optimal consumption and portfolio fractions in terms of $x - \frac{d}{r}$ under different situations. The dashed lines represent the optimal control fractions of different R_{pre} values with retirement option, while the solid lines represent the corresponding optimal control fractions without retirement option (and therefore with fixed liquidity constraint $R = R_{pre}$).¹ From all the dashed lines, we can see that the optimal consumption and portfolio fractions suffer a downward jump for various R_{pre} values. This is due to the discontinuity of the leisure rate at the retirement time, which leads to a shrinkage of labour income and reduces the resources allocated to the consumption and investment. Comparing the solid and dashed lines with the same colour, the agent with the additional retirement option tends to consume less and invest more in the risky asset for the motivation of arriving at the retirement wealth threshold and enjoying the full leisure rate faster. This kind of difference becomes more significant as the wealth approaches the critical level. Furthermore, the degree of this motivation is related to the liquidity constraint boundary. Observing the convexity of the pre-retirement part of different dashed lines, the optimal control fraction with a higher R_{pre} value takes a larger convexity, which is because stricter liquidity constraints give the agent a stronger motivation to achieve the critical wealth level to get rid of this restriction.

¹The theoretical solutions of optimal consumption-portfolio problem without retirement comes from [5, Section 5] by replacing the liquidity constraint boundary $F + \eta$ with R_{pre} .

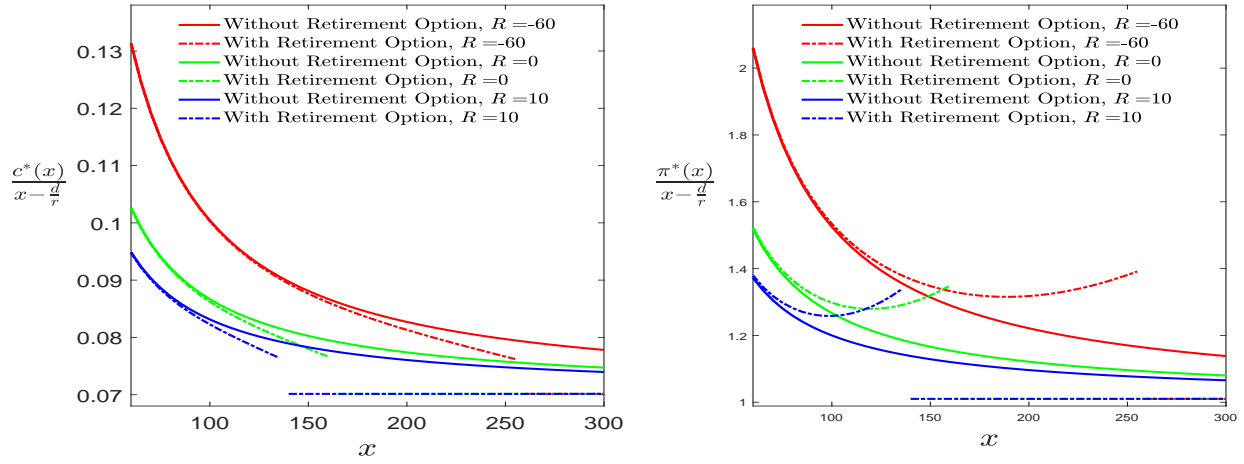
Figure 4.3: Optimal Control Fractions w.r.t. Liquidity Constraint Boundary of Post-Retirement Part



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Figure 4.4: Optimal Control Fractions w.r.t. Liquidity Constraint Boundary and Retirement Option.



Sensitivity of Optimal Retirement Problem to Liquidity Constraints - Online Appendix

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A Post Retirement Part

Assuming $\tau = 0^-$, we deal with the post-retirement problem, which is an infinite-time optimization problem with two control variables, the consumption and portfolio processes. Introducing $u_{PR}(c) \triangleq u(c, \bar{L}) = \frac{c^{\delta(1-k)} \bar{L}^{(1-\delta)(1-k)}}{\delta(1-k)}$, the corresponding value function, denoted as (P_{PR}) , is

$$V_{PR}(x) \triangleq \sup_{\{c(t), \pi(t)\} \in \mathcal{A}_{PR}(x)} J_{PR}(x; c, \pi). \quad (P_{PR})$$

The admissible control set $\mathcal{A}_{PR}(x)$ takes the compatible definition with $\mathcal{A}(x)$, except that the condition for stopping time is abolished, and the condition for liquidity constraint is given by $X(t) \geq R_{post}$, a.s., $\forall t \geq 0$. Then we derive the derivative function $u'_{PR}(c) = c^{\delta(1-k)-1} \bar{L}^{(1-k)(1-\delta)}$, which is positive and strictly decreasing and has the inverse function $I_{PR}(z) \triangleq z^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}$. Furthermore, referring to [4, Section 3, Definition 4.2], we define the Legendre-Fenchel transform of $u_{PR}(z)$ as $\tilde{u}_{PR}(z) \triangleq \sup_{c \geq 0} [u_{PR}(c) - cz]$, which has the explicit expression

$$\tilde{u}_{PR}(z) = u_{PR}(I_{PR}(z)) - zI_{PR}(z) = \frac{1 - \delta(1-k)}{\delta(1-k)} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}.$$

Proposition A.1. *The infinite horizon budget constraint of the post-retirement problem is*

$$\mathbb{E} \left[\int_0^\infty H(t)(c(t) + d) dt \right] \leq x. \quad (A.1)$$

Proof. The proof can be accomplished directly by replacing $l(t)$ in [5, Proposition 3.1] with the constant \bar{L} , meanwhile inserting a constant term d in the integral. \square

Depending on the value of R_{post} , the solution of Problem (P_{PR}) is divided into two different cases. One is $R_{post} = \frac{d}{r}$, in which the liquidity constraint has no restriction on the optimization, and the other is $R_{post} > \frac{d}{r}$, with the optimal solution being binded by the liquidity constraint.

As in [4, Chapter 3, Example 9.22], the optimal wealth process under the condition $R_{post} = \frac{d}{r}$ is $X^*(t) = (x - \frac{d}{r}) e^{\frac{1}{1-\delta(1-k)}(r - \gamma + \frac{\theta^2}{2})t + \frac{\theta}{1-\delta(1-k)}B(t)} + \frac{d}{r}$. The optimal consumption-portfolio polices are $c^*(t) = \frac{1}{K_1}[X^*(t) - \frac{d}{r}]$ and $\pi^*(t) = \frac{\theta}{\sigma(1-\delta(1-k))}[X^*(t) - \frac{d}{r}]$, with $K_1 \triangleq \frac{1-\delta(1-k)}{\gamma - r\delta(1-k) - \frac{\theta^2}{2} \frac{\delta(1-k)}{1-\delta(1-k)}} > 0$. And the value function of Problem (P_{PR}) can be obtained as

$$V_{PR}(x) = \left(x - \frac{d}{r}\right)^{\delta(1-k)} K_1^{1-\delta(1-k)} \frac{\bar{L}^{(1-k)(1-\delta)}}{\delta(1-k)}. \quad (A.2)$$

Remark A.1. *Analogous to the solution of the Merton problem, under the infinite time horizon, the optimal fraction invested in the risky asset in terms of the wealth minus the debt, i.e., $\frac{\pi^*(t)}{X^*(t) - \frac{d}{r}}$ keeps constant as*

$-\frac{\theta}{\sigma(\delta(1-k)-1)} = \frac{\mu-r}{\sigma^2(1-\delta(1-k))}$, and the optimal fractional consumption $\frac{c^(t)}{X^*(t) - \frac{d}{r}}$ takes a constant ratio as $\frac{1}{K_1}$.*

Hereafter, we impose a stricter liquidity constraint on the wealth process, $X(t) \geq R_{post} > \frac{d}{r}$. The following proposition provides the expectation form of the liquidity constraint, which will be accessible to deduce the duality problem subsequently.

Proposition A.2. *The infinite horizon liquidity constraint of the post-retirement problem is*

$$\mathbb{E} \left[\int_t^\infty \frac{H(s)}{H(t)} (c(s) + d) ds \middle| \mathcal{F}_t \right] \geq R_{post}. \quad (\text{A.3})$$

Proof. See [5, Proposition 3.2]. \square

Referring to [6, 7], we introduce a real number $\lambda_{PR} > 0$, the Lagrange multiplier, and a non-increasing process $D_{PR}(t) \geq 0$, then rewrite the post-retirement gain function as

$$\begin{aligned} J_{PR}(x; c, \pi) \leq & \mathbb{E} \left[\int_0^\infty e^{-\gamma t} (\tilde{u}_{PR}(\lambda_{PR} e^{\gamma t} H(t) D_{PR}(t)) - d \lambda_{PR} e^{\gamma t} D_{PR}(t) H(t)) dt \right] \\ & + \lambda_{PR} x + \lambda_{PR} \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}(t) \right]. \end{aligned}$$

The derivation of this inequality involves the budget constraint (A.1) and the liquidity constraint (A.3). In line with [7, Section 4], the post-retirement individual's dual shadow price problem, labelled (S_{PR}) , can be defined as

$$\begin{aligned} \tilde{V}_{PR}(\lambda_{PR}) \triangleq & \inf_{D_{PR}(t) \in \mathcal{D}} \mathbb{E} \left[\int_0^\infty e^{-\gamma t} (\tilde{u}_{PR}(\lambda_{PR} e^{\gamma t} H(t) D_{PR}(t)) - d \lambda_{PR} e^{\gamma t} D_{PR}(t) H(t)) dt \right] \\ & + \lambda_{PR} \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}(t) \right], \end{aligned} \quad (S_{PR})$$

where \mathcal{D} is the set of non-negative, non-increasing and progressively measurable processes. Then the duality between Problem (P_{PR}) and Problem (S_{PR}) is put forward.

Theorem A.1. (Duality Theorem) *Suppose $D_{PR}^*(t)$ is the optimal solution to the dual shadow price problem (S_{PR}) , then $c^*(t) = I_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t))$ is the optimal consumption solution to the problem (P_{PR}) . And we have the relation $V_{PR}(x) = \inf_{\lambda_{PR} > 0} [\tilde{V}_{PR}(\lambda_{PR}) + \lambda_{PR} x]$, with λ_{PR}^* attaining the infimum.*

Proof. See Appendix A.1. \square

The Duality Theorem enables us to transform the solution of Problem (P_{PR}) to its duality, (S_{PR}) . Besides, adopting the technique from [8], the subsequent assumption should be imposed for solving the problem explicitly.

Assumption A.1. *The non-increasing process $D_{PR}(t)$ is absolutely continuous with respect to t . Hence, there exists a process $\psi_{PR}(t)$ such that $dD_{PR}(t) = -\psi_{PR}(t) D_{PR}(t) dt$.*

Introducing $Z_{PR}(t) \triangleq \lambda_{PR} e^{\gamma t} D_{PR}(t) H(t)$, the value function of Problem (S_{PR}) is converted into

$$\tilde{V}_{PR}(\lambda_{PR}) = \inf_{\psi_{PR}(t) \geq 0} \mathbb{E} \left[\int_0^\infty e^{-\gamma t} (\tilde{u}_{PR}(Z_{PR}(t)) - dZ_{PR}(t) - \psi_{PR}(t) Z_{PR}(t) R_{post}) dt \right].$$

Then we define

$$\phi_{PR}(t, z) \triangleq \inf_{\psi_{PR}(t) \geq 0} \mathbb{E} \left[\int_t^\infty e^{-\gamma s} (\tilde{u}_{PR}(Z_{PR}(s)) - dZ_{PR}(s) - \psi_{PR}(s) Z_{PR}(s) R_{post}) ds \middle| Z_{PR}(t) = z \right],$$

and observe that $\tilde{V}_{PR}(\lambda_{PR}) = \phi_{PR}(0, \lambda_{PR})$. The associated Bellman equation to $\phi_{PR}(t, z)$ follows

$$\min_{\psi_{PR} \geq 0} \left\{ \tilde{\mathcal{L}}\phi_{PR}(t, z) + e^{-\gamma t}(\tilde{u}_{PR}(z) - dz) - \psi_{PR}z \left[\frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} \right] \right\} = \gamma\phi_{PR}(t, z),$$

with the operator $\tilde{\mathcal{L}} = (\gamma - r)z \frac{\partial}{\partial z} + \frac{1}{2}\theta^2 z^2 \frac{\partial^2}{\partial z^2}$. From the characterization of optimum ψ_{PR}^* :

$$\frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} = 0 \Rightarrow \psi_{PR}^* \geq 0; \quad \frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} \leq 0 \Rightarrow \psi_{PR}^* = 0,$$

the Bellman equation is equivalent to

$$\min \left\{ \tilde{\mathcal{L}}\phi_{PR}(t, z) - \gamma\phi_{PR}(t, z) + e^{-\gamma t}(\tilde{u}_{PR}(z) - dz), - \left[\frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} \right] \right\} = 0,$$

which results in the consequent modified variational inequalities: Find a free boundary $\hat{z}_{PR} > 0$, which makes R_{post} -wealth level, and a function $\phi_{PR}(\cdot, \cdot) \in C^2((0, \infty) \times \mathbb{R}^+)$ satisfying

$$\begin{cases} (V1) & \frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} = 0, & z \geq \hat{z}_{PR}, \\ (V2) & \frac{\partial \phi_{PR}}{\partial z}(t, z) + e^{-\gamma t}R_{post} \leq 0, & 0 < z < \hat{z}_{PR}, \\ (V3) & \tilde{\mathcal{L}}\phi_{PR}(t, z) - \gamma\phi_{PR}(t, z) + e^{-\gamma t}(\tilde{u}_{PR}(z) - dz) = 0, & 0 < z < \hat{z}_{PR}, \\ (V4) & \tilde{\mathcal{L}}\phi_{PR}(t, z) - \gamma\phi_{PR}(t, z) + e^{-\gamma t}(\tilde{u}_{PR}(z) - dz) \geq 0, & z \geq \hat{z}_{PR}, \end{cases} \quad (A.4)$$

for any $t \geq 0$, with the smooth fit conditions $\frac{\partial \phi_{PR}}{\partial z}(t, \hat{z}_{PR}) = -R_{post}e^{-\gamma t}$ and $\frac{\partial^2 \phi_{PR}}{\partial z^2}(t, \hat{z}_{PR}) = 0$.

Proposition A.3. *Under the assumption $\phi_{PR}(t, z) = e^{-\gamma t}v_{PR}(z)$, the variational inequalities (A.4) takes the solution*

$$v_{PR}(z) = \begin{cases} B_{2,PR}\hat{z}_{PR}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}\hat{z}_{PR}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r}\hat{z}_{PR} - R_{post}(z - \hat{z}_{PR}), & z \geq \hat{z}_{PR}, \\ B_{2,PR}z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r}z, & 0 < z < \hat{z}_{PR}, \end{cases}$$

with

$$n_2 = -\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2} + \sqrt{\left(\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2}\right)^2 + \frac{2\gamma}{\theta^2}},$$

$$\hat{z}_{PR} = \bar{L}^{(1-k)(1-\delta)} \left[\frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \frac{(R_{post} - \frac{d}{r})}{K_1} \right]^{\delta(1-k)-1} > 0,$$

and

$$B_{2,PR} = \frac{K_1^{\delta(1-k)-1}(n_2-1)\bar{L}^{(1-k)(1-\delta)(1-n_2)}}{n_2(n_2-1)(\delta(1-k)-1)} \left[\frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \left(R_{post} - \frac{d}{r} \right) \right]^{\delta(1-k)-n_2(\delta(1-k)-1)} < 0.$$

Furthermore, for a given initial wealth $x \geq R_{post}$, the value function of the post-retirement problem is

$$V_{PR}(x) = B_{2,PR}(\lambda_{PR}^*)^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}(\lambda_{PR}^*)^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r}\lambda_{PR}^* + \lambda_{PR}^*x, \quad (A.5)$$

with $-n_2 B_{2,PR}(\lambda_{PR}^*)^{n_2-1} + K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (\lambda_{PR}^*)^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r} = x$. Taking $Z_{PR}^*(t) \triangleq \lambda_{PR}^* e^{\gamma t} H(t)$, the optimal wealth process follows

$$X^*(t) = -n_2 B_{2,PR}(Z_{PR}^*(t))^{n_2-1} + K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r}, \quad 0 < Z_{PR}^*(t) \leq \hat{z}_{PR},$$

and the corresponding optimal consumption and portfolio strategies are

$$c^*(t) = I_{PR}(\lambda_{PR}^* e^{\gamma t} H(t)) = (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}},$$

$$\pi^*(t) = \frac{\theta}{\sigma} \left[n_2(n_2-1) B_{2,PR}(Z_{PR}^*(t))^{n_2-1} - K_1 \frac{1}{\delta(1-k)-1} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} \right].$$

Proof. See Appendix A.2. □

Then based on the dynamic programming principle, we can only consider a subset of the admissible control set of Problem (P), that is $\mathcal{A}_1(x) \subset \mathcal{A}(x)$, in which any policy achieves the maximum of the post-retirement problem's gain function. Hence, for any $(\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}_1(x)$, we have $\mathbb{E} \left[\int_{\tau}^{\infty} e^{-\gamma t} u(c(t), \bar{L}) dt \right] = \mathbb{E} \left[e^{-\gamma \tau} V_{PR}(X^{x,c,\pi,l}(\tau)) \mathbb{I}_{\{\tau < \infty\}} \right]$. Afterwards, the whole optimization problem can be rewritten as

$$V(x) = \sup_{(\tau, \{c(t), \pi(t), l(t)\}) \in \mathcal{A}_1(x)} \mathbb{E} \left[\int_0^{\tau} e^{-\gamma t} u(c(t), l(t)) dt + e^{-\gamma \tau} U(X^{x,c,\pi,l}(\tau)) \right],$$

denoting $U(X^{x,c,\pi,l}(\tau)) \triangleq \sup_{\{c(t), \pi(t), l(t)\} \in \mathcal{A}_1(x)} \mathbb{E} \left[\int_{\tau}^{\infty} e^{-\gamma(s-\tau)} u(c(s), \bar{L}) ds \mid \mathcal{F}_{\tau} \right] = V_{PR}(X^{x,c,\pi,l}(\tau))$. Finally, we summarize the two different forms of $U(x)$ and introduce its Legendre-Fenchel transform under the definition $\tilde{U}(z) \triangleq \sup_{x \geq R_{post}} [U(x) - xz]$, $0 < z < \infty$ following [4, Chapter 3, Definition 4.2].

Lemma A.1. *The post-retirement value function $U(x)$, for $x \geq R_{post}$, is given in two separate cases, the partition being based on the value of threshold in the liquidity constraint, i.e., R_{post} .*

$$U(x) = \begin{cases} \left(x - \frac{d}{r}\right)^{\delta(1-k)} K_1^{1-\delta(1-k)} \bar{L}^{(1-k)(1-\delta)} \frac{1}{\delta(1-k)}, & \text{if } R_{post} = \frac{d}{r}, \\ B_{2,PR}(\lambda_{PR}^*)^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (\lambda_{PR}^*)^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \lambda_{PR}^* + \lambda_{PR}^* x, & \text{if } R_{post} > \frac{d}{r}. \end{cases}$$

Furthermore, the Legendre-Fenchel transform of $U(x)$ is:

$$\bullet \quad \tilde{U}(z) = \frac{1-\delta(1-k)}{\delta(1-k)} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - \frac{d}{r} z, \quad z > 0, \text{ if } R_{post} = \frac{d}{r};$$

$$\bullet \quad \tilde{U}(z) = \begin{cases} B_{2,PR} \hat{z}_{PR}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \hat{z}_{PR} - R_{post}(z - \hat{z}_{PR}), & z \geq \hat{z}_{PR}, \\ B_{2,PR} z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z, & 0 < z < \hat{z}_{PR}, \end{cases} \quad \text{if } R_{post} > \frac{d}{r}.$$

Proof. See Appendix A.3. □

A.1 Proof of Theorem A.1

We first provide a lemma for proving Theorem A.1.

Lemma A.2. *For any given initial wealth $x > R_{post}$, and any given progressively measurable consumption process $c(t) \geq 0$ satisfying $\sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\int_0^\tau H(t)(c(t) + d)dt \right] \leq x - R_{post}$, with \mathcal{T} standing for the set of \mathcal{F} -stopping times, there exists a portfolio process $\pi(t)$ making $X^{x,c,\pi}(t) \geq R_{post}$, $\forall t \geq 0$, holds almost surely.*

Proof. Adopting the technique of [7, Appendix, Lemma 1], we introduce $K(t) \triangleq \int_0^t (c(s) + d)H(s)ds$ and show that $\{K(\tau)\}_{\tau \in \mathcal{T}}$ is uniformly integrable from the fact $\mathbb{E}[K(t)] < \infty$. Then, Dellacherie & Meyer (1982), Appendix I,¹ indicates that there exists a Snell envelope of $K(t)$ denoted as $\bar{K}(t)$, which is a super-martingale under the \mathbb{P} measure and satisfies $\bar{K}(0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[K(\tau)]$, $\bar{K}(\infty) = K(\infty)$. The Doob-Meyer Decomposition Theorem of

Karatzas & Shreve (1998), Section 1.4, Theorem 4.10,² enables us to represent the super-martingale $\bar{K}(t)$ as $\bar{K}(t) = \bar{K}(0) + \bar{M}(t) - \bar{A}(t)$, with a uniformly integrable martingale under the \mathbb{P} measure $\bar{M}(t)$ satisfying $\bar{M}(0) = 0$ and a strictly increasing process $\bar{A}(t)$ satisfying $\bar{A}(0) = 0$. Moreover, the Martingale Representation Theorem from Bjork (2009), Chapter 11, Theorem 11.2,³ makes $\bar{M}(t)$ take expression $\bar{M}(t) = \int_0^t \bar{\rho}(s)dB(s)$, where $\bar{\rho}(t)$ is an \mathbb{F} -adapted process satisfying $\int_0^\infty \bar{\rho}^2(s)ds < \infty$ a.s..

Defining a new process $\bar{X}(t) \triangleq \frac{1}{H(t)} [x - \bar{K}(0) + \bar{K}(t) - K(t) + \bar{A}(t)] - R_{post}$, it can be observed that $\bar{X}(t)$ is a non-negative process with the initial wealth $\bar{X}(0) = x - R_{post}$, because of

$$\bar{K}(0) = \sup_{\tau \in \mathcal{T}} \mathbb{E}[K(\tau)] = \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[\int_0^\tau H(t)(c(t) + d)dt \right] \leq x - R_{post}.$$

Then $\bar{X}(t)$ can be expressed with $\bar{M}(t)$ as

$$\bar{X}(t) = \frac{1}{H(t)} [x + \bar{M}(t) - K(t)] - R_{post} = \frac{1}{H(t)} \left[x + \int_0^t \bar{\rho}(s)dB(s) - \int_0^t (c(s) + d)H(s)ds \right] - R_{post}.$$

Applying the Itô's formula to $H(t)X^{x,c,\pi}(t)$, we can get

$$d(H(t)X^{x,c,\pi}(t)) = -H(t)X^{x,c,\pi}(t)\theta dB(t) - (c(t) + d)H(t)dt + \sigma\pi(t)H(t)dB(t).$$

Considering the portfolio strategy $\pi(t) = \frac{\bar{\rho}(t)}{\sigma H(t)} + \frac{\theta X^{x,c,\pi}(t)}{\sigma}$, the wealth process takes

$$X^{x,c,\pi}(t) = \frac{1}{H(t)} \left[x + \int_0^t \bar{\rho}(s)dB(s) - \int_0^t (c(s) + d)H(s)ds \right],$$

which indicates that $\bar{X}(t) = X^{x,c,\pi}(t) - R_{post}$, a.s.. The non-negativity of $\bar{X}(t)$ makes clear that $X^{x,c,\pi}(t) \geq R_{post}$, a.s., $\forall t \geq 0$. \square

Now we turn back to the proof of Theorem A.1. Following [7, Section 4, Theorem 1], the proof mainly contains two aspects: the first part is to show the admissibility of $c^*(t)$, and the second part is to claim that $c^*(t)$ is the optimal consumption strategy to Problem (P_{PR}) .

¹C. Dellacherie and P. Meyer, Probabilities and potential b, theory of martingales, North-Holland Mathematics Studies, 1982.

²I. Karatzas and S. E. Shreve, Brownian Motion and Stochastic Calculus. Second edition. Springer-Verlag, 1998.

³T. Bjork, Arbitrage theory in continuous time. Oxford university press, 2009.

(1) We first prove that $c^*(t) = I_{PR}^*(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t))$ is an admissible consumption policy. Taking any stopping time τ from \mathcal{T} and a positive constant ϵ , we can introduce $D_{PR}^\epsilon(t) \triangleq D_{PR}^*(t) + \epsilon \mathbb{I}_{[0, \tau)}(t)$, which evidently satisfies $D_{PR}^\epsilon(t) \in \mathcal{D}$. Then defining a function

$$\begin{aligned} \mathfrak{L}(D(t)) \triangleq & \mathbb{E} \left[\int_0^\infty e^{-\gamma t} (\tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D(t) H(t)) - d \lambda_{PR}^* e^{\gamma t} D(t) H(t)) dt \right] \\ & + \lambda_{PR}^* \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD(t) \right] + \lambda_{PR}^* (x - R_{post}) D(0), \end{aligned}$$

an inequality, $\mathfrak{L}(D_{PR}^*(t)) \leq \mathfrak{L}(D_{PR}^\epsilon(t))$, is obtained from the facts that $D_{PR}^*(t)$ is the optimal solution of Problem (S_{PR}) and $x \geq R_{post}$. This inequality gives us

$$\limsup_{\epsilon \downarrow 0} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} \frac{\tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^\epsilon(t) H(t)) - \tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t))}{\epsilon} - d \lambda_{PR}^* H(t) dt \right] + \lambda_{PR}^* (x - R_{post}) \geq 0,$$

considering $dD^\epsilon(t) = dD^*(t)$, $\forall t \in (0, \tau)$. The decreasing property of $\tilde{u}_{PR}(\cdot)$ and the Fatou's lemma endows us with

$$\begin{aligned} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} \tilde{u}'_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t)) \lambda_{PR}^* e^{\gamma t} H(t) dt \right] \geq \\ \limsup_{\epsilon \downarrow 0} \mathbb{E} \left[\int_0^\tau e^{-\gamma t} \frac{\tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^\epsilon(t) H(t)) - \tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t))}{\epsilon} dt \right]. \end{aligned}$$

Then $\tilde{u}'_{PR}(\cdot) = -I_{PR}(\cdot)$ indicates that $\mathbb{E} \left[\int_0^\tau H(t) (c^*(t) + d) dt \right] \leq x - R_{post}$. Since τ can be any stopping time in the set \mathcal{T} , Lemma A.2 claims that there exists a portfolio strategy $\pi^*(t)$ making the related wealth process satisfying $X^{x, c^*, \pi^*}(t) \geq R_{post}$, $\forall t \geq 0$.

(2) We move to show the optimality of $c^*(t)$ to Problem (P_{PR}) . The proof of Lemma A.2 indicates that for an arbitrary consumption strategy $c(t) \in \mathcal{A}_{PR}(x)$, there exists a process $\zeta(t)$ satisfying

$$\int_0^t (c(s) + d) H(s) ds + H(t) X^{x, c, \pi}(t) = x + \int_0^t \zeta(s) dB(s). \quad (\text{A.6})$$

The property $X^{x, c, \pi}(t) \geq R_{post}$ a.s. gives us the subsequent inequality with any process $D(t) \in \mathcal{D}$,

$$\int_0^T \int_0^t (c(s) + d) H(s) ds dD(t) + \int_0^T R_{post} H(t) dD(t) \geq \int_0^T \left[x + \int_0^t \zeta(s) dB(s) \right] dD(t),$$

where T is any time meeting $T \geq t$. Since $D(t)$ is bounded variational, integrating by parts gives us

$$\begin{aligned} \int_0^T D(s) (c(s) + d) H(s) ds - \int_0^T D(s) \zeta(s) dB(s) \leq \\ D(0)x + D(T) \left[\int_0^T (c(s) + d) H(s) ds - x - \int_0^T \zeta(s) dB(s) \right] + \int_0^T R_{post} H(s) dD(s). \end{aligned}$$

Then we can take the expectation under the \mathbb{P} measure on both sides and replace Equation (A.6) to get $\mathbb{E} \left[\int_0^T D(s) (c(s) + d) H(s) ds \right] \leq D(0)x + \mathbb{E} \left[\int_0^T R_{post} H(s) dD(s) \right]$. Then the Lebesgue's Monotone Convergence Theorem indicates that

$$\mathbb{E} \left[\int_0^\infty D(s) (c(s) + d) H(s) ds \right] \leq D(0)x + \mathbb{E} \left[\int_0^\infty R_{post} H(s) dD(s) \right],$$

which holds for any admissible consumption policy $c(t)$ and any non-negative, non-increasing process $D(t)$. Furthermore, it will be proved that the above inequality becomes equalized with the given $c^*(t)$ and $D_{PR}^*(t)$. Introducing $\bar{D}_{PR}^\epsilon(t) \triangleq D_{PR}^*(t)(1 + \epsilon) \in \mathcal{D}$ with a small enough constant ϵ and defining a new function as

$$\begin{aligned} \tilde{\mathcal{L}}(D(t)) \triangleq & \mathbb{E} \left[\int_0^\infty e^{-\gamma t} (\tilde{u}_{PR}(\lambda_{PR}^* e^{\gamma t} D(t) H(t)) - d \lambda_{PR}^* e^{\gamma t} D(t) H(t)) dt \right] \\ & + \lambda_{PR}^* \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD(t) \right] + \lambda_{PR}^* x D(0), \end{aligned}$$

we get $\tilde{\mathcal{L}}(\bar{D}_{PR}^\epsilon(t)) \geq \tilde{\mathcal{L}}(D_{PR}^*(t))$. Following the same argument with the first part, we apply the Fatou's lemma to obtain separately

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty D_{PR}^*(t) H(t) (c^*(t) + d) dt \right] & \leq x D_{PR}^*(0) + \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}^*(t) \right], \\ \mathbb{E} \left[\int_0^\infty D_{PR}^*(t) H(t) (c^*(t) + d) dt \right] & \geq x D_{PR}^*(0) + \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}^*(t) \right], \end{aligned}$$

which claims $\mathbb{E} \left[\int_0^\infty D_{PR}^*(t) H(t) (c^*(t) + d) dt \right] = x D_{PR}^*(0) + \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}^*(t) \right]$. Afterwards, we define a new optimization problem named (P'_{PR}) as

$$\begin{aligned} & \max_{c(t) \geq 0} \mathbb{E} \left[\int_0^\infty e^{-\gamma t} u_{PR}(c(t)) dt \right] \tag{P'_{PR}} \\ \text{s.t. } & \mathbb{E} \left[\int_0^\infty D_{PR}^*(t) H(t) (c(t) + d) dt \right] \leq x D_{PR}^*(0) + \mathbb{E} \left[\int_0^\infty R_{post} H(t) dD_{PR}^*(t) \right]. \end{aligned}$$

The Lagrange method implies that the optimal consumption solution of the above problem, denoted as $\tilde{c}^*(t)$, satisfies $e^{-\gamma t} u'_{PR}(\tilde{c}^*(t)) = \tilde{\lambda}_{PR} D_{PR}^*(t) H(t)$, with $\tilde{\lambda}_{PR} > 0$ as the Lagrange multiplier. The condition $\tilde{\lambda}_{PR} = \lambda_{PR}^*$ makes the constraint of Problem (P'_{PR}) takes equality. And the condition $u'_{PR}(\tilde{c}^*(t)) = \lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t)$ implies that $\tilde{c}^*(t) = I(\lambda_{PR}^* e^{\gamma t} D_{PR}^*(t) H(t)) = c^*(t)$, which shows that $c^*(t)$ is the optimal consumption policy of Problem (P'_{PR}) . Finally, since the maximum utility of Problem (P_{PR}) is upper bounded by the maximum utility of (P'_{PR}) , we can conclude that $c^*(t)$ is also the optimal consumption solution of the primal problem (P_{PR}) .

A.2 Proof of Proposition A.3

Referring to [2, Appendix A], the function $\phi_{PR}(t, z)$ is assumed to be time-independent, that is, $\phi_{PR}(t, z) = e^{-\gamma t} v_{PR}(z)$. Then the condition (V3) of (A.4) leads to a differential equation

$$-\gamma v_{PR}(z) + (\gamma - r) z v'_{PR}(z) + \frac{1}{2} \theta^2 z^2 v''_{PR}(z) + \tilde{u}_{PR}(z) - dz = 0, \quad 0 < z < \hat{z}_{PR}, \tag{A.7}$$

which has the solution

$$v_{PR}(z) = B_{1,PR} z^{n_1} + B_{2,PR} z^{n_2} + \frac{1 - \delta(1 - k)}{\delta(1 - k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z, \quad 0 < z < \hat{z}_{PR}.$$

n_1 and n_2 are the roots of the second-order equation $\frac{\theta^2}{2} n^2 + \left(\gamma - r - \frac{\theta^2}{2} \right) n - \gamma = 0$, and satisfy

$$n_{1,2} = -\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2} \mp \sqrt{\left(\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2} \right)^2 + \frac{2\gamma}{\theta^2}}, \quad n_1 < 0, \quad n_2 > 1.$$

Since $n_1 < 0$, the term z^{n_1} will suffer the explosion as z goes to 0. Therefore, we set the coefficient $B_{1,PR} = 0$ by the boundedness assumption. Considering the smooth conditions at \hat{z}_{PR} , we can construct a two-equations system to determine the parameters $B_{2,PR}$ and \hat{z}_{PR} .

- \mathcal{C}^1 condition at $z = \hat{z}_{PR}$: $n_2 B_{2,PR} \hat{z}_{PR}^{n_2-1} - K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} - \frac{d}{r} + R_{post} = 0$.
- \mathcal{C}^2 condition at $z = \hat{z}_{PR}$: $n_2(n_2 - 1) B_{2,PR} \hat{z}_{PR}^{n_2-2} - K_1 \frac{1}{\delta(1-k)-1} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} = 0$.

By multiplying the \mathcal{C}^2 condition with \hat{z}_{PR} and then adding with the \mathcal{C}^1 condition, we have

$$B_{2,PR} \hat{z}_{PR}^{n_2-1} = K_1 \frac{\delta(1-k)}{\delta(1-k)-1} \frac{1}{n_2} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} + \frac{1}{n_2} \left(\frac{d}{r} - R_{post} \right).$$

Then, substituting the above expression into the \mathcal{C}^1 condition, we get the exact value of \hat{z}_{PR} as

$$\hat{z}_{PR} = \bar{L}^{(1-k)(1-\delta)} \left[\frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \frac{(R_{post} - \frac{d}{r})}{K_1} \right]^{\delta(1-k)-1} > 0,$$

and $B_{2,PR}$ can also be solved by bringing \hat{z}_{PR} into the expression $B_{2,PR} \hat{z}_{PR}^{n_2-1}$,

$$B_{2,PR} = \frac{K_1^{\delta(1-k)-1} \bar{L}^{(1-k)(1-\delta)(1-n_2)}}{n_2(n_2-1)(\delta(1-k)-1)} \left[\frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \left(R_{post} - \frac{d}{r} \right) \right]^{\delta(1-k)-n_2(\delta(1-k)-1)} < 0.$$

Moreover, the piecewise function of $v_{PR}(z)$ is completely determined as

$$v_{PR}(z) = \begin{cases} B_{2,PR} \hat{z}_{PR}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \hat{z}_{PR} - R_{post}(z - \hat{z}_{PR}), & z \geq \hat{z}_{PR}, \\ B_{2,PR} z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z, & 0 < z < \hat{z}_{PR}. \end{cases}$$

Since $v_{PR}(z)$ is a piecewise polynomial function with smoothing merging conditions and differentiable everywhere, [6, Section 8, Theorem 8.5] indicates that $V_{PR}(x) = \inf_{\lambda_{PR} > 0} [\tilde{V}_{PR}(\lambda_{PR}) + \lambda_{PR}x]$ keeps true for any given initial wealth $x \geq R_{post}$. Thereafter, the closed-form of $V_{PR}(x)$ is

$$V_{PR}(x) = B_{2,PR} (\lambda_{PR}^*)^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (\lambda_{PR}^*)^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \lambda_{PR}^* + \lambda_{PR}^* x, \quad x \geq \hat{x}_{PR},$$

with $-n_2 B_{2,PR} (\lambda_{PR}^*)^{n_2-1} + K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (\lambda_{PR}^*)^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r} = x$, $x \geq \hat{x}_{PR}$. \hat{x}_{PR} is the critical wealth level corresponding to \hat{z}_{PR} and follows

$$\hat{x}_{PR} = -\frac{\partial v_{PR}}{\partial z} \Big|_{z=\hat{z}_{PR}} = -n_2 B_{2,PR} \hat{z}_{PR}^{n_2-1} + K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r}.$$

Moreover, the optimal wealth process takes the form

$$X^*(t) = -v'_{PR}(Z_{PR}^*(t)) = -n_2 B_{2,PR} (Z_{PR}^*(t))^{n_2-1} + K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r}, \quad 0 < Z_{PR}^*(t) \leq \hat{z}_{PR},$$

and the related optimal consumption-portfolio strategies are

$$c^*(t) = I_{PR}(\lambda_{PR}^* e^{\gamma t} H(t)) = (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}},$$

$$\pi^*(t) = \frac{\theta}{\sigma} \left[n_2(n_2-1) B_{2,PR} (Z_{PR}^*(t))^{n_2-1} - K_1 \frac{1}{\delta(1-k)-1} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} (Z_{PR}^*(t))^{\frac{1}{\delta(1-k)-1}} \right],$$

the optimal portfolio strategy is obtained from [7, Section 5, Theorem 3].

A.3 Proof of Lemma A.1

The form of function $U(x)$ is directly summarized from Equation (A.2) and (A.5), hence the proof here only focuses on the derivation of the Legendre-Fenchel transform of $U(x)$, which is also divided into two cases. We first extend the supremum in the definition of Legendre-Fenchel transform $\tilde{U}(z)$ by enlarging the range of x to \mathbb{R} , that is, $\tilde{U}(z) = \sup_{x \in \mathbb{R}} [U(x) - xz]$, for $0 < z < \infty$. Moreover, it can be proved the optimal solution x^* attaining the supremum automatically satisfies $x^* \geq R_{post}$.

(1) $R_{post} = \frac{d}{r}$: From the first-order condition, we have

$$z = U'(x^*) = K_1^{1-\delta(1-k)} \bar{L}^{(1-k)(1-\delta)} \left(x^* - \frac{d}{r} \right)^{\delta(1-k)-1},$$

which entails that $x^* = \left(z K_1^{\delta(1-k)-1} \bar{L}^{-(1-k)(1-\delta)} \right)^{\frac{1}{\delta(1-k)-1}} + \frac{d}{r}$. Then $x^* > R_{post} = \frac{d}{r}$ is obviously satisfied for $z > 0$. Taking the above relationship back to the dual transform definition, $\tilde{U}(z)$ is directly acquired after elementary calculation,

$$\tilde{U}(z) = \frac{1 - \delta(1-k)}{\delta(1-k)} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - \frac{d}{r} z.$$

(2) $R_{post} > \frac{d}{r}$: Considering the fact $V_{PR}(x) = \inf_{\lambda_{PR} > 0} [v_{PR}(\lambda_{PR}) + \lambda_{PR}x]$, it can be obtained that $v_{PR}(z)$ is the Legendre-Fenchel transform of $V_{PR}(x)$ from [4, Chapter 3, Lemma 4.3]. Then the identical forms of functions $V_{PR}(x)$ and $U(x)$ enable us to deduce the solution as $\tilde{U}(z) = v_{PR}(z)$. The last step is to claim $x^* \geq R_{post}$, which can be resorted to the condition $x^* = -v'_{PR}(z) \geq -v'_{PR}(\hat{z}_{PR}) = R_{post}$.

B Proof of Lemma 3.2

Following [6, Section 6, Lemma 6.3], we first define a new continuous process as

$$G(t) \triangleq \frac{1}{\xi(t)} \tilde{\mathbb{E}} \left[\int_t^\tau \xi(s)(c(s) + wl(s) + d - w\bar{L})ds + \xi(\tau)K \middle| \mathcal{F}_t \right], \quad \forall t \in [0, \tau],$$

where $\tilde{\mathbb{E}}[\cdot]$ representing the expectation under $\tilde{\mathbb{P}}$ measure. From the property of the random variable K , this process satisfies $G(t) = \frac{1}{\xi(t)} \tilde{\mathbb{E}} \left[\int_t^\tau \xi(s)(c(s) + wl(s))ds + \xi(\tau) \left(K - \frac{d-w\bar{L}}{r} \right) \middle| \mathcal{F}_t \right] + \frac{d-w\bar{L}}{r} \geq \frac{d-w\bar{L}}{r}$, a.s.. Then, making use of the condition $\mathbb{E} \left[\int_0^\tau H(t)(c(t) + wl(t) + d - w\bar{L})dt + H(\tau)K \right] = x$, we get $G(\tau) = K$ and

$$\begin{aligned} G(0) &= \tilde{\mathbb{E}} \left[\int_0^\tau \xi(s)(c(s) + wl(s) + d - w\bar{L})ds + \xi(\tau)K \right] \\ &= \mathbb{E} \left[\int_0^\tau H(s)(c(s) + wl(s) + d - w\bar{L})ds + H(\tau)K \right] = x, \end{aligned}$$

the above derivation involves changing the measure from the $\tilde{\mathbb{P}}$ measure with the pricing kernel as $\xi(t)$ to the \mathbb{P} measure with the pricing kernel as $H(t)$. Meanwhile, we define a new process

$$M(t) = \xi(t)G(t) + \int_0^t \xi(s)(c(s) + wl(s) + d - w\bar{L})ds, \quad \forall t \in [0, \tau].$$

Based on the fact

$$\begin{aligned}\tilde{\mathbb{E}}[M(t)] &= \tilde{\mathbb{E}} \left[\tilde{\mathbb{E}} \left[\int_t^\tau \xi(s)(c(s) + wl(s) + d - w\bar{L})ds + \xi(\tau)K \middle| \mathcal{F}_t \right] + \int_0^t \xi(s)(c(s) + wl(s) + d - w\bar{L})ds \right] \\ &= \tilde{\mathbb{E}} \left[\int_0^\tau \xi(s)(c(s) + wl(s) + d - w\bar{L})ds + \xi(\tau)K \right] = x = M(0),\end{aligned}$$

$M(t)$ is a $\tilde{\mathbb{P}}$ -martingale. According to the Martingale Representation Theorem from Bjork (2009), Chapter 11, Theorem 11.2, it can be expressed as $M(t) = x + \int_0^t \rho(s)d\tilde{B}(s)$, $\forall t \in [0, \tau]$, with an \mathbb{F} -adapted process $\rho(t)$ satisfying $\int_0^\infty \rho^2(s)ds < \infty$ a.s.. Furthermore, adopting the portfolio strategy $\pi(t) \triangleq \frac{\rho(t)}{\sigma\xi(t)}$, the wealth process becomes

$$\begin{aligned}dX^{x,c,\pi,l}(t) &= rX^{x,c,\pi,l}(t)dt + \pi(t)(\mu - r)dt - (c(t) + wl(t) + d - w\bar{L})dt + \sigma\pi(t)dB(t) \\ &= rX^{x,c,\pi,l}(t)dt - (c(t) + wl(t) + d - w\bar{L})dt + \sigma\pi(t)d\tilde{B}(t) \\ &= rX^{x,c,\pi,l}(t)dt - (c(t) + wl(t) + d - w\bar{L})dt + \frac{\rho(t)}{\xi(t)}d\tilde{B}(t),\end{aligned}$$

the second equality also comes from changing the measure by $\tilde{B}(t) \triangleq B(t) + \theta t$. We can observe that $G(t) = X^{x,c,\pi,l}(t)$ a.s. on $[0, \tau]$, which concludes the proof of this lemma.

C Proof of Lemma 3.3

Remark C.1. In this section we prove Lemma 3.3. Moreover, we also show that the conditions $\bar{z} < \tilde{y}$, defined below, and $\bar{z} < \hat{z}_{PR}$ hold true.

The proof here refers to Oksendal (2013) Section 10, Example 10.3.1.⁴ First of all, [5, Lemma 2.1] shows that

$$\tilde{u}(z) = \left[A_1 z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - wLz \right] \mathbb{I}_{\{0 < z < \tilde{y}\}} + \left[A_2 z^{-\frac{1-k}{k}} \right] \mathbb{I}_{\{z \geq \tilde{y}\}},$$

with $A_1 \triangleq \frac{1-\delta+\delta k}{\delta(1-k)} L^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}$, $A_2 \triangleq \frac{k}{\delta(1-k)} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}}$, and $\tilde{y} \triangleq L^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)}$.

Introducing two functions

$$g(t, z) \triangleq e^{-\gamma t} \tilde{U}(z), \quad G(t, z, \bar{w}) \triangleq g(t, z) + \bar{w} = e^{-\gamma t} \tilde{U}(z) + \bar{w},$$

and an operator $\mathcal{A}_P G(t, z, \bar{w}) \triangleq \frac{\partial G}{\partial t} + (\gamma - r)z \frac{\partial G}{\partial z} + \frac{\theta^2}{2} z^2 \frac{\partial^2 G}{\partial z^2} + e^{-\gamma t} \tilde{u}(z) - e^{-\gamma t} (d - w\bar{L})z$, we can determine the continuous region as $\Omega_1 = \{(t, z, \bar{w}) : \mathcal{A}_P G(t, z, \bar{w}) > 0\}$. Moreover, since

$$\mathcal{A}_P G(t, z, \bar{w}) = -\gamma e^{-\gamma t} \tilde{U}(z) + (\gamma - r)z e^{-\gamma t} \tilde{U}'(z) + \frac{\theta^2}{2} z^2 e^{-\gamma t} \tilde{U}''(z) + e^{-\gamma t} (\tilde{u}(z) - (d - w\bar{L})z),$$

defining a new function $h(z) = -\gamma \tilde{U}(z) + (\gamma - r)z \tilde{U}'(z) + \frac{\theta^2}{2} z^2 \tilde{U}''(z) + \tilde{u}(z) - (d - w\bar{L})z$, the continuous region can be rewritten as $\Omega_1 = \{z > 0 : h(z) > 0\}$. Since the function $\tilde{U}(z)$ takes two different forms based on the value of R_{post} , we split the remaining discussion also into two cases: $R_{post} = \frac{d}{r}$ and $R_{post} > \frac{d}{r}$.

⁴B. Oksendal, Stochastic differential equations: an introduction with applications. Springer Science & Business Media, 2013.

(1) For $R_{post} = \frac{d}{r}$, we have $\tilde{U}(z) = \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z$. After the basic calculation, we get

$$\begin{aligned} h(z) &= \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + w\bar{L}z + \tilde{u}(z) \\ &= \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + w\bar{L}z + \left[A_1 z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - wLz \right] \mathbb{I}_{\{0 < z < \tilde{y}\}} + \left[A_2 z^{-\frac{1-k}{k}} \right] \mathbb{I}_{\{z \geq \tilde{y}\}}. \end{aligned} \quad (C.1)$$

$h(z)$ inherits the piecewise form from the function $\tilde{u}(z)$. Afterwards, determining the continuous region corresponds to characterize the features of the zero of $h(z)$. We begin claiming its convexity by the second derivative function. On the interval $0 < z < \tilde{y}$, we can directly determine the sign of $h''(z)$ with $h''(z) = \frac{1}{\delta(1-k)-1} z^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} \left[\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - L^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \right] > 0$. As for the interval $z > \tilde{y}$, the corresponding second derivative function $h''(z)$ is

$$h''(z) = \frac{1}{\delta(1-k)-1} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} + \frac{1}{\delta k} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} z^{-\frac{1+k}{k}}.$$

By solving the inequality, $\frac{1}{\delta(1-k)-1} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} + \frac{1}{\delta k} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} z^{-\frac{1+k}{k}} > 0$, we get

$$z > \bar{L}^{-k} \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)}.$$

Since

$$\tilde{y} - \bar{L}^{-k} \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} = \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} \left[\bar{L}^{-k} - \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} \bar{L}^{-k} \right] > 0,$$

$h''(z) > 0$ keeps true for $z > \tilde{y}$. Besides, considering the condition $\lim_{z \uparrow \tilde{y}} h''(z) = \lim_{z \downarrow \tilde{y}} h''(z)$, we can conclude that the function $h(z)$ is strictly convex on the interval $z > 0$. Then we move to claim $h(\tilde{y}) > 0$: before this, a new function is introduced as

$$f(z) = \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{k}{\delta(1-k)} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} z^{-\frac{1-k}{k}} + w\bar{L}z, \quad z > 0,$$

and its derivative functions are

$$f'(z) = z^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - \frac{1}{\delta} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} z^{-\frac{1}{k}} + w\bar{L},$$

$$f''(z) = \frac{1}{\delta(1-k)-1} z^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} + \frac{1}{\delta k} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} z^{-\frac{1+k}{k}}.$$

Defining $\tilde{y}_1 = \bar{L}^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)}$, it can be obtained that

$$f(\tilde{y}_1) = \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{1-k} \left(\frac{1-\delta}{\delta w} \right)^{-\delta(1-k)} + \frac{k}{\delta(1-k)} \bar{L}^{1-k} \left(\frac{1-\delta}{\delta w} \right)^{-\delta(1-k)} + w\bar{L}^{1-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} = 0,$$

$$f'(\tilde{y}_1) = \left(\bar{L}^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} \right)^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - \frac{1}{\delta} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}} \left(\bar{L}^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} \right)^{-\frac{1}{k}} + w\bar{L} = 0.$$

For the second derivative, $f''(z) > 0$ is equivalent to $z > \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} \bar{L}^{-k}$. Since $0 < \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} < 1$, we have $\tilde{y}_1 > \left(\frac{1-\delta(1-k)}{\delta k} \right)^{\frac{k(\delta(1-k)-1)}{(1-k)(\delta-1)}} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} \bar{L}^{-k}$, which results in $f''(z) > 0$ for $z > \tilde{y}_1$. Then the fact $f'(\tilde{y}_1) = 0$ indicates that $f'(z) > 0$ for $z > \tilde{y}_1$, which means $f(z)$ is strictly increasing on the corresponding interval. Considering the relationship $\tilde{y} = L^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)} > \tilde{y}_1$, we can observe the positive value of $h(\tilde{y})$ through

$$0 = f(\tilde{y}_1) < f(\tilde{y}) = h(\tilde{y}). \quad (\text{C.2})$$

Finally, in view of the limitations

$$\lim_{z \downarrow 0} h(z) = \lim_{z \downarrow 0} \left[\frac{\delta(1-k)-1}{\delta(1-k)} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} \left(\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - L^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \right) + wz(\bar{L} - L) \right] = 0,$$

$$\lim_{z \downarrow 0} h'(z) = \lim_{z \downarrow 0} \left[z^{\frac{1}{\delta(1-k)-1}} \left(\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} - L^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \right) + w(\bar{L} - L) \right] = -\infty,$$

and the properties $h''(z) > 0$ for $z > 0$, $h(\tilde{y}) > 0$, we can conclude that there is a unique zero of $h(z)$, which is denoted as \bar{z} , satisfying $\bar{z} < \tilde{y}$ and $h'(\bar{z}) = \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}} + w\bar{L} + \tilde{u}'(\bar{z})$. Hence, the continuous region is $\Omega_1 = \{(t, z, \bar{w}) : \mathcal{A}_P G(s, z, \bar{w}) > 0\} = \{z : h(z) > 0\} = \{z > \bar{z}\}$.

(2) For $R_{post} > \frac{d}{r}$, we have

$$\tilde{U}(z) = \begin{cases} B_{2,PR} \hat{z}_{PR}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \hat{z}_{PR} - R_{post}(z - \hat{z}_{PR}), & z \geq \hat{z}_{PR}, \\ B_{2,PR} z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} z, & 0 < z < \hat{z}_{PR}, \end{cases}$$

then the function $h(z)$ on the interval $0 < z < \hat{z}_{PR}$ is obtained as

$$h(z) = B_{2,PR} z^{n_2} \left[-\gamma + (\gamma - r)n_2 + \frac{\theta^2}{2} n_2(n_2 - 1) \right] + \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + w\bar{L}z + \tilde{u}(z).$$

From $n_1 + n_2 = -\frac{\gamma - r - \frac{\theta^2}{2}}{\frac{\theta^2}{2}}$ and $n_1 n_2 = -\frac{\gamma}{\frac{\theta^2}{2}}$, we can deduce $-\gamma + (\gamma - r)n_2 + \frac{\theta^2}{2} n_2(n_2 - 1) = 0$. Hence, the function $h(z)$ is reduced as

$$h(z) = \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + w\bar{L}z + \tilde{u}(z), \quad 0 < z < \hat{z}_{PR}.$$

Compared to Equation (C.1), we can observe that $h(z)$ adopts the same form but applies to the different intervals. As for the interval $z \geq \hat{z}_{PR}$, considering that $\tilde{U}(z) = v_{PR}(\hat{z}_{PR}) - R_{post}(z - \hat{z}_{PR})$, and the condition (A.7) is applicable at the point $z = \hat{z}_{PR}$, we get

$$\begin{aligned} h(z) &= -\gamma(v_{PR}(\hat{z}_{PR}) - R_{post}(z - \hat{z}_{PR})) - (\gamma - r)zR_{post} + \tilde{u}(z) - (d - w\bar{L})z \\ &= (rR_{post} - d)(z - \hat{z}_{PR}) - \tilde{u}_{PR}(\hat{z}_{PR}) + \tilde{u}(z) + w\bar{L}z. \end{aligned}$$

$h''(z) = \tilde{u}''(z) > 0$ shows that $h(z)$ is strictly convex on (\hat{z}_{PR}, ∞) . Then a contradiction is constructed to prove $\bar{z} < \hat{z}_{PR}$. We first use $r = \frac{\theta^2}{2}(1-n_2)(n_1-1)$ and $n_2 > 1 > \frac{\delta(1-k)}{\delta(1-k)-1} > 0$ to derive the condition

$$\frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \frac{1}{K_1} = r \frac{\gamma - r\delta(1-k) - \frac{\theta^2}{2} \frac{\delta(1-k)}{1-\delta(1-k)}}{\gamma - r\delta(1-k) + \frac{\theta^2}{2} n_2} < r,$$

which gives us

$$\hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} = \bar{L}^{\frac{(1-k)(1-\delta)}{\delta(1-k)-1}} \frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \frac{(R_{post} - \frac{d}{r})}{K_1} < \bar{L}^{\frac{(1-k)(1-\delta)}{\delta(1-k)-1}} (rR_{post} - d).$$

Assuming $\bar{z} \geq \hat{z}_{PR}$, we can observe the contradiction through

$$\begin{aligned} h'(\bar{z}) &= \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}} + w\bar{L} + \tilde{u}'(\bar{z}) \\ &\leq \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} + w\bar{L} + \tilde{u}'(\bar{z}) \\ &< \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{L}^{\frac{(1-k)(1-\delta)}{\delta(1-k)-1}} (rR_{post} - d) + w\bar{L} + \tilde{u}'(\bar{z}) \\ &= (rR_{post} - d) + w\bar{L} + \tilde{u}'(\bar{z}) = h'(\bar{z}). \end{aligned}$$

Then the condition $\bar{z} < \hat{z}_{PR}$ implies $h(\hat{z}_{PR}) > 0$ and

$$\lim_{z \uparrow \hat{z}_{PR}} h'(z) = \lim_{z \uparrow \hat{z}_{PR}} \left[\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} z^{\frac{1}{\delta(1-k)-1}} + \tilde{u}'(z) + w\bar{L} \right] = -\tilde{u}'_{PR}(\hat{z}_{PR}) + \tilde{u}'(\hat{z}_{PR}) + w\bar{L} \geq 0.$$

Afterwards, we have

$$\begin{aligned} \lim_{z \downarrow \hat{z}_{PR}} h'(z) &= rR_{post} - d + \tilde{u}'(\hat{z}_{PR}) + w\bar{L} \\ &\geq rR_{post} - d + \tilde{u}'_{PR}(\hat{z}_{PR}) \\ &= rR_{post} - d - \hat{z}_{PR}^{\frac{1}{\delta(1-k)-1}} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \\ &= rR_{post} - d - \frac{(1-n_2)(1-\delta(1-k))}{n_2(\delta(1-k)-1)-\delta(1-k)} \frac{1}{K_1} \left(R_{post} - \frac{d}{r} \right) > 0, \end{aligned}$$

which indicates that $h(z)$ is strictly increasing for $z > \hat{z}_{PR}$ regarding the convex property already shown. Therefore, \bar{z} is the unique zero of function $h(z)$, and satisfies $\bar{z} < \hat{z}_{PR}$. The last step is to claim $\bar{z} < \tilde{y}$ under this case, which is equivalent to $h(\tilde{y}) > 0$ and discussed in two different situations. If $\tilde{y} \leq \hat{z}_{PR}$, using the result (C.2), we have

$$h(\tilde{y}) = \frac{\delta(1-k)-1}{\delta(1-k)} \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \tilde{y}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + w\bar{L}\tilde{y} + \tilde{u}(\tilde{y}) = f(\tilde{y}) > 0,$$

otherwise, if $\tilde{y} > \hat{z}_{PR}$, we use the increasing property of $h(z)$ on (\hat{z}_{PR}, ∞) to directly obtain $h(\tilde{y}) > 0$.

D Calculation of Variational Inequalities (3.3)

The solution of (3.3) is split into two different cases based on the value of R_{post} , namely $R_{post} = \frac{d}{r}$ and $R_{post} > \frac{d}{r}$. Following [2, Appendix A], we take the time-separated form of function $\phi(t, z) = e^{-\gamma t} v(z)$ for solving the above variational inequalities explicitly.

We recall that [5, Lemma 2.1] shows that $\tilde{u}(z) = \left[A_1 z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - wLz \right] \mathbb{I}_{\{0 < z < \tilde{y}\}} + \left[A_2 z^{-\frac{1-k}{k}} \right] \mathbb{I}_{\{z \geq \tilde{y}\}}$, with $A_1 \triangleq \frac{1-\delta+\delta k}{\delta(1-k)} L^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}$, $A_2 \triangleq \frac{k}{\delta(1-k)} \left(\frac{1-\delta}{\delta w} \right)^{\frac{(1-k)(1-\delta)}{k}}$, and $\tilde{y} \triangleq L^{-k} \left(\frac{1-\delta}{\delta w} \right)^{1-\delta(1-k)}$. Moreover, n_1 and n_2 are the roots of the second-order equation $\frac{\theta^2}{2} n^2 + \left(\gamma - r - \frac{\theta^2}{2} \right) n - \gamma = 0$, and satisfy

$$n_{1,2} = -\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2} \mp \sqrt{\left(\frac{\gamma - r - \frac{\theta^2}{2}}{\theta^2} \right)^2 + \frac{2\gamma}{\theta^2}}, \quad n_1 < 0, \quad n_2 > 1.$$

Case 1. $R_{pre} = \frac{d-w\bar{L}}{r}$ & $R_{post} = \frac{d}{r}$

From the condition (V1) of (3.3), the following differential equation holds in the region $z > \bar{z}$,

$$-\gamma v(z) + (\gamma - r)zv'(z) + \frac{1}{2}\theta^2 z^2 v''(z) + \tilde{u}(z) - (d - w\bar{L})z = 0, \quad (\text{D.1})$$

whose solution takes a form as

$$v(z) = \begin{cases} B_{11}z^{n_1} + B_{21}z^{n_2} + \frac{A_1}{\Gamma_1} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)-d}{r}z, & \bar{z} < z < \tilde{y}, \\ B_{12}z^{n_1} + B_{22}z^{n_2} + \frac{A_2}{\Gamma_2} z^{-\frac{1-k}{k}} + \frac{w\bar{L}-d}{r}z, & z \geq \tilde{y}. \end{cases}$$

Since $n_2 > 0$, for the sake of avoiding the explosion of the term z^{n_2} as z goes to ∞ , we set $B_{22} = 0$. Then, the condition (V4) of (3.3) enables us to obtain

$$v(\bar{z}) = \tilde{U}(\bar{z}) = \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \bar{z},$$

the second equality results from the condition $R_{post} = \frac{d}{r}$. Furthermore, combining with the smooth condition at the point $z = \tilde{y}$, we can construct a four-equations system to determine the parameters B_{11} , B_{21} , B_{12} and \bar{z} .

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_{11}\bar{z}^{n_1} + B_{21}\bar{z}^{n_2} + \frac{A_1}{\Gamma_1} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r} \bar{z} = \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}};$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1 B_{11} \bar{z}^{n_1-1} + n_2 B_{21} \bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r} = -K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}};$$

- \mathcal{C}^0 condition at $z = \tilde{y}$

$$B_{11}\tilde{y}^{n_1} + B_{21}\tilde{y}^{n_2} + \frac{A_1}{\Gamma_1} \tilde{y}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{wL}{r} \tilde{y} = B_{12}\tilde{y}^{n_1} + \frac{A_2}{\Gamma_2} \tilde{y}^{-\frac{1-k}{k}};$$

- \mathcal{C}^1 condition at $z = \tilde{y}$

$$n_1 B_{11} \tilde{y}^{n_1-1} + n_2 B_{21} \tilde{y}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \tilde{y}^{\frac{1}{\delta(1-k)-1}} - \frac{wL}{r} = n_1 B_{12} \tilde{y}^{n_1-1} - \frac{1-k}{k} \frac{A_2}{\Gamma_2} \tilde{y}^{-\frac{1}{k}}.$$

Case 2. $R_{pre} = \frac{d-w\bar{L}}{r}$ & $R_{post} > \frac{d}{r}$

Then we move to the second case with a different condition $R_{post} > \frac{d}{r}$ compared to Case 1, which mainly affects the post-retirement part and leads to a different form of $\tilde{U}(z)$. Lemma 3.1 shows that the corresponding Legendre-Fenchel transform of post-retirement value function $\tilde{U}(z)$ is

$$\tilde{U}(z) = B_{2,PR}z^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r}z.$$

Meanwhile, the dual transform involving the pre-retirement part $\tilde{u}(z)$ stays the same; hence Equation (V1) from (3.3) takes the identical solution. Afterwards, using the smooth fit conditions at $z=\bar{z}$ and $z=\tilde{y}$, we construct a four-equations system to achieve the unknowns, B_{11} , B_{21} , B_{12} and \bar{z} .

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_{11}\bar{z}^{n_1} + B_{21}\bar{z}^{n_2} + \frac{A_1}{\Gamma_1}\bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r}\bar{z} = B_{2,PR}\bar{z}^{n_2} + \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}\bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}};$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1B_{11}\bar{z}^{n_1-1} + n_2B_{21}\bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1}\frac{A_1}{\Gamma_1}\bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r} = n_2B_{2,PR}\bar{z}^{n_2-1} - K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}\bar{z}^{\frac{1}{\delta(1-k)-1}};$$

- \mathcal{C}^0 condition at $z = \tilde{y}$

$$B_{11}\tilde{y}^{n_1} + B_{21}\tilde{y}^{n_2} + \frac{A_1}{\Gamma_1}\tilde{y}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{wL}{r}\tilde{y} = B_{12}\tilde{y}^{n_1} + \frac{A_2}{\Gamma_2}\tilde{y}^{-\frac{1-k}{k}};$$

- \mathcal{C}^1 condition at $z = \tilde{y}$

$$n_1B_{11}\tilde{y}^{n_1-1} + n_2B_{21}\tilde{y}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1}\frac{A_1}{\Gamma_1}\tilde{y}^{\frac{1}{\delta(1-k)-1}} - \frac{wL}{r} = n_1B_{12}\tilde{y}^{n_1-1} - \frac{1-k}{k}\frac{A_2}{\Gamma_2}\tilde{y}^{-\frac{1}{k}}.$$

After obtaining the closed forms of $v(z)$ separately in Case 1 and Case 2, and given the initial wealth $x \geq R_{pre}$, the optimal Lagrange multiplier λ^* can be acquired through solving the equation $x = -v'(\lambda^*)$, due to the fact that $V(x) = \inf_{\lambda>0} [\tilde{V}(\lambda) + \lambda x] = \inf_{\lambda>0} [v(\lambda) + \lambda x] = v(\lambda^*) + \lambda^*x$ holds under the differentiable property of $v(\cdot)$.

Then the optimal dual process of wealth follows $Z^*(t) = \lambda^*e^{\gamma t}H(t)$.

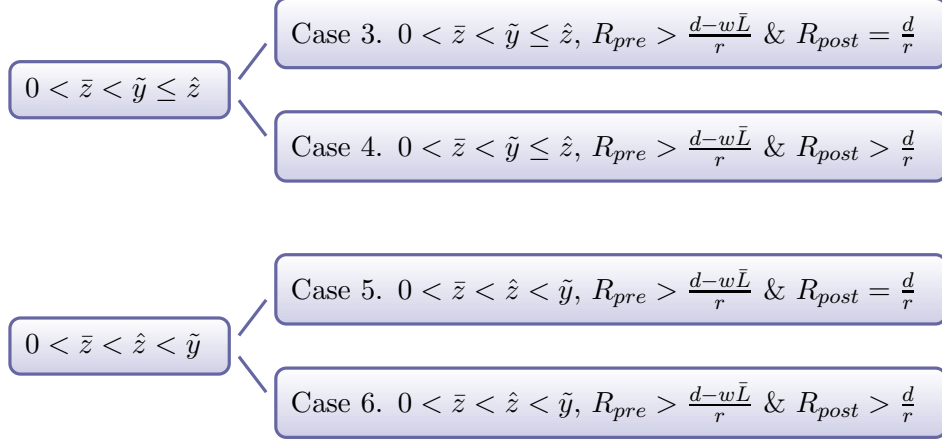
Proposition D.1. For Case 1 and Case 2, the optimal retirement time is $\tau^* = \inf_{t \geq 0} \{Z^*(t) \leq \bar{z}\}$, the optimal consumption-portfolio-leisure plan $\{c^*(t), \pi^*(t), l^*(t)\}$ before retirement is given by

$$\begin{aligned} c^*(t) &= \begin{cases} L^{-\frac{(1-k)(1-\delta)}{\delta(1-k)-1}}(Z^*(t))^{\frac{1}{\delta(1-k)-1}}, & \bar{z} < Z^*(t) < \tilde{y}, \\ \left(\frac{1-\delta}{\delta w}\right)^{\frac{(1-\delta)(1-k)}{k}}(Z^*(t))^{-\frac{1}{k}}, & Z^*(t) \geq \tilde{y}, \end{cases} \\ l^*(t) &= \begin{cases} L, & \bar{z} < Z^*(t) < \tilde{y}, \\ \left(\frac{1-\delta}{\delta w}\right)^{-\frac{\delta(1-k)-1}{k}}(Z^*(t))^{-\frac{1}{k}}, & Z^*(t) \geq \tilde{y}, \end{cases} \\ \pi^*(t) &= \begin{cases} \frac{\theta}{\sigma} \left[n_1(n_1-1)B_{11}(Z^*(t))^{n_1-1} + n_2(n_2-1)B_{21}(Z^*(t))^{n_2-1} \right. \\ \quad \left. + \frac{\delta(1-k)}{(\delta(1-k)-1)^2} \frac{A_1}{\Gamma_1} (Z^*(t))^{\frac{1}{\delta(1-k)-1}} \right], & \bar{z} < Z^*(t) < \tilde{y}, \\ \frac{\theta}{\sigma} \left[n_1(n_1-1)B_{12}(Z^*(t))^{n_1-1} + \frac{1-k}{k^2} \frac{A_2}{\Gamma_2} (Z^*(t))^{-\frac{1}{k}} \right], & Z^*(t) \geq \tilde{y}. \end{cases} \end{aligned}$$

Proof. The optimal consumption and leisure strategies come from [5, Lemma 2.1], and the optimal portfolio strategy is derived by $\pi^*(t) = \frac{\theta}{\sigma} Z^*(t) v''(Z^*(t))$ from [7, Section 5, Theorem 3]. \square

E Calculation of Variational Inequalities (3.5)

Recalling the condition $\bar{z} < \tilde{y}$ in Lemma 3.3, the problem to be solved is split into four different cases depending on the relationship between \tilde{y} with \hat{z} , and R_{post} with $\frac{d}{r}$. We provide a diagram for a clear classification.



Additionally, we assume that $\phi(t, z)$ takes the time-separated form, $\phi(t, z) = e^{-\gamma t} v(z)$, as in [2, Appendix A].

Case 3. $0 < \bar{z} < \tilde{y} \leq \hat{z}$, $R_{pre} > \frac{d-w\bar{L}}{r}$ & $R_{post} = \frac{d}{r}$

We begin with the condition (V3) in (3.5), the following differential equation is obtained,

$$-\gamma v(z) + (\gamma - r)zv'(z) + \frac{1}{2}\theta^2 z^2 v''(z) + \tilde{u}(z) - (d - w\bar{L})z = 0,$$

which is identical with Equation (D.1), hence shares the same solution as

$$v(z) = \begin{cases} B_{11}z^{n_1} + B_{21}z^{n_2} + \frac{A_1}{\Gamma_1}z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)-d}{r}z, & \bar{z} < z < \tilde{y}, \\ B_{12}z^{n_1} + B_{22}z^{n_2} + \frac{A_2}{\Gamma_2}z^{-\frac{1-k}{k}} + \frac{w\bar{L}-d}{r}z, & \tilde{y} \leq z < \hat{z}. \end{cases} \quad (\text{E.1})$$

As follows, a six-equations system is established to obtain the unknown parameters B_{11} , B_{21} , B_{12} , B_{22} , \bar{z} and \hat{z} .

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_{11}\bar{z}^{n_1} + B_{21}\bar{z}^{n_2} + \frac{A_1}{\Gamma_1}\bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r}\bar{z} = \frac{1-\delta(1-k)}{\delta(1-k)}K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}\bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1B_{11}\bar{z}^{n_1-1} + n_2B_{21}\bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1}\frac{A_1}{\Gamma_1}\bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L}-L)}{r} = -K_1\bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}}\bar{z}^{\frac{1}{\delta(1-k)-1}}.$$

- \mathcal{C}^0 condition at $z = \tilde{y}$

$$B_{11}\tilde{y}^{n_1} + B_{21}\tilde{y}^{n_2} + \frac{A_1}{\Gamma_1}\tilde{y}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{wL}{r}\tilde{y} = B_{12}\tilde{y}^{n_1} + B_{22}\tilde{y}^{n_2} + \frac{A_2}{\Gamma_2}\tilde{y}^{-\frac{1-k}{k}}.$$

- \mathcal{C}^1 condition at $z = \tilde{y}$

$$n_1 B_{11} \tilde{y}^{n_1-1} + n_2 B_{21} \tilde{y}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \tilde{y}^{\frac{1}{\delta(1-k)-1}} - \frac{wL}{r} = n_1 B_{12} \tilde{y}^{n_1-1} + n_2 B_{22} \tilde{y}^{n_2-1} - \frac{1-k}{k} \frac{A_2}{\Gamma_2} \tilde{y}^{-\frac{1}{k}}.$$

- \mathcal{C}^1 condition at $z = \hat{z}$

$$n_1 B_{12} \hat{z}^{n_1-1} + n_2 B_{22} \hat{z}^{n_2-1} - \frac{1-k}{k} \frac{A_2}{\Gamma_2} \hat{z}^{-\frac{1}{k}} + \frac{w\bar{L} - d}{r} + R_{pre} = 0.$$

- \mathcal{C}^2 condition at $z = \hat{z}$

$$n_1(n_1 - 1) B_{12} \hat{z}^{n_1-2} + n_2(n_2 - 1) B_{22} \hat{z}^{n_2-2} + \frac{1-k}{k^2} \frac{A_2}{\Gamma_2} \hat{z}^{-\frac{1+k}{k}} = 0.$$

Case 4. $0 < \bar{z} < \tilde{y} \leq \hat{z}$, $R_{pre} > \frac{d-w\bar{L}}{r}$ & $R_{post} > \frac{d}{r}$

The only difference between this case and the previous one occurs in $z = \bar{z}$. Since $R_{post} > \frac{d}{r}$, Lemma 3.1 shows that the Legendre-Fenchel transform of post-retirement value function $\tilde{U}(\bar{z})$ is

$$\tilde{U}(\bar{z}) = B_{2,PR} \bar{z}^{n_2} + \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} - \frac{d}{r} \bar{z}.$$

As the same before, we set up a six-equation system to achieve the unknowns, B_{11} , B_{21} , B_{12} , B_{22} , \bar{z} and \hat{z} . Compared with the first case, only \mathcal{C}^0 and \mathcal{C}^1 conditions at $z = \bar{z}$ change, whereas all the others keep true.

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_{11} \bar{z}^{n_1} + B_{21} \bar{z}^{n_2} + \frac{A_1}{\Gamma_1} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} \bar{z} = B_{2,PR} \bar{z}^{n_2} + \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1 B_{11} \bar{z}^{n_1-1} + n_2 B_{21} \bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} = n_2 B_{2,PR} \bar{z}^{n_2-1} - K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}}.$$

Case 5. $0 < \bar{z} < \hat{z} < \tilde{y}$, $R_{pre} > \frac{d-w\bar{L}}{r}$ & $R_{post} = \frac{d}{r}$

Firstly, the interval $0 < \bar{z} < z < \hat{z} < \tilde{y}$, where the condition (V3) of (3.5) holds, is considered. Also adopting the time-independent form of $\phi(t, z) = e^{-\gamma t} v(z)$, the following differential equation is obtained $-\gamma v(z) + (\gamma - r)zv'(z) + \frac{1}{2}\theta^2 z^2 v''(z) + \tilde{u}(z) - (d - w\bar{L})z = 0$. The dual transform of $u(c, l)$ is $\tilde{u}(z) = A_1 z^{\frac{\delta(1-k)}{\delta(1-k)-1}} - wLz$ in the considered interval; therefore, the above differential equation takes the identical form of the one in $0 < \bar{z} < z < \tilde{y} < \hat{z}$ of Case 3. The solution of $v(z)$ is given directly from (E.1), only changing the parameters' notations from B_{11} to B_1 and B_{21} to B_2 respectively,

$$v(z) = B_1 z^{n_1} + B_2 z^{n_2} + \frac{A_1}{\Gamma_1} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L} - L) - d}{r} z, \quad \bar{z} < z < \hat{z}.$$

Next, a four-equations system is set up to derive the desired parameters B_1 , B_2 , \bar{z} , \hat{z} . The same arguments with Case 3, only \mathcal{C}^1 and \mathcal{C}^2 conditions in $z = \hat{z}$ changes.

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_1 \bar{z}^{n_1} + B_2 \bar{z}^{n_2} + \frac{A_1}{\Gamma_1} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} \bar{z} = \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1 B_1 \bar{z}^{n_1-1} + n_2 B_2 \bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} = -K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \hat{z}$

$$n_1 B_1 \hat{z}^{n_1-1} + n_2 B_2 \hat{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \hat{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L} - L) - d}{r} + R_{pre} = 0.$$

- \mathcal{C}^2 condition at $z = \hat{z}$

$$n_1(n_1 - 1) B_1 \hat{z}^{n_1-2} + n_2(n_2 - 1) B_2 \hat{z}^{n_2-2} + \frac{\delta(1-k)}{(\delta(1-k) - 1)^2} \frac{A_1}{\Gamma_1} \hat{z}^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} = 0.$$

Case 6. $0 < \bar{z} < \hat{z} < \tilde{y}$, $R_{pre} > \frac{d-w\bar{L}}{r}$ & $R_{post} > \frac{d}{r}$

We now move to Case 6. The only difference from the previous case happens on the condition $R_{post} > \frac{d}{r}$, which is mainly involved in the post-retirement part; hence, the solution of the partial differential equation corresponding to Condition (V3) in (3.5) remains unchanged, that is,

$$v(z) = B_1 z^{n_1} + B_2 z^{n_2} + \frac{A_1}{\Gamma_1} z^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L} - L) - d}{r} z, \quad \bar{z} < z < \hat{z}.$$

Considering the smooth fit conditions at \bar{z} and \hat{z} , we construct a four-equations system to deduce the values of unknown parameters B_1 , B_2 , \bar{z} and \hat{z} .

- \mathcal{C}^0 condition at $z = \bar{z}$

$$B_1 \bar{z}^{n_1} + B_2 \bar{z}^{n_2} + \frac{A_1}{\Gamma_1} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} \bar{z} = B_{2,PR} \bar{z}^{n_2} + \frac{1 - \delta(1-k)}{\delta(1-k)} K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{\delta(1-k)}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \bar{z}$

$$n_1 B_1 \bar{z}^{n_1-1} + n_2 B_2 \bar{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \bar{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L} - L)}{r} = n_2 B_{2,PR} \bar{z}^{n_2-1} - K_1 \bar{L}^{\frac{(1-k)(1-\delta)}{1-\delta(1-k)}} \bar{z}^{\frac{1}{\delta(1-k)-1}}.$$

- \mathcal{C}^1 condition at $z = \hat{z}$

$$n_1 B_1 \hat{z}^{n_1-1} + n_2 B_2 \hat{z}^{n_2-1} + \frac{\delta(1-k)}{\delta(1-k)-1} \frac{A_1}{\Gamma_1} \hat{z}^{\frac{1}{\delta(1-k)-1}} + \frac{w(\bar{L} - L) - d}{r} + R_{pre} = 0.$$

- \mathcal{C}^2 condition at $z = \hat{z}$

$$n_1(n_1 - 1) B_1 \hat{z}^{n_1-2} + n_2(n_2 - 1) B_2 \hat{z}^{n_2-2} + \frac{\delta(1-k)}{(\delta(1-k) - 1)^2} \frac{A_1}{\Gamma_1} \hat{z}^{\frac{2-\delta(1-k)}{\delta(1-k)-1}} = 0.$$

Same argument with Case 1 and Case 2 in Appendix D, given the initial wealth $x \geq R_{pre}$ and solving $x = -v'(\lambda^*)$, we can obtain the optimal Lagrange multiplier λ^* and then the optimal process $Z^*(t) = \lambda^* e^{\gamma t} H(t)$.

Proposition E.1. *Under the condition $\tilde{y} \leq \hat{z}$, corresponding to Case 3 and Case 4, the optimal consumption-portfolio-leisure plan $\{c^*(t), \pi^*(t), l^*(t)\}$ before retirement is given by*

$$c^*(t) = \begin{cases} \left(\frac{1-\delta}{\delta w}\right)^{\frac{(1-\delta)(1-k)}{k}} (Z^*(t))^{-\frac{1}{k}}, & \tilde{y} \leq Z^*(t) \leq \hat{z}, \\ L^{-\frac{(1-k)(1-\delta)}{\delta(1-k)-1}} (Z^*(t))^{\frac{1}{\delta(1-k)-1}}, & \bar{z} < Z^*(t) < \tilde{y}, \end{cases}$$

$$l^*(t) = \begin{cases} \left(\frac{1-\delta}{\delta w}\right)^{-\frac{\delta(1-k)-1}{k}} (Z^*(t))^{-\frac{1}{k}}, & \tilde{y} \leq Z^*(t) \leq \hat{z}, \\ L, & \bar{z} < Z^*(t) < \tilde{y}, \end{cases}$$

$$\pi^*(t) = \begin{cases} \frac{\theta}{\sigma} \left[n_1(n_1-1)B_{12}(Z^*(t))^{n_1-1} + n_2(n_2-1)B_{22}(Z^*(t))^{n_2-1} \right. \\ \quad \left. + \frac{1-k}{k^2} \frac{A_2}{\Gamma_2} (Z^*(t))^{-\frac{1}{k}} \right], & \tilde{y} \leq Z^*(t) \leq \hat{z}, \\ \frac{\theta}{\sigma} \left[n_1(n_1-1)B_{11}(Z^*(t))^{n_1-1} + n_2(n_2-1)B_{21}(Z^*(t))^{n_2-1} \right. \\ \quad \left. + \frac{\delta(1-k)}{(\delta(1-k)-1)^2} \frac{A_1}{\Gamma_1} (Z^*(t))^{\frac{1}{\delta(1-k)-1}} \right], & \bar{z} < Z^*(t) < \tilde{y}. \end{cases}$$

Meanwhile, under the condition $\hat{z} < \tilde{y}$, corresponding to Case 5 and Case 6, the optimal consumption-portfolio-leisure plan $\{c^*(t), \pi^*(t), l^*(t)\}$ before retirement is given by

$$c^*(t) = L^{-\frac{(1-k)(1-\delta)}{\delta(1-k)-1}} (Z^*(t))^{\frac{1}{\delta(1-k)-1}}, \quad l^*(t) = L,$$

$$\pi^*(t) = \frac{\theta}{\sigma} \left[n_1(n_1-1)B_1(Z^*(t))^{n_1-1} + n_2(n_2-1)B_2(Z^*(t))^{n_2-1} + \frac{\delta(1-k)}{(\delta(1-k)-1)^2} \frac{A_1}{\Gamma_1} (Z^*(t))^{\frac{1}{\delta(1-k)-1}} \right].$$

Proof. Follow the lines of Proposition D.1. □