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Taking into account form variations in polyhedral approach in tolerancing analysis

Denis Teissandier[a,*], Vincent Delos[b], Santiago Arroyave-Tobon[a], Yann Ledoux[a]

[a]Univ. Bordeaux, I2M, UMR 5295, F-33400 Talence, France
[b]CNRS, I2M, UMR 5295, F-33400 Talence, France

* corresponding author. Tel.: +33-(0)55-684-6393. *E-mail address:* denis.teissandier@u-bordeaux.fr

Abstract

The general scope of tolerancing analysis with polytopes, more precisely by polytopes in double description: with half-spaces (half-spaces= constrains) and with vertex (vertex= extremal displacement), consists in modeling the displacement restrictions of a surface in a tolerance zone or between two surfaces of two distinct parts potentially in contact.
Initially these works were developed to simulate the variations of ideal surfaces with no form variation. This article presents an original methodology to overcome this limitation. The evolution of the polytope topology and the associated quantification will be presented. This approach will be applied on double descriptions.

Keywords: Tolerancing analysis; form defects; polytopes

1. Introduction

Modeling geometrical variations induced by manufacturing processes with sets of constraints was initiated at the end of the eighties [1]. In general, this kind of approach is suitable to define the position restrictions of a toleranced surface inside a tolerance zone as well as the position restrictions between two surfaces potentially in contact [2–5].
In this way, all the possible extreme positions of the feature according to the geometry of the boundary are considered simultaneously by domains, T-maps and polytopes.
Domains are defined from nonlinear constraints and T-maps are defined from areal coordinates: both take into account the exact nominal surfaces and generate volumes in a deviation space: these volumes do not have a linear boundary. A review of some of these methods is presented in [6].
Polytopes are defined from linear constraints coming from the discretization of features' boundaries and are always conformed into 6d polyedral objects [7].
By intersections and/or Minkowski sums on these sets of constraints, it is possible to determine the relative position between any couple of surfaces from a mechanical system [7–9]. The main advantage of this strategy is the ability to simulate over-constrained systems. Some works have been proposed to solve these operations by determinist methods or reliability methods [10–12].
Historically, these works are based on the following physical hypotheses: no defect in the shape of the surfaces, no local deformations on surfaces in contact and no flexible parts.
This paper focuses on sets of constrains defined by finite sets of half-spaces in the 6 dimensional deviation space $\mathbb{R}^6$. In general, these sets are polyhedra of $\mathbb{R}^6$. By limiting the degrees of freedom with some cap half-spaces these polyhedra can be conformed to 6-dimensional polytope [13]. This strategy allows to manipulate only bounded objects in dimension 6 to facilitate the computations of the required operations (Minkowski sums, intersections).

In a first section, this paper will define the correlation between the geometrical properties of a toleranced feature

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or a joint with the topology of the corresponding polytope. The main properties will be stated. This first part of the study is dedicated to surfaces with no form defects.
In a second section, the form defects of the manufactured surfaces will be taken into account in the polytope definition. Then, the main properties of the polytope resulting from a set of constraints will be analysed.
We will conclude on the main consequences of the implementation of such a problem with or without the consideration of the form defects.

2. Modeling the geometrical variations by HV-polytopes

In geometrical tolerancing, the H-description of a polytope P is deduced from the geometrical restrictions of a surface inside a tolerance zone or between two surfaces potentially in contact. In general, the H-description of P is defined by the intersection of a finite set of n half-spaces $\overline{H_i^-}$ according eq. (1) and eq. (2), [14].

$$\overline{H_i^-} = \left\{ \mathbf{x} \in \mathbb{R}^6 : a_{i1}.x_1 + \ldots + a_{i6}.x_6 \le b_i \right\} \quad (1)$$

$$\begin{aligned} P &= \bigcap_{i=1}^{n} \overline{H_i^-} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^6 : \mathbf{a}_i^T.\mathbf{x} \le b_i,\ i = 1,\ldots,n \right\} \\ &= \left\{ \mathbf{x} \in \mathbb{R}^6 : \mathbf{A}.\mathbf{x} \le \mathbf{b} \right\}, \mathbf{A} \in \mathbb{R}^{n\times 6}, \mathbf{b} \in \mathbb{R}^n \end{aligned} \quad (2)$$

Each half-space $\overline{H_i^-}$ derives from the limitation of the translation $\mathbf{t}_{Ni}$ of a surface 1 at a point N_i along its local normal $\mathbf{n}_i$ with respect to surface 2, see eq. (3), [13]:

$$\begin{aligned} \mathbf{t}_{Ni}.\mathbf{n}_i \le b_i &\Leftrightarrow \left(\mathbf{t}_M + N_iM \times \mathbf{r} \right)\mathbf{n}_i \le b_i \\ &\Leftrightarrow a_{i1}.x_1 + \ldots + a_{i6}.x_6 \le b_i \\ \text{with} : \mathbf{r} = \left(x_1, x_2, x_3\right)^T, &\mathbf{t}_M = \left(x_4, x_5, x_6\right)^T \end{aligned} \quad (3)$$

$\mathbf{r}$ and $\mathbf{t}_M$ are respectively the rotation and the translation vector expressed in point M of surface 1 with respect to surface 2 respectively. M is rigidly attached to point N_i.
b_i is the local distance between the surfaces 1 and 2 at point N_i along the normal $\mathbf{n}_i$.
The points N_i are the nodes of the mesh derived from the discretization of the surface 1 to generate a finite a linear set of half-spaces $\overline{H_i^-}$.
The extremal displacements of surface 1 with respect to surface 2 are obtained by the computation of the intersection of the n half-spaces $\overline{H_i^-}$ by a truncation algorithm [7,15]. The extremal displacements are the vertices v_k of the polytope P. The vertices v_k define the convex hull of P. The theorem of Minkowski-Weyl stated that these two definitions are equivalent, see eq. (4).

$$P = \bigcap_{i=1}^{n} \overline{H_i^-} \Leftrightarrow P = Conv\left\{v_k\right\}, Conv : \text{convex hull} \quad (4)$$

The HV-description allows the definition of a set of constrains and its resulting extremal displacements in a polyhedral model P by managing the dependencies between:

- the half-spaces and the geometrical constrains,
- the vertices and the extremal positions.

We use an algorithm that gives in practice good results even if its complexity remains theoretically unknown as stated by Fukuda [16]: "*the double description method is a simple and useful algorithm for enumerating all extreme rays of a general polyhedral cone in $\mathbb{R}^n$, despite the fact that we can hardly state any interesting theorems on its time and space complexities*."

This property is illustrated in Fig. 1 in the particular case of a pin 1 in contact with a hole 2 assuming $b_i = c\ \ \forall i \in \left(1,\ldots,n\right)$.

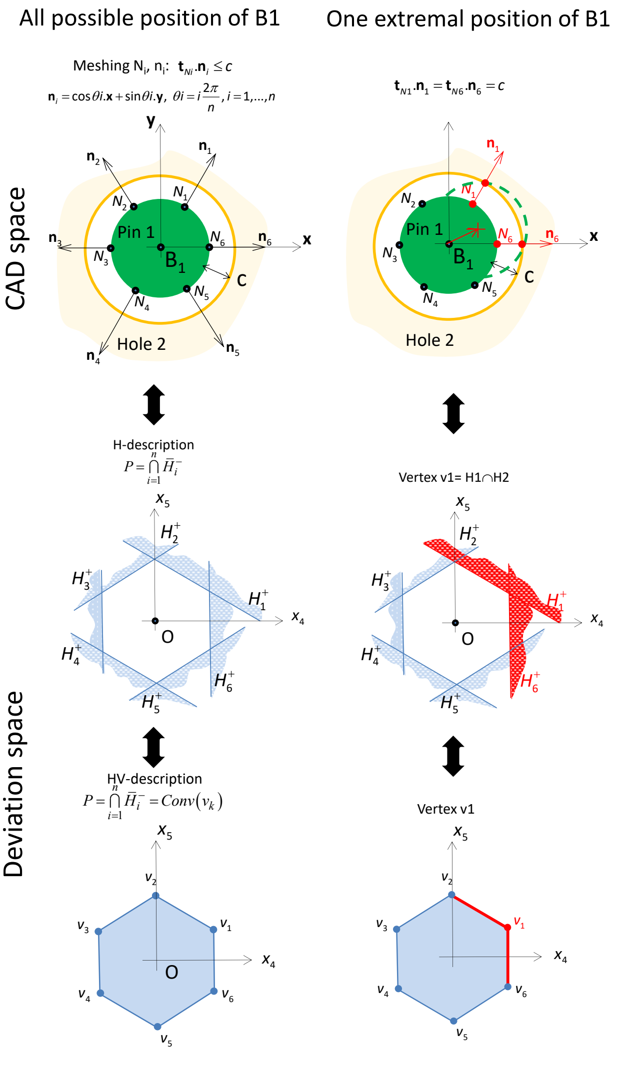

Fig. 1. Dependencies between geometrical constrains, extremal position, half-spaces and vertices.

The contact type is ball-and-cylinder and the associated polytope P is defined according to eq.(5).

$$P = \bigcap_{i=1}^{n} \overline{H_i^-} = \bigcap_{i=1}^{6} \left\{ \mathbf{t}_{Ni}.\mathbf{n_i} \le c \right\} \quad i \in (1,\ldots,6) \quad (5)$$

In this case, the dimension of the polytope P is 2. P defines

all the position of B1 (center of the pin 1 inside the hole 2). Only the translations along x-axis (x_4) and y-axis (x_5) are bounded; the 4 other displacements (x_1, x_2, x_3, x_6) are non-bounded. In order to manipulate only bounded polyhedron in $\mathbb{R}^6$, one method consists in limiting each non-bounded displacement with two cap half-spaces [13].
The adjunction of two half-spaces to limit an unbounded displacement consists in:

- increasing the dimension of the polytope by 1,
- increasing the number of facets by 2,
- multiplying the number of vertices by 2.

For example, Fig. 2 corresponds to the polytope P of the Fig. 1 in $\mathbb{R}^3$ (increase of the dimension from 2 to 3 and added caps)

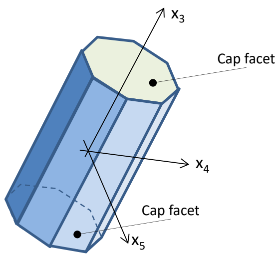

Fig. 2. Polytope of dimension 2 bounded by two cap half-spaces in $\mathbb{R}^3$

Let d be the dimension of the affine subspace of the bounded displacement of a surface. d depends on the degree of freedom of a joint or the invariance degree of a surface, see Fig. 3.

Class of toleranced surface	Kinematic joint	d
Spherical surface	Spherical pair	3
Plane surface	Planar pair	3
Cylindrical surface	Cylindrical pair	4
Surface of revolution	Revolute pair	5
Prismatic surface	Prismatic pair	5
Complex surface	N/A	6
N/A	Ball-and-cylinder pair	2
N/A	Cylinder-and-plane pair	2
N/A	Ball-and-plane pair	1

Fig. 3. Dimension of the sub affine space of the bounded displacements according the class or surface and the kinematic joint.

Let n_H be the number of facets of a polytope P associated to surface or a joint in the affine subspace of the bounded displacement.
Let n_V be the number of vertices of a polytope P associated to surface or a joint in the affine subspace of the bounded displacement. Then the HV-description of P in $\mathbb{R}^6$ has:

$n_H + 2(6-d)$ facets,

$n_V.2^{(6-d)}$ vertices.

3. Enumeration of the number of facets and vertices of a polytope according to the geometric properties of a surface with no form variations

A facet of a polytope P is included in a hyperplane H_i, the boundary of $\bar{H}_i^-$, and n_H is such as: $n_H \leq n$ because of the redundancy of some half-spaces; n depends on the number of the nodes of the mesh.

Let $\bar{H}_g^-$ be a half-space such a facet of P is included in H_g. $\bar{H}_g^-$ is called a generator half-space. A generator half-space derives from a restriction of the translation of surface 1 with respect to surface 2 at a point N_i along its local normal $\mathbf{n}_i$, such that N_i can reach the limit of the local restriction when surface 1 reaches an extremal position with respect to surface 2.

The number of generator half-spaces $\bar{H}_g^-$ is equal to n_H.

With no form variations, the second members b_i are constant (see eq.(3)). The second members depend on the clearance (contact constrains) or on the tolerance (geometrical constrains). For example, in Fig. 1: $b_i = c, \forall i \in \{1,....,n\}$.

Let us consider a polytope of dimension 3 defining the position of a planar surface inside a tolerance zone. If the contour of the plane is rectangular, the polytope is an octahedron (Fig. 4) defined through 8 generator half-spaces.

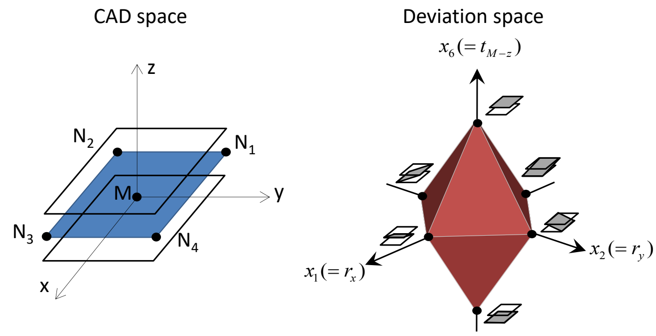

Fig. 4. Generator half-spaces of a polytope associated to a rectangular plane surface.

In this case each vertex of the polytope is generated by 4 facets, i.e. 4 generator half-spaces. These 4 generator half-spaces derive from 4 points $N_i, i = 1,...,4$. The 4 points N_i, $i = 1,...,4$ are located on the boundary of the tolerance zone simultaneously for each extremal position of the planar surface inside the tolerance zone.
If the toleranced surface can be decomposed into convex subsets, it is possible to determine or to estimate an upper bound of n_H.

If the toleranced surface is a plane surface, the generator half-spaces are induced by points located on the contour line of the plane. This can be justified by the fact that a plane and its tolerance zone are convex subsets. The points located on the convex hull of the contour produce the generator half-spaces. If the convex hull is a polygon, then

n_H is the number of vertices of the polygon multiplied by 2. If the convex hull is not a polygon, it is necessary to discretize the curves in a finite number of points with a mesh algorithm. In the particular case of the plane surface with a circular contour line discretized in k nodes evenly distributed $n_H = 2k$, see Fig. 4.
If k is an even number, the number of vertices is equal to $k+2$. The meshed contour has a center of symmetry, the number of vertices is then minimal.
If k is an odd number, the number of vertices is equal to $2k+2$. The meshed contour has no center of symmetry and the number of vertices is maximal.

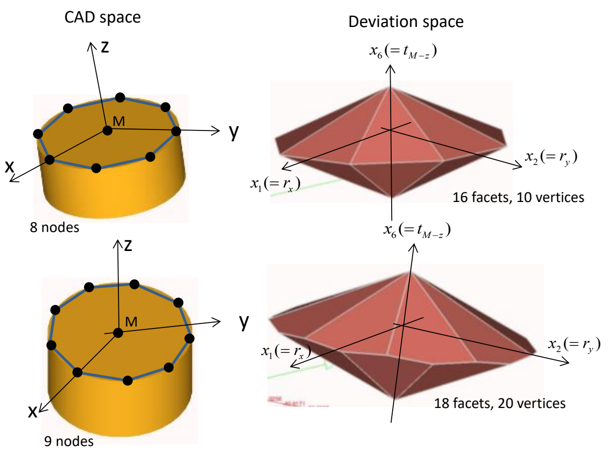

Fig. 5. Generator half-spaces of a polytope associated to a circular plane surface.

Finally, the enumeration of the facets and vertices of polytope associated to a planar surface is summarized in Fig. 6 by taking into account the adjunction of 6 cap half-spaces from $\mathbb{R}^3$ to $\mathbb{R}^6$.

Property of the contour line	$\mathbb{R}^3$	$\mathbb{R}^6$
Circular contour line, k nodes	2k facets k even : k+2 vertices k odd : 2k+2 vertices	2k + 6 facets k even : 8(k+2) vertices k odd : 8(2k+2) vertices
Convex hull with a center of symmetry, k nodes	2k facets k+2 vertices	2k + 6 facets 8(k+2) vertices
Convex hull with no center of symmetry, k nodes	2k facets nb of vertices $\leq$ 2k+2	2k + 6 facets nb of vertices ≤ 8(2k+2)

Fig. 6. Enumeration of the topologic properties of a polytope associated to a plane surface.

The inclusion of a cylindrical surface inside a tolerance zone made up of two coaxial cylinders is equivalent to the inclusion of its axis inside a cylinder. The axis of the surface is modeled as a straight segment [B_1, T_1], i.e. a convex subset of the CAD space.
The intersection between polytopes P_{B1} and P_{T1} define the geometrical constrains induced by the inclusion of the straight segment [B_1, T_1] inside a cylinder $\varnothing t$ according to eq. (6), see Fig. 7:

$$\begin{aligned} P_{B1} &= \bigcap_{i=1}^{n} \bar{H}_{B1i} = \left\{ \mathbf{t}_{B1}.\mathbf{n_i} \leq \frac{t}{2},\ i=1,\ldots,n \right\} \\ P_{T1} &= \bigcap_{i=1}^{n} \bar{H}_{T1i} = \left\{ \mathbf{t}_{T1}.\mathbf{n_i} \leq \frac{t}{2},\ i=1,\ldots,n \right\} \end{aligned} \quad (6)$$

The definitions of P_{B1} and P_{T1} are similar to the definition of polytope P in Fig. 1.

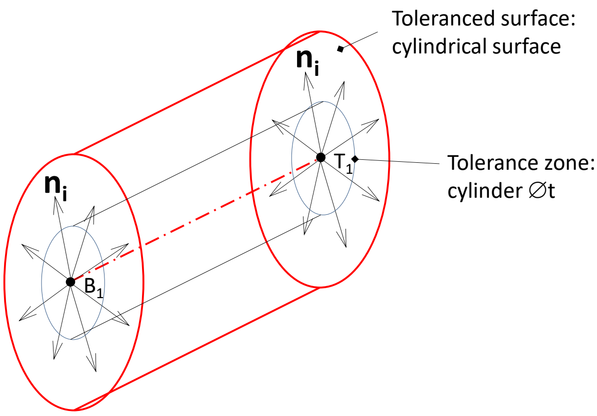

Fig. 7. Geometrical constrains associated to a cylindrical surface.

In this case, the determination of n_H, the number of generator half-space of $P_{B1} \cap P_{T1}$ is trivial: $n_H = 2n$.
The number of vertices of $P_{B1} \cap P_{T1}$ is n^2.
Finally, the enumeration of the facets and the vertices of $P_{B1} \cap P_{T1}$ are summarized in Fig. 8 according to (6).

$\mathbb{R}^3$	$\mathbb{R}^6$
2n facets n^2 vertices	2n + 4 facets $4n^2$ vertices

Fig. 8. Enumeration of the topologic properties of a polytope associated to a cylindrical surface.

The polytope illustrated in Fig. 1 defines the contact constrains of a ball-and-cylinder pair. In this case, the enumeration of the facets and the vertices are summarized in Fig. 9.

$\mathbb{R}^3$	$\mathbb{R}^6$
n facets n vertices	n + 8 facets 16n vertices

Fig. 9. Enumeration of the topologic properties of a polytope associated to a ball-and-cylinder pair

The identification of the generator half-spaces allows adapting the mesh on the CAD model of the tolerance surface in order to generate an optimized initial H-description before to launch the truncation algorithm to compute the HV-description of the polytope.
Therefore, it reduces the computational time of the vertices of the polytope with the truncation algorithm implemented in the open source politopix solver [15].
Furthermore, the mesh can be adapted on the CAD model by taking into account the class of the surface or the type of kinematic joint.

The properties stated on planar surface and cylindrical surface are valid on planar pair and cylindrical pair with no restriction.

4. Taking into account of the defect of form in the H-description

Let us consider the same joint defined in Fig. 1 between pin 1 and hole 2 with form variations, see Fig. 10.

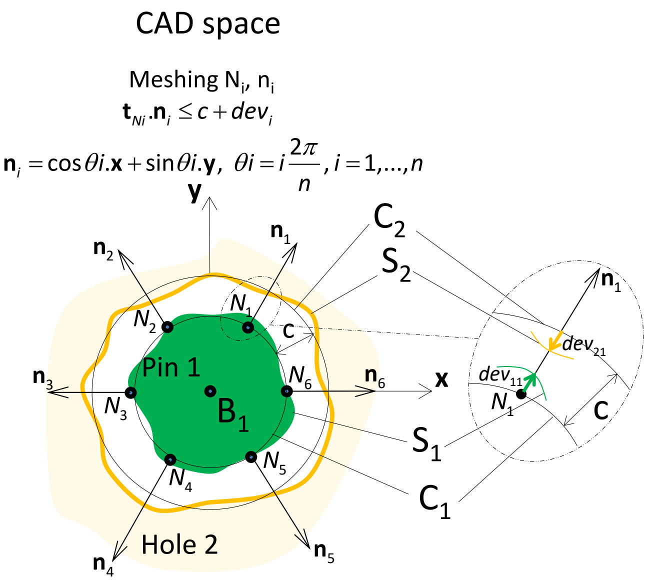

Fig. 10. Geometrical constrains associated to a ball-and-cylinder pair.

The profiles of pin 1 and hole 2 are not circular anymore. Therefore, the distance between these two profiles is not constant anymore. A circle C_1 has been fitted on the profile of pin 1 according the least square criterion. A circle C_2 has been fitted on the profile of hole 2 according the least square criterion. These two circles allow filtering of the form deviations from the other deviations (dimension and position). These contact constrains in Fig. 1 has been defined between these two circles.

In this case the polytope P modeling the relative position between the two profiles at point B_1 is defined by eq. (7):

$$P = \bigcap_{i=1}^{n} \bar{H}_i^- = \left\{\mathbf{t}_{Ni}.\mathbf{n}_i \le c + dev_i,\ i = 1,\ldots,n\right\} \tag{7}$$

dev_i is the contributive form variation of the profiles of pin 1 and hole 2 at point N_i along $\mathbf{n}_i$ according to eq.(8).

$$\begin{aligned}
&dev_i = dev_{2i} - dev_{1i}\\
&\text{with:}\\
&dev_{2i} > 0 : \text{profile outside } C_2\\
&dev_{2i} < 0 : \text{profile inside } C_2\\
&dev_{1i} > 0 : \text{profile outside } C_1\\
&dev_{1i} < 0 : \text{profile inside } C_1
\end{aligned} \tag{8}$$

A graphical representation of P is given in fig. 11.

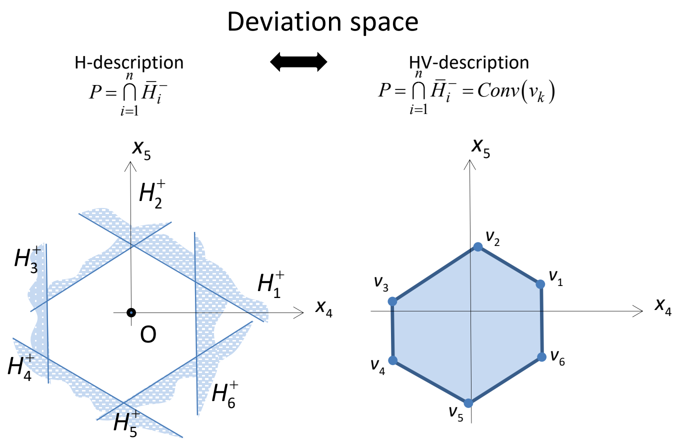

Fig. 11. Polytope P associated to a ball-and-cylinder pair with form variations.

Only the second members of the half-spaces $\bar{H}_i^-$ are different between the H-description of P according to (5) and (7). Therefore, the half-spaces of (7) result from translations of half-spaces of (5) respectively according to $dev_i.\mathbf{n_i}$. These translations are induced by the form variations. One consequence is the convex hull of the vertices is not a regular polygon, even if the topology remains invariant. If the form deviations of S_1 or S_2 have the same order of magnitude as the distance c , the topology of P could change.

If $dev_i \ll c$, the enumeration of the facets and the vertices remain identical to the results presented in Fig . 9. If not, some half-spaces $\bar{H}_i^-$ will not be generator half-spaces.

Let us consider a cylindrical surface with form variation, see Fig. 12. The real axis is not a convex subset. It is necessary to define several sets of constrains in several orthogonal planes to the axis of the tolerance zone to ensure the inclusion of the real axis in the tolerance zone. In this case it is not possible to estimate the number of generate half-spaces and the number of vertices of the polytope associated to this surface. The number of half-spaces of the H-description is much larger than the same surface with no form variations.

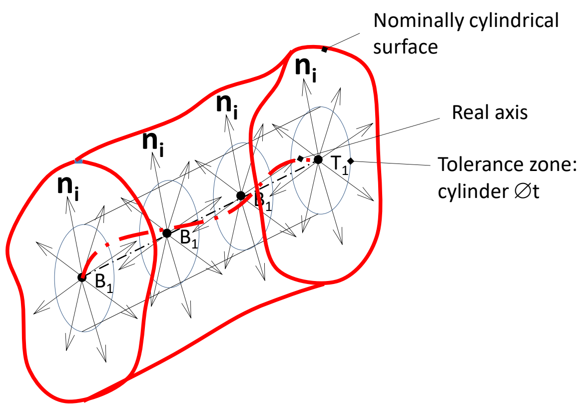

Fig. 12. Generation of the H-description of a cylindrical surface with form variation.

5. Discussion and Conclusion

This paper examined the entire process of determining a double-description polytope (HV-polytope) simulating positional variations between two surfaces. This process is suitable for the simulation of the position of a surface inside a tolerance zone as well as the relative position between two surfaces potentially in contact.

The concept of generator half-spaces has been introduced to ensure the complete traceability between the extremal displacements and the vertices of the polytope one the one hand and the points on the CAD model inducing the generator half-spaces on the other hand.

For planar and cylindrical surfaces with no form variations, the number of facets and the number of vertices have been estimated according the number of discretisation points. Furthermore, the locations of points inducing generator half-spaces have been identified. It allows optimizing the meshing process of the CAD model and the computation process of an operand polytope. In addition, it makes it possible to estimate the complexity of a tolerance analysis problem by operations (intersections and Minkowski sums) on polytopes.

Finally in the last part, the main properties of a polytope defining the positional variations between two surfaces with form variations have been defined. This shows that the estimation of the number of facets and vertices is very complex in general cases.

The next step of this work is to correlate the number of discretisation points of the CAD model with the precision of a simulation with or without the integration of form variations.

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