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Risk-Aware Control and Games in Engineering

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***Abstract*— We present a brief tutorial on risk-aware control and game theory applied to engineering problems by solving a backward-forward partial-integro differential system composed of the Hamilton-Jacobi-Bellman and Fokker-Planck coupled equations. First, we discuss about the role that risk terms play in the engineering field. Then, both the risk-aware control and game problems are stated and the computation of the corresponding solutions is presented. We mainly focus on a basic propagation regulation of the coronavirus disease by using mean-field-type control. Hence, we discuss other engineering applications that can be addressed by using the same risk-aware perspective. Among such applications, we discuss about the electric vehicles, and the bio-inspired collective decision-making.**

***Index Terms*— Risk-aware control and game theory, mean-field-type control and game theory, COVID-19, engineering applications**

I. Introduction

Risk-aware control and game theory studies the design of strategic interactions under risk considerations. The main theory to solve these types of problems is known as mean-field-type control or mean-field-type game theory depending on the number of decision-makers in the setup. This theory is tightly related to the mean-variance paradigm and risk-averse problems in economics introduced by the Markowitz (1990 Prize in Economic Sciences) [1].

Mean-field-type games are a class of stochastic games that incorporate the distribution of the system states and/or control inputs into the problem. In this regard, the mean-field-type games might involve risk terms such as the mean, variance, and higher order moments (e.g., skewness, kurtosis). In this class of games, either a finite or infinite number of decision-makers might take place. The interaction of a finite number of decision-makers can be analyzed in the context of atomic games, and the players' decisions can affect the mean-field term. When there is a unique decision-maker, then the problem becomes a mean-field-type control problem, which is also known as a risk-aware controller due to the risk terms that are taken into account. An extensive discussion about these approaches is developed in [2].

A very well known approach is the so-called mean-field games. Under this perspective, which could be either deterministic or stochastic, quite large or infinite number of homogeneous decision-makers are considered in the strategic interaction. Then, it is suggested to consider that each decision-maker plays against the entire mass of decision-makers in the large population. To this end, it is assumed that each decision-maker interacts against an aggregative term coming from the remaining decision-makers. Such aggregated behavior is normally computed by an average and it is known as the mean-field term, which is associated to the empirical expectation [3], [4], [5], [6].

Regarding the mean-field-type game theory, *it is not indispensable to consider a large or infinite number of decision-makers since the mean-field terms are given by means of the distribution of the variables*. It is relevant to mention that real engineering applications are characterized by having a finite number of interactive entities, components, or decision-makers. In fact, the mean-field-type control theory involves a unique decision-maker. Several methods have been studied to solve this class of mean-field-type (risk-aware) game problems. Solutions for mean-field-type control problems have been discussed by using methods based on either Hamilton-Jacobi-Bellman and Fokker-Planck coupled equations or stochastic maximum principle in [7]. In [8] and [9], optimal control problems, involving systems governed by mean-field-type Stochastic Differential Equations (SDE), are studied by applying the Stochastic Maximum Principle (SMP); in this work, both the system dynamics and cost functional are allowed to be of mean-field-type. Other reported works studying the SMP method to solve control and game problems of mean-field-type are [10], [11], [12], [13], [14]. In contrast, [15] proposes to solve mean-field-type control problems by applying dynamic programming method. Hence, new results on risk-sensitive mean-field-type games have been reported in [16] extending the control work presented in [13], and sufficient optimality equations are established via infinite-dimension dynamic programming principle. Indefinite mean-field stochastic linear-quadratic control is studied in [17]. In [18], mean-field problems in discrete time are addressed. Other method, known as direct, has also been succesfully applied to the solution of these risk-aware problems [19]. Other works following this approach has been reported. For example, the incorporation of jump-diffusion processes and switching regimes has been presented in [20]. Recent works on mean-field-type games are [21]

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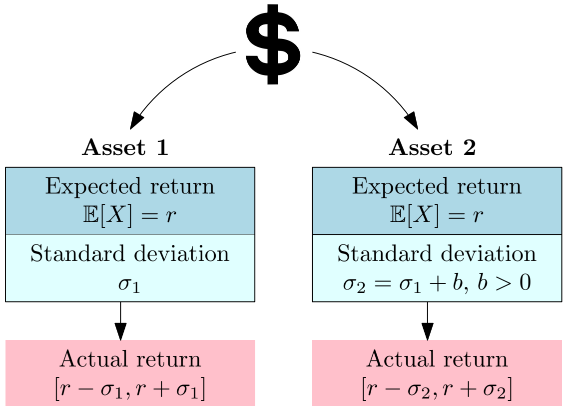

Fig. 1. Diversification problem with two assets and a total investment denoted by "$".

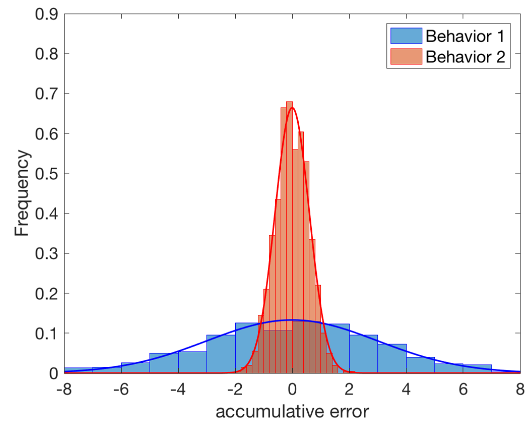

Fig. 2. Comparison of the behavior of two different controllers.

and [22], discussing non-linear continuous-time problems, and discrete-time settings involving information limitations, respectively.

The contribution of this paper consists on presenting a tutorial on risk-aware control and games in engineering. First, we present the role that risk terms play in the engineering field. We introduce the mean-field-type problems corresponding to risk-aware control and game problems, and present their solution by means of a backward-forward partial-integro differential system composed of the Hamilton-Jacobi-Bellman and Fokker-Planck coupled equations. Hence, we analyze the modeling and control of the coronavirus disease (COVID-19), and other engineering applications where the presented theory can be implemented are also exhibited.

The reminder of this paper is organized as follows. Section II introduces the risk terms from the economics perspective and motivates the analysis, measurement and minimization of risk terms in the engineering field. Sections III and IV introduce the risk-aware control and game problems together with the computation of their solutions, respectively. Section V presents a simple virus propagation model and risk-aware control, and Section VI introduces other engineering applications that can be addressed by using a risk-aware technique either with control or games. Finally, concluding remarks are made in Section VII.

II. Role of Risk in Engineering

Let us discuss about the role that the risk terms play in an optimization problem. To this end, we firstly introduce the risk concept from the economics perspective. Suppose you are about to invest a certain amount of money, denoted by "$", pursuing to obtain a revenue. For simplicity, let us consider there are two different assets where such money can be invested: asset 1 and asset 2 as presented in Figure 1. The assets are mainly characterized by two parameters, i.e., the expected return denoted by $\mathbb{E}[X_i]$, and its volatility expressed by means of a standard deviation σ_i. As an example, and for simplicity, let us consider both assets to have the same expected return given by $r \in \mathbb{R}^+$, but with different volatilities $\sigma_1 < \sigma_2$. Therefore, the actual return that can be obtain from assets 1 and 2 belong to the ranges $[r - \sigma_1, r + \sigma_1]$ and $[r - \sigma_2, r + \sigma_2]$, respectively. As a conclusion to highlight, the asset 2 can potentially return much more money than asset 1. However, asset 2 involves also a higher risk.

This problem is known as a diversification introduced by Harry Markowitz (Nobel Memorial Prize in Economic Sciences 1990) in [1] where the objective consists in maximizing the total expected return whereas minimizing the risk, i.e., for n assets we have the following optimization problem:

$$\max_{X_1,\dots,X_n} \sum_{i=1}^{n} \left(\mathbb{E}[X_i] - \sqrt{\text{var}(X_i)} \right).$$

Once the interpretation of the volatility in economics has been presented, then we are interested in investigating how this risk concept can be used in the design of controllers and/or game theoretical mechanisms in order to solved engineering problems. Consider the behavior of two different stochastic controllers for the same system state $x \in \mathbb{R}$ to be driven towards a desired set point denoted by $d \in \mathbb{R}$, and let $e = d - x$ be the error. Figure 2 shows the comparison of the two behaviors by means of the accumulative error along the time exhibiting different volatilities, i.e., the behavior 1 shows that the variation around the desired set point is higher than the variations obtained with the behavior 2. Equivalently, the volatility/risk in the behavior of the controller 2 is smaller than in the behavior of controller 1.

The risk terms in the optimization-based controllers and games are captured by means of the distribution of the variables-of-interest either for the system states or for the control inputs. These latter mentioned distribution terms are known as mean-field terms and the resulting optimization problems are known to be of mean-field-type [7].

III. Risk-Aware Control Theory

Risk-aware optimal control problems are a class of stochastic techniques that take into consideration the distribution of the variables-of-interest in both the cost functional and in the system dynamics. As a first alternative to deal with stochastic behaviors, the expectation can be used, e.g., $\mathbb{E}[x]$. However, this measurement fails to take into consideration the volatility. To overcome this limitation, the variance can be introduced, e.g., $\mathbb{E}[x-\mathbb{E}[x]]^2$, which is able to capture the volatility [20]. Hence, higher order moments could also be incorporated into the analysis and design of risk-aware control and games [13], [23]. Let us consider a scalar-valued system whose dynamics are given by a stochastic differential equation as follows:

$$dx(t) = b(x,m,u)dt + \sigma(x,m,u)dB(t), \tag{1}$$

where $x \in \mathbb{X}$ denotes the system state, $u \in \mathcal{U} = \mathbb{R}$ denotes the control input, $m(t,\cdot)$ is the probability measure of x on $\mathbb{R}$, and B is a standard Brownian motion. Moreover, $b : \mathbb{X}\times\mathbb{R}\times\mathcal{U} \to \mathbb{R}$ and $\sigma : \mathbb{X}\times\mathbb{R}\times\mathcal{U} \to \mathbb{R}$ are state and control-dependent drift, and diffusion terms, respectively. The control objective consists in minimizing the following cost functional:

$$L(x,m,u) = h(x(T),m(T)) + \int_0^T \ell(x(t),m(t),u(t))dt,$$

where $h : \mathbb{X}\times\mathbb{R} \to \mathbb{R}$ denotes the terminal cost and $\ell : \mathbb{X}\times\mathbb{R}\times\mathcal{U} \to \mathbb{R}$ denotes the running cost. Then, the risk-ware control problem is as follows:

$$\min_{u\in\mathcal{U}} \ \mathbb{E}\left[L(x,m,u)\right], \tag{2}$$

subject to (1) with $x(0) \triangleq x_0$. The problem in (2) is of mean-field-type, or equivalently, it is a risk-aware control problem involving mean-field terms, which are computed by using the probability measure of the system state and the the optimal control input is expressed in terms of the system state and its probability measure. The solution of the risk-aware control problem is found by solving the following backward-forward partial-integro-differential system as presented in [7], [24]:

$$m_t = -(mb)_x + \frac{1}{2}(m\sigma^2)_{xx}, \tag{3a}$$

$$0 = V_t(t,m) + \int_{\mathbb{R}} H(x,m,V_{xm},V_{xxm})m(dx). \tag{3b}$$

with initial boundary condition $m(0) = m_0$ making the Fokker-Plank-Kolmogorov equation in (3a) go forward, and with terminal boundary condition $V(T,m) = \int_{\mathbb{R}} h(y,m)m(T,dy)$ making the Hamilton-Jacobi-Bellman equation in (3b) go backward. Hence,

$$V(t,m) = \min_{u\in\mathcal{U}} \ \mathbb{E}[L(x,m,u)]$$

denotes the optimal cost and

$$H(x,m,V_{xm},V_{xxm}) = \inf_{u\in\mathcal{U}} \left\{ \ell + bV_{xm} + \frac{\sigma^2}{2}V_{xxm} \right\}$$

denotes the Hamiltonian. Existence and uniqueness of equilibria is studied in [5] for the infinite population case in the context of linear-quadratic systems [25], non-linear systems, PDE methods [26] and the master equation method [27]; and in [24] existence and uniqueness is discussed also by incorporating jumps and regime switching (random coefficients). Besides, there are several problems for which it is possible to find a semi-explicit, or even explicit, solution for the system in (3), e.g., [28], or for non-linear and non-quadratic cases, e.g., [21].

IV. Risk-Aware Game Theory

The risk-aware game can be studied in either an atomic or non-atomic context, i.e., it is possible to have a risk-aware problem in which there are a finite number of decision-makers whose decisions are non-negligible, or to have a large number of decision-makers with risk-aware terms divided into classes where some of their decisions might be non-negligible in the individual mean-field terms but negligible in the population mean-field terms.

We focus on the atomic game case. Let $\mathcal{N} = \{1,\dots,n\}$ denote the set of n decision-makers in a risk-aware strategic interaction. Let us consider now that the system state is being affected by all the decision-makers $\mathcal{N}$ as follows:

$$dx(t) = b(x,m,\{u_j\}_{j\in\mathcal{N}})dt + \sigma(x,m,\{u_j\}_{j\in\mathcal{N}})dB(t).$$

Each decision-maker plays in a strategical manner pursuing to minimize its cost functional given by:

$$L_i(x,m,u_1,\dots,u_n) = h_i(x(T),m(T)) + \int_0^T \ell_i(x(t),m(t),\{u_j(t)\}_{j\in\mathcal{N}})dt,$$

Let $\mathcal{U}_i$ be the set of feasible control inputs corresponding to the decision-maker $i \in \mathcal{N}$. The risk-aware game thepretical problem is stated as follows:

$$\min_{u_i\in\mathcal{U}_i} \ \mathbb{E}\left[L_i(x,m,u_1,\dots,u_n)\right],$$

subject to

$$dx(t) = b\left(x,m,\{u_j\}_{j\in\mathcal{N}}\right)dt + \sigma\left(x,m,\{u_j\}_{j\in\mathcal{N}}\right)dB(t),$$
$$x(0) \triangleq x_0.$$

Once the risk-aware game problem has been stated, we define the set of best-response strategies. Any feasible $u_i^* \in \mathcal{U}_i$ such that

$$u_i^* \in \arg\min_{u_i\in\mathcal{U}_i} \mathbb{E}\left[L_i(x,m,u_1,\dots,u_n)\right]$$

is a *mean-field-type best-response strategy* of $i \in \mathcal{N}$, i.e., $u_i^* \in \mathbb{BR}_i(u_{-i}^*)$, against $u_{-i} = \{u_1,\dots,u_{i-1},u_{i+1},\dots,u_n\}$ by $\mathcal{N}\setminus\{i\}$. Moreover, the game theoretical solution concept for the aforementioned problem is known as a risk-aware Nash equilibrium. Any feasible strategy profile

$$[u_1^* \quad \dots \quad u_n^*]^\top \in \prod_{j\in\mathcal{N}} \mathcal{U}_j,$$

such that $u_i^* \in \mathbb{BR}_i(u_{-i}^*)$, for all $i \in \mathcal{N}$, is a risk-aware Nash equilibrium of the mean-field-type game (risk-aware game).

Similar to the computation for the risk-aware control counterpart, the solution of the risk-aware game problem is found by solving a backward-forward partial-integro-differential system for all the decision makers with the following Hamilton-Jacobi-Bellman system:

$$0 = V_{i,t}(t,m) \tag{4a}$$

$$+\int_{\mathbb{R}} H_i(x,m,V_{i,xm},V_{i,xxm})m(t,dx),\ \forall i\in\mathcal{N},$$

$$V_i(T,m)=\int_{\mathbb{R}} h_i(y,m)m(T,dy),\ \forall i\in\mathcal{N}, \tag{4b}$$

where

$$H_i(x,m,V_{xm},V_{xxm})=\inf_{u_i\in\mathcal{U}_i}\left\{\ell_i+bV_{i,xm}+\frac{\sigma^2}{2}V_{i,xxm}\right\}$$

denotes the Hamiltonian corresponding to the decision-maker $i\in\mathcal{N}$. The Hamilton-Jacobi-Bellman system for the risk-aware game is deduced in the same manner and following the same reasoning as it was done for the risk-aware control.

In the next sections we discuss about some engineering problems that can be addressed by using risk aware techniques, either control or game theoretical perspective. First, we present a problem that has earned special interest in the recent months due to its worldwide impact, i.e., the propagation control of the COVID-19 by designing appropriate governmental policies. To this end, we propose a simple and standard SEIRD-based model. Next, we discuss other applications where the risk quantification and minimization must be considered. We present the electric vehicles and bio-inspired collective decision-making applications.

V. COVID-19 Propagation Control

At the end of 2019, coronavirus disease (COVID-19) appeared in Wuhan, China. After very few months, by April 23rd 2020, more than 2,500,000 infected cases have been reported in the entire globe affecting about 200 countries throughout all the continents. This COVID-19 pandemic has left near 200,000 deaths so far, and only about 750,000 people have been recovered. Among the most affected countries we find the following: The US, Spain, Italy, France, Germany, United Kingdom, Turkey, Iran, and China. In order to reduce the propagation of the virus and achieve an effective monitoring of its progress, governments have imposed some rules/policies such as cancelling both international and domestic flights, imposing lockdowns, decreasing the congestion in the public transportation systems, and/or recommending social distance/isolation. All these decisions have a tremendous economic implications and, in some cases, are not effective enough [29]. There is a quite urgent necessity in studying how to model and control the spread of the virus. Several studies have been already reported worldwide as presented next.

Regarding the behavior of the virus, some researchers have studied its response against different temperature conditions in order to identify in which regions the virus could spread in an easier manner [30]; and in [31], the relationship between the exposition with the pollution and the COVID-19 infection is studied. Other studies are oriented to identify how the immune system mechanisms and response against the COVID-19 can be improved [32]. In contrast, there are other reported analysis focussing on the economic impact of the virus. For instance, in [33], it is studied if the COVID-19 affects the stock market and how the amount of infected and death cases have a negative implication over the stock returns. In [34], the economic and social impact of the COVID-19 in the financial markets and institutions is analyzed by considering either a direct or indirect way. Finally, as a third main approach, researchers are working on the appropriate modelling for the propagation of the COVID-19 by using, adapting and/or enhancing pandemic mathematics. For instance, the different types of SIS and SIR stochastic epidemic models presented in [35], or the Auto-Regressive Integrated Moving Average (ARIMA) models, which are used to predict the epidemiological tendency of the virus as in [36], where the study is mainly focused on the Italian, Spanish and French cases. Hence, some deterministic and stochastic SI models considering a transmission rate [37], or the so-called Kolmogorov-forward equation as it is made in [38] by means of several methods oriented to stochastic disease dynamics modelling. Other models based on ordinary differential equations have been presented for specific regions as in [39], where the Italian case is considered. In [40], a data-driven COVID-19 propagation model is proposed in the context of mean-field-type games. The author in [40] incorporates more states in comparison to the existing SIR and SEIRD-based models in order to capture further details about the behavior of the pandemic. As an example, and to motivate the importance of modeling and studying how to control the propagation of a pandemic, Figure 3 shows the quick propagation of the virus in Colombia when some domestic flights are allows throughout the country during just 400 iterations representing possible interactions in the population.

A. Single-Player Problem

In order to model the propagation of the COVID-19 we consideration that the individuals can belong to any of the following possible states:

- Exposed: in this state the individual is healthy and exposed to get the disease
- Quarantined: in this state, we assume that the individual is healthy and is in quarantine in order to reduce the probability to get infected
- Infected: in this state the individual is sick by the COVID-19 and is susceptible to either passing away or recovering from the disease
- Recovered: in this state, the individual is healthy after having suffered from the disease
- Death: this state indicates that the individual has passed away because of the COVID-19

Let us consider a Markov process $\{X(t)\}_{t\geq 0}$ with the discrete state space $\mathcal{S}$ given by the possible states previously introduced, i.e., $\mathcal{S}=\{E,Q,I,R,D\}$ corresponding to Exposed, Quarantined, Infected, Recovered, and Death, respectively. Let $\mathbb{P}(X(t)=s)\in[0,1]$ denote the probability that the state $X(t)$ is $s\in\mathcal{S}$ at time t. Hence, let

$$m_s(t)=\left[\mathbb{P}(X(t)=s)\right]_{s\in\mathcal{S}}\in[0,1]^{|\mathcal{S}|}$$

denote the probability distribution of $X(t)$, from which an histogram may be constructed.

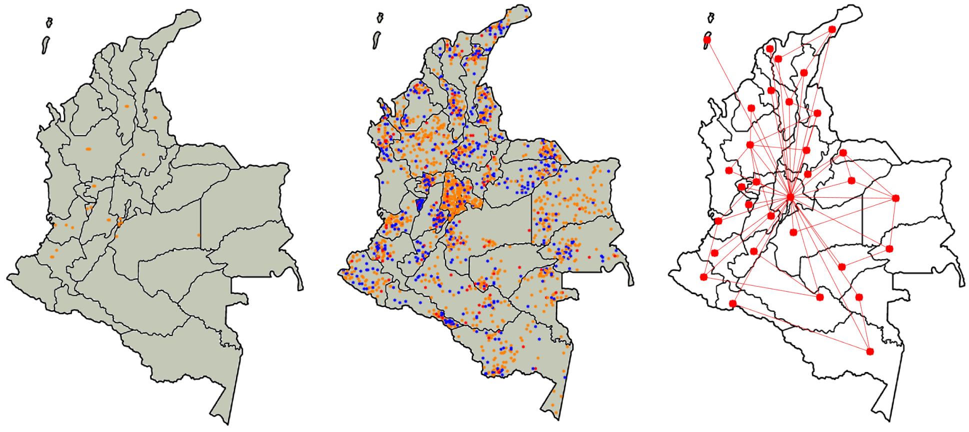

Fig. 3. Propagation of the virus in Colombia with migration constraints among the 32 departments without quarantine prevention (orange = infected, red = dead, blue = recovered). (a) initial condition, (b) spread of the virus after 400 iterations (interactions), (c) allowed air traffic.

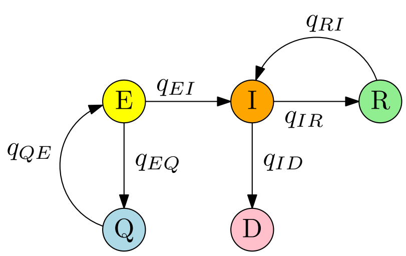

Fig. 4. Transition rates among the finite states $\mathcal{S}$.

The government and/or authorities impose some laws and/or policies in order to control the propagation of the virus, e.g., close the borders, cancel domestic and international flights, close restaurants, suspend concerts, impose compulsory quarantine, etc. These decisions are given by means of control inputs denoted by $u \in \mathbb{R}^{|\mathcal{S}|}$ and are normally economically costly.

Let $q_{ss'}(m(t), u(t)) \geq 0$, denote the transition rate (also known as jump intensities) from state $s \in \mathcal{S}$ to state $s' \in \mathcal{S}$ such that $q_{ss} = -\sum_{s' \in \mathcal{S}\setminus\{s\}} q_{ss'}$. Figure 4 shows the transition rates among the different states in the COVID-19 modeling. The transition rates $q_{ss'} : [0,1]^{|\mathcal{S}|} \times \mathbb{R}^{|\mathcal{S}|} \rightarrow \mathbb{R}_{\geq 0}$, for all $s, s' \in \mathcal{S}$, are functions depending on the probabilities of being occupying a state from $\mathcal{S}$ at certain time t, and the imposed policies and/or laws form the government in order to mitigate the propagation of the virus. Next, we introduce two different approaches. First, we present the atomic case that models an individual changing from one state to another according to the transition rates. Second, we present a mean-field-type approach that models the probabilities of being occupying a certain state. Notice that if a large (or infinite) number of individuals is considered, then the probabilities can be interpreted as a proportion of individuals and the mean-field term is given by the distribution throughout the finite states $\mathcal{S}$.

The authorities are interested in having control over the propagation of the virus by imposing laws/policies by means of the signals $u \in \mathbb{R}^{|\mathcal{S}|}$, which directly affect the transition rates among the states $\mathcal{S}$ in the Markov chain as shown in Figure 4. At the same time, it is important to guarantee that the designed policies do not considerably affect the economy of the country/region under study. There is therefore a dilemma, or in other words, there are two contradictory objectives that have to be appropriately balanced. On one hand, even though it is well known that, so far, the isolation is the best strategy to mitigate the propagation of the virus, it implies the freeze of the market affecting the whole population from the economic point of view. Let us consider the Kolmogorov-forward equation, which describes the evolution of the probabilities to be occupying a certain state along the time. In fact, when considering a large population with finite number of individuals, these probabilities can be interpreted as the proportion of individuals occupying a certain state in $\mathcal{S}$. The problem is given by

$$\min_{u \in \mathbb{R}^{|\mathcal{S}|}} \; L(m(t), u(t)), \tag{5a}$$

subject to

$$\begin{aligned}\frac{d}{dt} m_s(t) = & \sum_{s' \in \mathcal{S}\setminus\{s\}} q_{s's}(m(t), u(t)) \; m_{s'}(t) \\ & - \sum_{s' \in \mathcal{S}\setminus\{s\}} q_{ss'}(m(t), u(t)) \; m_s(t), \\ & \forall s \in \mathcal{S},\end{aligned} \tag{5b}$$

$$q_{ss}(m(t), u(t)) = -\sum_{s' \in \mathcal{S}\setminus\{s\}} q_{ss'}(m(t), u(t)), \tag{5c}$$

$$m(0) = m_0. \tag{5d}$$

Once the control problem has been introduced. The mitigation of the COVID-19 can be studied by considering multiple decision-makers. When there are multiple entities deciding then we call the problem to be a game or a strategic interaction. Next, we address the game theoretical perspective.

B. Multiple-Player Problem

The problem with several involved players in the interactive decision is analyzed, e.g., multiple regions, states, countries or continents. The set of players is denoted by $\mathcal{P} = \{1, \dots, n\}$, and the system dynamics describing the evolution of the COVID-19 for the different players are coupled to each other. The evolution of the COVID-19 for each player is modeled by means of a Markov chain with a finite number of states $\mathcal{S}^p = \{E^p, Q^p, I^p, R^p, D^p\}$, for all $p \in \mathcal{P}$.

The transit of an individual among the regions, which are given by players, is made by means of a connected undirected graph, i.e., exposed, infected and recovered individuals can travel from one region to another following a constrained graph denoted by $\mathcal{G}^E = (\mathcal{V}^E, \mathcal{E}^E)$, $\mathcal{G}^I = (\mathcal{V}^I, \mathcal{E}^I)$, and $\mathcal{G}^R = (\mathcal{V}^R, \mathcal{E}^R)$, respectively. The nodes in the graphs and sets of edges are given by

$$\mathcal{V}^E = \{E^p\}_{p\in\mathcal{P}}, \qquad \mathcal{E}^E \subseteq \{(i,j) : i,j \in \mathcal{V}^E\}, \tag{6a}$$
$$\mathcal{V}^I = \{I^p\}_{p\in\mathcal{P}}, \qquad \mathcal{E}^I \subseteq \{(i,j) : i,j \in \mathcal{V}^I\}, \tag{6b}$$
$$\mathcal{V}^R = \{R^p\}_{p\in\mathcal{P}}, \qquad \mathcal{E}^R \subseteq \{(i,j) : i,j \in \mathcal{V}^R\}. \tag{6c}$$

Thus, there is a new bigger aggregated Markov process emerging denoted by $\{\tilde{X}(t)\}_{t\geq 0}$, where the set of finite states is given by $\tilde{\mathcal{S}} = \bigcup_{p\in\mathcal{P}} \mathcal{S}^p$. Besides, there are some migration constraints representing the air-traffic restrictions among the players. Let $A \in [0,1]^{\sum_{p\in\mathcal{P}} |\mathcal{S}^p|}$ be the adjacency matrix for the aggregated graph $\mathcal{G} = \bigcup_{s\in\mathcal{S}} \mathcal{G}^s$ with $\mathcal{V} = \bigcup_{p\in\mathcal{P}} \mathcal{S}^p$ and $\mathcal{E} \subset \bigcup_{s\in\mathcal{S}} \mathcal{E}^s$.

The transition rates (jump intensities) in the overall Markov process $\{\tilde{X}(t)\}_{t\geq 0}$ satisfy the following condition. For two states $s \in \mathcal{S}^p$, and $s \in \mathcal{S}^q$, with $p \neq q$,

$$q_{ss'} \begin{cases} = 0 & \text{if } (s,s') \notin \mathcal{E}, \\ > 0 & \text{otherwise.} \end{cases}$$

In addition, each player imposes local policies in order to have control over the COVID-19 and also assigns heterogeneous prioritization to the factors composing the cost functional, i.e., the balance between the minimization of the virus and the economical crisis.

Each player establishes policies in order to minimize its own cost functional denoted by $L^p(m,u)$, where $m = (m^p)_p$, and m^p, denotes the probability measure corresponding to the player p, for all $p \in \mathcal{P}$. Similarly we denote $u = (u^p)_p$,

$$L^p(m(t),u(t)) = h^p\left(\{m^p(T)\}_{p\in\mathcal{P}}\right) + \int_0^T \ell^p\left(m(t),u(t)\right) dt,$$

where $h^p(m) \in \mathbb{R}$ denotes the terminal cost, and $\ell^p(m,u) \in \mathbb{R}$ denotes the running cost. The non-atomic game problem is

$$\min_{u^p\in\mathcal{U}^p} \; L^p(m(t),u(t)), \tag{7a}$$

subject to

$$\frac{d}{dt} m_s^p(t) = \sum_{s'\in\mathcal{S}^p} q_{s's}^p(m(t),u(t)) \; m_{s'}^p(t) - \sum_{s'\in\mathcal{S}^p} q_{ss'}^p(m(t),u(t)) \; m_s^p(t), \; \forall s \in \mathcal{S}^p, \tag{7b}$$
$$m^p(0) = m_0^p, \quad \forall p \in \mathcal{P}. \tag{7c}$$

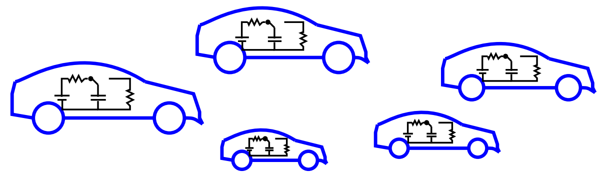

Fig. 5. Population of EVs. assimilated to capacitors

The solution of the problem in (7) is given by a Nash equilibrium as introduced in Section IV.

VI. Other Engineering Applications

Generally speaking, any engineering application involving risk terms can be addressed by using the mean-field-type theory. In fact, due to the presence of uncertainties in plenty of the engineering problems we have to solve nowadays, there is a large spectrum where the risk-aware techniques discussed in this paper can be performed.

Applications of the mean-field-type game theory in concrete engineering systems are: blockchains [41], water distribution systems [42], [43], storage in microgrids [44], modeling of pedestrian [45], financial engineering [12], production of an exhaustible resource [46], power networks [47], filtering [48], network security [49], smart grids [50], electricity price [51], opinion dynamics [52], evacuation [53]. To name some other examples, we describe some engineering applications next.

A. Electric Vehicles

In this section, we introduce responsive loads for a population of electric vehicles (EVs). There is no pre-programming of the EVs to implement specific charging policies. On the contrary, the EV's are designed to obtain their optimal charging policies in an autonomous fashion by calculating best-responses to the population behavior. The population behavior is measured by the each appliance through the mains frequency state. We formulate the problem as a mean-field game and discuss mean-field equilibrium points and investigate ways in which we can obtain an approximation of such equilibrium point via simple calculations. The methodology involves turning the game into a sequence of data-driven infinite-horizon receding horizon optimization problems that each EV solves online.

1) When electric vehicles (EVs) are large in number: Let us view EVs as capacitors and model their charging and discharging policies, as depicted in Fig. 5. The idea is to design a mean-field game to obtain social optimal behaviors on the part of the EVs.

Let us model each EV by using a continuous state $x(t)$ to indicate the level of charge, and a binary state $\pi(t) \in \{0,1\}$ to represent whether the EV is charging or discharging for each time $t \in [0,T]$, where $[0,T]$ is the time horizon window.

In the first case, we say that the EV is in on state and $\pi(t) = 1$, whereas in the second case the EV is in off state and $\pi(t) = 0$. When in on state, there is an exponential increase of the charge up to a fixed maximum level x_{on}. Differently, in the off state one observes an exponential decrease of the charge to a minimum level of charge x_{off}. We can model the level of charge of each single EV in $[0, T)$ as:

$$\dot{x}(t) = \begin{cases} -\alpha(x(t) - x_{on}) & \text{if } \pi(t) = 1 \\ -\beta(x(t) - x_{off}) & \text{if } \pi(t) = 0 \end{cases}, \tag{8}$$

where $x(0) = x$ is the initial state, and where $\alpha, \beta > 0$ are the charging and discharging rates respectively.

In the spirit of stochastic models as provided in [54], [55], [56], each EV is in states on or off with probability $\pi \in [0, 1]$. Then, the controlled input represents the transition rate u_{on} from state off to state on and the transition rate u_{off} from state on to state off.

Let $y(t) = \pi(t)$ and consider any x, y in the

$$\text{“set of feasible states”} \quad \mathcal{S} :=]x_{ff}, x_{on}[\times]0, 1[.$$

The model of each EV is then

$$\begin{aligned} \dot{x}(t) &= \Big(y(t)\Big[-\alpha(x(t) - x_{on})\Big] \\ &\quad + (1 - y(t))\Big[-\beta(x(t) - x_{off})\Big] \\ &=: f(x(t), y(t)),\ t \in [0, T), \quad x(0) = x, \\ \dot{y}(t) &= \Big(u_{on}(t) - u_{off}(t)\Big) \\ &=: g(u(t)),\ t \in [0, T), \quad y(0) = y. \end{aligned} \tag{9}$$

To capture the macroscopic evolution of the population let us consider the probability density function $m : [x_{on}, x_{off}] \times [0, 1] \times [t, T] \to [0, +\infty[$, $(x, y, t) \mapsto m(x, y, t)$, for which it holds

$$\int_{x_{on}}^{x_{off}} \int_{[0,1]} m(x, y, t) dx dy = 1$$

for every t in $[0, T)$. Furthermore, let

$$m_{on}(t) := \int_{x_{on}}^{x_{off}} \int_{[0,1]} y m(x, y, t) dx dy.$$

Analogously, let $m_{off}(t) := 1 - m_{on}(t)$.

Note that the mains frequency is a linear function of the deviation between the population of EVs in state on and a nominal value. Such deviation is referred to as *error* and is denoted as $e(t) = m_{on}(t) - \overline{m}_{on}$. Also we denote by $\overline{m}_{on}$ the nominal value. The physical intuition is that when more EVs are in state on than the ones indicated by the nominal value, the grid frequency presents negative deviation from the nominal value. The error captures the difference between the power supplied and consumed. In the following we make the simplifying assumption that the power supply is constant and equal to the nominal power consumption all the time.

2) Forecasting based on Holt's model: The EVs adapt their charging policies as best responses to the measured error $e(t)$ available at discrete times t_k. Forecasting is then based on the Holt's model. Based on the latest sampled error the players forecast the future value of the error and compute their best-response strategies over an infinite planning horizon. In other words they solve the asymptotic case for $T \to \infty$. In doing this, the players also assume that the forecasted error remains fixed throughout the planning horizon. This procedure returns a sequence of optimal controls. Then, the players implement their first controls until a new sample becomes available. This procedure resembles *receding horizon* techniques involving a multi-step ahead *action* horizon.

3) Receding horizon: Let the length of the interval between consecutive samples be $\delta = t_{k+1} - t_k$. For sake of simplicity and without loss of generality we can take $\delta = 1$. At each sampling time t_k the Holt's model returns $\hat{e}(t_{k+1})$, i.e., the forecast of the error at time t_{k+1}. The running cost for each player introduces the dependence on the distribution $m(x, y, t)$ through the error $\hat{e}(t_{k+1})$. Then for the running cost we have:

$$\begin{aligned} &c(\hat{x}(\tau, t_k), \hat{y}(\tau, t_k), \hat{u}(\tau, t_k), \hat{e}(t_{k+1})) \\ &= \frac{1}{2}\Big(q\hat{x}(\tau, t_k)^2 + r_{on}\hat{u}_{on}(\tau, t_k)^2 + r_{off}\hat{u}_{off}(\tau, t_k)^2\Big) \\ &+ \hat{y}(\tau, t_k)(S\hat{e}(t_{k+1}) + W), \end{aligned} \tag{10}$$

where q, r_{on}, r_{off}, and S are opportune positive scalars.

Cost (10) involves four terms. The first term is a penalty when the EVs' levels of charge deviate from the nominal value, which we set to zero. The second and third terms are a penalty on fast switching. The last term penalizes the EVs that are charging in peak times and discharging in off-peak times. We also consider a terminal cost $g : \mathbb{R} \to [0, +\infty[$, $x \mapsto g(x)$ to be yet designed.

Let the following update times be given, $t_k = t_0 + \delta k$, where $k = 0, 1, \ldots$. Let $\hat{x}(\tau, t_k)$, $\hat{y}(\tau, t_k)$ and $\hat{e}(t_{k+1})$, $\tau \geq t_k$ be the predicted state of player i and of the error $e(t)$ for $t \geq t_k$, respectively. Then, for all players and times $t_k, k = 0, 1, \ldots$, given the initial state $x(t_k)$, and $y(t_k)$ and $\hat{e}(t_{k+1})$ find

$$\hat{u}^\star(\tau, t_k) = \arg\min \ L(\hat{x}(t_k), \hat{y}(t_k), \hat{e}(t_{k+1})), \hat{u}(\tau, t_k)),$$

where

$$\begin{aligned} &L(\hat{x}(t_k), \hat{y}(t_k), \hat{e}(t_{k+1}), \hat{u}(\tau, t_k)) \\ &= \lim_{T \longrightarrow \infty} \int_{t_k}^{T} c(\hat{x}(\tau, t_k), \hat{y}(\tau, t_k), \hat{u}(\tau, t_k), \hat{e}(t_{k+1})) d\tau \end{aligned} \tag{11}$$

subject to the following constraints:

$$\begin{aligned} \dot{\hat{x}}(\tau, t_k) &= \Big(\hat{y}(\tau, t_k)\Big[-\alpha(\hat{x}(\tau, t_k) - x_{on})\Big] \\ &\quad + (1 - \hat{y}(\tau, t_k))\Big[-\beta(\hat{x}(\tau, t_k) - x_{off})\Big] \\ &=: f(\hat{x}(\tau, t_k), \hat{y}(\tau, t_k)),\ \tau \in [t_k, T), \\ \dot{\hat{y}}(\tau, t_k) &= \Big(\hat{u}_{on}(\tau, t_k) - \hat{u}_{off}(\tau, t_k)\Big) \\ &=: g(\hat{u}(\tau, t_k)),\ \tau \in [t_k, T). \end{aligned} \tag{12}$$

The above set of constraints are based on the forecast state dynamics of the individual player and of the rest of the population through $\hat{e}(t_{k+1})$. The constraints are characterized by boundary conditions at time t_k. Each player assumes that the error is constant over the planning horizon. When a new sample arrives at t_{k+1} the players update their best-response strategies, which we refer to as *receding horizon control policies*. The resulting problem for each player is described by the *closed-loop* system

$$\begin{aligned}\dot{\hat{x}}(\tau,t_k) &= \Big(\hat{y}(\tau,t_k)\Big[-\alpha(\hat{x}(\tau,t_k)-x_{on})\Big] \\ &\quad + (1-\hat{y}(\tau,t_k))\Big[-\beta(\hat{x}(\tau,t_k)-x_{off})\Big] \\ &=: f(\hat{x}(\tau,t_k),\hat{y}(\tau,t_k)),\ \tau\in[t_k,T), \\ \dot{\hat{y}}(\tau,t_k) &= \Big(u^{RH}_{on}(\tau)-u^{RH}_{off}(\tau)\Big) \\ &=: g(u^{RH}(\tau)),\ \tau\in[t_k,T), \end{aligned} \tag{13}$$

where the receding horizon control law $u^{RH}(\tau)$ satisfies

$$u^{RH}(\tau)=\hat{u}^{\star}(\tau,t_k),\quad \tau\in[t_k,t_{k+1}).$$

B. Bio-inspired Collective Decision-making

Motivated by honeybee swarms, the bio-inspired collective decision-making considers a population of homogeneous players that have to reach an agreement on two equally favourable options and where risk terms can also be considered. The problem results in a continuous-time discrete-state mean-field game model in which the players optimize their transition rates to minimize a cost functional. Model misspecifications are accounted by an additional adversarial disturbance. We present a consistent micro-macro model and solve it as an initial-terminal value problem (ITVP). We also discuss the existence and computation of stationary solutions. The framework can be extended to a structured environment with interaction topology.

We consider a large population of players. Let the probability vector $x(t)=[x_1,x_2,x_3]^\top\in\mathcal{S}^3$ be the population distribution across three states and let $\beta(t)\in\mathbb{R}^{3\times3}$ be the transition rate matrix, which depends on the state $x(t)$, for the corresponding continuous-time Markov process. Each element of matrix $\beta(t)$, i.e. β_{ij}, represents the transition rate from state i to state j, for $i,j\in\mathcal{I}^3$, and each column can be expressed in terms of two components as $\beta_i=\rho_i+w_i$. The players have control on the first component, namely $\rho_i\in(\mathbb{R}_0^+)^3$, whereas the second one, i.e. $w_i\in(\mathbb{R}_0^+)^3$, is controlled by an adversarial disturbance. In the mean-field limit, the corresponding macroscopic model is described by the following Kolmogorov equations:

$$\begin{aligned}\dot{x}_1 &= x_3\beta_{31}-x_1\beta_{13},\\ \dot{x}_2 &= x_3\beta_{32}-x_2\beta_{23},\\ \dot{x}_3 &= x_1\beta_{13}+x_2\beta_{23}-x_3\beta_{31}-x_3\beta_{32}.\end{aligned} \tag{14}$$

We study the mean-field response for a reference player, and model the microscopic dynamics under the assumption that the population distribution over the time horizon is given. The state of the reference player is denoted by $i\in\mathcal{I}^3$ and its evolution is described via a continuous-time Markov chain, depicted in Fig. 6 (top). The reference player chooses the transition rates in order to minimize a cost, which consists of two components: a running cost and a terminal penalty. The running cost is defined as to match the conditions in [57] to ensure contractivity of the mean field game, namely $g(i,x,\rho_i)=\frac{1}{2}\sum_{j\neq i}\rho_{ij}^2R_{ij}+f_i(x)$, where $\rho_i=[\rho_{i1}\,\rho_{i2}\,\rho_{i3}]^\top$, ρ_{ij} is the transition rate and $R_{ij}>0$. Let the terminal penalty be defined as $\psi(i,x):\mathcal{I}^3\times\mathcal{S}^3\to\mathbb{R}$, and assume it is Lipschitz continuous in $x\in\mathcal{S}_3$. The cost functional that each player seeks to minimize is:

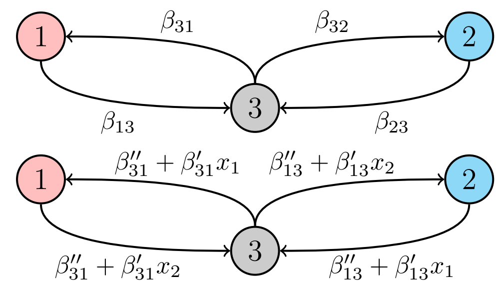

Fig. 6. Markov chain representations corresponding to the macroscopic model in (14) (top), and to the specialised honeybee model in (25) (bottom), describing the transition rates for each model.

$$\begin{aligned}&L_x^i(\rho,w,t)\\ &\quad=\mathbb{E}^{\rho,w}_{i_t=i}\Bigg[\int_t^\top\Big[g(i_\tau,x(\tau),\rho_{i_\tau})-\frac{1}{2}\sum_{j\neq i_\tau}w^2_{i_\tau j}\Gamma_{i_\tau j}\Big]d\tau\\ &\quad+\psi(i_T,x(T))\Bigg],\end{aligned} \tag{15}$$

where $\Gamma_{ij}>0$ and $\mathbb{E}^{\rho,w}_{i_t=i}$ is the expectation for the event $i_t=i$. The positive term on the control vector $\rho_i=[\rho_{i1},\rho_{i2},\rho_{i3}]^\top\in(\mathbb{R}_0^+)^3$, and the negative term on the disturbance vector $w_i=[w_{i1},w_{i2},w_{i3}]^\top\in(\mathbb{R}_0^+)^3$ give the functional the structure of a robust mean-field game in spirit with H_∞-optimal control, see [58].

Given $x(t):[0,T]\to\mathcal{S}^3$, the problem for the reference player is to find the optimal control which minimises the cost functional as:

$$v_i(x,t)=\inf_{\rho_i}\sup_{w_i}L_x^i(\rho_i,w_i,t), \tag{16}$$

where $v_i(x,t)$ is the value function, in the rest denoted also $v_i(t)$ or v_i for brevity, and the minimization is performed over the Markovian controls for the reference player control problem, i.e. $\beta_{ij_\tau}=\rho_{ij_\tau}+w_{ij_\tau}$, $i\neq j_\tau$.

The corresponding Hamiltonian function is given by

$$\mathcal{H}(x,v,i)=\inf_{\rho_i}\sup_{w_i}g(\cdot)-\frac{1}{2}\sum_{j\neq i}w_{ij}^2\Gamma_{ij}+(\rho_i+w_i)^\top\Delta_i v. \tag{17}$$

where $\Delta_i v=(v_1-v_i,v_2-v_i,v_3-v_i)^\top$ is the difference operator on i. Because of the superlinearity and uniform convexity of the cost function $g(\cdot)$, the function

$$\eta_i^*(x,v,i)=\operatorname*{argmin}_{\rho_i}\sup_{w_i}\ g(\cdot)-\frac{1}{2}\sum_{j\neq i}w_{ij}^2\Gamma_{ij}+(\rho_i+w_i)^\top\Delta_i v$$

is well defined and its solution is unique. We can now introduce the Hamilton-Jacobi-Bellman ODEs as in the following:

$$\begin{cases} -\dot{v}_i = \mathcal{H}(x, v, i), \\ v_i(T) = \psi(i, x(T)). \end{cases} \tag{18}$$

A system of coupled ODEs with a terminal condition like the one in (18) is referred to as terminal value problem. When v is the value function associated with the distribution x, then optimal Markovian control is $\beta_i^* = \rho_i^* + w_i^*$:

$$\rho_i^* = -R_i^{-1}\Big[\Delta_i v\Big]^- = -\begin{bmatrix} R_{i1}^{-1}(v_1 - v_i)^- \\ R_{i2}^{-1}(v_2 - v_i)^- \\ R_{i3}^{-1}(v_3 - v_i)^- \end{bmatrix}, \tag{19}$$

$$w_i^* = \Gamma_i^{-1}\Big[\Delta_i v\Big]^+ = \begin{bmatrix} \Gamma_{i1}^{-1}(v_1 - v_i)^+ \\ \Gamma_{i2}^{-1}(v_2 - v_i)^+ \\ \Gamma_{i3}^{-1}(v_3 - v_i)^+ \end{bmatrix}, \tag{20}$$

where $R_i = \text{diag}(R_{ij})$ and $\Gamma_i = \text{diag}(\Gamma_{ij})$, for $j = 1, 2, 3$. For values where $i = j$, we use the following formulae: $\rho_{ii}^*(x, v, i) = -\sum_{j \neq i} \rho_{ij}^*(x, v, i)$ and $w_{ii}^*(x, v, i) = -\sum_{j \neq i} w_{ij}^*(x, v, i)$.

When background players use strategy $\rho*$ and the best response for the reference player is also $\rho*$, we say that this solution is a mean-field Nash equilibrium and we can formulate a consistent micro-macro model by bringing together (14) and (18) as in the following:

$$\dot{x}_i(t) = (1 - x_1 - x_2)\beta_{3i} - x_i\beta_{i3}, \ \forall i \in \mathcal{I}^2 \tag{21a}$$

$$\dot{x}_3(t) = -\dot{x}_1 - \dot{x}_2, \tag{21b}$$

$$-\dot{v}_i(t) = \mathcal{H}(x, \Delta_i v, t), \ \forall i \in \mathcal{I}^3 \tag{21c}$$

$$x(0) = x_0, \tag{21d}$$

$$v_{i_T}(T) = \psi(i_T, x(T)). \tag{21e}$$

The above system is called *initial-terminal value problem* (ITVP) for the mean-field game. The macroscopic dynamics are modelled by (14), while the microscopic best response of each player is given by (18).

Let us consider the collective decision-making problem in honeybee swarms as in in [59]. The proposed mean-field model is a more general case of the classic bio-inspired collective decision-making, when the parameters are chosen in an appropriate way. Indeed, let the difference in the value function between the uncommitted state and state 1 and 2 be given by (19) as:

$$\rho_3^* = -\begin{bmatrix} v_1 - v_3 \\ v_2 - v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_{31}'' - \beta_{31}' x_1 \\ \beta_{31}'' - \beta_{31}' x_2 \\ 0 \end{bmatrix},$$

when $R_{31}, R_{32} = 1$. Likewise, from (20), we get:

$$w_1^* = \begin{bmatrix} 0 \\ 0 \\ \Gamma_{13}^{-1}(v_3 - v_1)^+ \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \beta_{13}'' + \beta_{13}' x_2 \end{bmatrix}, \tag{22}$$

where the second equality is obtained by setting

$$\Gamma_{13} = (\beta_{31}'' + \beta_{31}' x_1)/(\beta_{13}'' + \beta_{13}' x_2). \tag{23}$$

coincide with the transition rates from 3 to 1 and 2, respectively, Furthermore, when $R_{31}, R_{32} = 1$, from (19), we have

$$\rho_3^* = -\begin{bmatrix} v_1 - v_3 \\ v_2 - v_3 \\ 0 \end{bmatrix} = \begin{bmatrix} \beta_{31}'' - \beta_{31}' x_1 \\ \beta_{31}'' - \beta_{31}' x_2 \\ 0 \end{bmatrix}. \tag{24}$$

We provide a physical interpretation of the above parameters and link it to the symmetric case studied in [60], whose model can be rewritten in light of the previous calculations as:

$$\begin{cases} \dot{x}_1 = x_3(x_1\beta_{31}' + \beta_{31}'') - x_1(x_2\beta_{13}' + \beta_{13}''), \\ \dot{x}_2 = x_3(x_2\beta_{31}' + \beta_{31}'') - x_2(x_1\beta_{13}' + \beta_{13}''), \\ \dot{x}_3 = -\dot{x}_1 - \dot{x}_2, \end{cases} \tag{25}$$

where β_{3i}' represents the so-called *waggle dance*, which is used by the scout bees to convince others to choose their option i; β_{3i}'' considers the spontaneous commitment of bees to option i; β_{i3}' models the so-called *cross-inhibitory stop signal* that bees committed to an option different from i use to stop the waggle dance of bees committed to i; and β_{i3}'' considers those bees that spontaneously leave their commitment to option i. The corresponding Markov chain is depicted in Fig. 6 (bottom).

Finally, we derive a microscopic model with a finite number of players interacting by means of a network topology. This model approximate system (14), of which system (25) represents the evolutionary dynamics of the honeybee swarm model. To avoid confusion between the number of the state and the number of the player, we denote the probability of player i to be in states 1, 2 or 3 as r_i, s_i, z_i, respectively. The networked model involves a population of N players, where each player corresponds to a node of the network. The interaction topology is described by a fully connected, undirected and complete communication graph $G = \{\mathcal{V}, \mathcal{E}\}$, with adjacency matrix A, i.e.

$$A = \frac{1}{N-1}\left(\mathbf{1}_N \mathbf{1}_N^\top - I_N\right).$$

Similarly to what done before, we indicate with β_{ij}' the linear transition rates and with β_{ij}'' the constant transition rates. The model describing the time evolution of the players' state is given by the following system of equations:

$$\begin{cases} \dot{r}_i = -\beta_{13}' r_i \sum_{j=1}^{N} a_{ij} s_j + \beta_{31}' z_i \sum_{j=1}^{N} a_{ij} r_j \\ \qquad -\beta_{13}'' r_i + \beta_{31}'' z_i. \\ \dot{s}_i = -\beta_{23}' s_i \sum_{j=1}^{N} a_{ij} r_j + \beta_{32}' z_i \sum_{j=1}^{N} a_{ij} s_j \\ \qquad -\beta_{23}'' s_i + \beta_{32}'' z_i, \\ \dot{z}_i = \beta_{13}' r_i \sum_{j=1}^{N} a_{ij} s_j - \beta_{31}' z_i \sum_{j=1}^{N} a_{ij} r_j \\ \qquad + \beta_{23}' s_i \sum_{j=1}^{N} a_{ij} r_j - \beta_{32}' z_i \sum_{j=1}^{N} a_{ij} s_j \\ \qquad + \beta_{13}'' r_i - \beta_{31}'' z_i + \beta_{23}'' s_i - \beta_{32}'' z_i, \end{cases} \tag{26}$$

We are ready to study the impact of the interaction topology to the collective decision-making process, in the case

$$\beta'_{31} = k\beta'_{13},\ \beta''_{31} = k\beta''_{13},\ \beta'_{32} = k\beta'_{23},\ \beta''_{32} = k\beta''_{23}, \quad (27)$$

where k is a parameter that corresponds to a measure of the connectivity in terms of the strength of the waggle dance and the committed states. It can be proved that all the players converge to the same asymptotically stable equilibrium point defined as

$$\bar{r} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{s} = \frac{k}{2k+1}\mathbf{1}_N, \quad \bar{z} = \frac{1}{2k+1}\mathbf{1}_N \quad (28)$$

if the following condition holds true:

$$\nu = \frac{\beta''_{13}\beta''_{23}}{\beta'_{13}\beta'_{23}} > \frac{k}{4(k+1)}.$$

Remark 1: It is possible to formalise a problem for a differential game for system (26) similar to the initial-terminal value problem studied before, with the main difference that the control parameters are linked by k.

Example 1: Given $N = 20$ players and $N = 20$ random initial conditions such that $r_i(0) + s_i(0) + z_i(0) = 0$, let

$$\begin{aligned} \beta'_{13} &= 1,\ \beta''_{13} = 1,\ \beta'_{23} = 1, \\ \beta''_{23} &= 1,\ \beta'_{31} = 0.5,\ \beta''_{31} = 0.5, \\ \beta'_{32} &= 0.5,\ \beta''_{32} = 0.5,\ k = 0.5. \end{aligned}$$

The networked system (26) has an asymptotically stable equilibrium given by (28) as:

$$\bar{r} = 0.25\,\mathbf{1}_N, \quad \bar{s} = 0.25\,\mathbf{1}_N, \quad \bar{z} = 0.5\,\mathbf{1}_N.$$

Figure 7 depicts the time evolution of the states for $N = 20$ players. Each player is represented by 3 lines, one for each state, of the same color: dashed lines correspond to $r_i(t)$, dashed-dotted lines to $s_i(t)$ and solid lines to $z_i(t)$, a triple for each player.

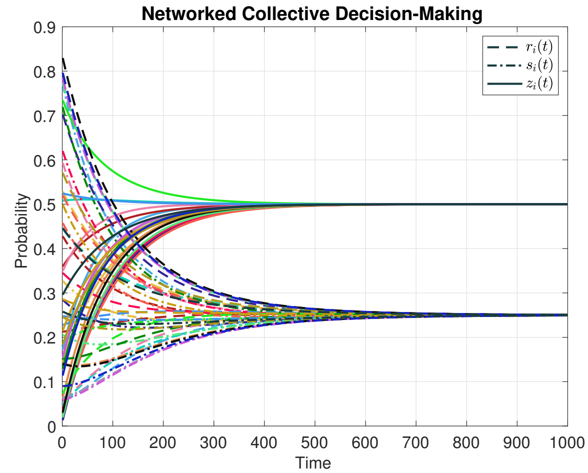

Fig. 7. Evolution of the networked system (26). $N = 20$ players are considered, with different initial conditions $r_i(0) + s_i(0) + z_i(0) = 0$. An identical color corresponds to the same player for the dashed line, $r_i(t)$, the dashed-dotted line, $s_i(t)$, and the solid line, $z_i(t)$.

VII. Conclusions

We have presented a brief tutorial on risk-aware control and game problems in engineering. These risk-aware problems have been presented by means of the mean-field-type control and game theory. We have shown that the solution for such problems can be obtained from an emerging backward-forward partial integro-differential system that is composed of both the Hamilton-Jacobi-Bellman and Fokker-Plank equations, which are coupled. Furthermore, we have presented several applications, i.e., the propagation control of the COVID-19 in a region and /or several coupled regions by means of a simple SEIRD-based model, electric vehicles, and bio-inspired collective decision-making.

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