Mathematics Education with Conjectures and Refutations

Gianni ARIOLI
Department of Mathematics, Politecnico di Milano
Milano, 20133, Italy

ABSTRACT

In this paper we discuss how Popper's epistemology and Lakatos' view on the development of mathematics may provide some interesting ideas to mathematics educators in both primary and secondary education. We present a teaching strategy which can complement more traditional approaches and we illustrate it with an example.

Keywords: Mathematics education, Popper, Lakatos.

1. INTRODUCTION

In two previous papers [1, 2] we discussed the issue of problem solving, which, according to many scholars in mathematics education, is a central theme of the discipline. We recall a thought of George Polya [7]: "Solving problems means finding a way out of a difficulty, a way to get around an obstacle, to achieve a goal that is not readily accessible. Solving problems is a specific feature of intelligence, and the intelligence is the specific gift of mankind: you can consider the problem-solving activities as the most peculiar feature of the human race..... Then, a math teacher has a great opportunity. Obviously, if he will use his lesson hours to have his students perform calculations, he will end up crushing their interest, slowing down their mental development and wasting the opportunities that present themselves. Instead, if he arouses the curiosity of the students proposing problems of difficulty proportionate to their knowledge and he helps them to solve the problems by asking appropriate questions, he will be able to inspire in them a taste for original reasoning". Following Polya, many researcher in mathematics education have reiterated and elaborated the importance of proposing challenging problems and situations taken from real life to the students. Unfortunately, however, the teaching in our schools mostly compresses the students in the mere execution of repetitive task. Usually teachers, when facing a new topic in the mathematics program, show on the blackboard the procedure and then require the students to solve similar problems in the same way. It is then necessary here to introduce the distinction between problem and exercise. The student carries out an exercise when he merely applies rules previously outlined and exhaustively covered by the teacher. Therefore, the solution is just the repetition of procedures already seen. A teaching-oriented problem solving promotes the cognitive construction of mathematical concepts and skills, and stimulates rational, logical and strategic learners. When the teacher submits to the pupils, individually or as group, a task for which there is no procedure immediately accessible, they are faced with the problem of finding a solution. They cannot use skills already acquired or apply concepts already learned. They must resort to creative thinking to formulate new hypotheses, draw on their knowledge, exploit heuristic methods. Furthermore, the pupil that is forced to find a solution to the problem proposed by the teacher, proceeds by trial and error. More precisely, the pupil must think to some strategy, some "theory" that he can/must test to see if it can be used to achieve the solution of the problem. If it does not, then he/she must start over with a new strategy, or possibly by adapting the previous strategy.

2. POPPER'S EPISTEMOLOGY

This "trial and error" approach to learning mathematics presents a strong resemblance with Popper's critical rationalism [8, 9]. He wrote that scientific knowledge is made of conjectures, is fallible, falsifiable and scientific research can only be based on falsification, that is on conjectures and attempts to refute them. More precisely, in [9] he writes: "The method of science is the method of bold conjectures and ingenious and severe attempts to refute them". We certainly will not enter the details of Popper's epistemology, but we are interested in deriving from it some interesting ideas and suggestions that can be exploited in science and in particular in mathematics education. Popper wrote: "…critical method... It is a method of trial and the elimination of errors, of proposing theories and submitting them to the severest tests we can design". Studying mathematics, physics, chemistry, biology..., scientists try to come closer to the truth: "what we attempt in science is to describe and (so far as possible) explain reality. We do so with the help of conjectural theories; that is, theories which we hope are true (or near the truth), but which we cannot establish as certain or even though they are the best theories which we are able to produce, and may therefore be called probable, as long as this term is kept free from any association with the calculus of probability. Since we are looking for the truth, but we cannot achieve it, Popper introduces the idea of verisimilitude: "the verisimilitude of a statement will be explained as increasing with its truth content and decreasing with its falsity content". Verisimilitude can be used as a criterion to choose among different scientific theories; in fact, we can never state the absolute truth of any of these, but we can perform always stricter tests in order to discard theories that do not pass the tests and therefore treat the surviving theory as the best approximation of the truth. Thus, aim and method of science are linked by the idea of the approximation of the truth: science aims at increasing the verisimilitude and its method is the rational way to achieve it.

3. LAKATOS' VIEW ON THE METHODOLOGY OF MATHEMATICS

A similar idea has been adopted by Lakatos in his three essays (later collected in his seminal book [6]) on the methodology of mathematics. He wrote: "I use the word 'methodology' in a sense akin to Polya's and Bernays' "heuristic" and Popper's "logic of discovery"." And also: "Their (the essays) modest aim is to elaborate the point that informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations." The essays are written as a Socratic dialogue involving a group of students who debate the proof of the Euler characteristic for the polyhedron. Lakatos points out that mathematical definitions are not immutable, but often need to be adapted in the light of later insights or failed proofs. Lakatos summarizes his ideas on mathematical discoveries by writing in [6, Appendix 1] that they consist of the following stages: primitive conjecture, proof, counterexamples, re-examination of the proof. This gives mathematics a somewhat experimental flavour.

4. TEACHING WITH CONJECTURE AND REFUTATIONS

In this paper we are not concerned about discovering new mathematics, but about teaching well established results. But we claim that Popper's and Lakatos' approach to scientific discoveries can be very effective also in mathematics education. Here we present a possible plan of laboratory activities based on conjectures and refutations, to complement the traditional teaching in the classroom, and to serve as a methodological basis. The plan is easily adaptable to both primary and secondary education. Any computer lab or classroom with recent PCs and equipped with internet access and a browser suffices to realise our proposal. The work can be done by the teacher of mathematics (organiser and manager) and his class without the need for any special technical skills other than the ability to open a website and download a plugin. A lesson in the laboratory can last an hour or even less, depending on the subject matter, and each topic can be dealt with separately from the others. Therefore, it is possible to program in a flexible way the number of lessons. Here is the outline of the proposal.

1. Planning. The teacher chooses a problem or rather a problematic situation to address. The teacher, as the person who promotes the initiative, plans and directs the activity, and plays a central and indispensable role. First of all, the teacher cannot ignore the effect, also in this specific field of educational technologies, of the didactic transposition, the complex transition from mathematical knowledge to know to mathematical knowledge to teach. He/she has the task of adapting the mathematical knowledge (Savoir Savant) in order to transform it into what Chevellard [3] calls "knowledge to be taught" and then, learned by the pupils. He/she must be aware that the learning environment, which is made with the integration of a specific instrument such as the symbolic computation software, requires a rethinking of mathematical problems to be proposed to the students.

2. Introduction. The teacher introduces or recalls to his students the basic knowledge needed to cope with the process of resolution of the problem under consideration.

3. Brainstorming. Formulation of the theory. The teacher presents the problem to be solved to the students, encouraging them to make free use of all their resources (knowledge, heuristics, strategic and creative application of the theorems and methods of calculation). At this point, each member of the class group is directly responsible of the burden of solving the problematic situation. He/she analyses the situation by breaking it down into its various elements or aspects. He/she makes assumptions, processes possible solving frameworks, creates theories. All this is the result of his/her creative thinking. In this problem-solving process, the student is influenced by cognitive, meta-cognitive and emotional factors. In fact, he/she puts in play all his/her knowledge regarding the topic under study; he/she makes use of the strategies that he/she knows (heuristics); takes decisions about how to manage the resources at his/her disposal; assesses the degree of difficulty of the task and compares it with his/her ability to deal with it; has his/her own personal convictions (on him/herself, on the issue under study, on the objectives which arise in carrying out this activity, on mathematics in general). Once he/she formulates one or more attempts to find a solution and/or an explanation of the problem/phenomenon under investigation, he/she applies the method of critical discussion, namely the method of trial and elimination of error correction, or by conjecture, refutation and self-correction (auto-correction). Thus, he/she shifts from the stage of development of bold conjectures to that immediately and necessarily consequent stage, which consists in subjecting these conjectures to ingenious and severe attempts to refute them.

4. Experimentation. The students uses the software to test his/her theory. The interaction with the software facilitates the acquisition by the student of the competence in mathematics, because it allows him/her to see and experience that specific topic in a quick, yet accurate and complete fashion.

5. Socialization. After individual experimentation of the problem through testing carried out by the software, the teacher assumes the role of moderator and starts the dialectical confrontation among the students, who discuss together their different theories and experiments: what is the best theory that explains the problem? or, what are the principles which govern and define the particular phenomenon under study? Also during this analysis the students gradually develop their mathematical competence, that is, one might say, they learn to look at reality with the eyes of mathematics, because they pay attention and think about quantitative data and the causal and logical connections that exist between the constituent elements of that problem or that phenomenon.

6. Institutionalization. At the end of the discussion, when the personal knowledge built by each student individually has been debated with the class and, therefore, has become a common knowledge shared by the whole class, the teacher summarises the results and institutionalise the knowledge achieved. That is, he/she reassembles into an orderly and exhaustive framework, rigorously reconstructing all the steps and logical links, the knowledge collectively produced and shared by the students. By virtue of what Brousseau calls "didactic contract", they feel entitled to consider official their cognitive achievement.

7. Checking. The teacher analyses the experiences and feedback collected during the activity to evaluate its effects on the participants (the results in terms of increased cognitive background and critical/rational skills, but also the impact on the affective/motivational sphere) in relation to targets that had been set from the beginning.

5. EXAMPLE

In order to explain how this approach can help students learn mathematics, we consider an example based on a very famous puzzle, the Monty Hall problem, see e.g. [10]. Suppose you're on a game show, and you are given the choice of three doors. Behind one door is a car, behind the others, goats. You pick a door, say #1, and the host, who knows what's behind the doors, opens another door, say #3, which has a goat. He says to you, "Do you want to pick door #2?" Is it to your advantage to switch your choice of doors? We think that this is a good example because the statement is very simple and can be understood even by a kid in primary school, it is easy to play the game and see the results, it is easy to build an applet that simulate the game, and yet the logic behind the solution may not be too easy to grasp. We built an applet with Mathematica to test the possibilities of the educational method that we are proposing. This applet can be embedded in any web page, and only requires a free plug-in to be used, thus it is very easily set up for use in a computer room in any school. To make things more interesting and easier to grasp at the same time, we may ask the pupils the following question: Assume you play the game 100 times, and you always choose the strategy of sticking to your original choice, how many times do you expect to win the car? In this way, we focus on a specific problem without referring to the calculus of probability at all. The pupils are invited to formulate a theory, and then the applet allows them to test it. Here is how the applet works: one chooses a number between 0 and 100, this is the conjecture. Following Popper's method, one does not need to observe anything to formulate the conjecture. At this stage, anything goes. The conjecture is inserted in the applet by moving the slider. Then one can play the game as many times as one wants: each time the button "Play" is clicked, the computer plays 100 times and then displays the result, that is how many times it has won. By playing a few times, the computer collects a sample of the results and can decide whether the conjecture can be accepted or should be refuted.

6. CONCLUSIONS

We think that a teaching strategy based on conjectures and refutation can be an effective complement to more traditional approaches. Here are some advantages.

(1) It is a natural way of proposing problem solving situations, and most mathematics educators agree that that is a sensible approach.

(2) It stimulates the interest and participation of the pupils, and we feel that is something that should always be pursued. In particular, it helps considering mathematics as a useful tool for solving problems, rather than a boring list of rules.

(3) It forces the pupils to think out of the box, enhancing their creativity. This point was particularly stressed by Popper, albeit with different words. He insisted that new theories are not based on observations, but they are creation of the human mind.

(4) It applies to pupils of all ages, from primary school to college.

(5) It only requires a basic computer laboratory, with recent pc's with an internet connection and a browser.

We also think that this strategy presents one difficulty, that is the lack of availability of ready to use applets or similar tools. The planning and writing of an applet (or a full fledged computer application) requires a careful study and some programming skills, it is not a task that can be tackled by a teacher in his/her spare time. This is not a drawback to the whole idea, only a difficulty in the first stages of experimentation. Also, we believe that this strategy is particularly suitable for some specific topic and not for other ones. This point also cannot be viewed as a drawback, since we are certainly not suggesting conjectures and refutations as a main teaching strategy, but as a complementary strategy. In a forthcoming paper we will develop further the idea, presenting more specific topics and corresponding applets. At that point it will be feasible to test this strategy in front of a classroom.

REFERENCES

[1] G. Arioli, Insegnare matematica con Mathematica, L'insegnamento della matematica e delle scienze integrate, 38 (2015) 8-32

[2] G. Arioli, Teaching mathematics with automatic symbolic computation, J. of Appl. Math. and Engineerings 8 (2016) 9-23

[3] Y. Chevellard, La trasposition didactique. Du savoir savant au savoir enseigné, Revue française de pédagogie. 76, (1986). 89-91

[4] W. Derske, Popper's Epistemology as a Pedagogic and Didactic Principle, or: Let Them Make More "Mistakes", Journal of Chemical Education, 58 (1981) 565-567

[5] K. Duncker, Psychologie des produktive Denkens. Berlin: Springer (1935)

[6] I. Lakatos, Proofs and refutations, Cambridge University Press (1976)

[7] G. Polya, How to solve it. Princeton University Press, (1945)

[8] K.R. Popper, Conjectures and refutations: the growth of scientific knowledge. London: Routledge, (1963)

[9] K.R. Popper, Objective Knowledge: An Evolutionary Approach, Oxford University Press, (1979)

[10] Behind Monty Hall's Doors: Puzzle, Debate and Answer, New York Times, July 21, (1991)