REGULARITY RESULTS FOR THE NONLOCAL CAHN-HILLIARD EQUATION WITH SINGULAR POTENTIAL AND DEGENERATE MOBILITY

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Abstract. We consider the nonlocal Cahn-Hilliard equation with singular potential and degenerate mobility in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$. We first prove the existence of maximal strong solutions in weighted (in time) L^p spaces. Then we establish further regularity properties of the solution through maximal regularity theory.

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1. Introduction

The so-called nonlocal Cahn-Hilliard (CH) equation was obtained by Giacomin and Lebowitz as a macroscopic model of phase segregation in binary alloys which accounts for long range interactions (see [13, 14], cf. also [12]). This was done by performing a hydrodynamic limit on a suitable microscopic model on a lattice gas evolving via Kawasaki exchange dynamics. The associated (nonlocal) Helmholtz free energy is given by (all the constants are taken equal to one)

$$\mathcal{E}(\varphi) = -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\varphi(x)\varphi(y)dxdy + \int_\Omega F(\varphi)dx. \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \leq 3$, φ denotes the relative concentration of one components of the alloy, J is an interaction kernel such that $J(x) = J(-x)$ and may include both

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short range (local) and long range (nonlocal) interactions (see [13, Section 3]). Moreover, the potential density F is an entropic term defined for all $s \in (-1,1)$ by

$$F(s) = (1+s)\log(1+s) + (1-s)\log(1-s). \tag{1.2}$$

Defining the flux $\mathbf{j}$ as follows

$$\mathbf{j}(\varphi) = -m(\varphi)\nabla\mu,$$

where $\mu = \frac{\delta\mathcal{E}(\varphi)}{\delta\varphi}$ is the so-called chemical potential and m is the (degenerate) mobility, namely,

$$m(s) = 1 - s^2\,, \tag{1.3}$$

the nonlocal CH equation deduced in [13] can then be viewed as a conservation law for $\mathbf{j}$, that is,

$$\varphi_t + \operatorname{div}(-m(\varphi)\nabla\mu) = 0,\;\; \mu = -J*\varphi + F'(\varphi), \quad \text{ in } \Omega\times(0,T). \tag{1.4}$$

subject to the boundary condition

$$m(\varphi)\partial_n\mu = 0, \quad \text{on } \partial\Omega\times(0,T), \tag{1.5}$$

where n is the outward normal to $\partial\Omega$. This condition ensures the conservation of mass.

Observe that problem (1.4)-(1.5) can (formally) be written as follows

$$\varphi_t = \operatorname{div}\left(-m(\varphi)\nabla J*\varphi + m(\varphi)F''(\varphi)\nabla\varphi\right), \quad \text{ in } \Omega\times(0,T), \tag{1.6}$$

$$(-m(\varphi)\nabla J*\varphi + m(\varphi)F''(\varphi)\nabla\varphi)\cdot n = 0, \quad \text{on } \partial\Omega\times(0,T). \tag{1.7}$$

On account of (1.2) and (1.3), the term $m(\varphi)F''(\varphi)$ is a constant. Therefore we are dealing with a diffusive equation with a nonlocal term which accounts for segregation. Interestingly, such nonlocal equations are also structurally related to other important models for biological aggregations (see [9]).

Existence of a weak solution in the case of periodic boundary conditions was proven in [14] through a fixed point argument. Uniqueness was obtained through a suitable reformulation of the equation. It is worth noting that these results also hold when mF'' is not necessarily constant but uniformly positive. The nonlocal effects are modeled through a sufficiently smooth, fast decaying kernel J. Typical examples are the Newtonian or Bessel potential. Such potentials are of essential interest in phase-segregation phenomena which exhibit competition between nonlocal aggregation and the dispersal of particles due to the diffusion (see [13]). The same equation was considered in [8] with no-flux boundary conditions and well-posedness was proven as well. Moreover, the authors studied the convergence of a solution to a stationary state along a time sequence.

The equation introduced in [8] was further analyzed in [15] where a major step was taken, namely, the proof of the so-called (uniform or strict) separation property. Namely, any (weak) solution will stay instantaneously away from the pure states (± 1 in our case), uniformly with respect to the mass of the initial datum which is supposed to belong to $(-1,1)$ (i.e. the initial datum cannot be a pure state itself). In particular, this result allowed the authors to show some smoothness properties of weak solutions and their convergence to single equilibria, thanks to a suitable non-smooth version of the Łojasiewicz-Simon inequality. This was done by supposing mF'' to be a positive constant. This requirement was then weakened and the proof of the uniform separation property was simplified (see [16]). We also refer the reader to [11] for the existence of global and exponential attractors and to [18] for the presence of a reaction term.

It is worth observing that in the case of constant mobility the separation property appears to be less trivial to prove. Indeed, its validity in three dimensions is still an open issue (cf. [10] for a proof in the two dimensional case).

Another variant of equation (1.4) which is related to the one proposed in [1] has been considered, for instance, in [5] and [7] (see also the references therein and [10, Introduction] for a comparison).

Problem (1.6)-(1.7) with a given initial condition and an additional convective term, has recently been analyzed in [6, Sec. 6]. Making rather general assumptions on m and F the existence of a strong solution (i.e. in $L^2(0,T;H^2(\Omega))$ if $d=3$, in $L^\infty(0,T;H^2(\Omega))$ if $d=2$) has been established. We recall that in [14] the authors claim that a smooth solution can be obtained by using the results contained in [17]. However, this does not seem so straightforward. In [6] the goal was achieved through a nontrivial approximation strategy based on a suitable time discretization scheme. It is worth noting that solving problem (1.6)-(1.7) is not equivalent to solve (1.4)-(1.5). This equivalence is guaranteed, for instance, if the initial datum φ_0 is such that $F'(\varphi_0) \in L^2(\Omega)$ (see [7, Thm.3]). We remind that the original gradient flow structure as well as the separation property are essential to prove the convergence of a solution to a single stationary state (see [15, 16]).

In this contribution we first establish the existence of regular solutions in weighted (in time) L^p spaces to problem (1.6)-(1.7) (plus initial condition) in any space dimension using maximal regularity theory instead of energy estimates. This result allows us, in particular, to recover the $L^\infty(0,T;H^2(\Omega))$-regularity in three dimensions even when mF'' is not constant, thus, closing the gap left open in [6, Rem. 6.3, Sec. 6]. Also, we present a slightly more general and simpler proof of the separation property. We recall that, for the classical fourth-order CH equation with degenerate mobility and singular potential (see [2, 3]), the existence of a suitable notion of global weak solution is known (see [4]), but uniqueness and regularity results are still open issues.

The paper is organized as follows. In Section 2 we introduce the functional framework and we recall some known results. The existence of maximal solutions and their further regularity properties are investigated in Section 3 and in Section 4. Some technical lemmas are proven in Sections 5,6, and 7. The Appendix is devoted to the separation property.

2. Preliminaries and known results

We denote by $W^r_p(\Omega)$, $r \in \mathbb{N}$, the Sobolev space of functions in $L^p(\Omega)$ with distributional derivative of order less or equal to r in $L^p(\Omega)$ and by $\|\cdot\|_{W^r_p(\Omega)}$ its norm. For an arbitrary $r \in \mathbb{N}$, $H^r(\Omega) = W^r_2(\Omega)$ is a Hilbert space with respect to the scalar product $(u,v)_r = \sum_{|\kappa|\le r} \int_\Omega D^\kappa u(x) D^\kappa v(x)\, dx$ (κ being a multi-index) and the induced norm $\|u\|_r = \sqrt{(u,u)_r}$. For simplicity of notation, we indicate $H = L^2(\Omega)$ and the inner product as well as the norm in H are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively (even for vector-valued functions). In the case of non-integer differentiability for $W^s_p(\Omega)$, $s \notin \mathbb{N}_0$ we may consider these spaces as interpolation spaces. If $s = [s] + s_* \notin \mathbb{N}_0$ with $[s] \in \mathbb{N}_0$ and $s_* \in (0,1)$, then $W^s_p = (W^{[s]}_p, W^{[s]+1}_p)_{s_*,p}$, where $(\cdot,\cdot)_{s_*,p}$ denotes real interpolation. Moreover, it is well-known that

$W_p^s = (L^p, W_p^2)_{s,p}$ for $s \in (0,2)$, $s \neq 1$. For $p \in (1,\infty)$ and $s > 1/p$, the trace $\mathrm{tr}_\Omega(u) = u|_{\partial\Omega}$ extends to a continuous operator $\mathrm{tr}_\Omega : W_p^s(\Omega) \to W_p^{s-1/p}(\partial\Omega)$. Here we exclude the case $s - 1/p \in \mathbb{N}$ for $p \neq 2$. We also set $V = H^1(\Omega)$ endowed with the obvious norm $\|u\|_V^2 = \|\nabla u\|^2 + \|u\|^2$ and we will often refer to the well-known Poincaré-Wirtinger inequality

$$C_\Omega^{-1}\|u\|_V \leq \|\nabla u\| + |\overline{u}|, \quad \forall u \in V,$$

where $\overline{u} := |\Omega|^{-1}\int_\Omega u(x)\,dx$. From now on, we indicate by V' the dual space of V and by $\|\cdot\|_{-1}$ its norm.

The interaction kernel and the singular potential are required to verify the following assumptions:

(H.1) $J \in W_1^1(\mathbb{R}^d)$ with $J(x) = J(-x)$;

(H.2) $F \in C([-1,1]) \cap C^2(-1,1)$ fulfills

$$F''(s) \geq \alpha > 0, \quad \forall s \in (-1,1),$$

and there exists some $\epsilon_0 > 0$ such that F'' is nondecreasing in $[1-\epsilon_0, 1)$ and nonincreasing in $(-1, -1+\epsilon_0]$.

In addition to assumptions (H.1) and (H.2), we shall also assume (see [7])

(H.3) $m \in C^1([-1,1])$, $m \geq 0$ with $m(s) = 0$ if and only if $s = -1$ or $s = 1$, and there exists $\varsigma_0 > 0$ such that m is non-increasing in $[1-\varsigma_0, 1]$ and non-decreasing in $[-1, -1+\varsigma_0]$. Furthermore, we assume that

$$\gamma := mF'' \in C([-1,1]).$$

By weak solution we mean the following.

Definition 2.1. *Let φ_0 be a measurable function such that $F(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A function φ is called weak solution to* (1.6)-(1.7) *on* $[0,T]$ *corresponding to the initial condition φ_0 if it satisfies*

$$\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^\infty(\Omega \times (0,T)),$$
$$\partial_t\varphi \in L^2(0,T;V'),$$

with

$$|\varphi(x,t)| \leq 1, \quad a.e.\ (x,t) \in \Omega \times (0,T) \tag{2.1}$$

and, for all $v \in V$ and almost every $t \in (0,T)$, there holds

$$\langle \partial_t\varphi, v\rangle_{V',V} + (m(\varphi)F''(\varphi)\nabla\varphi, \nabla v) - (m(\varphi)\nabla J * \varphi, \nabla v) = 0, \tag{2.2}$$

with

$$\varphi(0,\cdot) = \varphi_0. \tag{2.3}$$

Remark 2.2. Let us observe the following facts.

(1) $F(\varphi_0) \in L^1(\Omega)$ implies that $\varphi_0 \in L^\infty(\Omega)$ and $|\varphi_0(x)| \leq 1$, for almost any $x \in \Omega$.
(2) The conservation of mass is a straightforward consequence of Definition 2.1. Indeed, taking $v = 1$, we get $\langle \partial_t\varphi, v\rangle_{V',V} = 0$, so that $\overline{\varphi}(t) = \overline{\varphi}_0$ for all $t \geq 0$.
(3) Let $T > 0$ be arbitrary. Note that $\varphi \in L^\infty(\Omega \times (0,T))$ with $|\varphi(x,t)| \leq 1$, for almost any $(x,t) \in \Omega \times (0,T)$ implies $\varphi \in L^\infty(0,T;L^p(\Omega))$, for all $p \geq 1$, and $\|\varphi\|_{L^\infty(0,T;L^p(\Omega))} \leq |\Omega|^{\frac{1}{p}}$.

(4) As a direct consequence of Definition 2.1, any weak solution φ belongs to $C([0,T];H)$ so that the initial condition is well defined.

We now recall a result established in [7] (see also [6]).

Theorem 2.3. *Let (H.1)-(H.3) hold and let φ_0 be a measurable function such that $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$, where $M \in C^2(-1,1)$ is defined by $m(s)M''(s) = 1$ for all $s \in (-1,1)$ and $M(0) = M'(0) = 0$. Then, there exists a weak solution in the sense of Definition 2.1 which satisfies the energy equality for almost every $t > 0$,*

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|^2 + \int_\Omega m(\varphi)F''(\varphi)|\nabla\varphi|^2 dx - \int_\Omega m(\varphi)(\nabla J * \varphi)\cdot\nabla\varphi dx = 0. \tag{2.4}$$

Moreover, under the additional assumption

$$\gamma(s) \geq \theta > 0, \quad \textit{for all } s \in [-1,1], \tag{2.5}$$

the weak solution is unique and the following continuous dependence estimate holds for all $t \in [0,T]$,

$$\|\varphi_1(t) - \varphi_2(t)\|_{-1}^2 + \int_0^t \|\varphi_1(s) - \varphi_2(s)\|^2 ds \leq Ce^{Kt}\|\varphi_1(0) - \varphi_2(0)\|_{-1}^2. \tag{2.6}$$

Here, φ_1 and φ_2 are two weak solutions on $[0,T]$ with initial data φ_{01} and φ_{02}, respectively, and C and K are two positive constants.

Remark 2.4. The existence of a weak solution (in the sense of Definition 2.1) can be proven without requiring for F a singular behavior at the endpoints $s = \pm 1$ (cf. (H.2) and (H.3)). Instead, the key role is played by the degenerate mobility, i.e., by condition (H.3), with F being also in $C^2([-1,1])$. This is enough to ensure the bound $|\varphi| \leq 1$ almost everywhere in Q_T. However, concerning uniqueness and regularity results (cf. Theorem 2.7 below), assumption (2.5) is crucial, but it implies that F must have some singular behavior at the endpoints, in the sense that, at least, $F''(s) \to \infty$, as $s \to \pm 1$. Moreover, we point out that the existence of a weak solution does not depend on the spatial dimension.

Recalling [7, Section 6, Theorem 5] and [6, Lemma 4.2], we also have

Proposition 2.5. *Let all the assumptions of Theorem 2.3 be satisfied and assume* (2.5)*. Then, any weak solution belongs to $C^{\beta/2,\beta}([\delta,T]\times\overline{\Omega})$ for $T > \delta > 0$, and fulfils the dissipative estimate for all $t \geq 0$,*

$$\|\varphi(t)\|^2 + \int_t^{t+1} \|\varphi(\tau)\|_V^2 \, d\tau \leq \|\varphi(0)\|^2 e^{-\omega t} + C, \tag{2.7}$$

where ω and C are positive constants independent of the initial condition and on time.

Next, we define what we mean by a strong solution to our problem. We begin with the following definition.

Definition 2.6. *Let $\varphi_0 \in V \cap C^\beta\left(\overline{\Omega}\right)$ such that $F(\varphi_0) \in L^1(\Omega)$ and $T > 0$ be given. A function φ is called a strong solution to problem* (1.6)-(1.7) *on $[0,T]$ corresponding to $\varphi(0) = \varphi_0$ if it is a weak solution in the sense of Definition 2.1 and, in addition,*

$$\varphi \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)),$$

$$\varphi \in C\left([0,T];V\right) \cap C^{\beta,\beta/2}\left(\overline{\Omega}\times[0,T]\right).$$

In particular, φ satisfies

$$\partial_t\varphi = div\left(\gamma(\varphi)\nabla\varphi - m(\varphi)\nabla J * \varphi\right), \qquad a.e.\ \ in\ \Omega\times(0,T), \tag{2.8}$$

$$\gamma(\varphi)\partial_n\varphi = m\left(\varphi\right)\left(\nabla J\cdot n * \varphi\right), \qquad a.e.\ \ on\ \partial\Omega\times(0,T), \tag{2.9}$$

$$\varphi(0,\cdot) = \varphi_0, \qquad a.e.\ \ in\ \Omega. \tag{2.10}$$

Suppose that $\Omega\subset\mathbb{R}^d$, $d\le 3$ has a boundary $\partial\Omega$ of class $\mathcal{C}^2$. Concerning the interaction kernel J we assume the following

(H.4) *Either* $J\in W_1^2(\mathcal{B}_\delta)$, where $\mathcal{B}_\delta=\{x\in\mathbb{R}^d : |x|<\delta\}$ with $\delta\sim diam(\Omega)$ such that $\mathcal{B}_\delta\supset\overline{\Omega}$ *or* J is admissible in the sense of [5, Definition 2].

We recall that both Newtonian and Bessel potentials are still admissible due to the second part of (H.4). Also, as a consequence of (H.1) and (H.4), J satisfies the inequality

$$\left\|\nabla J * \varphi\right\|_{W_p^1(\Omega)} \le C_J\left\|\varphi\right\|_{L^p(\Omega)}. \tag{2.11}$$

Moreover, we assume

(H.5) $F\in C^3\left(-1,1\right)$;
(H.6) $\gamma\in C^1([-1,1])$;
(H.7) $\gamma(s)\ge\theta>0$ for all $s\in[-1,1]$.

We recall the following result that was proved in [6], among others.

Theorem 2.7. *Let the assumptions (H.1)-(H.7) hold. In addition, let $\varphi_0\in V\cap C^\beta\left(\overline{\Omega}\right)$ be such that $F(\varphi_0)\in L^1(\Omega)$ and $M(\varphi_0)\in L^1(\Omega)$, where M is defined as in Theorem 2.3. Then there exists a (unique) strong solution in the sense of Definition 2.6.*

3. Maximal solutions and Sobolev regularity

Here we state and prove the existence of a smooth solution which possesses $W_p^s(\Omega)$-regularity for some $s>1$ and $p>d$, for any fixed spatial dimension $d\ge 1$. Such a solution cannot be obtained by energy methods. Instead we shall employ a method that exploits maximal regularity results for parabolic equations with inhomogeneous Robin boundary conditions. A basic role will also be played by a Hölder bound in order to establish the existence of a *globally* $W_p^{2-2/p}(\Omega)$-bounded solution.

Let us briefly describe the function spaces that are used in this section. More details and information on them can be found in [19, 20, 21, 22]. We work below with weighted and unweighed vector-valued function spaces. To this end, let $p\in(1,\infty)$, $\rho\in(1/p,1]$ and let X be a (real) Banach space and $T\in(0,\infty]$. We set

$$L_{p,\rho}(0,T;X) := \big\{u:(0,T)\to X\ :\ \|u\|_{L_{p,\rho}(0,T;X)}^p = \int_0^T t^{p(1-\rho)}\|u(t)\|_X^p\,dt<\infty\big\},$$

$$W_{p,\rho}^1(0,T;X) := \big\{u\in L_{p,\rho}(0,T;X)\ :\ \exists\,u'\in L_{p,\rho}(0,T;X)\big\}.$$

Note that $\rho=1$ yields the unweighed case, i.e., $L^p=L_{p,1}$ and $W_p^1=W_{p,1}^1$. For instance, it is shown in [22, Lemma 2.1] that $W_{p,\rho}^1(0,T;X)\to W_1^1(0,T;X)$ is continuous, and thus the (temporal) trace $\mathrm{tr}_0\left(u\right)=u|_{t=0}$ is continuous on $W_{p,\rho}^1(0,T;X)$.

We can rewrite our nonlocal problem (see (1.6)-(1.7)) in the following form[1]

$$\varphi_t - \partial_i(m(\varphi)F''(\varphi)\partial_i\varphi) = -\partial_i\left(m(\varphi)\partial_i J * \varphi\right), \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

subject to the boundary condition

$$m(\varphi)\,F''(\varphi)\,n_i\partial_i\varphi = m(\varphi)\,n_i\partial_i J * \varphi, \quad \text{on } \partial\Omega \times (0,T), \tag{3.2}$$

and to the initial condition

$$\varphi_{|t=0} = \varphi_0, \quad \text{in } \Omega. \tag{3.3}$$

We also observe preliminarily that by assumption (H.5) (see below) we can further rewrite the boundary condition (3.2) into the equivalent form

$$n_i\partial_i\varphi + l(\varphi)\,n_i\partial_i J * \varphi = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{3.4}$$

which can then be treated as an inhomogeneous nonlinear Robin boundary condition. Our aim is to first show the global existence of smooth solutions in a scale of nonlinear spaces for (3.1)-(3.3) by exploiting maximal regularity results developed in [19, 21]. Our second aim is to use the inherent smoothing effect of the weighted spaces to show that if $\varphi(t_1) \in C^{\beta}(\overline{\Omega})$ for some time $t_1 > 0$ then $\varphi(t_2) \in W_p^{2-2/p}(\Omega)$ for $p \in (d+2,\infty)$ at a later time $t_2 > t_1$.

Here we actually assume that the bounded domain $\Omega \subset \mathbb{R}^d$ has a boundary of class $\mathcal{C}^2$. In addition, besides (H.1)-(H.7), we also suppose

(H.8) $\gamma \in C^2([-1,1])$, $l := -1/F'' \in C^2([-1,1])$. [2]

In order to rigorously introduce our notion of smooth solution for (3.1)-(3.3), we define the maximal regularity class[3]

$$E_{\varphi,\rho}(I) = W^1_{p,\rho}(I; L^p(\Omega)) \cap L_{p,\rho}\left(I; W^2_p(\Omega)\right),$$

where $I = (0,T)$ is a finite interval, $p \in (d+2,\infty)$ and $\rho \in (1/p, 1]$, as well as the boundary class[4]

$$F_\rho(I) = W_{p,\rho}^{\frac{1}{2}-\frac{1}{2p}}(I; L^p(\partial\Omega)) \cap L_{p,\rho}(I; W_p^{1-1/p}(\partial\Omega)).$$

Besides, we will also need the following weighted space $E_{0,\rho}(I) = L_{p,\rho}(I; L^p(\Omega))$. We recall that embedding results in weighted spaces[5] (see [21]; cf. also [19, Theorem 1.3.6]) yield

$$E_{\varphi,\rho}(I) \hookrightarrow C(\overline{I}; W_p^{2(\rho-1/p)}(\Omega)) \hookrightarrow C(\overline{I}; C^1(\overline{\Omega})) \tag{3.5}$$

where the last embedding in (3.5) holds if and only if $2(\rho - 1/p) > 1 + d/p$. Similarly, we have

$$F_\rho(I) \hookrightarrow C(\overline{I}; W_p^{2(\rho-1/p)-1-1/p}(\partial\Omega)),$$

provided that $2(\rho - 1/p) > 1 + 1/p$, so that

$$F_\rho(I) \hookrightarrow C(\overline{I}; C(\partial\Omega)), \text{ if } 2(\rho - 1/p) > 1 + d/p. \tag{3.6}$$

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[1]The summation convention is used.

[2]Note that by (H.5), $\gamma \in C^2([-1,1])$ and $l \in C^2([-1,1])$ is equivalent to having $\gamma \in C^2([-1,1])$ and $m \in C^2([-1,1])$.

[3]By the classical convention in Sobolev function theory, $E_{\varphi,\rho}(I) = W^{1,2}_{p,\rho}(I \times \Omega)$. We recall our meaning that $W^{s,t}_{p,\rho}(I \times X) := W^s_{p,\rho}(I; L^p(X)) \cap L_{p,\rho}\left(I; W^t_p(X)\right)$, where X is either Ω or $\partial\Omega$.

[4]Also, we make the following convention, $F_\rho(I) = W^{1/2-1/2p,1-1/p}_{p,\rho}(I \times \partial\Omega)$.

[5]We note that when restricting to ${}_0E_{\varphi,\mu}(I)$ and ${}_0F_\mu(I)$-spaces, the corresponding embeddings have the constants independent of the size of the interval I. For a definition of these spaces, see [19, 21].

Having defined these spaces, it is actually convenient to further convert (3.1)-(3.3) into the following abstract form

$$\begin{cases} \partial_t \varphi + A(\varphi) = 0, \text{ in } \Omega \times I, \\ B(\varphi) = 0, \text{ on } \partial\Omega \times I, \\ \varphi_{|t=0} \left(= \varphi(0,\cdot)\right) = \varphi_0, \text{ in } \Omega, \end{cases} \tag{3.7}$$

where, for $\varphi \in E_{\varphi,\rho}(I)$, the nonlinear operators A, B are defined as follows[6]:

$$\begin{aligned} A(\varphi) &:= -\partial_i (m(\varphi) F''(\varphi) \partial_i \varphi - m(\varphi) \partial_i J * \varphi), \\ B(\varphi) &:= n_i \mathrm{tr}_\Omega \partial_i \varphi + l(\mathrm{tr}_\Omega \varphi) n_i \mathrm{tr}_\Omega (\partial_i J * \varphi). \end{aligned} \tag{3.8}$$

Notice that, since the condition $|\varphi| \leq 1$ is not included in the definition of the space $E_{\varphi,\rho}(I)$, here and henceforth (cf. Lemma 3.2, Lemma 3.3, Theorem 3.4 and Lemma 3.5 below) we shall tacitly assume that the nonlinear functions m, γ, l are replaced my some fixed smooth extensions (which, for simplicity, we still denote by m, γ, l) outside the physical interval $|s| \leq 1$ over the whole real line.

Definition 3.1. *Let $\varphi_0 \in \mathcal{F} = \{\varphi_0 \in L^\infty(\Omega) : F(\varphi_0), M(\varphi_0) \in L^1(\Omega)\}$ and $T > 0$ be given. Assume further that $\varphi_0 \in M_p^s$, where*

$$M_p^s = \left\{\varphi_0 \in W_p^s(\Omega) : n_i \partial_i \varphi_0 = -l(\varphi_0) n_i \partial_i J * \varphi_0 \text{ a.e. on } \partial\Omega\right\}, \tag{3.9}$$

where $p \in (d+2, \infty)$, $\rho \in (1/2 + (d+2)/2p, 1]$ and $s = 2(\rho - 1/p)$. We say that φ is a smooth solution to problem (3.1)-(3.3) on the interval $I = (0,T)$ if it satisfies (2.1) and

$$\varphi \in E_{\varphi,\rho}(I) \cap C\left([0,T]; M_p^s \cap \mathcal{F}\right),$$

where the set $M_p^s \cap \mathcal{F}$ is endowed with the topology of $W_p^s(\Omega)$. In this case, the equations (3.1) and (3.2) are satisfied almost everywhere in $\Omega \times I$ and on $\partial\Omega \times I$, respectively, while the initial condition $\varphi_{|t=0} = \varphi_0$ holds in a strong sense.

In order to prove the existence and uniqueness of smooth solutions in the class of Definition 3.1, we shall rely on maximal $L_{p,\rho}$-regularity results for a linearized problem associated with (3.7) and the application of the Banach contraction principle. Indeed, it was noted in [19, Remark 4.3.7] that as long as the corresponding operators A, B in (3.7) are C^1 and a version of the maximal $L_{p,\rho}$-regularity result applies to the corresponding linearized problem, the proof of *local* existence and uniqueness is independent of the concrete form of the operators A and B. Henceforth, our aim is to establish first that $A, B \in C^1$ for our operators defined by (3.8) on the corresponding spaces. These properties are stated in the following two lemmas whose proofs are postponed in Sections 5 and 6.

In the sequel, in order to further simplify the estimates, we shall make use of the notation $a \lesssim b$ to mean that there exists a constant $C > 0$ such that $a \leq Cb$. This will be done only if the explicit value of C is irrelevant or tedious to write down. However, we shall point out various properties of the constant $C > 0$ when necessary.

Lemma 3.2. *Assume (H.1)-(H.8). Set $I = (0,T)$, $T > 0$, and let $p \in (d+2, \infty)$, $\rho \in (1/2 + (d+2)/2p, 1]$. Then $A \in C^1(E_{\varphi,\rho}(I), E_{0,\rho}(I))$ and for $\varphi \in E_{\varphi,\rho}(I)$, we have*

$$A'(\varphi) h = -\partial_i \left(\gamma(\varphi) \partial_i h + \gamma'(\varphi) \partial_i \varphi h - m'(\varphi) (\partial_i J * \varphi) h - m(\varphi) \partial_i J * h\right),$$

[6] Recall that $\mathrm{tr}_\Omega(\varphi) = \varphi_{|\partial\Omega}$.

for $h\in E_{\varphi,\rho}(I)$. *For* $T_0,R>0$ *given, there exists a continuous function* $\varepsilon:[0,\infty)\to[0,\infty)$, $\varepsilon(0)=0$, *such that for each* $T_0\le T$, *it holds*

$$||A(\varphi+h)-A(\varphi)-A'(\varphi)h||_{E_{0,\rho}(I)}\le\varepsilon(\|h\|_{E_{\varphi,\rho}(I)})\|h\|_{E_{\varphi,\rho}(I)}, \tag{3.10}$$

for all $\varphi,h\in E_{\varphi,\rho}(I)$ *with* $h(0,\cdot)=0$, *such that*

$$\|\varphi\|_{C(\bar{I};C^1(\bar{\Omega}))},\|\varphi\|_{E_{\varphi,\rho}(I)},\|h\|_{E_{\varphi,\rho}(I)}\le R. \tag{3.11}$$

Lemma 3.3. *Assume (H.1)-(H.8). Set* $I=(0,T)$, $T>0$, *and let* $p\in(d+2,\infty)$, $\rho\in(1/2+(d+2)/2p,1]$. *Then* $B\in C^1(E_{\varphi,\rho}(I),F_\rho(I))$ *and for* $\varphi\in E_{\varphi,\rho}(I)$, *we have*

$$B'(\varphi)h=n_i tr_\Omega\partial_i h+l'(tr_\Omega\varphi)n_i tr_\Omega(\partial_i J*\varphi)tr_\Omega h+l(tr_\Omega\varphi)tr_\Omega(\partial_i J*h)n_i,$$

for $h\in E_{\varphi,\rho}(I)$. *For* $T_0,R>0$ *given, there exists a continuous function* $\varepsilon:[0,\infty)\to[0,\infty)$, $\varepsilon(0)=0$, *such that for each* $T_0\le T$, *it holds*

$$\|B(\varphi+h)-B(\varphi)-B'(\varphi)h\|_{{}_0F_\rho(I)}\le\varepsilon(\|h\|_{E_{\varphi,\rho}(I)})\|h\|_{E_{\varphi,\rho}(I)}, \tag{3.12}$$

for all $\varphi,h\in E_{\varphi,\rho}(I)$ *with* $h(0,\cdot)=0$, *such that*

$$\|\varphi\|_{C(\bar{I};C^1(\partial\Omega))},\|\varphi\|_{E_{\varphi,\rho}(I)},\|h\|_{E_{\varphi,\rho}(I)},\|\varphi(0,\cdot)\|_{W_p^{2(\rho-1/p)}(\Omega)}\le R. \tag{3.13}$$

The existence of a smooth solution is given by the following

Theorem 3.4. *Let* $s=2(\rho-1/p)>1+d/p$, *with* $\rho\in(1/p,1]$ *and* $p\in(d+2,\infty)$. *Assume (H.1)-(H.8). Then, for each* $\varphi_0\in M_p^s\cap\mathcal{F}$, *there exists a time* $t^+=t^+(\varphi_0)>0$ *such that (3.1)-(3.3) has a unique maximal bounded solution in the sense of Definition 3.1 on* $I=(0,t^+)$. *Furthermore, if* $\varphi_0\in M_p^{2-2/p}\cap\mathcal{F}$ *the solution is global, i.e.,* $t^+=\infty$.

Proof. Step 1. (**Local well-posedness**) We can now consider the linearized problem associated with (3.7) and show that it enjoys maximal $L_{p,\rho}$-regularity for any $\varphi\in E_{\varphi,\rho}(I)$. The linearized problem takes the following form:

$$\begin{cases}\partial_t\psi+A'(\varphi)\psi=f,\text{ in }\Omega\times I,\\ B'(\varphi)\psi=g,\text{ on }\partial\Omega\times I,\\ \psi_{|t=0}=\psi_0,\text{ in }\Omega.\end{cases} \tag{3.14}$$

Let $\varphi\in E_{\varphi,\rho}(I)$ be given and let $f\in E_{0,\rho}(I)$, $g\in F_\rho(I)$ and $\psi_0\in W_p^s(\Omega)$ such that $B'(\varphi(0,\cdot))\psi_0=g(0,\cdot)$ on $\partial\Omega$, where $\rho\in(1/p,1]$, $p\in(d+2,\infty)$ and

$$s=2(\rho-1/p)>1+d/p.$$

We claim that there exists a unique bounded solution ψ to problem (3.14) such that

$$\|\psi\|_{E_{\varphi,\rho}(I)}\lesssim\|f\|_{E_{0,\rho}(I)}+\|g\|_{F_\rho(I)}+\|\psi_0\|_{W_p^s(\Omega)}.$$

Given $T_0>0$, with g belonging to ${}_0F_\rho(I)$, the above estimate is uniform in $T\le T_0$. Let us next consider the operator pair $(\widetilde{A}(\varphi),\widetilde{B})$ given by

$$\widetilde{A}(\varphi)h:=\partial_i(\gamma(\varphi)\partial_i h),\ \widetilde{B}h=n_i tr_\Omega\partial_i h,$$

for $\varphi\in E_{\varphi,\rho}(I)$. Then since

$$(A'(\varphi),B(\varphi))=(\widetilde{A}(\varphi),\widetilde{B})+\text{lower-order terms},$$

the claim follows from [19, Lemma 4.3.1] on the basis of Lemmas 3.2 and 3.3. In this case, as we have mentioned at the beginning of this section the proof for the *local* well-posedness of problem (3.1)-(3.3) on some maximal interval $I=[0,t^+)$, for some $t^+=t^+(\varphi_0)>0$, does not require a concrete form of the operators A, B, and thus follows from [19, Proposition 4.3.2]. The Lipschitz continuous dependence with respect to the initial datum φ_0, in the weaker L^2-metric is a consequence of Theorem 2.3.

Step 2. ($t^+=\infty$) Assume to the contrary that $t^+<\infty$. Then by application of Lemma 3.5 below, it follows that

$$\sup_{t\in[0,t^+)}\|\varphi\|_{W_p^{2-2/p}(\Omega)}\lesssim 1+\sup_{t\in[0,t^+/2)}\|\varphi\|_{C^\beta(\overline{\Omega})}\leq C,$$

where the last bound is a consequence of Proposition 2.5 and the fact that

$$\varphi\in C^{\beta,\beta/2}\left(\overline{\Omega}\times[0,T]\right),$$

for any $T>0$. Therefore, $\varphi(t)$ is bounded in $W_p^{2-2/p}(\Omega)$ for $t\in[0,t^+)$, and therefore it contains a convergent subsequence in $W_p^s(\Omega)$ for $s\in(1+d/p,2-2/p)$, which is in contradiction with the existence of the maximal time t^+; thus, we must have $t^+=\infty$. The proof is finished. □

We now state and prove the C^β-$W_p^{2-2/p}$ smoothing effect of the solution to our problem. Indeed we have

Lemma 3.5. *For $\varphi_0\in M_p^{2-2/p}\cap\mathcal{F}$, $p\in(d+2,\infty)$, let φ be a smooth maximal solution on $(0,t^+)$ in the sense of Definition 3.1. Let $t_1,t_2\in(0,t^+)$ such that $t_1<t_2$, with $\tau:=t_2-t_1$. Then for $\beta\in(0,1)$ there exists a constant $C>0$, depending only on τ, p, $\delta_\beta:=\|\varphi\|_{C([t_1,t_2];C^\beta(\overline{\Omega}))}$ and J, such that*

$$\|\varphi(t_2)\|_{W_p^{2-2/p}(\Omega)}\leq C\left(1+\|\varphi(t_1)\|_{C^\beta(\overline{\Omega})}\right). \tag{3.15}$$

Proof. We argue along the lines of [19, Lemma 4.4.1]. Let us set $I:=(0,\tau)$ and define $\psi(t):=\varphi(t+t_1)$, $t\in I$, and note that $\psi\in E_{\varphi,\rho}(I)$ since $\varphi\in E_{\varphi,1}(0,t^+)$. Furthermore, since the weight $t^{p(1-\rho)}$ only has an effect at $t=0$, we have

$$\|\varphi(t_2)\|_{W_p^{2-2/p}(\Omega)}=\|\psi(\tau)\|_{W_p^{2-2/p}(\Omega)}\leq C(\tau)\|\psi\|_{E_{\varphi,\rho}(I)}. \tag{3.16}$$

Observe that ψ solves the inhomogeneous linear problem

$$\begin{cases}\partial_t\xi-\gamma(\varphi)\Delta\xi=\gamma'(\varphi)(\partial_i\psi)^2-m(\psi)\partial_i(\partial_iJ*\psi)-m'(\psi)\partial_i\psi\partial_iJ*\psi,\text{ in }\Omega\times I,\\ n_i\partial_i\xi=l(\psi)\partial_iJ*\psi n_i,\text{ on }\partial\Omega\times I,\\ \xi_{|t=0}=\psi(0)=\varphi(t_1),\text{ in }\Omega,\end{cases} \tag{3.17}$$

so we can apply [19, Theorem 2.1.4 and Proposition 2.3.1] and infer the existence of a constant $C>0$, independent of δ_β and τ, such that

$$\begin{aligned}\|\psi\|_{E_{\varphi,\rho}(I)}&\leq C\left(||\gamma'(\psi)(\partial_i\psi)^2||_{E_{0,\rho}(I)}+||m(\psi)\partial_i(\partial_iJ*\psi)||_{E_{0,\rho}(I)}\right) \\ &\quad+C||m'(\psi)\partial_i\psi\partial_iJ*\psi||_{E_{0,\rho}(I)}\\ &\quad+C\left(||l(\psi)\partial_iJ*\psi n_i||_{F_\rho(I)}+\|\varphi(t_1)\|_{W_p^\sigma(\Omega)}\right),\end{aligned} \tag{3.18}$$

where $\sigma := 2(\rho - 1/p)$ with $\rho \in (1/p, 1]$. We estimate all summands on the right-hand side of (3.18) as follows. By Holder's inequality we first have

$$\begin{aligned} ||\gamma'(\psi)(\partial_i\psi)^2||^p_{E_{0,\rho}(I)} &\leq C(\delta_\beta)||(\partial_i\psi)^2||^p_{E_{0,\rho}(I)} \\ &\leq C(\delta_\beta)|| \, \|\partial_i\psi\|_{L^{2p}(\Omega)} \|\partial_i\psi\|_{L^{2p}(\Omega)} ||^p_{L_{p,\rho}(I)} \\ &\leq C(\delta_\beta)\int_I t^{p(1-\rho)} \|\psi(t)\|^p_{W^1_{2p}(\Omega)} \, dt. \end{aligned} \tag{3.19}$$

Let $r \in (1, \infty)$ and $\theta, \alpha > 0$ such that $1 - d/2p < 1/2(\alpha - d/r) + 1/2(\theta - d/p)$. Then, applying the Gagliardo-Nirenberg inequality [19, Proposition A.6.2], we get

$$\begin{aligned} \|\psi\|^{2p}_{W^1_{2p}(\Omega)} &\lesssim \|\psi\|^p_{W^\theta_p(\Omega)} \|\psi\|^p_{W^\alpha_r(\Omega)} \lesssim \|\psi\|^p_{W^\theta_p(\Omega)} \|\psi\|^p_{C^\beta(\overline{\Omega})} \\ &\leq \epsilon\|\psi\|^p_{W^2_p(\Omega)} + C(\epsilon, \delta_\beta)\|\varphi\|_{C([t_1,t_2];C(\overline{\Omega}))}, \end{aligned} \tag{3.20}$$

for any $\epsilon > 0$. Here in the second inequality we have used the fact that $C^\beta(\overline{\Omega}) \hookrightarrow W^\alpha_r(\Omega)$ for any $\alpha \in (0, \beta)$ and any $r \in (1, \infty)$, whereas in the last inequality we have exploited an interpolation inequality together with the Young inequality since one can take $\theta < 2$ sufficiently close to 2 and a sufficiently large $r \geq r_0$. Combining (3.19) with (3.20), we find that

$$||\gamma'(\psi)(\partial_i\psi)^2||_{E_{0,\rho}(I)} \leq \epsilon\|\psi\|_{E_{\varphi,\rho}(I)} + C(\epsilon, \delta_\beta). \tag{3.21}$$

For the second summand in (3.19), we have

$$\begin{aligned} ||m(\psi)\partial_i(\partial_i J * \psi)||^p_{E_{0,\rho}(I)} &\leq C(\delta_\beta)||\partial_i(\partial_i J * \psi)||^p_{E_{0,\rho}(I)} \\ &= \int_I t^{p(1-\rho)} \|\partial_i(\partial_i J * \psi)\|^p_{L^p(\Omega)} \, dt \\ &\leq C(\delta_\beta)\|\psi\|^p_{E_{0,\rho}(I)} \\ &\leq \epsilon\|\psi\|^p_{E_{\varphi,\rho}(I)} + C(\epsilon, \delta_\beta)\|\varphi\|_{C([t_1,t_2];C(\overline{\Omega}))}, \end{aligned} \tag{3.22}$$

owing to (5.35). On the other hand, the third summand can be controlled as follows

$$\begin{aligned} ||m'(\psi)\partial_i\psi\partial_i J * \psi||^p_{E_{0,\rho}(I)} &\leq C(\delta_\beta)||\partial_i\psi\partial_i J * \psi||^p_{E_{0,\rho}(I)} \\ &\leq C(\delta_\beta)|| \, \|\partial_i\psi\|_{L^p(\Omega)} \|\partial_i J * \psi\|_{L^\infty(\Omega)} ||^p_{L_{p,\rho}(I)} \\ &\leq C\left(\delta_\beta, \|J\|_{W^1_1}\right) \int_I t^{p(1-\rho)} \|\psi\|^p_{W^1_p(\Omega)} \, dt \\ &\leq \epsilon\|\psi\|^p_{E_{\varphi,\rho}(I)} + C\left(\epsilon, \delta_\beta, \|J\|_{W^1_1}\right)\|\varphi\|_{C([t_1,t_2];C(\overline{\Omega}))}, \end{aligned} \tag{3.23}$$

for any $\epsilon > 0$, owing to the Young convolution theorem, interpolation and the Young inequality. We may summarize from estimates (3.22)-(3.23) that

$$\begin{aligned} &||m(\psi)\partial_i(\partial_i J * \psi)||_{E_{0,\rho}(I)} + ||m'(\psi)\partial_i\psi\partial_i J * \psi||_{E_{0,\rho}(I)} \\ &\leq \epsilon\|\psi\|_{E_{\varphi,\rho}(I)} + C(\epsilon, \delta_\beta). \end{aligned} \tag{3.24}$$

For boundary summand in (3.18), we recall that $n = n(\cdot) \in C^1(\overline{\Omega})$ and (6.3), which states that $\mathrm{tr}_\Omega(\cdot) : W^{1/2,1}_{p,\rho}(I \times \Omega) \to F_\rho(I)$ is continuous and its operator norm depends on τ. In particular, we have

$$||l(\psi)\partial_i J * \psi n_i||_{F_\rho(I)} \leq C(\tau)\,||l(\psi)\partial_i J * \psi n_i||_{W^{1/2,1}_{p,\rho}(I\times\Omega)}. \tag{3.25}$$

We use the intrinsic norm for $W^{1/2}_{p,\rho}(I; L^p(\Omega))$ to estimate the right-hand side in (3.25). Exploiting the mean value theorem for $l \in C^1([-1,1])$, the Young convolution theorem and the Hölder inequality, we obtain

$$\begin{aligned}
&[l(\psi)\partial_i J * \psi n_i]^p_{W^{1/2}_{p,\rho}(I;L^p(\Omega))} \\
&= \int_0^T\int_0^s \frac{t^{p(1-\rho)}}{(s-t)^{1+(1/2)p}}\,\|l(\psi(t))\partial_i J * \psi(t)\,n_i - l(\psi(s))\partial_i J * \psi(s)\,n_i\|^p_{L^p(\Omega)}\,dtds \\
&\leq \int_0^T\int_0^s \frac{t^{p(1-\rho)}}{(s-t)^{1+(1/2)p}}\,\|(l(\psi(t)) - l(\psi(s)))\partial_i J * \psi(t)\,n_i\|^p_{L^p(\Omega)}\,dtds \\
&+ \int_0^T\int_0^s \frac{t^{p(1-\rho)}}{(s-t)^{1+(1/2)p}}\,\|l(\psi(s))\partial_i J * (\psi(t) - \psi(s))n_i\|^p_{L^p(\Omega)}\,dtds \\
&\leq C\left(\delta_\beta, \|J\|_{W^1_1}\right)[\psi]^p_{W^{1/2}_{p,\rho}(I;L^p(\Omega))} + C\left(\|J\|_{W^1_1}\right)\|l(\varphi)\|_{C([t_1,t_2];C(\overline{\Omega}))}[\psi]^p_{W^{1/2}_{p,\rho}(I;L^p(\Omega))} \\
&\leq C\left(\delta_\beta, \|J\|_{W^1_1}\right)[\psi]^p_{W^{1/2}_{p,\rho}(I;L^p(\Omega))}.
\end{aligned} \tag{3.26}$$

Similarly, using (5.37) we have

$$\begin{aligned}
||l(\psi)\partial_i J * \psi n_i||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} &\leq \|l(\varphi)\|_{C([t_1,t_2];C(\overline{\Omega}))}\|\partial_i J * \psi n_i\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\
&+ \|\partial_i J * \psi n_i\|_{L_{p,\rho}(I;L^\infty(\Omega))}\|l(\psi)\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\
&\leq C\left(\delta_\beta, \|J\|_{W^1_1}\right)\|\psi\|_{L_{p,\rho}(I;L^p(\Omega))} \\
&+ C\left(\|J\|_{W^1_1}, \tau, \delta_\beta\right)\left(1 + \|\psi\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)}\right) \\
&\leq C\left(\|J\|_{W^1_1}, \tau, \delta_\beta\right)\left(1 + \|\psi\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)}\right),
\end{aligned} \tag{3.27}$$

owing to (5.35) and the Young convolution theorem. Combining (3.26)-(3.27) together with (3.25), we obtain by interpolation and the Young inequality,

$$\begin{aligned}
||l(\psi)\partial_i J * \psi n_i||_{F_\rho(I)} &\leq C(J,\tau,\delta_\beta)\left(1 + \|\psi\|_{W^{1/2,1}_{p,\rho}(I\times\Omega)}\right) \\
&\leq \epsilon\,\|\psi\|_{W^{1,2}_{p,\rho}(I\times\Omega)} + C(\epsilon, J, \tau, \delta_\beta)\,\|\varphi\|_{C([t_1,t_2];C(\overline{\Omega}))},
\end{aligned} \tag{3.28}$$

for any $\epsilon > 0$. Since $E_{\varphi,\rho}(I) = W^{1,2}_{p,\rho}(I\times\Omega)$, we collect all the estimates from (3.21), (3.24) and (3.28), and then we choose a sufficiently small $\epsilon < 1/3$, to find from (3.16) and (3.18) that

$$\|\varphi(t_2)\|_{W^{2-2/p}_p(\Omega)} \leq C(\tau)\,\|\psi\|_{E_{\varphi,\rho}(I)} \leq C(J,\tau,\delta_\beta)\left(1 + \|\varphi(t_1)\|_{W^\sigma_p(\Omega)}\right),$$

where $\sigma := 2(\rho - 1/p)$ with $p \in (d+2,\infty)$ and $\rho \in (1/p, 1]$. Let us now choose $\rho = 1/p + \nu$, for some $\nu < \min(1 - 1/p, \beta/2)$, for $\beta \in (0,1)$. Then, since $C^\beta\left(\overline{\Omega}\right) \hookrightarrow W^\sigma_p(\Omega)$ for $\sigma \in (0,\beta)$ and any $p \in (1,\infty)$, we can easily arrive at inequality (3.15). The proof is complete. □

Lemma 3.5 entails the following

Corollary 3.6. *Let the assumptions of Lemma 3.5 hold. Then, for any $T > \zeta > 0$, we have*

$$\|\varphi(T)\|_{W_p^{2-2/p}(\Omega)} + \|\varphi\|_{E_{\varphi,\rho}(T-\zeta,T)} \leq C\left(\zeta, J, \|\varphi\|_{C\left([T-\zeta,T];C^\beta\left(\overline{\Omega}\right)\right)}\right),$$

for some sufficiently small $\rho \in (1/p, 1]$.

Remark 3.7. Thanks to Theorem 3.4, system (1.6)-(1.7) generates a family of closed semigroups on the phase space $\mathcal{Q}_p := \mathcal{F} \cap M_p^{2-2/p}$, for $p \in (d+2, \infty)$,

$$\mathcal{S}(t) : \mathcal{Q}_p \to \mathcal{Q}_p, \quad S(t)\varphi_0 = \varphi(t), \quad \forall t \geq 0,$$

where φ is the smooth solution in the sense of Definition 3.1. Furthermore, the dynamical system $(\mathcal{Q}_p, \mathcal{S}(t))$ is dissipative since it possesses a bounded absorbing set $\mathcal{B}_p \subset W_p^{2-2/p}(\Omega) \cap \mathcal{F}$ thanks to Corollary 3.6. More precisely, it holds

$$\sup_{t>2}\left(\|\varphi(t)\|_{W_p^{2-2/p}(\Omega)} + \|\varphi(t)\|_{C^{1+\delta_0}\left(\overline{\Omega}\right)}\right) \leq C_0, \tag{3.29}$$

for some $C_0 > 0$ independent of time and the initial datum, also owing to the embedding $W_p^{2-2/p}(\Omega) \hookrightarrow C^{1+\delta_0}\left(\overline{\Omega}\right)$, for some $\delta_0 > 0$. This is the starting point to investigate the existence of global and exponential attractors (cf. [11]).

4. Further regularity properties

In this section we exploit the previous results to show that φ belongs to $W_\infty^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ if $d \leq 3$, without the restriction γ constant in dimension three (cf. [6]).

Before stating and proving the main result of this section, we introduce a technical lemma whose proof is given in the last section.

Lemma 4.1. *Let (H.1)-(H.8) hold. Then there exists a constant $C > 0$ independent of t, T, φ and $\varphi(0)$, such that*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C\left(\|\partial_t\varphi(t)\| + \|\varphi(t)\|_V + \|\varphi(t)\|_{C^\beta(\overline{\Omega})}^{2(1-\theta)/(1-2\theta)}\right), \tag{4.30}$$

for almost any $t \in (0,T)$ and for some $\theta \in (0, 1/2)$.

The main result reads

Theorem 4.2. *Let (H.1)-(H.8) hold. Given an initial datum $\varphi_0 \in H^2(\Omega)$ that satisfies the compatibility condition*

$$\gamma(\varphi_0)\partial_n\varphi_0 = m(\varphi_0)(n \cdot \nabla J * \varphi_0) \ \text{ a.e. on } \partial\Omega, \tag{4.31}$$

the corresponding strong solution also has the following regularity

$$\varphi \in W_\infty^1\left(0,T;L^2(\Omega)\right) \cap L^\infty\left(0,T;H^2(\Omega)\right). \tag{4.32}$$

Proof. Let $\varphi_0 \in H^2(\Omega) \cap \mathcal{F}$ that satisfies (4.31) and let φ be the corresponding strong solution in the sense of Definition 2.6. Since $H^2(\Omega) \subset W_p^{2(1-1/p)}(\Omega)$ for *any* $p \in (d+2, 6]$ and $d \leq 3$, we also have $\varphi_0 \in M_p^{2-2/p}$ (see (3.9)) in this range for p (we have also taken $\rho = 1$ so all space-time integrals are no longer weighted). Thus Theorem 3.4 implies that the (global) strong solution satisfies

$$\varphi \in W_p^1(0,T;L^p(\Omega)) \cap L_p\left(0,T;W^{2,p}(\Omega)\right) \cap C([0,T];M_p^{2-2/p}) \tag{4.33}$$

for any $p \in (d+2, 6]$ with $d \le 3$. The improved maximal regularity (4.33) allows us to gain the desired regularity claim in (4.32) provided that we can show that $\partial_t \varphi \in L^\infty(0,T; L^2(\Omega))$. Indeed, the latter bound together with Holder regularity for φ (see Definition 2.6) allows to deduce the desired claim owing to the application of the elliptic estimate (4.30) (which also holds for φ), see Lemma 4.1.

Let us introduce the difference in time of a function v by $T_h v(t) = v(t+h) - v(t)$, for any $h > 0$. Being φ a solution to (2.8)-(2.9), $T_h\varphi$ solves

$$\begin{aligned}(\partial_t T_h\varphi, v) &+ (\gamma(\varphi(\cdot + h))T_h\nabla\varphi, \nabla v) + (T_h\gamma(\varphi)\nabla\varphi, \nabla v) \\ &= (m(\varphi(\cdot + h))\nabla J * T_h\varphi, \nabla v) + (T_h m(\varphi)\nabla J * \varphi, \nabla v),\end{aligned}$$

for any $v \in H$. Taking $v = T_h\varphi$, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|T_h\varphi\|^2 + \theta\|\nabla T_h\varphi\|^2 &\le -(T_h\gamma(\varphi)\nabla\varphi, \nabla T_h\varphi) \\ &+ (m(\varphi(\cdot + h))\nabla J * T_h\varphi, \nabla T_h\varphi) + (T_h m(\varphi)\nabla J * \varphi, \nabla T_h\varphi).\end{aligned}$$

Since γ' is bounded,

$$(T_h\gamma(\varphi)\nabla\varphi, \nabla T_h\varphi) \le \frac{\theta}{4}\|\nabla T_h\varphi\|^2 + C\|\nabla\varphi\|^2_{L^\infty(\Omega)}\|T_h\varphi\|^2.$$

On the other hand, we easily find

$$(m(\varphi(\cdot + h))\nabla J * T_h\varphi, \nabla T_h\varphi) + (T_h m(\varphi)\nabla J * \varphi, \nabla T_h\varphi) \le \frac{\theta}{4}\|\nabla T_h\varphi\|^2 + C\|T_h\varphi\|^2.$$

Thus, we arrive at the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|T_h\varphi\|^2 + \frac{\theta}{2}\|\nabla T_h\varphi\|^2 \le C(1 + \|\varphi\|^2_{W^{2,p}(\Omega)})\|T_h\varphi\|^2,$$

where $p > 3$. Thanks to the enhanced regularity (4.33), an application of the Gronwall lemma yields

$$\sup_{t\in[0,T]}\|T_h\varphi(t)\|^2 \le C\|T_h\varphi(0)\|^2, \tag{4.34}$$

where C depends on T but is independent of h. In order to pass to the limit as h goes to 0, we need to find a uniform control of $T_h\varphi(0)$. To this aim, recalling that φ_0 satisfies (4.31), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi(t) - \varphi_0\|^2 &= (\varphi_t, \varphi - \varphi_0) \\ &= -(\gamma(\varphi)\nabla\varphi, \nabla(\varphi - \varphi_0)) + (m(\varphi)\nabla J * \varphi, \nabla(\varphi - \varphi_0)) \\ &= -(\gamma(\varphi)\nabla(\varphi - \varphi_0), \nabla(\varphi - \varphi_0)) + (m(\varphi)\nabla J * \varphi, \nabla(\varphi - \varphi_0)) \\ &\quad + (\gamma(\varphi)\nabla\varphi_0, \nabla(\varphi - \varphi_0)).\end{aligned}$$

By standard computations, we have

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t) - \varphi_0\|^2 + \frac{\theta}{2}\|\nabla(\varphi - \varphi_0)\|^2 \le C(1 + \|\nabla\varphi_0\|^2).$$

For any $t > 0$, integrating the above inequality on the time interval $(0,t)$, we find

$$\|\varphi(t) - \varphi_0\|^2 \le C(1 + \|\nabla\varphi_0\|^2)t.$$

Hence, taking $t = h$, we obtain

$$\|T_h\varphi(0)\|^2 \leq Ch.$$

Combining the above estimate with (4.34), we have

$$\sup_{t\in[0,T]} h^{-1}\|T_h\varphi(t)\| \leq C.$$

Exploiting the convergence $\frac{T_h v}{h} \to \partial_t\varphi$ in weak sense, we end up with

$$\|\partial_t\varphi\|_{L^\infty(0,T;H)} \leq C.$$

The proof is finished. □

5. Proof of Lemma 3.2

Let us recall first that, thanks to (H.4), we have

$$\left\|\partial_i\left(\partial_j J * \varphi\right)\right\|_{L^p(\Omega)} \leq C\left(p\right)\left\|\varphi\right\|_{L^p(\Omega)}, \ \forall p \in (1,\infty), \tag{5.35}$$

and all $i, j \in \{1, ..., d\}$. This estimate will be crucial for most of the estimates performed in this section, and shall be used repeatedly. We also recall that

$$\left\|\partial_i J * \varphi\right\|_{L^q(\Omega)} \leq C(\left\|\partial_i J\right\|_{L^1\left(\mathbb{R}^d\right)})\left\|\varphi\right\|_{L^q(\Omega)}, \tag{5.36}$$

for $i \in \{1, ..., d\}$, $q \in (1, \infty]$. Secondly, for the sake of notational simplicity, we set

$$\|\cdot\|_{0,\infty} = \|\cdot\|_{C\left(\bar{I};C\left(\overline{\Omega}\right)\right)} \text{ and } \|\cdot\|_{1,\infty} = \|\cdot\|_{C\left(\bar{I};C^1\left(\overline{\Omega}\right)\right)}.$$

We shall also rely on the fact that $W^1_p(\Omega)$ is a Banach algebra for $p > d$. Thus we have

$$\|\varphi\psi\|_{W^1_p(\Omega)} \leq \|\varphi\|_{W^1_p(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)}\|\psi\|_{W^1_p(\Omega)}, \quad \forall\varphi,\psi \in W^1_p(\Omega). \tag{5.37}$$

We use (5.37) together with the embedding (3.5) and (5.35) to estimate, for $\varphi, h \in E_{\varphi,\rho}(I)$ satisfying (3.11), the linearization of A as follows

$$\begin{aligned}
&||A(\varphi+h) - A(\varphi) - A'(\varphi)h||_{E_{0,\rho}(I)} \qquad (5.38)\\
&\lesssim ||\left(\gamma(\varphi+h) - \gamma(\varphi) - \gamma'(\varphi)h\right)\partial_i(\varphi+h)||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)}\\
&+ ||\gamma'(\varphi)h\partial_i h||_{L_{p,\rho}(I;W^{1,p}(\Omega))}\\
&+ ||\left(m(\varphi+h) - m(\varphi) - m'(\varphi)h\right)\partial_i J*(\varphi+h)||_{L_{p,\rho}(I;W^{1,p}(\Omega))}\\
&+ ||m'(\varphi)h\partial_i J*h||_{L_{p,\rho}(I;W^{1,p}(\Omega))}\\
&\lesssim ||\gamma(\varphi+h) - \gamma(\varphi) - \gamma'(\varphi)h||_{1,\infty}\left(\|\varphi\|_{E_{\varphi,\rho}(I)} + \|h\|_{E_{\varphi,\rho}(I)}\right)\\
&+ ||\gamma'(\varphi)||_{1,\infty}\|h\|^2_{E_{\varphi,\rho}(I)}\\
&+ ||m(\varphi+h) - m(\varphi) - m'(\varphi)h||_{1,\infty}\left(\|\varphi\|_{E_{\varphi,\rho}(I)} + \|h\|_{E_{\varphi,\rho}(I)}\right)\\
&+ ||m'(\varphi)||_{1,\infty}\|h\|^2_{E_{\varphi,\rho}(I)}.
\end{aligned}$$

We note that for $h(0) = 0$ the above estimates are uniform in $T_0 \leq T$. Indeed to get the third summand in the second inequality of (5.38), by (5.35) and (5.36), we have

$$\begin{aligned}
&||\partial_i \left(\left(m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h\right)\partial_i J*\left(\varphi+h\right)\right)||_{E_{0,\rho}(I)} \qquad (5.39)\\
&\leq ||\partial_i \left(m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h\right)\partial_i J*\left(\varphi+h\right)||_{E_{0,\rho}(I)}\\
&+||\left(m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h\right)\partial_i\left(\partial_i J*\left(\varphi+h\right)\right)||_{E_{0,\rho}(I)}\\
&\lesssim ||m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h||_{1,\infty}\left\|\partial_i J*\left(\varphi+h\right)\right\|_{E_{0,\rho}(I)}\\
&+||m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h||_{0,\infty}\left\|\partial_i\left(\partial_i J*\left(\varphi+h\right)\right)\right\|_{E_{0,\rho}(I)}\\
&\lesssim ||m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h||_{1,\infty}\left\|\varphi+h\right\|_{E_{0,\rho}(I)}.
\end{aligned}$$

Similarly, we get the fourth summand as follows

$$\begin{aligned}
&||\partial_i(m'\left(\varphi\right)h\partial_i J*h)||_{E_{0,\rho}(I)} \qquad (5.40)\\
&\leq ||\partial_i(m'\left(\varphi\right))h\partial_i J*h||_{E_{0,\rho}(I)}+||m'\left(\varphi\right)\partial_i h\partial_i J*h||_{E_{0,\rho}(I)}\\
&+||m'\left(\varphi\right)h\partial_i\left(\partial_i J*h\right)||_{E_{0,\rho}(I)}\\
&\lesssim ||m'\left(\varphi\right)||_{1,\infty}\left\|h\right\|_{0,\infty}\left\|\partial_i J*h\right\|_{E_{0,\rho}(I)}+||m'\left(\varphi\right)||_{0,\infty}\left\|\partial_i h\right\|_{0,\infty}\left\|\partial_i J*h\right\|_{E_{0,\rho}(I)}\\
&+||m'\left(\varphi\right)||_{0,\infty}\left\|h\right\|_{0,\infty}\left\|\partial_i\left(\partial_i J*h\right)\right\|_{E_{0,\rho}(I)}\\
&\lesssim ||m'\left(\varphi\right)||_{1,\infty}\left\|h\right\|_{1,\infty}\left\|h\right\|_{E_{0,\rho}(I)}\\
&\lesssim ||m'\left(\varphi\right)||_{1,\infty}\left\|h\right\|^2_{E_{\varphi,\rho}(I)}.
\end{aligned}$$

For the first summand in (5.38), we use a basic uniform estimate for smooth functions (see [19, Lemma 4.2.1]) and the embedding (3.5), to deduce

$$||\gamma\left(\varphi+h\right)-\gamma\left(\varphi\right)-\gamma'\left(\varphi\right)h||_{0,\infty}\leq\varepsilon\left(\left\|h\right\|_{0,\infty}\right)\left\|h\right\|_{0,\infty}\leq\varepsilon\left(\left\|h\right\|_{E_{\varphi,\rho}(I)}\right)\left\|h\right\|_{E_{\varphi,\rho}(I)}.$$

On the other hand, we have

$$\begin{aligned}
&||\nabla\left(\gamma\left(\varphi+h\right)-\gamma\left(\varphi\right)-\gamma'\left(\varphi\right)h\right)||_{0,\infty} \qquad (5.41)\\
&\leq ||\gamma''\left(\varphi\right)\nabla hh||_{0,\infty}\\
&+||\gamma'\left(\varphi+h\right)-\gamma'\left(\varphi\right)-\gamma''\left(\varphi\right)h||_{0,\infty}\left(\left\|\varphi\right\|_{1,\infty}+\left\|h\right\|_{1,\infty}\right)\\
&\lesssim ||\gamma''\left(\varphi\right)||_{0,\infty}\left\|h\right\|^2_{E_{\varphi,\rho}(I)}+\varepsilon\left(\left\|h\right\|_{0,\infty}\right)\left\|h\right\|_{0,\infty}\\
&\leq\varepsilon\left(\left\|h\right\|_{E_{\varphi,\rho}(I)}\right)\left\|h\right\|_{E_{\varphi,\rho}(I)},
\end{aligned}$$

where in the second summand of the first inequality we have applied once again [19, Lemma 4.2.1] to the function γ'. Furthermore, the third summand that involves the mobility m, in the second inequality of (5.38), can be estimated exactly *verbatim*, so that

$$||m\left(\varphi+h\right)-m\left(\varphi\right)-m'\left(\varphi\right)h||_{1,\infty}\leq\varepsilon\left(\left\|h\right\|_{E_{\varphi,\rho}(I)}\right)\left\|h\right\|_{E_{\varphi,\rho}(I)}.$$

Clearly, since both $||\gamma'(\varphi)||_{1,\infty}$ and $||m'(\varphi)||_{1,\infty}$ are bounded by a constant in terms of $R > 0$, according to (3.11) and the fact that $m, \gamma \in C^2([-1,1])$, we have

$$||m'(\varphi)||_{1,\infty} \|h\|^2_{E_{\varphi,\rho}(I)} + ||\gamma'(\varphi)||_{1,\infty} \|h\|^2_{E_{\varphi,\rho}(I)} \leq \varepsilon\left(\|h\|_{E_{\varphi,\rho}(I)}\right)\|h\|_{E_{\varphi,\rho}(I)}. \tag{5.42}$$

Combining all the foregoing estimates into (5.38), we arrive at the uniform estimate (3.10) for as long as $\varphi, h \in E_{\varphi,\rho}(I)$ are as in (3.11). Therefore, A is also differentiable in each $\varphi \in E_{\varphi,\rho}(I)$ with derivative $A'(\varphi)$.

It remains to show that $A' : E_{\varphi,\rho}(I) \to \mathcal{B}(E_{\varphi,\rho}(I), E_{0,\rho}(I))$ is continuous. To this end, we take $\varphi, \psi, h \in E_{\varphi,\rho}(I)$ with $\|h\|_{E_{\varphi,\rho}(I)} \leq 1$, and $\|\psi\|_{E_{\varphi,\rho}(I)}, \|\varphi\|_{E_{\varphi,\rho}(I)} \leq R$, for arbitrary but fixed $R > 0$. We exploit (5.37) and the embedding (3.5) in what follows to find

$$\begin{aligned} &||(A'(\varphi) - A'(\psi))h||_{E_{0,\rho}(I)} \\ &\leq \|(\gamma(\varphi) - \gamma(\psi))\partial_i h\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} + ||(\gamma'(\varphi)\partial_i\varphi - \gamma'(\psi)\partial_i\psi)h||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &+ \|(m(\varphi) - m(\psi))\partial_i J * h\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &+ ||(m'(\varphi)\partial_i J * \varphi - m'(\psi)\partial_i J * \psi)h||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &=: S_1 + S_2 + S_3. \end{aligned} \tag{5.43}$$

Furthermore, for $\|h\|_{E_{\varphi,\rho}(I)} \leq 1$, we have

$$\begin{aligned} S_1 &:= \|(\gamma(\varphi) - \gamma(\psi))\partial_i h\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} + ||(\gamma'(\varphi)\partial_i\varphi - \gamma'(\psi)\partial_i\psi)h||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\lesssim \|\gamma(\varphi) - \gamma(\psi)\|_{1,\infty} + ||\gamma'(\varphi)\partial_i\varphi - \gamma'(\psi)\partial_i\psi||_{0,\infty} \\ &+ ||\gamma'(\varphi)\partial_i\varphi - \gamma'(\psi)\partial_i\psi||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\lesssim \|\gamma(\varphi) - \gamma(\psi)\|_{1,\infty} + ||\gamma'(\varphi)\partial_i\varphi - \gamma'(\psi)\partial_i\psi||_{0,\infty} \\ &+ ||\gamma'(\varphi)(\partial_i\varphi - \partial_i\psi)||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} + ||(\gamma'(\varphi) - \gamma'(\psi))\partial_i\psi||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\to 0, \end{aligned} \tag{5.44}$$

due to (3.5), as $\varphi \to \psi$ in the strong $E_{\varphi,\rho}(I)$-norm sense. Similarly, it follows that

$$\begin{aligned} S_3 &:= ||(m'(\varphi)\partial_i J * \varphi - m'(\psi)\partial_i J * \psi)h||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\lesssim ||m'(\varphi)\partial_i J * \varphi - m'(\psi)\partial_i J * \psi||_{0,\infty} + ||m'(\varphi)\partial_i J * \varphi - m'(\psi)\partial_i J * \psi||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\lesssim ||m'(\varphi)\partial_i J * (\varphi - \psi)||_{0,\infty} + ||(m'(\varphi) - m'(\psi))\partial_i J * \psi||_{0,\infty} \\ &+ ||m'(\varphi)\partial_i J * (\varphi - \psi)||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} + ||(m'(\varphi) - m'(\psi))\partial_i J * \psi||_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \\ &\leq C\left(R, \|J\|_{W^1_1}\right)\|\varphi - \psi\|_{E_{\varphi,\rho}(I)} + ||m'(\varphi) - m'(\psi)||_{1,\infty} \\ &\to 0, \end{aligned} \tag{5.45}$$

due to (3.5), (5.35) and $m' \in C^1([-1,1])$, for as long as $\varphi \to \psi$ in $E_{\varphi,\rho}(I)$. Finally, exploiting once again (5.37), on account of (5.35) we have

$$\begin{aligned} S_2 &:= \|(m(\varphi) - m(\psi))\,\partial_i J * h\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)} \qquad (5.46) \\ &\lesssim \|m(\varphi) - m(\psi)\|_{0,\infty}\|h\|_{E_{\varphi,\rho}(I)} + \|m(\varphi) - m(\psi)\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)}\|h\|_{0,\infty} \\ &\lesssim \|m(\varphi) - m(\psi)\|_{E_{\varphi,\rho}} \\ &\to 0, \end{aligned}$$

as $\varphi \to \psi$ in $E_{\varphi,\rho}(I)$, due to the continuity of m. Summarizing, from estimates (5.44)-(5.46), we have shown in (5.43) that indeed we have

$$\|A'(\varphi) - A'(\psi)\|_{\mathcal{B}(E_{\varphi,\rho}(I),E_{0,\rho}(I))} = \sup_{h\in E_{\varphi,\rho}(I),\|h\|_{E_{\varphi,\rho}(I)}\leq 1} ||(A'(\varphi) - A'(\psi))h||_{E_{0,\rho}(I)} \to 0\,,$$

as $\varphi \to \psi$ in $E_{\varphi,\rho}(I)$, which is the required continuity of A'. The proof is now finished.

6. Proof of Lemma 3.3

We first recall that, since we have assumed that $\partial\Omega$ of class $\mathcal{C}^2$, then the boundary can be locally "flattened". Fix a point $x_0 \in \partial\Omega$. Then there is an open ball $B = B(x_0)$ and a bijection $\pi : B \to D \subset \mathbb{R}^d$ such that $\pi(B \cap \Omega) \subset \mathbb{R}^d_+$, $\pi(\partial\Omega \cap B) \subset \partial\mathbb{R}^d_+$ and $\pi \in C^2(B)$, $\pi^{-1} \in C^2(D)$. In particular, for $\pi = (\pi_1, ..., \pi_d)$ we have $\pi_d \equiv 0$ on $B \cap \partial\Omega$. In this case, $n = n(x_0) = -\nabla\pi_d(x_0)/\|\nabla\pi_d(x_0)\|$ is a well defined outer-normal vector to $x_0 \in \partial\Omega$. In particular, $n = n(x_0) \in C^1\left(B \cap \overline{\Omega}\right)$. Next, we observe that for $\varphi \in E_{\varphi,\rho}(I)$, there holds that

$$\begin{aligned} \mathrm{tr}_\Omega\varphi \in W^{1-1/2p,2-1/p}_{p,\rho}(I \times \partial\Omega) &\hookrightarrow C(\overline{I}; W^{2(\rho-1/p)-1/p}_p(\partial\Omega)) \cap L_{p,\rho}(I; W^{2-1/p}_p(\partial\Omega)) \\ &\hookrightarrow C\left(\overline{I} \times \partial\Omega\right) \cap L_{p,\rho}\left(I; C^1(\partial\Omega)\right), \qquad (6.1) \end{aligned}$$

where [19, Proposition 1.3.12] in the first part of (6.1), and [19, Theorem 1.3.6], together with the condition $2(\rho - 1/p) > d/p$, in the second part have been used. Notice also that $W^{2-1/p}_p(\partial\Omega) \hookrightarrow C^1(\partial\Omega)$, since $p \in (d+2, \infty)$. Therefore, by [19, Lemma 4.2.3, part (a)] we have

$$l'(\mathrm{tr}_\Omega\varphi),\ l(\mathrm{tr}_\Omega\varphi) \in F_\rho(I) \cap C\left(\overline{I} \times \partial\Omega\right).$$

Furthermore,

$$n_i \mathrm{tr}_\Omega(\partial_i J * \varphi) \in F_\rho(I) \cap C\left(\overline{I} \times \partial\Omega\right), \qquad (6.2)$$

due to the regularity of J and $n = (n_1, ..., n_d) \in C^1$. Indeed, it suffices to show it in the norm of $F_\rho(I)$ due to (3.6) since $2(\rho - 1/p) > 1 + d/p$.

Instead of applying the Young convolution theorem in boundary spaces (in fact, this would surely require further regularity of J in addition to (H.4)), we recall the fact that the spatial trace

$$\mathrm{tr}_\Omega(\cdot) : W^{1/2,1}_{p,\rho}(I \times \Omega) \to F_\rho(I) = W^{1/2-1/2p,1-1/p}_{p,\rho}(J \times \partial\Omega) \qquad (6.3)$$

is continuous, and that the operator norm of tr_Ω in ${}_0W^{1/2,1}_{p,\rho}(I \times \Omega)$ is independent of the length of I (cf. [19, Proposition 1.3.12]). Noticing that

$$W^{1/2,1}_{p,\rho}(I \times \Omega) = W^{1/2}_{p,\mu}(I; L^p(\Omega)) \cap L_{p,\mu}\left(I; W^1_p(\Omega)\right)$$

we get

$$||n_i \mathrm{tr}_\Omega (\partial_i J * \varphi)||_{F_\rho(I)} \lesssim \max_{i=1,\ldots,d} ||\mathrm{tr}_\Omega (\partial_i J * \varphi)||_{F_\rho(I)} \lesssim \max_{i=1,\ldots,d} ||\partial_i J * \varphi||_{W^{1/2,1}_{p,\rho}(I\times\Omega)} \tag{6.4}$$
$$\lesssim ||\varphi||_{W^{1/2,1}_{p,\rho}(I\times\Omega)} \leq \|\varphi\|_{E_{\varphi,\rho}(I)},$$

owing to (5.35) and (5.36). Here, we point out again that the embedding constant in (6.4) is independent of the length of $I = (0,T)$ if $F_\rho(I)$ and $W^{s,1}_{p,\rho}(I\times\Omega)$ in (6.4) are replaced by ${}_0F_\rho(I)$ and ${}_0W^{s,1}_{p,\rho}(I\times\Omega)$, respectively. Application of [19, Lemma 1.3.23] yields that $F_\rho(I)$ is a Banach algebra if $2(\rho - 1/p) > 1 + d/p$ so that by (3.6) the pointwise multiplications

$$l'(\mathrm{tr}_\Omega\varphi)(n_i \mathrm{tr}_\Omega(\partial_i J * \varphi)),\ l(\mathrm{tr}_\Omega\varphi)(n_i \mathrm{tr}_\Omega(\partial_i J * h)) \in F_\rho(I) \cap C\left(\overline{I} \times \partial\Omega\right),$$

and once again by (6.1),

$$l'(\mathrm{tr}_\Omega\varphi)(n_i \mathrm{tr}_\Omega(\partial_i J * \varphi))\mathrm{tr}_\Omega h \in \mathcal{B}(E_{\varphi,\rho}(I), F_\rho(I)).$$

We now show the differentiability of B at $\varphi \in E_{\varphi,\rho}(I)$. For this it suffices to show that the superposition nonlinear operator

$$g(\mathrm{tr}_\Omega\varphi) := l(\mathrm{tr}_\Omega\varphi) n_i \mathrm{tr}_\Omega(\partial_i J * \varphi) \tag{6.5}$$

is differentiable with derivative $g' : E_{\varphi,\rho}(I) \to \mathcal{B}(E_{\varphi,\rho}(I), F_\rho(I))$, given by[7]

$$g'(\varphi) h = l'(\varphi)(n_i \partial_i J * \varphi) h + l(\varphi)(\partial_i J * h) n_i. \tag{6.6}$$

Moreover, we will show that g' is continuous, and that the uniform estimate holds

$$||g(\varphi + h) - g(\varphi) - g'(\varphi) h||_{{}_0F_\rho(I)} \leq \varepsilon(\|h\|_{E_{\varphi,\rho}(I)}) \|h\|_{E_{\varphi,\rho}(I)}, \tag{6.7}$$

for $\varphi, h \in E_{\varphi,\rho}(I)$, $h(0,\cdot) = 0$ equipped with the property (3.13). Arguing as in the proof of Lemma 3.2 (see, in particular, (5.38)-(5.42)), we obtain

$$\begin{aligned} &||g(\varphi + h) - g(\varphi) - g'(\varphi) h||_{L_{p,\rho}\left(I; W^{1-1/p}_p(\partial\Omega)\right)} \\ &\leq ||(l(\varphi + h) - l(\varphi) - l'(\varphi) h)(\partial_i J * (\varphi + h)) n_i||_{L_{p,\rho}\left(I; W^{1-1/p}_p(\partial\Omega)\right)} \\ &+ ||(l'(\varphi) h \partial_i J * h n_i)||_{L_{p,\rho}\left(I; W^{1-1/p}_p(\partial\Omega)\right)} \\ &\leq ||(l(\varphi + h) - l(\varphi) - l'(\varphi) h)(\partial_i J * (\varphi + h)) n_i||_{L_{p,\rho}\left(I; W^1_p(\Omega)\right)} \\ &+ ||(l'(\varphi) h \partial_i J * h n_i)||_{L_{p,\rho}\left(I; W^1_p(\Omega)\right)} \\ &\leq \varepsilon(\|h\|_{E_{\varphi,\rho}(I)}) \|h\|_{E_{\varphi,\rho}(I)}, \end{aligned} \tag{6.8}$$

since the spatial trace $\mathrm{tr}_\Omega : W^1_p(\Omega) \to W^{1-1/p}_p(\partial\Omega)$ is continuous. Observe that this estimate is always uniform in $T_0 \leq T$ and $R > 0$ if (3.13) holds. In order to bound the same quantity in $W^{1/2-1/2p}_{p,\rho}(I; L^p(\partial\Omega))$, we use its intrinsic norm given as

$$\|\varphi\|^p_{W^{1/2-1/2p}_{p,\rho}(I; L^p(\partial\Omega))} = [\varphi]^p_{W^{1/2-1/2p}_{p,\rho}(I; L^p(\partial\Omega))} + \|\varphi\|^p_{L_{p,\rho}(I; L^p(\partial\Omega))} \tag{6.9}$$

[7] In the sequel, we drop the spatial trace tr_Ω to simplify readability.

with

$$[\varphi]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} := \int_0^T \int_0^s \frac{t^{p(1-\rho)}}{(s-t)^{1+(1/2-1/2p)p}} \left\|\varphi(t) - \varphi(s)\right\|^p_{L^p(\partial\Omega)} dtds, \tag{6.10}$$

to estimate the following expression

$$\Sigma(t,x) := g(\varphi(t,x) + h(t,x)) - g(\varphi(t,x)) - g'(\varphi(t,x))\, h(t,x).$$

For the nonlinear operators given by (6.5) and (6.6), by a simple computation we can further split the difference $\Sigma(t,x) - \Sigma(s,x)$ into three summands $\Sigma_1 + \Sigma_2 + \Sigma_3$, where

$$\begin{aligned}
\Sigma_1 :=& [(l(\varphi+h) - l(\varphi) - l'(\varphi)h)(t) - (l(\varphi+h) - l(\varphi) - l'(\varphi)h)(s)]\,\partial_i J * (\varphi+h)(t)\, n_i\,,\\
\Sigma_2 :=& (l(\varphi+h) - l(\varphi) - l'(\varphi)h)(s)\,[\partial_i J * ((\varphi+h)(t) - (\varphi+h)(s))\, n_i]\ ,\\
\Sigma_3 :=& [l'(\varphi)\, h(\partial_i J * h)(t) - l'(\varphi)\, h(\partial_i J * h)(s)] n_i\,.
\end{aligned}$$

Using [19, Lemma 4.2.1], the embeddings (3.5), (3.6) and assumption (H.6) together with (6.4), we find

$$\begin{aligned}
&[\Sigma]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} \qquad\qquad (6.11)\\
&\lesssim \left(\|h\|^p_{E_{\varphi,\rho}(I)} + \|\varphi\|^p_{E_{\varphi,\rho}(I)}\right)[l(\varphi+h) - l(\varphi) - l'(\varphi)h]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))}\\
&+ \|l(\varphi+h) - l(\varphi) - l'(\varphi)h\|^p_{C(\bar{I}\times\partial\Omega)}\left(\|\varphi\|^p_{E_{\varphi,\rho}(I)} + \|h\|^p_{E_{\varphi,\rho}(I)}\right)\\
&+\|h\|^p_{C(\bar{I}\times\partial\Omega)}\|h\|^p_{C(\bar{I}\times\Omega)}[l'(\varphi)]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} + \|l'(\varphi)\|^p_{C(\bar{I}\times\partial\Omega)}[h(\partial_i J * h)n_i]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))}\\
&\lesssim \varepsilon(\|h\|_{C(\bar{I}\times\partial\Omega)})\left([h]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} + \|h\|^p_{C(\bar{I}\times\partial\Omega)}[\varphi]^p_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))}\right)\\
&\times\left(\|h\|^p_{E_{\varphi,\rho}(I)} + \|\varphi\|^p_{E_{\varphi,\rho}(I)}\right)\\
&+ \varepsilon(\|h\|_{C(\bar{I}\times\partial\Omega)})\,\|h\|^p_{C(\bar{I}\times\partial\Omega)}\left(\|\varphi\|^p_{E_{\varphi,\rho}(I)} + \|h\|^p_{E_{\varphi,\rho}(I)}\right)\\
&+\|l'(\varphi)\|^p_{C(\bar{I}\times\partial\Omega)}\left(\|h\|^{2p}_{C(\bar{I}\times\Omega)} + \|h\|^p_{F_\rho(I)}\|\partial_i J * hn_i\|^p_{L^\infty(I\times\partial\Omega)} + \|\partial_i J * hn_i\|^p_{F_\rho(I)}\|h\|^p_{L^\infty(I\times\partial\Omega)}\right)\\
&\lesssim \varepsilon(\|h\|_{E_{\varphi,\rho}(I)})\,\|h\|^p_{E_{\varphi,\rho}(I)} + \|h\|^{2p}_{F_\rho(I)} + \|h\|^p_{F_\rho(I)}\|h\|^p_{E_{\varphi,\rho}(I)} \lesssim \varepsilon(\|h\|_{E_{\varphi,\rho}(I)})\,\|h\|^p_{E_{\varphi,\rho}(I)}\ ,
\end{aligned}$$

where the first summand was estimated again using [19, Lemma 4.2.3, part (b)] and, for the last term, we used (6.4) and

$$\|\varphi\psi\|_{F_\rho(I)} \lesssim \|\varphi\|_{F_\rho(I)}\,\|\psi\|_{L^\infty(I\times\partial\Omega)} + \|\psi\|_{F_\rho(I)}\,\|\varphi\|_{L^\infty(I\times\partial\Omega)}\,. \tag{6.12}$$

Estimate (6.11) is also valid if we replace I by $\mathbb{R}_+$. Finally, this estimate together with (6.8) implies that g is differentiable at each $\varphi \in E_{\varphi,\rho}(I)$. However, the intrinsic norm (6.10) does not yield a uniform estimate in $T > 0$ if (6.11) involves the ${}_0W^{1/2-1/2p}_{p,\rho}(I; L^p(\partial\Omega))$-norm, whenever (3.13) is assumed. More precisely, the embedding constant $C > 0$ in (6.11) may depend on $T > 0$ and typically becomes large as T becomes small, see [19, Remark 1.1.15]. In order to overcome this obstacle, when working over small intervals $I = (0,T)$ one has to equip the space ${}_0W^{1/2-1/2p}_{p,\rho}(I; L^p(\partial\Omega))$ with an interpolation norm via suitable extension and restriction operators. Arguing then as in the proof of [19, Lemma 4.2.3, Step (IV)], by virtue of (6.11) and (6.4) we can then show the estimate

$$||g(\varphi+h)-g(\varphi)-g'(\varphi)h||_{{}_0W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} \tag{6.13}$$
$$\leq \varepsilon(\|h\|_{E_{\varphi,\rho}(I)})\,\|h\|_{E_{\varphi,\rho}(I)}\left(R+\|\varphi(0)\|_{W_p^{2(\rho-1/p)}(\Omega)}\right),$$

where now the function ε is uniform in $T_0 \leq T$ and $R>0$, for as long as $\varphi, h \in E_{\varphi,\rho}(I)$ are as in (3.13).

It remains to show that g' is continuous. We apply the statement of [19, Lemma 1.3.23] repeatedly, due to the embedding (3.6), by also employing (6.12). To this end, we take $\varphi, \psi, h \in E_{\varphi,\rho}(I)$ with $\|h\|_{E_{\varphi,\rho}(I)} \leq 1$, and estimate according to (6.4) using also (3.6). This gives

$$\begin{aligned} ||(l(\varphi)-l(\psi))\partial_i J * hn_i||_{F_\rho(I)} &\lesssim \|l(\varphi)-l(\psi)\|_{L^\infty(I\times\partial\Omega)}\,\|\partial_i J * hn_i\|_{F_\rho(I)} \\ &\quad+\|l(\varphi)-l(\psi)\|_{F_\rho(I)}\,\|\partial_i J * hn_i\|_{L^\infty(I\times\partial\Omega)} \\ &\lesssim \|l(\varphi)-l(\psi)\|_{L^\infty(I\times\partial\Omega)}+\|l(\varphi)-l(\psi)\|_{F_\rho(I)} \\ &\lesssim \|l(\varphi)-l(\psi)\|_{L^\infty(I\times\partial\Omega)}+\|l(\varphi)-l(\psi)\|_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} \\ &\quad+\|l(\varphi)-l(\psi)\|_{L_{p,\rho}\left(I;W^1_p(\Omega)\right)}, \end{aligned} \tag{6.14}$$

owing to the regularity of J and the fact that the spatial trace $\mathrm{tr}_\Omega : W^1_p(\Omega) \to W^{1-1/p}_p(\partial\Omega)$ is continuous. We observe that the first summand in the last inequality of (6.14) goes to zero as $\varphi \to \psi$ in $E_{\varphi,\rho}(I)$, and so does the last summand due to the continuity of $l \in C^2([-1,1])$. For the second summand, we use [19, Lemma 4.2.1, part (c)] to estimate

$$\begin{aligned} [l(\varphi)-l(\psi)]_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} &\lesssim \varepsilon(\|\varphi-\psi\|_{C(\bar I\times\partial\Omega)})\,[\psi]_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))} \\ &\quad+\|\varphi-\psi\|_{W^{1/2-1/2p}_{p,\rho}(I;L^p(\partial\Omega))}, \end{aligned}$$

and notice once again that the right-hand side also goes to zero as φ (strongly) converges to in $E_{\varphi,\rho}(I)$. Collecting these estimates, we obtain

$$||(l(\varphi)-l(\psi)\,\partial_i J * hn_i)||_{F_\rho(I)} \to 0 \text{ as } \varphi \to \psi \text{ in } E_{\varphi,\rho}(I). \tag{6.15}$$

Next, using (6.4), (6.12) and arguing as in (6.14) we want to bound the following quantity (recall that $\|h\|_{E_{\varphi,\rho}(I)} \leq 1$):

$$\begin{aligned} &||(l'(\varphi)\,(\partial_i J * \varphi n_i)-l'(\psi)\,(\partial_i J * \psi n_i))h||_{F_\rho(I)} \\ &\leq ||(l'(\varphi)-l'(\psi))\,(\partial_i J * \psi n_i)\,h||_{F_\rho(I)}+||(l'(\psi)\,\partial_i J * (\varphi-\psi)\,n_i)h||_{F_\rho(I)} \\ &\lesssim ||(l'(\varphi)-l'(\psi))\,(\partial_i J * \psi n_i)\,||_{L^\infty(I\times\partial\Omega)}\,\|h\|_{F_\rho(I)} \\ &+||(l'(\varphi)-l'(\psi))\,(\partial_i J * \psi n_i)\,||_{F_\rho(I)}\,\|h\|_{L^\infty(I\times\partial\Omega)} \\ &+C(R)\,||\partial_i J * (\varphi-\psi)\,n_i||_{F_\rho(I)}+\|l'(\psi)h\|_{F_\rho(I)}\|\partial_i J * (\varphi-\psi)n_i\|_{L^\infty(I\times\partial\Omega)} \\ &\lesssim C(J,R)\,||l'(\varphi)-l'(\psi)||_{L^\infty(I\times\partial\Omega)}+||l'(\varphi)-l'(\psi)||_{L^\infty(I\times\partial\Omega)}||\partial_i J * \psi n_i||_{F_\rho(I)} \\ &+||l'(\varphi)-l'(\psi)||_{F_\rho(I)}||\partial_i J * \psi n_i||_{L^\infty(I\times\partial\Omega)}+C(R,J)\,||\varphi-\psi||_{E_{\varphi,\rho}(I)} \end{aligned} \tag{6.16}$$

$$+\left(\|l'(\psi)\|_{L^\infty(I\times\partial\Omega)}\|h\|_{F_\rho(I)}+\|l'(\psi)\|_{F_\rho(I)}\|h\|_{L^\infty(I\times\partial\Omega)}\right)\|\partial_i J*(\varphi-\psi)n_i\|_{L^\infty(I\times\partial\Omega)} \tag{6.17}$$
$$\lesssim C\,(J,R)\left(||l'(\varphi)-l'(\psi)||_{L^\infty(I\times\partial\Omega)}+||(l'(\varphi)-l'(\psi))||_{F_\rho(I)}+||\varphi-\psi||_{E_{\varphi,\rho}(I)}\right).$$

Notice that first and last summands in the last inequality of (6.16) go to zero as $\varphi\to\psi$ in $E_{\varphi,\rho}(I)$. For the second summand, we argue exactly as in the uniform bound for (6.14) so that this also goes to zero owing to $l'\in C^1([-1,1])$. Thus, we have also shown that

$$||(l'(\varphi)\,(\partial_i J*\varphi n_i)-l'(\psi)\,(\partial_i J*\psi n_i))h||_{F_\rho(I)}\to 0 \text{ as } \varphi\to\psi \text{ in } E_{\varphi,\rho}\,(I)\ , \tag{6.18}$$

uniformly for $\|h\|_{E_{\varphi,\rho}(I)}\leq 1$. This estimate together with (6.15) implies that, as $\varphi\to\psi$ in $E_{\varphi,\rho}\,(I)$, we have

$$\|g'(\varphi)-g'(\psi)\|_{\mathcal{B}(E_{\varphi,\rho}(I),F_\rho(I))}=\sup_{h\in E_{\varphi,\rho}(I)\,,\|h\|_{E_{\varphi,\rho}(I)}\leq 1}||(g'\,(\varphi)-g\,(\psi))h||_{F_\rho(I)}\to 0\,,$$

which entails the desired continuity of g'. This concludes the proof of lemma.

7. Proof of Lemma 4.1

Observe that, for almost any $t\in(0,T)$, we have

$$\begin{cases} -\gamma(\varphi)\Delta\varphi=\gamma'(\varphi)\nabla\,(\varphi)\cdot\nabla\varphi-\operatorname{div}\,(m(\varphi)\nabla J*\varphi)-\partial_t\varphi, \text{ a.e. in } \Omega\times(0,T) \\ \partial_n\varphi=b\,(\varphi)\,(\nabla J\cdot n*\varphi)\,, \text{ a.e. on } \partial\Omega. \end{cases} \tag{7.19}$$

The idea is to apply an elliptic regularity result to the equation $-\Delta\varphi=f$ in Ω with $\partial_n\varphi=g$ on $\partial\Omega$, where

$$f:=\gamma'(\varphi)\gamma^{-1}(\varphi)\nabla\,(\varphi)\cdot\nabla\varphi-\gamma^{-1}(\varphi)\operatorname{div}\,(m(\varphi)\nabla J*\varphi)-\gamma^{-1}(\varphi)\partial_t\varphi$$

and

$$g:=b\,(\varphi)\,(\nabla J\cdot n*\varphi)\,.$$

Observe that, by (H.1) and (2.11), we have

$$\begin{aligned}\|f\|_H &\leq C\left(||\gamma'||_{C([-1,1])},\theta\right)\|\nabla\varphi\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^4(\Omega)} \\ &\quad+C\,(\theta)\,\|\operatorname{div}\,(m(\varphi)\nabla J*\varphi)\,\|_{L^2(\Omega)}+C\,(\theta)\,\|\partial_t\varphi\|_{L^2(\Omega)} \\ &\leq C\left(||\gamma'||_{C[-1,1]},\theta\right)\|\nabla\varphi\|^2_{L^4(\Omega)}+C\,(\theta)\,\|\partial_t\varphi\|_{L^2(\Omega)} \\ &\quad+C\left(||\gamma'||_{C[-1,1]},\|\nabla J\|_{L^1}\right)\|\varphi\|^2_{W^1_4(\Omega)}+C\,(m_0)\,\|\nabla\,(\nabla J*\varphi)\,\|_{L^2(\Omega)} \\ &\leq C\left(\gamma,m_0,\theta,\|J\|_{W^1_1}\right)\|\varphi\|^2_{W^1_4(\Omega)}+C_J\|\varphi\|_{L^2(\Omega)}+C\,(\theta)\,\|\partial_t\varphi\|_{L^2(\Omega)}.\end{aligned} \tag{7.20}$$

In order to estimate the $W^1_4\,(\Omega)$-norm on the right-hand side of (7.20) we exploit a proper form of the Sobolev interpolation inequality (see [19, Proposition A.6.2]) with the choices

$$s=1,\ p=4,\ s_1=2,\ p_1=2,\ s_0=\beta-\delta,\ p_0=r,$$

where β is the Hölder exponent of Definition 2.6 and $\delta>0$, $1<r<\infty$. We find

$$\|\varphi\|_{W^1_4(\Omega)}\leq C\|\varphi\|^\theta_{H^2(\Omega)}\|\varphi\|^{1-\theta}_{W^{s_0}_{p_0}(\Omega)}, \tag{7.21}$$

if $0<\theta<1$ satisfies the following inequalities

$$\begin{cases} \frac{1}{4} \le \frac{\theta}{2} + \frac{1-\theta}{r}, \\ 1-\frac{3}{4} < \theta(2-\frac{3}{2}) + (1-\theta)(\beta-\delta-\frac{3}{r}). \end{cases} \tag{7.22}$$

Our claim is that there exist δ, r and $\theta<0.5$ which satisfy these conditions. Indeed, the first one is equivalent to $\frac{1}{2}\frac{r-4}{r-2} \le \theta$, so then there exists $\theta<0.5$ which satisfies the first condition of (7.22) for any $r>4$. On the other hand, choosing δ and r such that $\beta-\delta-\frac{3}{r}=\beta/2$, we get in the second condition that

$$\frac{1}{4}-\frac{\beta}{2} < \theta\left(\frac{1}{2}-\frac{\beta}{2}\right) \Leftrightarrow \frac{1-2\beta}{4} < \theta\left(\frac{1-\beta}{2}\right) \Leftrightarrow \frac{1-2\beta}{2(1-\beta)} < \theta.$$

We note that the first term on left-hand side is less than 0.5 for any $\beta\in(0,1)$, so we conclude that there exists $\theta<0.5$ which satisfies (7.22). Then, exploiting the Gagliardo-Nirenberg inequality (7.21) together with the continuous embedding $C^\beta(\overline{\Omega}) \hookrightarrow W^{\alpha,r}(\Omega)$, for any $\alpha\in(0,\beta)$ and $r\in(1,\infty)$, we have

$$\|\varphi\|_{W^1_4(\Omega)} \le C\|\varphi\|^{\theta}_{H^2(\Omega)}\|\varphi\|^{1-\theta}_{C^\beta(\overline{\Omega})}, \qquad \text{with } \theta<1/2. \tag{7.23}$$

Thus, using (7.23), from (7.20) we infer

$$\|f\|_H \le C\left(\gamma,m_0,\theta,\Omega,J\right)\left(\|\varphi\|^{2\theta}_{H^2(\Omega)}\|\varphi\|^{2(1-\theta)}_{C^\beta(\overline{\Omega})} + \|\varphi\|_{L^2(\Omega)} + \|\partial_t\varphi\|_{L^2(\Omega)}\right),$$

for some $\theta\in(0,1/2)$. This yields for any $\delta>0$ that

$$\|f\|_H \le \delta\|\varphi\|_{H^2(\Omega)} + C_\delta\left(\|\varphi\|^{2(1-\theta)/(1-2\theta)}_{C^\beta(\overline{\Omega})} + \|\varphi\|_V + \|\partial_t\varphi\|_{L^2(\Omega)}\right). \tag{7.24}$$

Concerning the boundary term g, using the classical trace theorem, recalling (2.11) and observing that $||b'||_{C([-1,1])}<\infty$, we similarly deduce

$$\begin{aligned} \|g\|_{H^{1/2}(\partial\Omega)} &\le C\|b\left(\varphi\right)\left(\nabla J\cdot n * \varphi\right)\|_V \\ &\le C\left(\|n\|_{C^1}\right)\max_{i=1,..,d}\left(\|b\left(\varphi\right)\partial_i J * \varphi\|_H + \|\nabla\left(\varphi\right)\partial_i J * \varphi)\|_H\right) \\ &\le C\left(\alpha,\|n\|_{C^1},J,||b'||_{C[-1,1]}\right)\left(\|\varphi\|_V + \|\varphi\|^2_{W^1_4(\Omega)}\right). \end{aligned}$$

Arguing as above in (7.20)-(7.24) we find

$$\|g\|_{H^{1/2}(\partial\Omega)} \le \delta\|\varphi\|_{H^2(\Omega)} + C_\delta\left(\|\varphi\|^{2(1-\theta)/(1-2\theta)}_{C^\beta(\overline{\Omega})} + \|\varphi\|_V\right). \tag{7.25}$$

Finally, recalling the well-known elliptic estimate

$$\|\varphi\|_{H^2(\Omega)} \le C\left(\Omega\right)\left(\|\varphi\|_V + \|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial\Omega)}\right), \tag{7.26}$$

combining together (7.24) and (7.25) into (7.26) and choosing a sufficiently small $\delta<1/2$, we deduce from (7.26) the desired (4.30).

8. Appendix: on the separation property

Here we present a proof of the separation property for the nonlocal Cahn-Hilliard equation with degenerate mobility (see [15, 16]) for the original proofs). More precisely, we establish the validity of the following

Theorem 8.1. *Let $\kappa \in (0,1)$. Assume that the assumptions of Theorem 2.3 are satisfied, together with* (2.5)*. In addition, let the mobility m be such that*

$$m(s) \leq M(1-s^2)\,, \qquad \forall s \in [-1,1]\,, \tag{8.1}$$

for some $M>0$. Then, there exists $\delta = \delta(\kappa) > 0$ such that, for any measurable initial data φ_0 with $|\overline{\varphi}_0| \leq 1-\kappa$, it holds

$$\|\varphi(t)\|_{L^\infty(\Omega)} \leq 1-\delta, \quad \forall t \geq 2. \tag{8.2}$$

The proof of Theorem 8.1 will be carried out by means of two lemmas which are slight generalization of the ones contained in [15, Sec.3]. In particular, here we assume that $mF'' \geq \theta > 0$ on $[-1,1]$, while in [15] they assume $mF'' = a$, for some positive constant a albeit the authors claim that this is just a simplifying assumption.

The first lemma is an L^1-bound of a suitable entropy function, while the second one will enable us to use a Moser-Alikakos argument to get an L^∞-bound and conclude the proof.

Lemma 8.2. *Under the assumptions of Theorem 8.1, there exists $C(\kappa)>0$ such that*

$$\|\eta(\varphi(t))\|_{L^1(\Omega)} + \int_t^{t+1} \|\nabla\varphi\,(s)\|^2_{L^2(\Omega)}\,ds \leq C(\kappa), \quad \forall t \geq 1.$$

Proof of Lemma 8.2. Our aim is to obtain a suitable differential inequality for the L^1-norm of $\log(1-\varphi)$ and $\log(1+\varphi)$. For simplicity, we consider just one of these two function, say $\log(1+\varphi)$, since the argument for the other one is similar. By employing equation (2.2) we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Big|\log\Big(\frac{1+\varphi}{2}\Big)\Big|\,dx &= \frac{d}{dt}\int_\Omega -\log(1+\varphi)\,dx \\
&= \int_\Omega \frac{-1}{1+\varphi}\,\varphi_t\,dx \\
&= \int_\Omega \frac{-\nabla\varphi}{(1+\varphi)^2}\cdot\big[m(\varphi)F''(\varphi)\nabla\varphi - m(\varphi)(\nabla J * \varphi)\big]\,dx \\
&= -\int_\Omega \Big|\nabla\log\Big(\frac{1+\varphi}{2}\Big)\Big|^2 m(\varphi)F''(\varphi)\,dx \\
&\quad + \int_\Omega \frac{m(\varphi)}{1+\varphi}\nabla\log\Big(\frac{1+\varphi}{2}\Big)\cdot(\nabla J * \varphi)\,dx\,.
\end{aligned} \tag{8.3}$$

Therefore, by using assumptions (2.4) and (8.1) we get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Big|\log\Big(\frac{1+\varphi}{2}\Big)\Big|\,dx + \theta\Big\|\nabla\log\Big(\frac{1+\varphi}{2}\Big)\Big\|^2 &\leq 2M\int_\Omega \Big|\nabla\log\Big(\frac{1+\varphi}{2}\Big)\Big||\nabla J * \varphi|\,dx \\
&\leq \frac{\theta}{2}\Big\|\nabla\log\Big(\frac{1+\varphi}{2}\Big)\Big\|^2 + N\,,
\end{aligned} \tag{8.4}$$

where the constant N is given by $N = 8M^2 J_0^2 |\Omega| \theta^{-1}$, with $J_0 := \sup_{x\in\Omega} \int_\Omega |\nabla J(x-y)| dy$. Hence, we get

$$\frac{d}{dt}\int_\Omega |\eta(\varphi)|\,dx + \frac{\theta}{2}\|\nabla\eta(\varphi)\|^2 \le N\,, \tag{8.5}$$

where we have set

$$\eta(s) := \log\Big(\frac{1+s}{2}\Big)\,. \tag{8.6}$$

The second term on the right hand side of (8.5) is now estimated according to the following generalized Poincaré inequality

$$\Big\|\eta(\varphi) - \frac{1}{|\Omega_1|}\int_{\Omega_1}\eta(\varphi)\,dx\Big\| \le \frac{C_P}{|\Omega_1|}\|\nabla\eta(\varphi)\|\,, \tag{8.7}$$

where $C_P = C_P(\Omega)$ and $\Omega_1 \subset \Omega$ is any subset of Ω such that $|\Omega_1| > 0$. We now choose $\Omega_1 = \Omega_{1,t}$ given by

$$\Omega_{1,t} := \Big\{x\in\Omega\,:\,\varphi(x,t) \ge -\frac{1-\overline{\varphi}}{2}\Big\}\,. \tag{8.8}$$

Then, we can see that

$$|\Omega_{1,t}| \ge \frac{1+\overline{\varphi}}{4}|\Omega|\,. \tag{8.9}$$

Indeed, if this were not true, we would have

$$\begin{aligned}\overline{\frac{1+\varphi}{2}} &= \frac{1}{|\Omega|}\int_{\Omega_{1,t}}\frac{1+\varphi}{2}\,dx + \frac{1}{|\Omega|}\int_{\Omega-\Omega_{1,t}}\frac{1+\varphi}{2}\,dx\\ &\le \frac{|\Omega_{1,t}|}{|\Omega|} + \frac{|\Omega-\Omega_{1,t}|}{|\Omega|}\,\frac{1+\overline{\varphi}}{4} < \frac{1+\overline{\varphi}}{2}\end{aligned} \tag{8.10}$$

which is a contradiction.

By employing (8.7), we have

$$\begin{aligned}\|\eta(\varphi)\|^2_{L^1(\Omega)} &\le |\Omega|\|\eta(\varphi)\|^2 \le 2\,\frac{|\Omega|\,C_P^2}{|\Omega_{1,t}|^2}\,\|\nabla\eta(\varphi)\|^2 + 2\,\frac{|\Omega|^2}{|\Omega_{1,t}|^2}\Big(\int_{\Omega_{1,t}}|\eta(\varphi)|\,dx\Big)^2\\ &\le \frac{32\,C_P^2}{|\Omega|}\,\frac{1}{(1+\overline{\varphi})^2}\|\nabla\eta(\varphi)\|^2 + 2\,|\Omega|^2\log^2\Big(\frac{1+\overline{\varphi}}{4}\Big)\end{aligned} \tag{8.11}$$

Inserting this estimate in (8.5), we therefore obtain the following differential inequality

$$\frac{d}{dt}\int_\Omega|\eta(\varphi)|\,dx + \kappa(\overline{\varphi}_0)\Big(\int_\Omega|\eta(\varphi)|\,dx\Big)^2 \le K\,, \tag{8.12}$$

where the positive constants κ, K are given by

$$\kappa(\overline{\varphi}_0) := \frac{\theta|\Omega|}{64\,C_P^2}(1+\overline{\varphi}_0)^2\,,\quad K := N + \frac{\theta\,|\Omega|^3}{32\,C_P^2}(1+\overline{\varphi}_0)^2\,\log^2\Big(\frac{1+\overline{\varphi}_0}{4}\Big)\,. \tag{8.13}$$

Notice that, on the right hand side of (8.12), K can be replaced by a constant (that we can still denote by K) which does not depend on $\overline{\varphi}_0$.

Due to the lack of regularity of the initial condition ($\eta(\varphi_0) \notin L^1(\Omega)$), we consider an approximation sequence of initial data $\varphi_{0,n}$ fulfilling

$$\eta(\varphi_{0,n}) \in L^1(\Omega) \quad \text{and} \quad \varphi_{0,n} \to \varphi_0 \text{ in } \ L^2(\Omega).$$

We observe that

$$|\overline{\varphi}_{0,n}| \leq \|\varphi_{0,n} - \varphi_0\|_{L^1(\Omega)} + |\overline{\varphi}_0| \leq 1 - \frac{\kappa}{2},$$

for n large enough. Since any weak solution related to $\varphi_{0,n}$ satisfies the differential inequality (8.12), we can control $\eta(\varphi_n(t))$ by means of the solution $\Lambda(t)$ of the differential equation

$$\dot{\Lambda}(t) + \omega_1 \Lambda^2(t) = K, \quad \Lambda(0) = \|\eta(\varphi_{0,n})\|_{L^1(\Omega)}, \tag{8.14}$$

where ω_1 and K are exactly the constants of (8.12), and we obtain

$$\|\eta(\varphi_n(t))\|_{L^1(\Omega)} \leq \hat{K}, \quad \forall t \geq 1,$$

where $\hat{K}$ is a positive constant depending only on ω_1 and K. Therefore, on account of the continuous dependence with respect to the initial conditions, for all $t \geq 1$, $\varphi_n(t) \to \varphi(t)$ in $L^2(\Omega)$ and so $\varphi_n(t) \to \varphi(t)$ for almost every $x \in \Omega$, up to a subsequence. In particular, this also implies that $\eta(\varphi_n(t)) \to \eta(\varphi(t))$ for almost every $x \in \Omega$ and, by the Fatou's Lemma, we finally deduce

$$\int_\Omega |\eta(\varphi(t))| dx \leq \liminf_{n\to\infty} \int_\Omega |\eta(\varphi_n(t))| dx \leq K.$$

Finally integrating (8.5) once more over $(t, t+1)$ with $t \geq 1$, the claim in the statement of Lemma 8.2 follows immediately. □

The second lemma contains the proof of Theorem 8.1. Indeed we have

Lemma 8.3. *Let the assumptions of Theorem 8.1 hold. Then there exists $C(\kappa)$ such that*

$$\|\eta(\varphi(t))\|_{L^\infty(\Omega)} \leq C(\kappa), \quad \forall t \geq 2. \tag{8.15}$$

Proof. We argue as in Lemma 8.2 to prove a differential inequality in the L^p-norm. We only consider the function η defined as in (8.6), since the argument for η defined as $\log((1-\varphi)/2)$ is similar. Performing a differentiation with respect to time and using the equation (2.2), we have

$$\begin{aligned} \frac{1}{p+1}\frac{d}{dt}\int_\Omega |\eta(\varphi)|^{p+1} dx &+ \int_\Omega m(\varphi) F''(\varphi) \nabla\varphi \cdot \nabla \left(|\eta(\varphi)|^{p-1}\eta(\varphi)\frac{1}{1+\varphi}\right) dx \\ &= \int_\Omega m(\varphi)\left(\nabla J * \varphi\right) \cdot \nabla \left(|\eta(\varphi)|^{p-1}\eta(\varphi)\frac{1}{1+\varphi}\right) dx\,. \end{aligned} \tag{8.16}$$

Exploiting the following relation

$$\begin{aligned} \nabla\left(|\eta(\varphi)|^{p-1}\eta(\varphi)\frac{1}{1+\varphi}\right) &= p|\eta(\varphi)|^{p-1}\frac{\nabla\varphi}{(1+\varphi)^2} + |\eta(\varphi)|^{p-1}\eta(\varphi)\frac{-\nabla\varphi}{(1+\varphi)^2} \\ &= p|\eta(\varphi)|^{p-1}\frac{\nabla\varphi}{(1+\varphi)^2} + |\eta(\varphi)|^{p}\frac{\nabla\varphi}{(1+\varphi)^2}\,, \end{aligned} \tag{8.17}$$

we can control from below the second term on the left-hand side of (8.16) as

$$\int_\Omega m(\varphi)F''(\varphi)\nabla\varphi\cdot\nabla\left(|\eta(\varphi)|^{p-1}\eta(\varphi)\frac{1}{1+\varphi}\right)dx \ge p\theta\int_\Omega |\eta(\varphi)|^{p-1}\frac{|\nabla\varphi|^2}{(1+\varphi)^2}\,dx\,.$$

Concerning the right-hand side of (8.16), we split it into two terms accordingly to (8.17) and we control them as follows:

$$\begin{aligned}|\mathcal{J}_1| &\le p\int_\Omega m(\varphi)|\nabla J*\varphi||\eta(\varphi)|^{p-1}\frac{|\nabla\varphi|}{(1+\varphi)^2}\,dx\\ &\le Cp\int_\Omega\left(|\eta(\varphi)|^{\frac{p-1}{2}}\frac{|\nabla\varphi|}{1+\varphi}\right)\left(|\eta(\varphi)|^{\frac{p-1}{2}}\frac{m(\varphi)}{1+\varphi}\right)dx\\ &\le \varepsilon p\int_\Omega|\eta(\varphi)|^{p-1}\frac{|\nabla\varphi|^2}{(1+\varphi)^2}\,dx+\frac{Cp}{4\varepsilon}\int_\Omega|\eta(\varphi)|^{p-1}dx\,,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{J}_2| &\le \int_\Omega m(\varphi)|\nabla J*\varphi||\eta(\varphi)|^{p}\frac{|\nabla\varphi|}{(1+\varphi)^2}\,dx \le C\int_\Omega|\eta(\varphi)|^{\frac{p-1}{2}}\frac{|\nabla\varphi|}{1+\varphi}|\eta(\varphi)|^{\frac{p+1}{2}}\\ &\le \varepsilon p\int_\Omega|\eta(\varphi)|^{p-1}\frac{|\nabla\varphi|^2}{(1+\varphi)^2}\,dx+\frac{C}{4\varepsilon p}\int_\Omega|\eta(\varphi)|^{p+1}dx\,,\end{aligned}$$

where we have used (8.1) and the fact that $|\varphi|\le 1$. Finally, collecting the above estimates together and recalling that

$$|\eta(\varphi)|^{p-1}\frac{|\nabla\varphi|^2}{(1+\varphi)^2}=\frac{4}{(p+1)^2}|\nabla|\eta(\varphi)|^{\frac{p+1}{2}}|^2, \tag{8.18}$$

we deduce the differential inequality

$$\frac{d}{dt}\int_\Omega|\eta(\varphi)|^{p+1}dx+\frac{2p\theta}{p+1}\int_\Omega|\nabla|\eta|^{\frac{p+1}{2}}|^2\,dx\le C(p+1)^2\left(1+\int_\Omega|\eta(\varphi)|^{p+1}dx\right),$$

Starting from this differential inequality, we exploit an iterative argument (see [11], [10] and references therein) together with Lemma 8.2, to infer that

$$\sup_{t\ge 2}\|\eta(\varphi(t))\|_{L^\infty(\Omega)}\le C\sup_{t\ge 1}\|\eta(\varphi(t))\|_{L^1(\Omega)}\le C(\kappa).$$

This inequality, together with the similar bound involving the function η defined as $\log((1-\varphi)/2)$, entail (8.2).

Remark 8.4. A careful look at the above proofs shows that the separation property holds instantaneously (i.e. for any time $t_0>0$). Therefore, if the initial datum is already separated (i.e. $-1+\kappa\le\varphi_0\le 1-\kappa$ in Ω) then the (strong) solution is separated for any time $t\ge 0$ due to its space-time continuity.

As a consequence of the separation property, the weak solution gets more regular in finite time. For instance, we have the following uniform-in-time bound

Corollary 8.5. *Under the assumptions of Theorem 8.1, there exists $C(\kappa) > 0$ such that*

$$\|\mu(t)\|_{L^\infty(\Omega)} + \int_t^{t+1} \|\nabla\mu(\tau)\|^2_{L^2(\Omega)} d\tau \le C(\kappa), \quad \forall t \ge 2. \tag{8.19}$$

Furthermore, it follows that

$$\sup_{t\ge 3}\left(\|\varphi(t)\|_V + \|\partial_t\varphi\|_{L^2([t,t+1];L^2(\Omega))}\right) \le C(\kappa). \tag{8.20}$$

Proof. The first estimate (8.19) is an immediate consequence of (8.15). For the second estimate (8.20), let us recall again that

$$\frac{1}{2}\|\partial_t\varphi\|^2_{L^2(\Omega)} + \frac{d}{dt}E(\varphi) \le C_*\left(\|\varphi\|^{4(1-\theta)/(1-2\theta)}_{C^\beta(\overline{\Omega})} + E(\varphi)\right),$$

for some constant $C_* > 0$ independent of time, $T > 0$ and the initial datum φ_0. We can then apply the uniform Gronwall lemma taking advantage of the estimate of Lemma 8.2, as well as the fact that $\sup_{t\ge 2}\|\varphi(t)\|_{C^\beta(\overline{\Omega})} \le C$, for some $C > 0$ independent of time and φ_0. This quickly yields (8.20) and finishes the proof.

Remark 8.6. Note that formulation (2.2) (or its strong form (3.1)) also includes pure phases, namely constant solutions $\varphi = \pm 1$. However, if the initial datum is not a pure phase (i.e. its total mass belongs to $(-1, 1)$) then the solution gets separated from pure states in finite time due to Theorem 8.1. Therefore, it is possible to show that this solution becomes a smooth solution in finite time in the sense of Definition 3.1. In particular, further uniform-in-time bounds hold (cf. Remark 3.7).

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