Integral Sliding-Mode Control with Internal Model: A Separation

Gian Paolo Incremona, Leonid Mirkin, and Patrizio Colaneri

***Abstract*— This note proposes a novel architecture of integral sliding mode control, in which a special "ideal control" part is introduced. This part incorporates a fairly general form of internal model to deal with regular (i.e., modeled) persistent disturbances. A key property of this architecture is a *complete separation* of the internal model from the design of the "discontinuous" component of the controller. In particular, the latter component is designed for systems whose dimension coincides with that of the plant, even if the internal model is infinite dimensional.**

***Index Terms*— Sliding mode control, internal-model principle, repetitive control, time-delay systems.**

I. Introduction

High-gain feedback is an efficient mean to attenuate unmeasured load (i.e., matched input) disturbances, even in the presence of classes of modeling uncertainty. An example of a successful use of this idea is the sliding mode control (SMC), whose discontinuous nonlinear control law can be viewed as a high gain followed by a saturation element, see [1]–[3] and the references therein. The price of attaining highly efficient disturbance attenuation is the introduction of chattering effects, as well as sensitivity to measurement imperfections and unmodeled loop lags.

A way to introduce less costly high-gain elements, albeit at the expense of limiting the class of compensated disturbances and resorting to asymptotic requirements, is the use of the internal model principle (IMP) [4]. If disturbances belong to a class of signals with fixed unstable, mostly pure imaginary, modes, then the inclusion of these modes into the feedback loop produces infinite loop gains at required frequencies and perfect asymptotic disturbance rejection. A classical example of this approach is the use of controllers with an integral action [5], which are capable of rejecting all constant disturbances by smooth control signals. Another example is the repetitive control [6], in which infinite-dimensional internal models are employed to reject arbitrary periodic disturbances of a given period.

The idea of combining SMC with internal models to have the best of both worlds is not new, especially regarding the use of integral actions, see [2, Sec. 2.5] for an introduction and [7] for a more recent take on the idea. The concept, loosely speaking, is to employ smooth control to handle a regular, e.g., that generated by a known model under unknown initial conditions, component of the disturbance and complement it by a SMC element to handle its unmodeled part. If irregular components of disturbances are relatively small, then their suppression by SMC has a lower price tag. Technically, this combination involves augmenting the plant by dynamics of the internal model. The resulting dimension increases and the need in additional measurements may be tolerable in the integrator or a harmonic oscillator cases. However, it might not be manageable in the repetitive control case, where the model includes a time delay.

In this note we put forward an alternative approach to amalgamate SMC and the IMP. We propose a nontrivial *twist* on the integral sliding mode (ISM) scheme of [8], where the "ideal control" is a specially designed state-feedback controller with an internal model. Our scheme includes also a compensation element, whose role is to make the internal controller *invisible* for the design of the discontinuous part of the ISM controller. Consequently, the latter needs to be designed for the very same "ideal system" as in the case with no internal models. The only difference is that now the discontinuous design should cope only with a deviation of the disturbance signal from its nominal component used in the internal model. As such, the proposed architecture is readily applicable to high-dimensional internal models and even to infinite-dimensional models, like those in repetitive control. To the best of our knowledge, this is the first continuous-time SMC architecture enabling that.

The paper is organized as follows. Section II reviews the conventional ISM control configuration and explains its limitations in the studied context. Section III introduces the proposed control architecture, its justification and main properties, in particular, separation. In Section IV a numerical example illustrating advantages of the proposed approach is presented. The paper ends with concluding remarks in Section V.

Notation: The transpose of a matrix M is denoted by M'. The sets of non-negative reals is notated as $\mathbb{R}_+$. Signals in the time and Laplace domains are denoted by lower- and upper-case letters, respectively, like $x(t)$, or just x, and $X(s)$. Linear systems in the time domain are denoted by capital letters with no argument, like G, and $G(s)$ stands for the corresponding transfer function. The L_∞ norm of a signal $f : \mathbb{R}_+ \to \mathbb{R}$ is $\|f\|_\infty := \sup_{t\in\mathbb{R}_+}|f(t)|$.

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II. Preliminary: Integral Sliding Mode Control

We start with reviewing the integral sliding mode (ISM) control approach, originated in [8]. To streamline the exposition, we consider a stripped-down disturbance attenuation problem for a linear plant with matched load disturbances and fully measured state. Extensions to some more complex settings should be standard, see [2, Sec. 2.7] or [3, Sec. 1.6].

So consider an LTI system described by the state equation

$$\dot{x}(t) = Ax(t) + Bd(t) + Bu(t), \tag{1}$$

where $u(t) \in \mathbb{R}$ is the control input, $d(t) \in \mathbb{R}$ is the load disturbance, and $x(t) \in \mathbb{R}^n$ is the (measurable) state. The transfer function of the plant, which is the $u \mapsto x$ part of (1), is $P(s) = (sI - A)^{-1}B$. The control problem is to attenuate, or even reject, the effect of d on x.

The idea of the ISM control is to amalgamate a sliding-mode control (SMC) loop with a smooth control law, representing an "ideal" nominal behavior of the controlled system. In this way, the discontinuous SMC loop is designed for a specially selected sliding surface, which follows the nominal evolution of the plant. Moreover, it guarantees that the sliding mode is enabled from the initial time instant, thus removing the so-called reaching phase.

Applying to (1), the approach may be presented as follows. Choose the control signal as composed of two components,

$$u(t) = u_0(t) + u_1(t), \tag{2}$$

where u_0 is the "ideal control" to be designed for the nominal dynamics, i.e., (1) without d, and u_1 is a (discontinuous) SMC add-on, aiming at handling deviations from the nominal case. Consider the choice

$$u_0(t) = Kx(t), \tag{3}$$

for a gain $K \in \mathbb{R}^{1\times n}$ such that $A + BK$ is Hurwitz and the nominal dynamics $\dot{x}(t) = (A + BK)x(t)$ are satisfactory, in whatever appropriate sense.

The component u_1 is then chosen according to the standard SMC law

$$u_1(t) = -\rho(t)\operatorname{sign}(\sigma(x(t))), \tag{4}$$

for some gain ρ (constant / varying / adaptive) and sliding variable σ. The trick now is to select the sliding variable, for which the sliding mode $\sigma \equiv 0$ achieved by (4) attempts to keep the closed-loop system evolving as the nominal one. This property can be attained by the choice

$$\sigma(x(t)) = Sx(t) - z(t), \tag{5a}$$

where the *transient function* $z(t)$ satisfies

$$\dot{z}(t) = S(A + BK)x(t), \quad z(0) = Sx(0) \tag{5b}$$

and $S \in \mathbb{R}^{1\times n}$ is a design variable. With this choice,

$$\dot{\sigma}(x(t)) = SB\big(d(t) + u_1(t)\big), \quad \sigma(0) = 0. \tag{6}$$

If S is chosen so that $SB \neq 0$ and $S(sI - A - BK)^{-1}B$ is minimum phase, which is always possible, then $\sigma \equiv 0$ yields $\dot{\sigma} \equiv 0$ and leads to the so-called *equivalent control* corresponding to u_1 given by $\tilde{u}_1(t) = -d(t)$, [3, Ch. 1]. Consequently, u_1 cancels the load disturbance in (1) and causes the closed-loop dynamics to evolve according to the ideal response shaped by $A + BK$.

Conceptually, the ISM control approach does not limit the choice of the ideal control component by (3). A more sophisticated nominal design can be incorporated. For example, if d contains some "regular" component, like a constant or a periodic signal, it would make sense to include a model of that component into the nominal design, potentially leading to a smaller gain ρ in (4) and less dominant discontinuous component u_1. The inclusion of internal models is not new to SMC, see [2, Sec. 2.5] for the use of an integral action, or [9] for a more general finite-dimensional disturbance model. The dynamics of the internal model can be routinely combined with those of the plant, bringing us back to (1), but now for a higher-dimensional augmented plant. This increased complexity is an artefact of the approach, affecting the choice of the discontinuous part in (4). Moreover, the approach is not readily extendable to include infinite-dimensional disturbance models, like those used in repetitive control.

III. The Proposed ISM Control Architecture

Motivated by the reasonings at the end of the previous section, we now present an alternative ISM control architecture to include an internal model of d. This architecture keeps the spirit of the ISM approach, but also includes a compensating element in the ideal loop to simplify the design and properties of the discontinuous component (4).

A. Disturbance model

Return to system (1) and assume now that the disturbance can be decomposed as

$$d(t) = d_M(t) + d_\delta(t), \tag{7}$$

where d_M is a "regular" part, satisfying some known evolution law (model), and d_δ is an "irregular" part, which is only known to be bounded. We assume that $\|d_\delta\|_\infty \ll \|d_M\|_\infty$, i.e., that the regular component is dominant. This is a logical assumption, justifying a special treatment of the regular part.

It is conventional to describe regular disturbances in the deterministic setting as the output of a known model (*exosystem*), represented by unknown initial conditions [4]. Accordingly, we assume that the Laplace transform of d_M satisfies

$$D_M(s) = M(s)D_0(s) \tag{8}$$

for a not-necessarily stable *model* $M(s)$ and "initial conditions" $d_0(t)$, whose Laplace transform $D_0(s)$ is holomorphic and bounded in the right-half plane $\{s \in \mathbb{C} \mid \operatorname{Re} s > 0\}$. We need the following assumption about the model:

$\boldsymbol{\mathcal{A}_1}$: $M(s) = 1/(1 - N(s))$ for a stable strictly proper $N(s)$.

This assumption, saying that $M(s)$ is minimum phase and has the unit high-frequency gain, is technical and nonrestrictive. Some common particular cases are presented below.

- Any constant $d_M(t) = \mu$ can be presented via

$$M(s) = \frac{s + a}{s} \quad \text{and} \quad d_0(t) = \mathrm{e}^{-at}\mu, \tag{9a}$$

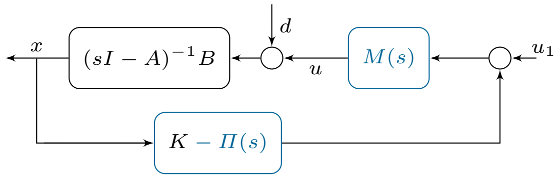

Fig. 1. Proposed ISM control architecture

for every $a>0$. This choice corresponds to the admissible $N(s)=a/(s+a)$.

- Any harmonic $d_M(t)=\mu\sin(\theta t+\phi)$ with a given θ corresponds to

$$M(s)=\frac{(s+a)^2}{s^2+\theta^2}\quad\text{and}\quad d_0(t)=\mathrm{e}^{-at}(\alpha t+\beta)\mu,\tag{9b}$$

for $\alpha=\theta\cos\phi-a\sin\phi$ and $\beta=\sin\phi$. This yields the admissible $N(s)=(2as+a^2-\theta^2)/(s+a)^2$.

- A pragmatic model in repetitive control is

$$M(s)=\frac{1}{1-F(s)\mathrm{e}^{-\tau s}}\quad\text{and}\quad d_0:[0,\tau]\to\mathbb{R},\tag{9c}$$

for a strictly proper stable $F(s)$. This corresponds to the admissible $N(s)=F(s)\mathrm{e}^{-\tau s}$. If $F(s)=1$, this model would represent all τ-periodic signals, those equal to d_0 over each period. However, the internal model in that case leads normally to a non-stabilizable system [6]. Thus, a low-pass F is introduced such that $F(\mathrm{j}\omega_i)\approx 1$ over a sufficiently large number of frequencies $\omega_i:=i\pi/\tau$.

The generating signal d_0 can be made decaying arbitrarily fast if a is chosen sufficiently large in (9a) and (9b) and it vanishes after $t=\tau$ in (9c). Thus, we may indeed regard d_0 as standard nonzero initial conditions.

If the disturbance satisfied (8), then the inclusion of $M(s)$ into the controller would guarantee its asymptotic rejection. However, presuming that $d=d_M$ would not be realistic. This is why we assume (7), whose irregular term d_δ may, for example, account for slow drifts or occasional jumps in the case of d_M from class (9a). Alternatively, it may represent deviations of (9c) from periodic disturbances in repetitive control, as well as aperiodic components and mismatches due to uncertain periods. The very presence of d_δ motivates the introduction of the SMC add-on term u_1.

B. ISM architecture and the separation

Consider the block-diagram in Fig. 1. It presents the proposed "ideal" component of the proposed ISM control configuration. Instead of adding up the ideal and discontinuous components, as done by (2), it injects the discontinuous component u_1 between two added elements. The block $M(s)$ is the exosystem, which corresponds to the model of the regular part of the disturbance in (7) and is the conventional internal model helping to reject d_M.

The $1\times n$ system Π, which together with the state-feedback gain K as in (3) constitutes the feedback gain of this system, is less orthodox. It is introduced to cancel out the effect of M on the design of u_1, while still keeping the attenuation of the regular part of d by M. Technically, $\Pi(s)$ may be any *stable* transfer function, satisfying

$$\Pi(s)P(s)=N(s)=1-1/M(s),\tag{10}$$

where the strictly proper and stable $N(s)$ is defined as in $\boldsymbol{\mathcal{A}_1}$. A required Π always exists. One particular solution is

$$\Pi(s)=N(s)B^{\#}(sI-A),$$

where $B^{\#}$ is a left inverse of B, although it is not unique and other alternatives may be used if they offer implementational benefits. The choice above in the particular case of the PI $M(s)$ as in (9a) yields

$$\Pi(s)=aB^{\#}-\frac{a}{s+a}B^{\#}(aI+A).$$

By multiplying it by the delay $\mathrm{e}^{-\tau s}$ we obtain $\Pi(s)$ for the repetitive control as in (9c) under

$$F(s)=\frac{a}{s+a}.\tag{11}$$

The structure of the closed-loop system that connects the available component of the control signal, u_1, with the state x is derived below.

Theorem 1: Let $\boldsymbol{\mathcal{A}_1}$ hold. If $A+BK$ is Hurwitz and Π satisfying (10) is stable, then the system in Fig. 1 satisfies

$$\dot{x}(t)=(A+BK)x(t)+B\delta(t)+Bu_1(t),\tag{12}$$

where $\delta=M^{-1}d$ is bounded. Moreover, u stabilizes (1) iff u_1 stabilizes (12).

Proof: First, note that the stability of Π implies that all cancellations in the system in Fig. 1 are stable. Hence, the internal stability arguments of [10, Prop. 1], which are based on loop shifting, can be applied, so that the stability statement (the last sentence) of the theorem follows. Now,

$$U(s)=\frac{1}{1-N(s)}\big(U_1(s)+(K-\Pi(s))X(s)\big)$$
$$X(s)=P(s)\big(D(s)+U(s)\big),$$

where uppercase letters denote the Laplace transforms of corresponding signals. Combining these equations,

$$\big(1-N(s)-(K-\Pi(s))P(s)\big)U(s)=U_1(s)+(K-\Pi(s))P(s)D(s).$$

At this point condition (10) pays off, leading to the following simpler relation:

$$\big(1-KP(s)\big)U(s)=(KP(s)-N(s))D(s)+U_1(s),$$

from which

$$U(s)=\frac{1}{1-KP(s)}\big((1-N(s))D(s)+U_1(s)\big)-D(s)$$

and then $x=P(d+u)$ reads

$$X(s)=T_{\mathrm{d}}(s)\big(\Delta(s)+U_1(s)\big),$$

where $\Delta(s)=M^{-1}(s)D(s)$ and

$$T_{\mathrm{d}}(s):=P(s)(1-KP(s))^{-1}=(sI-A-BK)^{-1}B.$$

By $\mathcal{A}_1$, $1/M$ is stable, so the signal δ is bounded. ∎

The new closed-loop system (12) is of the same form as the system for which the conventional ISM with the ideal control (3) has to be designed. The only difference is that the original disturbance d is replaced with $\delta = M^{-1}d_\delta + d_0$ (cf. (8)). We can expect that if d is indeed dominated by d_M, then δ is substantially smaller than d. This is because the model M is expected to have high gains at dominant frequencies of d, so that its inverse suppresses those frequencies. This is indeed the situation in the examples discussed in §III-A.

- If M is an integrator as in (9a), then $1/M(s) = s/(s+a)$ is a high-pass filter, whose gain is strictly contractive at all finite frequencies. Hence, $|\Delta(\mathrm{j}\omega)| < |D(\mathrm{j}\omega)|$ at every finite frequency. Moreover, $|\Delta(0)| < \infty$ whenever d is bounded, meaning that δ has a zero DC component, as expected.
- If M is as in (9b), then $1/M(s) = (s^2+\theta^2)/(s+a)^2$ and its gain is also strictly contractive at all finite frequencies, provided $a > \theta$. Also, $|\Delta(\mathrm{j}\theta)| < \infty$, so the dominant harmonic at $\omega = \theta$ is eliminated in δ.
- In the repetitive control case, with M as in (9c), we have $1/M(s) = 1 - F(s)\mathrm{e}^{-\tau s}$ and its gain is upperbounded by 2, provided $|F(\mathrm{j}\omega)| \le 1$ for all frequencies. Also, at each $\omega_i = 2i\pi/\tau$ such that $F(\mathrm{j}\omega_i) \approx 1$ we have $1/|M(\mathrm{j}\omega_i)| \ll 1$, so that those harmonics are substantially attenuated in δ.

Thus, we may expect that the design of a discontinuous u_1 for the ISM architecture in Fig. 1 requires a lower amplitude of the discontinuous control signal than the design of u for the original system (1).

Remark 3.1 (choice of K): Apart from the stability of $A+BK$, the gain K is unconstrained. This can be exploited to various purposes, like the reduction of the sensitivity of system (12) to its disturbance inputs. Pursuing this direction is beyond the scope of this note. ▽

C. Discontinuous SMC design

There is a complete freedom in the choice of the sliding mode component u_1. We may consider the conventional (4) or even choose high-order SMC algorithms, see [2, Ch. 4 and 6] or [3, Ch. 2]. What should be emphasized is that, since we may expect $\|\delta\|_\infty \ll \|d\|_\infty$, then the gain ρ required in (4) to dominate the disturbance may be substantially smaller than that required in the direct SMC of (1).

In fact, it appears logical to tune $\rho(t)$ adaptively, in which case the gain can be reduced if d_δ in (7) becomes small and increased if some spikes arise. There is a number of available adaptation methods, see [11] and the references therein. Below we slightly modify the approach of [12], with the following gain adaptation law:

$$\dot\rho(t) = \begin{cases} \bar\rho\,|\sigma(t)|\,\mathrm{sign}(|\sigma(t)| - \varepsilon) & \text{if } \rho(t) \ge \rho_0 \\ c & \text{if } \rho(t) < \rho_0 \end{cases}, \qquad (13)$$

where $\rho_0 = \rho(0) > 0$ is an upper bound on $\|\delta\|_\infty$, $\bar\rho > 0$, $0 < \varepsilon \ll 1$, and $c > \varepsilon$ are design parameters. In particular, ε represents an index of robustness, as it determines how far the sliding variable is allowed to deviate from the ideal sliding motion $\sigma \equiv 0$ in the steady state. We refer the reader to [12, Sec. 4] for further details on practices of tuning this parameter. Note that the choice (13) is motivated mostly by our experience with this law in simulations, but other adaptation mechanisms can be used as well.

The result below shows the boundedness of all involved variables in the resulting closed-loop system. It extends the results in [12], by exploiting an integral sliding surface.

Proposition 2: Let $SB \neq 0$ and $S(sI - A - BK)^{-1}B$ be minimum phase. If in (12) $\|\delta\|_\infty < \rho_0$ and the control gain is tuned according to (13), then u_1 in (4) guarantees that there is $t_\sigma > 0$ such that $\sigma(t) = 0$ for all $t \ge t_\sigma$, resulting in the evolution of x according to

$$\dot x(t) = (A + BK)x(t),$$

and $\rho(t)$ is bounded.

Proof: If $|\sigma| > \varepsilon$, then in line with arguments of [12], we can select a suitable Lyapunov function V_1, which depends on sizes of σ and ρ, such that

$$\dot V_1(t) \le -\beta\sqrt{V_1(t)}$$

for some $\beta > 0$. This implies that there is a finite time instant at which the boundary $|\sigma| = \varepsilon$ is reached and that ρ is bounded during this stage.

Now, consider the case of $|\sigma| < \varepsilon$ and assume, without loss of generality, that $SB = 1$. Equation (13) can then be rewritten as $\dot\rho = -\kappa\,\mathrm{sign}(\rho - \rho_0)$, where $\kappa(t) \ge 0$ equals either $\bar\rho|\sigma(t)|$ or c, according to (13). Consider the Lyapunov candidate $V_2 = (\sigma^2 + (\rho - \rho_0)^2)/2$. As $\dot\sigma = \delta - \rho\,\mathrm{sign}\,\sigma$ now,

$$\begin{aligned}\dot V_2 &= \sigma(\delta - \rho\,\mathrm{sign}\,\sigma) - \kappa|\rho - \rho_0| \\ &= -(\rho_0 - \delta\,\mathrm{sign}\,\sigma)|\sigma| - (\kappa + \mathrm{sign}(\rho - \rho_0)|\sigma|)|\rho - \rho_0|.\end{aligned}$$

Let $\gamma = \rho_0 - \|\delta\|_\infty$. Because $\|\delta\|_\infty < \rho_0$, we have that $|\delta(t)| \le \rho_0 - \gamma$ for all t and

$$\dot V_2 \le \begin{cases} -\gamma|\sigma| - (\bar\rho + 1)|\sigma||\rho - \rho_0| & \text{if } \rho \ge \rho_0 \\ -\gamma|\sigma| - (c - |\sigma|)|\rho - \rho_0| & \text{if } \rho < \rho_0 \end{cases}$$

Taking into account that $c > \varepsilon > |\sigma|$, we have that $\dot V_2 \le -\gamma|\sigma|$. This implies that $\sigma \to 0$. Moreover,

$$\sigma\dot\sigma = \sigma(\delta - \rho\,\mathrm{sign}\,\sigma) = -(\rho - \delta\,\mathrm{sign}\,\sigma)|\sigma| < -\gamma|\sigma|,$$

where the last inequality follows by the facts that ρ cannot drop below ρ_0 in (13) and $|\delta(t)| \le \rho_0 - \gamma$ for all t. Thus, the reachability condition [1] holds and we have σ converging in finite time. Therefore, exploiting the concept of equivalent control, defined as $\tilde u_1$, computing $\dot\sigma$ and posing it equal to zero, we have $\tilde u_1 + \delta = 0$, that is $\tilde u_1 = -\delta$, which makes system (12) invariant with respect to δ. ∎

IV. Illustrative Example

In this section, the proposed control architecture is assessed in simulation on an academic example given by a DC electric motor with negligible electric dynamics. We consider a current-controlled DC motor, described by

$$\dot x(t) = \begin{bmatrix} 0 & 1 \\ 0 & -h/J \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1/J \end{bmatrix} d(t),$$

where $x(t) \in \mathbb{R}^2$ is the state, whose first component is the shaft position and the second component is its angular velocity, $u(t) \in \mathbb{R}$ is the input torque (proportional to the armature current), and $d(t) \in \mathbb{R}$ is the resistance torque. The motor shaft is connected with a rigid mechanical load having the damping coefficient $h = 1\,\mathrm{N\,m\,s\,rad^{-1}}$ and the moment of inertia $J = 1\,\mathrm{N\,m\,s^2\,rad^{-1}}$. The transfer function of this system is

$$P(s) = \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s(s+1)}.$$

We assume that the regular component d_M in (7) is a periodic unit-amplitude signal, a square wave filtered by $1/(0.3s+1)$, whose period $\tau = \pi$. Its waveform over a period is as in the signals in Fig. 2 for $t \in [0, \tau]$. The disturbance contains also an irregular parasitic component d_δ such that $\|d_\delta\|_\infty < 0.3\,\mathrm{N\,m}$. The latter is not persistent. Rather, it is assumed to act only during some finite time intervals.

A. The ideal control

The need to cope with a periodic disturbance signal fits the repetitive control framework presented by model (9c). We choose the first-order $F(s)$ as in (11) with $a = \omega_1\alpha$, where $\omega_1 = 2\pi/\tau = 2$ is the fundamental frequency of π-periodic signals. A possible choice of the function $\mathit{\Pi}(s)$ is then

$$\mathit{\Pi}(s) = \mathrm{e}^{-\tau s} \frac{\omega_1\alpha(s+1)}{s+\omega_1\alpha} \begin{bmatrix} 0 & 1 \end{bmatrix},$$

which is stable and satisfies (10). We select $K = -\begin{bmatrix} 2 & 2 \end{bmatrix}$, which places the eigenvalues of $A + BK$ to $\{-1, -2\}$.

The equivalent disturbance $\delta = M^{-1}d$ for model (12) is shown in Fig. 2 for $\alpha = 10$ (thin gray line) and $\alpha = 100$ (thick blue line). The curves for $t \in [0, \tau]$, the same

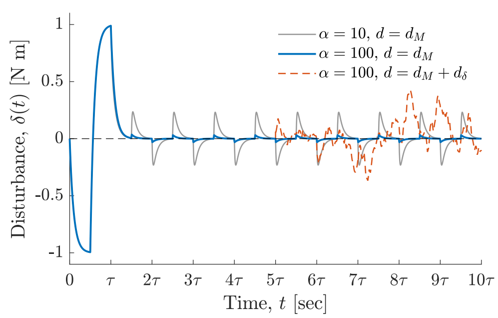

Fig. 2. Equivalent disturbance $\delta(t)$ for the filtered square wave $d_M(t)$.

for all α, correspond to the initial condition d_0, which is d_M over one period. In steady state, for $t \geq 2\tau$, we can see residual oscillations, which are caused by the use of the filter F in the model M. As α increases, $F(\mathrm{j}\omega) \approx 1$ in a wider frequency range, thus rendering residual oscillations smaller and $\delta \ll d$, cf. the discussion after the proof of Theorem 1. Consequently, the steady-state response of the motor for $\alpha = 100$ under only the linear repetitive control, which corresponds to $u_1 = 0$ in (12), is satisfactory, see the solid blue line in Fig. 3.

The situation changes when d_δ, chosen as a random zero-mean signal acting in the interval $t \in [5\tau, 10\tau]$, is added to

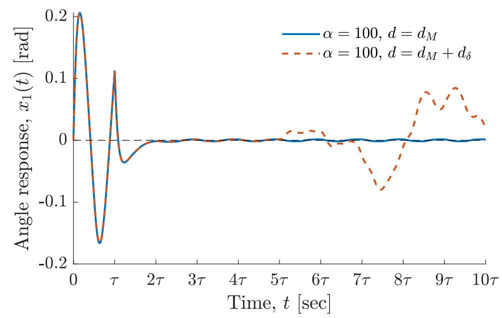

Fig. 3. Disturbance response under only linear repetitive controller.

the disturbance d. The equivalent disturbance δ, shown by the dashed red line in Fig. 2, also for $\alpha = 100$, contains then substantial components after $t = 5\tau$. This is because the spectrum of d_δ is not limited to the frequencies $\omega_i = i\pi/\tau$, so it might be amplified by M^{-1}. As a result, disturbance attenuation properties of the linear controller are not satisfactory, see the dashed red line in Fig. 3. This motivates the introduction of an adaptive discontinuous control discussed in §III-C.

B. SMC add-ons to repetitive control

Now add the discontinuous loop (4) under the ISM (5) and the adaptation law (13). We choose $S = \begin{bmatrix} 1 & 1 \end{bmatrix}$, for which $SB = 1 \neq 0$ and $S(sI - A - BK)^{-1}B = \frac{1}{s+2}$ is minimum phase. The parameters of (13) are selected as $\bar{\rho} = 10000$, $\rho_0 = 0.2$, $c = 0.1$, and $\varepsilon = 0.01$.

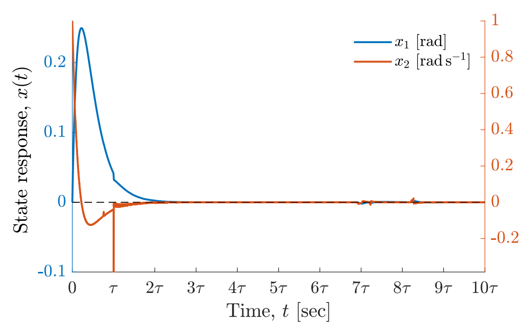

Fig. 4. Disturbance response under $d(t) = d_M(t) + d_\delta(t)$ and the proposed control architecture with the adaptive SMC loop.

Fig. 4 shows the response of the shaft position and its angular velocity in this case. It is evident that the disturbance rejection capabilities are substantially improved in the interval $t \in [5\tau, 10\tau]$, where the irregular disturbance d_δ acts. According to Proposition 2, the sliding variable is steered inside the layer of size ε and kept there as far as the disturbance is smaller than the amplitude of the adaptive control gain. This property can be seen in Fig. 5. At the beginning of the interval $t \in [5\tau, 10\tau]$, where the amplitude of $\delta = M^{-1}d$ is small (see the dashed red line in Fig. 2), the sliding variable is kept equal to zero as can be seen in Fig. 6. The SMC add-on amplitude is equal to ρ_0, and the disturbance deviation is completely dominated. When δ increases, the sliding mode is lost and the sliding variable increases, exceeding the threshold ε. As a consequence,

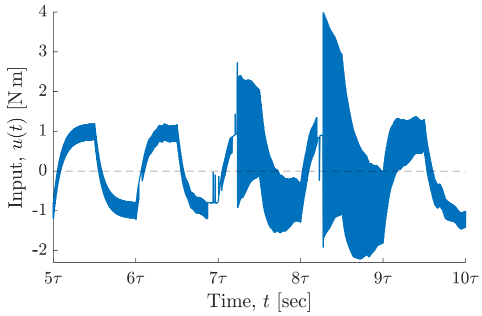

Fig. 5. The control signal $u(t)$ under $d(t) = d_M(t) + d_\delta(t)$ and the proposed control architecture for $t \in [5\tau, 10\tau]$.

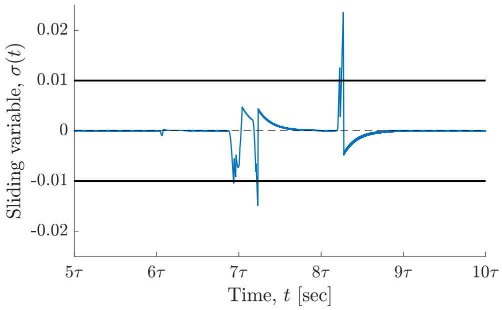

Fig. 6. The sliding variable $\sigma(t)$ steered to the boundary layer of size ε for $t \in [5\tau, 10\tau]$.

the control amplitude increases in order to dominate the disturbance and the sliding variable is then steered again inside the layer, as shown in Fig. 6 accordingly.

C. Comparisons

We conclude this section with a brief comparison of the proposed control algorithm with some other available options. As discussed in Section II, our main motivation for the use of the ISM approach with the internal model is to reduce the discontinuous component of u. This is also the incentive for adapting the SMC gain by (13). It thus makes sense to compare the proposed algorithm with other approaches to generate the discontinuous control signal u_1.

Our comparison criteria are:

1) the disturbance attenuation performance, measured by the root mean square of the shaft position x_1, termed oRMS;
2) the the root mean square of the discontinuous control component u_1, termed iRMS.

Apart from the adaptive algorithm discussed throughout this section, we select two additional strategies. One is the conventional fixed-gain SMC with $\rho = 1.25$, which is sufficient to dominate the equivalent disturbance δ. Another one is the dead-zone SMC algorithm, for which the control law with fixed gain is activated only outside the boundary layer $[-0.01, 0.01]$. The comparison is then carried out in the interval $t \in [5\tau, 10\tau]$.

The results reported in Table I show that the adaptive strategy allows about 34% reduction of the discontinuous control effort with respect to the fixed-gain strategy. Although the dead-zone approach yields an even smaller u_1, about 75% of the iRMS level of the fixed-gain SMC, it results the worst

TABLE I
PERFORMANCE OF THE SMC LOOP

strategy	oRMS	iRMS
adaptive-gain	4.33×10^{-4}	0.83
fixed-gain	9.75×10^{-6}	1.25
dead-zone	5.8×10^{-3}	0.31

disturbance attenuation among all three. As expected, the oRMS achieved by using the fixed-gain strategy is the best one. Still, the adaptive integral sliding mode control law allows a good trade-off between disturbance attenuation and control effort.

V. Concluding Remarks

The note has proposed a novel architecture of integral sliding mode control, in which the “ideal control” part can incorporate an internal model without affecting the “discontinuous control” part. This separation is enabled by the use of a special compensation element and facilitates the use of high- or even infinite-dimensional models, like those used in repetitive control.

Although only the relatively simple linear state-feedback setup with matched disturbances has been addressed, the presented ideas appear to be extendable to more general settings. For example, it extends to MIMO case seamlessly. Less immediate extensions, where some nonlinearities, modeling uncertainty, and unmatched disturbances are present and only a part of the state variable can be measured, are currently under investigation. The use of alternative adaptation mechanisms is also of interest.

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