MULTIPHASE FREE DISCONTINUITY PROBLEMS: MONOTONICITY FORMULA AND REGULARITY RESULTS

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Abstract. The purpose of this paper is to analyze regularity properties of local solutions to free discontinuity problems characterized by the presence of multiple phases. The key feature of the problem is related to the way in which two neighboring phases interact: the contact is penalized at jump points, while no cost is assigned to no-jump interfaces which may occur at the zero level of the corresponding state functions. Our main results state that the phases are open and the jump set (globally considered for all the phases) is essentially closed and Ahlfors regular. The proof relies on a multiphase monotonicity formula and on a sharp collective Sobolev extension result for functions with disjoint supports on a sphere, which may be of independent interest.

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1. Introduction and statement of the main result

Free discontinuity problems characterized by the presence of multiple phases arise naturally in different contexts, such as image reconstruction or models in thermo-elasticity.

The aim of the paper is to study a class of these *multiphase free discontinuity problems* which, in view of the interaction of the different phases, exhibit also some features similar to those of *free boundary problems*. In particular we will focus on regularity properties of local solutions.

Before entering into the description of the results, let us briefly address two model problems. It is not our purpose to solve or describe these problems in full details, but just to serve them as meaningful motivations to attack the study of local minimizers to multiphase free discontinuity problems. Further examples can be drawn from the analysis of different phenomena in applied sciences, such as for instance quasistatic crack evolution (Francfort-Marigo [20]) or columnar jointing

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of cooling lava (Jungen [23]), clusters of Cheeger sets (Caroccia [10]) or spectral partition problems related to the Robin Laplacian [3].

Multiphase Mumford-Shah problem. As a first prototype example, one can consider the celebrated segmentation problem by Mumford and Shah (see [21, 18] for a review). Actually, the original formulation of the problem presented in [25] was of multiphase type, the primary underlying idea being that of decomposing the domain of an input image into different regions, and then reconstructing a new image allowed to have jumps across the boundaries of such regions. The ensuing developments via different approaches (by De Giorgi-Carriero-Leaci [17] or Dal Maso-Morel-Solimini [15]) led genuinely to a single phase problem, in which the boundaries of the different regions are seen as the jump set of a single state function. By this way the notion of phase is fatally lost, because the jump set of the state function has no a priory reason to decompose the original image into disconnected regions. Nevertheless, the necessity to extract distinct objects (possibly exhibiting themselves inner jumps) is highly present in image reconstruction, as testified by the abundant literature devoted to the attempt of forcing phase separation in the original spirit by Mumford and Shah. With no attempt of completeness, see [11, 26, 12, 22, 24, 27, 28] and references therein. It is however an evidence that these models refer essentially to the simplistic case of piecewise constant functions, and are mainly focused on the numerical side. Thus, a satisfactory mathematical study of the Mumford-Shah functional in a multiphase context seemed to be missing. To fill this gap, in our recent paper [4] we introduced a *relaxed multiphase Mumford-Shah problem*; given an open bounded set $\Omega \subseteq \mathbb{R}^d$, positive coefficients α_i, β_i, and a function f on Ω with values into $[0,1]$, it reads as follows:

$$\inf\left\{\overline{MMS}(\omega, \mathbf{U}) \; : \; (\omega, \mathbf{U}) \in \mathcal{A}_k(\Omega) \times \mathcal{F}(\omega)\right\}, \tag{1.1}$$

where

$$\mathcal{A}_k(\Omega) := \Big\{\omega := (\Omega_1, \ldots, \Omega_k) \; : \; (\Omega_1, \ldots, \Omega_k) \text{ is a Caccioppoli partition of } \Omega \Big\}$$

$$\mathcal{F}(\omega) = \Big\{\mathbf{U} := (u_1, \ldots, u_k) \in (SBV(\mathbb{R}^d))^k \; : \; u_i = 0 \text{ a.e. on } \Omega_i^c \text{ for every } i = 1, \ldots, k \Big\},$$

with

$$\overline{MMS}(\omega, \mathbf{U}) := \sum_{i=1}^k \left(\int_{\Omega_i} \alpha_i |\nabla u_i|^2 + \beta_i \mathcal{H}^{d-1}\left((\partial^e \Omega_i \cup J_{u_i}) \cap \Omega\right)\right) + \sum_{i=1}^k \mathcal{E}_i(\Omega_i, u_i).$$

Here $\partial^e\Omega_i$ and J_{u_i} denote respectively the essential boundary of Ω_i and the jump set of u_i, while the energies $\mathcal{E}_i$ are fidelity terms to the image f (see [4]). $SBV(\mathbb{R}^d)$ stands for the space of special functions of bounded variation in $\mathbb{R}^d$.

We point out that, in problem (1.1), the full perimeter of each phase is penalised in the energy. In spite, the purpose of this paper is to analyse *hybrid* interactions between phases, in which the jump part of the contact between the neighbouring phases is penalised, while interfaces occuring at the zero level of the state function are not. A multiphase Mumford-Shah problem which is essentially different from (1.1) and fits the framework of this paper can be formulated as

$$\min_{\substack{u_i \in SBV(\Omega) \\ u_i \cdot u_j = 0 \text{ for } i \neq j}} \quad \sum_{i=1}^k \left[\alpha_i \int_\Omega |\nabla u_i|^2 dx + \beta_i \mathcal{H}^{d-1}(J_{u_i})\right] + \int_\Omega \Big(\sum_{i=1}^k u_i - f\Big)^2 dx. \tag{1.2}$$

Notice that, precisely because in (1.2) the contact at zero level is not penalized in terms of energy interface, the situation is locally similar to the interaction of the *free boundaries*, and such a problem is appropriate to reconstruct images having some constant background.

Multiphase thermal insulation problem. As a second example, let us address the thermal insulation of multiple obstacles, in the spirit of Caffarelli-Kriventsov [9] (see also [19] for a related problem). As above, we consider a bounded open set $\Omega \subseteq \mathbb{R}^d$, in which we now place a family of compact, pairwise disjoint subsets $G_1, \dots, G_k$ representing the obstacles. We assume that the background temperature is 0, while the temperature of the obstacles is 1. Then, if each obstacle is isolated by a conducting material with unitary cost $m > 0$ which is enveloped with an insulating thin layer of characteristic $\beta > 0$ and of unitary cost l, the multiphase insulation problem reads

$$\min \sum_{i=1}^{k} \left[\int_\Omega |\nabla u_i|^2 dx + \beta \int_{J_{u_i}} (u_i^+)^2 + (u_i^-)^2 d\mathcal{H}^{d-1} + l\mathcal{H}^{d-1}(J_{u_i}) + m|\{u_i > 0\}| \right], \tag{1.3}$$

the minimization being carried in the class

$$\Big\{ u_i \in SBV(\Omega) : u_i = 1 \text{ on } G_i,\ u_i \cdot u_j = 0 \text{ for } i \neq j \Big\}.$$

As well as in case of problem (1.2), let us point out that, in problem (1.3), the interaction between the neighbouring phases involves no penalty of the interface when the contact occurs at the zero level of the state function.

Let us now get to the heart of the matter, and drive our attention to the generic, local formulation of the problem. Since we deal with multiple phases, each one with possible free discontinuities, the natural functional framework is given by the class

$$\mathcal{U}(\Omega) := \Big\{ u = (u_1, \dots, u_k) \in (SBV(\Omega))^k \ : \ u_i \cdot u_j = 0 \quad \text{in } \Omega \text{ for } i \neq j \Big\}. \tag{1.4}$$

Here, as in the models described above, Ω is a bounded open subset of $\mathbb{R}^d$, and $SBV(\Omega)$ denotes the space of special functions of bounded variation in Ω introduced by De Giorgi-Ambrosio [16]; we refer to [1] for a detailed account of the mathematical background.

Minimizers of the previous model problems fit naturally into the following notion of multiphase local minimizers.

Definition 1.1 (**Multiphase local almost quasi-minimizers**). *We say that $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ is a multiphase local almost-quasi minimizer at the point $x \in \Omega$ (with parameters $(\Lambda, \alpha, c_\alpha)$), if there exist constants $\Lambda \geq 1$, $\alpha > 0$, and $c_\alpha \geq 0$ such that, for every ball $B_\rho(x) \subset \Omega$ and every $(v_1, \dots, v_k) \in \mathcal{U}(\Omega)$ such that $\bigcup_i \{v_i \neq u_i\} \subseteq B_\rho(x)$, it holds*

$$\sum_{i=1}^{k} \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \leq \sum_{i=1}^{k} \left(\int_{B_\rho(x)} |\nabla v_i|^2 \, dx + \Lambda \mathcal{H}^{d-1}(J_{v_i} \cap \overline{B}_\rho(x)) \right) + c_\alpha \rho^{d-1+\alpha}.$$

The words "almost" and "quasi" in the terminology refer respectively to the presence of the coefficient $\Lambda \geq 1$ in front of the jump terms and to power decay of order higher than $(d-1)$ of the deviation from minimality. The definition above is a natural multiphase analogue of the one introduced in [5] for a *single* phase problem.

The main result of the paper is the following.

Theorem 1.2 (**Regularity of multiphase local almost-quasi minimizers**). *Let $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ be a local almost-quasi minimizer of a multiphase free discontinuity problem at every point $x \in \Omega$.*

- *The function u is Hölder continuous on $\Omega \setminus \overline{\bigcup_i J_{u_i}}$, so that each phase*

$$\Omega_i := \left\{ x \in \Omega \setminus \overline{\bigcup_j J_{u_j}} \ : \ u_i \neq 0 \right\}$$

is open in $\mathbb{R}^d$.

– *The union of the jump sets $\bigcup_i J_{u_i}$ is essentially closed and Ahlfors regular, meaning respectively that*

$$\mathcal{H}^{d-1}\left(\overline{\bigcup_i J_{u_i}} \setminus \bigcup_i J_{u_i}\right) = 0\,,$$

and that there exist $c > 0$ and $\rho_0 > 0$ such that, for every $x \in \bigcup_i J_{u_i}$ and every $B_\rho(x) \subset \Omega$ with $\rho < \rho_0$, it holds

$$c\rho^{d-1} \le \mathcal{H}^{d-1}\left(\bigcup_i J_{u_i} \cap B_\rho(x)\right) \le \frac{1}{c}\rho^{d-1}\,. \tag{1.5}$$

Before commenting on the proof of Theorem 1.2, some words are in order about "no-jump interfaces". To clarify what we mean, if we associate with an element $u = (u_1, \dots, u_k)$ in $\mathcal{U}(\Omega)$ the k phases $\Omega_i = \{u_i \neq 0\}$, the interaction between two distinct phases Ω_i, Ω_j may be of different nature, according to whether it occurs at jump points of the state functions u_i, u_j, or at their zero level. A complete analysis of local minimizers to multiphase free discontinuity problems should include also the study of the regularity of such no-jump interfaces. At this level, since the length of the boundaries of the phases are not counted in our notion of multiphase local almost-quasi minimizers, the problem becomes of free boundary type, with the distinguished feature of dealing with quasi-minimality instead of minimality. However, if we go back to the model problems (1.2) and (1.3), the regularity of the no-jump interfaces can be derived from the analysis of multiphase free boundary problems (see for instance [7], [8], [14]).

We prove Theorem 1.2 by extending to the multiphase context the techniques developed in [5] concerning the *monotonicity formula* for the Mumford-Shah functional, and the *decay estimate* of [17] for local minimizers. These tools lead quickly, as in the one phase case, to the closedness and the Ahlfors regularity of the (global) jump set: however, in view of the interactions of the phases, these extensions turn out to involve quite delicate issues.

A pivotal ingredient in the proof of the monotonicity formula of [5] is the following sharp estimate:

$$\int_{B_\rho} |\nabla \tilde{w}|^2 dx \le \frac{\rho}{d-1} \int_{\partial B_\rho} |\nabla_\tau w|^2 d\mathcal{H}^{d-1}, \tag{1.6}$$

where B_ρ is a ball of radius ρ, $w \in H^1(\partial B_\rho)$, $\tilde{w}$ is the harmonic extension of w in B_ρ, and ∇_τ stands for the tangential gradient. A multiphase version of the previous inequality is also fundamental to address the monotonicity issue for the associated multiphase free discontinuity problem. Establishing such an inequality, interesting in itself, turns out to be very delicate. In Theorem 2.1, we prove that for every $(w_1, \dots, w_k) \in (H^1(\partial B_\rho))^k$, with $w_i \cdot w_j = 0$ on ∂B_ρ, we can extend them to functions $(\tilde{w}_1, \dots, \tilde{w}_k) \in (H^1(B_\rho))^k$, with $\tilde{w}_i \cdot \tilde{w}_j = 0$ inside B_ρ, in such a way that

$$\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 dx \le \frac{\rho}{d-1} \sum_{i=1}^k \int_{\partial B_\rho} |\nabla_\tau w_i|^2 d\mathcal{H}^{d-1}. \tag{1.7}$$

The difficult part in the proof of the previous result is to show that the inequality holds true with the sharp constant $\rho/(d-1)$ for every given setting of data on ∂B_ρ. In this respect, let us stress that the radial extensions are not enough accurate to obtain the inequality, as there are equality cases in which the optimal functions $\tilde{w}_i$'s are not radial (equality occurs for $k = 1, 2$, corresponding to positive and negative parts of an affine function). Actually, the optimal extension of the boundary functions is based on the solution of a multiphase free boundary problem with homogeneous Dirichlet conditions (see Definition 2.2), which has been firstly considered in [14]. We get the result by manipulating some integral identities for the solution to this problem, which take into account in a subtle way their fine regularity properties established by Conti-Terracini-Verzini [14] and Caffarelli-Lin [8, 7].

The interaction of the various phases plays a delicate role also in adapting in our version of the decay estimate of De Giorgi, Carriero and Leaci [17], which is the content of our Theorem 4.1.

In the classical case, the decay estimate follows from the analysis of the asymptotic behavior of sequences of configurations with vanishing jump set and vanishing deviation from minimality: through a suitable truncation procedure (involving a Poincaré estimate in SBV), a limit configuration arises which turns out to be a local minimizer of the Dirichlet energy (jumps are disappearing, and the deviation vanishes), and this provides the crucial energetic information for the conclusion.

In the multiphase setting, the analysis requires to distinguish the cases when the sequence of functions mantains its multiphase character in the limit (see Proposition 4.5) from that in which one phase prevails onto the others (see Proposition 4.6). The crucial information in the first alternative is provided by the analysis of local multiphase minimizers of the Dirchlet energy (see Proposition 4.3).

The paper is organized as follows. In Section 2, we establish the sharp extension estimate (1.7) (see Theorem 2.1). Section 3 contains the monotonicity formula for multiphase local almost quasi-miminizers. The decay estimate is addressed in Section 4, while the proof of Theorem 1.2 is contained in Section 5.

Notation. Throughout the paper, for every pair of closed sets $A, B \subseteq \mathbb{R}^d$ we will denote with $dist(A, B)$ the distance between A and B. We will also write $A \subset\subset B$ if the closure $\bar{A}$ is compact and contained in B. If $E \subseteq \mathbb{R}^d$, $|E|$ will denote its Lebesgue measure, while $\mathcal{H}^\alpha(E)$ will stand for its α-dimensional Hausdorff measure. $B_\rho(x)$ will stand for the open ball of center $x \in \mathbb{R}^d$ and radius $\rho > 0$. When $x = 0$, we will write simply B_ρ: we set $\omega_d := |B_1|$.

Concerning functional spaces, if $A \subseteq \mathbb{R}^d$ is open, $H^1(A)$ will stand for the usual space Sobolev functions which are square integrable together with their weak partial derivatives, while $SBV(A)$ will denote the space of special functions of bounded variation on A. We will consider also functions in H^1 or SBV on open subsets of spheres, which are defined locally through coordinate systems.

Finally for $a, b \in \mathbb{R}$, we set $a \wedge b := \min\{a, b\}$.

2. The multiphase extension theorem

The present Section is devoted to establish the following sharp multiphase gradient extension estimate.

Theorem 2.1 (Multiphase sharp gradient estimate). *For every $w = (w_1, \dots w_k) \in (H^1(\partial B_\rho))^k$ with $w_i \cdot w_j = 0$ on ∂B_ρ, it holds*

$$\min\left\{\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2\,dx \,:\, (\tilde{w}_1, \dots, \tilde{w}_k) \in (H^1(B_\rho))^k,\ \tilde{w}_i|_{\partial B_\rho} = w_i\,,\ \tilde{w}_i \cdot \tilde{w}_j = 0 \text{ in } B_\rho\right\}$$
$$\leq \frac{\rho}{d-1} \sum_{i=1}^k \int_{\partial B_\rho} |\nabla_\tau w_i|^2\,d\mathcal{H}^{d-1}\,,$$

where ∇_τ denotes the tangential gradient.

Up to rescaling and by considering positive and negative parts, it is not restrictive in the proof of Theorem 2.1 to assume

$$\rho = 1 \qquad \text{and} \qquad w_i \geq 0 \quad \text{for } i = 1, \dots, k. \tag{2.1}$$

Notice indeed that if $w_i w_j = 0$, then $w_i^+ w_j^+ = w_i^+ w_j^- = w_i^- w_j^+ = w_i^- w_j^- = 0$.

In order to prove Theorem 2.1, we focus our attention on the following variational problem. Below, we say that w is an *admissible datum* if $w = (w_1, \dots w_k) \in (H^1(\partial B_1))^k$, with $w_i \geq 0$ and $w_i \cdot w_j = 0$ on ∂B_1.

Definition 2.2 (**Multiphase extension problem**). *For every admissible datum* $w = (w_1, \dots w_k) \in (H^1(\partial B_1))^k$, *i.e., such that* $w_i \geq 0$ *and* $w_i \cdot w_j = 0$ *on* ∂B_1, *we denote with* $\mathcal{P}(w)$ *the following minimal energy extension problem*

$$\min\Big\{ \sum_{i=1}^{k} \int_{B_1} |\nabla \tilde{w}_i|^2 \, dx \,:\, \tilde{w}_i \in H^1(B_1)\,,\, \tilde{w}_i|_{\partial B_1} = w_i\,,\, \tilde{w}_i \cdot \tilde{w}_j = 0 \ in\ B_1 \Big\}. \tag{2.2}$$

Notice that the family of extensions is not empty: indeed one can consider for example radial extensions of the form

$$\hat{w}_i(x) := |x| w_i \left(\frac{x}{|x|} \right) \qquad x \neq 0$$

which however, as noticed in the Introduction, turn out to be not optimal.

The proposition below collects some (known) properties of the solution to problem (2.2).

Proposition 2.3 (**Properties of the solution to the multiphase extension problem**). *Let* w *be an admissible datum. Then the following items hold true.*

(i) *Problem* $\mathcal{P}(w)$ *admits a unique solution.*

(ii) *The solution* $\tilde{w}$ *to problem* $\mathcal{P}(w)$ *is Lipschitz in* B_1, *and it is Lipschitz up to the boundary in case* w *is smooth.*

(iii) *For every* $i = 1, \dots, k$, *the set* $\Omega_i := \{x \in B_1 \,:\, \tilde{w}_i(x) > 0\}$ *is open and*

$$\Delta \tilde{w}_i = 0 \quad in \quad \Omega_i.$$

(iv) *Let* $x \in B_1$ *be such that* $\{i \,:\, |\Omega_i \cap B_r(x)| > 0$ *for* r *small enough*$\} = \{i_1, i_2\}$. *Then*

$$\lim_{\Omega_{i_1} \ni y \to x} \nabla \tilde{w}_{i_1}(y) = - \lim_{\Omega_{i_2} \ni y \to x} \nabla \tilde{w}_{i_2}(y)\,.$$

(v) *The family of walls* $(\cup_{i=1}^{k} \partial \Omega_i) \cap B_1$ *is the disjoint union of a finite number of analytic hypersurfaces, and a relatively closed set having Hausdorff dimension at most* $d - 2$ *and zero capacity. We shall refer to the latter as to the* singular set of $\tilde{w}$.

(vi) *If a sequence of admissible boundary data* w^n *converges strongly to some* w *in* $H^{1/2}(\partial B_1)$, *the sequence of solutions* $\tilde{w}^n$ *to problems* $\mathcal{P}(w^n)$ *converges strongly in* $H^1(B_1)$ *to the solution* $\tilde{w}$ *to problem* $\mathcal{P}(w)$.

Proof. For items (i)-(ii)-(iii)-(iv), we refer to [14], see respectively Theorem 3.1, Theorem 4.2, Theorem 5.1, Theorems 8.3 and 8.4, Remark 6.4. For item (v), we refer to [7, Section 1.6]. For item (vi), see again [14], Theorem 3.2. □

In order to establish Theorem 2.1 in the non restrictive situation (2.1), it suffices to prove that for the solution $\tilde{w}$ of problem $\mathcal{P}(w)$ we have

$$\sum_{i=1}^{k} \int_{B_1} |\nabla \tilde{w}_i|^2 \, dx \leq \frac{1}{d-1} \sum_{i=1}^{k} \int_{\partial B_1} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1}. \tag{2.3}$$

The proof of the previous inequality relies essentially on integration by parts of the full gradient term on each phase. More precisely, we start from the following observations:

(a) Thanks to point (v), each phase Ω_i has a smooth boundary inside B_1 except for a set with zero capacity, which consequently plays a negligible role in the computation. The contributions of the inner interfaces to the integration by parts turn out to cancel in view of property (iv).

(b) The sets $\partial \Omega_i \cap \partial B_1$ do not enjoy a priori regularity properties, since they depend on the given data $(w_1, \dots, w_k)$. Even if the data are smooth, the separation "lines" $\partial \Omega_i \cap \partial \Omega_j \cap \partial B_1$ between the phases on the external boundary could be highly irregular, so that removing them through capacity arguments is not feasible.

In view of the above remarks, we are going to proceed as follows:

– In a first step, we prove inequality (2.3) for the class of smooth admissible data which determine a smooth partition of ∂B_1 except for a set with zero capacity, see Subsection 2.1 (we prove indeed in this case a stronger property, *cf.* identity (2.9) below).
– Then we prove that each admissible data can be approximated by elements of this class, see Subsection 2.2.
– Finally we can obtain (2.3) in its full generality, see Subsection 2.3.

2.1. **The case of regular admissible data.** In this section we are going to focus attention on a particular class of admissible data. Namely we will consider

$$v \in C^\infty(\partial B_1)^k, \qquad v_i \geq 0 \text{ on } B_1 \tag{2.4}$$

such that, setting

$$S_i := \{x \in \partial B_1 \,:\, v_i(x) > 0\}, \tag{2.5}$$

we have

$$\partial B_1 = E \cup \bigcup_{i=1}^k S_i, \quad S_i \cap S_j = \emptyset, \quad E \text{ is a smooth submanifold of dimension } d-2. \tag{2.6}$$

The following result holds true.

Proposition 2.4. *Let v be an admissible datum satisfying* (2.4) *and* (2.6)*, and let $\tilde v = (\tilde v_1, \dots, \tilde v_k)$ be the solution to $\mathcal{P}(v)$. Set $\Omega_i := \{\tilde v_i > 0\}$, and denote by n the outer normal to ∂B_1. Then*

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} = d \sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx + \sum_{i=1}^k \int_{\Omega_i} (1-|x|^2)|D^2 \tilde v_i|^2 \, dx\,. \tag{2.7}$$

and

$$(d-2)\sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx = \sum_{i=1}^k \int_{\partial B_1} |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} - 2\sum_{i=1}^k \int_{\partial B_1} \left|\frac{\partial \tilde v_i}{\partial n}\right|^2 \, d\mathcal{H}^{d-1}, \tag{2.8}$$

As a consequence, denoting by ∇_τ the tangential gradient, we have

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau v_i|^2 \, d\mathcal{H}^{d-1} = (d-1)\sum_{i=1}^k \int_{B_1} |\nabla \tilde v_i|^2 \, dx + \frac{1}{2}\sum_{i=1}^k \int_{\Omega_i} (1-|x|^2)|D^2 \tilde v_i|^2 \, dx. \tag{2.9}$$

In particular, we get

$$\int_{\Omega_i} (1-|x|^2)|D^2 \tilde v_i|^2 \, dx < +\infty \qquad \textit{for every } i = 1, \dots, k.$$

Proof. Let us set up a geometric construction which will be exploited in the proof of identities (2.7) and (2.8). Let Σ be the singular set of $\tilde v$ according to Proposition 2.3 (v). From point (ii) of Proposition 2.3, we infer that $\overline{\Sigma} \cap \partial B_1 \subset E$: indeed if $x \in S_i$ so that $v_i(x) > 0$, then by Lipschitz continuity of $\tilde v_i$ on $\bar B_1$ we infer $\tilde v_i > 0$ in $B_r(x) \cap B_1$ for r small enough, which yields $x \notin \overline{\Sigma}$.

We deduce in particular that $\overline{\Sigma}$ has zero capacity. Consequently, we can find a sequence $(\mathcal{U}_\varepsilon, \varphi_\varepsilon)_{\varepsilon>0}$ satisfying the following conditions:

– $\mathcal{U}_\varepsilon$ is a sequence of open neighborhoods of $\overline{\Sigma}$ (in $\mathbb{R}^d$), monotone decreasing with respect to inclusions with $\bigcap_{\varepsilon>0} \mathcal{U}_\varepsilon = \overline{\Sigma}$, such that $\partial \mathcal{U}_\varepsilon$ is a smooth hypersurface and $\partial \mathcal{U}_\varepsilon \cap \partial \Omega_i$ is $\mathcal{H}^{d-1}$-negligible;
– φ_ε is a sequence of piecewise smooth functions in $\overline{B}_2$, pointwise monotone increasing, such that, for every ε:

$$\varphi_\varepsilon = 1 \text{ on } \partial B_2, \qquad \varphi_\varepsilon \equiv 0 \text{ in } \mathcal{U}_\varepsilon\,, \qquad \Delta \varphi_\varepsilon = 0 \text{ in } B_2 \setminus \overline{\mathcal{U}_\varepsilon}\,,$$

and, in the limit as $\varepsilon \to 0^+$,

$$\varphi_\varepsilon \to 1 \text{ in } H^1(B_2)\,.$$

In particular, thanks to the regularity of $\partial\Omega_i \setminus \overline{\Sigma}$ and since $\tilde{v}_i$ is harmonic on Ω_i, for every $\varepsilon > 0$ we have

$$\tilde{v}_i \in C^\infty(\overline{\Omega_i \setminus \overline{\mathcal{U}}_\varepsilon}). \tag{2.10}$$

Indeed, $\partial\Omega_i \setminus \overline{\Sigma}$ is composed of relatively open inner regular hypersurfaces (see point (v) of Proposition 2.3) on which $\tilde{v}_i$ has zero trace, and the relatively open subset S_i of ∂B_1, on which $\tilde{v}_i$ has trace given by v_i. Then by classical elliptic regularity, $\tilde{v}_i$ is smooth on $B_r(x) \cap \overline{\Omega}_i$ for every x belonging to these parts of $\partial\Omega_i$ provided that r is sufficiently small. Then (2.10) follows. In particular, we see that the integrals involving the full gradient and the normal derivative on the right hand side of (2.8) are well defined.

Proof of identity (2.7). Let i be a fixed index in $\{1, \dots, k\}$. We are going to compute, for every $\varepsilon > 0$,

$$\int_{\Omega_i} \varphi_\varepsilon (1 - |x|^2)\, |D^2 \tilde{v}_i|^2 \, dx = \int_{\Omega_i \setminus \overline{\mathcal{U}}_\varepsilon} \varphi_\varepsilon (1 - |x|^2)\, |D^2 \tilde{v}_i|^2 \, dx, \tag{2.11}$$

where $\Omega_i := \{\tilde{v}_i > 0\}$, while $\mathcal{U}_\varepsilon$ and φ_ε are defined as above.

We point out that, by construction (due to Proposition 2.3 (v) and the choice of $\mathcal{U}_\varepsilon$), we have

$$\partial\Omega_i \setminus \overline{\mathcal{U}_\varepsilon} \subset S_i \cup \Gamma_i,$$

where $S_i \subseteq \partial B_1$ is given in (2.5) while $\Gamma_i \subset B_1$ is a smooth hypersurface.

In the following computations, we adopt Einstein convention on repeated indices. Moreover, for simplicity and with an abuse of notation, until otherwise stated we set $v := \tilde{v}_i$, $S = S_i$, $\Gamma := \Gamma_i$, $\mathcal{U} := \mathcal{U}_\varepsilon$, and $\varphi := \varphi_\varepsilon$; further, we denote by n the unit outer normal vector defined (everywhere) on $\partial\Omega \setminus \overline{\mathcal{U}}$.

Notice that we can perform integration by parts on the right hand side of (2.11) thanks to (2.10) and since $\Omega \setminus \overline{\mathcal{U}}$ is piecewise regular with a singular part which is $\mathcal{H}^{d-1}$-negligible (in particular it has finite perimeter): since boundary integrals take place only on $S \cup \Gamma$ (as $\varphi = 0$ in $\overline{\mathcal{U}}$), we obtain

$$\begin{aligned}
&\int_\Omega \varphi(1 - |x|^2)\, |D^2 v|^2 \, dx = \int_{\Omega \setminus \overline{\mathcal{U}}} \varphi(1 - |x|^2)\, |D^2 v|^2 \, dx \\
= \;& \int_S \varphi(1 - x_k x_k)\, \partial_{ij} v \, \partial_j v \, n_i \, d\mathcal{H}^{d-1} + \int_\Gamma \varphi(1 - x_k x_k)\, \partial_{ij} v \, \partial_j v \, n_i \, d\mathcal{H}^{d-1} \\
& - \int_\Omega \partial_i \varphi (1 - x_k x_k)\, \partial_{ij} v \, \partial_j v \, dx + 2 \int_\Omega \varphi x_i \, \partial_{ij} v \, \partial_j v \, dx - \int_\Omega \varphi(1 - x_k x_k)\, \partial_{iij} v \, \partial_j v \, dx \,.
\end{aligned}$$

The first and last integrals in the latter sum vanish, respectively because $|x| = 1$ on $S \subseteq \partial B_1$, and because v is harmonic in Ω. Thus we have

$$\int_\Omega \varphi(1 - |x|^2)\, |D^2 v|^2 \, dx = I_A + I_B + I_C \,, \tag{2.12}$$

being

$$\begin{aligned}
I_A &:= \int_\Gamma \varphi(1 - |x|^2)\, \partial_{ij} v \, \partial_j v \, n_i \, d\mathcal{H}^{d-1} \\
I_B &:= - \int_\Omega \partial_i \varphi (1 - |x|^2)\, \partial_{ij} v \, \partial_j v \, dx \\
I_C &:= 2 \int_\Omega \varphi x_i \, \partial_{ij} v \, \partial_j v \, dx \,.
\end{aligned}$$

We now analyze separately the three integrals above.

(a) *Computation of* I_A. Recall the general formula

$$\Delta v = \Delta_\Gamma v + (d-1) \frac{\partial v}{\partial n} H_\Gamma + \frac{\partial^2 v}{\partial n^2} \qquad \text{on } \Gamma,$$

being H_Γ the scalar mean curvature. Since v is harmonic in Ω and vanishes on Γ, we get

$$\nabla v = -|\nabla v| n \qquad \text{and} \qquad (d-1)\frac{\partial v}{\partial n} H_\Gamma + \frac{\partial^2 v}{\partial n^2} = 0 \quad \text{on } \Gamma.$$

Moreover we may write

$$\partial_{ij} v \partial_j v n_i = -|\nabla v| (\partial_{ij} v) n_i n_j = -|\nabla v| \frac{\partial^2 v}{\partial n^2},$$

so that

$$I_A = -(d-1) \int_\Gamma \varphi (1-|x|^2) H_\Gamma \, |\nabla v|^2 \; d\mathcal{H}^{d-1} . \tag{2.13}$$

(b) *Computation of* I_B. We perform a further integration by parts. Exploiting the equality $|x| = 1$ on ∂B_1, and the fact that φ is harmonic in $\Omega \setminus \mathcal{U}$, we get

$$I_B = -\int_\Gamma \partial_i \varphi \, n_i (1-|x|^2) \partial_j v \partial_j v \, d\mathcal{H}^{d-1} - 2 \int_\Omega \partial_i \varphi \, x_i \partial_j v \partial_j v \, dx - I_B \, ,$$

so that

$$I_B = -\frac{1}{2} \int_\Gamma \frac{\partial \varphi}{\partial n} (1-|x|^2) |\nabla v|^2 \, d\mathcal{H}^{d-1} - \int_\Omega (\nabla \varphi \cdot x) |\nabla v|^2 \, dx \, . \tag{2.14}$$

(c) *Computation of* I_C. Also in this case, we perform a further integration by parts. Exploiting the equality $x = n$ on ∂B_1, and the identity $\operatorname{div}(x) = d$, we get

$$I_C = 2 \int_S \varphi \partial_j v \partial_j v n_i n_i \, d\mathcal{H}^{d-1} + 2 \int_\Gamma \varphi \partial_j v \partial_j v x_i n_i \, d\mathcal{H}^{d-1} - 2 \int_\Omega \partial_i \varphi \, x_i |\nabla v|^2 \, dx \\ - 2d \int_\Omega \varphi |\partial_j v|^2 \, dx - I_C$$

so that

$$I_C = \int_S \varphi |\nabla v|^2 \, d\mathcal{H}^{d-1} + \int_\Gamma \varphi |\nabla v|^2 (x \cdot n) \, d\mathcal{H}^{d-1} \\ - \int_\Omega (\nabla \varphi \cdot x) |\nabla v|^2 \, dx - d \int_\Omega \varphi |\nabla v|^2 \, dx \, . \tag{2.15}$$

Now, we sum up the equalities (2.12) over all the phases, that we resume to denote by $\tilde v_i$, for $i = 1, \dots, k$. Notice carefully that, in doing so, we have cancellations of all the integrals over the hypersurfaces Γ_i coming from the expressions of I_A, I_B and I_C as computed respectively in (2.13), (2.14), and (2.15). Namely

$$(d-1) \sum_{i=1}^k \int_{\Gamma_i} \varphi (1-|x|^2) H_{\Gamma_i} \, |\nabla \tilde v_i|^2 \; d\mathcal{H}^{d-1} = \sum_{i=1}^k \int_{\Gamma_i} \frac{\partial \varphi}{\partial n} (1-|x|^2) |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} \\ = \sum_{i=1}^k \int_{\Gamma_i} \varphi |\nabla \tilde v_i|^2 (x \cdot n) \, d\mathcal{H}^{d-1} = 0 \, .$$

Indeed, for two adjacent phases, all the integrands above have the same modulus (thanks to Proposition 2.3 (iv)), and opposite sign (due to the sign change respectively of the terms H_{Γ_i}, $\frac{\partial \varphi}{\partial n}$, and $(x \cdot n)$ when passing from a phase to an adjacent one).

So far, we have obtained

$$\sum_{i=1}^k \int_{\Omega_i} \varphi (1-|x|^2) |D^2 \tilde v_i|^2 \, dx = \sum_{i=1}^k \int_{\partial B_1} \varphi |\nabla \tilde v_i|^2 \, d\mathcal{H}^{d-1} - d \sum_{i=1}^k \int_{B_1} \varphi |\nabla \tilde v_i|^2 \, dx \\ -2 \sum_{i=1}^k \int_{\Omega_i} (\nabla \varphi \cdot x) |\nabla \tilde v_i|^2 \, dx \, .$$

Finally, we recall that $\varphi = \varphi_\varepsilon$, and obtain the lemma by passing to the limit as $\varepsilon \to 0$ in the identity above. Indeed, since the sequence $\{\varphi_\varepsilon\}$ converges increasingly to 1 as $\varepsilon \to 0^+$, by monotone convergence we can pass to the limit in the first three integrals above. On the other hand, since $\nabla\varphi_\varepsilon \to 0$ strongly in $L^2(B_1)$ and $|\nabla\tilde{v}_i| \le M$ (thanks to the smoothness of the boundary data v and to Proposition 2.3 (iii)), we can pass to the limit also in the fourth integral and see that it is infinitesimal as $\varepsilon \to 0$.

Proof of identity (2.8). In the case of a single harmonic function in the ball, the proof of the lemma is standard and can be found, for instance, in [13, Appendix A]. In the multiphase context, the proof is still based on the same key, namely the minimality of the total Dirichlet energy, but contains some additional technical difficulties, due to the presence of the interfaces and their singularities.

For $t \in (-\tau, \tau)$, consider a one-parameter family of bi-Lipschitz homeomorphisms of B_1 into itself of the form

$$\Phi_t(x) = x + t\Psi_{\varepsilon,\delta}(x)\,, \qquad x \in B_1,$$

where

$$\Psi_{\varepsilon,\delta}(x) = \varphi_\varepsilon(x)\gamma_\delta(|x|)\,x\,, \qquad x \in B_1.$$

Above φ_ε is defined as done just above Lemma 2.4, while $\gamma_\delta : \mathbb{R} \to \mathbb{R}$ is an even function defined, for $\delta \in (0, 1/2)$, by

$$\gamma_\delta(t) := \begin{cases} 1 & \text{if } 0 \le t \le 1-\delta \\ \frac{1}{\delta}(1-t) & \text{if } 1-\delta \le t \le 1 \\ 0 & \text{if } t \ge 1\,. \end{cases}$$

Throughout the proof, for simplicity and by abuse of notation we simply write v_i in place of $\tilde{v}_i$. Set

$$v_i^t(x) := v_i(\Phi_t(x))\,, \qquad x \in B_1\,.$$

By the minimality of $v = (v_1, \dots, v_k)$, the map $t \mapsto E(t) := \sum_{i=1}^k \int_{B_1} |\nabla v_i^t|^2\,dx$ has a critical point at $t = 0$. In order to compute $E'(0)$, we can argue in the same way as in the classical case of harmonic functions (see [13]). Indeed, the functions v_i are smooth on the support of $\Psi_{\varepsilon,\delta}$ (by the presence of the cut-off function φ_ε in the definition of $\Psi_{\varepsilon,\delta}$ itself). Thus, starting from the identity

$$\nabla v_i^t(x) = \nabla v_i(\Phi_t(x))\big(I + tD\Psi_{\varepsilon,\delta}(x)\big)\,, \quad x \in B_1\,,$$

and, letting $\Omega_i := \{v_i > 0\}$, we get

$$\begin{aligned} E'(0) &= \sum_{i=1}^k \int_{B_1} \frac{d}{dt}|\nabla v_i^t|^2\Big|_{t=0}\,dx \\ &= 2\sum_{i=1}^k \int_{\Omega_i} (D^2 v_i \cdot \nabla v_i)\cdot \Psi_{\varepsilon,\delta}\,dx + 2\sum_{i=1}^k \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta}\cdot \nabla v_i)\,dx \\ &= -\sum_{i=1}^k \int_{B_1} |\nabla v_i|^2 \operatorname{div}\Psi_{\varepsilon,\delta}\,dx + 2\sum_{i=1}^k \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta}\cdot \nabla v_i)\,dx\,. \end{aligned}$$

(The last equality follows by noticing that $\sum_{i=1}^k 2\int_{B_1}(D^2 v_i \cdot \nabla v_i \cdot \Psi_{\varepsilon,\delta})\,dx$ is the derivative at $t = 0$ of the map $t \mapsto \sum_{i=1}^k \int_{B_1} |\nabla v_i(\Phi_t(x))|^2\,dx = \sum_{i=1}^k \int_{B_1} |\nabla v_i|^2 \det(D\Phi_t)^{-1}(x)\,dx$).

Thus the stationarity condition $E'(0) = 0$ gives us the identity

$$\sum_{i=1}^k \int_{B_1} |\nabla v_i|^2 \operatorname{div}\Psi_{\varepsilon,\delta}\,dx = 2\sum_{i=1}^k \int_{B_1} \nabla v_i \cdot (D\Psi_{\varepsilon,\delta}\cdot \nabla v_i)\,dx\,. \tag{2.16}$$

It is straightforward to compute $\operatorname{div}\Psi_{\varepsilon,\delta}$ and $D\Psi_{\varepsilon,\delta}$ as

$$\operatorname{div}\Psi_{\varepsilon,\delta} = d\,\varphi_\varepsilon(x)\gamma_\delta(|x|) + \varphi_\varepsilon(x)\gamma_\delta'(|x|)|x| + \gamma_\delta(|x|)\big(x\cdot\nabla\varphi_\varepsilon(x)\big)$$

$$D\Psi_{\varepsilon,\delta} = \varphi_\varepsilon(x)\gamma_\delta(|x|)I + \gamma_\delta(|x|)(x\otimes\nabla\varphi_\varepsilon(x)) + \gamma_\delta'(|x|)\varphi_\varepsilon(x)\left(x\otimes\frac{x}{|x|}\right).$$

Since $\gamma_\delta'(|x|) = -1/\delta$ on $B_1\setminus B_{1-\delta}$ and $\gamma_\delta'(|x|)=0$ on $B_{1-\delta}$, we infer that

$$\begin{aligned}\sum_{i=1}^k\int_{B_1}|\nabla v_i|^2\operatorname{div}\Psi_{\varepsilon,\delta}\,dx \quad &= d\sum_{i=1}^k\int_{B_1}\varphi_\varepsilon(x)\gamma_\delta(|x|)|\nabla v_i(x)|^2\,dx\\ &+\sum_{i=1}^k\int_{B_1}\gamma_\delta(|x|)\big(x\cdot\nabla\varphi_\varepsilon(x)\big)|\nabla v_i(x)|^2\,dx\\ &-\frac{1}{\delta}\sum_{i=1}^k\int_{B_1\setminus B_{1-\delta}}\varphi_\varepsilon(x)|x||\nabla v_i(x)|^2\,dx,\end{aligned}$$

and

$$\begin{aligned}2\sum_{i=1}^k\int_{B_1}\nabla v_i D\Psi_{\varepsilon,\delta}\cdot\nabla v_i\,dx \quad &= 2\sum_{i=1}^k\int_{B_1}\varphi_\varepsilon(x)\gamma_\delta(|x|)|\nabla v_i(x)|^2\,dx\\ &+2\sum_{i=1}^k\int_{B_1}\gamma_\delta(|x|)\big(x\cdot\nabla v_i(x)\big)\big(\nabla v_i(x)\cdot\nabla\varphi_\varepsilon(x)\big)\,dx\\ &-\frac{2}{\delta}\sum_{i=1}^k\int_{B_1\setminus B_{1-\delta}}\varphi_\varepsilon(x)\frac{1}{|x|}\big(\nabla v_i(x)\cdot x\big)^2\,dx\,.\end{aligned}$$

Passing to the limit as $\delta\to 0$ in (2.16), we obtain

$$\begin{aligned}&\sum_{i=1}^k d\int_{B_1}\varphi_\varepsilon(x)|\nabla v_i(x)|^2\,dx + \sum_{i=1}^k\int_{B_1}\big(x\cdot\nabla\varphi_\varepsilon(x)\big)|\nabla v_i(x)|^2\,dx\\ &-\sum_{i=1}^k\int_{\partial B_1}\varphi_\varepsilon(x)|\nabla v_i(x)|^2\,d\mathcal{H}^{d-1} = \sum_{i=1}^k 2\int_{B_1}\varphi_\varepsilon(x)|\nabla v_i(x)|^2\,dx\\ &+\sum_{i=1}^k 2\int_{B_1}\big(x\cdot\nabla v_i(x)\big)\big(\nabla v_i(x)\cdot\nabla\varphi_\varepsilon(x)\big) - \sum_{i=1}^k 2\int_{\partial B_1}\varphi_\varepsilon(x)\Big(\frac{\partial v_i}{\partial n}\Big)^2 d\mathcal{H}^{d-1}\,.\end{aligned}$$

Eventually, we obtain the lemma by passing to the limit as $\varepsilon\to 0$ in the above equality, since $\varphi_\varepsilon\to 1$ strongly in $H^1(B_1)$, and since $|\nabla v_i|\le M$ (thanks to the smoothness of the boundary data v and to Proposition 2.3 (ii)).

Proof of identity (2.9)**.** Setting

$$T := \sum_{i=1}^k\int_{\partial B_1}|\nabla_\tau\tilde v_i|^2\,d\mathcal{H}^{d-1}\,,\qquad N := \sum_{i=1}^k\int_{\partial B_1}\left|\frac{\partial\tilde v_i}{\partial n}\right|^2\,d\mathcal{H}^{d-1}$$

from Lemma 2.4 we have respectively

$$T+N = d\sum_{i=1}^k\int_{B_1}|\nabla\tilde v_i|^2\,dx + \sum_{i=1}^k\int_{\Omega_i}(1-|x|^2)|D^2\tilde v_i|^2\,dx$$

$$T-N = (d-2)\sum_{i=1}^k\int_{B_1}|\nabla\tilde v_i|^2\,dx\,.$$

Solving in (T,N), we find (2.9). $\square$

2.2. **Approximation of admissible data.** In this section we prove that any admissible data $w = (w_1, \dots, w_k) \in (H^1(\partial B_1))^k$ for the extension Problem 2.2 can be approximated through the regular configurations analyzed in Subsection 2.1, satisfying (2.4) and (2.6).

We shall need the following technical lemma, which employs the tool of γ-convergence of quasi-open sets. Here we limit ourselves to recall that, by definition, a sequence of equi-bounded quasi-open sets Ω_n γ-converges to Ω if the corresponding torsion functions w_{Ω_n}, namely the unique minimizers in $H^1_0(\Omega_n)$ of the functional

$$J(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla u|^2 - u \right) dx$$

converge in $L^1(\mathbb{R}^d)$ to w_Ω. More details on the notion of γ-convergence and its applications in variational problems can be found in [2].

Lemma 2.5. *Let D be an open bounded subset of $\mathbb{R}^d$. Given two quasi-open subsets A_1, A_2 of D such that $\mathrm{cap}(A_1 \cap A_2) = 0$, there exist two sequences (A_1^n, A_2^n) of open subsets of D such that $A_1^n \cap A_2^n = \emptyset$ for every n and, as $n \to +\infty$, $(A_1^n, A_2^n) \to (A_1, A_2)$ in the sense of γ-convergence.*

Proof. For $i = 1, 2$, let w_i denote the torsion function of the quasi-open set A_i, defined as above. By [6, Proposition 2.1], every point of $\mathbb{R}^d$ is a Lebesgue point of w_i and (the Lebsgue representative of) w_i is upper-semicontinuous on $\mathbb{R}^d$.

For a fixed $\varepsilon > 0$, we consider the set

$$F_1^\varepsilon := \{w_1 \geq \varepsilon\}.$$

By the upper semicontinuity of w_1, this set is closed. Moreover, since $F_1^\varepsilon \subset A_1$, $\mathrm{cap}(A_1 \cap A_2) = 0$, and $w_2 = 0$ q.e. on $D \setminus A_2$, we have that $w_2 = 0$ q.e. on F_1^ε, or equivalently $w_2 \in H^1_0(D \setminus F_1^\varepsilon)$.

Then, since the set $D \setminus F_1^\varepsilon$ is open, the quasi-open set A_2 can be approximated in the sense of γ-convergence by a sequence $A_2^{\varepsilon,n}$ of open sets contained into $D \setminus F_1^\varepsilon$ (see e.g. [2, Lemma 4.3.15]). Further, possibly passing to smaller open sets $\widetilde{A}_2^{\varepsilon,n} \subset A_2^{\varepsilon,n}$ such that $\mathrm{dist}(\widetilde{A}_2^{\varepsilon,n}, \partial A_2^{\varepsilon,n}) > 0$ and still $\widetilde{A}_2^{\varepsilon,n} \to A_2$ in the sense of γ-convergence, we can assume that

$$\{w_1 > \varepsilon\} \subset D \setminus \overline{A_2^{\varepsilon,n}}.$$

Then the quasi-open set $\{w_1 > \varepsilon\}$ can be approximated in the sense of γ-convergence by a sequence $A_1^{\varepsilon,n}$ of open sets contained into $D \setminus \overline{A_2^{\varepsilon,n}}$. Again, possibly replacing $A_1^{\varepsilon,n}$ by smaller open sets $\widetilde{A}_1^{\varepsilon,n}$, we can assume that

$$\mathrm{dist}(\partial A_1^{\varepsilon,n}, \partial A_2^{\varepsilon,n}) > 0.$$

So far, for every $\varepsilon > 0$ we have found two sequences $(A_1^{\varepsilon,n}, A_2^{\varepsilon,n})$ of open subsets of D such that $A_1^{\varepsilon,n} \cap A_2^{\varepsilon,n} = \emptyset$ for every n and, as $n \to +\infty$, γ-converge to $(\{w_1 > \varepsilon\}, A_2)$.

The proof is achieved by letting $\varepsilon \to 0$ and taking diagonal sequences. □

Proposition 2.6. *Any admissible datum $w = (w_1, \dots, w_k)$ can be approximated, in the strong topology of $(H^1(\partial B_1))^k$, by a sequence $v^n = (v_1^n, \dots, v_k^n)$ of admissible data satisfying* (2.4) *and* (2.6).

Proof. We prove the statement for $k = 2$ phases, being the proof the same if $k > 2$. So we consider the case of two functions $(w_1, w_2) \in (H^1(\partial B_1))^2$, with $w_i \geq 0$ and $w_1 \cdot w_2 = 0$ on ∂B_1.

Step 1: There exist two sequences $(w_1^n, w_2^n) \in (\mathcal{C}^\infty(\partial B_1))^2$ such that $w_i^n \geq 0$ on ∂B_1 $(i = 1, 2)$,

$$\mathrm{dist}\big(\{w_1^n > 0\}, \{w_2^n > 0\}\big) > 0$$

and

$$(w_1^n, w_2^n) \to (w_1, w_2) \qquad \text{strongly in } (H^1(\partial B_1))^2.$$

This is obtained as an immediate consequence of Lemma 2.5 (applied replacing D with ∂B_1 by local coordinates). Indeed, the sets $A_i := \{w_i > 0\}$ are quasi-open and satisfy the condition $\mathrm{cap}(A_1 \cap A_2) = 0$ (since $w_1 \cdot w_2 = 0$ on ∂B_1). Then Lemma 2.5 ensures the existence of two sequences of open sets (A_1^n, A_2^n) such that $A_1^n \cap A_2^n = \emptyset$ for every n and, as $n \to +\infty$, $(A_1^n, A_2^n) \to (A_1, A_2)$ in the sense of γ-convergence. In particular, since $w_i \in H_0^1(A_i)$, by the Mosco-convergence of the spaces $H_0^1(A_i^n)$ to $H_0^1(A_i)$ (*cf.* [2, Proposition 4.5.3]), we can find $(v_1^n, v_2^n) \in H_0^1(A_1^n) \times H_0^1(A_2^n)$ which converge to (w_1, w_2) strongly in $(H^1(\partial B_1))^2$. Possibly passing to $\max\{v_i^n, 0\}$ it is not restrictive to assume that $v_i^n \geq 0$ $(i = 1, 2)$. In turn, since the sets A_i^n are open, for every fixed n we can find sequences $(\varphi_1^{n,k}, \varphi_2^{n,k}) \subset \mathcal{C}_c^\infty(A_1^n) \times \mathcal{C}_c^\infty(A_2^n)$ (still with non-negative values), which converge to (v_1^n, v_2^n) strongly in $H_0^1(A_1^n) \times H_0^1(A_2^n)$ as $k \to +\infty$. Passing to a diagonal sequence $(w_1^n, w_2^n) := (\varphi_1^{n,k(n)}, \varphi_2^{n,k(n)})$ we get the claim of Step 1.

Step 2: Let $(w_1^n, w_2^n) \in (\mathcal{C}^\infty(\partial B_1))^2$ be as in Step 1. For every fixed n, there exists $(\varphi_1^{n,k}, \varphi_2^{n,k})_{k\in\mathbb{N}} \in (\mathcal{C}^\infty(\partial B_1))^2$ with

$$(\varphi_1^{n,k}, \varphi_2^{n,k}) \to (w_1^n, w_2^n) \qquad \text{strongly in } (H^1(\partial B_1))^2$$

such that, for every k, setting $S_i^{n,k} := \{\varphi_i^{n,k} > 0\}$ it holds

$$\partial B_1 = S_1^{n,k} \cup S_2^{n,k} \cup E^{n,k},$$

where $E^{n,k}$ is a smooth $(d-2)$-dimensional submanifold, and the sets $S_1^{n,k}, S_2^{n,k}, E^{n,k}$ are mutually disjoint.

Indeed, we can proceed as follows. Considering the well separated compact sets $\overline{\{w_i^n > 0\}}_{i=1,2}$, we can find a relatively open set with smooth boundary $S_1^n \subseteq \partial B_1$ such that

$$\overline{\{w_1^n > 0\}} \subset S_1^n \qquad \text{and} \qquad \overline{\{w_2^n > 0\}} \subset \partial B_1 \setminus \overline{S_1^n} := S_2^n.$$

Through a partition of unity argument, we can construct $\psi_i^n \in \mathcal{C}^\infty(\partial B_1)$ with $\psi_i^n \geq 0$ and such that

$$S_i^n = \{\psi_i^n > 0\} \qquad i = 1, 2.$$

The conclusion follows by setting

$$\varphi_i^{n,k} := w_i^n + \frac{1}{k}\psi_i^n, \qquad S_i^{n,k} := S_i^n \qquad \text{and} \qquad E^{n,k} := \partial S_i^n,$$

where the boundary is taken clearly in the relative topology of ∂B_1.

Step 3: Conclusion. Let $(\varphi_1^{n,k}, \varphi_2^{n,k}) \in (\mathcal{C}^\infty(\partial B_1))^2$ be as in Step 2. Passing to a diagonal sequence, namely setting

$$(v_1^n, v_2^n) := (\varphi_1^{n,k(n)}, \varphi_2^{n,k(n)})$$

we obtain the proposition. Indeed, by construction, (v_1^n, v_2^n) are admissible data satisfying (2.4) and (2.6), and converging strongly to (w_1, w_2) in $H^1(\partial B_1)$. □

2.3. **Proof of the multiphase extension theorem.** Let $w = (w_1, \ldots, w_k) \in (H^1(\partial B_1))^k$ be an admissible datum. Let $v^n = (v_1^n, \ldots, v_k^n) \subset (\mathcal{C}^\infty(\partial B_1))^k$ be a sequence as given by Proposition 2.6. By Proposition 2.4, for every n a solution $\tilde{v}^n = (\tilde{v}_1^n, \ldots, \tilde{v}_k^n)$ to problem $\mathcal{P}(v_n)$ satisfies

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau v_i^n|^2 \, d\mathcal{H}^{d-1} = (d-1) \sum_{i=1}^k \int_{B_1} |\nabla \tilde{v}_i^n|^2 \, dx + \frac{1}{2} \sum_{i=1}^k \int_{\Omega_i^n} (1 - |x|^2) |D^2 \tilde{v}_i^n|^2 \, dx \,, \tag{2.17}$$

where $\Omega_i^n := \{\tilde{v}_i^n > 0\}$. Moreover, by Proposition 2.3 (vi), the sequence $\{\tilde{v}^n = (\tilde{v}_1^n, \ldots, \tilde{v}_k^n)\}$ converges strongly in $(H^1(B_1))^k$ to the solution $\tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_k)$ to problem $\mathcal{P}(v)$.

Then we pass to the limit as $n \to +\infty$ in (2.17). Since the quantities

$$\int_{\partial B_1} |\nabla_\tau v_i|^2 \, dx \qquad \text{and} \qquad \int_{B_1} |\nabla \tilde{v}_i|^2 \, dx$$

are strongly continuous respectively in $H^1(\partial B_1)$ and in $H^1(B_1)$, we obtain

$$\sum_{i=1}^k \int_{\partial B_1} |\nabla_\tau w_i|^2 \, d\mathcal{H}^{d-1} = (d-1) \sum_{i=1}^k \int_{B_1} |\nabla \tilde{w}_i|^2 \, dx + \frac{1}{2} \liminf_n \sum_{i=1}^k \int_{\Omega_i^n} (1-|x|^2)|D^2 \tilde{v}_i^n|^2 \, dx$$
$$\geq (d-1) \min \Big\{ \sum_{i=1}^k \int_{B_1} |\nabla \tilde{w}_i|^2 \, dx \, : \, \tilde{w}_i|_{\partial B_1} = w_i \, , \, \tilde{w}_i \cdot \tilde{w}_j = 0 \text{ in } B_1 \Big\} .$$

3. The monotonicity formula

In this section we establish the following multiphase version of the monotonicity formula in [5].

Theorem 3.1 (Multiphase monotonicity formula). *There exists a dimensional constant c_d such that, if $(u_1, \ldots, u_k) \in \mathcal{U}(\Omega)$ is a multiphase local almost-quasi minimizer at $x \in \Omega$, with parameters $(\Lambda, \alpha, c_\alpha)$ according to Definition 1.1, then the mapping*

$$\rho \mapsto F_u(\rho) := \left[\frac{1}{\rho^{d-1}} \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2 \, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \right] \wedge \frac{c_d \Lambda^{2-d}}{d-1} + (d-1) \frac{c_\alpha}{\alpha} \rho^\alpha$$

is non decreasing on $(0, dist(x, \partial\Omega))$.

As mentioned in the Introduction, the above result is obtained along the same proof line of [5]. In particular, one establishes the result for $d = 2$, and then obtains the general case by induction on the space dimension. We emphasize that the key point is to replace inequality (1.6) by the sharp gradient multiphase estimate given by Theorem 2.1. With this weapon in hand, the proof given in [5] can be followed line by line, making purely formal adjustments. For this reason we omit the proof in its full length, and we limit ourselves to present it in the $2d$ case, both for the sake of the reader, and because the $2d$ setting includes already several interesting applications, for instance in image segmentation.

Below we assume $d = 2$. In this case we can set $c_2 := 1$. Moreover, it is not restrictive to consider $x = 0$. Let $(u_1, \ldots, u_k) \in \mathcal{U}(\Omega)$ be a local $(\Lambda, \alpha, c_\alpha)$-almost-quasi minimizer of a multiphase free discontinuity problem at 0. For $\rho \in I := (0, dist(0, \partial\Omega))$, we can write $F_u(\rho)$ as

$$F_u(\rho) = \frac{E_u(\rho)}{\rho} \wedge 1 + \frac{c_\alpha}{\alpha} \rho^\alpha,$$

where

$$E_u(\rho) := \sum_{i=1}^k \left(\int_{B_\rho(0)} |\nabla u_i|^2 \, dx + \mathcal{H}^1(J_{u_i} \cap \overline{B}_\rho(0)) \right).$$

In the remaining of the proof, we write for brevity F, E in place of F_u, E_u.

Since the map $\rho \mapsto E(\rho)$ is non decreasing on I, we have that E belongs to $BV_{\rm loc}(I)$. As a consequence the distributional derivative of E is a nonnegative measure DE such that

$$DE = E' \, d\rho + \mu_E,$$

where E' denotes the density of the absolutely continuous part of DE, and

$$\mu_E = [E^+ - E^-] \mathcal{H}^0 \llcorner J_E + D^c E$$

stands for its singular part.

From the chain rule in BV (see [1, Theorem 3.99]), we have that $F \in BV_{\rm loc}(I)$, with

$$DF = F' \, d\rho + \mu_F,$$

where

$$F'(\rho) = \begin{cases} c_\alpha \rho^{\alpha-1} & \mathcal{L}^1\text{-a.e. on } I_+ := \Big\{ E(\rho) > \rho \Big\} \\ \dfrac{E'(\rho)}{\rho} - \dfrac{E(\rho)}{\rho^2} + c_\alpha \rho^{\alpha-1} & \mathcal{L}^1\text{-a.e. on } I_- := \Big\{ E(\rho) < \rho \Big\} . \end{cases}$$

and

$$\mu_F = \left(\frac{E^+(\rho) - E^-(\rho)}{\rho} \wedge c_2 \right) \mathcal{H}^0 \llcorner J_E + \left(\frac{E(\rho)}{\rho} \wedge c_2 \right) D^c E \geq 0.$$

Hence, in order to show that F is non decreasing, it is enough to prove that the density $F' \geq 0$ a.e. on I.

Clearly $F'(\rho) \geq 0$ $\mathcal{L}^1$-a.e. on I_+. Let us show that the same holds true also on I_-. Assume by contradiction that

$$F'(\rho) < 0 \qquad \mathcal{L}^1 \text{ -a.e. on a subset } J,$$

where $J \subset I_-$ with $\mathcal{L}^1(J) > 0$. In this case, for $\mathcal{L}^1$-a.e. $\rho \in J$, we have

$$E'(\rho) < \frac{E(\rho)}{\rho} - c_\alpha \rho^\alpha < 1. \tag{3.1}$$

In view of the coarea formula for rectifiable sets (see [1, Theorem 2.93]), we obtain that for a.e. $\rho \in J$

$$\begin{aligned} 1 > E'(\rho) &= \sum_{i=1}^k \left[\int_{\partial B_\rho} |\nabla u_i|^2 \, d\mathcal{H}^1 + \lim_{h \to 0^+} \frac{\mathcal{H}^1(J_{u_i} \cap \overline{B}_{\rho+h}) - \mathcal{H}^1(J_{u_i} \cap \overline{B}_\rho)}{h} \right] \\ &\geq \sum_{i=1}^k \left[\int_{\partial B_\rho} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1 + \lim_{h \to 0^+} \frac{1}{h} \int_\rho^{\rho+h} \mathcal{H}^0(J_{u_i} \cap \partial B_s) \, ds \right] \\ &= \sum_{i=1}^k \left[\int_{\partial B_\rho} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1 + \mathcal{H}^0(J_{u_i} \cap \partial B_\rho) \right], \end{aligned} \tag{3.2}$$

from which we get $\sum_{i=1}^k \mathcal{H}^0(J_{u_i} \cap \partial B_\rho) = 0$ and consequently

$$u_{|\partial B_\rho} \in (H^1(\partial B_\rho))^k.$$

By Theorem 2.1, we get the existence of a function $\tilde{w} \in (H^1(B_\rho))^k$ with $\tilde{w}_{|\partial B_\rho} = u_{|\partial B_\rho}$, $\tilde{w}_i \cdot \tilde{w}_j = 0$ and such that

$$\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx \leq \rho \sum_{i=1}^k \int_{\partial B_\rho} |\nabla_\tau u_i|^2 \, d\mathcal{H}^1.$$

Coming back to (3.2) and (3.1) we deduce

$$\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx \leq \rho E'(\rho) < E(\rho) - c_\alpha \rho^{\alpha+1},$$

that is

$$\sum_{i=1}^k \int_{B_\rho} |\nabla \tilde{w}_i|^2 \, dx + c_\alpha \rho^{\alpha+1} < E(\rho).$$

But this is against the fact the u is a local almost-quasi minimizer (by taking the trial function $z = (z_1, \ldots, z_k)$ defined by $z_i = \tilde{w}_i \chi_{B_\rho} + u_i \chi_{\Omega \setminus B_\rho}$), so that the result follows.

4. The decay estimate

In this section we establish a multiphase extension of the decay estimate of De Giorgi, Carriero and Leaci [17].

For any $u := (u_1, \ldots, u_k) \in \mathcal{U}(\Omega)$ (see (1.4)), $A \subseteq \Omega$, and $c \geq 0$, we define

$$\Phi(u, c, A) := \sum_{i=1}^k \left[\int_A |\nabla u_i|^2 \, dx + c\mathcal{H}^{d-1}(J_{u_i} \cap A) \right],$$

and we set $Dev(u, c, A)$ the minimum $\lambda \geq 0$ such that

$$\Phi(u, c, A) \leq \Phi(v, c, A) + \lambda \tag{4.1}$$

for every competitor $v \in \mathcal{U}(\Omega)$ such that $\{u \neq v\} \subset\subset A$. For $c = 1$ we write simply $\Phi(u, A)$ and $Dev(u, A)$.

The decay estimate is the following.

Theorem 4.1 (Multiphase decay estimate). *There exists $C_d > 0$ such that for every $\tau \in]0, 1[$ there exist $\varepsilon(\tau), \vartheta(\tau)$ such that if $u = (u_1, \dots, u_k) \in \mathcal{U}(\Omega)$, $B_\rho(x) \subset \Omega$ and*

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) < \varepsilon(\tau)\rho^{d-1}, \qquad Dev(u, B_\rho(x)) \leq \vartheta(\tau)\Phi(u, B_\rho(x)),$$

then

$$\Phi(u, B_{\tau\rho}(x)) \leq C_d \tau^d \Phi(u, B_\rho(x)). \tag{4.2}$$

Loosely speaking, the argument to prove the decay estimate in the classical case is as follows (see the presentation of [1, Chapter 7]). One proceeds by contradiction: by rescaling, one is led to consider a sequence $(u_n)_{n\in\mathbb{N}}$ on B_1 with vanishing jump set and vanishing deviation along which (4.2) is violated. If (u_n) admits a limit configuration u, one expects u to be a Sobolev function (jumps are vanishing) which is a local minimizer of the Dirichlet energy (deviation is going to zero). In particular u is harmonic on B_1, so that the energy in B_τ behaves like $\hat{C}_d \tau^d$ with $\hat{C}_d$ a dimensional constant, and this yields to a contradiction if (4.2) is assumed to be violated. The limit configuration is obtained through suitable truncations, employing Poincaré type inequalities in SBV.

We adapt the same procedure to the multiphase setting as follows.

(a) In Section 4.1 (Proposition 4.3), we prove an auxiliary result about a subharmonicity property of local multiphase minimizers for the Dirichlet energy on B_1.

(b) In Section 4.2, we describe the asymptotic behaviour of a sequence of elements in $\mathcal{U}(B_1)$ for which the deviation from minimality and the total length of the jumps go to zero: we will treat separately the cases where more than one or just one phase are prevailing in the limit (see Propositions 4.5 and 4.6). Limit configurations are recovered which are local minimizers of the multiphase or the scalar Dirichlet energy.

(c) Finally in Section 4.3 we give the proof of Theorem 4.1.

4.1. **A subarmonicity result for multiphase local minimizers of the Dirichlet energy.** We set the following Definition.

Definition 4.2 (Multiphase local minimizer of the Dirichlet energy). *We say that $u = (u_1, \dots, u_m) \in H^1(B_1; \mathbb{R}^m)$ with $u_i \cdot u_j = 0$ for $i \neq j$ is a multiphase local minimizer of the Dirichlet energy if*

$$\sum_{i=1}^{m} \int_{B_1} |\nabla u_i|^2 \, dx \leq \sum_{i=1}^{m} \int_{B_1} |\nabla v_i|^2 \, dx$$

for every $v = (v_1, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$ with $v_i \cdot v_j = 0$ and $\{v \neq u\} \subset\subset B_1$.

In the case $m = 1$, the preceding definition reduces to the usual one of local minimizer of the Dirichlet energy.

We will need the following subarmonicity result.

Proposition 4.3. *Let $(v_1, v_2, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$ be a local multiphase minimizer of the Dirichlet energy. Then the function $\sum_{i=1}^{m} |\nabla v_i|^2$ is subharmonic in B_1.*

Proof. It is not restrictive to assume that the functions are positive (considering positive and negative parts, we obtain a $(2m)$-multiphase local minimizer of the Dirichlet energy). To get the conclusion, it suffices to check that

$$-\Delta \left(\sum_{i=1}^{m} |\nabla v_i|^2 \right) \leq 0 \qquad \text{in the sense of distributions on } B_1,$$

i.e.,

$$\sum_{i=1}^{m} \int_{B_1} \Delta\varphi |\nabla v_i|^2 \, dx \geq 0 \tag{4.3}$$

for every $\varphi \in C_c^\infty(B_1)$ with $\varphi \geq 0$.

Let us denote with Γ the interfaces in B_1 associated to $(v_1, v_2, \dots, v_m)$. By Proposition 2.3 we have that Γ is the disjoint union of analytic hypersurfaces and a relatively closed set Γ_0 having Hausdorff dimension at most $d-2$ and zero capacity: indeed for every $\rho < 1$, $(v_1, v_2, \dots, v_m)$ is a solution to problem (2.2) on B_ρ with respect to its own boundary data. We may therefore proceed integrating by parts on each phase, disregarding the set Γ_0 which plays no role since it has zero capacity.

We may write integrating by parts

$$\begin{aligned}\sum_{i=1}^{m} \int_{B_1} \Delta\varphi |\nabla v_i|^2 \, dx &= \sum_{i=1}^{m} \sum_{k,j=1}^{d} \int_{B_1} |\partial_k v_i|^2 \partial_{jj}\varphi \, dx \\ &= \sum_{i=1}^{m} \sum_{k,j=1}^{d} \left[\int_\Gamma |\partial_k v_i|^2 \partial_j \varphi n_j \, d\mathcal{H}^{d-1} - 2 \int_{B_1} \partial_k v_i \partial_{kj} v_i \partial_j \varphi \, dx \right].\end{aligned}$$

Notice that

$$\sum_{i=1}^{m} \sum_{k,j=1}^{d} \int_\Gamma |\partial_k v_i|^2 \partial_j \varphi n_j \, d\mathcal{H}^{d-1} = \sum_{i=1}^{m} \int_\Gamma |\nabla v_i|^2 \frac{\partial \varphi}{\partial n} \, d\mathcal{H}^{d-1} = 0$$

since at the interface between v_i and v_j we have $\nabla v_i = -\nabla v_j$ (see Proposition 2.3), while the normals are oppositely oriented. We infer integrating again by parts, taking into account that each v_i is harmonic on its phase

$$\sum_{i=1}^{m} \int_{B_1} \Delta\varphi |\nabla v_i|^2 \, dx = -2 \sum_{i=1}^{m} \sum_{k,j=1}^{d} \left[\int_\Gamma \partial_k v_i \partial_{kj} v_i \varphi n_j \, d\mathcal{H}^{d-1} + \int_{B_1} |\partial_{kj} v_i|^2 \varphi \, dx \right]. \tag{4.4}$$

Since $v_i = 0$ on Γ, we have

$$\partial_k v_i = \frac{\partial v_i}{\partial n} n_k \qquad \text{and} \qquad \sum_{k,j=1}^{d} \partial_{kj} v_i n_k n_j = -(d-1) \frac{\partial v_i}{\partial n} H_\Gamma \qquad \text{on } \Gamma,$$

where H_Γ stands for the mean curvature. Again by cancellation due to the different sign of H_Γ on the two sides of the interfaces we deduce

$$\begin{aligned}\sum_{i=1}^{m} \sum_{k,j=1}^{d} \int_\Gamma \partial_k v_i \partial_{kj} v_i \varphi n_j \, d\mathcal{H}^{d-1} &= \sum_{i=1}^{m} \sum_{k,j=1}^{d} \int_\Gamma \frac{\partial v_i}{\partial n} n_k \partial_{kj} v_j \varphi n_j \, d\mathcal{H}^{d-1} \\ &= -(d-1) \sum_{i=1}^{m} \int_\Gamma H_\Gamma \left(\frac{\partial v_i}{\partial n} \right)^2 \varphi \, d\mathcal{H}^{d-1} = 0.\end{aligned}$$

Equality (4.4) thus yields

$$\sum_{i=1}^{m} \int_{B_1} \Delta\varphi |\nabla v_i|^2 \, dx = -2 \sum_{i=1}^{m} \sum_{k,j=1}^{d} \int_{B_1} |\partial_{kj} v_i|^2 \varphi \, dx \leq 0$$

so that (4.3) follows, and the proof is complete. □

4.2. **Asymptotic behaviour of sequences with vanishing jumps and deviation.** We begin with a useful lemma:

Lemma 4.4. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBV(B_1)$ with $u_n \geq 0$,*

$$\int_{B_1} |\nabla u_n|^2\, dx \leq C, \qquad \mathcal{H}^{d-1}(J_{u_n}) \to 0$$

and such that

$$\liminf_n |\{u_n = 0\}| > 0. \tag{4.5}$$

Then there exist a subsequence $(u_{n_k})_{k\in\mathbb{N}}$, $\tau_{n_k} \geq 0$, $u \in H^1(B_1)$ and $c > 0$ such that setting

$$\bar{u}_{n_k} := u_{n_k} \wedge \tau_{n_k}$$

we have

$$|\{u_{n_k} \neq \bar{u}_{n_k}\}| \leq c \left(\mathcal{H}^{d-1}(J_{u_{n_k}})\right)^{d/d-1},$$

$$\bar{u}_{n_k} \to u \qquad \text{strongly in } L^2(B_1),$$

and

$$\int_{B_1} |\nabla u|^2\, dx \leq \liminf_k \int_{B_1} |\nabla \bar{u}_{n_k}|^2\, dx.$$

Proof. The result is a variant of [1, Proposition 7.5], where the conclusion is proved for

$$[(u_n \wedge \tau^+(u_n, B_1) \vee \tau^-(u_n, B_1)] - m_n,$$

where m_n is a median for u_n, while $\tau^\pm(u_n, B_1)$ are truncation levels associated to the Poincaré inequality in SBV (see [1, Theorem 4.14]). In our situation, truncation from below is not necessary since the functions are positive, and the use of the median can be avoided because u_n vanishes on a nontrivial part of B_1.

We proceed in two steps.

Step 1. Up to a subsequence we may assume that for every $n \in \mathbb{N}$

$$|\{u_n = 0\}| \geq \delta > 0. \tag{4.6}$$

Let us set following [17]

$$\tau_n := \tau^+(u_n, B_1) := \inf\left\{ t \geq 0 \,:\, |\{u < t\}| > \omega_d - [2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n})]^{\frac{d}{d-1}} \right\},$$

where γ_δ is the constant for which the isoperimetric inequality

$$\gamma_\delta Per(E, B_1) \geq |E|^{\frac{d-1}{d}} \tag{4.7}$$

holds true for every set $E \subset B_1$ such that $|E| \leq \omega_d - \delta$: here $Per(E, B_1)$ denotes the perimeter of E in B_1.

Let us set $\bar{u}_n := u_n \wedge \tau_n$ so that

$$|\{u_n \neq \bar{u}_n\}| \leq [2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n})]^{\frac{d}{d-1}}. \tag{4.8}$$

We may write

$$|D\bar{u}_n|(B_1) \leq \int_{B_1} |\nabla \bar{u}_n|\, dx + \tau_n \mathcal{H}^{d-1}(J_{u_n}), \tag{4.9}$$

while using the coarea formula and the isoperimetric inequality (4.7) we have

$$|D\bar{u}_n|(B_1) = \int_0^{+\infty} Per(\{\bar{u}_n > t\}, B_1)\, dt = \int_0^{\tau_n} Per(\{u_n > t\}, B_1)\, dt$$
$$\geq \frac{1}{\gamma_\delta} \int_0^{\tau_n} |\{u_n > t\}|^{\frac{d-1}{d}}\, dt.$$

Notice that the use of the isoperimetric inequality is allowed since

$$|\{u_n > t\}| = \omega_d - |\{u_n \le t\}| \le \omega_d - |\{u_n = 0\}| \le \omega_d - \delta.$$

Since by the very definition of τ_n we have

$$|\{u_n > t\}|^{\frac{d-1}{d}} \ge 2\gamma_\delta \mathcal{H}^{d-1}(J_{u_n}) \qquad \text{for every } 0 \le t < \tau_n,$$

we conclude

$$|D\bar{u}_n|(B_1) \ge 2\tau_n \mathcal{H}^{d-1}(J_{u_n})$$

which together with (4.9) yields

$$|D\bar{u}_n|(B_1) \le 2\int_{B_1} |\nabla \bar{u}_n|\, dx. \tag{4.10}$$

In view of (4.6), Poincaré-Sobolev inequality in BV holds true for $\bar{u}_n$ (without subtracting a median), so that we may write

$$\left(\int_{B_1} \bar{u}_n^{\frac{d}{d-1}}\, dx\right)^{\frac{d-1}{d}} \le \eta_\delta |D\bar{u}_n|(B_1) \le 2\eta_\delta \int_{B_1} |\nabla \bar{u}_n|\, dx \le 2\eta_\delta \int_{B_1} |\nabla u_n|\, dx \tag{4.11}$$

for some $\eta_\delta > 0$.

Let $d \ge 3$. Applying the previous inequality to $v_n := u_n^{\frac{2(d-1)}{d-2}}$, being $\tau^+(v_n, B_1) = [\tau^+(u_n, B_1)]^{\frac{2(d-1)}{d-2}}$ we deduce by a straightforward calculation

$$\|\bar{u}_n\|_{L^{2^*}} \le c_\delta \|\nabla u_n\|_{L^2}. \tag{4.12}$$

for some $c_\delta > 0$, where as usual $2^* := \frac{2d}{d-2}$.

If $d = 2$, for any $q > 2$ Poincaré-Sobolev inequality together with (4.10) yields

$$\|\bar{u}_n\|_{L^q} \le \eta_{\delta,q} |D\bar{u}_n|(B_1) \le 2\eta_{\delta,q} \int_{B_1} |\nabla \bar{u}_n|\, dx \le 2\eta_{\delta,q} |B_1|^{1/2} \|\nabla u_n\|_{L^2}. \tag{4.13}$$

Step 2. Thanks to (4.10), (4.11) and (4.12) ((4.13) if $d = 2$), in view of the compact embedding of BV into L^1, there exist $u \in BV(B_1) \cap L^2(B_1)$ and $\{u_{n_k}\}_{k \in \mathbb{N}}$ such that

$$\bar{u}_{n_k} \to u \qquad \text{strongly in } L^2(B_1).$$

Using Ambrosio's theorem on $\bar{u}_n \wedge M$ for any $M > 0$ as in [1, Proposition 7.5], we infer that $u \in H^1(B_1)$, so that the conclusion follows (take into account (4.8)).

□

Proposition 4.5. *Let $u_n := (u_1^n, u_2^n, \ldots, u_k^n) \in \mathcal{U}(B_1)$ and $c_n > 0$ be such that*

$$\sum_{i=1}^{k} \mathcal{H}^{d-1}\left(J_{u_i^n}\right) \to 0, \qquad \Phi(u_n, c_n, B_1) \le C, \qquad Dev(u_n, c_n, B_1) \to 0. \tag{4.14}$$

Assume that, for some $2 \le m \le k$,

$$\liminf_n |\{u_i^n \ne 0\}| > 0 \qquad i = 1, \ldots, m \tag{4.15}$$

and

$$\lim_n |\{u_i^n \ne 0\}| = 0 \qquad i = m+1, \ldots, k. \tag{4.16}$$

Then up to a subsequence

$$(u_1^n, \ldots, u_m^n) \to (u_1, \ldots, u_m) \in H^1(B_1; \mathbb{R}^m) \qquad \text{a.e. in } B_1,$$

where $(u_1, \ldots, u_m)$ is a multiphase local minimizer for the Dirichlet energy with

$$\sum_{i=1}^{m} \int_{B_\rho} |\nabla u_i|^2\, dx = \lim_n \Phi(u_n, c_n, B_\rho) \tag{4.17}$$

for every $0 \le \rho < 1$.

Proof. We follow [1, Theorem 7.7] adapting the arguments to our multiphase setting. We divide the proof in several steps.

Step 1: Compactness. By Helly's theorem we may assume that up to a subsequence for every $\rho \in [0,1]$

$$\lim_n \Phi(u_n, c_n, B_\rho) = \alpha(\rho),$$

where $\alpha : [0,1] \to [0,+\infty[$ is non decreasing. Moreover we may assume $c_n \to c_\infty \in [0,+\infty]$.

In view of (4.14), (4.15) and (4.16), by Lemma 4.4 (applied to the positive and negative parts of u_n) we get that up to a subsequence, for $i = 1, \ldots, m$

$$\bar{u}_i^n \to u_i \in H^1(B_1) \qquad \text{strongly in } L^2(B_1),$$

while for $i = m+1, \ldots, k$ (notice that the set on which they vanish converges to B_1 in measure)

$$\bar{u}_i^n \to 0 \qquad \text{strongly in } L^2(B_1).$$

More precisely, $\bar{u}_i^n := (u_n^i)^+ \wedge \tau_{i,+}^n - (u_n^i)^- \wedge \tau_{i,-}^n$ with $\tau_{i,\pm}^n > 0$ and

$$|\{u_i^n \neq \bar{u}_i^n\}| \leq c_d \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1} \to 0.$$

This means that

$$u_n \to (u_1, \ldots, u_m, 0, \ldots, 0) \qquad \text{a.e. in } B_1,$$

with $u_i \cdot u_j = 0$ a.e. in B_1 for $i \neq j$, while

$$\bar{u}_n := (\bar{u}_1^n, \ldots, \bar{u}_k^n) \to (u_1, \ldots, u_m, 0, \ldots, 0) \qquad \text{strongly in } L^2(B_1; \mathbb{R}^k).$$

Finally, thanks again to Lemma 4.4 for every $\rho \in [0,1]$

$$\sum_{i=1}^m \int_{B_\rho} |\nabla u_i|^2 \, dx \leq \liminf_n \sum_{i=1}^m \int_{B_\rho} |\nabla \bar{u}_i^n|^2 \, dx \leq \alpha(\rho). \tag{4.18}$$

Step 2: Local minimality for $\bar{u}_n$. We may write thanks to (4.14)

$$c_n \int_0^1 \mathcal{H}^{d-1}(\{u_i^n \neq \bar{u}_i^n\} \cap \partial B_\rho) \, d\rho = c_n |\{u_i^n \neq \bar{u}_i^n\} \cap B_1| \leq c c_n \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1}$$
$$= c c_n \mathcal{H}^{d-1}(J_{u_i^n}) \left[\mathcal{H}^{d-1}(J_{u_i^n})\right]^{1/d-1} \to 0.$$

Thus up to a further subsequence, we may assume that for every $i = 1, \ldots, k$ and for a.e. $\rho \in [0,1]$

$$c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar{u}_i^n\} \cap \partial B_\rho) \to 0.$$

Since $\mathcal{H}^{d-1}(J_{\bar{u}_i^n} \cap \partial B_\rho) = 0$ for a.e. $\rho \in [0,1]$, by comparing u_n with

$$v_n := (v_1^n, \ldots, v_k^n)$$

where

$$v_i^n := \bar{u}_i^n 1_{B_\rho} + u_i^n 1_{B_1 \setminus B_\rho}, \qquad \rho \in [0,1[,$$

we get

$$\Phi(\bar{u}_n, c_n, B_\rho) \leq \Phi(u_n, c_n, B_\rho)$$
$$\leq \Phi(\bar{u}_n, c_n, B_\rho) + \sum_{i=1}^k c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar{u}_i^n\} \cap \partial B_\rho) + Dev(u_n, c_n, B_1).$$

We conclude that for a.e. $\rho \in [0,1[$

$$\Phi(\bar{u}_n, c_n, B_\rho) \to \alpha(\rho), \tag{4.19}$$

and since (see e.g. [1, Lemma 7.3])

$$Dev(\bar u_n, c_n, B_\rho) \le \Phi(\bar u_n, c_n, B_\rho) - \Phi(u_n, c_n, B_\rho) + \sum_{i=1}^k c_n \mathcal{H}^{d-1}(\{u_i^n \neq \bar u_i^n\} \cap \partial B_\rho) + Dev(u_n, c_n, B_1),$$

we infer

$$Dev(\bar u_n, c_n, B_\rho) \to 0 \qquad \text{for a.e. } \rho \in [0,1[. \tag{4.20}$$

Step 3: Multiphase local minimality in the limit. Let us consider

$$(v_1, \dots, v_m) \in H^1(B_1; \mathbb{R}^m)$$

admissible multiphase competitor for $(u_1, \dots, u_m)$. Let us choose $\rho < \rho' < 1$ such that

$$\{(v_1, \dots, v_m) \neq (u_1, \dots, u_m)\} \subset\subset B_\rho,$$

the function α is continuous at ρ', the convergences (4.19) and (4.20) hold true at ρ and ρ'.

Let us set

$$w_i^n := \varphi v_i + (1-\varphi)\bar u_i^n \qquad i = 1, \dots, m$$

and

$$w_j^n := (1-\varphi)\bar u_j^n \qquad j = m+1, \dots, k,$$

where φ is a smooth cut-off function such that $\varphi = 1$ on B_ρ, the support of φ is contained in $B_{\rho'}$, and such that $|\nabla \varphi| \le C/(\rho' - \rho)$.

Notice that $w_n := (w_1^n, \dots, w_k^n)$ is not a priori a good competitor for $\bar u_n$ since we have no control on the supports of the components. Following [14] we set

$$v_i^n := \left((w_i^n)^+ - \sum_{j \neq i} |w_j^n| \right)^+ - \left((w_i^n)^- - \sum_{j \neq i} |w_j^n| \right)^+.$$

Notice that $v^n := (v_1^n, \dots, v_k^n)$ coincides with $\bar u_n$ on $B_1 \setminus B_\rho$. Moreover the supports of v_i^n are mutually disjoint. Indeed if for example $v_i^n(x) > 0$, then for $h \neq i$

$$|w_i^n(x)| = (w_i^n)^+(x) > \sum_{i \neq j} |w_j^n|(x) \ge |w_h^n|(x) = (w_h^n)^+(x) + (w_h^n)^-(x),$$

which yields immediately $v_h^n(x) = 0$.

We conclude that v^n is an admissible competitor for $\bar u_n$ with

$$v^n \to (v_1, \dots, v_m, 0, \dots, 0) \qquad \text{strongly in } L^2(B_1)$$

and

$$v^n = (v_1, \dots, v_m, 0, \dots, 0) \qquad \text{in } B_\rho.$$

Comparing v^n with $\bar u_n$ we get easily for some $c > 0$ (see e.g. [1, Lemma 7.4])

$$\Phi(\bar u_n, c_n, B_\rho) \le \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx + c \left[\Phi(\bar u_n, c_n, B_{\rho'} \setminus B_\rho) + \sum_{i=1}^m \int_{B_{\rho'} \setminus B_\rho} |\nabla v_i|^2 \, dx + \frac{1}{(\rho' - \rho)^2} \sum_{i=1}^k \int_{B_{\rho'} \setminus B_\rho} |\bar u_i^n - v_i^n|^2 \, dx \right] + Dev(\bar u_n, c_n, B_{\rho'}),$$

so that, in the limit as $n \to +\infty$,

$$\alpha(\rho) \le \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx + c\left[\alpha(\rho') - \alpha(\rho) + \sum_{i=1}^m \int_{B_{\rho'}\setminus B_\rho} |\nabla v_i|^2 \, dx + \frac{1}{(\rho'-\rho)^2} \sum_{i=1}^m \int_{B_{\rho'}\setminus B_\rho} |u_i - v_i|^2 \, dx\right].$$

Letting $\rho \to \rho'$, and since $u_i = v_i$ on $B_{\rho'} \setminus B_\rho$, we infer

$$\alpha(\rho') \le \sum_{i=1}^m \int_{B_{\rho'}} |\nabla v_i|^2 \, dx.$$

Choosing $v_i = u_i$ and recalling (4.18) we deduce that

$$\sum_{i=1}^m \int_{B_{\rho'}} |\nabla u_i|^2 \, dx = \alpha(\rho'),$$

which yields in particular the multiphase local minimality of the limit configuration $(u_1, \dots, u_m)$.

If $\rho \in [0,1[$, we can choose $\rho' > \rho$ satisfying the relation above, so that by monotonicity

$$\alpha(\rho) \le \alpha(\rho') = \sum_{i=1}^m \int_{B_{\rho'}} |\nabla u_i|^2 \, dx.$$

For $\rho' \to \rho$ we deduce relation (4.17) and the proof is concluded. □

Proposition 4.6. *Let $u_n := (u_1^n, u_2^n, \dots, u_k^n) \in \mathcal{U}(B_1)$ and $c_n > 0$ be such that*

$$\sum_{i=1}^k \mathcal{H}^{d-1}\left(J_{u_i^n}\right) \to 0, \qquad \Phi(u_n, c_n, B_1) \le C, \qquad Dev(u_n, c_n, B_1) \to 0.$$

Assume that for every $2 \le i \le k$

$$\lim_n |\{u_i^n \neq 0\}| = 0.$$

Then up to a subsequence

$$u_1^n - m_n \to u_1 \in H^1(B_1) \qquad a.e. \text{ in } B_1,$$

where m_n is a median of u_1^n, and u_1 is a local minimizer for the Dirichlet energy with

$$\int_{B_\rho} |\nabla u_1|^2 \, dx = \lim_n \Phi(u_n, c_n, B_\rho)$$

for every $0 \le \rho < 1$.

Proof. The proof is a variant of that of Proposition 4.5. We need to employ a median for u_1^n since we have no control on the size of its zero set.

By Helly's theorem we may assume that up to a subsequence for every $\rho \in [0,1]$

$$\lim_n \Phi(u_n, c_n, B_\rho) = \alpha(\rho),$$

where $\alpha : [0,1] \to [0,+\infty[$ is non decreasing. Moreover we may assume $c_n \to c_\infty \in [0,+\infty]$.

We divide the proof in several steps.

Step 1: Truncation for $(u_2^n, \dots, u_k^n)$. We can repeat Step 1 and Step 2 of the proof of Proposition 4.5 working on $(u_2^n, \dots, u_k^n)$ for which we know that the corresponding zero set is converging in measure the entire B_1. We infer that

$$\bar{u}_n := (u_1^n, \bar{u}_2^n, \dots, \bar{u}_k^n)$$

is such that for a.e. $\rho \in [0,1[$

$$\Phi(\bar{u}_n, c_n, B_\rho) \to \alpha(\rho) \tag{4.21}$$

and

$$Dev(\bar{u}_n, c_n, B_\rho) \to 0. \tag{4.22}$$

Here $\bar{u}_i^n := (u_n^i)^+ \wedge \tau_{i,+}^n - (u_n^i)^- \wedge \tau_{i,-}^n$ with $\tau_{i,\pm}^n > 0$,

$$\bar{u}_i^n \to 0 \qquad \text{strongly in } L^2(B_1)$$

and

$$|\{u_i^n \neq \bar{u}_i^n\}| \leq c_d \left(\mathcal{H}^{d-1}(J_{u_i^n})\right)^{d/d-1} \to 0.$$

Step 2: Local minimality for u_1^n. Let us fix $\rho \in [0,1[$ satisfying (4.21), (4.22) and such that α is continuous at ρ. Let $v_n \in SBV(B_1)$ be such that

$$\{v_n \neq u_1^n\} \subset\subset B_\rho.$$

Let us consider $\rho' > \rho$ satisfying (4.21), (4.22), and let us compare $\bar{u}_n$ with $(v_n, \varphi\bar{u}_2^n, \dots, \varphi\bar{u}_k^n)$, where φ is a smooth cut-off function such that $\varphi = 0$ on B_ρ, $\varphi = 1$ on $B_1 \setminus B_{\rho'}$, and $|\nabla\varphi| \leq 2/(\rho'-\rho)$. We get

$$\begin{aligned}\Phi(\bar{u}_n, c_n, B_{\rho'}) \quad &\leq \int_{B_{\rho'}} |\nabla v_n|^2\,dx + 2\sum_{i=2}^k \int_{B_{\rho'}} [\varphi^2|\nabla\bar{u}_i^n|^2 + (\bar{u}_i^n)^2|\nabla\varphi|^2]\,dx \\ &\quad + c_n\mathcal{H}^{d-1}(J_{v_n} \cap B_{\rho'}) + c_n\sum_{i=2}^k \mathcal{H}^{d-1}(J_{\varphi\bar{u}_i^n} \cap B_{\rho'}) + Dev(\bar{u}_n, c_n, B_{\rho'}),\end{aligned}$$

so that

$$\Phi(\bar{u}_n, c_n, B_\rho) \leq \int_{B_\rho} |\nabla v_n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + e_n(\rho, \rho'), \tag{4.23}$$

where

$$e_n(\rho,\rho') := 2\Phi(\bar{u}_n, c_n, B_{\rho'} \setminus B_\rho) + \frac{8}{(\rho'-\rho)^2}\sum_{i=2}^k \int_{B_1} |\bar{u}_i^n|^2\,dx + Dev(\bar{u}_n, c_n, B_{\rho'}).$$

In particular we may write

$$\int_{B_\rho} |\nabla u_1^n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \leq \int_{B_\rho} |\nabla v_n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + e_n(\rho,\rho').$$

Notice that in view of (4.21) and (4.22) we have

$$\limsup_n e_n(\rho,\rho') \leq 2[\alpha(\rho') - \alpha(\rho)].$$

Choosing ρ' of the form $\rho' := (1+a_n)\rho$ with a suitable $a_n > 0$, we deduce that for a.e. $\rho \in [0,1[$

$$\int_{B_\rho} |\nabla u_1^n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \leq \int_{B_\rho} |\nabla v_n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{v_n} \cap B_\rho) + \hat{e}_n(\rho)$$

with $\hat{e}_n(\rho) \to 0$ and (choose $v_n = u_1^n$ in (4.23))

$$\int_{B_\rho} |\nabla u_1^n|^2\,dx + c_n\mathcal{H}^{d-1}(J_{u_1^n} \cap B_\rho) \to \alpha(\rho).$$

Step 3: Conclusion. In view of Step 2, the function u_1^n enjoys a local minimality property for the Mumford Shah energy with constant c_n which is independent of the other phases $(u_2^n, \dots, u_k^n)$. We are thus in the classical setting of [1, Theorem 7.7]: truncating (from above and below) and translating with a median m_n, we get the convergence almost everywhere to some function $u \in H^1(B_1)$ which is a local minimizer for the Dirichlet energy. The proof is thus concluded. □

4.3. **Proof of the decay estimate.** If $1/2 \le \tau < 1$, the result follows by choosing $C_d \ge 2^d$. Let us thus consider the case $0 < \tau < 1/2$, and let $C_d > 0$ to be fixed below.

Assume by contradiction that there exist $\varepsilon_n, \vartheta_n \to 0$, $B_{\rho_n}(x_n) \subset \Omega$ and $u_n = (u_1^n, \dots, u_k^n) \in \mathcal{U}(\Omega)$ such that

$$\sum_{i=1}^k \mathcal{H}^{d-1}(J_{u_i^n} \cap B_{\rho_n}(x_n)) = \varepsilon_n \rho_n^{d-1}, \qquad Dev(u_n, B_{\rho_n}(x_n)) = \vartheta_n \Phi(u_n, B_{\rho_n}(x_n)),$$

and

$$\Phi(u_n, B_{\tau\rho_n}(x_n)) > C_d \tau^d \Phi(u_n, B_{\rho_n}(x_n)).$$

Setting

$$v_n(y) := \sqrt{\frac{c_n}{\rho_n}} u_n(x_n + \rho_n y), \qquad y \in B_1, \qquad c_n := \frac{\rho_n^{d-1}}{\Phi(u_n, B_{\rho_n}(x_n))},$$

we obtain $v_n = (v_1^n, \dots, v_k^n) \in \mathcal{U}(B_1)$ with

$$\Phi(v_n, c_n, B_1) = 1, \qquad \sum_{i=1}^k \mathcal{H}^{d-1}(J_{v_i^n}) = \varepsilon_n, \qquad Dev(v_n, c_n, B_1) = \vartheta_n$$

and

$$\Phi(v_n, c_n, B_\tau) > C_d \tau^d. \tag{4.24}$$

If the phases vary according to Proposition 4.5, up to a subsequence we have

$$(v_1^n, \dots, v_m^n) \to (v_1, \dots, v_m) \in H^1(B_1; \mathbb{R}^m) \qquad \text{a.e. in } B_1$$

with

$$\sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx = \lim_n \Phi(v_n, c_n, B_\rho) \le 1$$

for every $\rho \in [0, 1[$, where $(v_1, \dots, v_m)$ is a local multiphase minimizer of the Dirichlet energy. Since in view of Proposition 4.3

$$\sum_{i=1}^m |\nabla v_i|^2 \text{ is subharmonic on } B_1,$$

we deduce that for every $\rho \in [0, 1/2]$

$$\frac{1}{\omega_d \rho^d} \sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \le \sum_{i=1}^m \int_{B_1} |\nabla v_i|^2 \, dx \le 1$$

so that

$$\sum_{i=1}^m \int_{B_\rho} |\nabla v_i|^2 \, dx \le \omega_d \rho^d.$$

Passing to the limit in (4.24) we get

$$\sum_{i=1}^m \int_{B_\tau} |\nabla v_i|^2 \, dx \ge C_d \tau^d$$

which yields a contradiction if we choose $C_d > \omega_d$.

If the phases vary according to Proposition 4.6, up to a subsequence we have

$$u_1^n - m_n \to u_1 \in H^1(B_1) \qquad \text{a.e. in } B_1,$$

with

$$\int_{B_\rho} |\nabla u_1|^2 \, dx = \lim_n \Phi(v_n, c_n, B_\rho) \le 1$$

for every $\rho \in [0,1[$, where m_n is a median of u_1^n, and u_1 is a local minimizer for the Dirichlet energy. This means that u_1 is harmonic in B_1, so that the function $|\nabla u_1|^2$ is subharmonic in B_1. We can thus adapt the previous arguments to get again a contradiction provided that $C_d > \omega_d$.

□

5. Proof of Theorem 1.2

In this section we finally prove our main result. Let $u = (u_1, \dots u_k) \in \mathcal{U}(\Omega)$ be a local almost-quasi minimizer of a multiphase free discontinuity problem at every point $x \in \Omega$.

We proceed in three steps.

Step 1: Hölder continuity of u and openness of the phases. We denote by $J^r_{u_i}$ the set of regular points of J_{u_i} (namely points of density 1), and we set

$$C := \overline{\bigcup_i J^r_{u_i}}\,.$$

Since C is a closed set such that $\mathcal{H}^{d-1}(J_{u_h} \setminus C) = 0$ for every $h = 1, \dots, k$, we deduce that $u_i \in H^1_{loc}(\Omega \setminus C)$. Moreover, thanks to the local minimality, we get that, for every $B_\rho(x) \subset \Omega \setminus C$,

$$\int_{B_\rho(x)} |\nabla u_i|^2\,dx \le \Lambda d \omega_d \rho^{d-1} + c_\alpha \rho^{d-1+\alpha}.$$

By Poincaré inequality we infer

$$\int_{B_\rho(x)} |u_i - (u_i)_{x,\rho}|^2\,dx \le C_d \rho^2 \int_{B_\rho(x)} |\nabla u_i|^2\,dx \le C_d(\Lambda d\omega_d \rho^{d+1} + c_\alpha \rho^{d+1+\alpha}),$$

where $(u_i)_{x,\rho}$ denotes the integral mean of u_i on $B_\rho(x)$. Thanks to Campanato's criterion (see e.g. [1, Theorem 7.51]), we deduce that u_i is locally Hölder continuous (with exponent 1/2) in $\Omega \setminus C$. In particular we obtain $J_{u_i} \setminus C = \emptyset$, so that

$$\bigcup_i J_{u_i} \subseteq C \subseteq \overline{\bigcup_i J_{u_i}},$$

which entails (C is closed)

$$\overline{\bigcup_i J_{u_i}} = C. \tag{5.1}$$

From the local Hölder continuity of u in $\Omega \setminus \overline{\bigcup_i J_{u_i}}$, we infer that the phases $\Omega_i := \{x \in \Omega \setminus \overline{\bigcup_j J_{u_j}} : u_i(x) \neq 0\}$ are open sets, whose boundary is composed either of jump points of u_i or of regular points for which $u_i = 0$.

Step 2: Essential closedness of the union of jump sets. In view of (5.1), in order to obtain the essential closedness of $\bigcup_i J_{u_i}$, it is enough to show that

$$\mathcal{H}^{d-1}\Big(\overline{\bigcup_i J^r_{u_i}} \setminus \bigcup_i J_{u_i}\Big) = 0\,. \tag{5.2}$$

Let $x \in \bigcup_i J^r_{u_i}$, and let $F_u(\rho)$ be defined as in Theorem 3.1. By Theorem 3.1, comparing $F_u(\rho)$ with its behaviour as $\rho \to 0^+$, we obtain, for $\rho \in (0, dist(x, \partial\Omega))$

$$\left[\frac{1}{\rho^{d-1}} \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\,dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x))\right)\right] + (d-1)\frac{c_\alpha}{\alpha}\rho^\alpha \ge \omega_{d-1} \wedge \frac{c_d \Lambda^{2-d}}{d-1}\,.$$

Hence there exists $\rho_0 > 0$ and $C_0 > 0$ (independent of x) such that for every $\rho < dist(x, \partial\Omega) \wedge \rho_0$

$$\sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\,dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x))\right) \ge C_0 \rho^{d-1}. \tag{5.3}$$

By continuity, we can extend the above inequality to points $\overline{x} \in \overline{\bigcup_i J^r_{u_i}}$.

Now we recall the well-known fact that, since $u_i \in SBV(\Omega)$, it holds

$$\lim_{\rho\to 0^+} \rho^{1-d} \sum_{i=1}^k \Big(\int_{B_\rho(y)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(y)) \Big) = 0$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \Omega \setminus \bigcup_i J_{u_i}$ (see for instance [17, Theorem 3.6]). In view of this fact, the validity of inequality (5.3) for points $\overline{x} \in \overline{\bigcup_i J^r_{u_i}}$ tells us that

$$\overline{\bigcup_i J^r_{u_i}} \subseteq \bigcup_i J_{u_i} \cup A\,,$$

for some set A with $\mathcal{H}^{d-1}(A) = 0$, which implies (5.2).

Step 3: Ahlfors regularity of the union of jump sets. Firstly observe that there exist $c' > 0$ and $\rho'_0 > 0$ such that, for every $x \in \bigcup_i J_{u_i}$ and every $B_\rho(x) \subset \Omega$ with $\rho < \rho'_0$, it holds

$$c'\rho^{d-1} \le \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap \overline{B}_\rho(x)) \right) \le \frac{1}{c'}\rho^{d-1}\,.$$

Namely, the estimate from above follows immediately from the almost-quasi minimality of u, whereas the estimate from below follows by arguing as in Step 2 (*cf.* (5.1) and (5.3)). A simple monotonicity by inclusion argument yields also

$$c'\rho^{d-1} \le \sum_{i=1}^k \left(\int_{B_\rho(x)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap B_\rho(x)) \right) \le \frac{1}{c'}\rho^{d-1}\,. \tag{5.4}$$

As a consequence of (5.4), we immediately obtain the upper bound inequality in (1.5). In order to show the lower bound inequality in (1.5), we argue by contradiction. Assume there exist sequences $\{x_n\} \subset \bigcup_i J_{u_i}$, $\rho_n \to 0$, $c_n \to 0$ such that

$$\mathcal{H}^{d-1}\Big(\bigcup_i J_{u_i} \cap B_{\rho_n}(x_n) \Big) \le c_n \rho_n^{d-1}$$

and hence

$$\sum_{i=1}^k \mathcal{H}^{d-1}\Big(J_{u_i} \cap B_{\rho_n}(x_n) \Big) \le k c_n \rho_n^{d-1}\,. \tag{5.5}$$

In view of (5.4), this implies that there exists a sequence $\varepsilon_n \to 0$ such that

$$\sum_{i=1}^k \mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho_n}(x_n)) \le \varepsilon_n \sum_{i=1}^k \int_{B_{\rho_n}(x_n)} |\nabla u_i|^2\, dx \le \frac{\varepsilon_n}{c'}\rho_n^{d-1}.$$

Using the above inequality and the almost-quasi minimality of u, we get, for every $(v_1, \dots, v_k) \in \mathcal{U}(\Omega)$ such that $\bigcup_i \{v_i \ne u_i\} \subset\subset B_{\rho_n}(x_n)$

$$\begin{aligned}
&\sum_{i=1}^k \Big(\int_{B_{\rho_n}(x_n)} |\nabla u_i|^2\, dx + \Lambda \mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho_n}(x_n)) \Big) \\
&\quad = \sum_{i=1}^k \Big(\int_{B_{\rho_n}(x_n)} |\nabla u_i|^2\, dx + \mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho_n}(x_n)) \Big) + \sum_{i=1}^k (\Lambda - 1)\mathcal{H}^{d-1}(J_{u_i} \cap B_{\rho_n}(x_n)) \Big) \\
&\qquad \le \sum_{i=1}^k \Big(\int_{B_{\rho_n}(x_n)} |\nabla v_i|^2\, dx + \Lambda \mathcal{H}^{d-1}(J_{v_i} \cap B_{\rho_n}(x_n)) \Big) + c_\alpha \rho_n^{d-1+\alpha} + (\Lambda - 1)\frac{\varepsilon_n}{c'}\rho_n^{d-1}.
\end{aligned}$$

Setting

$$\Phi(u,\Lambda,A):=\sum_{i=1}^{k}\left[\int_A|\nabla u_i|^2\,dx+\Lambda\mathcal{H}^{d-1}(J_{u_i}\cap A)\right],$$

the previous inequality reads

$$\Phi(u,\Lambda,B_{\rho_n}(x_n))\le\Phi(v,\Lambda,B_{\rho_n}(x_n))+c_\alpha\rho_n^{d-1+\alpha}+(\Lambda-1)\frac{\varepsilon_n}{c'}\rho_n^{d-1},$$

which means that the associated deviation from minimality (with coefficient Λ) of u in $B_{\rho_n}(x_n)$ (see (4.1)) satisfies

$$Dev(u,\Lambda,B_{\rho_n}(x_n))\le\left[(\Lambda-1)\frac{\varepsilon_n}{c'}+c_\alpha\rho_n^\alpha\right]\rho_n^{d-1}\,.$$

Recalling (5.4), this implies

$$Dev(u,\Lambda,B_{\rho_n}(x_n))\le\theta_n\Phi(u,\Lambda,B_{\rho_n}(x_n))\,,\tag{5.6}$$

for an infinitesimal sequence θ_n. By (5.5) and (5.6), applying Theorem 4.1 (Λ is fixed), for every $\tau\in(0,1)$ and n large enough, we obtain that

$$\Phi(u,\Lambda,B_{\tau\rho_n}(x_n))\le C_d\tau^d\Phi(u,\Lambda,B_{\rho_n}(x_n))\,.$$

This contradicts the energy estimate (5.4) as soon as τ is chosen small enough.

□

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