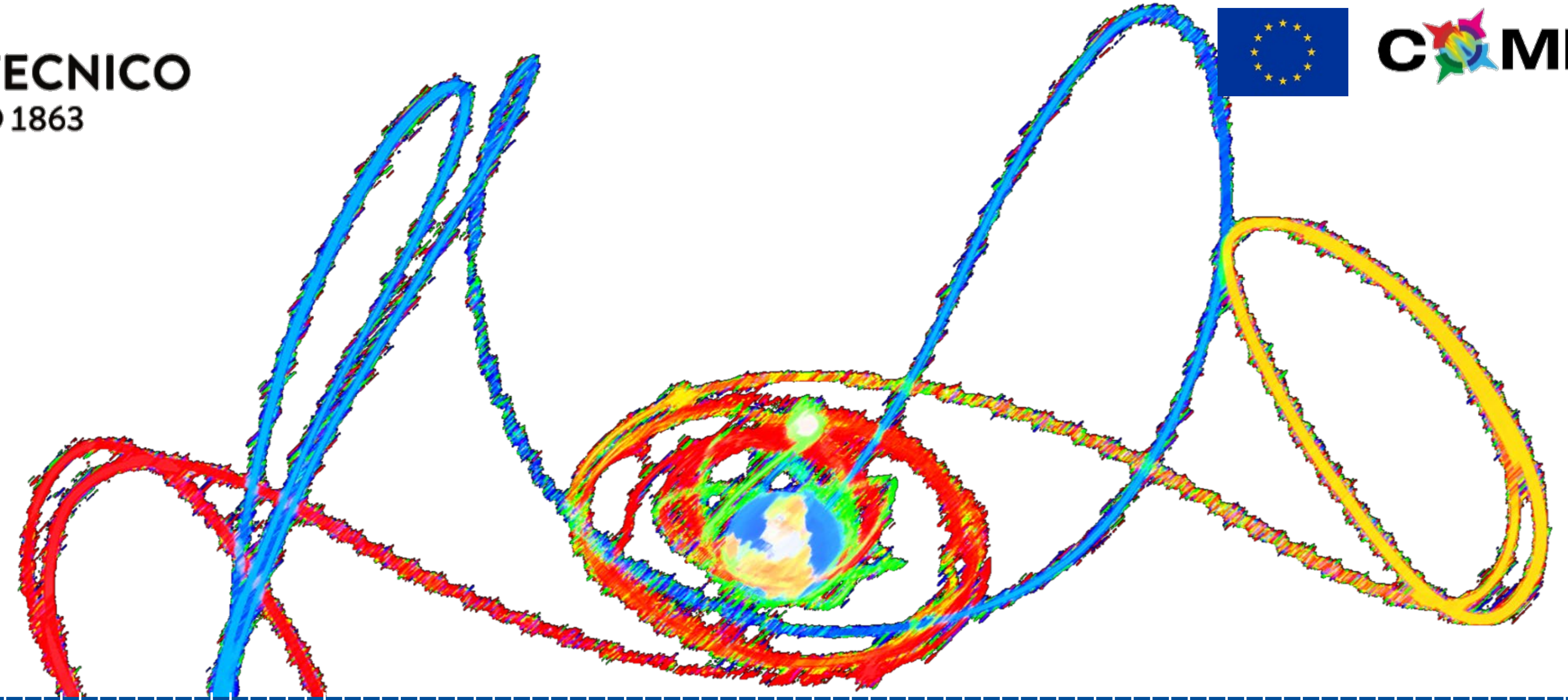


Keplerian map in the restricted N-body problem

Lorenzo Giudici, Camilla Colombo

International workshop on co-orbital motion

IMATI-CNR, Milano

Motivation

- Develop an alternative model to the classical N-body problem in cartesian coordinates, which ensures:
  1. High **accuracy** under known conditions
  2. A well-defined and **broad field of application**
  3. Competitiveness in terms of computational **efficiency**
  4. An easier and deeper **dynamical interpretation**

Keplerian map theory

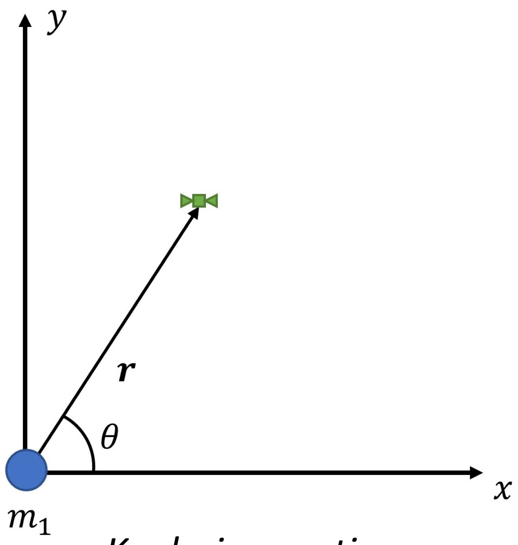
Features

- **Model:** Lagrange planetary equations-based perturbation approach applied to the three-body potential
- **Formulation:**
  1. Trajectory described through Keplerian elements relative to the centre-of-mass
  2. The reference Keplerian motion is fictitious and assumes the primary body in the centre-of-mass
  3. Disturbing function from the differential gravitational potential between the reference Keplerian motion and the three-body system

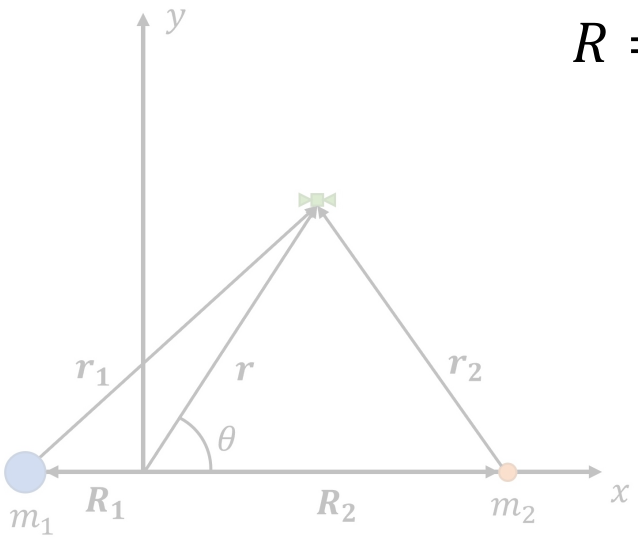
Keplerian map theory

Disturbing function

- Disturbing function from the differential gravitational potential between the reference Keplerian motion and the three-body system



Keplerian motion



Perturbed motion (three-body system)

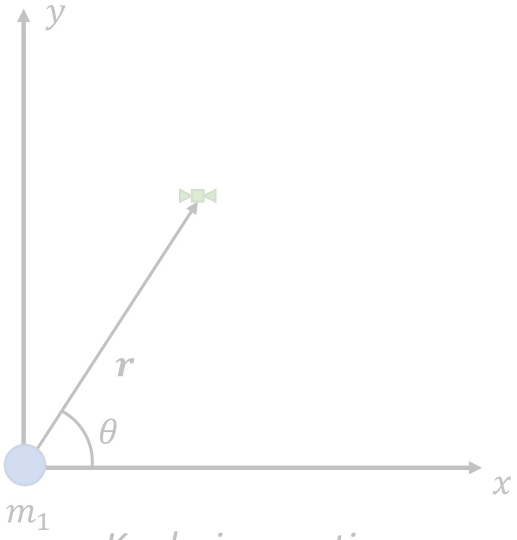
$$R = -\delta U = -\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) + \underbrace{\left(-\frac{1-\mu}{r}\right)}_{\text{Fictitious two-body potential}}$$

Fictitious two-body potential

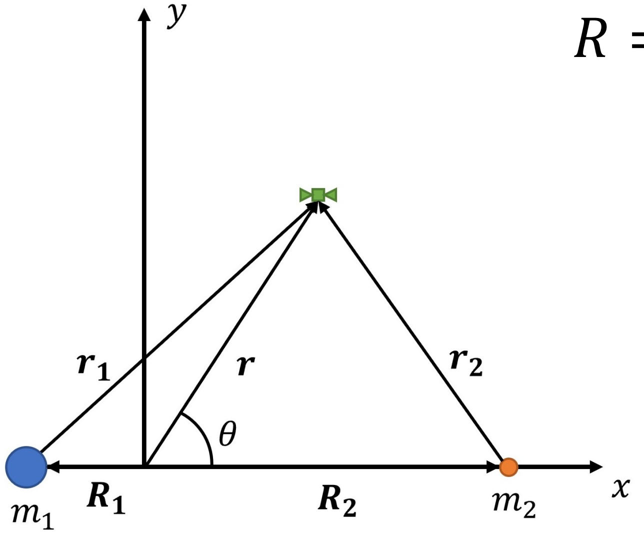
Keplerian map theory

Disturbing function

- Disturbing function from the differential gravitational potential between the reference Keplerian motion and the three-body system



Keplerian motion



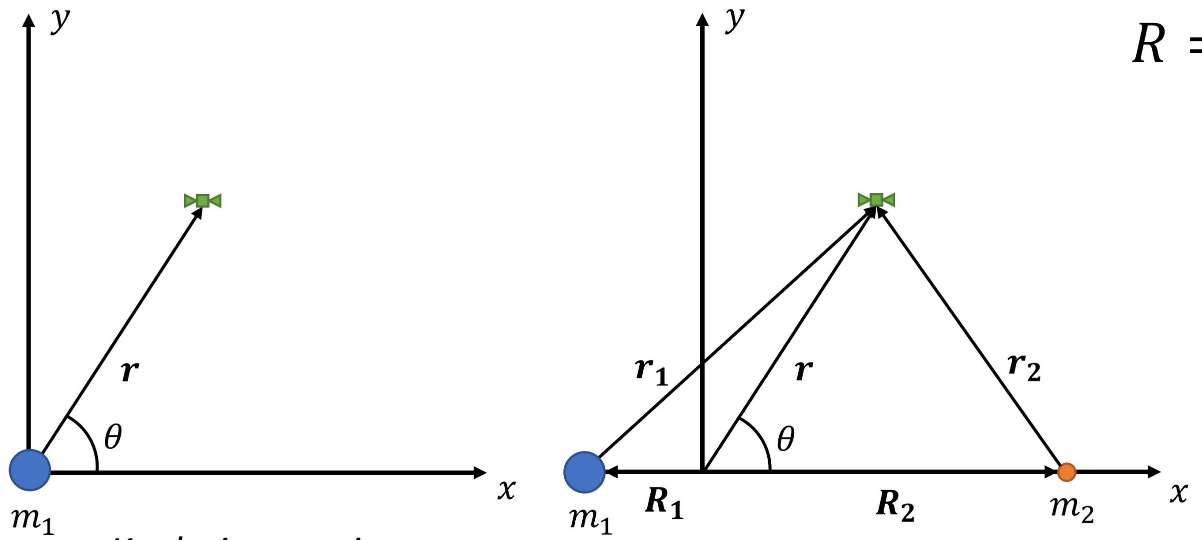
Perturbed motion (three-body system)

$$R = -\delta U = -\underbrace{\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right)}_{\textit{Three-body potential}} + \left(-\frac{1-\mu}{r}\right)$$

Keplerian map theory

Disturbing function

- Disturbing function from the differential gravitational potential between the reference Keplerian motion and the three-body system



$$R = -\delta U = -\left(-\frac{1-\mu}{r_1} - \frac{\mu}{r_2}\right) + \left(-\frac{1-\mu}{r}\right)$$

Dependencies:

1. $r_1 = r_1(r, R_2, \theta),\ r_2 = r_2(r, R_2, \theta)$
2. $r = r(a, e, \nu),\ R_2 = R_2(a_{3B}, e_{3B}, \nu_{3B}),$

   $\theta = \theta(i, \Omega, \omega, \nu, i_{3B}, \Omega_{3B}, \omega_{3B}, \nu_{3B})$

a = semi-major axis, e = eccentricity, i = inclination, Ω = right ascension of ascending nodes, ω = argument of periapsis, ν = true anomaly

Keplerian map theory

Differentiation of the disturbing function

- The disturbing function must depend on six constant orbital elements and time (according to the Lagrangian brackets derivation)

Initial disturbing function:

$$R = f(a, e, i, \Omega, \omega, \nu)$$

Keplerian map theory

Differentiation of the disturbing function

- The disturbing function must depend on six constant orbital elements and time (according to the Lagrangian brackets derivation)

Initial disturbing function:

$$R = f(a, e, i, \Omega, \omega, \nu)$$

Relation true-anomaly – time:

$$\nu = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right)$$

$$E - e\sin E = \sqrt{\frac{1-\mu}{a^3}}t + M_0$$

Keplerian map theory

Differentiation of the disturbing function

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Final disturbing function:

$$R = f\left(a, e, i, \Omega, \omega, \nu\big(e, E(a, e, M_0, t)\big)\right)$$

Keplerian map theory

Differentiation of the disturbing function

- The disturbing function must depend on six constant orbital elements and time (according to the Lagrangian brackets derivation)

Initial disturbing function:

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Final disturbing function:

$$R = f\left(a, e, i, \Omega, \omega, \nu\left(e, E(a, e, M_0, t)\right)\right)$$

- Derivatives of the disturbing function for LPE:

$$\frac{dR}{da} = \frac{\partial R}{\partial a} + \frac{\partial R}{\partial \nu}\left(\frac{\partial \nu}{\partial E}\frac{\partial E}{\partial a}\right)$$

$$\frac{dR}{de} = \frac{\partial R}{\partial e} + \frac{\partial R}{\partial \nu}\left(\frac{\partial \nu}{\partial e} + \frac{\partial \nu}{\partial E}\frac{\partial E}{\partial e}\right)$$

$$\frac{dR}{d(i, \Omega, \omega)} = \frac{\partial R}{\partial(i, \Omega, \omega)}$$

$$\frac{dR}{dM_0} = \frac{\partial R}{\partial \nu}\left(\frac{\partial \nu}{\partial E}\frac{\partial E}{\partial M_0}\right)$$

Accuracy and efficiency analysis

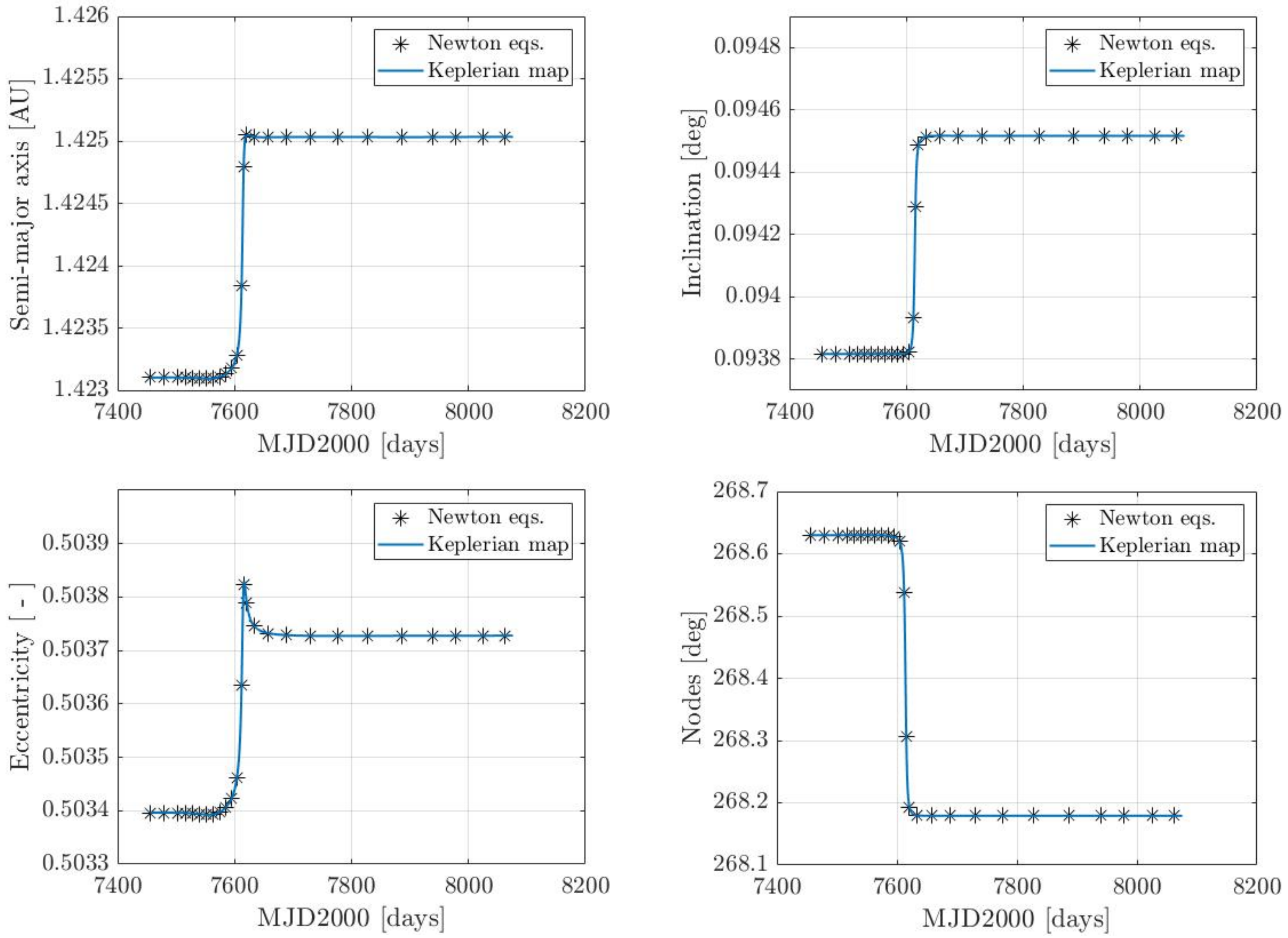
Validation approach

- **Two validation scenarios:**
  1. Propagation of the orbit of a Near-Earth asteroid; initial Keplerian elements are tuned to test the accuracy of the model in a wide range of conditions
  2. Propagation of the orbit of the JUICE spacecraft under the simultaneous attraction of Jupiter and the Galilean moons

- **Objective:**
  1. Find the boundaries of applicability of the Keplerian map theory (in particular, the dependency on the relative distance particle – third body)
  2. Have a hint on the computational efficiency of the model
  3. Verify the applicability to many-body systems

Accuracy and efficiency analysis

Near-Earth asteroid orbit propagation



Nominal Keplerian elements propagation, one orbital period

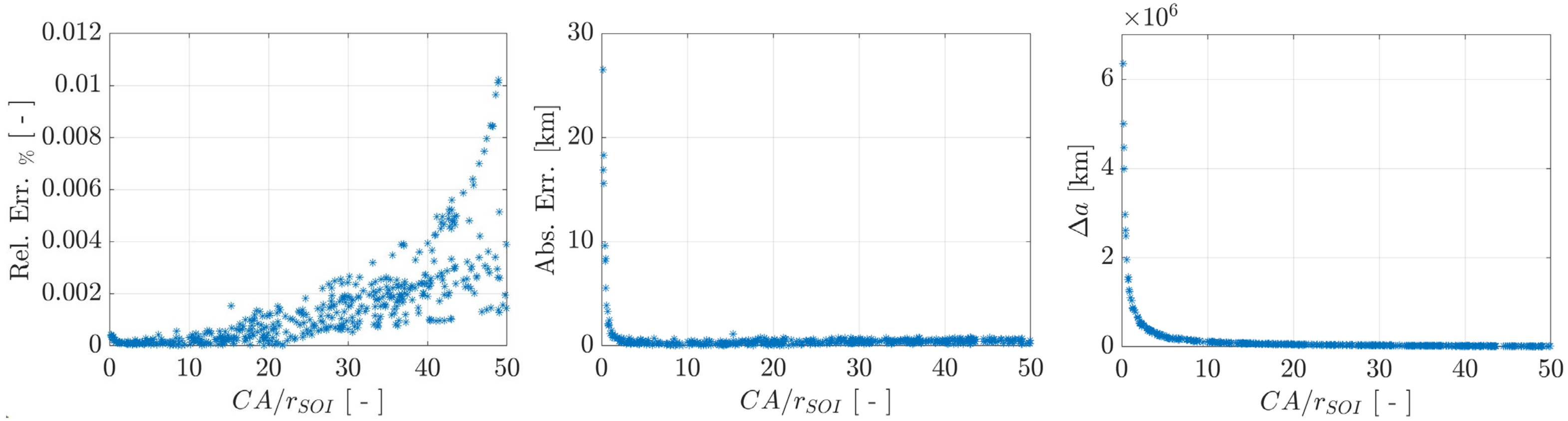
Accuracy and efficiency analysis

Near-Earth asteroid orbit propagation

- Keplerian elements extracted from probability distributions
- Quality index:

$$\mathrm{Err.} = |\Delta a_{\mathrm{KM}} - \Delta a_{\mathrm{Newton}}|$$

1000 sampled Keplerian elements propagations, one orbital period – semi-major axis kick and Keplerian map error



✓ Accurate at any distance from the third body

✓ Computational time reduced by 17%

Method	Computational time* [s]	Average n° time steps
Newton eqs.	38.5	479
Keplerian map	33.5	170

*Processor: Intel(R) Core(TM) i7-10700 CPU @ 2.90 GHz, Matlab® R2020b ode45

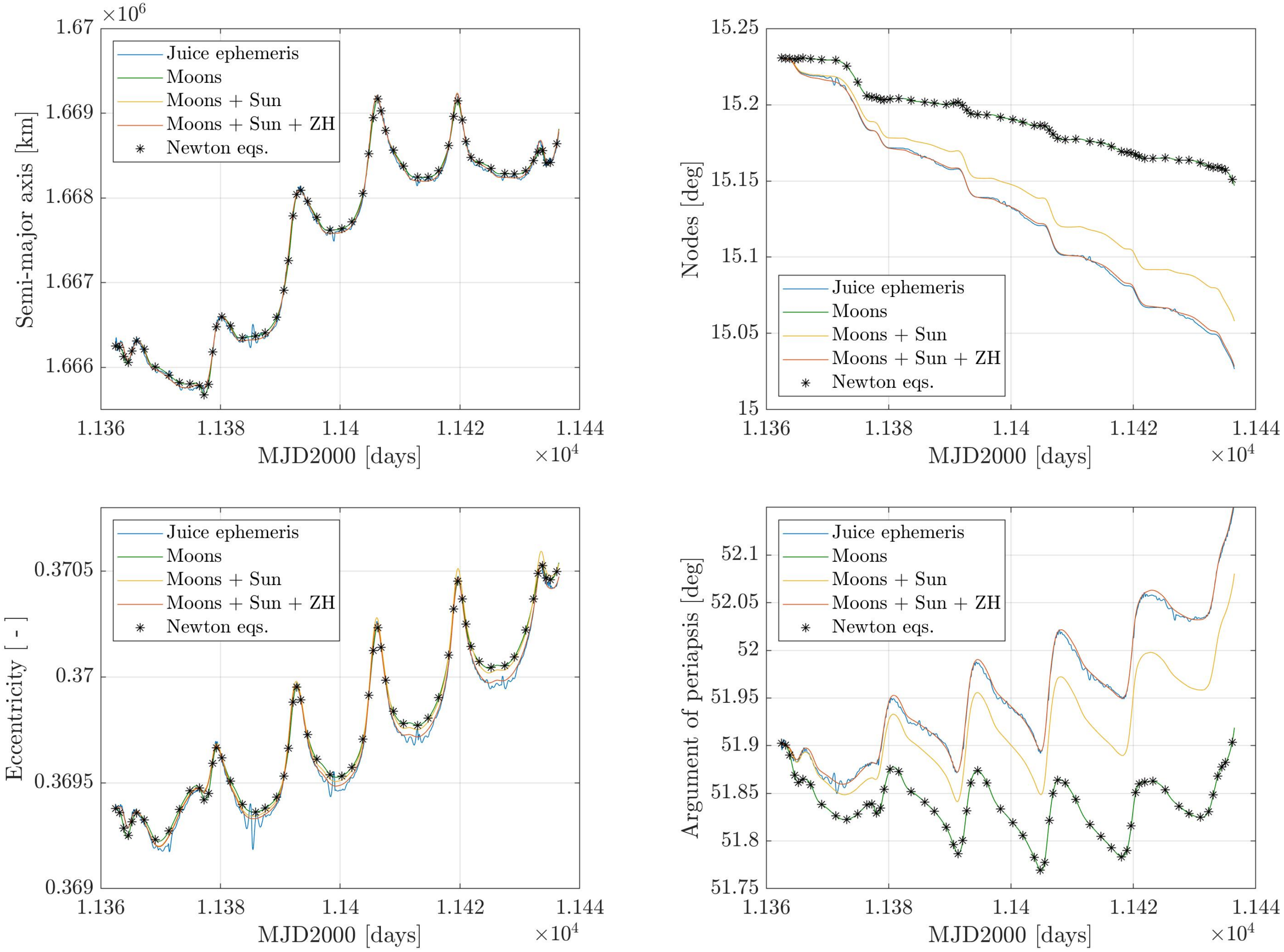
Accuracy and efficiency analysis

JUICE spacecraft orbit propagation

- Mission phase without flybys with the Galilean moons
- Sun gravitational disturbance and Jupiter zonal harmonics perturbation included

✓ Accurate in the many-body scenario

JUICE spacecraft Keplerian elements propagation, 12/02/2031 – 25/04/2031



Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Mission baseline:

1. Departure from Halo orbit around L_2
2. Injection on an unstable manifold
3. Low-thrust trajectory to target the asteroid

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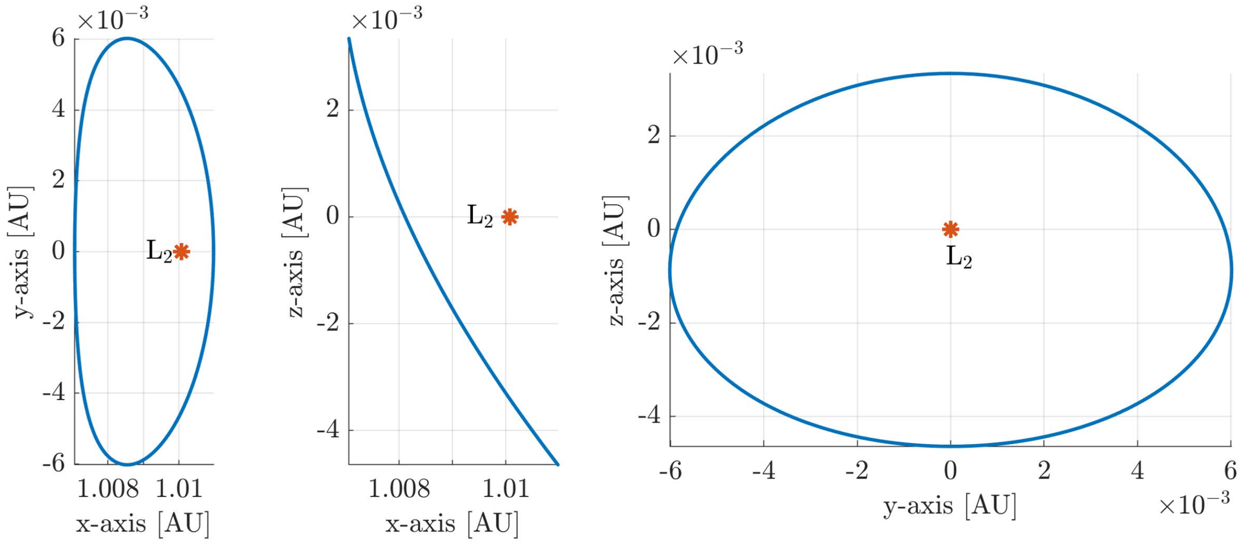

Departure Halo orbit: $500\,000 \times 1\,500\,000 \times 800\,000\ km$

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

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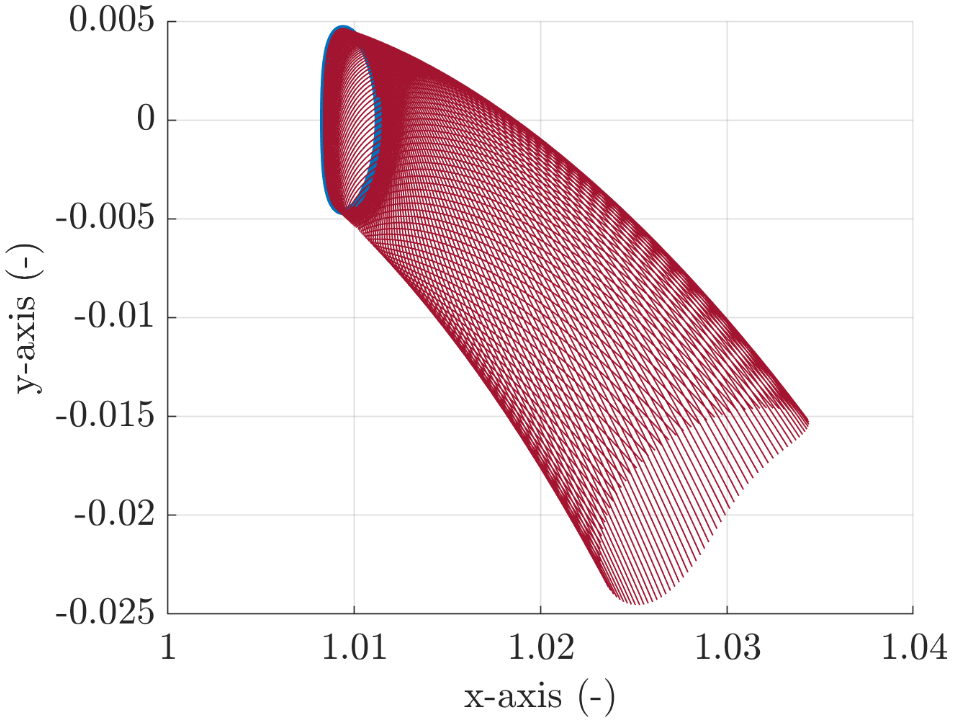

Unstable manifold of the departure Halo orbit around the Sun-Earth L_2 point

Keplerian map-based trajectory design

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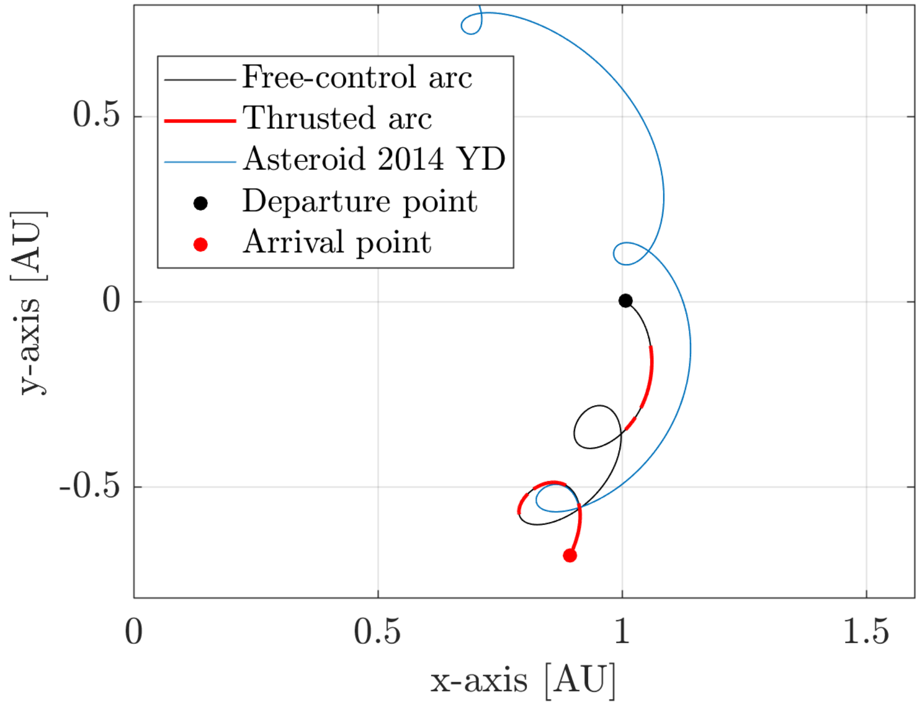

Low-thrust trajectory in the synodic reference frame

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Mission baseline:

1. Departure from Halo orbit around L_2
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Mission constraints (M-Argo mission):

CubeSat dry mass	22.6 kg
Maximum fuel mass	2.8 kg
Specific impulse	3022 s
Maximum thrust	1.89 mN
Earliest departure date	01/01/2023
Latest departure date	01/01/2025
Maximum TOF	3 years

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Two-steps optimisation strategy:

	Step 1	Step 2
Optimisation method	Indirect	Direct
Assumptions	Fixed propellant mass	Fixed departure and arrival dates
Dynamical model	CR3BP	R3BP (Keplerian map)
Mission profile	1. Free-control unstable manifold phase 2. Thrusted arc to the asteroid	Bang-bang – like thrusted arc
Objective	Minimise thrusting period	Minimise fuel mass

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Step 1 – Indirect optimisation:

Optimal control problem: $\min_{\boldsymbol{u} \in \mathcal{U}} J = \int_0^{t_f} 1 \, \mathrm{d}t$ subject to:

$$
\begin{cases}
\begin{Bmatrix} \dot{\boldsymbol{r}} \\ \dot{\boldsymbol{v}} \\ \dot{m} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{v} \\ \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) + \frac{F_T}{m}\boldsymbol{d} \\ -\frac{F_T^2}{2P} \end{Bmatrix} \\
\theta(t_0, \boldsymbol{x}_0) = 0 \\
\psi\left(t_f, \boldsymbol{x}(t_f)\right) = 0
\end{cases}
$$

[1] Senent J. and Ocampo C., "Low-Thrust Variable-Specific-Impulse Transfers and Guidance To Unstable Periodic Orbits," *JGCD*, 2005

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

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- Hamiltonian: $H = \boldsymbol{\lambda_r}^T \boldsymbol{v} + \boldsymbol{\lambda_v}^T \left(\boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) + \frac{T}{m}\boldsymbol{d}\right) + \lambda_m \left(-\frac{T^2}{2P}\right) + 1$

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Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

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Hamiltonian:

$$H = \boldsymbol{\lambda_r}^T v + \boldsymbol{\lambda_v}^T \left(\boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{h}(\boldsymbol{v}) + \frac{T}{m}\boldsymbol{d}\right) + \lambda_m \left(-\frac{T^2}{2P}\right) + 1$$

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \lambda} \\ \dot{\lambda} = -\dfrac{\partial H}{\partial x} \\ \dfrac{\partial H}{\partial u} = 0 \\ \dfrac{\partial H}{\partial t} = 0 \rightarrow H = 1 \end{cases}$$

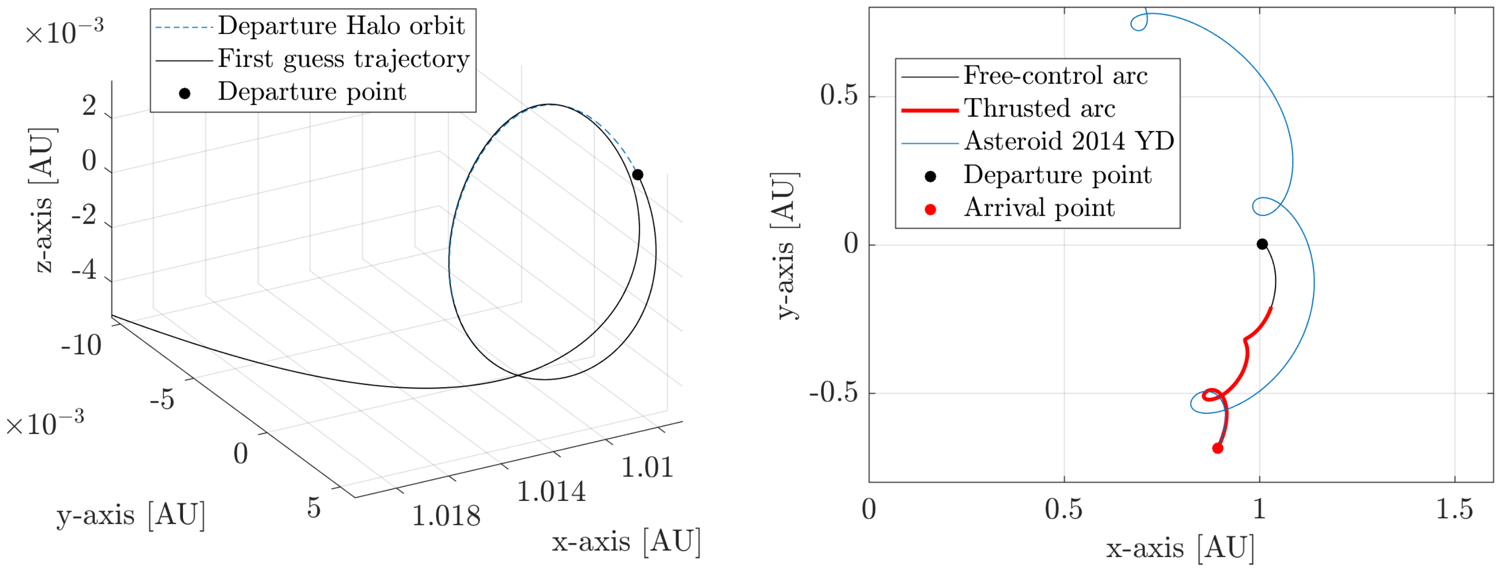
$+$ tranversality condition → **Two-points boundary value problem**

[1] Senent J. and Ocampo C., "Low-Thrust Variable-Specific-Impulse Transfers and Guidance To Unstable Periodic Orbits," *JGCD*, 2005

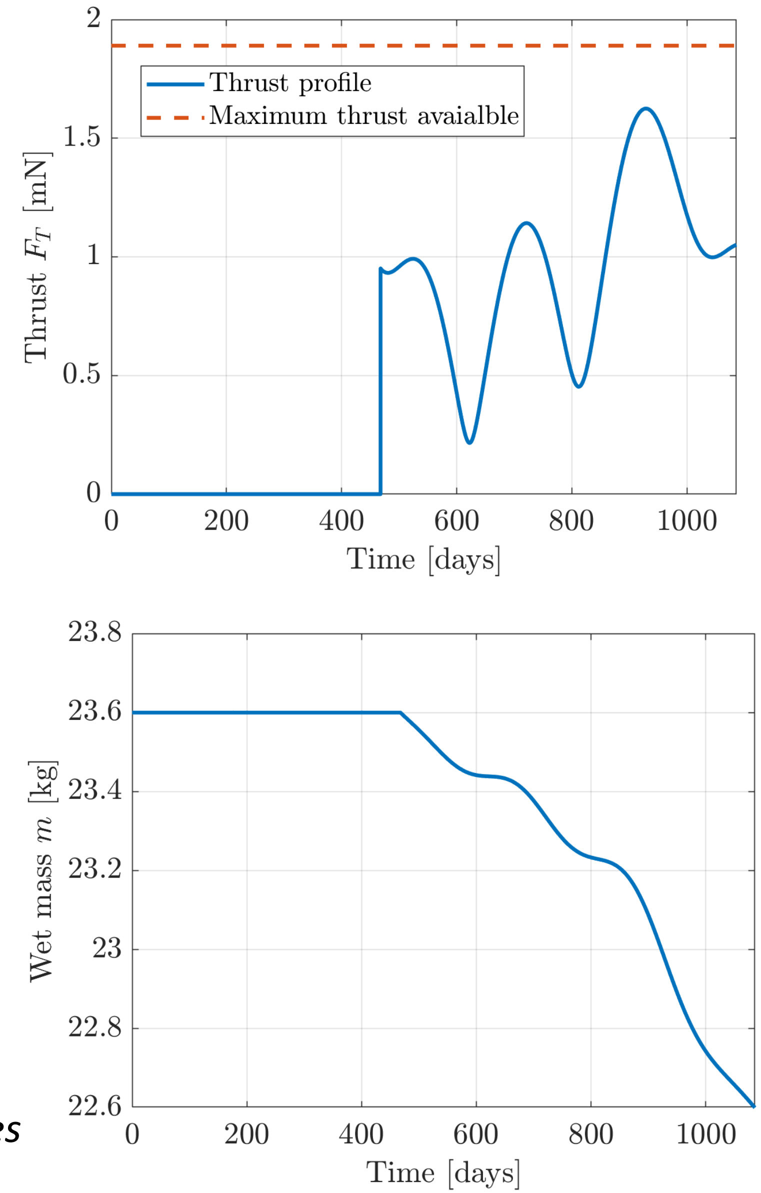
Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Step 1 – Indirect optimisation:



First guess trajectory – synodic reference frame



Thrust and mass profiles

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

Step 2 – Direct optimisation:

Optimal control problem: $\min_{\boldsymbol{u}\in\boldsymbol{\mathcal{U}}} J = \int_0^{t_f} \frac{\mathrm{d}m}{\mathrm{d}t}\,\mathrm{d}t$ subject to:

$$\begin{cases} \begin{Bmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{m} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F}(t,\boldsymbol{\alpha},\boldsymbol{u}) \\ -\frac{F_T}{I_s g_0} \end{Bmatrix} \\ \boldsymbol{\alpha}(t_0) = \boldsymbol{\alpha}_0 \\ \boldsymbol{\alpha}(t_f) = \boldsymbol{\alpha}_f \\ m(t_f) = m_{dry} \end{cases}$$

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

LPE (Keplerian map) + GPE (Thrust)

Step 2 – Direct optimisation:

Optimal control problem: $\min_{\boldsymbol{u} \in \boldsymbol{\mathcal{U}}} J = \int_0^{t_f} \frac{\mathrm{d}m}{\mathrm{d}t}\,\mathrm{d}t$ subject to:

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Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

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Discretisation: $\boldsymbol{x}(t),\ \boldsymbol{u}(t),\ m(t)\ \ t \in [0, t_f] \ \rightarrow\ \boldsymbol{\alpha}(t),\ \boldsymbol{u}(t),\ m(t)\ \ t \in [0, t_f] \ \rightarrow\ \boldsymbol{\alpha}_k, \boldsymbol{u}_k, m_k\ \ k = 1, \dots, N_p$

Keplerian map-based trajectory design

Low-thrust trajectory to near-Earth asteroid 2014 YD

LPE (Keplerian map) + GPE (Thrust)

Step 2 – Direct optimisation:

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$$\boldsymbol{y}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3, \quad t\in[0,h]$$

$$\downarrow$$

Hermite-Simpson collocation: $\Delta = \dot{\boldsymbol{y}}_c - f(t_c,\boldsymbol{y}_c,\boldsymbol{u}_c) = 0$, with:

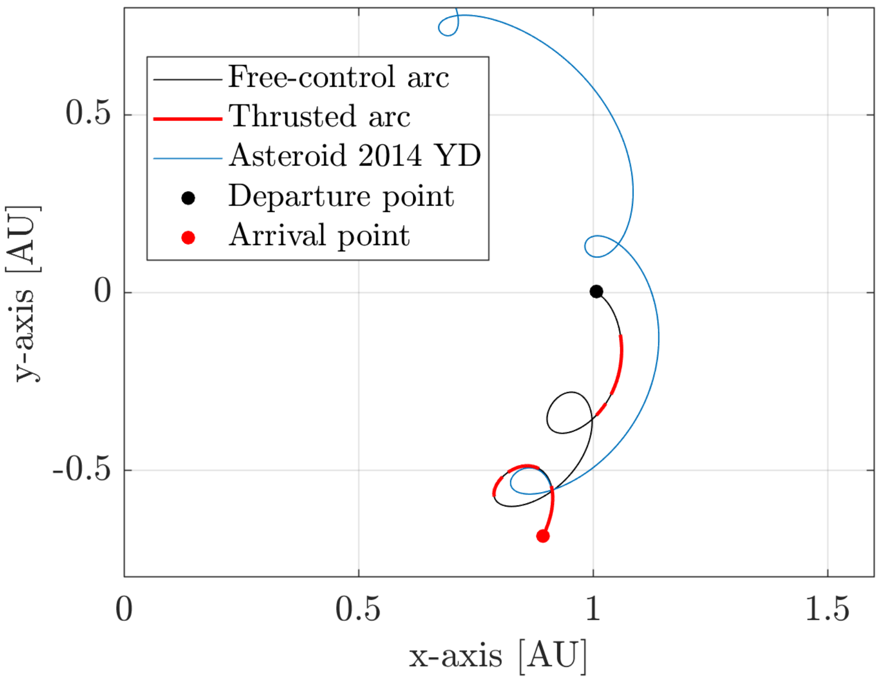
$$\boldsymbol{y}_c = \boldsymbol{y}\left(\frac{h}{2}\right),\quad \dot{\boldsymbol{y}}_c = \dot{\boldsymbol{y}}\left(\frac{h}{2}\right)$$

$$\boldsymbol{u}_c = \frac{\boldsymbol{u}_k+\boldsymbol{u}_{k+1}}{2},\quad t_c = \frac{t_k+t_{k+1}}{2}$$

Keplerian map-based trajectory design

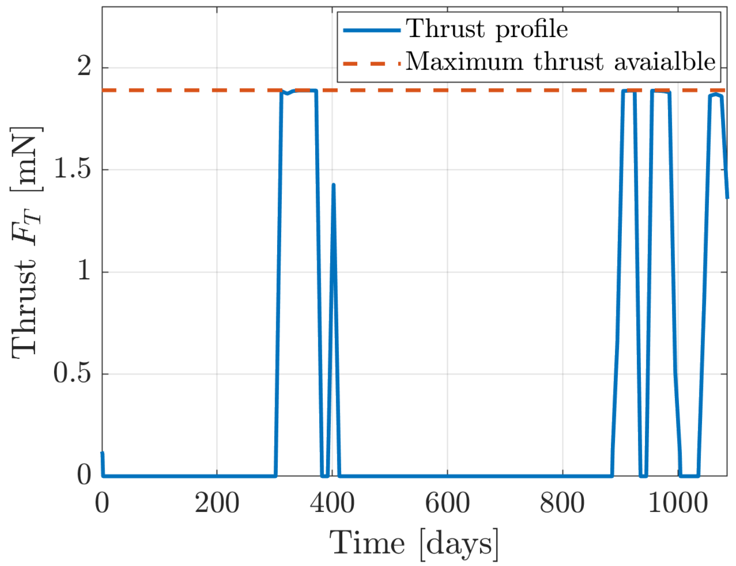
Low-thrust trajectory to near-Earth asteroid 2014 YD

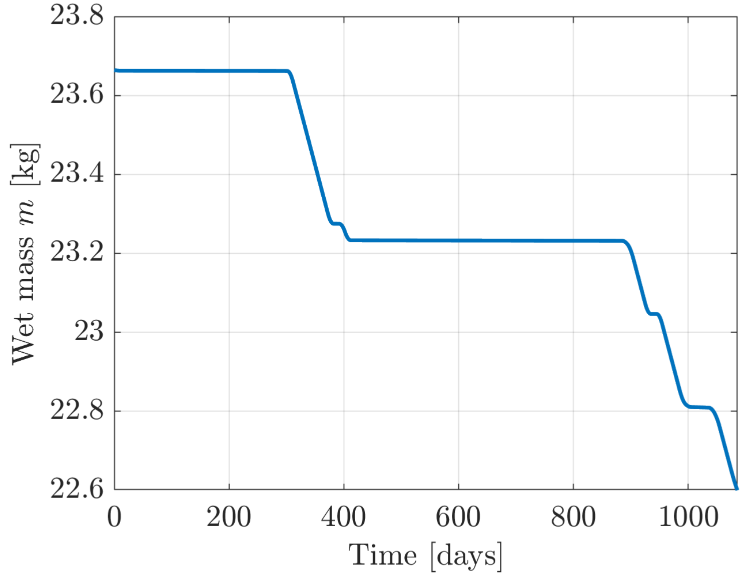
Step 2 – Direct optimisation:



Optimal trajectory – synodic reference frame

Thrust and mass profiles





Departure date	02/09/2023
Arrival date	23/08/2026
TOF	1086 days
Thrusting period	271.4 days
Fuel mass	1.064 kg

Conclusions and future works

Conclusions

- The Keplerian map theory proved **high accuracy**:
    1. Independently of the relative position between particle and third body
    2. When the particle moves inside the third body sphere of influence
    3. When several celestial bodies exert a simultaneous and comparable force on the particle
- The Keplerian map demonstrated to be **competitive** from a **computational point of view**
- The Keplerian map proved **efficacy in trajectory design and optimisation**

Future works

- Application of the Keplerian map theory to the Earth-Moon system, which is characterised by a high mass ratio

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Keplerian map in the restricted N-body problem

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