AEROSPACE PHD DAY

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Low-thrust Optimal Trajectories using Differential Dynamic Programming enhancing the effects of Orbital Perturbations

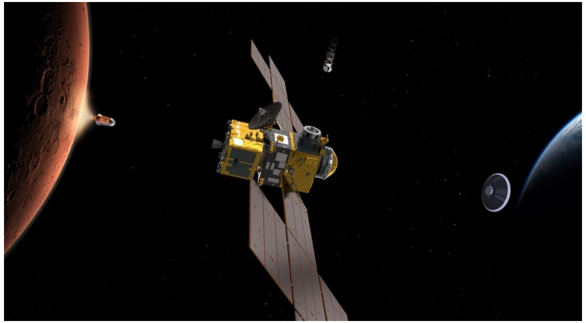
Outline

- Background
- Differential Dynamic Programming
  - Bellman's principle of optimality
  - Dynamics formulation
  - Constraints formulation
- Results
- Conclusions

Background

Low-thrust and Differential Dynamic Programming

- Low thrust satellite missions
  - Interplanetary: Mars sample return orbiter
  - Earth-orbiting satellites: OneWeb, Starlink
- Nonlinear optimal control problems
  - Direct/Indirect methods
  - Dynamic Programming
    - Differential Dynamic Programming
- Robustness
  - Sensitivity to uncertainties and failures
  - Correction manoeuvres



Mars sample return orbiter. Credit: ESA/ATG medialab

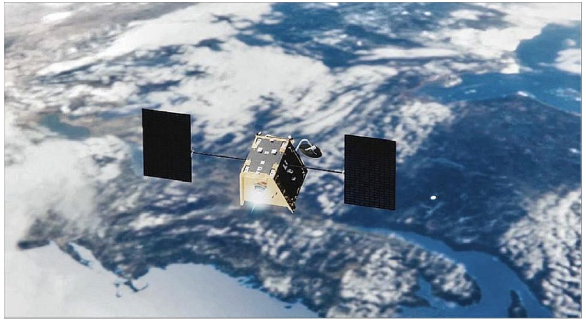


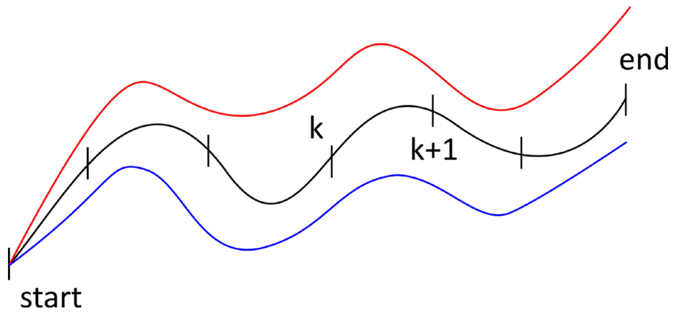
Illustration of a OneWeb satellite. Credit: OneWeb

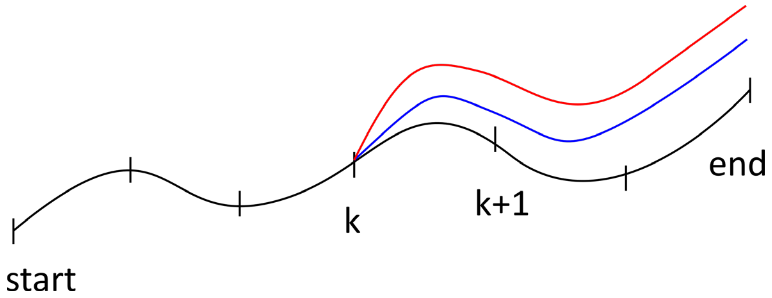
Differential Dynamic Programming

Low-thrust and Differential Dynamic Programming

- The Differential Dynamic Programming (DDP) is a technique used for solving nonlinear optimal control problem [1].
- It is based on the application of Bellman's principle of optimality [2]:

"An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision."





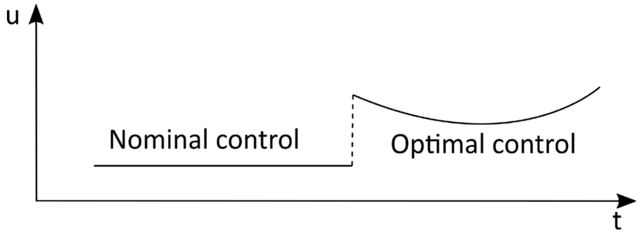
[1] Jacobson, D. H., and Mayne, D. Q., *Differential Dynamic Programming*, Elsevier, 1970.

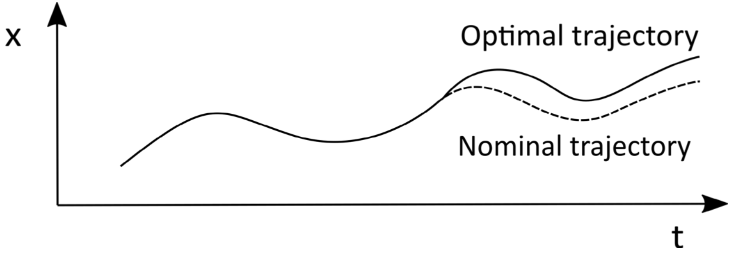
[2] Bellman, R., *Dynamic Programming.*, Princeton University Press, 1957.

Differential Dynamic Programming

Mathematical definition

- The DDP can be summarised in two steps:
  - A backward sweep where the optimality necessary conditions are solved for the control law.
  - A forward integration where the new control law is used to check the cost reduction.





- The optimality condition is mathematically expressed using **Hamilton-Jacobi-Bellman** (HJB) equation:
  - Continuous form: $$-\frac{\partial V(\boldsymbol{x};t)}{\partial t} = \min_{\boldsymbol{u}}[J(\boldsymbol{x},\boldsymbol{u};t) + \langle V_x(\boldsymbol{x};t), \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u};t)\rangle]$$
  - Discrete form: $$V_k^*(\boldsymbol{x}_k) = \min_{\boldsymbol{u}_k}[J_k(\boldsymbol{x}_k,\boldsymbol{u}_k;t_k) + V_{k+1}^*(\boldsymbol{x}_{k+1};t_{k+1})]$$

Differential Dynamic Programming

Analytical derivation

- The DDP is based on a linear-quadratic expansion of the HJB equation starting from a nominal nonoptimal solution.

$$\boldsymbol{x} = \overline{\boldsymbol{x}} + \delta\boldsymbol{x} \qquad \boldsymbol{u} = \overline{\boldsymbol{u}} + \delta\boldsymbol{u} \qquad \boldsymbol{b} = \overline{\boldsymbol{b}} + \delta\boldsymbol{b}$$

- Two versions of the DDP algorithm exist according to the initial point of the Taylor expansion:
  - Local: the initial point is coincident with a nominal solution used as initial guess.
  - Global: the initial point is obtained from the minimisation of the expanded HJB with all the variations set equal to zero.

$$\boldsymbol{u}^* = \min_{\boldsymbol{u}}[J(\overline{\boldsymbol{x}}, \boldsymbol{u}, t) + \langle V_x(\boldsymbol{x}; t), \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}; t)\rangle]$$

- The optimal feedback control law is obtained by means of differentiation and it is replaced inside the expanded HJB equation.

$$\delta\boldsymbol{u} = \beta_1\delta\boldsymbol{x} + \beta_2\delta\boldsymbol{b}$$

Differential Dynamic Programming

Complete cycle

$$H(\boldsymbol{x},\boldsymbol{u},V_x;t) = J(\boldsymbol{x},\boldsymbol{u};t) + \langle V_x, \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u};t)\rangle$$

Using a nominal control $\bar{\boldsymbol{u}}(t)$ run a nominal $\bar{\boldsymbol{x}}(t)$ trajectory. Calculate the nominal cost $\bar{V}(\bar{\boldsymbol{x}}_0, \bar{\boldsymbol{b}}, t_0)$.

$$\begin{cases} -\dot{a} = H(\bar{\boldsymbol{x}},\boldsymbol{u}^*,V_x;t) - H(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}},V_x;t) \\ -\dot{V}_x = H_x(\bar{\boldsymbol{x}},\boldsymbol{u}^*,V_x;t) + V_{xx}\left(\boldsymbol{f}(\bar{\boldsymbol{x}},\boldsymbol{u}^*;t) - \boldsymbol{f}(\bar{\boldsymbol{x}},\bar{\boldsymbol{u}};t)\right) \\ -\dot{V}_{xx} = H_{xx} + \boldsymbol{f}_x^T V_{xx} + V_{xx}\boldsymbol{f}_x - (H_{ux} + \boldsymbol{f}_u^T V_{xx})^T H_{uu}^{-1}(H_{ux} + \boldsymbol{f}_u^T V_{xx}) \end{cases}$$

Using the final boundary conditions solve the system of differential equations backwards from t_f to t_0, all the while minimising H with respect to $\boldsymbol{u}$ to obtain $\boldsymbol{u}^*$, and storing $\boldsymbol{u}^*(t), \beta_1(t)$.

If $t_{eff} = t_0$, integrate backwards from the final boundary conditions the other two differential equations. Calculate $\delta\boldsymbol{b}$. Integrate state equations with the new control.

$$\begin{cases} -\dot{V}_{xb} = [\boldsymbol{f}_x^T(\boldsymbol{x}^*,\boldsymbol{u}^*;t) + \beta_1^T(t)\boldsymbol{f}_u^T(\boldsymbol{x}^*,\boldsymbol{u}^*;t)]V_{xb} \\ -\dot{V}_{bb} = -V_{xb}^T\boldsymbol{f}_u(\boldsymbol{x}^*,\boldsymbol{u}^*;t)H_{uu}^{-1}\boldsymbol{f}_u^T(\boldsymbol{x}^*,\boldsymbol{u}^*;t)V_{xb} \end{cases}$$

$$\delta\boldsymbol{b} = -\varepsilon V_{bb}^{-1}(t_0)\varphi(\bar{\boldsymbol{x}}(t_f);t_f)$$

$$\beta_2(t) = -H_{uu}^{-1}(\boldsymbol{x}^*,\boldsymbol{u}^*,V_x;t)\boldsymbol{f}_u^T(\boldsymbol{x}^*,\boldsymbol{u}^*;t)V_{xb}$$

Apply the ''step-size adjustment method'' to obtain a new improved trajectory. If the current control is optimal, halts the computation.

If an improved trajectory is obtained, replace the old nominal $\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}$ and $\bar{V}$ by these new values.

Orbit Dynamics

Gauss' planetary equations

- The orbit dynamics is expressed by Gauss' equations in $[\hat{\boldsymbol{t}}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{h}}]$ and is made adimensional:

$$\frac{d\tilde{a}}{d\tilde{t}} = 2\sqrt{\frac{\tilde{a}^3(1+2e\cos f + e^2)}{(1-e^2)}}\frac{\tilde{u}_t}{\tilde{m}}$$

$$\frac{de}{d\tilde{t}} = \sqrt{\frac{\tilde{a}(1-e^2)}{(1+2e\cos f + e^2)}}\left[2(e+\cos f)\frac{\tilde{u}_t}{\tilde{m}} - \frac{(1-e^2)\sin f}{1+e\cos f}\frac{\tilde{u}_n}{\tilde{m}}\right]$$

$$\frac{di}{d\tilde{t}} = \sqrt{\tilde{a}(1-e^2)}\frac{\cos(\omega+f)}{1+e\cos f}\frac{\tilde{u}_h}{\tilde{m}}$$

$$\frac{d\Omega}{d\tilde{t}} = \sqrt{\tilde{a}(1-e^2)}\frac{\sin(\omega+f)}{(1+e\cos f)\sin i}\frac{\tilde{u}_h}{\tilde{m}}$$

$$\frac{d\omega}{d\tilde{t}} = \frac{1}{e}\sqrt{\frac{\tilde{a}(1-e^2)}{(1+2e\cos f + e^2)}}\left[2\sin f\frac{\tilde{u}_t}{\tilde{m}} + \left(2e + \frac{1-e^2}{1+e\cos f}\cos f\right)\frac{\tilde{u}_n}{\tilde{m}}\right] - \sqrt{\tilde{a}(1-e^2)}\frac{\sin(\omega+f)\sin i}{(1+e\cos f)\cos i}\frac{\tilde{u}_h}{\tilde{m}}$$

$$\frac{df}{d\tilde{t}} = \sqrt{\frac{1}{\tilde{a}^3(1-e^2)^3}}(1+e\cos f)^2 - \frac{1}{e}\sqrt{\frac{\tilde{a}(1-e^2)}{(1+2e\cos f + e^2)}}\left[2\sin f\frac{\tilde{u}_t}{\tilde{m}} + \left(2e + \frac{1-e^2}{1+e\cos f}\cos f\right)\frac{\tilde{u}_n}{\tilde{m}}\right]$$

$$\frac{d\tilde{m}}{d\tilde{t}} = -\sqrt{\frac{\mu}{L_{ref}}}\frac{(\tilde{u}_t^2 + \tilde{u}_n^2 + \tilde{u}_h^2)^{\frac{1}{2}}}{Isp \cdot g_0}$$

- Reference quantities used for the adimensionalisation:
  - $L_{ref} = a_0 \; or \; a_f$
  - $t_{ref} = \sqrt{L_{ref}^3/\mu}$
  - $m_{ref} = \frac{m_0}{n}$
  - $U_{ref} = \mu m_{ref}/L_{ref}^2$

Constraints Formulation

Equivalence with Cartesian representation

- In the Cartesian representation the following endpoint constraints are imposed:
  - Position vector
  - Velocity vector
  - Position and velocity vectors

$$\varphi = \begin{cases} \dfrac{\tilde{a}(1-e^2)}{1+e\cos f}\left[\cos(\omega+f)\cos\Omega - \sin(\omega+f)\cos i \sin\Omega\right] - \tilde{x}_f \\ \dfrac{\tilde{a}(1-e^2)}{1+e\cos f}\left[\cos(\omega+f)\sin\Omega - \sin(\omega+f)\cos i \cos\Omega\right] - \tilde{y}_f \\ \dfrac{\tilde{a}(1-e^2)}{1+e\cos f}\sin(\omega+f)\sin i - \tilde{z}_f \\ \dfrac{e\sin f}{\sqrt{\tilde{a}(1-e^2)}}\left[\cos(\omega+f)\cos\Omega - \sin(\omega+f)\cos i\sin\Omega\right] + \dfrac{1+e\cos f}{\sqrt{\tilde{a}(1-e^2)}}\left[-\sin(\omega+f)\cos\Omega - \cos(\omega+f)\cos i\sin\Omega\right] - \tilde{v}_{x_f} \\ \dfrac{e\sin f}{\sqrt{\tilde{a}(1-e^2)}}\left[\cos(\omega+f)\sin\Omega + \sin(\omega+f)\cos i\cos\Omega\right] + \dfrac{1+e\cos f}{\sqrt{\tilde{a}(1-e^2)}}\left[-\sin(\omega+f)\sin\Omega + \cos(\omega+f)\cos i\cos\Omega\right] - \tilde{v}_{y_f} \\ \dfrac{e\sin f}{\sqrt{\tilde{a}(1-e^2)}}\sin(\omega+f)\sin i + \dfrac{1+e\cos f}{\sqrt{\tilde{a}(1-e^2)}}\cos(\omega+f)\sin i - \tilde{v}_{z_f} \end{cases}$$

Constraints Formulation

Orbits elements formulation

- In the research the Euclidean norm of the difference between the final state and the prescribed final orbital elements is used.

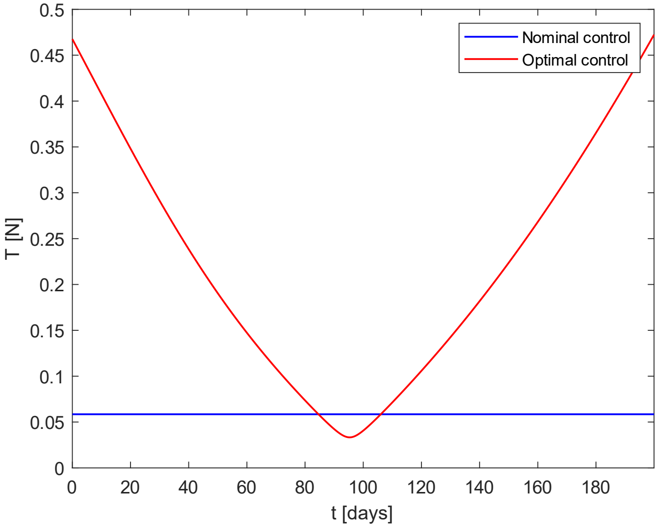
$$\varphi = \begin{cases} \left[\tilde{a}\left(\boldsymbol{x}\left(t_f\right), t_f\right) - \tilde{a}_f\right]^2 \\ \left[e\left(\boldsymbol{x}\left(t_f\right), t_f\right) - e_f\right]^2 \\ \left[i\left(\boldsymbol{x}\left(t_f\right), t_f\right) - i_f\right]^2 \\ \left[\Omega\left(\boldsymbol{x}\left(t_f\right), t_f\right) - \Omega_f\right]^2 \\ \left[\omega\left(\boldsymbol{x}\left(t_f\right), t_f\right) - \omega_f\right]^2 \\ \left[f\left(\boldsymbol{x}\left(t_f\right), t_f\right) - f_f\right]^2 \end{cases}$$

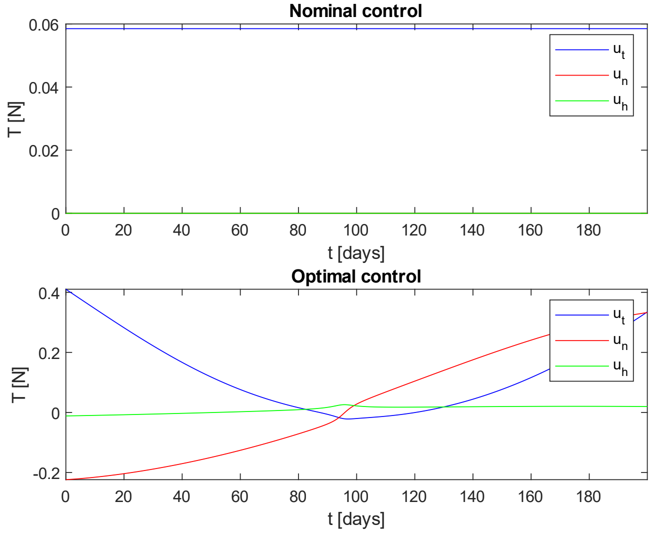
- The constraints are adjoint to the cost function using Lagrange multipliers.
- The Euclidean norm formulation ensures that the value function is always positive.
- Orbital elements introduce angles as state variables which are periodic and limited in [0,2π].

Results

Mars interplanetary transfer

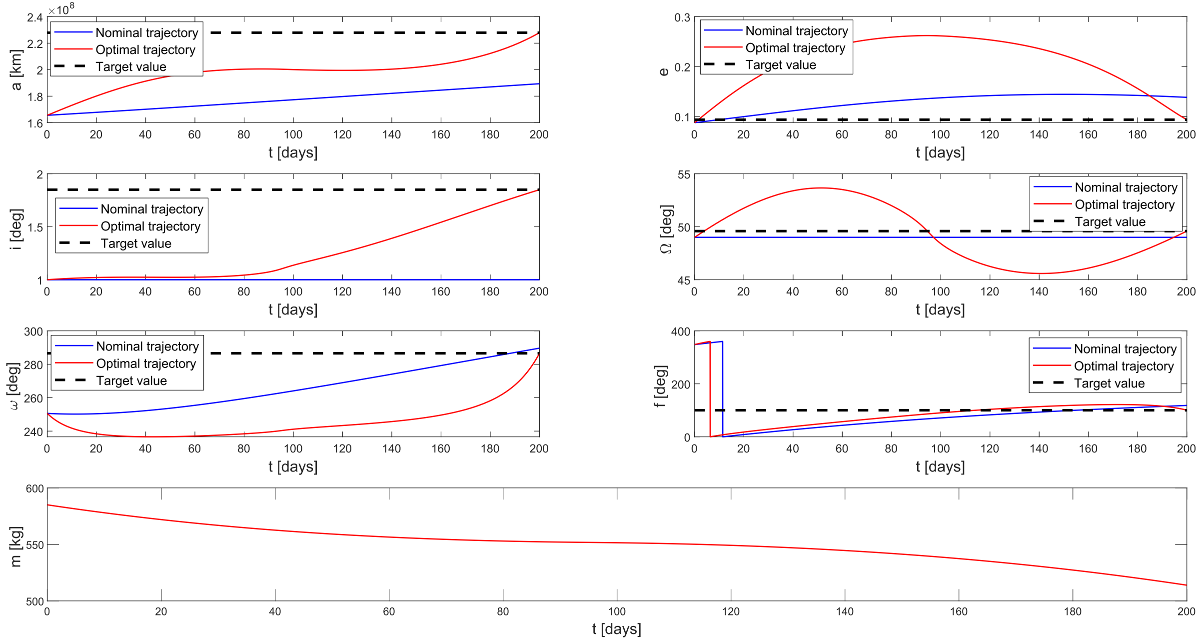
- A Mars interplanetary transfer has been considered as first reference scenario.
- The initial condition has been slightly modified to avoid the singularity for the inclination.





Results

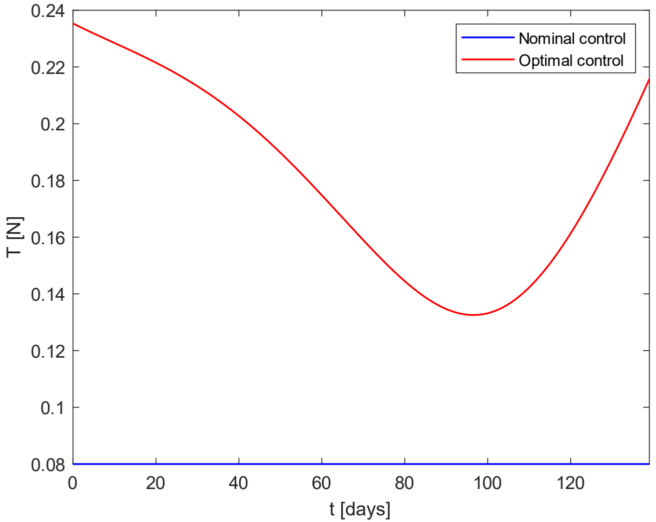
Mars interplanetary transfer: optimal trajectory

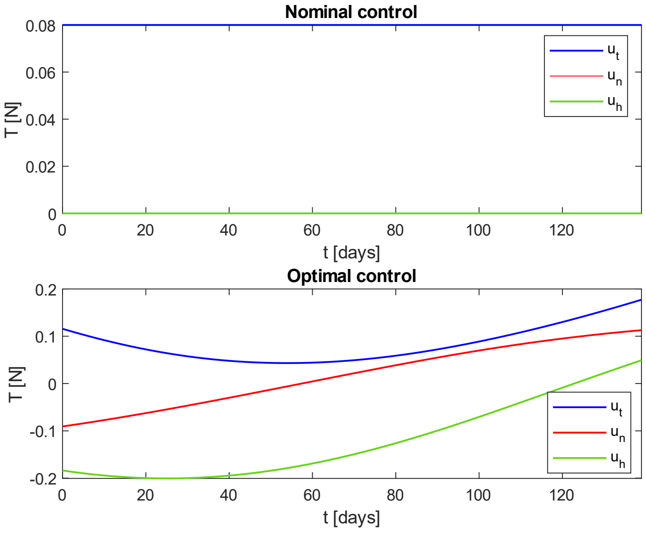


Results

Apophis interplanetary transfer

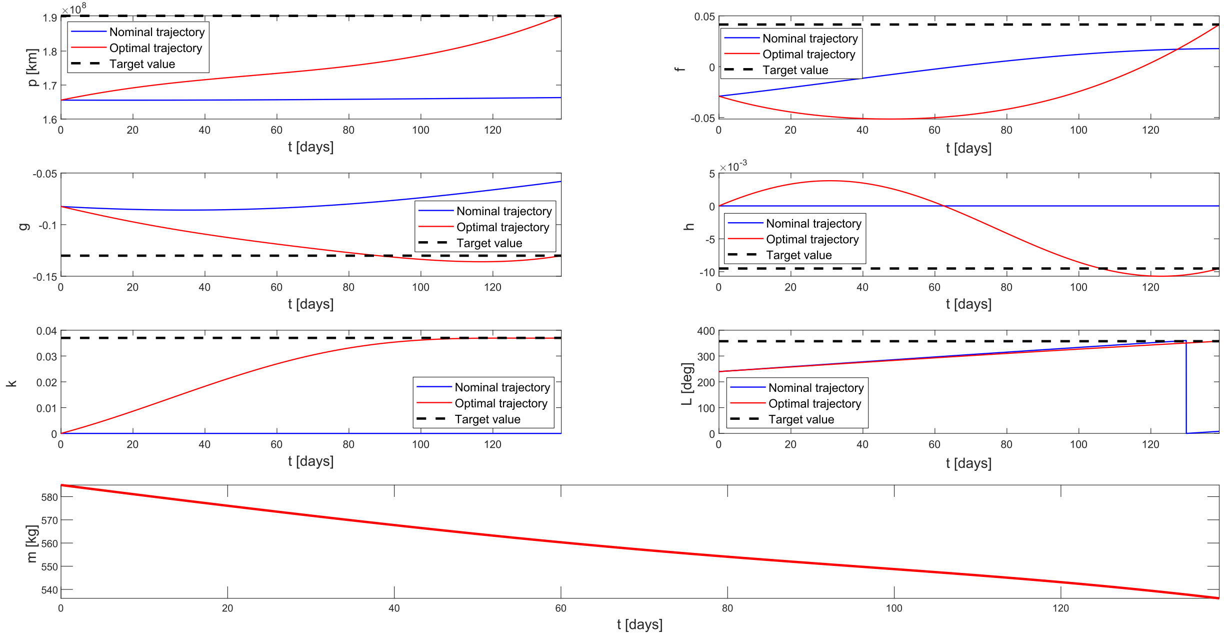
- A near-Earth asteroid interplanetary transfer has been considered as second reference scenario.
- The problem has been solved using non-singular elements to remove the inclination singularity.





Results

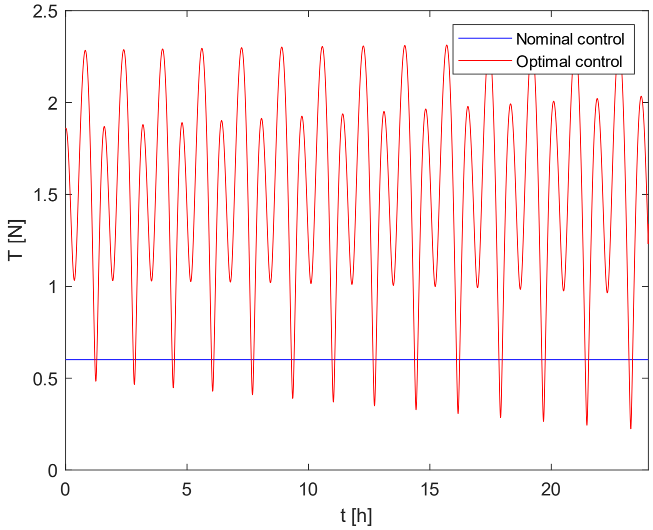
Apophis interplanetary transfer: optimal trajectory

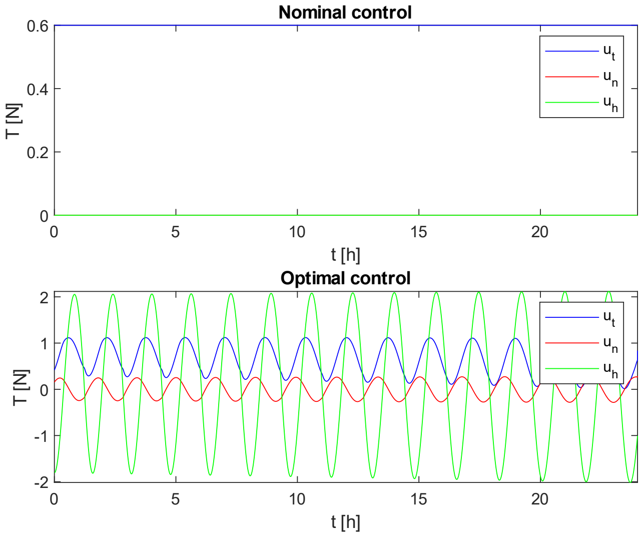


Results

Earth-satellite orbit raising

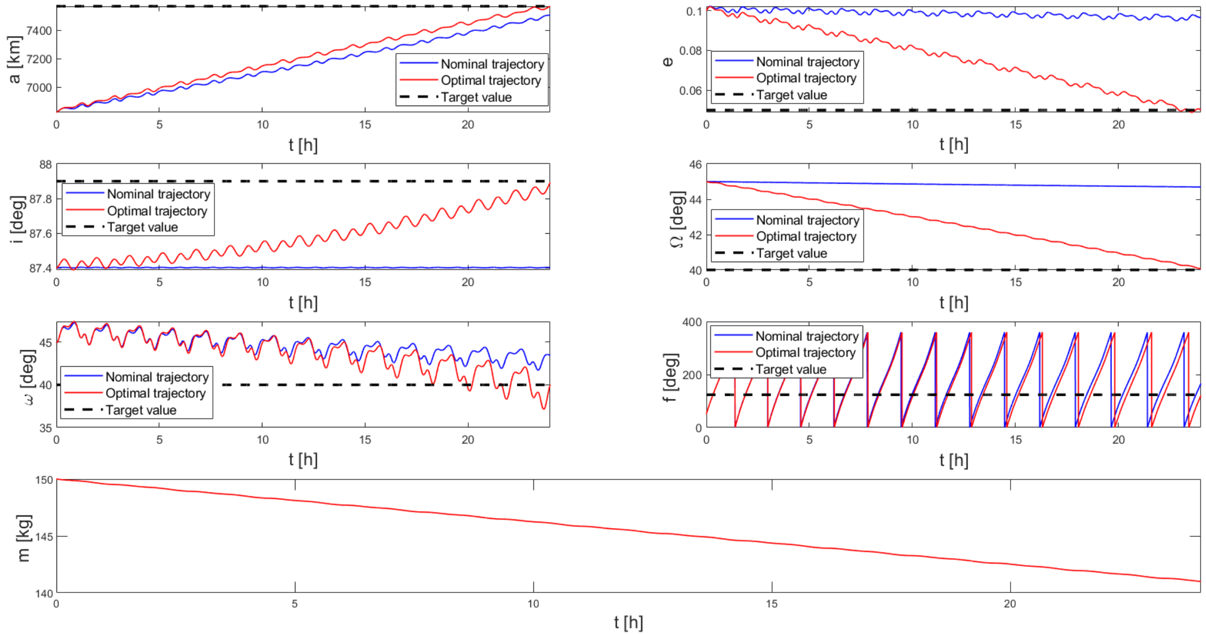
- An Earth-satellite orbit raising has been considered as last example.
- The problem has been solved considering the effect of J_2 orbital perturbation inside the dynamics.





Results

Earth-satellite orbit raising: optimal trajectory



Conclusions

- The DDP optimisation algorithm in terms of orbital elements as state representation has been presented.
- The algorithm works very well removing the fast variable (true anomaly). This suggests the application to averaged equations.
- New techniques for the computation of the partials are to be investigated (e.g., State Transition Matrix).
- Coupling of the DDP algorithm with classic indirect methods for Lagrange multipliers initialisation.

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Thank you for your attention!

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