

# Piola's approach to the equilibrium problem for bodies with second gradient energies. Part I: first gradient theory and differential geometry.

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*Dedicated to the 90<sup>th</sup> birthday of Professor Giulio Maier.*

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**Abstract** In this study some pioneering contributions, envisaged in the works of Gabrio Piola, were developed through tools of the modern differential geometry and applied to the second gradient continua. Part I introduced the variational approach for the equilibrium problem according to the first gradient theory and exploited differential geometric perspectives for the present scenario. By prescribing the stationarity of the Lagrangian energy functional, the virtual work equations for a Cauchy's medium were recovered. The focus was on the deformation process regarded as a diffeomorphism between Riemannian embedded submanifolds, emphasizing the roles of the pullback metrics and of the covariant differentiation. Novel transport formulae were provided for normal and tangent vectors in the neighborhood of a boundary edge. The divergence theorem for curved surfaces with border was revisited, providing remarkable relationships between Lagrangian and Eulerian expressions involving projectors.

**Keywords** Continuum mechanics · Second Gradient material · Differential Geometry · Piola transformation · Lagrangian and Eulerian formulation

## 1 Introduction

The second gradient materials, namely materials whose constitutive equations involve the second derivatives of the placement map, represent since a couple of decades a growing area of interest in theoretical and applied mechanics. The attractiveness of the second gradient modelling is motivated basically by its capability to detect a localized mechanical response, possibly affected by size dependence, without losing the long range modes, in all those scenarios in which the separation of scales is not so sharp. Thanks also to the progress in experimental mechanics,

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the paradigm related to Cauchy has revealed its constitutive limitations: nowadays it can be no more presented as the unique valid choice for any material at each observation scale. In fact, the first gradient theory rests on the assumption that the characteristic length scale is negligible with respect to the representative volume element [1]: accordingly, the properties averaged over any representative element must be statistically independent from its location (homogeneity assumption). Moreover, the contact pressures must depend only on the point and on the outward normal (Cauchy's postulate). These assumptions are not necessarily true when the structure size becomes comparable to the characteristic length scale, as in the case of MEMS and NEMS, or also of the so called metamaterials. In fact, the acoustic and mechanical properties of the metamaterials, which do not exist in nature, see e.g. [2–7], rest on a properly designed microstructure, constituted of cells or of a lattice with large values of the surface-to-volume ratio: they frequently exhibit surface and corner effects markedly distinct from the bulk behavior. High frequency or short wavelength perturbations (see e.g. [8]) brings out dramatic discrepancies with respect to the conventional behavior.

For the above typologies of materials and structures, the spatial interactions and a microstructural length scale are needed to describe effectively the mechanical behavior, analogously to the nonlocal approaches [9–11]. The constitutive response should include what happens in a sufficiently small neighborhood of the point: therefore, the contact pressures may depend not only on the normal vector but also on other features of the local geometry, such as, for instance, the curvature of the face [12]. As pointed out in [13], a not trivial task is to correlate the involved length scale parameters with a specific microstructure. Even when an equivalent continuum at the macroscale is derived through the homogenization of a three dimensional Cauchy elastic medium, the second gradient models represent always a richer limit: from the mathematical standpoint, in fact, they constitute the closure in the sense of Gamma-convergence for any possible homogenized elastic behaviour [14]. Recently, variational strategies have been proposed for the second gradient modelling of periodic heterogeneous materials, see e.g. [10,15].

As deeply discussed in [16–18,14], when the deformation energy depends on the second gradient of the placement map, the Cauchy stress tensor alone is no more capable of governing the internal work. This circumstance can be understood from the fact that, within a dual perspective, the materials equipped only with the Cauchy stress tensor cannot sustain external forces concentrated along any edges of the boundary, or double forces distributed over the outer faces. The peculiar features of the second gradient materials require new ingredients, like a third rank hyperstress tensor, and supplementary boundary conditions, which at a first sight may seem less intuitive. Thanks to this enrichment, the second gradient models can deal with specific loading conditions in crack and dislocation problems (see e.g. [19,20]) where the first gradient theory predicted singularities. By including length scale parameters, they are capable of describing the size dependence, surface and corner effects, and thus can compete, for instance, with the atomistic simulations of a crystal lattice in view of the first principles calculations [2]. In the recent literature, the second gradient theories were effectively applied to fibrous composite reinforcements, constituted at the mesoscale of woven yarns [21,22], and were able to describe the onset of the boundary layers, namely the transition zones between two regions with diverse deformation modes, see e.g. [23]. The macroscopic response of pantographic structures, special case of a network of

beam, was investigated from both the analytical and the numerical standpoint. Within the framework of the second gradient continua, mechanical models with damage, plasticity or viscosity were developed and applied to the study of the granular and porous materials, even biological such as bone [24–26]. In addition, the second gradient theory offered interesting perspectives to improve the modelling of the surface tension and adhesion for the capillary fluids (see e.g. [17]), especially in the presence of interfaces and interphases<sup>1</sup>.

The present study can be regarded as a development of [27], where a framework for a fully variational derivation of the second gradient theory by an action functional was established. In that paper, much attention was devoted to the capillary fluids, but the boundary terms were neither developed in detail nor transported. In his studies on the second gradient models [16, 17], Mindlin made exclusive reference to the linearized strains. Surprisingly also Germain in [28] preferred the Eulerian approach. Moreover, differently from [29], focusing on the existence and uniqueness criteria when reconstructing the manifolds from their fundamental forms, and [30], dealing with the elasticity problems over Riemannian manifolds, herein the differential geometric structure of the problem was exploited in view of the transformation of the second gradient equations from the Eulerian to the Lagrangian form.

The content of this study, divided into two Parts, aimed at enabling the readers to a fully understanding of advanced mathematical perspectives as a prerequisite for the novel mechanical developments. As for the present Part I, in Sections 2 and 3 kinematics for a deformable continuum was outlined, and the first gradient solution for the equilibrium problem was recalled through a variational approach. In Section 4 an accurate geometrical description of the boundary faces and of their border edges was provided. Section 5 introduced a differential geometric interpretation of the placement map as a diffeomorphism between manifolds, at the light of which the pullback metrics and the covariant derivative were discussed. In Section 6 novel transport formulae were provided for the moving frame vectors in the neighborhood of a border edge. In Section 7 the divergence theorem for Riemannian embedded submanifolds was revisited, providing useful transformation formulae between the Lagrangian and the Eulerian configurations. Finally, an Appendix highlighted the main properties of the surface projectors and provided for them novel transport formulae. Part II was entirely devoted to the variational derivation of the second gradient equations and to their transport from the Eulerian to the Lagrangian configuration. Detailed expressions of virtual work contributions were provided, in terms of volume, face and edge terms. The transformation of the governing equations revealed an unexpected coupling of terms, transversely to the involved domains. Preliminary results were announced in [2].

**Notation:** In what follows recourse will be made to index, componentwise notation for the involved equations, although sometimes the relevant matrix or tensorial expressions will be reported for the reader’s convenience, see [31]. Classical syntaxis of the tensor algebra will be adopted, in agreement with Levi-Civita and Ricci-Curbastro absolute calculus (see [32, 33]) and with the Einstein conven-

<sup>1</sup> In materials science the sharp surface between two homogeneous phases (or interface) is usually distinguished from an intermediate phase (or interphase), possibly existing within a system as the transition zone between two phases

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tion on the implicit sum of repeated indices. As far as possible, the Lagrangian quantities will be denoted by uppercase symbols, and their Eulerian counterparts by lowercase symbols. In the tensor calculus, to distinguish the valences acting on Lagrangian vectors from those specifying Eulerian spaces, e.g. as in  $F_A^a$ , the former will be indicated by uppercase letters, i.e.  $A, B, \dots$ , the latter by lowercase ones,  $a, b, \dots$ . The symbols will be specified along the manuscript.

## 2 Kinematics and Piola's tinted glass

<sup>3</sup> Gabrio Piola's formulations in continuum mechanics, as outlined especially in his works [34,35] with a language not far from our scientific context but still lacking of the matrix formalism, started from a Lagrangian prejudice. He was convinced that, as already established in analytical mechanics for the discrete systems, also for a continuous medium the mechanical models must rest on a rigorous kinematical description. By means of it the response of the entire body can be controlled and the solution can be straightforwardly achieved through the calculus of variations (see also [36–39]). Kinematics represented for Piola as the green tinted glass mentioned by Immanuel Kant in “The critique of pure reasoning” (1781), founding his *Weltanschauung*.

We observe in everyday life the deformation of several bodies, such as the structural elements, which experience a change of place, for instance during the transport and the building phases, and even vary in their shape along a time sequence, e.g. when subjected to external loading. Hence it is natural considering two connected spatial regions occupied by the same deformable body, the reference (not necessarily the initial, at instant  $t_0$ ) and the current configuration at instant  $t$ , denoted by  $\Omega_*$  and  $\Omega \subseteq \mathcal{R}^3$ , respectively, and comparing them. The above configurations are often referred to as Lagrangian (or material), and Eulerian (or spatial), respectively, and can be dealt with as three dimensional differential manifolds with boundary. Symbol  $\mathcal{R}^3$  denotes herein the three dimensional Euclidean space with a reference frame, which naturally can be identified with the underlying vector space, and is the ambient space of each volume. Through an atlas of local homeomorphisms (or coordinate “charts”), with differentiable transition functions in the regions of mutual overlap, the structure of  $\mathcal{R}^3$  is transferred *par morceaux* (piecewise) to the specific manifold. Herein the domain and the image space of such charts coincide with subsets of the ambient space. Both the configurations are equipped at each point with coordinate lines, whose tangents constitute the basis vectors. In particular, two distinct origins and orthonormal bases are adopted in the ambient space, namely  $\{O, \mathbf{e}_A\}$  and  $\{o, \mathbf{e}_a\}$  ( $A, a = 1, 2, 3$ ), in turn for the reference and the current domain. In what follows points and vectors will be denoted by the same symbols: point coordinates, referred to the origin, will coincide with the position vector components. Then, a one-to-one mathematical correspondence covering the entire domains, i.e. a surjection, is naturally specified, relating points  $\mathbf{P}$  and  $\mathbf{p}$  belonging to the above volumes, usually referred to as *placement* map. By formulae

$$\chi : \mathbf{P} \in \Omega_* \subseteq \mathcal{R}^3 \rightarrow \mathbf{p} \in \Omega \subseteq \mathcal{R}^3 \quad (1)$$

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<sup>3</sup> Born in Milan, 15 July 1794; died in Giussano, 9 November 1850.

Each material point of the deformable continuum, along its trajectory, continues being labelled by the same set of Lagrangian coordinates. Position vectors corresponding through  $\chi$  can then be assigned the following components:  $\mathbf{P} = X^A \mathbf{e}_A$ , in  $\Omega_*$ ;  $\mathbf{p} = x^a \mathbf{e}_a = \chi^a(X^A) \mathbf{e}_a$ , in  $\Omega$ . Thus  $\chi^i(\mathbf{X})$  ( $i = 1, 2, 3$ ) represents an Eulerian vector, functionally dependent on the Lagrangian, material coordinates in the reference configuration. Of course the vector-valued map  $\chi(\mathbf{X})$  must satisfy some basic regularity conditions. Each component of the placement map must be continuous up to its first order (weak) derivatives. The Sobolev space  $[\mathcal{H}^1(\Omega_*)]^3$  turns out to be a suitable framework for the present purposes, see e.g. [40, 30]. Moreover, to guarantee the placement map to be one-to-one, its Jacobian matrix, referred to as deformation gradient,  $F_A^a = \frac{\partial \chi^a}{\partial X^A}$ , must be nonsingular. In particular, to prevent the interpenetration of matter, a nonnegative determinant is prescribed, namely  $J = \det(\mathbf{F}) > 0$ . The two conditions specified above imply that such a vector map is invertible, with a continuous inverse. It represents therefore what modern differential geometry calls *diffeomorphism*, i.e. a map continuous and differentiable acting between two differential manifolds, its inverse also being continuous and differentiable, apt to share not only the coordinates, but also their topological and differential structures [41]. Deformation gradient  $\mathbf{F}(\mathbf{X})$  represents the tangent map of such a diffeomorphism, transforming tangent vectors of the reference configuration into tangent vectors of the current configuration. Hence,  $F_A^a$  is a mixed contravariant covariant second rank tensor, and can be thought as an exotic object with two legs, one in the Lagrangian configuration and the other one in the Eulerian configuration, see e.g. [42, 43]. Once transferred the structure of  $\mathcal{R}^3$  to such three dimensional manifolds, by utilizing the usual inner product in the ambient space one can specify the metric tensors for the reference and the spatial configuration, in turn  $g_{AB} = \langle \mathbf{e}_A, \mathbf{e}_B \rangle$  and  $g_{ab} = \langle \mathbf{e}_a, \mathbf{e}_b \rangle$ , herein equal to the unit tensors. In particular, this choice implies null Christoffel symbols everywhere, see e.g. [44]. Symbol  $\langle \mathbf{a}, \mathbf{b} \rangle_g = g_{rs} a^r b^s$  denotes the inner product according to the metrics  $\mathbf{g}$ . It is worth emphasizing that both the Lagrangian and the Eulerian configurations are embedded in a flat ambient space, in which the Riemann-Christoffel curvature tensor identically vanishes [45, 46, 41]. At a first sight, the assumption of an orthonormal basis with a unit metric tensor may seem over restrictive. On the contrary, among the different possibilities one can select for the metric tensor, i.e. unit, constant or varying (in a climax of increasing generality), this choice brings out more clearly the features naturally emanating from the deformation process as a diffeomorphism in view of the equation transport.

The polar decomposition theorem, see e.g. [47], allows one to express uniquely any deformation gradient with positive determinant as the product of a proper orthogonal tensor and a symmetric, positive-definite tensor, namely

$$F_C^a = g^{BA} R_A^a U_{BC} \quad (2)$$

where only the left decomposition was considered. Thus, a quota of tangent map usually corresponds to isometries, namely rigid body motions represented by an orthogonal transformation (forming the group  $SO(3)$ ), which fulfills the conditions  $g_{ab} R_A^a R_B^b = \delta_{AB}$  and  $\det(\mathbf{R}) = +1$ . When the doubly covariant tensor  $C_{AB} = g_{ab} F_A^a F_B^b = g^{DC} U_{DA} U_{CB}$ , referred to as right Cauchy-Green tensor, locally equals  $\delta_{AB}$ , and the (right or material) stretch tensor with it,  $U_{AB} = \delta_{AB}$ , deformation gradient reduces to an isometry,  $F_A^a = R_A^a$ . In this case the material line elements can be recognized in the current configuration without any variation of length or

of their mutual angle. A deformation measure, which must vanish in the presence of rigid body motions, arises naturally by comparing the metric tensors in the two configurations, namely

$$E_{AB} \equiv \frac{1}{2} \left( g_{ab} F_A^a F_B^b - \delta_{AB} \right) = \frac{1}{2} (g_{AB}^* - g_{AB}) \quad (3)$$

The second rank, doubly covariant tensor  $E_{AB}$ , referred to as Green-Lagrange-Saint Venant finite strain tensor, can be expressed, up to the factor  $\frac{1}{2}$ , as the difference between the Eulerian metric tensor transported to the Lagrangian configuration (in the jargon *pulledback*), and its counterpart in the reference configuration (coincident with  $\delta_{AB}$  for an orthonormal basis). The important role played by the pullback metrics will be widely discussed in both the parts of this study.

### 3 In the beginning was the energy

Once specified the infinite dimensional kinematics of a continuous body, Piola introduced an energy functional, apt to measure by a scalar value any variation of configuration in terms of stored energy, say  $\mathcal{E}^{\text{DEF}}$ . The real-valued functional  $\mathcal{E}^{\text{DEF}}$  is expressed by an integral over the reference configuration  $\Omega_*$  of a scalar energy density that we assume in the form  $W(\mathbf{X}, \mathbf{F}(\mathbf{X}))$ . Such an energy density must be nonnegative and objective. The objectivity (in the sense of material frame indifference) implies the invariance under any orthogonal transformation  $\mathbf{R}$  ( $\in \text{Orth}^+$ ): regarding the deformation gradient as an Eulerian vector, one obtains the condition  $W(\mathbf{R}\mathbf{F}) = W(\mathbf{F})$ . By formulae one has:

$$\mathcal{E}_I^{\text{DEF}} = \int_{\Omega_*} W(\mathbf{X}, \mathbf{F}(\mathbf{X})) d\Omega_* \quad (4)$$

where the subscript  $I$  emphasizes the dependence on the the first gradient only.

When seeking for an equilibrium configuration, the above contribution  $\mathcal{E}_I^{\text{DEF}}$  in terms of the stored energy must compete with the work performed by the external loading, say  $\mathcal{E}^{\text{EXT}}(\mathbf{X}, \boldsymbol{\chi}(\mathbf{X}))$ , which in turn may depend on a potential. In practice, Piola utilized the total potential energy before Menabrea and Castigliano. The equilibrium configuration of a given body under the prescribed loading corresponds to the deformation which minimizes its total energy functional, assumed that such a minimum exists and is unique. By symbols

$$\hat{\boldsymbol{\chi}} = \arg \min_{\boldsymbol{\chi} \in \mathcal{K}} \left\{ \mathcal{E}^{\text{TOT}}(\boldsymbol{\chi}) = \mathcal{E}_I^{\text{DEF}} - \mathcal{E}^{\text{EXT}} \right\} \quad (5)$$

where the feasible functional subset  $\mathcal{K} \subset [\mathcal{H}^1(\Omega_*)]^3$  must incorporate the essential boundary conditions.

We prescribe the stationarity by setting null the first variation  $\delta$  of the functional Eq. (5). Through the additivity of the first variation, one obtains the equation of virtual work

$$\delta \mathcal{E}_I^{\text{DEF}} - \delta \mathcal{E}^{\text{EXT}} = 0 \quad (6)$$

From the above relationship, it is clear that not all the contributions of the external work are consistent with the energy modes allowed by the assumed kinematics with the specific energy density. To compute the variation of the first addend, recourse is made to the Fréchet apparatus, see e.g. [48]. One has

$$\delta \mathcal{E}_I^{\text{DEF}} = \int_{\Omega_\star} \underbrace{\{W(\mathbf{F} + \delta\mathbf{F}) - W(\mathbf{F})\}}_{=\delta W} d\Omega_\star \simeq \int_{\Omega_\star} \frac{\partial W}{\partial \mathbf{F}} : \delta \mathbf{F} d\Omega_\star \quad (7)$$

where the symbol  $:$  denotes the usual double dot product between second rank tensors [31]. By prescribing the dependence of the energy density on the right Cauchy-Green tensor  $\mathbf{C}$  (or on its inverse), the objectivity of the energy is guaranteed. Hence, the energy derivatives must be intended as follows

$$\frac{\partial W}{\partial F_A^a} = \frac{\partial W}{\partial C_{DE}} \frac{\partial C_{DE}}{\partial F_A^a}; \quad (8)$$

To develop the term  $\delta \mathbf{F}$  in Eq. (7), let us consider that the variation and the derivative operators can commute if  $\chi$  map and its derivatives remain small. This requirement is met when the placement map belongs to the Sobolev space  $[\mathcal{H}^1(\Omega_\star)]^3$ , which also ensures trace regularity over the boundary, see e.g. [40,30]. Then one has

$$\begin{aligned} \delta \mathcal{E}_I^{\text{DEF}} &= \int_{\Omega_\star} \frac{\partial W}{\partial F_A^i} \delta F_A^i d\Omega_\star = \int_{\Omega_\star} \frac{\partial W}{\partial F_A^i} \delta \left( \frac{\partial \chi^i}{\partial X^A} \right) d\Omega_\star = \\ &= \int_{\Omega_\star} \frac{\partial W}{\partial F_A^i} \frac{\partial}{\partial X^A} (\delta \chi^i) d\Omega_\star \end{aligned} \quad (9)$$

The variation of the placement map  $\delta \chi$  corresponds to elementary or *virtual* displacements, which must vanish over the boundary. In what follows the frontier of the reference domain  $\partial \Omega_\star$  will be denoted by symbol  $\Sigma_\star$ . Through integration by parts of Eq. (9), one finds

$$\begin{aligned} \delta \mathcal{E}_I^{\text{DEF}} &= \int_{\Omega_\star} \left\{ \frac{\partial}{\partial X^A} \left( \frac{\partial W}{\partial F_A^i} \delta \chi^i \right) - \frac{\partial}{\partial X^A} \left( \frac{\partial W}{\partial F_A^i} \right) \delta \chi^i \right\} d\Omega_\star = \\ &= \int_{\Sigma_\star} \frac{\partial W}{\partial F_A^i} \delta \chi^i N_A d\Sigma_\star - \int_{\Omega_\star} \frac{\partial}{\partial X^A} \left( \frac{\partial W}{\partial F_A^i} \right) \delta \chi^i d\Omega_\star \end{aligned} \quad (10)$$

For the first addend we utilized the Gauss-Ostrogradsky divergence theorem (see e.g. [42]), being  $\mathbf{N} = \{N_A\}$  the covariant component of the outward normal vector to the boundary in the Lagrangian configuration. Equation (9) is now represented as the sum of two addends, a volume and a surface integral, which cannot be further reduced since both include  $\delta \chi^i$  as test functions. Such an internal virtual work representation can be orderly set equal to the relevant contributions defining the external virtual work Eq. (6), namely

$$\delta \mathcal{E}^{\text{EXT}} = \int_{\Omega_\star} \mathcal{F}_{\Omega_\star^i}^{\text{ext}} \delta \chi^i d\Omega_\star + \int_{\Sigma_\star} \mathcal{F}_{\Sigma_\star^i}^{\text{ext}} \delta \chi^i d\Sigma_\star; \quad (11)$$

where symbols  $\mathcal{F}_{\Omega_\star^i}^{\text{ext}}(\mathbf{X})$  and  $\mathcal{F}_{\Sigma_\star^i}^{\text{ext}}(\mathbf{X})$  denote Eulerian vectors defined in the Lagrangian domain and over its boundary surface, dimensionally equal to force densities per unit volume and per unit surface, respectively. It is worth emphasizing

that, if the material boundary is constituted of piecewise regular *faces* with disjoint interior parts, the surface integral above can be straightforwardly split into a finite sum of integrals, each one extended to a single face. The border of each face, constituted of piecewise regular curves referred to as *edges*, the support of which is shared between two *faces* (see Section 4), represents for the surface integral a null measure subset, and has no effect on the evaluation of Lebesgue integrals. Accordingly, the external forces sustainable by a first gradient material are bulk and surface loading, and nothing else.

From Eqs. (10)-(11) one can derive the strong form of the equilibrium equations by selecting test functions  $\delta\chi$  with their compact support localized within the volume interior or over the boundary (fundamental lemma of the calculus of variations). Hence, one obtains

$$\frac{\partial}{\partial X^A} \left( \frac{\partial W}{\partial F_A^i} \right) + \mathcal{F}_{\Omega_\star^i}^{\text{ext}} = 0 \quad \mathbf{X} \in \mathring{\Omega}_\star; \quad \left( \frac{\partial W}{\partial F_A^i} \right) N_A - \mathcal{F}_{\Sigma_\star^i}^{\text{ext}} = 0 \quad \mathbf{X} \in \Sigma_\star; \quad (12)$$

It is worth noting that the stress-like Lagrangian tensor between parentheses, referred to as first Piola-Kirchhoff stress, borrows from  $F_A^i$  a hybrid, amphibian existence between the tangents spaces of two different manifolds. In fact, as resulting from Eq. (12), such a tensor, multiplied by the normal vector  $N_A$  of the material surface  $\Sigma_\star$ , provides the traction vector acting over the Eulerian face  $\Sigma$  (at varying the covariant index  $i$ ), see e.g. [49]. The equilibrium equations for Cauchy's first gradient continuum, presented so far, were expressed in the Lagrangian form. In principle, it is possible to transform them into the Eulerian form. If one multiplies the bulk equation in Eq. (12) by  $J^{-1}$ , transforms by the chain rule the Lagrangian partial derivative into an Eulerian gradient, and refers all the variables to  $\Omega$  by composition with the inverse placement map  $\chi^{-1}$ , one obtains (at varying  $i$ )

$$\begin{aligned} & - J^{-1} F_A^a \frac{\partial}{\partial x^a} \left( \frac{\partial W}{\partial F_A^i} \circ \chi^{-1} \right) - J^{-1} \mathcal{F}_{\Omega_\star^i}^{\text{ext}} \circ \chi^{-1} = \\ & = - \frac{\partial}{\partial x^a} \left( J^{-1} F_A^a \frac{\partial W}{\partial F_A^i} \circ \chi^{-1} \right) - \left( J^{-1} \mathcal{F}_{\Omega_\star^i}^{\text{ext}} \circ \chi^{-1} \right) = \\ & = - \frac{\partial}{\partial x^a} \left( T_i^a(\mathbf{x}) \right) - \mathcal{F}_{\Omega^i}^{\text{ext}}(\mathbf{x}) = 0; \end{aligned} \quad (13)$$

In the above equation the Piola's bulk transformation was utilized to shift the functional group  $J^{-1} F_A^a$  inside the derivative, being  $\partial(J^{-1} F_A^a)/\partial x^a = 0$ , see e.g. [42,50,27]. Symbol  $T_i^a(\mathbf{x})$  denotes a second rank Eulerian tensor (in mixed form), referred to as Cauchy stress, and  $\mathcal{F}_{\Omega^i}^{\text{ext}}(\mathbf{x}) = J^{-1} \mathcal{F}_{\Omega_\star^i}^{\text{ext}}$  indicates the Eulerian counterpart of the Lagrangian external loading per unit volume. When one tries to apply the same procedure to the equilibrium condition over  $\Sigma$  in Eq. (12), additional difficulties are met to transform the Lagrangian normal vector  $N_A$  into its Eulerian form (see Section 6). For instance, one could utilize the so called Nanson's formula (see e.g. [43]) to write

$$\left( \frac{\partial W}{\partial F_A^i} \right) J^{-1} F_A^a n^a \frac{d\Sigma}{d\Sigma_\star} - \mathcal{F}_{\Sigma_\star^i}^{\text{ext}} = 0 \quad (14)$$

where  $d\Sigma_\star/d\Sigma = \|J^{-1}F_A^a n^a\|$  denotes the ratio between the area elements in the Lagrangian and in the Eulerian configuration, respectively. Hence, the correct Eulerian form of the Lagrangian boundary condition in Eq. (12) turns out to be

$$\left(\frac{\partial W}{\partial F_A^i}\right) J^{-1}F_A^a n^a - J^{-1}\mathcal{F}_{\Sigma_\star}^{\text{ext}} \|F_A^a n^a\| = 0 \quad (15)$$

It should appear clear that, when one manipulates the strong form of the governing equations, it is extremely hard to verify the goodness of the procedure (excepted one knows in advance the result). The recourse to the weak integral form can be extremely beneficial.

#### 4 Edge anatomy

When considering the structural elements utilized in the engineering practice, one can notice that the material volumes are limited by the union of regular surfaces with border exhibiting disjoint interior parts. The curved frontier of these regular surfaces represents discontinuity loci for the normal vector, and include piecewise regular *edges*, separated by a finite number of vertices or *wedges*. *Wedges* in turn represents discontinuity points for the edge tangent vector. At least three concurring edges are needed to specify a *wedge*, point of intersection of at least three faces having (the support of) one edge in common two by two.

In the reference configuration, the frontier of our three dimensional volume  $\Omega_\star$  can be oriented by assigning a positive side for the normal vector field, usually outwards. We indicate the oriented boundary by  $\vec{\Sigma}_\star$ . If such a boundary results from the union of  $n_{\text{face}}$  regular surfaces, the individual faces must be oriented in a globally consistent fashion, namely

$$\partial\vec{\Omega}_\star = \vec{\Sigma}_\star = \bigcup_{f=1}^{n_{\text{face}}} \vec{\Sigma}_\star^{(f)} = \vec{\Sigma}_\star^{(1)} \cup \vec{\Sigma}_\star^{(2)} \cup \dots \cup \vec{\Sigma}_\star^{(n_{\text{face}})} \quad (16)$$

For each one of these  $n_{\text{face}}$  faces  $\Sigma_\star^{(f)}$ , run by index  $f$ , the outward normal vector will be denoted by symbol  $\mathbf{N}^{(f)}$ . The border of the  $(f)$ -th face must be oriented consistently with the face and its normal vector, for instance according to the right hand grip rule (the palm suggesting a counterclockwise rotation when the thumb points the positive normal direction). The border of the  $f$ -th face is constituted of  $n_{\text{edge}}^{(f)}$  edges. The  $e$ -th edge, positively oriented as belonging to the  $(f)$ -face, will be denoted by  $\vec{L}_\star^{(fe)}$ . Hence one can write

$$\partial\vec{\Sigma}_\star^{(f)} = \vec{L}_\star^{(f)} = \bigcup_{e=1}^{n_{\text{edges}}^{(f)}} \vec{L}_\star^{fe} = \vec{L}_\star^{(f1)} \cup \vec{L}_\star^{(f2)} \cup \dots \cup \vec{L}_\star^{(fn_{\text{edges}}^{(f)})} \quad (17)$$

The support  $L_\star^{(fe)}$ , indicated by the same symbol without the arrow, belongs simultaneously to the border of two contiguous faces, but in each of them the edge possesses a diverse orientation, specified by one tangent vector  $\mathbf{T}^{(fe)}$  or by its opposite. Besides  $\mathbf{N}^{(f)}$  and  $\mathbf{T}^{(ef)}$ , oriented according to the right hand rule, one

can consider a third vector orthogonal to both of them, say  $\mathbf{B}^{(fe)}$ , referred to as edge normal vector. Such a vector, normal to the edge  $L^{(fe)}$  and tangent to the face, can be generated by the cross product, namely  $\mathbf{B}^{(fe)} = \mathbf{T}^{(fe)} \wedge \mathbf{N}^{(f)}$ , thus it points outwards with respect to the face border. In conclusion,  $\mathbf{B}^{(fe)}$ ,  $\mathbf{T}^{(fe)}$ ,  $\mathbf{N}^{(f)}$  constitute a moving frame for the border edge of a boundary face, different from the Frenet-Serret frame of the edge when regarded as a solitary curve. Analogous considerations can be repeated for the Eulerian frame  $(\mathbf{b}, \mathbf{t}, \mathbf{n})$  in the current configuration.

At the end of this Section, we intend to specify a tensor field along the border edge which plays an important role for the second gradient equations. To this purpose, let us introduce the following symbols and conventions. Let the apex  $+$  (or  $-$ ) indicate the face at the left of each edge, when crossed in the positive direction  $\mathbf{T}^+$ , so that,  $\mathbf{B}^+$ ,  $\mathbf{T}^+$ ,  $\mathbf{N}^+$  constitute a right handed (counterclockwise) frame along the edge, regarded as belonging to the oriented face at its left. Analogously, the apex  $-$  indicates the face at the right of the edge, with relevant vectors. The tensor-valued function in point is defined as follows

$$\left( [N^+]_P [B^+]_Q + [N^-]_P [B^-]_Q \right) \quad (18)$$

Since  $\mathbf{T}^+ = -\mathbf{T}^-$ , making explicit the cross product by the Levi-Civita symbol  $\epsilon_{PRS}$  [31], one has

$$\begin{aligned} [N^+]_Q [B^+]_P + [N^-]_Q [B^-]_P &= [N^+]_Q (\mathbf{T}^+ \wedge [\mathbf{N}^+])_P + N^-_Q ([\mathbf{T}^-] \wedge [\mathbf{N}^-])_Q = \\ [N^+]_Q (\epsilon_{PRS} [T^+]^R [N^+]^S) + [N^-]_Q (\epsilon_{PR'S'} [T^-]^{R'} [N^-]^{S'}) &= \\ [N^+]_Q (\epsilon_{PRS} [T^+]^R [N^+]^S) - [N^-]_Q (\epsilon_{PR'S'} [T^+]^{R'} [N^-]^{S'}) &; \end{aligned} \quad (19)$$

The above sum Eq. (19) is symmetric with respect to the indices  $P$  and  $Q$ . Equivalently, if one permutes such indices and equals the relevant contributions,

$$\begin{aligned} [N^+]_Q (\epsilon_{PRS} [T^+]^R [N^+]^S) - [N^-]_Q (\epsilon_{PR'S'} [T^+]^{R'} [N^-]^{S'}) &= \\ = [N^+]_P (\epsilon_{QRS} [T^+]^R [N^+]^S) - [N^-]_P (\epsilon_{QR'S'} [T^+]^{R'} [N^-]^{S'}) &; \end{aligned} \quad (20)$$

a vanishing difference must be found, namely

$$\begin{aligned} [T^+]^R [N^+]^S \left( \epsilon_{RSP} [N^+]_Q - \epsilon_{RSQ} [N^+]_P \right) + \\ - [T^+]^{R'} [N^-]^{S'} \left( \epsilon_{R'S'P} [N^-]_Q - \epsilon_{R'S'Q} [N^-]_P \right) &= 0; \end{aligned} \quad (21)$$

Let us notice that the edge tangent vector  $\mathbf{T}^+$  can be expressed through the cross product of the normals belonging to the faces at the left and at the right, respectively (both orthogonal to  $\mathbf{T}^+$ ), as follows

$$\mathbf{T}^+ = \mathbf{N}^+ \wedge \mathbf{N}^-; \quad [T^+]^R = \epsilon^{RMV} [N^+]_M [N^-]_V; \quad (22)$$

Substituting Eq. (22) into Eq. (21), and recalling that  $\epsilon_{RSP} \epsilon^{RMV} = \delta_S^M \delta_P^V - \delta_P^M \delta_S^V$ , one finds

$$\begin{aligned}
& \left( \delta_S^M \delta_P^V - \delta_P^M \delta_S^V \right) [N^+]^S [N^+]_Q [N^+]_M [N^-]_V + \\
& - \left( \delta_S^M \delta_Q^V - \delta_Q^M \delta_S^V \right) [N^+]^S [N^+]_P [N^+]_M [N^-]_V + \\
& - \left( \delta_{S'}^M \delta_P^V - \delta_P^M \delta_{S'}^V \right) [N^-]^{S'} [N^-]_Q [N^+]_M [N^-]_V + \\
& + \left( \delta_{S'}^M \delta_Q^V - \delta_Q^M \delta_{S'}^V \right) [N^-]^{S'} [N^-]_P [N^+]_M [N^-]_V = 0; \quad (23)
\end{aligned}$$

Developing the above products we obtain

$$\begin{aligned}
& + [N^+]^S [N^+]_Q [N^+]_S [N^-]_P - [N^+]^S [N^+]_Q [N^+]_P [N^-]_S + \\
& - [N^+]^S [N^+]_P [N^+]_S [N^-]_Q + [N^+]^S [N^+]_P [N^+]_Q [N^-]_S + \\
& - [N^-]^{S'} [N^-]_Q [N^+]_{S'} [N^-]_P + [N^-]^{S'} [N^-]_Q [N^+]_P [N^-]_{S'} + \\
& + [N^-]^{S'} [N^-]_P [N^+]_{S'} [N^-]_Q - [N^-]^{S'} [N^-]_P [N^+]_Q [N^-]_{S'} = 0; \quad (24)
\end{aligned}$$

and hence, considering that  $[N^-]^{S'} [N^-]_{S'} = 1$  and  $[N^+]^{S'} [N^+]_{S'} = 1$ , finally we have

$$\begin{aligned}
& \underbrace{+[N^+]_Q [N^-]_P}_{\nearrow} - \underbrace{[N^+]^S [N^+]_Q [N^+]_P [N^-]_S}_{\searrow} + \\
& - \underbrace{[N^+]_P [N^-]_Q}_{\nwarrow} + \underbrace{[N^+]^S [N^+]_P [N^+]_Q [N^-]_S}_{\searrow} + \\
& - \underbrace{[N^-]^{S'} [N^-]_Q [N^+]_{S'} [N^-]_P}_{\swarrow} + \underbrace{[N^-]_Q [N^+]_P}_{\nwarrow} + \\
& + \underbrace{[N^-]^{S'} [N^-]_P [N^+]_{S'} [N^-]_Q}_{\swarrow} - \underbrace{[N^-]_P [N^+]_Q}_{\nearrow} = 0; \quad (25)
\end{aligned}$$

where the contributions equal opposite were marked underneath by the same symbol. Finally, let us remark that, to evaluate Eq. (18) along the edges, the choice of the edge positive orientation is irrelevant and in practice can be left to the user.

## 5 Pullback metrics and covariant derivatives

The transformation of the virtual work equations for a second gradient continuum from the Eulerian to the Lagrangian form (and vice versa), represents a rather complex task, which, as far as the Author knows, was tackled directly and solved entirely for the first time by the Author and his coworkers, see the footnote (1). However, in the present context such a task is expected not only to confirm the equations derived independently for the material and the spatial configuration, but also to reveal important differential geometric perspectives for the equilibrium problem in point. The ambient space  $\mathcal{R}^3$ , in which our body is embedded, is a flat

space, with an identically null Riemann-Christoffel curvature tensor, see e.g. [43, 41]. Since we adopted for the material and the current domains unit metric tensors ( $g_{AB}$  and  $g_{rs}$ , resp.), this choice implies Christoffel symbols identically vanishing, everywhere. These results represent a pillar of the absolute calculus as formalized by Levi-Civita and Ricci-Curbastro [32,33]. However, the deformation process of our body specifies a diffeomorphism between the Lagrangian and the Eulerian domains, associated with the placement map  $\chi^i(\mathbf{X})$ . Such a surjection, continuous and differentiable with its inverse, allows the topological and differential features be shared between the two configurations. Surprisingly, although we continue dealing with orthonormal frames in both the configurations, novel curvilinear coordinates are naturally drawn by the deformation process. In fact, when in the Lagrangian configuration starting from a point  $\bar{\mathbf{X}}$  we let varying the coordinate  $X^Q$  (with the other coordinates  $\bar{X}^{I \neq Q}$  held constant), thus generating a perturbation parallel to the basis unit vector  $\mathbf{e}_Q$ , in the Eulerian configuration one observes, in a neighborhood of the point  $\bar{\mathbf{x}} = \chi(\bar{\mathbf{X}})$ , a novel coordinate curve obeying to the equation  $x^r = \chi^r(X^Q, \bar{X}^{I \neq Q})$  ( $r = 1, 2, 3$ ). The same procedure can be repeated at varying  $Q$ , up to draw three families of coordinate curves in the Eulerian configuration. Hence, at each point  $x^r = \chi^r(\mathbf{X})$  of the spatial configuration, curvilinear basis vectors  $\boldsymbol{\theta}_Q(\mathbf{X}) \equiv F_Q^i(\mathbf{X})$  ( $Q = 1, 2, 3$ ) are tangent to the above coordinate lines, and result to be neither mutually orthogonal nor normalized (except for the case of an isometric map). Analogously, if one considers the relationship induced by  $\chi$  between the cotangent spaces (i.e. the dual spaces of the tangent spaces defined at corresponding points of the two manifolds), the contravariant basis vectors are expressed for each  $Q$  as  $\boldsymbol{\theta}^Q(\mathbf{X}) \equiv (\mathbf{F}^{-1})_i^Q$ . Covariant and contravariant basis vectors must satisfy the reciprocity condition  $(\boldsymbol{\theta}_P)^i(\boldsymbol{\theta}^Q)_i = \delta_P^Q$ . Once introduced the above curvilinear coordinates and the relevant tangents, which are defined in the Lagrangian configuration, an alternative Lagrangian metrics is generated by them, namely  $g_{DS}^*(\mathbf{X}) = \langle \boldsymbol{\theta}_D, \boldsymbol{\theta}_S \rangle_g = g_{rs} F_D^r F_S^s$ , where the reader is invited to distinguish carefully the dummy indices. Symbol  $\langle \cdot, \cdot \rangle_g$  denotes the inner product in the tangent space of the Eulerian configuration, according to the metric tensor  $g_{rs}$ . In the jargon of the differential geometry (see e.g. [42]), tensor  $g_{DS}^*(\mathbf{X})$  is referred to as *pullback* of the Eulerian metric tensor  $g_{rs}$  to the Lagrangian configuration, since generated as moving upstream along the flow of the placement map. Such a Lagrangian metric tensor coincides with the right Cauchy-Green tensor in doubly covariant form, see Eq. (3). Analogously, one can consider the contravariant tensor  $g^{*DS}(\mathbf{X}) = \langle \boldsymbol{\theta}^D, \boldsymbol{\theta}^S \rangle_g = g^{rs} (\mathbf{F}^{-1})_r^D (\mathbf{F}^{-1})_s^S$ , coinciding with  $\mathbf{C}^{-1}$  in doubly contravariant form. For the readers' convenience we report synoptically the expression of such a *pullback* metric tensor in both doubly covariant and doubly contravariant forms (often denoted by the musical symbols  $\flat$  and  $\sharp$ , respectively), in index and matrix notation:

$$\begin{aligned} g_{DS}^* &= \left( g^{rs} (\mathbf{F}^{-1})_r^D (\mathbf{F}^{-1})_s^S \right)^{-1} = g_{rs} F_D^r F_S^s; & \mathbf{g}^{*\flat} &= \mathbf{C} = \mathbf{F}^T \mathbf{F}; \\ g^{*DS} &= g^{rs} (\mathbf{F}^{-1})_r^D (\mathbf{F}^{-1})_s^S; & \mathbf{g}^{*\sharp} &= \mathbf{C}^{-1} = \mathbf{F}^{-1} \mathbf{F}^{-T}; \\ g^{*DS} g_{SL}^* &= \delta_L^D; \end{aligned} \tag{26}$$

It is worth emphasizing that the tensors  $\mathbf{g}^{*\sharp}$  and  $\mathbf{g}^{*\flat}$  satisfy the reciprocity condition, and hence they can be used to raise or lower the indices of any tensor: the result, however, will hold with exclusive reference to such a *pullback* metrics, if not

otherwise proved. To better understand the action of the pullback metric tensor  $g^{*RS}$  on the Lagrangian normal vector  $N_S$ , one can consider its product with the vectors  $N_R$ ,  $T_R$  and  $B_R$ , constituting an orthonormal basis over each curved face. By the transformation formula for the covariant components of the normal vector, see [46] (to be discussed in Section 6), one has

$$\frac{g^{*RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S N_R = g^{rs} \underbrace{\frac{(\mathbf{F}^{-1})^R_r N_R}{\|\mathbf{F}^{-T}\mathbf{N}\|}}_{=n_r} \underbrace{\frac{(\mathbf{F}^{-1})^S_s N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|}}_{=n_s} = \frac{\|\mathbf{F}^{-T}\mathbf{N}\|^2}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} = 1; \quad (27)$$

whilst for the remaining cases one obtains unexpectedly

$$\begin{aligned} \frac{g^{*RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S T_R &= g^{rs} n_s \frac{(\mathbf{F}^{-1})^R_r T_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \frac{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{T} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} \neq 0; \\ \frac{g^{*RS}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_S B_R &= g^{rs} n_s \frac{(\mathbf{F}^{-1})^R_r B_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \frac{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{B} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} \neq 0; \end{aligned} \quad (28)$$

where  $\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g = \langle \mathbf{N}, \mathbf{N} \rangle_{g^{*\sharp}} = \|\mathbf{N}\|_{g^{*\sharp}}^2$ . When it will appear clear from the context, we will omit to specify symbol  $\sharp$  or  $\flat$  as a subscript of the inner product. We underline also that  $\langle \boldsymbol{\theta}^R T_R, \boldsymbol{\theta}^S N_S \rangle_g = \langle \boldsymbol{\theta}^R, \boldsymbol{\theta}^S \rangle_g T_R N_S = g^{*RS} T_R N_S$ .

Now, let us consider the Eulerian vector  $\mathbf{F}^{-T}\mathbf{N} = \{\boldsymbol{\theta}^Q N_Q\}$  (proportional but not coincident with the normal), and let us evaluate its conventional partial derivative with respect to the Lagrangian coordinate  $X^D$ , namely

$$\begin{aligned} \frac{\partial}{\partial X^D} \left[ (\mathbf{F}^{-1})^Q_d N_Q \right] &= \frac{\partial}{\partial X^D} \left( (\mathbf{F}^{-1})^Q_d \right) N_Q + (\mathbf{F}^{-1})^Q_d \frac{\partial N_Q}{\partial X^D} = \\ &= \left[ -(\mathbf{F}^{-1})^Q_i (\mathbf{F}^{-1})^M_d F^i_{MD} N_Q + (\mathbf{F}^{-1})^Q_d \frac{\partial N_Q}{\partial X^D} \right] = \\ &= (\mathbf{F}^{-1})^M_d \left[ -\Gamma^Q_{MD} N_Q + \frac{\partial N_M}{\partial X^D} \right] = (\mathbf{F}^{-1})^M_d N_{M|D} \end{aligned} \quad (29)$$

where the Lagrangian derivative of the inverse matrix components was computed according to the formula (see e.g. [27])

$$\frac{\partial}{\partial X^D} \left( (\mathbf{F}^{-1})^Q_d \right) = \frac{\partial}{\partial F^i_M} \left( (\mathbf{F}^{-1})^Q_d \right) \frac{\partial F^i_M}{\partial X^D} = -(\mathbf{F}^{-1})^Q_i (\mathbf{F}^{-1})^M_d F^i_{MD};$$

and comma was omitted in  $F^i_{M,D}$  due to the Schwarz theorem. The symbol  $N_{M|D}$  denotes the covariant derivative of the Lagrangian covariant vector  $N_Q$  with respect to the coordinate  $X^D$ , defined as follows (see [44, 41, 29])

$$N_{Q|D} = \frac{\partial N_Q}{\partial X^D} - (\mathbf{F}^{-1})^B_i F^i_{QD} N_B = \frac{\partial N_Q}{\partial X^D} - \Gamma^B_{QD} N_B; \quad (30)$$

whilst  $\Gamma^B_{QD}$  denotes the (symmetric, torsionless) Christoffel symbols of the second kind associated to the metric tensor  $\mathbf{g}^{*b}$ , defined as (see e.g. [41–43])

$$\begin{aligned} \Gamma^B_{QD} &= \Gamma^B_{DQ} = \frac{\partial x^i}{\partial X^Q \partial X^D} \frac{\partial X^B}{\partial x^i} = \frac{\partial (\boldsymbol{\theta}_Q)^i}{\partial X^D} (\boldsymbol{\theta}^B)_i = \\ &= \frac{1}{2} (\mathbf{C}^{-1})^{BL} \left\{ \frac{\partial C_{LD}}{\partial X^Q} + \frac{\partial C_{LQ}}{\partial X^D} - \frac{\partial C_{QD}}{\partial X^L} \right\} \end{aligned} \quad (31)$$

In the expression of the covariant derivative Eq. (30), the first addend at rhs represents the usual partial derivative wrt the ambient space variables, quantifying the change of the normal vector  $N_Q$ , whilst the second addend represents a correction, taking into account the change, with respect to same coordinate basis  $\mathbf{e}_Q$ , of the pulled-back Eulerian basis vectors. It is worth emphasizing that the above definition in a sense generalizes the common use of the covariant derivatives in differential geometry, according to which the change of coordinates concerns the same manifold: herein, instead, the Lagrangian and the Eulerian configurations are identified by a diffeomorphism induced by  $\chi$ . The covariant derivative reduces to the first addend when the distortion is isometric. The Christoffel symbol can be expressed also as a function of the associated metric tensor (here  $\mathbf{g}^{*b}$ ) and of its derivatives, as indicated in the second line of Eq. (31), see [42]. For contravariant vectors, the expression of the covariant derivatives includes differences as for the sign and the contracted indices of the Christoffel symbol. In the presence of a higher rank tensor, the covariant differentiation must include as many correction terms as the available valences, see e.g. [43].

If we differentiate the Eulerian norm  $\|\mathbf{F}^{-T}\mathbf{N}\|_g$  by the conventional partial derivative with respect to the Lagrangian coordinate  $X^D$ , following Eq. (29) one finds

$$\begin{aligned} \frac{\partial}{\partial X^D} \left( \|\mathbf{F}^{-T}\mathbf{N}\| \right) &= \frac{\partial}{\partial X^D} (\|\mathbf{N}\|_{g^*}) = \frac{1}{2\|\mathbf{F}^{-T}\mathbf{N}\|} \frac{\partial}{\partial X^D} \left( \|\mathbf{F}^{-T}\mathbf{N}\|^2 \right) = \\ &= \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left\{ g^{cd} (\mathbf{F}^{-1})_c^R N_R \frac{\partial}{\partial X^D} \left( (\mathbf{F}^{-1})_d^S N_S \right) \right\} = \\ &= \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left\{ g^{cd} (\mathbf{F}^{-1})_c^R N_R \left[ (\mathbf{F}^{-1})_d^M N_{M|D} \right] \right\} = \\ &= \frac{1}{\|\mathbf{N}\|_{g^*}} \left\{ g^{*RM} N_R N_{M|D} \right\} \end{aligned} \quad (32)$$

We can recognize in the above equation the *pullback* metric tensor  $\mathbf{g}^{*\sharp}$ , see Eq. (26). Moreover, let us notice that the inner product according to  $\mathbf{g}^{*\sharp}$  of the covariant derivative  $N_{M|D}$  with the normal vector  $N_R$  (dummy index  $M$  being relevant to the normal vector component) does not vanish. This result is different from what occurs with the usual metrics  $g_{AB}$  in the Lagrangian configuration, between the normal vector and its gradient. By comparing the expressions in Eq. (32), it appears clear that the metric tensor  $\mathbf{g}^*$  can be “shifted out” of the covariant derivative, namely

$$\left( \|\mathbf{N}\|_{g^*}^2 \right)_{|R} = \left( g^{*DS} N_S N_D \right)_{|R} = g^{*DS} (N_S N_D)_{|R} = 2 g^{*DS} N_S N_{D|R}; \quad (33)$$

being indeed null its covariant derivative. This important property can be proved also by covariantly differentiating  $\mathbf{g}^*$  as a second rank tensors.

For the reciprocal of the norm  $\|\mathbf{F}^{-T}\mathbf{N}\|$ , one finds

$$\frac{\partial}{\partial X^D} \left( \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = - \frac{1}{\|\mathbf{N}\|_{g^*}^3} \left\{ g^{*RM} N_R N_{M|D} \right\} \quad (34)$$

with the additional property (being  $\frac{\partial}{\partial X^D}(1) = 0$ )

$$\frac{\partial}{\partial X^D} \left( \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \|\mathbf{F}^{-T}\mathbf{N}\| = - \frac{\partial}{\partial X^D} \left( \|\mathbf{F}^{-T}\mathbf{N}\| \right) \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \quad (35)$$

If we evaluate the directional (ordinary) derivative of the norm along the normal direction in the Lagrangian configuration, through Eq. (34) we obtain

$$\begin{aligned}
N^D \frac{\partial}{\partial X^D} \left( \frac{1}{\|\mathbf{F}^{-T} \mathbf{N}\|} \right) &= -\frac{1}{\|\mathbf{N}\|_{g^*}^3} \left\{ g^{*RM} N_R N_{M|D} N^D \right\} = \\
&= -\frac{1}{\|\mathbf{N}\|_{g^*}^3} \left\{ g^{*RM} N_R \left[ -\Gamma_{DM}^S N_S N^D + \underbrace{\frac{\partial N_M}{\partial X^D} N^D}_{=0} \right] \right\} = \\
&= \frac{1}{\|\mathbf{N}\|_{g^*}^3} \left\{ \Gamma_{DM}^S N_S N^D g^{*RM} N_R \right\} \tag{36}
\end{aligned}$$

Surprisingly in the above directional derivative only the addend including the Christoffel symbol did not vanish.

To illustrate the features of the covariant derivative, we can provide another meaningful example, which involves the Piola's bulk transformation [43,46,42]. If one considers in the Eulerian configuration the ordinary divergence operator of the functional group  $J^{-1} F_A^a V^A$ , being  $V^A$  a Lagrangian vector field, the following relationship holds ( $\forall \mathbf{F}, J > 0$ ):

$$\begin{aligned}
&\frac{\partial}{\partial x^a} \left( J^{-1} F_A^a V^A \right) = \\
&= \frac{\partial}{\partial x^a} \left( J^{-1} \right) F_A^a V^A + J^{-1} \frac{\partial}{\partial x^a} \left( F_A^a \right) V^A + J^{-1} F_A^a \frac{\partial}{\partial x^a} \left( V^A \right) = \\
&= \left( -\frac{J}{J^2} \right) \left( \mathbf{F}^{-1} \right)_i^M F_{MD}^i \underbrace{\left( \mathbf{F}^{-1} \right)_a^D F_A^a V^A}_{\delta_A^D} + J^{-1} \underbrace{F_{AD}^a \left( \mathbf{F}^{-1} \right)_a^D}_{=\Gamma_{AD}^D} V^A + \\
&+ J^{-1} \frac{\partial V^A}{\partial X^D} \underbrace{F_A^a \left( \mathbf{F}^{-1} \right)_a^D}_{=\delta_A^D} = \\
&= J^{-1} \left\{ -\underbrace{\left( \mathbf{F}^{-1} \right)_i^M F_{MD}^i V^D}_{=\Gamma_{MD}^M} + \underbrace{F_{AD}^a \left( \mathbf{F}^{-1} \right)_a^D V^A + \frac{\partial V^A}{\partial X^A}}_{=V_{|A}^A} \right\} = J^{-1} \frac{\partial V^A}{\partial X^A}; \tag{37}
\end{aligned}$$

where the correction provided by the Christoffel symbol was considered with the positive sign since vector  $V^A$  was contravariant. In this case the covariant derivative  $V_{|A}^A$  shrank to an ordinary partial derivative since the addend with the Christoffel symbols  $\Gamma_{(DM)}^M$  was equal opposite to the first addend, resulting from the ordinary derivative of  $J = \det(\mathbf{F})$ , and cancelled out. The formula for the Jacobian determinant differentiation was based on a cofactor decomposition, see e.g. [46,27].

At this point, we can consider the transformation of the Eulerian gradient of the normal vector (by conventional differentiation) into its Lagrangian counterpart. Recourse was made to the transport formula for the covariant normal vector (see

e.g. [43, 27]), to be discussed in Section 6. Exploiting Eqs. (29) and (32), one finds

$$\begin{aligned}
\frac{\partial n_k}{\partial x^c} &= \frac{\partial}{\partial x^c} \left( \frac{(\mathbf{F}^{-1})^R_k N_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = \\
&= \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} (\mathbf{F}^{-1})^B_k N_{B|D} (\mathbf{F}^{-1})^D_c + \\
&+ (\mathbf{F}^{-1})^R_k N_R \frac{\partial}{\partial x^c} \left( \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = \\
&= \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left\{ (\mathbf{F}^{-1})^B_k N_{B|D} (\mathbf{F}^{-1})^D_c + \right. \\
&+ (\mathbf{F}^{-1})^B_k N_B (\mathbf{F}^{-1})^D_c \left. \left( -\frac{1}{\|\mathbf{N}\|_{g^*}^2} g^{*RM} N_R N_{M|D} \right) \right\} = \\
&= \frac{(\mathbf{F}^{-1})^B_k (\mathbf{F}^{-1})^D_c}{\|\mathbf{N}\|_{g^*}} \left\{ N_{B|D} - N_B \left( \frac{g^{*RM} N_R N_{M|D}}{\|\mathbf{N}\|_{g^*}^2} \right) \right\}; \quad (38)
\end{aligned}$$

It can be noticed that the Lagrangian counterpart of the Eulerian gradient is constituted of two addends, due to the Leibniz product rule, one for the numerator and another one for the denominator. Both of them involve the covariant derivatives, which as expected transform like a doubly covariant tensor. Analogously, the Eulerian divergence of the normal vector can be transformed as follows

$$\begin{aligned}
\frac{\partial n^r}{\partial x^r} &= \frac{\partial}{\partial x^r} \left( g^{rk} \frac{(\mathbf{F}^{-1})^R_k N_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = \\
&= \frac{g^{*BD}}{\|\mathbf{F}^{-T}\mathbf{N}\|} \left\{ N_{B|D} + N_B \frac{\partial}{\partial X^D} \left( \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \|\mathbf{F}^{-T}\mathbf{N}\| \right\} = \\
&= \frac{1}{\|\mathbf{N}\|_{g^*}} g^{*BD} N_{B|D} - \frac{1}{\|\mathbf{N}\|_{g^*}^3} g^{*BD} N_B \left( g^{*RS} N_R N_{S|D} \right) \quad (39)
\end{aligned}$$

In the first addend, the pullback metric tensor in contravariant form plays the role of raising index  $B$  of  $N_{B|D}$ , needed to generate the divergence operator. The same operation is repeated two times for the second addend, scaled by a power of the surface element norm: the result is to contract both the indices of the covariant derivative.

As a further step, let us consider in the Eulerian configuration the vanishing inner products between the normal vector gradient and the normal vector itself, in the two possible forms: namely with a dummy index belonging to the spatial coordinate at the denominator (next point i), or to the vector at the numerator (point ii).

(i) Exploiting previous results, one obtains

$$\begin{aligned}
0 &= \frac{\partial n_v}{\partial x^r} n^r = g^{rs} \frac{(\mathbf{F}^{-1})^S_s N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|} \frac{\partial}{\partial x^r} \left( \frac{(\mathbf{F}^{-1})^V_v N_V}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = \\
&= (\mathbf{F}^{-1})^M_v \left\{ g^{*DS} N_S N_{M|D} \frac{1}{\|\mathbf{N}\|_{g^*}^2} \right\} + \\
&+ (\mathbf{F}^{-1})^W_v N_W \left[ -\frac{1}{\|\mathbf{N}\|_{g^*}^4} g^{*RT} N_R \left( g^{*SD} N_S N_{T|D} \right) \right] = 0 \quad (40)
\end{aligned}$$

The above equality is actually vector valued, since the covariant index  $v$  is free. We can multiply it by  $F_Q^v$  (norms at denominator are left intentionally)

$$\begin{aligned} & \frac{1}{\|\mathbf{N}\|_{g^*}^2} \underbrace{F_Q^v \left( \mathbf{F}^{-1} \right)_v^M}_{=\delta_Q^M} \left\{ g^{*DS} N_S N_{M|D} \right\} + \\ & - \frac{1}{\|\mathbf{N}\|_{g^*}^4} \underbrace{F_Q^v \left( \mathbf{F}^{-1} \right)_v^W}_{=\delta_Q^W} N_W \left\{ g^{*RT} N_R \left( g^{*SD} N_S N_{T|D} \right) \right\} = 0; \end{aligned} \quad (41)$$

and then by  $N^Q$

$$\begin{aligned} & \frac{1}{\|\mathbf{N}\|_{g^*}^2} N^Q \left\{ g^{*DS} N_S N_{Q|D} \right\} + \\ & - \frac{1}{\|\mathbf{N}\|_{g^*}^4} \underbrace{N^Q N_Q}_{=1} \left\{ g^{*RT} N_R \left( g^{*SD} N_S N_{T|D} \right) \right\} = 0; \end{aligned} \quad (42)$$

finally obtaining

$$\frac{1}{\|\mathbf{N}\|_{g^*}^2} \left\{ N^Q g^{*DS} N_S N_{Q|D} \right\} = \frac{1}{\|\mathbf{N}\|_{g^*}^4} \left\{ g^{*RT} N_R \left( g^{*SD} N_S N_{T|D} \right) \right\}; \quad (43)$$

By comparing both sides of the above Equation, one can notice that the spatial coordinate  $X^D$  of the covariant derivative is contracted through the pullback metric tensor  $\mathbf{g}^{*DS}$  with the normal vector itself  $N_S$ . As for the remaining valence  $T$  of the covariant derivative  $N_{T|D}$ , there is an important difference. At rhs the pullback metric tensor appears for a second time, scaled by a squared norm, multiplying the covariant components of the normal vector, i.e.  $g^{*RT} N_R$ . At lhs, instead, a contravariant vector  $N^Q$  plays the same role. Hence, at the rhs the action of the pullback metric tensor, normalized by the squared norm, is effective in raising the index of the normal vector: however, such a result holds exclusively with reference to the present expression, and cannot be generalized to other vectors. If one substitutes the scalar relationship Eq. (43) back into the previous vector-valued Equation (42), and the squared norm at denominator is finally simplified, the following noteworthy equation is found

$$g^{*DS} N_S N_{Q|D} = N_Q \left( N^Y g^{*D'S'} N_{S'} N_{Y|D'} \right); \quad (44)$$

At a first sight, the above equality may seem counterintuitive, since expressed as a covector. We can notice that the covariant derivative of the normal vector, multiplied through the pullback metric tensor by the normal vector (in covariant form), provides as result the same normal vector (in covariant form) multiplied by a scalar coefficient. Such a coefficient (i.e. the term within parentheses at rhs) equals the same expression at lhs but exhibits the remaining valence contracted with the normal vector in contravariant form. If we multiply both sides by the contravariant component  $N^Q$ , we attain a self evident scalar identity.

(ii) As a closing exercise, let us consider the transformation of the Eulerian expression which follows

$$\begin{aligned}
0 &= \frac{\partial n_v}{\partial x^r} n^v = g^{vs} n_s \frac{\partial n_v}{\partial x^r} = g^{vs} \frac{(\mathbf{F}^{-1})^S_N N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|} \frac{\partial}{\partial x^r} \left( \frac{(\mathbf{F}^{-1})^V_V N_V}{\|\mathbf{F}^{-T}\mathbf{N}\|} \right) = \\
&= \frac{(\mathbf{F}^{-1})^D_r}{\|\mathbf{N}\|_{g^*}^2} \left\{ g^*{}^{MS} N_S N_{M|D} \right\} + \\
&\quad - \underbrace{\frac{g^*{}^{VS} N_S N_V}{\|\mathbf{N}\|_{g^*}^2}}_{=1} \left\{ \frac{(\mathbf{F}^{-1})^D_r}{\|\mathbf{N}\|_{g^*}^2} \left( g^*{}^{RT} N_R N_{T|D} \right) \right\} = 0;
\end{aligned} \tag{45}$$

When transformed into the Lagrangian form, the above dot product gives rise to a vanishing difference of addends including covariant derivatives: the former included the derivative of  $\mathbf{F}^{-T}\mathbf{N}$ , the latter the derivative of the norm at the denominator. Vanishing of the above difference was conditioned to the equality  $g^*{}^{VS} N_S N_V = \|\mathbf{N}\|_{g^*}^2$ .

Obviously, when passing from the Lagrangian to the Eulerian configuration, the tangent and the normal to the border edges, belonging to the tangent space, do not obey to the same transport rule of the vector normal to the face: moreover, the covariant and the contravariant components of the same vector are expected to transform in different ways.

## 6 Transport of edge vectors

The placement map  $\chi$  specifies a diffeomorphism between the Lagrangian and the Eulerian manifolds: since such a surjection is continuous and differentiable with its inverse, it transforms connected volumes, their boundary faces with relevant border edges possibly including a finite number of wedges, into their counterparts in the current configuration. As well known, the tangent maps preserve neither lengths nor angles, except the case of isometric transformations. Now let us consider a (generally curved) boundary face in the Lagrangian configuration, diffeomorphic to a face in the Eulerian configuration in the sense specified above. Any curve with parametric representation  $\boldsymbol{\mu}(s)$  (being  $s$  the curvilinear abscissa) drawn over that face is transformed into a curve  $\boldsymbol{\chi}(\boldsymbol{\mu}(s))$  over the diffeomorphic Eulerian face. Vector  $\mathbf{T}(\mathbf{P}_o)$  tangent to the Lagrangian curve at the point  $\mathbf{P}_o \equiv \boldsymbol{\mu}(s_o)$ , when multiplied by the tangent map  $\mathbf{F}(\mathbf{P}_o)$  is transformed into another vector no more with a unit modulus, but tangent to the diffeomorphic Eulerian curve at the image point  $\mathbf{p}_o = \boldsymbol{\chi}(\mathbf{P}_o) \equiv \boldsymbol{\chi}(\boldsymbol{\mu}(s_o))$ . Hence the following Eulerian-Lagrangian transformation formula holds for the contravariant tangent vector:

$$\begin{aligned}
t^r &= \frac{F_R^r T^R}{\|\mathbf{F}\mathbf{T}\|} = F_R^r T^R \|\mathbf{F}^{-1}\mathbf{t}\|; \\
\left( t_r = g_{rs} t^s = g_{rs} \frac{F_R^s T^R}{\|\mathbf{F}\mathbf{T}\|} = g_{rs} \frac{F_R^s g^{RS} T_S}{\|\mathbf{F}\mathbf{T}\|} \right)
\end{aligned} \tag{46}$$

In the above equation, the transformation rule for the covariant component  $t_r$  is obtained from the contravariant one through the Eulerian metric tensors  $g^{rs}$ , playing as usual the role of raising or lowering indices. To attain a formula relating the covariant components for both the configurations, the Lagrangian metric tensor  $g^{RS}$  must be involved. However, such a representation is not unique: alternative representations may exist, which can be useful for analytical developments and can reveal features of the underlying differential geometric structure.

Another widely adopted transport formula concerns the covariant normal vector to the face (see e.g. [46, 43, 27]), namely

$$\begin{aligned} n_r &= \frac{(\mathbf{F}^{-1})^R_r N_R}{\|\mathbf{F}^{-T}\mathbf{N}\|} = (\mathbf{F}^{-1})^R_r N_R \|\mathbf{F}^T \mathbf{n}\|; \\ &\left( n^r = g^{rs} n_s = g^{rs} (\mathbf{F}^{-1})^R_r g_{RS} N^S \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \right) \end{aligned} \quad (47)$$

To the author's knowledge, it is not yet available any formula relating the Eulerian edge normal  $b^r$  in contravariant form to its Lagrangian counterpart  $B^R$ . In this research, we will adopt the following strategy: leaving to the end the requirement on the unit modulus, we indicate by  $\tilde{b}^r$  a not normalized ansatz for the edge normal, which must satisfy the orthogonality constraints with respect to the remaining vectors of the Eulerian local frame, namely  $\tilde{b}^r n_r = 0$  and  $\tilde{b}^r t_r = 0$ . The unit modulus requirement ( $b^r b_r = 1$ ) has also the meaning of reciprocity condition between the covariant and the contravariant components of the same vector. The image of  $B^R$  through the tangent map continues belonging to the tangent space in the Eulerian configuration (see the Appendix A), but in general it is no more orthogonal to the edge tangent  $\mathbf{t}$ , and therefore a correction term must be considered. Our ansatz has the expression  $\tilde{b}^r = F^r_R (B^R + S^r)$ . Since a transport formula for the covariant normal  $n_r$  is already available from Eq. (47), one finds

$$\begin{aligned} 0 &= \tilde{b}^r n_r = \left( F^r_R B^R + F^r_Q S^Q \right) \frac{(\mathbf{F}^{-1})^S_r N_S}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \\ &= \underbrace{F^r_R (\mathbf{F}^{-1})^S_r}_{=\delta^S_R} N_S B^R + \underbrace{F^r_Q (\mathbf{F}^{-1})^S_r}_{=\delta^S_Q} N_S S^Q = \\ &= \underbrace{N_S B^S}_{=0} + N_Q S^Q = 0; \end{aligned} \quad (48)$$

being mutually orthogonal the Lagrangian vectors  $\mathbf{N}$  and  $\mathbf{B}$ . Hence, the unknown Lagrangian correction  $\mathbf{S}$  results to be orthogonal to  $\mathbf{N}$  and must be expressed as a linear combination of vectors spanning the tangent space, namely  $\mathbf{S} = \alpha \mathbf{T} + \beta \mathbf{B}$ , being  $\alpha$  and  $\beta$  real coefficients. Since our ansatz must be orthogonal also to the edge tangent, by entering the expression of vector  $\mathbf{S}$  as a function of the real coefficients  $\alpha$  and  $\beta$ , one has

$$\tilde{b}^r t_r = \left( F^r_R B^R + F^r_S S^S \right) g_{cr} F^c_S T^S = \langle \mathbf{FB} + \alpha \mathbf{FT} + \beta \mathbf{FB}, \mathbf{FT} \rangle_g = 0; \quad (49)$$

obtaining

$$\alpha = -(\beta + 1) \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g}; \quad (50)$$

Then at varying  $\beta$  we can provide  $\infty^1$  solutions, namely

$$\begin{aligned}\tilde{b}^r &= F_R^r B^R + F_W^r S^W = F_R^r B^R + F_W^r \left( \beta B^W - (\beta + 1) \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} T^W \right) = \\ &= (\beta + 1) \left( F_R^r B^R - \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} F_W^r T^W \right)\end{aligned}\quad (51)$$

Up to the normalization factor, the contravariant components of the edge normal vector  $\tilde{\mathbf{b}}$  can be expressed as a function of their Lagrangian counterparts as follows

$$\frac{\tilde{\mathbf{b}}}{(\beta + 1)} = \mathbf{FB} - \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} \mathbf{FT} = \mathbf{FB} - \langle \mathbf{FB}, \mathbf{t} \rangle_g \mathbf{t} \quad ; \quad (52)$$

or by index notation

$$\frac{\tilde{b}^r}{(\beta + 1)} = F_R^r B^R - \frac{(g_{rs} F_R^r B^R F_S^s T^S)}{(g_{rs} F_R^r T^R F_S^s T^S)} F_R^r T^R \quad (53)$$

The above transport formula finally has appeared as a trivial application of Gram Schmidt procedure, when interpreting the (normalized) inner product  $\langle \mathbf{FB}, \mathbf{FT} \rangle_g / \|\mathbf{FT}\|$  as the orthogonal projection of the Eulerian vector  $\mathbf{FB}$  onto the direction tangent to the edge, being  $\mathbf{t} = \mathbf{FT} / \|\mathbf{FT}\|$  and  $\langle \mathbf{FT}, \mathbf{FT} \rangle_g = \|\mathbf{FT}\|^2$ . Let us now evaluate the modulus of  $\tilde{\mathbf{b}}$  through well known properties of the cross product, namely

$$\begin{aligned}\frac{\|\tilde{\mathbf{b}}\|^2}{(\beta + 1)^2} &= \langle \tilde{\mathbf{b}}, \tilde{\mathbf{b}} \rangle_g = \langle \mathbf{FB}, \mathbf{FB} \rangle_g - 2 \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g^2}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} + \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g^2}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} = \\ &= \langle \mathbf{FB}, \mathbf{FB} \rangle_g - \frac{\langle \mathbf{FB}, \mathbf{FT} \rangle_g^2}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} = \\ &= \left( \langle \mathbf{FB}, \mathbf{FB} \rangle_g \langle \mathbf{FT}, \mathbf{FT} \rangle_g - \langle \mathbf{FB}, \mathbf{FT} \rangle_g^2 \right) \frac{1}{\langle \mathbf{FT}, \mathbf{FT} \rangle_g} = \\ &= \left( \|\mathbf{FB} \wedge \mathbf{FT}\|^2 \right) \frac{1}{\|\mathbf{FT}\|^2} = \left( \frac{\|\mathbf{FB} \wedge \mathbf{FT}\|}{\|\mathbf{FT}\|} \right)^2 = \left( \frac{\|\mathbf{JF}^{-T} \mathbf{N}\|}{\|\mathbf{FT}\|} \right)^2 ;\end{aligned}\quad (54)$$

Finally, by substituting the square root of Eq. (54) into Eq. (53) and simplifying the proportionality coefficient, the transport formula for the edge normal vector is provided

$$\tilde{b}^r = \left\{ F_R^r B^R - \frac{(g_{rs} F_R^r B^R F_S^s T^S)}{(g_{rs} F_R^r T^R F_S^s T^S)} F_R^r T^R \right\} \frac{\|\mathbf{FT}\|}{\|\mathbf{JF}^{-T} \mathbf{N}\|} ; \quad (55)$$

It is worth noting that the above inner products can be expressed through the pullback metric tensor in covariant form (see Eq. 26), namely  $g_{rs} F_R^r B^R F_S^s T^S = g_{RS}^* B^R T^S$  and  $g_{rs} F_R^r T^R F_S^s T^S = g_{RS}^* T^R T^S$ .

By the same strategy an analogous relationship can be sought for the covariant components of the edge normal. This time we assume as ansatz  $\tilde{b}_r = (\mathbf{F}^{-1})_r^R B_R + (\mathbf{F}^{-1})_r^R S_R$ , up to a normalization coefficient. Firstly, let us prescribe the orthogonality of the ansatz with respect to the edge tangent, as follows

$$\tilde{b}_r t^r = \left\{ (\mathbf{F}^{-1})_r^R B_R + (\mathbf{F}^{-1})_r^R S_R \right\} \frac{F_Q^r T^Q}{\|\mathbf{FT}\|} = 0 ; \quad (56)$$

which implies (neglecting the norm at the denominator)

$$\underbrace{B_Q T^Q}_{=0} + S_Q T^Q = 0; \quad (57)$$

being mutually orthogonal the Lagrangian unit vectors  $\mathbf{B}$  and  $\mathbf{T}$ . Hence, the unknown correction vector must be sought in the form:  $\mathbf{S} = \alpha \mathbf{N} + \beta \mathbf{B}$ , being  $\alpha$  and  $\beta$  real coefficients. When we exploit the last orthogonality constraint, we realize that unfortunately a direct transport formula for the contravariant face normal is not yet available and hence we must utilize the Eulerian metric tensor to raise the index of covariant normal vector Eq. (47). One has

$$\tilde{b}_r n^r = \left\{ \left( \mathbf{F}^{-1} \right)_r^R B_R + \left( \mathbf{F}^{-1} \right)_r^R S_R \right\} g^{rs} \frac{\left( \mathbf{F}^{-1} \right)_s^Q N_Q}{\| \mathbf{F}^{-T} \mathbf{N} \|} = 0; \quad (58)$$

By simplifying the norm at the denominator and entering the expression for  $\mathbf{S}$  as a linear combination of the coefficients  $\alpha$  and  $\beta$ , one obtains

$$\begin{aligned} & \left\{ \left( \mathbf{F}^{-1} \right)_r^R B_R (\beta + 1) + \alpha \left( \mathbf{F}^{-1} \right)_r^R N_R \right\} g^{rs} \left( \mathbf{F}^{-1} \right)_s^Q N_Q = \\ & = (\beta + 1) \langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g + \alpha \langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g = 0; \end{aligned} \quad (59)$$

and hence

$$\alpha = -(\beta + 1) \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g}; \quad (60)$$

Finally, a novel expression for the covariant edge normal  $\tilde{b}_r$  at varying the coefficient  $\beta$  is provided

$$\frac{\tilde{b}_r}{(\beta + 1)} = \left\{ \left( \mathbf{F}^{-1} \right)_r^R B_R - \left( \mathbf{F}^{-1} \right)_r^R N_R \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} \right\} \quad (61)$$

Also for the covariant component of the edge normal vector the transport formula can be interpreted as an application of Gram Schmidt procedure. In fact, in the Eulerian configuration the orthogonal projection of  $\mathbf{F}^{-T} \mathbf{B}$  onto the unit normal vector is provided by the expression

$$\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g \frac{1}{\| \mathbf{F}^{-T} \mathbf{N} \|_g} = \langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{n} \rangle_g; \quad (62)$$

being  $\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g = \|\mathbf{F}^{-T} \mathbf{N}\|_g^2$ . Up to a normalization coefficient, the squared norm of the covariant edge normal vector can be computed as follows:

$$\begin{aligned}
\frac{\|\tilde{\mathbf{b}}\|^2}{(\beta+1)^2} &= \langle \tilde{\mathbf{b}}, \tilde{\mathbf{b}} \rangle_g = \\
&= \langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{B} \rangle_g - 2 \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g^2}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} + \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g^2}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} = \\
&= \langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{B} \rangle_g - \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g^2}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} = \\
&= \frac{(\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{B} \rangle_g \langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g - \langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g^2)}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} = \\
&= \frac{\|\mathbf{F}^{-T} \mathbf{B} \wedge \mathbf{F}^{-T} \mathbf{N}\|^2}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} = \frac{\|J^{-1} \mathbf{F} (\mathbf{B} \wedge \mathbf{N})\|^2}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} = \left( \frac{\|J^{-1} \mathbf{F} \mathbf{T}\|}{\|\mathbf{F}^{-T} \mathbf{N}\|} \right)^2;
\end{aligned} \tag{63}$$

and one obtains

$$b_r = \left\{ \left( \mathbf{F}^{-1} \right)_r^R B_R - \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} \left( \mathbf{F}^{-1} \right)_r^R N_R \right\} \frac{\|\mathbf{F}^{-T} \mathbf{N}\|}{\|J^{-1} \mathbf{F} \mathbf{T}\|}; \tag{64}$$

It is worth noting that the above inner products can be expressed through the pullback metric tensor in contravariant form (see Eq. 26), namely  $\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g = g^{*RS} B_R N_S$  and  $\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g = g^{*RS} N_R N_S$ . Moreover, the reciprocity condition between the covariant and the contravariant components of the edge normal is satisfied, namely

$$\begin{aligned}
b_r b^r &= \left\{ \left( \mathbf{F}^{-1} \right)_r^R B_R - \frac{\langle \mathbf{F}^{-T} \mathbf{B}, \mathbf{F}^{-T} \mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T} \mathbf{N}, \mathbf{F}^{-T} \mathbf{N} \rangle_g} \left( \mathbf{F}^{-1} \right)_r^R N_R \right\} \times \\
&\times \left\{ F_S^r B^S - \frac{\langle \mathbf{F} \mathbf{B}, \mathbf{F} \mathbf{T} \rangle_g}{\langle \mathbf{F} \mathbf{T}, \mathbf{F} \mathbf{T} \rangle_g} F_S^r T^S \right\} \frac{\|\mathbf{F} \mathbf{T}\|}{\|J \mathbf{F}^{-T} \mathbf{N}\|} \frac{\|\mathbf{F}^{-T} \mathbf{N}\|}{\|J^{-1} \mathbf{F} \mathbf{T}\|} = 1;
\end{aligned} \tag{65}$$

At this stage, we propose two novel “direct” formulae for the transport of the covariant tangent vector  $t_r$  and of the contravariant normal vector  $n^r$ , which are lacking in the literature. By symbols

$$\begin{aligned}
t_r &= \frac{g_V^{*R} \left( \mathbf{F}^{-1} \right)_r^V T_R}{\|\mathbf{F} \mathbf{T}\|}; \\
n^r &= \frac{g_S^{*E} F_E^r N^S}{\|\mathbf{F}^{-T} \mathbf{N}\|};
\end{aligned} \tag{66}$$

As one can easily check, such expressions satisfy all the prescribed conditions with respect to the remaining vectors of the Eulerian frame  $(\mathbf{b}, \mathbf{t}, \mathbf{n})$ . Moreover, the expression for the contravariant normal vector in Eq. (66) turns out to be consistent with the novel transport formula for the orthogonal projector  $([m_\perp]_s^r = n^r n_s)$  provided in the Appendix A. In particular, we notice that the expression

$$\frac{g_S^{*E} N^S}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} = \frac{g^{tj} \left( \mathbf{F}^{-1} \right)_t^E \left( \mathbf{F}^{-1} \right)_j^P g_{PS} N^S}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} = \frac{n^t \left( \mathbf{F}^{-1} \right)_t^E}{\|\mathbf{F}^{-T} \mathbf{N}\|}; \tag{67}$$

gives rise to a skew Lagrangian vector, with non vanishing components in both the tangent and the normal space. In fact, recalling that  $[M_\perp]_E^Q = N^Q N_E$  and  $[M_\parallel]_E^Q = \delta_E^Q - [M_\perp]_E^Q$ , one has

$$[M_\perp]_E^Q \frac{n^t (\mathbf{F}^{-1})_t^E}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \underbrace{\left( n^t n_t \right)}_{=1} N^Q; \quad [M_\parallel]_E^Q \frac{n^t (\mathbf{F}^{-1})_t^E}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \frac{n^t (\mathbf{F}^{-1})_t^Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} - N^Q; \quad (68)$$

Instead, when we include  $F_E^r$  into Eq. (67), a vector proportional to the normal is attained. In fact

$$\frac{g_S^* F_E^r N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} = \frac{g^{tj} \overbrace{\left( \mathbf{F}^{-1} \right)_t^E}^{=\delta_t^r} \overbrace{F_E^r \left( \mathbf{F}^{-1} \right)_j^P}^{=n_j \|\mathbf{F}^{-T}\mathbf{N}\|} N^P}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} = \frac{g^{tj} \delta_t^r n_j}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \quad (69)$$

At the end of this Section, it is worth emphasizing that additional formulae, maybe less intuitive at a first sight but possibly useful for analytical developments, can be generated through the cross product by simply permuting the frame vectors. Moreover, the presence of the norm makes commutative this procedure. For instance, starting from the relationship

$$\begin{aligned} \|J^{-1}\mathbf{F}\mathbf{T}\|^2 &= \|J^{-1}\mathbf{F}(\mathbf{B} \wedge \mathbf{N})\|^2 = \|\mathbf{F}^{-T}\mathbf{B} \wedge \mathbf{F}^{-T}\mathbf{N}\|^2 = \\ &= \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{B} \rangle_g \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g^2; \end{aligned} \quad (70)$$

alternative expressions for the Jacobian determinant  $J$  can be provided, for instance

$$\begin{aligned} J^{-2} &= \frac{\|J^{-1}\mathbf{F}(\mathbf{B} \wedge \mathbf{N})\|^2}{\|\mathbf{F}\mathbf{T}\|^2} = \frac{\|\mathbf{F}^{-T}\mathbf{B} \wedge \mathbf{F}^{-T}\mathbf{N}\|^2}{\|\mathbf{F}\mathbf{T}\|^2} = \\ &= \frac{\|J^{-1}\mathbf{F}(\mathbf{N} \wedge \mathbf{T})\|^2}{\|\mathbf{F}\mathbf{B}\|^2} = \frac{\|\mathbf{F}^{-T}\mathbf{N} \wedge \mathbf{F}^{-T}\mathbf{T}\|^2}{\|\mathbf{F}\mathbf{B}\|^2} = \\ &= \frac{\|J^{-1}\mathbf{F}(\mathbf{T} \wedge \mathbf{B})\|^2}{\|\mathbf{F}\mathbf{N}\|^2} = \frac{\|\mathbf{F}^{-T}\mathbf{T} \wedge \mathbf{F}^{-T}\mathbf{B}\|^2}{\|\mathbf{F}\mathbf{N}\|^2}; \end{aligned} \quad (71)$$

Moreover, since Equation (70) must hold for any invertible deformation gradient (with  $J > 0$ ),  $\mathbf{F}$  can be formally changed into its inverse  $\mathbf{F}^{-1}$  (defined in the Eulerian configuration), and consistently  $J$  into  $J^{-1}$ , obtaining again a valid relationship this time involving Eulerian frame vectors, namely

$$\begin{aligned} \|J\mathbf{F}^{-1}\mathbf{t}\|^2 &= \|J\mathbf{F}^{-1}(\mathbf{b} \wedge \mathbf{n})\|^2 = \|\mathbf{F}^T\mathbf{b} \wedge \mathbf{F}^T\mathbf{n}\|^2 = \\ &= \langle \mathbf{F}^T\mathbf{b}, \mathbf{F}^T\mathbf{b} \rangle_g \langle \mathbf{F}^T\mathbf{n}, \mathbf{F}^T\mathbf{n} \rangle_g - \langle \mathbf{F}^T\mathbf{b}, \mathbf{F}^T\mathbf{n} \rangle_g^2; \end{aligned} \quad (72)$$

If in Eq. (72) one superimposes to the inversion of the deformation gradient the transpose operator, the following relationship is met among Lagrangian frame vectors

$$\begin{aligned} \|J\mathbf{F}^{-T}\mathbf{T}\|^2 &= \|J\mathbf{F}^{-T}(\mathbf{B} \wedge \mathbf{N})\|^2 = \|\mathbf{F}\mathbf{B} \wedge \mathbf{F}\mathbf{N}\|^2 = \\ &= \langle \mathbf{F}\mathbf{B}, \mathbf{F}\mathbf{B} \rangle_g \langle \mathbf{F}\mathbf{N}, \mathbf{F}\mathbf{N} \rangle_g - \langle \mathbf{F}\mathbf{B}, \mathbf{F}\mathbf{N} \rangle_g^2; \end{aligned} \quad (73)$$

## 7 The surface divergence theorem revisited

In the present Section, the divergence theorem for curved surfaces with border (see e.g. [41, 51, 27]) was revisited and provided of a novel formulation, suitable for the transport along diffeomorphisms.

Firstly, let us briefly recall such a theorem. It represents an important result of differential geometry, which can be applied to the Riemannian embedded submanifolds with codimension one, such as the bidimensional surfaces embedded in a three dimensional ambient space, see e.g. [41, 44]. We make here reference to the Lagrangian configuration, but the result equally holds for the Eulerian configuration. Hence, let us consider the boundary face  $\Sigma_\star \subset \mathcal{R}^3$ , and let  $L_\star = \partial\Sigma_\star$  denote its border edge, being  $\mathbf{B}$  the unit vector normal to the edge. At each point of this face a tangential projector  $[\mathbf{M}_\parallel]_A^C$  can be defined, whose properties were outlined in the Appendix. Such a linear operator projects onto the local tangent space any vector of the space environment. Moreover, it can be utilized to express the surface divergence with respect to the coordinates of the space environment, without making recourse to any intrinsic representation of the surface. Let  $W^B$  be a vector field defined over  $\Sigma_\star$ . Under such assumptions, the following equality holds:

$$\int_{\Sigma_\star} [\mathbf{M}_\parallel]_A^C \frac{\partial}{\partial X^C} \left( [\mathbf{M}_\parallel]_B^A W^B \right) d\Sigma_\star = \int_{L_\star} [\mathbf{M}_\parallel]_A^B W^A B_B dL_\star \quad (74)$$

It is worth emphasizing that the integrand at lhs has the meaning of surface divergence of a tangential vector, namely  $\text{DIV}_\parallel(\mathbf{W}_\parallel)$ .

Now let us consider in the Eulerian configuration the regular surface  $\Sigma$ , over which a vector field  $w^b$  is defined, and its border  $L$ . Let symbol  $[m_\parallel]_b^a$  denote the Eulerian counterpart of the tangential projector  $[\mathbf{M}_\parallel]_B^A$ , as outlined in the Appendix. For any test function  $\delta\chi$  with compact support included in  $\Sigma$  (its dimension is irrelevant to the present purposes), one can write:

$$(\blacktriangle) = \int_\Sigma \frac{\partial}{\partial x^c} \left( [m_\parallel]_b^a w^b \right) [m_\parallel]_a^c \delta\chi d\Sigma \quad (75)$$

Through integration by parts of the above Eulerian expression, one can apply the surface divergence theorem to the first addend (attaining form of Eq. 74), which thereafter vanishes over the border edge due to the compact support of  $\delta\chi(\mathbf{x})$ , namely

$$\begin{aligned} &= \int_\Sigma \left\{ \frac{\partial}{\partial x^c} \left( [m_\parallel]_b^a w^b \delta\chi \right) - [m_\parallel]_b^a w^b \frac{\partial \delta\chi}{\partial x^c} \right\} [m_\parallel]_a^c d\Sigma = \\ &= \underbrace{\int_L \left( [m_\parallel]_b^a w^b \delta\chi \right) b_c [m_\parallel]_a^c d\Sigma}_{=0} - \int_\Sigma [m_\parallel]_a^c [m_\parallel]_b^a w^b \frac{\partial \delta\chi}{\partial x^c} d\Sigma = (\blacklozenge) \quad (76) \end{aligned}$$

Due to the idempotence of the projector, one can write  $[m_\parallel]_b^c w^b = [m_\parallel]_a^c ([m_\parallel]_b^a w^b)$ , even if  $w^b$  is not tangent to the surface. After the above developments carried out in the Eulerian configuration, from Eq. (76) one can change variables passing to the Lagrangian configuration. The outer tangential projector  $[m_\parallel]_a^c$  can be transformed

as outlined in the Appendix A for the case of tangential vector field (i.e. neglecting the contribution of  $[M_\perp]$ ). One finds:

$$\begin{aligned}
(\diamond) &= \int_{\Sigma_\star} -[m_\parallel]_a^c \left( [m_\parallel]_b^a w^b \right) \frac{\partial \delta \chi}{\partial X^A} \frac{\partial X^A}{\partial x^c} \|J\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star = \\
&= \int_{\Sigma_\star} -F_S^c [M_\parallel]_R^S \left( \mathbf{F}^{-1} \right)_a^R \left( [m_\parallel]_b^a w^b \right) \frac{\partial \delta \chi}{\partial X^A} \left( \mathbf{F}^{-1} \right)_c^A \|J\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star = \\
&= \int_{\Sigma_\star} -[M_\parallel]_R^S \left( \mathbf{F}^{-1} \right)_a^R \delta_S^A \left( [m_\parallel]_b^a w^b \right) \frac{\partial \delta \chi}{\partial X^A} \|J\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star = \quad (77)
\end{aligned}$$

In the Lagrangian configuration, Equation (77) can be again integrated by parts, making recourse to the surface divergence theorem Eq. (74) and exploiting the compact support of  $\delta\chi(\mathbf{X})$ . It is worth emphasizing that the placement map, continuous and differentiable with its inverse, transforms compact sets over  $\Sigma_\star$  into compact sets over  $\Sigma$ . Due to idempotence of the Lagrangian projector, one can write

$$\begin{aligned}
&= \int_{\Sigma_\star} - \left( \mathbf{F}^{-1} \right)_a^R \left( [m_\parallel]_b^a w^b \right) \overbrace{[M_\parallel]_S^A [M_\parallel]_R^S}^{=[M_\parallel]_R^A} \frac{\partial \delta \chi}{\partial X^A} \|J\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star = \\
&= \underbrace{\int_{\Sigma_\star} -[M_\parallel]_S^A \frac{\partial}{\partial X^A} \left( \delta \chi \left( \mathbf{F}^{-1} \right)_a^R \left( [m_\parallel]_b^a w^b \right) [M_\parallel]_R^S \|J\mathbf{F}^{-T}\mathbf{N}\| \right)}_{=0} d\Sigma_\star + \\
&+ \int_{\Sigma_\star} \delta \chi [M_\parallel]_S^A \frac{\partial}{\partial X^A} \left( \|J\mathbf{F}^{-T}\mathbf{N}\| \left( \mathbf{F}^{-1} \right)_a^R \left( [m_\parallel]_b^a w^b \right) [M_\parallel]_R^S \right) d\Sigma_\star ; \quad (78)
\end{aligned}$$

Clearly, we must achieve the same result of Eq. (78) by following a different path: starting from the first addend of Equation (75), without any manipulation we pass directly to the material configuration by a change of variables. Hence, one finds

$$(\blacktriangle) = \int_{\Sigma_\star} [m_\parallel]_a^c \frac{\partial}{\partial x^c} \left( [m_\parallel]_b^a w^b \right) \delta \chi \|J\mathbf{F}^{-T}\mathbf{N}\| d\Sigma_\star \quad (79)$$

Equating the expressions of Eqs. (78) and (79), the following relationship must hold  $\forall \delta\chi$  and  $\forall \mathbf{F}$  (with  $J > 0$ ):

$$\begin{aligned}
&[m_\parallel]_a^c \frac{\partial}{\partial x^c} \left( [m_\parallel]_b^a w^b \right) \|J\mathbf{F}^{-T}\mathbf{N}\| = \\
&= [M_\parallel]_S^A \frac{\partial}{\partial X^A} \left( \|J\mathbf{F}^{-T}\mathbf{N}\| \left( \mathbf{F}^{-1} \right)_a^R \left( [m_\parallel]_b^a w^b \right) [M_\parallel]_R^S \right) ; \quad (80)
\end{aligned}$$

By applying to the Eulerian projector  $[m_{\parallel}]_b^a$  at rhs the complete transport formula outlined in the Appendix A, rhs of Eq. (80) can be developed as follows:

$$\begin{aligned}
&= [M_{\parallel}]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| \left(\mathbf{F}^{-1}\right)_a^R \left[ F_Q^a [M_{\parallel}]_W^Q \left(\mathbf{F}^{-1}\right)_b^W + \right. \right. \\
&+ \left. \left( F_Q^a \left(\mathbf{F}^{-1}\right)_b^W - \frac{g_Q^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^a \left(\mathbf{F}^{-1}\right)_b^W \right) [M_{\perp}]_W^Q \right] w^b [M_{\parallel}]_R^S \left. \right\} = \\
&= [M_{\parallel}]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| \left[ \delta_Q^R [M_{\parallel}]_W^Q \left(\mathbf{F}^{-1}\right)_b^W w^b [M_{\parallel}]_R^S + \right. \right. \\
&+ \left. \left( \delta_Q^R N^Q N_W \left(\mathbf{F}^{-1}\right)_b^W w^b [M_{\parallel}]_R^S + \right. \right. \\
&\left. \left. - [M_{\parallel}]_R^S \delta_E^R \frac{g_Q^{*E} N^Q}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_W \left(\mathbf{F}^{-1}\right)_b^W w^b \right) \right] \left. \right\} = \\
&= [M_{\parallel}]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| \left[ [M_{\parallel}]_W^S \left(\mathbf{F}^{-1}\right)_b^W w^b + \right. \right. \\
&+ \left. \left( N_W \left(\mathbf{F}^{-1}\right)_b^W w^b \underbrace{N^R [M_{\parallel}]_R^S}_{=0} - [M_{\parallel}]_R^S \frac{g_S^{*R} N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \underbrace{N_W \left(\mathbf{F}^{-1}\right)_b^W w^b}_{=(\mathbf{n}\cdot\mathbf{w})\|\mathbf{F}^{-T}\mathbf{N}\|} \right) \right] \left. \right\} = \\
&= [M_{\parallel}]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| [M_{\parallel}]_W^S \left(\mathbf{F}^{-1}\right)_b^W w^b \right\} + \\
&- [M_{\parallel}]_S^A \frac{\partial}{\partial X^A} \left\{ \|J\mathbf{F}^{-T}\mathbf{N}\| [M_{\parallel}]_R^S \frac{g_S^{*R} N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_W \left(\mathbf{F}^{-1}\right)_b^W w^b \right\}; \quad (81)
\end{aligned}$$

The above Equation (80) and its extended form (81) were derived without any a priori assumption on the vector field  $w^b$ . If such a vector field were purely tangential to the face, the second addend in Eq. (81) including the term  $\mathbf{n}\cdot\mathbf{w}$  would vanish. Such expressions have to be regarded as the surface counterparts of the Piola's volume transformation, which relates Eulerian and Lagrangian functional expressions including the same vector field. It is worth noting that, differently from the vector fields constant in a volume, here it is not trivial to provide a general field with a null surface divergence.

## 8 Intermediate remarks

This paper constitutes the Part I of a wide research, aiming to derive the governing equations for a second gradient continuum via variational calculus and to prove the feasibility of their transformation from the Eulerian to the Lagrangian form. In the previous pages the variational approach for the equilibrium problem was outlined with reference to Cauchy's first gradient theory. Mathematical tools such as projectors, pullback metrics and covariant differentiation, resting on the differential geometry of the Riemannian manifolds, were investigated with reference to the equilibrium problem, naturally (and unexpectedly) emanating from the deformation process regarded as a diffeomorphism. Novel and remarkable transport formulae were provided for the normal and tangent vectors over the boundary faces

and along the edges, and for the surface projectors. The theorem of the surface divergence was revisited providing novel relationships over the surface analogous to Piola's volume transformation, useful for the transport of the governing equations to be discussed in Part II.

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## A Properties of surface projectors and their transport formulae

In this Appendix basic properties of the surface projectors and novel formulae for their transport from the Eulerian to the Lagrangian form, were detailed, endowed with short proofs and remarks.

Let us consider the Lagrangian configuration. As well known, at each point  $\mathbf{P}$  of a curved surface  $\Sigma_*$ , one can specify a tangent space  $\mathcal{T}_P \Sigma_*$ , including all the vectors tangent at that point to curves drawn over that face, and a normal space  $\mathcal{N}_P \Sigma_*$ , i.e. its orthogonal complement with respect to the ambient space. Hence, the ambient space can be expressed as the direct sum of the above spaces, namely  $\mathcal{T}_P \Sigma_* \oplus \mathcal{N}_P \Sigma_* = \mathcal{R}^3$ .

Accordingly, at each point of the same curved surface two linear operators can be defined, apt to project any vector of the ambient space onto the tangential and normal spaces, referred to as the (Lagrangian) tangential and orthogonal projectors and denoted by symbols  $[M_{\parallel}]_B^A$  and  $[M_{\perp}]_B^A$ , respectively. The same considerations can be repeated for the Eulerian projectors, for which lowercase symbols were adopted.

The projectors possess the following noteworthy properties (in both index and matrix notation):

$$\begin{aligned}
[M_{\parallel}]_B^A + [M_{\perp}]_B^A &= \delta_B^A; & \mathbf{M}_{\parallel} + \mathbf{M}_{\perp} &= \mathbf{1}; \\
[M_{\perp}]_A^C &= N^C N_A; & [\mathbf{M}_{\perp}] &= (\mathbf{N} \otimes \mathbf{N}); \\
[M_{\parallel}]_A^C &= \delta_A^C - N^C N_A; & [\mathbf{M}_{\parallel}] &= \mathbf{1} - \mathbf{N} \otimes \mathbf{N}; \\
[M_{\parallel}]_B^A [M_{\parallel}]_C^B &= [M_{\parallel}]_C^A; & \mathbf{M}_{\parallel}^2 &= \mathbf{M}_{\parallel}; \\
[M_{\perp}]_B^A [M_{\perp}]_C^B &= [M_{\perp}]_C^A; & \mathbf{M}_{\perp}^2 &= \mathbf{M}_{\perp};
\end{aligned} \tag{82}$$

where the Kronecker symbol  $\delta_B^A = \mathbf{1} = g_B^A$  represents the identity operator, coincident with the mixed form of the metric tensor.

In the Eulerian configuration, the orthogonal projector can be expressed as a function of its Lagrangian counterpart by using the transformation formula for the normal vector, as follows

$$\begin{aligned}
[m_{\perp}]_s^r &= n^r n_s = g^{rt} \frac{(\mathbf{F}^{-1})_t^Q N_Q (\mathbf{F}^{-1})_s^V N_V}{\|\mathbf{F}^{-T} \mathbf{N}\|} = \frac{g^{rt} (\mathbf{F}^{-1})_t^Q g_{QS}}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} [N^W N_V] (\mathbf{F}^{-1})_s^V = \\
&= \frac{g^{rt} (\mathbf{F}^{-1})_t^Q g_{QS}}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V
\end{aligned} \tag{83}$$

The above expression can be further simplified through multiplying the rhs by  $\delta_y^r$  and then by expressing the Kronecker symbol as the product of the deformation gradient with its inverse, namely

$$\begin{aligned}
[m_{\perp}]_s^r &= \delta_y^r \frac{g^{yt} (\mathbf{F}^{-1})_t^Q g_{QS}}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V = \\
&= F_E^r (\mathbf{F}^{-1})_y^E \frac{g^{yt} (\mathbf{F}^{-1})_t^Q g_{QS}}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} [M_{\perp}]_V^Q (\mathbf{F}^{-1})_s^V = \\
&= \frac{1}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} g^{yt} (\mathbf{F}^{-1})_t^Q g_{QS} (\mathbf{F}^{-1})_y^E F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V = \\
&= \frac{1}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} g^{*EQ} g_{QS} F_E^r [M_{\perp}]_V^Q (\mathbf{F}^{-1})_s^V = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T} \mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V
\end{aligned} \tag{84}$$

where symbol  $g^{*EQ}$  denotes the *pullback* metric tensor, with  $g^{*EQ} N_E N_Q = \|\mathbf{F}^{-T} \mathbf{N}\|^2$ . It is worth noting that in Eq. (84) the symbol  $g_S^{*E}$  constitutes a slight abuse of notation, as

an abbreviated form of  $g^{*EQ} g_{QS}$ , admissible if its meaning appears clear from the context. In fact, the proper metric tensor apt to lower indices of the contravariant *pullback* metrics is the doubly covariant *pullback* tensor  $g_{EQ}^*$ , as outlined in Section 5. Moreover, since the *pullback* metrics  $g^{*EQ}$  often provides counterintuitive results with respect to the conventional Lagrangian metric tensor, it is convenient to write explicitly all the terms involved. Analogously, the Eulerian tangential projector can be expressed as follows

$$\begin{aligned}
[m_{\parallel}]_s^r &= \delta_s^r - n^r n_s = \delta_s^r - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V = \\
&= F_S^r \delta_V^S (\mathbf{F}^{-1})_s^V - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V = \\
&= F_S^r [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V + F_S^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V = \\
&= F_S^r [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V + \left[ F_S^r (\mathbf{F}^{-1})_s^V - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r (\mathbf{F}^{-1})_s^V \right] [M_{\perp}]_V^S = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_{\perp}]_V^S \right\} (\mathbf{F}^{-1})_s^V ; \tag{85}
\end{aligned}$$

In the above equation, the addends within square brackets provide a contribution purely tangential to the face. In fact, if one expresses the Lagrangian orthogonal projector as the tensor product of the normal vectors, namely  $[M_{\perp}]_V^S = N^S N_V$ , and decomposes additively the Eulerian vector  $F_S^r N^S$  into a tangential and a normal component, the normal component cancel out the second addend within the parentheses. By utilizing the transformation rule for the covariant normal vector  $n_r = (\mathbf{F}^{-1})_r^Q N_Q \|\mathbf{F}^{-T}\mathbf{N}\|^{-1}$ , from Eq. (85) one has

$$\begin{aligned}
&\left[ F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \right] = \\
&= F_S^b N^S N_V (\mathbf{F}^{-1})_s^V - \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} g^{tj} (\mathbf{F}^{-1})_t^E (\mathbf{F}^{-1})_j^P g_{PS} F_E^r N^S N_V (\mathbf{F}^{-1})_s^V = \\
&= F_S^b N^S n_s \|\mathbf{F}^{-T}\mathbf{N}\| - g^{tj} \delta_t^r n_j n_s = \\
[m_{\parallel}]_v^r F_S^b N^S n_s \|\mathbf{F}^{-T}\mathbf{N}\| &= + [m_{\perp}]_v^r F_S^b N^S n_s \|\mathbf{F}^{-T}\mathbf{N}\| - n_r n_s = \\
[m_{\parallel}]_v^r F_S^b N^S N_V (\mathbf{F}^{-1})_s^V &+ \underbrace{\frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|} n_s \|\mathbf{F}^{-T}\mathbf{N}\| - n_r n_s}_{=0} = [m_{\parallel}]_v^r F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \tag{86}
\end{aligned}$$

Thus one can write Eq. (85) in a shorter but implicit form, as follows

$$[m_{\parallel}]_s^r = F_S^r [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V + [m_{\perp}]_b^r \left[ F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \right] \tag{87}$$

One can easily check that the proposed transport rules are compatible with the idempotence of the projectors. Preliminarily we notice that, due to the idempotence, for any vector  $w^s$  one has

$$[m_{\parallel}]_r^t \left( [m_{\parallel}]_s^r w^s \right) = [m_{\parallel}]_r^t \left( w_{\parallel}^s \right); \tag{88}$$

being  $w_{\parallel}^s$  the tangential projection of  $w^s$ . Thereafter, considering that the outer projector acts on a purely tangential vector and thus the addend of Eq. (85) including the orthogonal projector vanishes, one finds

$$\begin{aligned}
&= F_R^t [M_{\parallel}]_Q^R (\mathbf{F}^{-1})_r^Q \left( F_S^r [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V w^s + [m_{\parallel}]_b^r \left[ F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \right] w^s \right) = \\
&= F_R^t [M_{\parallel}]_Q^R \left( \delta_S^Q [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V w^s + \left[ (\mathbf{F}^{-1})_r^Q [m_{\parallel}]_b^r F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \right] w^s \right) = \\
&= F_R^t [M_{\parallel}]_V^R (\mathbf{F}^{-1})_s^V w^s + [m_{\parallel}]_r^t [m_{\parallel}]_b^r F_S^b N^S \left[ (\mathbf{F}^{-1})_s^V N_V w^s \right] = \\
&= F_R^t [M_{\parallel}]_V^R (\mathbf{F}^{-1})_s^V w^s + [m_{\parallel}]_b^t \left[ F_S^b [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V w^s \right] \tag{89}
\end{aligned}$$

Analogously, for the orthogonal projector one has

$$\begin{aligned}
[m_\perp]_s^r [m_\perp]_t^s &= [m_\perp]_t^r ; \\
\frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S (\mathbf{F}^{-1})_s^V &\frac{g_G^{*D}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_D^s [M_\perp]_W^G (\mathbf{F}^{-1})_t^W = \\
&= \frac{g_S^{*E} g_G^{*D}}{\|\mathbf{F}^{-T}\mathbf{N}\|^4} \delta_D^V F_E^r [M_\perp]_V^S [M_\perp]_W^G (\mathbf{F}^{-1})_t^W = \\
&= \frac{g_S^{*E} g_G^{*D} N_D N^G}{\|\mathbf{F}^{-T}\mathbf{N}\|^4} F_E^r N^S N_W (\mathbf{F}^{-1})_t^W = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_W^S (\mathbf{F}^{-1})_t^W ; \tag{90}
\end{aligned}$$

The examples which follow provide some insight into the use of surface projectors, and are especially suitable to characterize the moving frame vectors over a boundary face (namely  $\mathbf{B}, \mathbf{T}, \mathbf{N}$ ) and their transformation.

(i) If we multiply both the projectors by the covariant normal vector expressed through its transport rule, we expect that the orthogonal projection coincides with the same vector, whilst the tangential projection vanishes. By formulae

$$\begin{aligned}
[m_\perp]_s^r n_r &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S (\mathbf{F}^{-1})_s^V \frac{(\mathbf{F}^{-1})_r^D N_D}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} [M_\perp]_V^S (\mathbf{F}^{-1})_s^V \delta_E^D N_D \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \\
&= \frac{g_S^{*E} N_E N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_V (\mathbf{F}^{-1})_s^V \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \\
&= \frac{N_V (\mathbf{F}^{-1})_s^V}{\|\mathbf{F}^{-T}\mathbf{N}\|} = n_s ; \tag{91}
\end{aligned}$$

and, by using the very last expression in Eq. (86), one has

$$\begin{aligned}
[m_\parallel]_s^r n_r &= (\delta_s^r - [m_\perp]_s^r) n_r = 0 = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_\perp]_V^S \right\} (\mathbf{F}^{-1})_s^V \frac{(\mathbf{F}^{-1})_r^D N_D}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ N^S N_E \delta_V^D N_D - \delta_E^D N_D N^S N_V \right\} (\mathbf{F}^{-1})_s^V \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|} = 0 ; \tag{92}
\end{aligned}$$

If instead we consider the Eulerian vector  $\tilde{n}^s = F_Q^s N_Q$ , we expect that both its projections do not vanish. In fact one has

$$\begin{aligned}
[m_\perp]_s^r \tilde{n}^s &= n^r n_s \tilde{n}^s = n^r n_s F_Q^s N_Q = \\
&= n^r \frac{(\mathbf{F}^{-1})_s^W N_W}{\|\mathbf{F}^{-T}\mathbf{N}\|} F_Q^s N_Q = n^r \frac{\delta_Q^W N_W N^Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|} ; \tag{93}
\end{aligned}$$

The same result is attained by utilizing Eq. (85), namely

$$\begin{aligned}
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S (\mathbf{F}^{-1})_s^V F_Q^s N_Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S \delta_Q^V N^Q = \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r N^S = \\
&= \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r N^S g^{ts} (\mathbf{F}^{-1})_t^E (\mathbf{F}^{-1})_s^O g_{OS} = \frac{1}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \delta_t^r g^{ts} (\mathbf{F}^{-1})_s^O N_O = \frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|} ; \tag{94}
\end{aligned}$$

As for the tangential projection, one has

$$\begin{aligned} [m_{\parallel}]_s^r \tilde{n}^s &= (\delta_s^r - [m_{\perp}]_s^r) F_Q^s N^Q = F_Q^r N^Q - n^r n_s F_Q^s N^Q = \\ &= F_Q^r N^Q - n^r \frac{(\mathbf{F}^{-1})_s^D N_D}{\|\mathbf{F}^{-T}\mathbf{N}\|} F_Q^s N^Q = F_Q^r N^Q - n^r \frac{\delta_Q^D N_D N^Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \tilde{n}^r - \frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \end{aligned} \quad (95)$$

or alternatively, through Eq. (85),

$$\begin{aligned} &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_{\perp}]_V^S \right\} (\mathbf{F}^{-1})_s^V F_Q^s N^Q = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E \delta_Q^V N^Q - F_E^r [M_{\perp}]_V^S \delta_Q^V N^Q \right\} = \\ &= \frac{g_S^{*E} N^S N_E}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^V \right\} - \frac{g_S^{*E} N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_E^r \right\} = \\ &= \tilde{n}^r - \frac{n^r}{\|\mathbf{F}^{-T}\mathbf{N}\|}; \end{aligned} \quad (96)$$

(ii) If we consider the vector  $t^r = F_Q^r T^Q \|\mathbf{F}\mathbf{T}\|^{-1}$ , we expect that its tangential projection equals the same vector, whilst the orthogonal projection vanishes. In fact,

$$\begin{aligned} [m_{\perp}]_s^r t^s &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \frac{F_Q^s T^Q}{\|\mathbf{F}\mathbf{T}\|} = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \delta_Q^V T^Q \frac{1}{\|\mathbf{F}\mathbf{T}\|} = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N^S \underbrace{N_V T^V}_{=0} (\mathbf{F}^{-1})_s^V \frac{1}{\|\mathbf{F}\mathbf{T}\|} = 0; \end{aligned} \quad (97)$$

whilst, for the tangential projection, through Eq. (86) one finds

$$\begin{aligned} [m_{\parallel}]_s^r t^s &= (\delta_s^r - [m_{\perp}]_s^r) t^s = t^r = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_{\perp}]_V^S \right\} (\mathbf{F}^{-1})_s^V \frac{F_Q^s T^Q}{\|\mathbf{F}\mathbf{T}\|} = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_{\perp}]_V^S \right\} \delta_Q^V T^Q \frac{1}{\|\mathbf{F}\mathbf{T}\|} = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r T^V N^S N_E - F_E^r [M_{\perp}]_V^S T^V \right\} \frac{1}{\|\mathbf{F}\mathbf{T}\|} = \\ &= \frac{g_S^{*E} N^S N_E}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r T^V \right\} \frac{1}{\|\mathbf{F}\mathbf{T}\|} = \frac{F_V^r T^V}{\|\mathbf{F}\mathbf{T}\|} = t^r; \end{aligned} \quad (98)$$

On the contrary, if we consider the Eulerian vector  $\check{t}_r = (\mathbf{F}^{-1})_r^Q T_Q$ , we expect that both its projections do not vanish. As for the orthogonal component, one finds

$$\begin{aligned} [m_{\perp}]_s^r \check{t}_r &= n^r n_s \check{t}_r = \langle \mathbf{n}, \check{\mathbf{t}} \rangle n_s = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V (\mathbf{F}^{-1})_r^Q T_Q = \\ &= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \delta_E^Q T_Q = \\ &= \frac{g_S^{*E} N^S T_E}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_V (\mathbf{F}^{-1})_s^V = \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{T} \rangle_g \frac{n_s}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \langle \mathbf{n}, \check{\mathbf{t}} \rangle_g n_s; \end{aligned} \quad (99)$$

whilst for the tangential projection one has

$$\begin{aligned}
[m_\parallel]_s^r \check{t}_r &= (\delta_s^r - [m_\perp]_s^r) \check{t}_r = \check{t}_s - \langle \mathbf{n}, \check{\mathbf{t}} \rangle_g n_s = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_\perp]_V^S \right\} (\mathbf{F}^{-1})_s^V (\mathbf{F}^{-1})_r^Q T_Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ \delta_V^Q T_Q N^S N_E - \delta_E^Q T_Q [M_\perp]_V^S \right\} = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ T_V N^S N_E - T_E [M_\perp]_V^S \right\} (\mathbf{F}^{-1})_s^V = \\
&= (\mathbf{F}^{-1})_s^V T_V - \frac{g_S^{*E} T_E N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} (\mathbf{F}^{-1})_s^V N_V = \\
&= \check{t}_s - \langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{N} \rangle_g \frac{n_s}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \check{t}_s - \langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{n} \rangle_g n_s; \tag{100}
\end{aligned}$$

(iii) If we consider the Eulerian vector  $\check{b}^r = F_Q^r B^Q$ , being  $B^Q$  the Lagrangian vector normal to the border edge which belongs to the tangent plane, we expect that the tangential projection equals the same vector, whilst its orthogonal projection trivially vanishes. In fact, through the transformation of the covariant normal vector one finds

$$[m_\perp]_s^r \check{b}^s = n^r n_s \check{b}^s = n^r \frac{(\mathbf{F}^{-1})_s^Q N_Q}{\|\mathbf{F}^{-T}\mathbf{N}\|} F_Q^s B^Q = 0; \tag{101}$$

and the same result is provided through Eq. (85), namely

$$\begin{aligned}
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S (\mathbf{F}^{-1})_s^V F_Q^s B^Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S \delta_Q^V B^Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r N^S N_V B^V = 0; \tag{102}
\end{aligned}$$

As for the tangential projection, one finds

$$\begin{aligned}
[m_\parallel]_s^r \check{b}^s &= (\delta_s^r - [m_\perp]_s^r) \check{b}^s = \check{b}^r - n^r n_s \check{b}^s = \check{b}^r = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_\perp]_V^S \right\} (\mathbf{F}^{-1})_s^V F_Q^s B^Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_\perp]_V^S \right\} \delta_Q^V B^Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r B^V N^S N_E - F_E^r N^S N_V B^V \right\} \frac{1}{\|\mathbf{F}\mathbf{T}\|} = \\
&= \frac{g_S^{*E} N^S N_E}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r B^V \right\} = F_V^r B^V = \check{b}^r; \tag{103}
\end{aligned}$$

If we consider the Eulerian vector  $\check{b}_r = (\mathbf{F}^{-1})_r^Q B_Q$ , both its projections do not vanish. As for the orthogonal component, one obtains

$$\begin{aligned}
[m_\perp]_s^r \check{b}_r &= n^r n_s \check{b}_r = \langle \mathbf{n}, \check{\mathbf{b}} \rangle_g n_s = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_\perp]_V^S (\mathbf{F}^{-1})_s^V (\mathbf{F}^{-1})_r^Q B_Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} [M_\perp]_V^S (\mathbf{F}^{-1})_s^V \delta_E^Q B_Q = \\
&= \frac{g_S^{*E} N^S B_E}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} N_V (\mathbf{F}^{-1})_s^V = \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{B} \rangle_g \frac{n_s}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \langle \mathbf{n}, \check{\mathbf{b}} \rangle_g n_s; \tag{104}
\end{aligned}$$

whilst for the tangential component one finds

$$\begin{aligned}
[m_{\parallel}]_s^r \check{b}_r &= (\delta_s^r - [m_{\perp}]_s^r) \check{b}_r = \check{b}_s - \langle \mathbf{n}, \check{\mathbf{b}} \rangle_g n_s = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ F_V^r N^S N_E - F_E^r [M_{\perp}]_V^S \right\} (\mathbf{F}^{-1})_s^V (\mathbf{F}^{-1})_r^Q B_Q = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ \delta_V^Q B_Q N^S N_E - \delta_E^Q B_Q [M_{\perp}]_V^S \right\} = \\
&= \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} \left\{ B_V N^S N_E - B_E [M_{\perp}]_V^S \right\} (\mathbf{F}^{-1})_s^V = \\
&= (\mathbf{F}^{-1})_s^V B_V - \frac{g_S^{*E} T_E N^S}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} (\mathbf{F}^{-1})_s^V N_V = \\
&= \check{b}_s - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g \frac{n_s}{\|\mathbf{F}^{-T}\mathbf{N}\|} = \check{b}_s - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{n} \rangle_g n_s; \tag{105}
\end{aligned}$$

Since the transport formulae are available for both the covariant and contravariant components of the edge normal vector, we can proceed with a further check. As for the orthogonal projection, one finds

$$\begin{aligned}
[m_{\perp}]_s^r b_r &= [m_{\perp}]_s^r \left\{ (\mathbf{F}^{-1})_r^R B_R - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} (\mathbf{F}^{-1})_r^R N_R \right\} \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} = \\
&= \left\{ \langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{B} \rangle_g \frac{n_s}{\|\mathbf{F}^{-T}\mathbf{N}\|} - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} (\mathbf{F}^{-1})_r^R N_R \right\} \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} = 0; \tag{106}
\end{aligned}$$

whilst for the tangential component one has

$$\begin{aligned}
[m_{\parallel}]_s^r b_r &= (\delta_s^r - [m_{\perp}]_s^r) \left\{ (\mathbf{F}^{-1})_r^R B_R - \frac{\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g}{\langle \mathbf{F}^{-T}\mathbf{N}, \mathbf{F}^{-T}\mathbf{N} \rangle_g} (\mathbf{F}^{-1})_r^R N_R \right\} \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} = \\
&= \left\{ (\mathbf{F}^{-1})_r^R B_R - \langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{n} \rangle_g n_s - 0 \right\} \frac{\|\mathbf{F}^{-T}\mathbf{N}\|}{\|J^{-1}\mathbf{F}\mathbf{T}\|} = b_r; \tag{107}
\end{aligned}$$

(iv) The following relationships, trivial to prove, are worth considering:

$$\begin{aligned}
\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}\mathbf{N} \rangle_g &= g_r^s (\mathbf{F}^{-1})_s^V B_V F_M^r N^M = \delta_V^M B_V N^M = 0; \\
\langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}\mathbf{N} \rangle_g &= g_r^s (\mathbf{F}^{-1})_s^V T_V F_M^r N^M = \delta_V^M T_V N^M = 0; \tag{108}
\end{aligned}$$

since the orthogonality of the above Eulerian vectors was not evident. But, for a generic tangent map (diverse from an isometric transformation), we expect

$$\begin{aligned}
\langle \mathbf{F}^{-T}\mathbf{B}, \mathbf{F}^{-T}\mathbf{N} \rangle_g &\neq 0; \\
\langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{N} \rangle_g &\neq 0; \\
\langle \mathbf{F}^{-T}\mathbf{T}, \mathbf{F}^{-T}\mathbf{B} \rangle_g &\neq 0; \\
\langle \mathbf{F}\mathbf{T}, \mathbf{F}\mathbf{B} \rangle_g &\neq 0; \tag{109}
\end{aligned}$$

(v) It can be useful to arrange the transport relationships for the two surface projectors in the form of a system of equations:

$$\begin{cases} [m_{\perp}]_s^r = \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V \\ [m_{\parallel}]_s^r = F_S^r [M_{\parallel}]_V^S (\mathbf{F}^{-1})_s^V + \left[ F_S^r (\mathbf{F}^{-1})_s^V - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r (\mathbf{F}^{-1})_s^V \right] [M_{\perp}]_V^S \end{cases} \tag{110}$$

Hence, to characterize the generic (contravariant) Eulerian vector with respect to the boundary face, one can multiply by it both sides of the above equations, visualizing simultaneously the results in both the Lagrangian and the Eulerian configuration. For instance, let us consider a vector  $w^s$  orthogonal to the face, such that it results  $[m_{\parallel}]_s^r w^s = 0$  in the Eulerian configuration. We have

$$\begin{cases} [m_{\perp}]_s^r w^s = \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S (\mathbf{F}^{-1})_s^V w^s \\ [m_{\parallel}]_s^r w^s = 0 = \left\{ F_S^r [M_{\parallel}]_V^S + \left[ F_S^r [M_{\perp}]_V^S - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S \right] \right\} (\mathbf{F}^{-1})_s^V w^s \end{cases} \quad (111)$$

By the position  $W^V = (\mathbf{F}^{-1})_s^V w^s$ , one finds

$$\begin{cases} [m_{\perp}]_s^r w^s = \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S W^V \\ [m_{\parallel}]_s^r w^s = 0 = \left\{ F_S^r [M_{\parallel}]_V^S W^V + \left[ F_S^r - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r \right] [M_{\perp}]_V^S W^V \right\} \end{cases} \quad (112)$$

It is clear that, since vector  $w^s$  is orthogonal to the face in the Eulerian configuration, when transported to the Lagrangian configuration by  $\mathbf{F}^{-1}$  ( $\forall \mathbf{F}$ ,  $J = \det(\mathbf{F}) > 0$ ) it possesses also a tangential component, namely  $[M_{\parallel}]_V^S W^V \neq \mathbf{0}$ . However, such a tangential component through Equation (112) must give rise to a contribution equal opposite to that generated by its normal component  $[M_{\perp}]_V^S W^V$ .

On the contrary, when we consider a vector tangent to the same face of the Eulerian configuration, one has

$$\begin{cases} [m_{\perp}]_s^r w^s = 0 = \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r [M_{\perp}]_V^S W^V \\ [m_{\parallel}]_s^r w^s = \left\{ F_S^r [M_{\parallel}]_V^S W^V + \left[ F_S^r - \frac{g_S^{*E}}{\|\mathbf{F}^{-T}\mathbf{N}\|^2} F_E^r \right] [M_{\perp}]_V^S W^V \right\} = \\ = F_S^r [M_{\parallel}]_V^S W^V = F_S^r W^S = w^s ; \end{cases} \quad (113)$$

Thus, the image of such a vector through the inverse tangent map ( $\forall \mathbf{F}$ ,  $J > 0$ ) must possess a purely tangential component in the Lagrangian configuration, namely  $[M_{\perp}]_V^S W^V = \mathbf{0}$ .