



# An Introduction to Hyperholomorphic Spectral Theories and Fractional Powers of Vector Operators

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**Abstract.** The aim of this paper is to give an overview of the spectral theories associated with the notions of holomorphicity in dimension greater than one. A first natural extension is the theory of several complex variables whose Cauchy formula is used to define the holomorphic functional calculus for  $n$ -tuples of operators  $(A_1, \dots, A_n)$ . A second way is to consider hyperholomorphic functions of quaternionic or paravector variables. In this case, by the Fueter-Sce-Qian mapping theorem, we have two different notions of hyperholomorphic functions that are called slice hyperholomorphic functions and monogenic functions. Slice hyperholomorphic functions generate the spectral theory based on the  $S$ -spectrum while monogenic functions induce the spectral theory based on the monogenic spectrum. There is also an interesting relation between the two hyperholomorphic spectral theories via the  $F$ -functional calculus. The two hyperholomorphic spectral theories have different and complementary applications. We finally discuss how to define the fractional Fourier's law for nonhomogeneous materials using the spectral theory on the  $S$ -spectrum.

**Mathematics Subject Classification.** 47A10, 47A60.

**Keywords.** Spectral theory,  $S$ -spectrum, Monogenic spectrum, Hyperholomorphic spectral theories, Fractional powers of vector operators.

## 1. Introduction

The problem to define functions of an operator  $A$  or of an  $n$ -tuple of operators  $(A_1, \dots, A_n)$  is very important both in mathematics and in physics and has been investigated with different methods starting from the beginning of the

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This article is part of the Topical Collection on Proceedings ICCA 12, Hefei, 2020, edited by Guangbin Ren, Uwe Kähler, Rafal Ablamowicz, Fabrizio Colombo, Pierre Dechant, Jacques Helmstetter, G. Stacey Staples, Wei Wang.

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last century. The spectral theorem is one of the most important tools to define functions of normal operators on a Hilbert space and it is of crucial importance in quantum mechanics as well as the Weyl functional calculus.

The theory of holomorphic functions plays a central role in operator theory. In fact, the Cauchy formula allows to define the holomorphic functional calculus (often called Riesz-Dunford functional calculus) in Banach spaces [14], and this calculus can be extended to unbounded operators. For sectorial operators the  $H^\infty$ -functional calculus, introduced by A. McIntosh in [71], turned out to be the most important extension.

Holomorphic functions of one complex variable  $f : \Omega \subseteq \mathbb{C} \rightarrow \mathbb{C}$  (denoted by  $\mathcal{O}(\Omega)$ ) have the following extensions:

(E1) Systems of Cauchy-Riemann equations, for functions

$$f : \Pi \subseteq \mathbb{C}^n \rightarrow \mathbb{C},$$

give the theory of several complex variables.

(E2) Holomorphicity of vector fields is connected with quaternionic-valued functions or more in general with Clifford algebra-valued functions. There are two different extensions that are obtained by the Fueter-Sce-Qian theorem, also called the Fueter-Sce-Qian construction, and gives two different notions of hyperholomorphic functions, see for more details [33].

Consider functions defined on an open set  $U$  in the quaternions  $\mathbb{H}$  or in  $\mathbb{R}^{n+1}$  for Clifford algebra-valued functions, then the Fueter-Sce-Qian extension consists of two steps.

Step (I) gives the class of slice hyperholomorphic functions (denoted by  $SH(U)$ ), these functions are also called slice monogenic for Clifford algebra-valued functions and slice regular in the quaternionic case.

Step (II) gives the monogenic functions (denoted by  $M(U)$ ) and Fueter regular functions in the case of the quaternions.

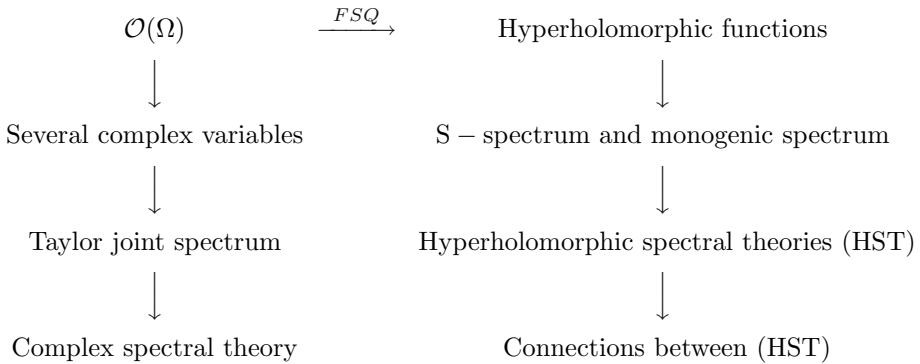
Both classes of hyperholomorphic functions have a Cauchy formula that can be used to define functions of quaternionic operators or of  $n$ -tuples of operators that do not necessarily commute.

(S) The Cauchy formula of slice hyperholomorphic functions generates the  $S$ -functional calculus for quaternionic linear operators or for  $n$ -tuples of not necessarily commuting operators, this calculus is based on the the notion of  $S$ -spectrum. The spectral theorem for quaternionic operators is also based on the  $S$ -spectrum.

(M) The Cauchy formula of monogenic functions generates the monogenic functional calculus that is based on the monogenic spectrum.

The hyperholomorphic functional calculi coincide with the Riesz-Dunford functional calculus when they are applied to a single complex operator.

If we denote by the symbol  $FSQ$  the Fueter-Sce-Qian construction then the following diagram illustrates the possible extensions:



The first mathematicians who understood the importance of hypercomplex analysis to define functions of noncommuting operators on Banach spaces were A. McIntosh and his collaborators, starting from preliminary results in [74]. Using the theory of monogenic functions they developed the monogenic functional calculus and several of its applications, see [73]. The  $S$ -functional calculus, and in general the spectral theory on the  $S$ -spectrum, started its development only in 2006 when F. Colombo and I. Sabadini discovered the  $S$ -spectrum. The discovery of the  $S$ -spectrum and of the  $S$ -functional calculus is well explained in the introduction of the book [24] with a complete list of the references and it is also described how hypercomplex analysis methods were used to identify the appropriate notion of quaternionic spectrum whose existence was suggested by quaternionic quantum mechanics.

If we denote by  $\mathcal{B}(V)$  the Banach space of all bounded right linear operators acting on a two sided quaternionic Banach space  $V$  then the appropriate notion of quaternionic spectrum, the  $S$ -spectrum, is defined in a very counterintuitive way because it involves the square of the quaternionic linear operator  $T$  and it is defined as:

$$\sigma_S(T) = \{s \in \mathbb{H} \mid T^2 - 2s_0T + |s|^2\mathcal{I} \text{ is not invertible in } \mathcal{B}(V)\}.$$

The  $S$ -spectrum for quaternionic operators can be naturally defined also for paravector operators when we work in a Clifford algebra, see [45] and the book [48].

The problem of defining the quaternionic spectrum for the quaternionic spectral theorem has been an open problem for long time even though several attempts have been done by several authors in the past decades, see e.g. [85, 87], however the correct definition of spectrum was not specified. Finally in [2] the spectral theorem on the  $S$ -spectrum was proved for both bounded and unbounded normal operators on a quaternionic Hilbert space.

The main problems with the quaternionic notion of spectrum can be described with the following considerations related to bounded linear operators just for the sake of simplicity. Let  $T : V \rightarrow V$  be a right linear bounded quaternionic operator acting on a two sided quaternionic Banach space  $V$ . If we readapt the notion of spectrum for a complex linear operator to the

quaternionic setting we obtain two different notions of spectra because of the noncommutativity of the quaternions. The left spectrum  $\sigma_L(T)$  of  $T$  is defined as

$$\sigma_L(T) = \{s \in \mathbb{H} \mid s\mathcal{I} - T \text{ is not invertible in } \mathcal{B}(V)\},$$

where the notation  $s\mathcal{I}$  in  $\mathcal{B}(V)$  means that  $(s\mathcal{I})(v) = sv$ . The right spectrum  $\sigma_R(T)$  of  $T$  is associated with the right eigenvalue problem, i.e., the search of those quaternions  $s$  such that there exists a nonzero vector  $v \in V$  satisfying

$$T(v) = vs.$$

In both spectral problems it is unclear how to associate to the spectrum a resolvent operator with the property of being an hyperholomorphic function operator-valued. In fact, for the left spectrum  $\sigma_L(T)$  it is not clear what notion of hyperholomorphicity is associated to the map  $s \rightarrow (s\mathcal{I} - T)^{-1}$ , for  $s \in \mathbb{H} \setminus \sigma_L(T)$  and for the right spectrum it is even more weird because the operator  $\mathcal{I}s - T$  (where  $\mathcal{I}s$  means  $(\mathcal{I}s)(v) = vs$ ) is not linear, so it is not clear which operator is the candidate to be the resolvent operator.

*Remark 1.1.* One of the main motivations that suggested the existence of the  $S$ -spectrum is the paper [13] by G. Birkhoff and J. von Neumann, where they showed that quantum mechanics can be formulated also on quaternionic numbers. Since that time, several papers and books treated this topic, however it is interesting, and somewhat surprising, that an appropriate notion of spectrum for quaternionic linear operators was not present in the literature. Moreover, in quaternionic quantum mechanics the right spectrum  $\sigma_R(T)$  is the most useful notion of spectrum to study the bounded states of a quantum systems. Before 2006 only in one case the quaternionic spectral theorem was proved specifying the spectrum and it is the case of quaternionic normal matrices, see [53], where the right spectrum  $\sigma_R(T)$  has been used.

Now we recall some research directions and applications of the hyperholomorphic function theories and related spectral theories.

The *first step of FSQ-construction* generates *slice hyperholomorphic functions* and the *spectral theory of the  $S$ -spectrum*, has the following research directions:

- The foundation of the quaternionic spectral theory on the  $S$ -spectrum are organized in the books [23, 24], and for paravector operators see [48].
- The mathematical tools for quaternionic quantum mechanics is the spectral theorem based on the  $S$ -spectrum [2, 57].
- Quaternionic evolution operators, Phillips functional calculus,  $H^\infty$ -functional calculus, see [23].
- Quaternionic approximation [56].
- The characteristic operator functions and applications to linear system theory [5].
- Quaternionic spectral operators [58].
- Quaternionic perturbation theory and invariant subspaces [16].
- Schur analysis in the slice hyperholomorphic setting [4].
- The theory of function spaces of slice hyperholomorphic functions [6].

- New classes of fractional diffusion problems based on fractional powers of quaternionic linear operators, see the book [23] and the more recent contributions [17–19, 31, 32].

In the last section of this paper we explain how to treat fractional diffusion problems using the quaternionic spectral theory on the  $S$ -spectrum and we show some of the results on fractional Fourier's law for nonhomogeneous materials recently obtained. An example of problems that we can treat is the following.

We warn the reader that in this paper, with an abuse of notation, we use the symbol  $\underline{x}$  for both the coordinates of a point  $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  or for the vector part of a quaternion or for the imaginary part of a paravector in a Clifford algebra.

Let  $\Omega$  be a bounded or an unbounded domain in  $\mathbb{R}^3$  and let  $\tau > 0$  and denote by  $v$  the temperature of the material contained in  $\Omega$ . Let  $\underline{x} = (x_1, x_2, x_3) \in \Omega$  and consider the evolution problem

$$\begin{cases} \partial_t v(\underline{x}, t) + \operatorname{div} T(\underline{x})v(\underline{x}, t) = 0, & (\underline{x}, t) \in \Omega \times (0, \tau], \\ v(\underline{x}, 0) = f(\underline{x}), & \underline{x} \in \Omega, \\ v(\underline{x}, t) = 0, & \underline{x} \in \partial\Omega \quad t \in [0, \tau], \end{cases}$$

where  $f$  is a given datum and the heat flux for the nonhomogeneous material contained in  $\Omega$ , is given by the vector differential operator:

$$T(\underline{x}) = a(\underline{x})\partial_{x_1}e_1 + b(\underline{x})\partial_{x_2}e_2 + c(\underline{x})\partial_{x_3}e_3, \quad \underline{x} \in \Omega.$$

We determine the conditions on the coefficients  $a, b, c : \Omega \rightarrow \mathbb{R}$  under which the operator  $T(\underline{x})$  generates the fractional powers  $P_\alpha(T(\underline{x}))$  of  $T(\underline{x})$ , for  $\alpha \in (0, 1)$ . The vector part of  $P_\alpha(T(\underline{x}))$  of  $T(\underline{x})$  is defined to be the nonlocal Fourier's law associated with  $T(\underline{x})$ .

The *second step of FSQ-construction* generates *Fueter or monogenic functions* and the *spectral theory on the monogenic spectrum*. We highlight some references for the research directions in this area:

- Monogenic spectral theory and applications [73]. Here one can also find the relations of the monogenic functional calculus with the Taylor functional calculus and the Weyl functional calculus see also some of the original contributions [68–70, 72, 79].
- Harmonic analysis in higher dimension, singular integrals and Fourier transform see the recent book [81].
- Algebraic Analysis of Dirac systems [44].
- The theory of spinor valued function [52].
- Boundary value problems treated with quaternionic techniques [64].
- The extension of Schur analysis in the Fueter setting and related topics [8–10].
- Dirac operator on manifolds and spectral theory [54, 65].

Prior to the recent developments on slice hyperholomorphic functions, this function theory was simply seen as an intermediate step in the Fueter-Sce-Qian construction. The literature on hyperholomorphic function theories and related spectral theories is nowadays very large. For the function theory

of slice hyperholomorphic functions the main books are [6, 46, 48, 56, 60], while for the spectral theory on the  $S$ -spectrum we mention the books [5, 23, 24, 48]. For the Fueter and monogenic function theory and related topics see the books [15, 44, 52, 65, 66, 73, 82].

## 2. Spectral Theory in the Complex Setting

In this section we discuss what is a functional calculus of a single operator on a Banach space and also for the case of several operators. When we consider a closed linear operator  $A$  with domain  $\mathcal{D}(A) \subset X$ , where  $X$  is a Banach space, the resolvent set  $\rho(A)$  of  $A$  is defined as

$$\rho(A) = \{\lambda \in \mathbb{C} \mid (\lambda\mathcal{I} - A)^{-1} \in B(X)\}$$

where  $B(X)$  in the space of all bounded linear operators on  $X$  and the spectrum of  $A$  is the set  $\sigma(A) = \mathbb{C} \setminus \rho(A)$  and for  $\lambda \in \rho(A)$  the map  $\lambda \rightarrow (\lambda\mathcal{I} - A)^{-1}$  is called the resolvent operator. We start with the following intuitive definition of what is a functional calculus.

*A functional calculus for a closed linear operator  $A$  on complex Banach space  $X$  is a mathematical technique that allows to construct in a meaningful way an operator  $f(A)$  for any function  $f$  in a certain class of functions  $\mathfrak{F}$  defined on sets that contain the spectrum  $\sigma(A)$  of  $A$ .*

The formulation *in a meaningful way* usually means that the functional calculus is compatible with formally plugging  $A$  into the function  $f$ , whenever this is possible. That is, whenever  $f(z)$  can be expressed by a formula so that formally replacing  $z$  by the operator  $A$  yields an expression that is meaningful, then  $f(A)$  should correspond to this expression. Several examples below illustrate this idea:

- (a) For any polynomial  $p(z) = a_n z^n + \dots + a_1 z + a_0 \in \mathfrak{F}$  with  $a_\ell \in \mathbb{C}$ , the operator  $p(A)$  should be given by

$$p(A) = a_n A^n + \dots + a_1 A + a_0 \mathcal{I}.$$

- (b) For any  $\lambda \in \rho(A)$  with  $R_\lambda(z) = (\lambda - z)^{-1} \in \mathfrak{F}$ , the functional calculus is compatible with the resolvent operator at  $\lambda$ . That is, we have  $R_\lambda(A) = (\lambda\mathcal{I} - A)^{-1}$ .

- (c) For any rational function  $r(z) = p(z)/q(z) \in \mathfrak{F}$  with polynomials

$$p(z) = a_n z^n + \dots + a_1 z + a_0$$

and

$$q(z) = b_m z^m + \dots + b_1 z + b_0$$

with  $a_\ell, b_\ell \in \mathbb{C}$ , the operator  $r(A)$  should be given by

$$r(A) = p(A)q(A)^{-1},$$

where

$$p(A) = a_n A^n + \dots + a_1 A + a_0 \mathcal{I}$$

and

$$q(A) = b_m A^m + \dots + b_1 A + b_0 \mathcal{I}.$$

- (d) If  $A$  is the infinitesimal generator of a strongly continuous group  $U_A(t), t \geq 0$  and  $\exp(tz) \in \mathfrak{F}$ , then  $\exp(tA) = U_A(t)$ .

Of course in the case of unbounded operators one has to pay attention to the domain of the operators. Usually the class  $\mathfrak{F}$  constitutes an algebra, often even a Banach algebra, and the *meaningfulness* of the functional calculus as described above follows from the compatibility of the functional calculus with the algebraic operation. Precisely, a functional calculus usually satisfies several (or all) of the following conditions:

- (I) The functional calculus is an algebra homomorphism that is  $(af + g)(T) = af(T) + g(T)$  for all  $f, g \in \mathfrak{F}$  and all  $a \in \mathbb{C}$ .
- (II) For  $f$  and  $g \in \mathfrak{F}$  such that  $fg \in \mathfrak{F}$  we expect

$$(fg)(A) = f(A)g(A).$$

- (III) For  $f(z) = 1$ , we have  $f(A) = 1(A) = \mathcal{I}$ .
- (VI) For the identity  $f(z) = z$ , we have  $f(A) = z(A) = A$ .
- (V) If  $X$  is a Hilbert space and both  $f$  and  $\overline{f}(z) = \overline{f(z)}$  belong to  $\mathfrak{F}$ , then  $\overline{f(A)} = f(A)^*$ .
- (VI) If  $\mathfrak{F}$  is normed, then the functional calculus defines a continuous mapping into the space of bounded linear operators  $\mathcal{B}(X)$  on  $X$ , that is  $\|f(A)\|_{\mathcal{B}(X)} \leq C\|f\|_{\mathfrak{F}}$ .

We restrict ourselves to the case  $\mathfrak{F}$  consists of functions that are at least continuous and to the case that the topology on  $\mathfrak{F}$  is coarser than the topology of locally uniform convergence. In particular, convergence in  $\mathfrak{F}$  implies pointwise convergence.

For the measurable functional calculus the main statements of this section hold true, but their justification is based on different arguments.

There are two main methods for defining a functional calculus.

**Method 2.1.** *One considers a subalgebra  $\mathfrak{F}_0$  of  $\mathfrak{F}$  that is dense in  $\mathfrak{F}$  such that  $f(A)$  can be defined easily for any  $f \in \mathfrak{F}_0$  (for instance the set of polynomials or the set of rational functions in  $\mathfrak{F}$ ). If  $f \in \mathfrak{F}$  is arbitrary, one chooses an approximating sequence  $(f_n)_{n \in \mathbb{N}}$  in  $\mathfrak{F}_0$  for  $f$  and defines*

$$f(A) := \lim_{n \rightarrow +\infty} f_n(A).$$

**Method 2.2.** *If any  $f \in \mathfrak{F}$  admits an integral representation*

$$f(z) = \int K(\xi, z)f(\xi) d\mu(\xi),$$

*and formally replacing  $z$  by  $A$  in  $K(\xi, z)$  yields a meaningful operator  $K(\xi, A)$ , then one may define*

$$f(A) := \int K(\xi, A)f(\xi) d\mu(\xi).$$

An example for method 2.1 is the continuous functional calculus. With method 2.2 we define for example the Riesz-Dunford-functional calculus or the Philips functional calculus.

There is also another concept behind the notion of functional calculus: *the operator  $f(A)$  should be defined by letting  $f$  act on the spectral values of  $A$ . In particular, this means that*

$$Ax = \lambda x \implies f(A)x = f(\lambda)x, \quad \text{for } x \in X. \tag{1}$$

This idea is usually not so much emphasized in the complex setting when one introduces and explains the concept of a functional calculus and we shall see here in the sequel the reason way this happens. However, it is this relation that explains why functional calculi are the fundamental techniques for investigating linear operators. If it does not hold, then a functional calculus does not provide any information about the operator even though it generates functions of operators.

Interestingly enough a deep difference between the theory of complex and the theory of quaternionic linear operators (or more in general for hyperholomorphic spectral theories) is revealed here so that in the latter, the relation (1) needs to be addressed explicitly.

Let us start our considerations by justifying the importance of the relation (1). We therefore recall the easiest result that is shown by an application of a functional calculus.

**Theorem 2.3.** *Let  $A \in \mathbb{C}^{m \times m}$ . Then  $A$  has an eigenvalue and for any polynomial  $p \in \mathbb{C}[n]$  with  $p(A) = 0$ , the set of eigenvalues of  $A$  is contained in the set of roots of  $p$ .*

It is obvious that even for the above, very easy, and fundamental result, the fact that the polynomial functional calculus satisfies the relation (1) is crucial.

The fact that the relation (1) trivially holds true for any known functional calculus in the complex setting is shown in the next two results, and this is the reason for which it is not explicitly mentioned.

**Theorem 2.4.** *Let  $\Phi : \mathfrak{F} \rightarrow \mathcal{B}(X)$  be a functional calculus for an operator  $A$  defined via method 2.1. If (1) holds true for any function in  $\mathfrak{F}_0$ , then the functional calculus  $\Phi$  is compatible with (1). This is in particular the case if  $\mathfrak{F}_0$  consists of polynomials or rational functions.*

**Theorem 2.5.** *Let  $\Phi : \mathfrak{F} \rightarrow \mathcal{B}(X)$  be a functional calculus for an operator  $A$  defined via method 2.2. If (1) holds true for  $K(\lambda, \cdot)$  for any  $\lambda$ , then the functional calculus  $\Phi$  is compatible with (1).*

We now recall that the spectral theorem works as a functional calculus. In the finite dimensional case when we pick an  $n \times n$  matrix  $A = (a_{i,j})$  for  $i, j = 1, \dots, n$  of complex numbers such that  $\overline{a_{i,j}} = a_{j,i}$  for all  $i, j = 1, \dots, n$ . The spectrum  $\sigma(A)$  consists of eigenvalues of  $A$ , that is, complex numbers  $\lambda$  for which the equation  $Av = \lambda v$  has a nonzero vector  $v \in \mathbb{C}^n$  as a solution. The hermitian matrix  $A$  has a unique decomposition as a finite sum

$$A = \sum_{j=1}^n \lambda_j E_{\lambda_j, A}$$

where  $E_{\lambda_j, A}$  is the orthogonal projection onto the eigenspace of the eigenvalue  $\lambda_j$ . In the case of bounded selfadjoint (or more in general normal operators) operators  $A$  acting in Hilbert space the spectral theorem is the most important tool for the complete description of such operators, in fact we have

$$A = \int_{\sigma(A)} \lambda dE_{\lambda; A}$$

with respect to a spectral measure  $E_{\lambda; A}$  associated with  $A$ . From the spectral theorem we can define  $f(A)$  by

$$f(A) = \int_{\sigma(A)} f(\lambda) dE_{\lambda; A}$$

for any continuous (but also bounded Borel measurable) function  $f : \sigma(A) \rightarrow \mathbb{C}$ . The mapping  $f \mapsto f(A)$  is an algebra homomorphism into the space of bounded linear operators. The spectral theorem can be generalized to the case of  $n$ -tuple of commuting bounded selfadjoint operators  $(A_1, \dots, A_n)$ , as

$$f(A_1, \dots, A_n) = \int_{\sigma(A_1, \dots, A_n)} f(\lambda) dE_{\lambda; A_1, \dots, A_n}$$

is valid for the joint spectral measure  $E_{\lambda; A_1, \dots, A_n}$  associated with  $A_1, \dots, A_n$ . The joint spectrum of  $A_1, \dots, A_n$  in  $\mathbb{R}^n$  is the support of  $E_{\lambda; A_1, \dots, A_n}$  and  $f : \sigma(A_1, \dots, A_n) \rightarrow \mathbb{C}$  is any bounded Borel measurable function. The theorem holds more in general for unbounded normal operators but one has to pay attention to the definition of commutativity in this case, see the book [84].

In the case we work in a Banach space the most natural way to define functions of bounded (and also of unbounded) operators is the Riesz-Dunford functional calculus

$$f(A) = \frac{1}{2\pi i} \int_C (\lambda \mathcal{I} - A)^{-1} f(\lambda) d\lambda$$

which holds for all holomorphic functions  $f$  defined in a neighborhood of  $\sigma(A)$  in the complex plane. The simple closed contour  $C$  surrounds  $\sigma(A)$  and is contained in the domain of the function  $f$ . There is a natural generalization of Riesz-Dunford functional calculus for  $n$ -tuples of bounded operators  $A_1, \dots, A_n$  as

$$f(A_1, \dots, A_n) = \frac{1}{(2\pi i)^n} \int_{C_1} \dots \int_{C_n} (\lambda_1 \mathcal{I} - A_1 \mathbb{1})^{-1} \dots (\lambda_n \mathcal{I} - A_n \mathbb{1})^{-1} f(\lambda_1, \dots, \lambda_n) d\lambda$$

where  $d\lambda = d\lambda_1 \dots d\lambda_n$  and  $f$  is any holomorphic function in a neighborhood of  $\sigma(A_1) \times \dots \times \sigma(A_n)$  in  $\mathbb{C}^n$ . For each  $j = 1, \dots, n$  the simple closed contour  $C_j$  surrounds  $\sigma(A_j)$  and  $C_1 \times \dots \times C_n$  is contained in the domain of  $f$  in  $\mathbb{C}^n$ . Also when the operators  $A_1, \dots, A_n$  do not commute with each other, the functional calculus makes sense with any change in the operator ordering of the function

$$(\lambda_1, \dots, \lambda_n) \mapsto (\lambda_1 \mathcal{I} - A_1 \mathbb{1})^{-1} \dots (\lambda_n \mathcal{I} - A_n \mathbb{1})^{-1}$$

in  $\mathbb{C}^n \setminus (\sigma(A_1) \times \dots \times \sigma(A_n))$ . For noncommuting operators, the results are in general more complicated and we will not enter into the details here (see, e.g., the book [73]).

The material discussed in this section can be found in several classical books, such as [86].

*Some remarks in view of the hyperholomorphic spectral theories.*

(I) In operator theory on the  $S$ -spectrum, the statements of Theorems 2.4, 2.5 still hold true but just for a subclass of functions. The conditions that the functions in the dense subspace  $\mathfrak{F}_0$  resp. the kernel  $K(\lambda, A)$  satisfy (1) is not true in this setting. It is neither satisfied by the  $S$ -resolvent operator, nor by the  $F$ -resolvent operator, nor by slice hyperholomorphic rational functions with non-real coefficients. In particular it is not satisfied, whenever the left-linear structure of the space has a prominent role.

(II) In general for the definitions of hyperholomorphic functional calculi ( $S$ -functional calculus, the  $F$ -functional calculus, the monogenic functional calculus) the product rule, the composition rule or the spectral mapping theorem do not hold. One needs additional arguments to show that functional calculi based on the  $S$ -spectrum satisfy (1) at least for a subclass of functions namely, the class of intrinsic functions and for these functions, the problems mentioned before do not occur.

### 3. Spectral Theories in the Hyperholomorphic Setting

At the beginning of the last century several authors started the study of hyperholomorphic functions and the most popular classes of functions are nowadays called Fueter (or Cauchy-Fueter) regular functions in the case of the quaternions and monogenic functions (or functions in the kernel of the Dirac operator) for Clifford algebra setting. The second class of hyperholomorphic functions have been developed more recently, just at the beginning of this century, and different definitions are possible even though they are not totally equivalent.

In the following we will discuss mainly the implications of the Fueter-Sce-Qian construction in the Clifford setting, the quaternionic setting is similar and we will use it for the fractional powers of vector operators in the last section of this paper.

Let  $\mathbb{R}_n$  be the real Clifford algebra over  $n$  imaginary units  $e_1, \dots, e_n$  satisfying the relations  $e_\ell e_m + e_m e_\ell = 0$ ,  $\ell \neq m$ ,  $e_\ell^2 = -1$ . An element in the Clifford algebra will be denoted by  $\sum_A e_A x_A$  where  $A = \{\ell_1 \dots \ell_r\} \in \mathcal{P}\{1, 2, \dots, n\}$ ,  $\ell_1 < \dots < \ell_r$  is a multi-index and  $e_A = e_{\ell_1} e_{\ell_2} \dots e_{\ell_r}$ ,  $e_\emptyset = 1$ . A point  $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$  will be identified with the element  $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j \in \mathbb{R}_n$  called paravector and the real part  $x_0$  of  $x$  will also be denoted by  $\text{Re}(x)$ . The imaginary part of  $x$  is defined by  $\text{Im}(x) = x_1 e_1 + \dots + x_n e_n$  and for the sake of simplicity we also use the notation  $\underline{x}$  for  $\text{Im}(x)$ . The conjugate of  $x$  is denoted by  $\bar{x} = x_0 - \text{Im}(x)$  and the Euclidean modulus of  $x$  is given by  $|x|^2 = x_0^2 + \dots + x_n^2$ . The sphere of purely imaginary paravectors with modulus 1, is defined by

$$\mathbb{S} = \{x = e_1 x_1 + \dots + e_n x_n \mid x_1^2 + \dots + x_n^2 = 1\}.$$

The element  $I \in \mathbb{S}$  are such that  $I^2 = -1$  so we will denote the complex space with imaginary unit  $I$  by  $\mathbb{C}_I$ . For this reason the elements of  $\mathbb{S}$  are also called

imaginary units. Given a non-real paravector  $x = x_0 + \text{Im}(x) = x_0 + J_x |\text{Im}(x)|$ ,  $J_x := \text{Im}(x)/|\text{Im}(x)| \in \mathbb{S}$ , we can associate to it the sphere defined by

$$[x] = \{x_0 + J|\text{Im}(x)| \mid J \in \mathbb{S}\}.$$

The set of quaternions will be denoted by  $\mathbb{H}$  and the above definitions adapt to this setting in a natural way.

**Definition 3.1.** Let  $U \subseteq \mathbb{R}^{n+1}$  (or  $U \subseteq \mathbb{H}$ ). We say that  $U$  is axially symmetric if, for every  $u + Iv \in U$ , all the elements  $u + Jv$  for  $J \in \mathbb{S}$  are contained in  $U$ .

For operator theory the most appropriate definition of slice hyperholomorphic functions is the one that comes from the Fueter-Sce-Qian mapping theorem because it allows to define functions on axially symmetric open sets.

**Definition 3.2.** Let  $U \subseteq \mathbb{R}^{n+1}$  be an axially symmetric open set and let  $\mathcal{U} \subseteq \mathbb{R} \times \mathbb{R}$  be such that  $x = u + Jv \in U$  for all  $(u, v) \in \mathcal{U}$ . We say that a function  $f : U \rightarrow \mathbb{R}_n$  of the form

$$f(x) = f_0(u, v) + Jf_1(u, v)$$

is left slice hyperholomorphic if  $f_0, f_1$  are  $\mathbb{R}_n$ -valued differentiable functions such that

$$f_0(u, v) = f_0(u, -v), \quad f_1(u, v) = -f_1(u, -v) \quad \text{for all } (u, v) \in \mathcal{U}$$

and if  $f_0$  and  $f_1$  satisfy the Cauchy-Riemann system

$$\partial_u f_0 - \partial_v f_1 = 0, \quad \partial_v f_0 + \partial_u f_1 = 0.$$

The above definition adapts naturally to the quaternionic setting. Since we will restrict just to left slice hyperholomorphic function on  $U$  we introduce the symbol  $SH_L(U)$  to denote them. The subset of intrinsic functions consist of those slice hyperholomorphic functions such that  $f_0, f_1$  are real-valued and is denoted by  $N(U)$ . We recall that right slice hyperholomorphic functions are of the form

$$f(x) = f_0(u, v) + f_1(u, v)J$$

where  $f_0, f_1$  satisfy the above conditions.

**Definition 3.3.** (*Monogenic functions*) Let  $f : U \rightarrow \mathbb{R}_n$  be a continuously differentiable function defined on an open subset  $U \subseteq \mathbb{R}^{n+1}$ . We say that  $f$  is (left) monogenic on  $U$ , if

$$Df(x) = 0$$

where  $D$  is the Dirac operator defined by

$$D = \partial_{x_0} + \sum_{j=1}^n e_j \partial_{x_j}.$$

The definition of slice hyperholomorphic functions and of monogenic functions can be seen as two steps in the Fueter-Sce-Qian constructions to extend holomorphic functions to dimension greater than one for the vector-valued functions (quaternionic or Clifford valued-functions).

In fact, starting from holomorphic functions, R. Fueter in 1935, see [55], showed an interesting way to generate Cauchy-Fueter regular functions. More then 20 years later in 1957 M. Sce, see [83], extended this result in a very pioneering and general way that includes Clifford algebras, see the English translation of his works in hypercomplex analysis with commentaries collected in the recent book [33].

In the original construction of R. Fueter the holomorphic functions are defined on open sets of the complex upper half plane. This condition can be relaxed by taking function

$$g(z) = g_0(u, v) + ig_1(u, v), \quad z = u + iv$$

defined in a set  $D \subseteq \mathbb{C}$ , symmetric with respect to the real axis such that

$$g_0(u, -v) = g_0(u, v) \quad \text{and} \quad g_1(u, -v) = -g_1(u, v)$$

namely if  $g_0$  and  $g_1$  are, respectively, even and odd functions in the variable  $v$ . Additionally the pair  $(g_0, g_1)$  satisfies the Cauchy-Riemann system. The above remark holds also for M. Sce's theorem that we state in the following for Clifford algebras.

**Theorem 3.4.** (Sce [83]) *Consider the Euclidean space  $\mathbb{R}^{n+1}$  whose elements are identified with paravectors  $x = x_0 + \underline{x}$ . Let*

$$\tilde{f}(z) = f_0(u, v) + if_1(u, v)$$

*be a holomorphic function defined in a domain (open and connected)  $D$  in the upper-half complex plane and let*

$$\Omega_D = \{x = x_0 + \underline{x} \mid (x_0, |\underline{x}|) \in D\}$$

*be the open set induced by  $D$  in  $\mathbb{R}^{n+1}$ . The following map*

$$f(x) = T_{FS1}(\tilde{f}) := f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|)$$

*takes the holomorphic functions  $\tilde{f}(z)$  and induces the Clifford-valued function  $f(x)$ . Then the function*

$$\check{f}(x) := T_{FS2}\left(f_0(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} f_1(x_0, |\underline{x}|)\right),$$

*where  $T_{FS2} := \Delta_{n+1}^{\frac{n-1}{2}}$  and  $\Delta_{n+1}$  is the Laplacian in  $n + 1$  dimensions, is in the kernel of the Dirac operator, i.e.,*

$$D\check{f}(x) = 0 \quad \text{on} \quad \Omega_D.$$

The case where the operator  $\Delta_{n+1}^{\frac{n-1}{2}}$  has a fractional index has been treated by T. Qian in [78]. Observe that for the Fueter's theorem the operator  $T_{FS2}$  is equal to the Laplacian  $\Delta$  in 4 dimensions. Further developments can be found in [75–77] see also the survey [80].

We can summarize the Fueter-Sce constructions as follows. Denoting by  $\mathcal{O}(D)$  the set of holomorphic functions on  $D$ , by  $N(\Omega_D)$  the set of induced functions on  $\Omega_D$  (which turn out to be intrinsic slice hyperholomorphic functions) and by  $AM(\Omega_D)$  the set of axially monogenic functions on  $\Omega_D$  the Fueter-Sce construction can be visualized by the diagram:

$$\mathcal{O}(D) \xrightarrow{T_{FS1}} N(\Omega_D) \xrightarrow{T_{FS2}=\Delta^{(n-1)/2}} AM(\Omega_D),$$

where  $T_{FS1}$  denotes the first linear operator of the Fueter-Sce construction and  $T_{FS2}$  the second one. The Fueter-Sce mapping theorem induces two spectral theories according to the two classes of hyperholomorphic functions it generates.

Recently also the problem of construction the inversion of the maps that appear in the Fueter-Sce-Qian extension has been treated, we mention the papers [11, 12, 34, 42, 43], while a different method to connect slice monogenic and monogenic functions is via the Radon and dual Radon transform, see [30].

*Remark 3.5.* The theory of slice hyperholomorphic functions was somewhat abandoned until 2006 when G. Gentili and D. C. Struppa (inspired by C. G. Cullen [51]) introduced in [59] the notion of slice regular functions for the quaternions. Further developments of the theory of slice regular functions were discussed also in [28] and the above definition was extended by F. Colombo, I. Sabadini and D.C. Struppa, in [47], (see also [35, 49, 50]) to the Clifford algebra setting. Slice regular functions as defined in [59] and their generalization to the Clifford algebra as in [47], called slice monogenic functions, possess good properties on specific open sets that are called axially symmetric slice domains. When it is not necessary to distinguish between the quaternionic case and the Clifford algebra case we call these functions slice hyperholomorphic. The extension of the notion of a slice hyperholomorphic function on real alternative algebras can be found in [62].

*Remark 3.6.* It is also possible to define slice hyperholomorphic functions, as functions in the kernel of the first order linear differential operator (introduced in [29])

$$Gf = \left( |x|^2 \frac{\partial}{\partial x_0} + \underline{x} \sum_{j=1}^n x_j \frac{\partial}{\partial x_j} \right) f = 0,$$

where  $\underline{x} = x_1 e_1 + \dots + x_n e_n$ . While, a fourth way to introduce slice hyperholomorphicity, done in 1998 by G. Laville and I. Ramadanoff in the paper [67], is inspired by the Fueter-Sce-Qian mapping theorem. They introduce the so called Holomorphic Cliffordian functions defined by the differential equation  $D\Delta^m f = 0$  over  $\mathbb{R}^{2m+1}$ , where  $D$  is the Dirac operator. Observe that the definition via the global operator  $G$  requires less regularity of the functions than the definition in [67].

We now recall the hyperholomorphic Cauchy formulas that are the heart of the hyperholomorphic spectral theories. It is important to remark that the hypotheses of the following Cauchy formula are related to the Definition 3.2 of slice hyperholomorphic functions.

**Theorem 3.7.** (*Cauchy formula for slice hyperholomorphic functions*) Let  $U \subseteq \mathbb{R}^{n+1}$  be an axially symmetric open set such that  $\partial(U \cap \mathbb{C}_I)$  is union of a finite number of continuously differentiable Jordan curves, for every  $I \in \mathbb{S}$ . Let  $f$  be an  $\mathbb{R}_n$ -valued slice hyperholomorphic function on an open set containing  $\bar{U}$  and, for any  $I \in \mathbb{S}$ , we set  $ds_I = -Ids$ . Then, for every  $x \in U$ , we have:

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, x) ds_I f(s), \tag{2}$$

where the slice hyperholomorphic Cauchy kernel is given by

$$S_L^{-1}(s, x) = -(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}(x - \bar{s}), \quad s \notin [x] \tag{3}$$

and the value of the integral (2) depends neither on  $U$  nor on the imaginary unit  $I \in \mathbb{S}$ . Moreover, the Cauchy kernel can be written in two ways as follows:

$$\begin{aligned} S_L^{-1}(s, x) &= -(x^2 - 2x\text{Re}(s) + |s|^2)^{-1}(x - \bar{s}) \\ &= (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-1}, \quad s \notin [x]. \end{aligned} \tag{4}$$

*Remark 3.8.* The notion of slice regularity was introduced in the paper [59], besides the definition, the authors treated power series centered at the origin and some consequences. Without any tools the Cauchy formula with slice hyperholomorphic kernel and the representation formula were originally determined via the following elementary considerations. To determine the slice hyperholomorphic Cauchy kernel  $S_L^{-1}(s, q)$  we observe that from the definition of slice regularity its expansion

$$S_L^{-1}(s, q) := \sum_{m=0}^{\infty} q^m s^{-1-m}, \quad |q| < |s|$$

is true when  $q$  and  $s$  belong to the same complex plane  $\mathbb{C}_I$ , for  $I \in \mathbb{S}$ . Then we ask ourselves what is the closed form of the series in the case  $q$  and  $s$  do not belong to the same complex plane  $\mathbb{C}_I$  observing that

$$\left( \sum_{m=0}^{\infty} q^m s^{-1-m} \right) s - q \left( \sum_{m=0}^{\infty} q^m s^{-1-m} \right) = 1 \tag{5}$$

is true also when  $s$  and  $q$  do not belong to the same complex plane  $\mathbb{C}_I$ . In the quaternionic case it was observed that the inverse  $S$  of  $S_L^{-1}(s, q)$  is the non trivial solution of the quaternionic equation

$$S^2 + Sq - sS = 0,$$

which easily follows from (5). The unknown  $S$  was determined using the Niven’s Algorithm as it is shown in the historical Note 4.18.3 in the book [48] and it gives

$$S(s, q) = (q - \bar{s})^{-1} s (q - \bar{s}) - q.$$

Taking the inverse of  $S(s, q)$  we have the Cauchy kernel defined in (3) for the quaternions. This strategy to determine the Cauchy kernel shows that in the

Clifford setting the Cauchy kernel remains the same if we consider

$$S_L^{-1}(s, x) := \sum_{m=0}^{\infty} x^m s^{-1-m}, \quad |x| < |s|$$

where  $x = x_0 + x_1 e_1 + \dots + x_n e_n$  and  $s = x_0 + s_1 e_1 + \dots + s_n e_n$  are paravectors. Moreover, observe that a direct computation of the integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, x) ds_I f(s)$$

by computing the residues of the singularities of the kernel  $S_L^{-1}(s, x)$  in the complex plane  $\mathbb{C}_I$ , gives:

$$\begin{aligned} & \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, x) ds_I f(s) \\ &= \frac{1}{2} [f(u + Jv) + f(u - Jv)] + I \frac{1}{2} [J[f(u - Jv) - f(u + Jv)]], \end{aligned}$$

choosing any  $J \in \mathbb{S}$  for all  $x = u + Iv \in U$ . From here one can clearly see the existence of the structure formula (or representation formula) for slice monogenic functions, see for example [35] (or [36]):

$$f(u + Iv) = \frac{1}{2} [f(u + Jv) + f(u - Jv)] + I \frac{1}{2} [J[f(u - Jv) - f(u + Jv)]].$$

The quaternionic setting is just a particular case and from these observations started a full development of the theory of slice hyperholomorphic functions.

The second ingredient for our discussion below is the Cauchy formula for monogenic functions.

**Theorem 3.9.** (Cauchy formula for monogenic functions) *Let  $U \subset \mathbb{R}^{n+1}$  be an open set with smooth boundary  $\partial U$  and let  $\eta(\omega)$  be the outer unit normal to  $\partial U$  and  $dS(\omega)$  be the scalar element of surface area on  $\partial U$ . Let  $f$  be a monogenic function on an open set that contains  $\bar{U}$  then*

$$f(x) = \int_{\partial U} G_\omega(x) \eta(\omega) f(\omega) dS(\omega)$$

for every  $x$  in  $U$ , where the monogenic Cauchy kernel is given by

$$G_\omega(x) := \frac{1}{\sigma_n} \frac{\overline{\omega - x}}{|\omega - x|^{n+1}}, \quad x, \omega \in \mathbb{R}^{n+1}, \quad x \neq \omega$$

and  $\sigma_n := 2\pi^{\frac{n+1}{2}} / \Gamma(\frac{n+1}{2})$  is the volume of unit  $n$ -sphere in  $\mathbb{R}^{n+1}$ .

Before to introduce the basic fact on the hyperholomorphic spectral theories we need some important considerations.

(I) Holomorphic functions of one complex variable and harmonic analysis are strongly connected since the Cauchy-Riemann operator factorizes the Laplace operator. The holomorphic functional calculus and the spectral theorem are based on the same notion of spectrum.

(II) In order to restore the analogy with the holomorphic functional calculus and the spectral theorem in the complex setting we have to replace the classical spectrum with the  $S$ -spectrum. In fact the  $S$ -functional calculus

and the quaternionic spectral theorem are both based on the  $S$ -spectrum. The monogenic functional calculus, based on the monogenic spectrum, has applications in harmonic analysis and in other related fields.

Let us consider a Banach space  $V$  over  $\mathbb{R}$  with norm  $\|\cdot\|$ . It is possible to endow  $V$  with an operation of multiplication by elements of  $\mathbb{R}_n$  which gives a two-sided module over  $\mathbb{R}_n$  and by  $V_n$  we indicate the two-sided Banach module over  $\mathbb{R}_n$  given by  $V \otimes \mathbb{R}_n$ .

We start with the definition of a functional calculus for  $(n + 1)$ -tuples of not necessarily commuting operators using slice hyperholomorphic functions. So we consider the paravector operator

$$T = T_0 + \sum_{j=1}^n e_j T_j,$$

where  $T_\mu \in \mathcal{B}(V)$  for  $\mu = 0, 1, \dots, n$ , and where  $\mathcal{B}(V)$  is the space of all bounded  $\mathbb{R}$ -linear operators acting on  $V$ . The notion of  $S$ -spectrum follows from the Cauchy formula of slice hyperholomorphic functions and from the fact that we can replace in the Cauchy kernel  $S_L^{-1}(s, x)$  the paravector  $x$  by the paravector operator  $T$  also in the case the components  $(T_0, T_1, \dots, T_n)$  of  $T$  do not commute among themselves.

*Remark 3.10.* We make a crucial observation which justifies the definition of  $S$ -spectrum and of  $S$ -resolvent operator. With the procedure of Remark 3.8 it is natural to replace the paravector  $x$  by the paravector operator  $T = T_0 + T_1 e_1 + \dots + T_n e_n$  with bounded not necessarily commuting components  $T_\ell$ ,  $\ell = 0, \dots, n$  in the Cauchy kernel series. We obtain

$$\sum_{m=0}^{\infty} T^m s^{-1-m} = -(T^2 - 2\operatorname{Re}(s)T + |s|^2 \mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \quad \|T\| < |s|.$$

even though the components of  $T$  do not commute. From this relation we justify the definition of the  $S$ -resolvent operator and of the  $S$ -spectrum. The quaternionic setting is just a particular case.

We have the following definition.

**Definition 3.11.** (*S-spectrum*) Let  $T \in \mathcal{B}(V_n)$  be a paravector operator. We define the  $S$ -spectrum  $\sigma_S(T)$  of  $T$  as:

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} : T^2 - 2\operatorname{Re}(s)T + |s|^2 \mathcal{I} \text{ is not invertible in } \mathcal{B}(V_n)\}$$

where  $\mathcal{I}$  denotes the identity operator. The  $S$ -resolvent set of  $T$  is defined as

$$\rho_S(T) = \mathbb{H} \setminus \sigma_S(T).$$

**Definition 3.12.** Let  $T \in \mathcal{B}(V_n)$  be a paravector operator and  $s \in \rho_S(T)$ . We define the left  $S$ -resolvent operator as

$$S_L^{-1}(s, T) := -(T^2 - 2\operatorname{Re}(s)T + |s|^2 \mathcal{I})^{-1}(T - \bar{s}\mathcal{I}). \tag{6}$$

A similar definition can be given for the right resolvent operator.

**Definition 3.13.** We denote by  $SH_{\sigma_S(T)}^L$  the set of slice hyperholomorphic functions defined on the axially symmetric set  $U$  that contains the  $S$ -spectrum of  $T$ .

A crucial result for the definition of the  $S$ -functional calculus is that integral

$$\frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, T) ds_I f(s), \quad \text{for } f \in SH_{\sigma_S}^L(T) \tag{7}$$

depends neither on  $U$  nor on the imaginary unit  $I \in \mathbb{S}$ , so the  $S$ -functional calculus turns out to be well defined.

**Definition 3.14.** ( $S$ -functional calculus) Let  $T \in B(V_n)$  and let  $U \subset \mathbb{R}^{n+1}$  be as above. We set  $ds_I = -Ids$  and we define the  $S$ -functional calculus as

$$f(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} S_L^{-1}(s, T) ds_I f(s), \tag{8}$$

for  $f \in SH_{\sigma_S}^L(T)$ .

Observe that the definition of the  $S$ -functional calculus is very natural for non commuting operators in noncommutative spectral theory. The heart of the general version of the  $S$ -functional calculus can be found in the original papers [1, 36, 39, 40] and its commutative version [38].

*Warning.* In the monogenic setting the natural functional calculus is for vector operators that is when we set  $T_0 = 0$  in the paravector operator  $T = T_0 + \sum_{j=1}^n e_j T_j$ . The reason will be clear in the sequel, but to point out this fact we use the symbol  $A = (A_1, \dots, A_n)$  or  $A = \sum_{j=1}^n e_j A_j$  instead of  $(T_1, \dots, T_n)$  or  $T = \sum_{j=1}^n e_j T_j$ .

Using the Cauchy integral formula for monogenic functions, we establish the monogenic functional calculus for the  $n$ -tuple  $A = (A_1, \dots, A_n)$  of bounded linear operators on a Banach space  $X$  by substituting the  $n$ -tuple  $A$  for the vector  $x \in \mathbb{R}^n$ .

In the following for the monogenic functional calculus we limit ourselves to the most simple case when  $n$  is odd and the  $n$ -tuple  $A = (A_1, \dots, A_n)$  of bounded linear operators commute among themselves. Such restrictions can be removed but one needs to do further considerations.

*Remark 3.15.* If  $n$  is odd,  $A$  is a commutative  $n$ -tuple, that is,  $A_j A_k = A_k A_j$  for  $j, k = 1, \dots, n$ , and each operator  $A_j$  has real spectrum  $\sigma(A_j) \subset \mathbb{R}$  for  $j = 1, \dots, n$ , then for suitable  $\omega \in \mathbb{R}^{n+1}$ , the expression

$$G_\omega(A) := \frac{1}{\sigma_n} \frac{\overline{\omega \mathcal{I} - A}}{|\omega \mathcal{I} - A|^{n+1}} \tag{9}$$

makes sense as an element of  $\mathcal{B}(V_n)$  and it is called the monogenic resolvent.

*Remark 3.16.* For an even integer  $m$  we have

$$|\omega \mathcal{I} - A|^{-m} = \left( \left( \omega_0^2 \mathcal{I} + \sum_{j=1}^n (\omega_j \mathcal{I} - A_j)^2 \right)^{-1} \right)^{m/2}$$

and

$$\overline{\omega \mathcal{I} - A} = \omega_0 I - \sum_{j=1}^n (\omega_j \mathcal{I} - A_j) e_j$$

for  $\omega = \omega_0 + \sum_{j=1}^n \omega_j$ . Observe that the operator

$$\omega_0^2 \mathcal{I} + \sum_{j=1}^n (\omega_j \mathcal{I} - A_j)^2$$

is invertible in  $\mathcal{B}(V)$  for each  $\omega_0 \neq 0$ .

**Definition 3.17.** (Monogenic spectrum) The function

$$\omega \mapsto G_\omega(A),$$

is defined on the set  $\mathbb{R}^{n+1} \setminus (\{0\} \times \gamma(A))$  where

$$\gamma(A) = \left\{ (\omega_1, \dots, \omega_n) \mid \sum_{j=1}^n (\omega_j \mathcal{I} - A_j)^2 \text{ is not invertible in } \mathcal{B}(V) \right\}$$

is called the monogenic spectrum.

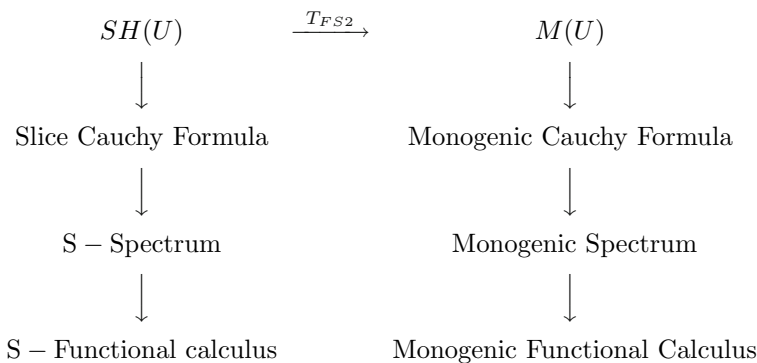
**Definition 3.18.** (*The monogenic functional calculus*) Let  $n$  be an odd number and let us assume that  $A = (A_1, \dots, A_n)$  is a commutative  $n$ -tuple of bounded linear operators (that is  $A_j A_k = A_k A_j$  for  $j, k = 1, \dots, n$ ), and each operator  $A_j$  has real spectrum  $\sigma(A_j) \subset \mathbb{R}$  for  $j = 1, \dots, n$ . If  $f$  is a monogenic function on an open set that contains  $\bar{U} \subset \mathbb{R}^{n+1}$  with  $\gamma(A) \subset U$ . Then we define the monogenic functional calculus as

$$f(A) = \int_{\partial U} G_\omega(A) \eta(\omega) f(\omega) dS(\omega)$$

where  $G_\omega(A)$  is the monogenic resolvent operator (9),  $\eta(\omega)$  is the outer unit normal to  $\partial U$  and  $dS(\omega)$  is the scalar element of surface area on  $\partial U$ .

In the case  $m = 2, 4, 6, \dots$  the operator  $|\omega \mathcal{I} - A|^{-m}$  needs to be defined in a suitable way. The direct formulation employs Taylor’s functional calculus, but by using the plane wave decomposition of the Cauchy kernel, the case of even  $n$  and noncommuting operators can be treated simultaneously. For an  $n$ -tuple  $(A_1, \dots, A_n)$  of commuting bounded linear operators on a Banach space  $V$  with real spectra, the nonempty compact subset  $\gamma(A)$  of  $\mathbb{R}^n$  coincides with Taylor’s joint spectrum defined in terms of the Koszul complex.

The Cauchy formula of slice hyperholomorphic functions allows to define the notion of  $S$ -spectrum, while the Cauchy formula for monogenic functions induces the notion of monogenic spectrum, as illustrated by the diagram:



In the above diagram we have replaced the set of intrinsic functions  $N$  by the larger set of slice hyperholomorphic functions  $SH$ . This is clearly possible because the map  $T_{FS2}$  is the Laplace operator or its powers.

We finally recall that the quaternionic spectral theorem is based on the  $S$ -spectrum and not on the monogenic spectrum. In 2015 (and published in 2016) the quaternionic spectral theorem for quaternionic normal operators was finally proved, see [2] (the case of unitary operators is treated in [3] and the case of compact normal operators is in [61]). Later on perturbation results of quaternionic normal operators were proved in [16]. Beyond the spectral theorem there are more recent developments in the direction of the characteristic operator functions, see [5] and the theory of quaternionic spectral operators was developed in [58].

Finally, we wish to give an idea of the structure of the quaternionic spectral theorem. For a complete treatment see [24]. If  $T \in \mathcal{B}(\mathcal{H})$  is a bounded normal quaternionic linear operator, on a quaternionic Hilbert space  $\mathcal{H}$ , then there exist three quaternionic linear operators  $A, \mathfrak{J}, B$  such that  $T = A + \mathfrak{J}B$ , where  $A$  is self-adjoint and  $B$  is positive,  $\mathfrak{J}$  is an anti self-adjoint partial isometry (called imaginary operator). Moreover,  $A, B$  and  $\mathfrak{J}$  mutually commute.

There exists a unique spectral measure  $E_I$  on  $\sigma_S(T) \cap \mathbb{C}_I^+$  so that for any slice continuous intrinsic function  $f = f_0 + f_1 I$  we have:

$$\begin{aligned} \langle f(T)x, y \rangle &= \int_{\sigma_S(T) \cap \mathbb{C}_I^+} f_0(q) d\langle E_I(q)x, y \rangle + \int_{\sigma_S(T) \cap \mathbb{C}_I^+} f_1(q) d\langle \mathfrak{J}E_I(q)x, y \rangle \end{aligned} \tag{10}$$

where  $x, y \in \mathcal{H}$ . This theorem extends to the case of unbounded operators as well and holds true for a larger class of functions that are not necessarily continuous. The continuous functional calculus is treated in [63].

### 4. Intersection of the Hyperholomorphic Spectral Theories

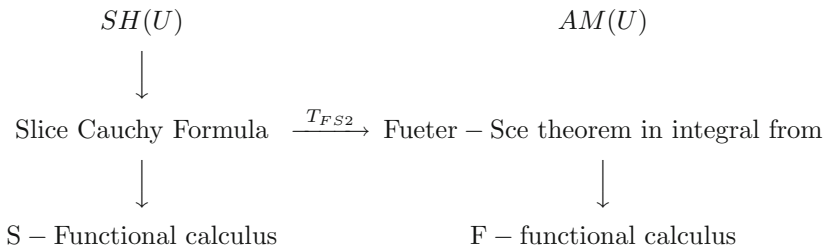
Now we formulate the Fueter-Sce-Qian theorem in integral form and we use it to define the  $F$ -functional calculus. This gives a version of the monogenic functional calculus for  $n$ -tuples of commuting operators but it is based on the  $S$ -spectrum instead of the monogenic spectrum. This calculus was introduced in [41] and further investigated in [21,37].

It is important to recall that the monogenic functional calculus is defined for  $n$ -tuples of operators  $A_j, j = 1, \dots, n$  that have real spectrum considered as operators  $A_j : V \rightarrow V$  on the real Banach space  $V$ . The  $F$ -functional calculus has advantages and disadvantages with respect to the monogenic functional calculus. Precisely, the  $F$ -functional calculus allows to consider a much larger class of operators because it does not require that the spectrum of the operators  $A_j, j = 1, \dots, n$  has to be real. Moreover, this calculus allows to consider paravector operators and not only vector operators as the monogenic functional calculus imposes.

On the other hand, from the hyperholomorphic functions point of view the  $F$ -functional calculus is less general with respect to the monogenic functional calculus because it works for the subset of monogenic function given by

$$\check{M} = \{\check{f} \mid \check{f}(x) = \Delta^{\frac{n-1}{2}} f(x) \text{ for } f \in SH(U)\}.$$

We now show how the Fueter-Sce mapping theorem provides an alternative way to define the functional calculus for monogenic functions. The main idea is to apply the Fueter-Sce operator  $T_{FS2}$  to the slice hyperholomorphic Cauchy kernel as illustrated by the diagram:



*Remark 4.1.* Observe that in the above diagram the arrow from the space of axially monogenic function  $AM(U)$  is missing because the  $F$ -functional calculus is deduced from the slice hyperholomorphic Cauchy formula.

This method generates an integral transform, called the Fueter-Sce mapping theorem in integral form, that allows to define the so called  $F$ -functional calculus. This calculus uses slice hyperholomorphic functions and the commutative version of the  $S$ -spectrum and now we show how it works. We point out that the operator  $T_{FS2}$  has a kernel and one has to pay attention to this fact with the definition of the  $F$ -functional calculus, more details are given in [24].

Now observe that one can apply the powers of the Laplace operators to both sides of (2) so that we have

$$\Delta^h f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \Delta^h S_L^{-1}(s, x) ds_I f(s).$$

In general, it is not easy to compute  $\Delta^h f$  and when we apply  $\Delta^h$  to the Cauchy kernel written in the form (3), we do not get a simple formula. However,  $S_L^{-1}(s, x)$  can be written in two equivalent ways as follows.

If we use the second expression for the Cauchy kernel (see formula (4)) we find a very simple expression for  $\Delta^h S_L^{-1}(s, x)$ . In fact, we have:

**Theorem 4.2.** *Let  $x, s \in \mathbb{R}^{n+1}$  be such that  $x^2 - 2x\text{Re}(s) + |s|^2 \neq 0$ . Let*

$$S_L^{-1}(s, x) = (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-1}$$

*be the slice monogenic Cauchy kernel and let  $\Delta = \sum_{i=0}^n \frac{\partial^2}{\partial x_i^2}$  be the Laplace operator in the variables  $(x_0, x_1, \dots, x_n)$ . Then, for  $h \geq 1$ , we have:*

$$\Delta^h S_L^{-1}(s, x) = C_{n,h} (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-(h+1)}, \tag{11}$$

where

$$C_{n,h} := (-1)^h \prod_{\ell=1}^h (2\ell) \prod_{\ell=1}^h (n - (2\ell - 1)).$$

The function  $\Delta^h S^{-1}(s, x)$  is slice hyperholomorphic in  $s$  for any  $h \in \mathbb{N}$  but is monogenic in  $x$  if and only if  $h = (n + 1)/2$ , namely if and only if  $h$  equals the Sce's exponent. We define the kernel

$$\mathcal{F}_L(s, x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) = \gamma_n (s - \bar{x})(s^2 - 2\text{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n - 1)! \left(\frac{n - 1}{2}\right)! \tag{12}$$

which can be used to obtain the Fueter-Sce mapping theorem in integral form.

**Theorem 4.3.** *Let  $n$  be an odd number. Let  $f$  be a slice hyperholomorphic function defined in an open set that contains  $\bar{U}$ , where  $U$  is a bounded axially symmetric open set. Suppose that the boundary of  $U \cap \mathbb{C}_I$  consists of a finite number of rectifiable Jordan curves for any  $I \in \mathbb{S}$ . Then, if  $x \in U$ , the function  $\check{f}(x)$ , given by*

$$\check{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$$

is monogenic and it admits the integral representation

$$\check{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_L(s, x) ds_I f(s), \quad ds_I = ds/I, \tag{13}$$

where the integral depends neither on  $U$  nor on the imaginary unit  $I \in \mathbb{S}$ .

In the sequel, we will consider bounded paravector operators  $T$ , with commuting components  $T_\ell \in \mathcal{B}(V)$  for  $\ell = 0, 1, \dots, n$ . Such subset of  $\mathcal{B}(V_n)$  will be denoted by  $\mathcal{BC}^{0,1}(V_n)$ . The  $F$ -functional calculus is based on the commutative version of the  $S$ -spectrum given by

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} : s^2 \mathcal{I} - (T + \bar{T})s + T\bar{T} \text{ is not invertible in } \mathcal{B}(V_n)\}$$

where the operator  $\bar{T}$  is defined by

$$\bar{T} = T_0 - T_1 e_1 - \dots - T_n e_n.$$

We observe that for historical reasons the commutative version of the  $S$ -spectrum is sometimes called  $F$ -spectrum because it is used for the  $F$ -functional calculus. So we define the  $F$ -resolvent operators.

**Definition 4.4.** ( $F$ -resolvent operators) Let  $n$  be an odd number and let  $T \in \mathcal{BC}^{0,1}(V_n)$ . For  $s \in \rho_S(T)$  we define the left  $F$ -resolvent operator by

$$F_L(s, T) := \gamma_n (sI - \bar{T})(s^2 \mathcal{I} - (T + \bar{T})s + T\bar{T})^{-\frac{n+1}{2}}, \tag{14}$$

where the constants  $\gamma_n$  are given in (12).

**Definition 4.5.** (*The  $F$ -functional calculus for bounded operators*) Let  $n$  be an odd number, let  $T = T_0 + T_1e_1 + \dots + T_n e_n \in \mathcal{BC}^{0,1}(V_n)$  and set  $ds_I = ds/I$ , for  $I \in \mathbb{S}$ . Let  $SH^L_{\sigma_S(T)}$  and  $U$  be as in Definition 3.13. We define

$$\check{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} F_L(s, T) ds_I f(s). \tag{15}$$

The definition of the  $F$ -functional calculus is well posed since the integral in (15) depends neither on  $U$  and nor on the imaginary unit  $I \in \mathbb{S}$ .

We conclude this section with several remarks on the hyperholomorphic functional calculi to stress the difference with respect to the complex case.

(I) The product rule holds for the  $S$ -functional calculus but only in the case one of the two functions is intrinsic function. For the monogenic functional calculus the product rule does not hold. This is due to the fact that the product of two monogenic functions is not monogenic. For the  $F$ -functional calculus the product rule does not hold.

(II) Regarding the compatibility with polynomials we have that the  $S$ -functional calculus and the monogenic functional calculus are compatible with slice hyperholomorphic polynomials and with monogenic polynomials, respectively. For the  $F$ -functional calculus the compatibility with polynomials holds if we consider

$$\check{P}(q) = \Delta P(q)$$

where  $\check{P}(q)$  is a monogenic (or Fueter) and  $P$  is a slice monogenic polynomial

$$q \rightarrow T \Rightarrow \check{P}(q) \Rightarrow \check{P}(T)$$

(III) The spectral properties of the operator  $T$  can be deduced by the  $S$ -functional calculus and the quaternionic spectral theorem for which

$$Tx = \lambda x \quad \implies \quad f(T)x = f(\lambda)x \tag{16}$$

when we use intrinsic functions.

### 5. The $S$ -Spectrum Approach to Fractional Diffusion Problems

An important extension of the  $S$ -functional calculus to unbounded sectorial operators is the  $H^\infty$ -functional calculus which is one of the ways to define functions of unbounded operators. The  $H^\infty$ -functional calculus has been used to define fractional powers of paravector operators and of quaternionic linear operators that define fractional Fourier laws for nonhomogeneous material in the theory of heat propagation. For the original contributions on fractional powers of vector operators and of quaternionic operators and of the  $H^\infty$ -functional calculus based on the  $S$ -spectrum see [7, 20, 22]. For a systematic and recent treatment of quaternionic spectral theory on the  $S$ -spectrum and the fractional diffusion problems based on techniques on the  $S$ -spectrum see the books [23, 24] published in 2019. Moreover, in the monograph [48], published 2011, one can find also the foundations of the spectral theory on the  $S$ -spectrum for  $n$ -tuples of noncommuting operators.

The theory on the fractional powers of quaternionic operators has been recently applied to physical problems and in particular to generate the fractional Fourier law for the heat equation that is collected in the papers [17–19, 31, 32], here we give an overview of some of our results.

We denote by  $\underline{x} := (x_1, x_2, x_3)$  a generic point in  $\mathbb{R}^3$ . Let  $\Omega \subset \mathbb{R}^3$  bounded or unbounded domain (with  $C^1$  boundary), the heat equation for nonhomogeneous materials with the associated initial-boundary conditions describes the evolution of the heat. Precisely, we determine  $v : \Omega \times (0, \tau] \rightarrow \mathbb{R}$  (for  $\tau > 0$ ) such that

$$\begin{cases} \partial_t v(\underline{x}, t) + \operatorname{div} T(\underline{x})v(\underline{x}, t) = 0, & (\underline{x}, t) \in \Omega \times (0, \tau], \\ v(\underline{x}, 0) = f(\underline{x}), & \underline{x} \in \Omega, \\ v(\underline{x}, t) = 0, & \underline{x} \in \partial\Omega \quad t \in [0, \tau], \end{cases} \tag{17}$$

where  $f$  is a given datum and

$$T(\underline{x}) = \begin{pmatrix} a_1(\underline{x})\partial_{x_1} \\ a_2(\underline{x})\partial_{x_2} \\ a_3(\underline{x})\partial_{x_3} \end{pmatrix} \tag{18}$$

where we suppose that the coefficients  $a_1, a_2, a_3 : \bar{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  of  $T$  belong to  $C^1(\bar{\Omega})$  and they are not necessarily constant. We also consider the heat equation for nonhomogeneous materials with Robin boundary conditions, that consists in finding  $v : \Omega \times (0, \tau] \rightarrow \mathbb{R}$  (for  $\tau > 0$ ) such that

$$\begin{cases} \partial_t v(\underline{x}, t) + \operatorname{div} T(\underline{x})v(\underline{x}, t) = 0, & (\underline{x}, t) \in \Omega \times (0, \tau], \\ v(\underline{x}, 0) = f(\underline{x}), & \underline{x} \in \Omega, \\ b(\underline{x})v(\underline{x}, t) + \sum_{\ell=1}^3 a_\ell(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}, t) = 0, & (\underline{x}, t) \in \partial\Omega \times (0, \tau], \end{cases} \tag{19}$$

where  $n = (n_1, n_2, n_3)$  is the outward unit normal vector to  $\partial\Omega$ , and  $b : \partial\Omega \rightarrow \mathbb{R}$  is a given continuous function. From the physical point of view, if we call  $q(\underline{x}, t)$  the flux of the quantity described by  $v(\underline{x}, t)$  at the instant  $t$ , the Fourier’s law states that  $q(\underline{x}, t) = T(\underline{x})(v(\underline{x}, t))$ .

The simpler case is when we consider  $\Omega = \mathbb{R}^3$  and the homogeneous diffusion problem is the consequence of Fourier’s law

$$q(\underline{x}, t) = -\nabla v(\underline{x}, t)$$

and of the conservation of the energy

$$\partial_t v(\underline{x}, t) + \operatorname{div}(q(\underline{x}, t)) = 0.$$

In this case  $T$  is reduced to the negative gradient operator  $T = -\nabla$  and observing that  $\operatorname{div} \circ \nabla = \Delta$  the fractional diffusion model is obtained by replacing in the heat equation the Laplace operator by its fractional powers

$$(-\Delta)^\alpha u(\underline{x}) := \int_{\mathbb{R}^3} \frac{u(\underline{x}) - u(\underline{y})}{|\underline{x} - \underline{y}|^{3+2\alpha}} dV(\underline{y}), \quad \text{for } \alpha \in (0, 1).$$

The fractional versions of the evolution equation in  $\mathbb{R}^3$  is given by

$$\partial_t v(\underline{x}, t) + (-\Delta)^\alpha v(\underline{x}, t) = 0. \tag{20}$$

We observe that the fractional diffusion problem modifies both the Fourier’s law and the conservation of the energy. Using the quaternionic functional calculus we are able to define the fractional powers of vector operators, such as  $\nabla$  or  $T$ , in a bounded or unbounded domain  $\Omega$  of  $\mathbb{R}^3$ . Denoting, just for the moment, by  $T^\alpha$  or  $\nabla^\alpha$  these fractional operators, we can define the fractional diffusion problem (20) in the divergence form

$$\partial_t v(\underline{x}, t) + \operatorname{div} T^\alpha(x)v(\underline{x}, t) = 0, \quad (\underline{x}, t) \in \Omega \times (0, \tau]. \tag{21}$$

The boundary conditions to associate with the fractional evolution problem are a very delicate issue and will not be discussed here. We just mention that the most natural boundary conditions are  $v = 0$  at infinity in the case  $\Omega = \mathbb{R}^3$ .

*Remark 5.1.* The boundary condition that we have to assume to generate the fractional powers of  $T$  are given by

$$a(\underline{x})v(\underline{x}, t) + \sum_{\ell=1}^3 a_\ell^2(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}, t) = 0, \quad (\underline{x}, t) \in \partial\Omega \times (0, \tau], \tag{22}$$

where  $a : \partial\Omega \rightarrow \mathbb{R}$  is a given continuous function. These Robin-like boundary condition differs from the boundary condition in (19) by a power two on the coefficients  $a_\ell$ ’s. This is due to the fact that in (19) the boundary condition rise from a physical condition on the flux through the boundary, instead the boundary condition in (22) naturally comes from the definition of  $T^\alpha$ . In any case, the two boundary conditions are related when  $T$  has coefficients that become constant on the boundary  $\partial\Omega$ . Indeed, suppose that there exists a constant  $\mu$  such that the functions  $a_1, a_2, a_3$  satisfy the conditions

$$a_1(\underline{x}) = a_2(\underline{x}) = a_3(\underline{x}) = \mu \text{ for all } x \in \partial\Omega \tag{23}$$

and the coefficients  $a$  and  $b$  are such that

$$a(\underline{x}) = \mu b(\underline{x}) \text{ for all } \underline{x} \in \partial\Omega. \tag{24}$$

Then the relation

$$\sum_{\ell=1}^3 a_\ell(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}) + b(\underline{x})v(\underline{x}) = 0$$

is equivalent to

$$\sum_{\ell=1}^3 a_\ell^2(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}) + a(\underline{x})v(\underline{x}) = 0$$

when  $x \in \partial\Omega$ . For, using (23) and (24), we have

$$\begin{aligned} \sum_{\ell=1}^3 a_\ell^2(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}) + a(\underline{x})v(\underline{x}) &= \mu^2 \sum_{\ell=1}^3 n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}) + \mu b(\underline{x})v(\underline{x}) \\ &= \mu \left( \sum_{\ell=1}^3 a_\ell(\underline{x})n_\ell(\underline{x})\partial_{x_\ell} v(\underline{x}) + b(\underline{x})v(\underline{x}) \right). \end{aligned}$$

This approach has several advantages.

- It generates the fractional Fourier law from the Fourier law

$$q(\underline{x}, t) = T^\alpha(\underline{x})v(x, t) \tag{25}$$

using the boundary conditions of the problem and without modifying the conservation of energy law.

- We can define the fractional heat equation for nonhomogeneous materials.
- The fractional differential equation remains in the divergence form, so the definition of a weak solution is obtained in a simple way.
- It turns out that the approach through the quaternionic functional calculus for defining the fractional heat equations is consistent with the classical one. Indeed, we have that for any  $\alpha \in (0, 1)$

$$2 \operatorname{div}(\nabla^\alpha) = (-\Delta)^{\frac{1}{2} + \frac{\alpha}{2}}.$$

Now we present how the quaternions can be used to describe the vector operators. Let  $e_\ell$ , for  $\ell = 1, 2, 3$ , be an orthogonal basis for the quaternions  $\mathbb{H}$ . We identify the vector operator  $T$ , described in (18), with the quaternionic gradient operator with non constant coefficients

$$\begin{pmatrix} a_1(\underline{x})\partial_{x_1} \\ a_2(\underline{x})\partial_{x_2} \\ a_3(\underline{x})\partial_{x_3} \end{pmatrix} \equiv \sum_{\ell=1}^3 e_\ell T_\ell, \tag{26}$$

where the components  $T_\ell$ ,  $\ell = 1, 2, 3$ , are defined by  $T_\ell := a_\ell(\underline{x})\partial_{x_\ell}$ ,  $\underline{x} \in \overline{\Omega}$ . From the physical point of view the operator  $T$ , defined in (26), can represent the Fourier law for nonhomogeneous materials, but it can represent also different physical laws. Our goal is to generate the fractional powers of  $T$ , that we denote with  $P_\alpha(T)$  for  $\alpha \in (0, 1)$ , when the operators  $T_\ell$ , for  $\ell = 1, 2, 3$  do not commute among themselves.

*Remark 5.2.* The notation  $P_\alpha(T)$ , for the fractional powers of  $T$ , is more precise with respect to the formal notation  $T^\alpha$  in a sense that will be clear just in the following with the precise definition. In simple words  $P_\alpha(T)$  is defined by a projector when there are points of the  $S$ -spectrum that belong to the negative real line (see for example (30)). The formal notation  $T^\alpha$  is used in (21) and (22) (see (25)).

**Definition 5.3.** The vector part of the fractional powers  $P_\alpha(T)$  is called the fractional Fourier law associated with  $T$ .

Now we present the general theory of the  $S$ -spectrum to construct the fractional power of a quaternionic right linear operator. Let  $V$  be a two-sided quaternionic Banach space and  $\mathcal{K}(V)$  the set of closed quaternionic right linear operators on  $V$ . Recall that the Banach space of all bounded right linear operators on  $V$  is indicated by the symbol  $\mathcal{B}(V)$  and is endowed with

the natural operator norm. For  $T \in \mathcal{K}(V)$ , we define the operator associated with the  $S$ -spectrum as:

$$\mathcal{Q}_s(T) := T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I}, \quad \text{for } s \in \mathbb{H} \tag{27}$$

where  $\mathcal{Q}_s(T) : \text{dom}(T^2) \rightarrow V$ , where  $\text{dom}(T^2)$  is the domain of  $T^2$ . We define the  $S$ -resolvent set of  $T$  as

$$\rho_S(T) := \{s \in \mathbb{H} : \mathcal{Q}_s(T) \text{ is invertible and } \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(V)\}$$

and the  $S$ -spectrum of  $T$  as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

The operator  $\mathcal{Q}_s(T)^{-1}$  is called the pseudo  $S$ -resolvent operator. For  $s \in \rho_S(T)$ , the left  $S$ -resolvent operator is defined as

$$S_L^{-1}(s, T) := \mathcal{Q}_s(T)^{-1}\bar{s} - T\mathcal{Q}_s(T)^{-1} \tag{28}$$

and the right  $S$ -resolvent operator is given by

$$S_R^{-1}(s, T) := -(T - \mathcal{I}\bar{s})\mathcal{Q}_s(T)^{-1}. \tag{29}$$

The fractional powers of  $T$ , denoted by  $P_\alpha(T)$ , are defined as follows: for any  $I \in \mathbb{S}$ , for  $\alpha \in (0, 1)$  and  $v \in \text{dom}(T)$  we set

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-I\mathbb{R}} S_L^{-1}(s, T) ds_I s^{\alpha-1}Tv, \tag{30}$$

or

$$P_\alpha(T)v := \frac{1}{2\pi} \int_{-I\mathbb{R}} s^{\alpha-1} ds_I S_R^{-1}(s, T)Tv, \tag{31}$$

where  $ds_j = ds/I$ . These formulas are a consequence of the quaternionic version of the  $H^\infty$ -functional calculus based on the  $S$ -spectrum, see the book [23] for more details. For the generation of the fractional powers  $P_\alpha(T)$  a crucial assumption on the  $S$ -resolvent operators is that, for  $s \in \mathbb{H} \setminus \{0\}$  with  $\text{Re}(s) = 0$ , the estimates

$$\|S_L^{-1}(s, T)\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|} \quad \text{and} \quad \|S_R^{-1}(s, T)\|_{\mathcal{B}(V)} \leq \frac{\Theta}{|s|}, \tag{32}$$

hold with a constant  $\Theta > 0$  that does not depend on the quaternion  $s$ . It is important to observe that the conditions (32) assure that the integrals (30) and (31) are convergent and so the fractional powers are well defined.

For the definition of the fractional powers of the operator  $T$  we can use equivalently the integral representation in (30) or the one in (31). Moreover, they correspond to a modified version of Balakrishnan’s formula that takes only spectral points with positive real part into account.

We want to apply the previous theory to the case  $V := L^2(\Omega, \mathbb{H})$  and  $T \in \mathcal{K}(L^2(\Omega, \mathbb{H}))$  defined as in (26) ( $\text{dom}(T) \subset L^2(\Omega, \mathbb{H})$  is a densely subset). A crucial problem is to determine the conditions on the coefficients  $a_1, a_2, a_3 : \bar{\Omega} \subset \mathbb{R}^3 \rightarrow \mathbb{R}$  such that (30) and (31) are convergent. This problem is split into two problems:

- The first is to find appropriate conditions for the coefficients  $a_i$ 's such that the purely imaginary quaternions are in the  $S$ -resolvent set  $\rho_S(T)$  (i.e.  $\mathcal{Q}_s(T) : \text{dom}(T^2) \rightarrow L^2(\Omega, \mathbb{H})$  is invertible and bounded). This is a necessary condition, see formulas (30) and (31). Then, observe that in the quaternionic case the map  $s \mapsto s^\alpha$ , for  $\alpha \in (0, 1)$  is not defined for  $s \in (-\infty, 0)$ . For this reason it is of great importance to assume the condition  $\text{Re}(s) \geq 0$  that avoids the half real line  $(-\infty, 0]$ .
- The second crucial fact is to determine the conditions on the coefficients  $a_i$ 's such that the estimate (32) for the  $S$ -resolvent operator of  $T$  holds true.

Both these problems are solved by considering the following approach. According to the initial condition of the boundary-value problems we invert the operator  $\mathcal{Q}_s(T)$  on the space  $H_0^1(\Omega, \mathbb{H})$ , when we consider the Dirichlet boundary condition, and on the space

$$\mathcal{H} := \{u \in H^1(\Omega) \mid \int_{\Omega} u(\underline{x}) dV(\underline{x}) = 0\}$$

when we consider the Robin boundary condition. The invertibility of  $\mathcal{Q}_s(T)$  is thus reduced to solve in a weak sense the following two partial differential equations: given  $F \in L^2(\Omega, \mathbb{H})$  and  $s \in \mathbb{H} \setminus \{0\}$  such that  $\text{Re}(s) = 0$

$$\begin{cases} \mathcal{Q}_s(T)(u) = (T^2 - 2s_0T + |s|^2T)u(\underline{x}) = F(\underline{x}), & \underline{x} \in \Omega, \\ u \in H_0^1(\Omega, \mathbb{H}), \end{cases} \tag{33}$$

and

$$\begin{cases} \mathcal{Q}_s(T)(u) = (T^2 - 2s_0T + |s|^2T)u(\underline{x}) = F(\underline{x}), & \underline{x} \in \Omega, \\ u \in \mathcal{H}(\Omega, \mathbb{H}), \\ b(\underline{x})v(\underline{x}) + \sum_{\ell=1}^3 a_\ell^2(\underline{x})n_\ell(\underline{x})\partial_{x_\ell}v(\underline{x}) = 0, & \underline{x} \in \partial\Omega. \end{cases} \tag{34}$$

To solve in the weak sense (33) (resp. (34)) means that for any  $F \in L^2(\Omega, \mathbb{H})$  we have to find  $u_F \in H_0^1(\Omega, \mathbb{H})$  (resp.  $u_F \in \mathcal{H}(\Omega, \mathbb{H})$ ) such that: for any  $v \in H_0^1(\Omega, \mathbb{H})$  (resp.  $v \in \mathcal{H}(\Omega, \mathbb{H})$ ) we have

$$\langle \mathcal{Q}_s(T)(u_F), v \rangle = (F, v)_{L^2} = \int_{\Omega} \overline{F}v dV(\underline{x}), \tag{35}$$

where the angle-brackets means that  $\mathcal{Q}_s(T)$  is applied to  $u_F$  in the sense of distribution. If we solve (35), we can define

$$\mathcal{Q}_s(T)^{-1}(F) := u_F$$

In order to solve (35) we apply the Lax-Milgram Lemma to the sesquilinear form:  $\langle \mathcal{Q}_s(T)(u_F), v \rangle$ . Thus it is crucial to prove an explicit formula for the left hand side of (35). This formula can be deduced from an arguments of

integration by parts and using the Dirichlet boundary condition for (33):

$$\begin{aligned}
 \langle Q_s(T)(u), v \rangle &= \sum_{\ell=1}^3 \int_{\Omega} \overline{a_{\ell}(\underline{x})\partial_{x_{\ell}}(u(\underline{x}))} a_{\ell}(\underline{x})\partial_{x_{\ell}}(v(\underline{x})) dV(\underline{x}) + \\
 &\frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_{\ell}}(u(\underline{x}))}\partial_{x_{\ell}}(a_{\ell}^2(\underline{x})) v(\underline{x}) dV(\underline{x}) + \\
 &(\text{Vect}(Q_s(T))u, v)_{L^2} + |s|^2(u, v)_{L^2} (:= b_{s,1}(u, v))
 \end{aligned} \tag{36}$$

or the Robin boundary condition for (34)

$$\begin{aligned}
 \langle Q_s(T)(u), v \rangle &= \sum_{\ell=1}^3 \int_{\Omega} \overline{a_{\ell}(\underline{x})\partial_{x_{\ell}}(u(\underline{x}))} a_{\ell}(\underline{x})\partial_{x_{\ell}}(v(\underline{x})) dV(\underline{x}) \\
 &+ \frac{1}{2} \sum_{\ell=1}^3 \int_{\Omega} \overline{\partial_{x_{\ell}}(u(\underline{x}))}\partial_{x_{\ell}}(a_{\ell}^2(\underline{x})) v(\underline{x}) dV(\underline{x}) \\
 &+ (\text{Vect}(Q_s(T))u, v)_{L^2} \\
 &+ \int_{\partial\Omega} a(\underline{x})\overline{u(\underline{x})}v(\underline{x}) dS(\underline{x}) + |s|^2(u, v)_{L^2} (:= b_{s,2}(u, v)).
 \end{aligned} \tag{37}$$

In [17–19, 31, 32] we found suitable conditions for the coefficients  $a_i$ 's such that the two sesquilinear forms  $b_{s,1}$  and  $b_{s,2}$  are coercive and continuous when  $\Omega$  is bounded or unbounded. In conclusion by the Lax–Milgram lemma we obtain the solvability of the equation (35) (i.e. the invertibility of  $Q_s(t)$ ) and the estimate (32) for the  $\mathcal{S}$ -resolvent operator. Thus the problem of the convergence of (30) and (31) is solved. In the next three paragraphs we summarize the conditions we found on the coefficients of  $T$  to obtain the convergence of (30) and (31).

The fractional Fourier's law in the problem (21) with  $\Omega$  bounded. In the paper [19] it was considered the commutative Fourier's law  $T_{com}$ , that is an operator of the form

$$T_{com} = a_1(x_1)\partial_{x_1}e_1 + a_2(x_2)\partial_{x_2}e_2 + a_3(x_3)\partial_{x_3}e_3 \tag{38}$$

where the real operators  $a_1(x)\partial_{x_1}$ ,  $a_2(x_2)\partial_{x_2}$  and  $a_3(x_3)\partial_{x_3}$  commute among themselves. It has been shown that if the coefficients  $a_{\ell} : \bar{\Omega} \rightarrow \mathbb{R}$ , for  $\ell = 1, 2, 3$  belong to  $C^1(\bar{\Omega}, \mathbb{R})$  and if  $a_{\ell}$ , for  $\ell = 1, 2, 3$  are suitably large and their derivative are suitably small then the integrals in (30) and (31) are convergent.

In [31] we replace the commutative Fourier's law  $T_{com}$  by the more general Fourier's law

$$T(x) = a_1(\underline{x})\partial_{x_1}e_1 + a_2(\underline{x})\partial_{x_2}e_2 + a_3(\underline{x})\partial_{x_3}e_3 \tag{39}$$

where now the real operators  $a_1(\underline{x})\partial_{x_1}$ ,  $a_2(\underline{x})\partial_{x_2}e_2$  and  $a_3(\underline{x})\partial_{x_3}$  do not commute among themselves. In this case the conditions for the existence of the fractional powers are more complicated.

The main result is summarized in the following theorem (see for more details Theorems 4.1, 4.4 and 4.5 in [31]).

**Theorem 5.4.** *Let  $\Omega$  be a bounded  $C^1$ -domain in  $\mathbb{R}^3$ , let*

$$T = \sum_{i=1}^3 a_i(\underline{x}) \partial_{x_i} e_i$$

with  $a_i \in C^1(\overline{\Omega})$  for any  $i = 1, 2, 3$  and set

$$F_{(a_1, a_2, a_3)} := \sum_{i=1}^3 e_i \partial_{x_i} (a_i).$$

Let  $a_1, a_2, a_3 \geq m > 0$ , and assume that

$$\begin{aligned} & \min\left\{ \inf_{\underline{x} \in \Omega} a_1^2, \inf_{\underline{x} \in \Omega} a_2^2, \inf_{\underline{x} \in \Omega} a_3^2 \right\} \\ & - (2 \max\left\{ \sup_{\underline{x} \in \Omega} a_1^2, \sup_{\underline{x} \in \Omega} a_2^2, \sup_{\underline{x} \in \Omega} a_3^2 \right\})^{1/2} C_\Omega \|F_{(a_1, a_2, a_3)}\|_{L^\infty} > 0 \end{aligned} \tag{40}$$

and

$$\begin{aligned} & 1 - 2 \|F_{(a_1, a_2, a_3)}\|_{L^\infty} \\ & \times \left( 1 + 4C_\Omega^2 \max\left\{ \sup_{\underline{x} \in \Omega} (1/a_1^2), \sup_{\underline{x} \in \Omega} (1/a_2^2), \sup_{\underline{x} \in \Omega} (1/a_3^2) \right\} \right) > 0, \end{aligned} \tag{41}$$

where  $C_\Omega$  in the Poincaré constant of  $\Omega$  and

$$\|F_{(a_1, a_2, a_3)}\|_{L^\infty} := \sup_{\underline{x} \in \Omega} (|\partial_{x_1}(a_1)| + |\partial_{x_2}(a_2)| + |\partial_{x_3}(a_3)|).$$

Then for any  $\alpha \in (0, 1)$  and for any  $v \in \text{dom}(T)$ , the integrals (30) and (31) converge absolutely.

The sesquilinear form  $b_{s,1}(u, v)$ , defined in (36), associated with the invertibility of the operator  $\mathcal{Q}_s(T) := T^2 - 2s_0T + |s|^2\mathcal{I}$ , with homogeneous Dirichlet boundary conditions, has to be considered with care. The next Remark 5.5 summarizes several facts concerning  $b_{s,1}(u, v)$ .

*Remark 5.5.* We point out some fact that appear in the application of the Lax–Milgram lemma according to the dimension  $n$  of  $\Omega$ .

(I) The quadratic form  $b_{s,1}(u, v)$  associated to the operator  $\mathcal{Q}_s(T)$  is in general degenerate on  $H_0^1(\Omega, \mathbb{H})$ .

(II) In dimension  $n = 3$ , when  $\Omega$  is a  $C^1$  bounded set in  $\mathbb{R}^3$  and  $a_1 \neq 0$ ,  $a_2 \neq 0$ ,  $a_3 \neq 0$ , it turns out that  $b_{s,1}(u, v)$  is continuous and coercive under suitable conditions on the coefficients of  $\mathcal{Y} \times \mathcal{Y}$ , where

$$\mathcal{Y} := \{v \in H_0^1(\Omega, \mathbb{H}) : v_0 = v_1 = v_2 = v_3\}$$

is a closed subspace of  $H_0^1(\Omega, \mathbb{H})$  and the  $S$ -resolvent operators satisfy suitable growth conditions which ensure the existence of the fractional powers.

(III) In dimension  $n = 2$ , when  $\Omega$  is a  $C^1$  bounded set in  $\mathbb{R}^2$  and  $a_1 \neq 0$ ,  $a_2 \neq 0$ , it turns out that  $b_{s,1}(u, v)$  is continuous and coercive under suitable conditions on the coefficients of  $\mathcal{X} \times \mathcal{X}$ , where

$$\mathcal{X} := \{v \in H_0^1(\Omega, \mathbb{H}) : v_0 = v_2 \text{ and } v_1 = v_3\}$$

is a closed subspace of  $H_0^1(\Omega, \mathbb{H})$  and the  $S$ -resolvent operators satisfy suitable growth conditions which ensure the existence of the fractional powers.

(IV) If we consider the quadratic form in dimension  $n = 3$ , that is when  $\Omega$  is a  $C^1$  bounded set in  $\mathbb{R}^3$  and  $a_1 \neq 0, a_2 \neq 0, a_3 = 0$ , then the quadratic form is not coercive because of  $a_3 = 0$ . It seems that this case cannot be treated using Lax–Milgram Lemma, but a suitable method for degenerate equations has to be used.

(V) In [31] we proved, under more restrictive hypothesis on the coefficients  $a_i$ 's, that the sesquilinear form  $b_{s,1}(u, v)$  is continuous and coercive in  $H_0^1(\Omega, \mathbb{H})$ .

(VI) From the physical point of view the case for  $a_1 \neq 0, a_2 \neq 0, a_3 = 0$ , in dimension  $n = 3$  is the case in which the conductivity in the direction  $z$  goes to zero.

(VII) The proofs for the continuity and coercivity are similar in any dimension, the estimate for the  $S$ -resolvent operators have some differences according to the fact that we work in  $\mathcal{Y}$  or in  $\mathcal{X}$ .

*The fractional Fourier's law in the problem (22) with  $\Omega$  bounded.* Regarding the initial boundary value problem of Robin-type, the conditions on the coefficients  $a_i$ 's depend on two positive constants which appear in the following two inequalities:

- for all  $u \in H^1(\Omega, \mathbb{R})$  the following inequality holds:

$$\|u\|_{H^{1/2}(\partial\Omega, \mathbb{R})} \leq C_{\partial\Omega} \|u\|_{H^1(\Omega, \mathbb{R})}. \tag{42}$$

where  $C_{\partial\Omega}$  does not depend on  $u$

- for all  $u \in H^1(\Omega, \mathbb{R})$  the following inequality holds:

$$\left\| u - |\Omega|^{-1} \int_{\Omega} u(x) dx \right\|_{L^2(\Omega, \mathbb{R})} \leq C_P \|\nabla u\|_{L^2(\Omega, \mathbb{R})}, \tag{43}$$

where  $C_P$  does not depend on  $u$ .

We obtained in [18] the following result (see Theorems 4.4, 5.1 and 5.2 in [18]).

**Theorem 5.6.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^3$  with boundary  $\partial\Omega$  of class  $C^1$ . Assume that  $a \in C^0(\partial\Omega, \mathbb{R})$  and let  $T$  be the operator defined in (18) with coefficients  $a_1, a_2, a_3 \in C^1(\bar{\Omega}, \mathbb{R})$ . Define the following constants:*

$$C_T := \min_{\ell=1,2,3} \inf_{x \in \Omega} (a_\ell^2(x)), \quad C'_T := \sum_{i,\ell=1}^3 \|a_\ell \partial_{x_\ell} a_i\|_\infty, \quad K_{a,\Omega} := C_{\partial\Omega}^2 \|a\|_\infty$$

where  $\|\cdot\|_\infty$  denotes the sup norm and  $C_{\partial\Omega}$  is the constant in (42). Moreover, assume that

$$C_T - C'_T C_P - K_{a,\Omega} (1 + C_P^2) > 0 \quad \text{and} \quad C_T > 0, \tag{44}$$

where  $C_P$  is the constant in (43). Then for any  $\alpha \in (0, 1)$  and for any  $v \in \text{dom}(T)$ , the integrals (30) and (31) converge absolutely.

*Remark 5.7.* In [17] we treated the case of the operator  $T$  with commutative coefficients and of  $\Omega$  bounded with Robin-type boundary condition.

The fractional Fourier's law in the problem (21) with  $\Omega$  unbounded. Regarding the initial boundary value problem of Dirichlet-type for the unbounded domains, we obtained in [18] the following result (see Theorems 4.8, 5.1 and 5.2 in [18]).

**Theorem 5.8.** *Let  $\Omega$  be an unbounded domain in  $\mathbb{R}^3$  with boundary  $\partial\Omega$  of class  $\mathcal{C}^1$ . Let  $T$  be the operator defined in (18) with coefficients  $a_1, a_2, a_3 \in \mathcal{C}^1(\overline{\Omega}, \mathbb{R}) \cap L^\infty(\Omega, \mathbb{R})$ . Suppose that*

$$M := \sum_{i,j=1}^3 \|a_i \partial_{x_i}(a_j)\|_{L^3(\Omega)} < +\infty \quad (45)$$

and

$$C_T := \min_{\ell=1,2,3} \inf_{\underline{x} \in \Omega} (a_\ell^2(\underline{x})) > 0, \quad C_T - 4M > 0. \quad (46)$$

Then for any  $\alpha \in (0, 1)$  and for any  $v \in \text{dom}(T)$ , the integrals (30) and (31) converge absolutely.

## 6. Concluding Remarks

During the pandemic period several efforts have been done by D.P Kimsey and I. Sabadini with the authors for further developments of the spectral theory on the  $S$ -spectrum and its applications. During the year 2020 part of this material has been collected in some papers. Precisely in [26] it was proved the spectral theorem for normal operators on a Clifford module, this paper has motivated the theory of slice monogenic functions of a Clifford variable introduced in [27] and the universality property of the  $S$ -functional calculus considered in [25].

### Acknowledgements

The authors we would like to thank D. P. Kimsey and both the referees for careful reading of the manuscript and for their comments. The first author is partially supported by the PRIN project Direct and inverse problems for partial differential equations: theoretical aspects and applications.

**Funding** Open access funding provided by Politecnico di Milano within the CRUI-CARE Agreement.

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Received: November 24, 2020.

Accepted: May 14, 2021.