Blow-up and global existence for solutions to the porous medium equation with reaction and fast decaying density

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Abstract

We are concerned with nonnegative solutions to the Cauchy problem for the porous medium equation with a variable density $\rho(x)$ and a power-like reaction term u^p with $p > 1$. The density decays *fast* at infinity, in the sense that $\rho(x) \sim |x|^{-q}$ as $|x| \to +\infty$ with $q \geq 2$. In the case when $q = 2$, if p is bigger than m, we show that, for large enough initial data, solutions blow-up in finite time and for small initial datum, solutions globally exist. On the other hand, in the case when $q > 2$, we show that existence of global in time solutions always prevails. The case of *slowly* decaying density at infinity, i.e. $q \in [0, 2)$, is examined in [41].

1 Introduction

We investigate global existence and blow-up of nonnegative solutions to problem

$$\begin{cases} \rho(x)u_t = \Delta(u^m) + \rho(x)u^p & \text{in } \mathbb{R}^N \times (0, \tau) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \times \{0\} \end{cases} \tag{1.1}$$

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where $N \geq 3, u_0 \in L^\infty(\mathbb{R}^N), u_0 \geq 0, \rho \in C(\mathbb{R}^N), \rho > 0, p > 1, m > 1$ and $\tau > 0$. We always assume that

$$\begin{cases} \text{(i)} \ \rho \in C(\mathbb{R}^N), \ \rho > 0 \text{ in } \mathbb{R}^N, \\ \text{(ii)} \ u_0 \in L^\infty(\mathbb{R}^N), u_0 \geq 0 \text{ in } \mathbb{R}^N, \end{cases} \qquad (H)$$

and that

there exist $k_1, k_2 \in (0, +\infty)$ with $k_1 \leq k_2, r_0 > 0, q \geq 2$ such that

$$k_1(|x| + r_0)^q \leq \frac{1}{\rho(x)} \leq k_2(|x| + r_0)^q \quad \text{for all } x \in \mathbb{R}^N. \qquad (1.2)$$

The parabolic equation in problem (1.1) is of the *porous medium* type, with a variable density $\rho(x)$ and a reaction term $\rho(x)u^p$. Clearly, such parabolic equation is degenerate, since $m > 1$. Moreover, the differential equation in (1.1) is equivalent to

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \qquad \text{in } \mathbb{R}^N \times (0, \tau);$$

thus the related diffusion operator is $\frac{1}{\rho(x)}\Delta$, and in view of (1.2), the coefficient $\frac{1}{\rho(x)}$ can positively diverge at infinity. The differential equation in (1.1), posed in the interval $(-1, 1)$ with homogeneous Dirichlet boundary conditions, has been introduced in [34] as a mathematical model of evolution of plasma temperature, where u is the temperature, $\rho(x)$ is the particle density, $\rho(x)u^p$ represents the volumetric heating of plasma. Indeed, in [34, Introduction] a more general source term of the type $A(x)u^p$ has also been considered; however, then the authors assume that $A \equiv 0$; only some remarks for the case $A(x) = \rho(x)$ are made in [34, Section 4]. Then in [32] and [33] the Cauchy problem (1.1) is dealt with in the case without the reaction term $\rho(x)u^p$.

In view of (1.2) the density ρ decays at infinity. Indeed,

$$\frac{1}{k_2(|x| + r_0)^q} \leq \rho(x) \leq \frac{1}{k_1(|x| + r_0)^q} \quad \text{for all } |x| > 1, \qquad (1.3)$$

with

$$q \geq 2.$$

Since we assume (1.2) with $q \geq 2$, we refer to $\rho(x)$ as a *fast decaying density* at infinity. On the other hand, in [41] it is studied problem (1.1) with a *slowly decaying density,* that is (1.2) is assumed with $q < 2$.

There is a huge literature concerning various problems related to (1.1). For instance, problem (1.1) with $\rho \equiv 1, m = 1$ is studied in [5, 6, 13, 14, 20, 24,

26, 35, 48, 52, 54, 58], problem (1.1) without the reaction term u^p is treated in [11, 12, 16, 17, 18, 27, 28, 29, 30, 31, 32, 33, 34, 45, 21, 49, 50, 51]. Moreover, problem (1.1) with $m = 1$ is addressed in [36] (see also [10]), where ρ satisfies (1.3) with $0 \le q < 2$. In particular, let us recall some results established in [53] for problem (1.1) with $\rho \equiv 1, m > 1, p > 1$ (see also [15, 44]). We have:

- ([53, Theorem 1, p. 216]) For any $p > 1$, for all sufficiently large initial data, solutions blow-up in finite time;

- ([53, Theorem 2, p. 217]) if $p \in \left(1, m + \frac{2}{N}\right)$, for *all* initial data, solutions blow-up in finite time;

- ([53, Theorem 3, p. 220]) if $p > m + \frac{2}{N}$, for all sufficiently small initial data with compact support, solutions exist globally in time and belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$.

Similar results for quasilinear parabolic equations, also involving p-Laplace type operators or double-nonlinear operators, have been stated in [1], [2], [3], [7], [8], [9], [22], [23], [37], [38], [39], [42], [43], [46], [55], [59] (see also [40] for the case of Riemannian manifolds); moreover, in [19] the same problem on Cartan-Hadamard manifolds has been investigated. In particular, in [37, Theorem 2] it is shown that if $\rho(x) = (1 + |x|)^{-q}$ with $0 < q < 2$, $p > m$, and u_0 is small enough (in an appropriate sense), then there exists a global solution; moreover, a smoothing estimate is given. Such result will be compared below with one of our results (see Remark 2.5).

In [41] the following results for problem (1.1) are established, assuming (1.2) with $0 \le q < 2$.

- ([41, Theorem 2.1]. If
$$p > \bar{p},$$
u_0 has compact support and is small enough, then there exist global in time solutions to problem (1.1) which belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$; here $\bar{p}$ is a certain exponent, which depends on N, m, q, k_1, k_2. In particular, for $k_1 = k_2$ we have
$$\bar{p} = m + \frac{2 - q}{N - q}.$$

- ([41, Theorem 2.3]). For any $p > 1$, if u_0 is sufficiently large, then solutions to problem (1.1) blow-up in finite time.

- ([41, Corollary 2.4]). If $1 < p < m$, then for any $u_0 \not\equiv 0$, solutions to problem (1.1) blow-up in finite time. Moreover ([41, Theorem 2.5]), if $m \le p < \underline{p}$, where $\underline{p} < \bar{p}$ is a certain exponent depending on N, m, q, k_1, k_2, then solutions

to problem (1.1) blow-up in finite time for any nontrivial initial datum. For $k_1 = k_2$, $\underline{p} = \bar{p}$.

Analogous results, proved by different methods, can be found also in [37, 38], where also more general double-nonlinear operators are treated.

1.1 Outline of our results

Let us now describe our main results. We distinguish between two cases: $q = 2$ and $q > 2$. First, assume that (1.2) holds with $q = 2$.

- (Theorem 2.1). If
$$p > m,$$
  u_0 has compact support and is small enough, then there exist global in time solutions to problem (1.1), which belong to $L^\infty(\mathbb{R}^N \times (0, +\infty))$;

- (Theorem 2.2). For any $p > m$, if u_0 is sufficiently large, then solutions to problem (1.1) blow-up in finite time.

The proofs mainly relies on suitable comparison principles and properly constructed sub- and supersolutions, which crucially depend on the behavior at infinity of the inhomogeneity term $\rho(x)$. More precisely, they are of the type

$$w(x,t) = C\zeta(t) \left[1 - \frac{\log(|x| + r_0)}{a} \eta(t) \right]_+^{\frac{1}{m-1}} \quad \text{for any } (x,t) \in \left[\mathbb{R}^N \setminus B_1(0) \right] \times [0, T),$$
(1.4)

for suitable functions $\zeta = \zeta(t)$, $\eta = \eta(t)$ and constants $C > 0$, $a > 0$. The presence of $\log(|x| + r_0)$ in w is strictly related to the assumption that $q = 2$. Observe that the barriers used in [41] for the case $0 \le q < 2$, which are of power type in $|x|$, do not work in the present situation. Furthermore, note that the exponent $\bar{p}$ introduced in [41] for $0 \le q < 2$, when $q = 2$ becomes $\bar{p} = m$. Hence Theorem 2.1 can be seen as a generalization of [41, Theorem 2.1] to the case $q = 2$.

Now, assume that $q > 2$. We have the following results (see Theorem 2.3 and Remark 2.4).

- Let $1 < p < m$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$ there exist global in time solutions to problem (1.1). We do not assume that u_0 has compact support, but we need that it fulfills a decay condition as $|x| \to +\infty$. However, u_0 in a compact subset of $\mathbb{R}^N$ can be arbitrarily large. We cannot deduce that the corresponding solution belongs to $L^\infty(\mathbb{R}^N \times (0, +\infty))$, but it is in $L^\infty(\mathbb{R}^N \times (0, \tau))$ for each $\tau > 0$.

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- Let $p > m \geq 1$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$, problem (1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$. We need that

$$0 \leq u_0(x) \leq CW(x) \quad \text{for all} \quad x \in \mathbb{R}^N,$$

where $C > 0$ is small enough and $W(x)$ is a suitable function, which vanishes as $|x| \to +\infty$. We should mention that, as recalled above, a similar result was been obtained in [37, Theorem 2], where also double-non linear operators are treated; see Remark 2.5 below.

- Let $p = m > 1$. Then for suitable $u_0 \in L^\infty(\mathbb{R}^N)$, problem (1.1) admits a solution in $L^\infty(\mathbb{R}^N \times (0, +\infty))$, provided that $r_0 > 0$ in (1.2) is big enough.

Such results are very different with respect to the cases $0 \leq q < 2$ and $q = 2$. In fact, we do not have finite-time blow-up, but global existence prevails, for suitable initial data. The results follow by comparison principles, once we have constructed appropriate supersolutions, that have the form

$$w(x, t) = \zeta(t)W(x) \quad \text{for all} \quad (x, t) \in \mathbb{R}^N \times (0, +\infty),$$

for suitable $\zeta(t)$ and $W(x)$. When $p \geq m$, $\zeta(t) \equiv 1$. Observe that we can also include the linear case $m = 1$, whenever $p > m$. In this respect, our result complement the results in [36], where only the case $q < 2$ is addressed. Finally, let us mention that it remains to be understood whether in the case $1 < p < m$ solutions can blow-up in infinite time or not.

2 Statements of the main results

For any $x_0 \in \mathbb{R}^N$ and $R > 0$ we set

$$B_R(x_0) = \{x \in \mathbb{R}^N : \|x - x_0\| < R\}.$$

When $x_0 = 0$, we write $B_R \equiv B_R(0)$.

For the sake of simplicity, sometimes instead of (2.1), we suppose that

there exist $k_1, k_2 \in (0, +\infty)$ with $k_1 \leq k_2$, $q \geq 2$, $R > 0$ such that

$$k_1|x|^q \leq \frac{1}{\rho(x)} \leq k_2|x|^q \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_R. \tag{2.1}$$

In view of (H)-(i),

for any $R > 0$ there exist $\rho_1(R), \rho_2(R) \in (0, +\infty)$ with $\rho_1(R) \leq \rho_2(R)$

such that $\rho_1(R) \leq \dfrac{1}{\rho(x)} \leq \rho_2(R) \quad \text{for all} \quad x \in \overline{B_R}. \tag{2.2}$

Obviously, (1.2) is equivalent to (2.1) and (2.2).

In the sequel we shall refer to q as the order of decaying of $\rho(x)$ as $|x| \to +\infty$.

2.1 Order of decaying: $q = 2$

Let $q = 2$. The first result concerns the global existence of solutions to problem (1.1) for $p > m$. We assume that

$$r_0 > e, \quad \frac{k_2}{k_1} < (N-2)(m-1)\frac{p-m}{p-1}\log r_0. \tag{2.3}$$

Theorem 2.1. *Assume (H), (1.2) for $q = 2$ and (2.3). Suppose that*

$$p > m,$$

and that u_0 is small enough and has compact support. Then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$.
More precisely, if $C > 0$ is small enough, $a > 0$ is so that

$$0 < \omega_0 \leq \frac{C^{m-1}}{a} \leq \omega_1$$

for suitable $0 < \omega_0 < \omega_1$, $T > 0$,

$$u_0(x) \leq CT^{-\frac{1}{p-1}}\left[1 - \frac{\log(|x|+r_0)}{a}T^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \quad \textit{for any } x \in \mathbb{R}^N, \tag{2.4}$$

then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$. Moreover,

$$u(x,t) \leq C(T+t)^{-\frac{1}{p-1}}\left[1 - \frac{\log(|x|+r_0)}{a}(T+t)^{-\frac{p-m}{p-1}}\right]_+^{\frac{1}{m-1}} \textit{for any } (x,t) \in \mathbb{R}^N \times (0, +\infty). \tag{2.5}$$

Observe that if u_0 satisfies (2.4), then

$$\operatorname{supp} u_0 \subseteq \{x \in \mathbb{R}^N : \log(|x|+r_0) \leq aT^{\frac{p-m}{p-1}}\}.$$

From (2.5) we can infer that

$$\operatorname{supp} u(\cdot, t) \subseteq \{x \in \mathbb{R}^N : \log(|x|+r_0) \leq a(T+t)^{\frac{p-m}{p-1}}\} \quad \text{for all } t > 0. \tag{2.6}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 4.2.

The next result concerns the blow-up of solutions in finite time, for every $p > m > 1$, provided that the initial datum is sufficiently large. We assume that hypothesis (2.1) holds with the choice

$$q = 2, \quad R = e. \tag{2.7}$$

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So we fix, in assumption (2.2),

$$\rho_1(R) = \rho_1(e) =: \rho_1 , \quad \rho_2(R) = \rho_2(e) =: \rho_2 .$$

Let

$$\mathfrak{s}(x) := \begin{cases} \log(|x|) & \text{if} \quad x \in \mathbb{R}^N \setminus B_e, \\[2mm] \dfrac{|x|^2 + e^2}{2e^2} & \text{if} \quad x \in B_e . \end{cases}$$

Theorem 2.2. *Let assumption (H), (2.1) and (2.7). For any*

$$p > m$$

and for any $T > 0$, if the initial datum u_0 is large enough, then the solution u of problem (1.1) blows-up in a finite time $S \in (0, T]$, in the sense that

$$\|u(t)\|_\infty \to \infty \ \text{as} \ t \to S^- . \tag{2.8}$$

More precisely, if $C > 0$ and $a > 0$ are large enough, $T > 0$,

$$u_0(x) \geq CT^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} T^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} , \quad \text{for any } x \in \mathbb{R}^N , \tag{2.9}$$

then the solution u of problem (1.1) blows-up and satisfies the bound from below

$$u(x,t) \geq C(T-t)^{-\frac{1}{p-1}} \left[1 - \frac{\mathfrak{s}(x)}{a} (T-t)^{\frac{m-p}{p-1}} \right]_+^{\frac{1}{m-1}} , \quad \text{for any } (x,t) \in \mathbb{R}^N \times (0, S) . \tag{2.10}$$

Observe that if u_0 satisfies (2.9), then

$$\text{supp}\, u_0 \supseteq \left\{ x \in \mathbb{R}^N \ : \ \mathfrak{s}(x) < aT^{\frac{p-m}{p-1}} \right\} .$$

From (2.10) we can infer that

$$\text{supp}\, u(\cdot, t) \supseteq \left\{ x \in \mathbb{R}^N \ : \ \mathfrak{s}(x) < a(T-t)^{\frac{p-m}{p-1}} \right\} \quad \text{for all } t \in [0, S) . \tag{2.11}$$

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 5.2.

2.2 Order of decaying: $q > 2$

Let $q > 2$. The first result concerns the global existence of solutions to problem (1.1) for any $p > 1$ and $m > 1$, $p \neq m$. Let us introduce the parameter $\bar{b} \in \mathbb{R}$ such that

$$0 < \bar{b} < \min\{N - 2, \ q - 2\}. \tag{2.12}$$

Moreover, we can find $\bar{c} > 0$ such that

$$(r + r_0)^{-\frac{\bar{b}p}{m}} \leq \bar{c} \quad \text{for any } r \geq 0, \tag{2.13}$$

with $r_0 > 0$ as in hypothesis (1.2).

Theorem 2.3. *Let assumptions (H), (1.2) and (2.12) be satisfied with $q > 2$. Suppose that*

$$1 < p < m, \quad \text{or } p > m \geq 1,$$

and that u_0 is small enough. Then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, \tau))$ for any $\tau > 0$. More precisely, we have the following cases.

(a) Let $1 < p < m$. If $C > 0$ is big enough, $r_0 > 0$, $T > 1$, $\alpha > 0$,

$$u_0(x) \leq CT^\alpha \left(|x| + r_0\right)^{-\frac{\bar{b}}{m}} \quad \text{for any } x \in \mathbb{R}^N, \tag{2.14}$$

then problem (1.1) admits a global solution u, which satisfies the bound from above

$$u(x, t) \leq C(T + t)^\alpha \left(|x| + r_0\right)^{-\frac{\bar{b}}{m}} \text{ for any } (x, t) \in \mathbb{R}^N \times (0, +\infty). \tag{2.15}$$

(b) Let $p > m \geq 1$. If $C > 0$ is small enough, $r_0 > 0$, $T > 0$ and (2.14) holds with $\alpha = 0$, then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$, which satisfies the bound from above (2.15) with $\alpha = 0$.

Remark 2.4. *Observe that, in the case when $p = m$, if $C > 0$ is small enough, $r_0 > 0$ big enough to have*

$$\left(\frac{1}{r_0}\right)^{\frac{\bar{b}p}{m}} \leq \bar{b}k_1(N - 2 - \bar{b}),$$

$T > 0$ and (2.14) holds with $\alpha = 0$, then problem (1.1) admits a global solution $u \in L^\infty(\mathbb{R}^N \times (0, +\infty))$ which satisfies the bound from above (2.15) for $\alpha = 0$.

Note that in Theorem 2.3 we do not require that supp u_0 is compact.

The choice of the parameters $C > 0, T > 0$ and $a > 0$ is discussed in Remark 4.4.

Remark 2.5. *The statement in Theorem 2.3-(b) is in agreement with [37, Theorem 2], where it is assumed that $p > m$, $\rho(x) = (1 + |x|)^{-q}$ with $q > 2$, $\int_{\mathbb{R}^N} \rho(x) u_0(x) dx < +\infty$, $\int_{\mathbb{R}^N} \rho(x) [u_0(x)]^{\bar{q}} dx < \delta$, for some $\delta > 0$ small enough and $\bar{q} > \frac{N}{2}(p - m)$.*

Note that the assumption on u_0 is of a different type. In particular, in view of (2.14) and (2.12), the initial datum u_0 considered in Theorem 2.3-(b) not necessarily satisfies $\int_{\mathbb{R}^N} \rho(x) u_0(x) dx < +\infty$.

In [37] the proofs are based on the energy method, so they are completely different with respect to our approach.

3 Preliminaries

In this section we give the precise definitions of solutions of all problems we address, then we state a local in time existence result for problem (1.1). Moreover, we recall some useful comparison principles. The proofs of such auxiliary results can be found in [41, Section 3].

Throughout the paper we deal with *very weak* solutions to problem (1.1) and to the same problem set in different domains, according to the following definitions.

Definition 3.1. *Let $u_0 \in L^\infty(\mathbb{R}^N)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a solution of problem (1.1) if*

$$
\begin{aligned}
-\int_{\mathbb{R}^N} \int_0^\tau \rho(x) u \varphi_t \, dt \, dx = &\int_{\mathbb{R}^N} \rho(x) u_0(x) \varphi(x, 0) \, dx \\
&+ \int_{\mathbb{R}^N} \int_0^\tau u^m \Delta \varphi \, dt \, dx \\
&+ \int_{\mathbb{R}^N} \int_0^\tau \rho(x) u^p \varphi \, dt \, dx
\end{aligned}
\tag{3.1}
$$

for any $\varphi \in C_c^\infty(\mathbb{R}^N \times [0, \tau))$, $\varphi \geq 0$. Moreover, we say that a nonnegative function $u \in L^\infty(\mathbb{R}^N \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.1) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ " with $\varphi \geq 0$.

For every $R > 0$, we consider the auxiliary problem

$$
\begin{cases}
u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p & \text{in } B_R \times (0, \tau) \\
u = 0 & \text{on } \partial B_R \times (0, \tau) \\
u = u_0 & \text{in } B_R \times \{0\} .
\end{cases}
\tag{3.2}
$$

Definition 3.2. *Let $u_0 \in L^\infty(B_R)$ with $u_0 \geq 0$. Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a solution of problem (3.2) if*

$$- \int_{B_R} \int_0^\tau \rho(x)u\,\varphi_t\,dt\,dx = \int_{B_R} \rho(x)u_0(x)\varphi(x,0)\,dx$$
$$+ \int_{B_R} \int_0^\tau u^m \Delta\varphi\,dt\,dx \qquad (3.3)$$
$$+ \int_{B_R} \int_0^\tau \rho(x)u^p\varphi\,dt\,dx$$

for any $\varphi \in C_c^\infty(\overline{B_R} \times [0, \tau))$ with $\varphi|_{\partial B_R} = 0$ for all $t \in [0, \tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(B_R \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.3) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ ", with $\varphi \geq 0$.

Proposition 3.3. *Let hypothesis (H) be satisfied. Then there exists a solution u to problem (3.2) with*

$$\tau \geq \tau_R := \frac{1}{(p-1)\|u_0\|_{L^\infty(B_R)}^{p-1}}.$$

Moreover, the following comparison principle for problem (3.2) holds (see [4] for the proof).

Proposition 3.4. *Let assumption (H) hold. If u is a subsolution of problem (3.2) and v is a supersolution of (3.2), then*

$$u \leq v \quad a.e. \ in \ B_R \times (0, \tau).$$

Proposition 3.5. *Let hypothesis (H) be satisfied. Then there exists a solution u to problem (1.1) with*

$$\tau \geq \tau_0 := \frac{1}{(p-1)\|u_0\|_\infty^{p-1}}.$$

Moreover, u is the minimal solution, *in the sense that for any solution v to problem (1.1) there holds*

$$u \leq v \quad in \ \mathbb{R}^N \times (0, \tau).$$

In conclusion, we can state the following two comparison results, which will be used in the sequel.

Proposition 3.6. *Let hypothesis (H) be satisfied. Let $\bar{u}$ be a supersolution to problem (1.1). Then, if u is the minimal solution to problem (1.1) given by Proposition 3.5, then*

$$u \leq \bar{u} \quad a.e. \ in \ \mathbb{R}^N \times (0, \tau) \,. \tag{3.4}$$

In particular, if $\bar{u}$ exists until time τ, then also u exists at least until time τ.

Proposition 3.7. *Let hypothesis (H) be satisfied. Let u be a solution to problem (1.1) for some time $\tau = \tau_1 > 0$ and $\underline{u}$ a subsolution to problem (1.1) for some time $\tau = \tau_2 > 0$. Suppose also that*

$$\operatorname{supp} \underline{u}|_{\mathbb{R}^N \times [0, S]} \ \text{is compact for every} \ S \in (0, \tau_2) \,.$$

Then

$$u \geq \underline{u} \quad in \ \mathbb{R}^N \times (0, \min\{\tau_1, \tau_2\}) \,. \tag{3.5}$$

In what follows we also consider solutions of equations of the form

$$u_t = \frac{1}{\rho(x)} \Delta(u^m) + u^p \quad \text{in} \ \Omega \times (0, \tau), \tag{3.6}$$

where $\Omega \subseteq \mathbb{R}^N$. Solutions are meant in the following sense.

Definition 3.8. *Let $\tau > 0$, $p > 1, m > 1$. We say that a nonnegative function $u \in L^\infty(\Omega \times (0, S))$ for any $S < \tau$ is a solution of problem (3.2) if*

$$
\begin{aligned}
-\int_\Omega \int_0^\tau \rho(x) u \, \varphi_t \, dt \, dx = {} & \int_\Omega \int_0^\tau u^m \Delta \varphi \, dt \, dx \\
& + \int_\Omega \int_0^\tau \rho(x) u^p \varphi \, dt \, dx
\end{aligned}
\tag{3.7}
$$

for any $\varphi \in C_c^\infty(\overline{\Omega} \times [0, \tau))$ with $\varphi|_{\partial\Omega} = 0$ for all $t \in [0, \tau)$. Moreover, we say that a nonnegative function $u \in L^\infty(\Omega \times (0, S))$ for any $S < \tau$ is a subsolution (supersolution) if it satisfies (3.3) with the inequality " $\leq$ " (" $\geq$ ") instead of " $=$ ", with $\varphi \geq 0$.

Finally, let us recall the following well-known criterion, that will be used in the sequel. Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$, and that $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$ is of class C^1. Let n be the unit outwards normal to Ω_1 at Σ. Let

$$u = \begin{cases} u_1 & \text{in} \ \Omega_1 \times [0, T), \\ u_2 & \text{in} \ \Omega_2 \times [0, T) \,, \end{cases} \tag{3.8}$$

where $\partial_t u \in C(\Omega_1 \times (0, T))$, $u_1^m \in C^2(\Omega_1 \times (0, T)) \cap C^1(\overline{\Omega}_1 \times (0, T))$, $\partial_t u_2 \in C(\Omega_2 \times (0, T)))$, $u_2^m \in C^2(\Omega_2 \times (0, T)) \cap C^1(\overline{\Omega}_2 \times (0, T))$.

Lemma 3.1. *Let assumption (H) be satisfied.*
(i) Suppose that

$$\partial_t u_1 \geq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \text{for any } (x,t) \in \Omega_1 \times (0,T),$$
$$\partial_t u_2 \geq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \text{for any } (x,t) \in \Omega_2 \times (0,T), \tag{3.9}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \geq \frac{\partial u_2^m}{\partial n} \quad \text{for any } (x,t) \in \Sigma \times (0,T). \tag{3.10}$$

Then u, defined in (3.8), is a supersolution to equation (3.6), in the sense of Definition 3.8.
(ii) Suppose that

$$\partial_t u_1 \leq \frac{1}{\rho}\Delta u_1^m + u_1^p \quad \text{for any } (x,t) \in \Omega_1 \times (0,T),$$
$$\partial_t u_2 \leq \frac{1}{\rho}\Delta u_2^m + u_2^p \quad \text{for any } (x,t) \in \Omega_2 \times (0,T), \tag{3.11}$$

$$u_1 = u_2, \quad \frac{\partial u_1^m}{\partial n} \leq \frac{\partial u_2^m}{\partial n} \quad \text{for any } (x,t) \in \Sigma \times (0,T). \tag{3.12}$$

Then u, defined in (3.8), is a subsolution to equation (3.6), in the sense of Definition 3.8.

4 Global existence: proofs

In what follows we set $r \equiv |x|$. We construct a suitable family of supersolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,+\infty). \tag{4.1}$$

4.1 Order of decaying: $q = 2$

We assume (H), (1.2) with $q = 2$ and (2.3). In order to construct a suitable family of supersolutions of (4.1), we define, for all $(x,t) \in \mathbb{R}^N \times (0,+\infty)$,

$$\bar{u}(x,t) \equiv \bar{u}(r(x),t) := C\zeta(t)\left[1 - \frac{\log(r+r_0)}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{4.2}$$

where $\eta, \zeta \in C^1([0,+\infty);[0,+\infty))$ and $C > 0$, $a > 0$, $r_0 > e$.

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p.$$

To this aim, let us set

$$F(r,t) := 1 - \frac{\log(r + r_0)}{a}\eta(t)\,,$$

and define

$$D_1 := \left\{(x,t) \in [\mathbb{R}^N \setminus \{0\}] \times (0, +\infty) \mid 0 < F(r,t) < 1\right\}.$$

For any $(x,t) \in D_1$, we have:

$$\bar{u}_t = C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}F^{\frac{1}{m-1}-1}\left(-\frac{\log(r + r_0)}{a}\eta'\right)$$

$$= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\left(1 - \frac{\log(r + r_0)}{a}\eta\right)\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}$$

$$= C\zeta' F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}.$$

$$(4.3)$$

$$(\bar{u}^m)_r = -\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r + r_0)}\eta.$$

$$(4.4)$$

$$(\bar{u}^m)_{rr} = -\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\left(1 - \frac{\log(r + r_0)}{a}\eta\right)\eta\frac{1}{(r + r_0)^2\log(r + r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r + r_0)^2\log(r + r_0)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r + r_0)^2}\eta$$

$$= -\frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta\frac{1}{(r + r_0)^2\log(r + r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r + r_0)^2\log(r + r_0)} + \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r + r_0)^2}\eta.$$

$$(4.5)$$

$$\Delta(\bar{u}^m) = \frac{(N-1)}{r}(\bar{u}^m)_r + (\bar{u}^m)_{rr}$$

$$= \frac{(N-1)}{r}\left(-\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)}\eta\right)$$

$$- \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta\frac{1}{(r+r_0)^2\log(r+r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta \qquad (4.6)$$

$$\leq \frac{N-1}{r+r_0}\left(-\frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)}\eta\right)$$

$$- \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}}\eta\frac{1}{(r+r_0)^2\log(r+r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{(r+r_0)^2\log(r+r_0)}$$

$$+ \frac{C^m}{a}\zeta^m\frac{m}{m-1}F^{\frac{1}{m-1}}\frac{1}{(r+r_0)^2}\eta$$

We also define

$$K := \left[\left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}}\right] > 0,$$

$$\bar{\sigma}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_1(N-2), \qquad (4.7)$$

$$\bar{\delta}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + C^{m-1}\zeta^m\frac{m}{(m-1)^2}\frac{\eta}{a}\frac{1}{\log(r_0)}k_2,$$

$$\bar{\gamma}(t) := C^{p-1}\zeta^p.$$

Proposition 4.1. *Let $\zeta = \zeta(t)$, $\eta = \eta(t) \in C^1([0,+\infty);[0,+\infty))$. Let K, $\bar{\sigma}$, $\bar{\delta}$, $\bar{\gamma}$ be as defined in (4.7). Assume (H), (1.2) with $q = 2$, (2.3) and that, for all $t \in (0,+\infty)$,*

$$-\frac{\eta'}{\eta^2} \geq \frac{1}{\log(r_0)}\frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}k_2 \qquad (4.8)$$

and

$$\zeta' + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[(N-2)k_1 - \frac{k_2}{(m-1)\log(r_0)}\right] - C^{p-1}\zeta^p \geq 0. \qquad (4.9)$$

then $\bar{u}$ defined in (4.2) is a supersolution of equation (4.1).

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Proof of Proposition 4.1. In view of (4.3), (4.4), (4.5) and (4.6), for any $(x, t) \in D_1$,

$$
\bar{u}_t - \frac{1}{\rho} \Delta(\bar{u}^m) - \bar{u}^p
$$

$$
\geq C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} \frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C\zeta \frac{1}{m-1} \frac{\eta'}{\eta} F^{\frac{1}{m-1}-1}
$$
$$
+ \frac{C^m}{a} \zeta^m \frac{m}{m-1} \eta \frac{1}{\rho(r+r_0)^2} F^{\frac{1}{m-1}} \left(\frac{1}{(m-1)\log(r+r_0)} + N - 2 \right)
$$
$$
- \frac{C^m}{a} \zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1} \frac{\eta}{\log(r+r_0)} \frac{1}{\rho(r+r_0)^2} - C^p \zeta^p F^{\frac{p}{m-1}}.
$$

$$(4.10)$$

Thanks to hypothesis (H), (1.2) and (2.3), we have

$$
\frac{1}{\log(r+r_0)} \geq 0, \quad -\frac{1}{\log(r+r_0)} \geq -\frac{1}{\log(r_0)} \quad \text{for all} \ \ x \in \mathbb{R}^N, \qquad (4.11)
$$

$$
\frac{1}{\rho(r+r_0)^2} \geq k_1, \quad -\frac{1}{\rho(r+r_0)^2} \geq -k_2 \quad \text{for all} \ \ x \in \mathbb{R}^N. \qquad (4.12)
$$

From (4.10), (4.11) and (4.12) we get,

$$
\bar{u}_t - \frac{1}{\rho} \Delta(\bar{u}^m) - \bar{u}^p
$$

$$
\geq C F^{\frac{1}{m-1}-1} \left\{ F \left[\zeta' + \zeta \frac{1}{m-1} \frac{\eta'}{\eta} + \frac{C^{m-1}}{a} \zeta^m \frac{m}{m-1} \eta(N-2)k_1 \right] \right.
$$
$$
\left. -\zeta \frac{1}{m-1} \frac{\eta'}{\eta} - \frac{C^{m-1}}{a} \zeta^m \frac{m}{(m-1)^2} \frac{1}{\log(r_0)} \eta k_2 - C^{p-1} \zeta^p F^{\frac{p+m-2}{m-1}} \right\}
$$

$$(4.13)$$

From (4.13) and (4.7), we have

$$
\bar{u}_t - \frac{1}{\rho} \Delta(\bar{u}^m) - \bar{u}^p \geq C F^{\frac{1}{m-1}-1} \left[\bar{\sigma}(t) F - \bar{\delta}(t) - \bar{\gamma}(t) F^{\frac{p+m-2}{m-1}} \right]. \qquad (4.14)
$$

For each $t > 0$, set

$$
\varphi(F) := \bar{\sigma}(t) F - \bar{\delta}(t) - \bar{\gamma}(t) F^{\frac{p+m-2}{m-1}}, \quad F \in (0, 1).
$$

Now our goal is to find suitable C, a, ζ, η such that, for each $t > 0$,

$$
\varphi(F) \geq 0 \quad \text{for any} \ \ F \in (0, 1).
$$

We observe that $\varphi(F)$ is concave in the variable F. Hence it is sufficient to have $\varphi(F)$ positive in the extrema of the interval $(0, 1)$. This reduces, for any $t > 0$, to the conditions

$$
\varphi(0) \geq 0,
$$
$$
\varphi(1) \geq 0.
$$

$$(4.15)$$

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These are equivalent to

$$-\bar{\delta}(t) \geq 0\,, \quad \bar{\sigma}(t) - \bar{\delta}(t) - \bar{\gamma}(t) \geq 0\,,$$

that is

$$-\frac{\eta'}{\eta^2} \geq \frac{C^{m-1}}{a}\zeta^{m-1}\frac{m}{m-1}\frac{1}{\log(r_0)}k_2\,,$$

$$\zeta' + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\left[(N-2)\,k_1 - \frac{k_2}{(m-1)\log(r_0)}\right] - C^{p-1}\zeta^p \geq 0\,,$$

which are guaranteed by (2.3), (4.8) and (4.9). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in } D_1\,.$$

Now observe that

$$\bar{u} \in C(\mathbb{R}^N \times [0,+\infty))\,,$$

$\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0,+\infty))\,,$ and by the definition of $\bar{u}\,,$

$\bar{u} \equiv 0$ in $[\mathbb{R}^N \setminus D_1] \times [0,+\infty))\,.$

Hence, by Lemma 3.1 (applied with $\Omega_1 = D_1$, $\Omega_2 = \mathbb{R}^N \setminus D_1$, $u_1 = \bar{u}$, $u_2 = 0$, $u = \bar{u}$), $\bar{u}$ is a supersolution of equation

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p = 0 \quad \text{in } (\mathbb{R}^N \setminus \{0\}) \times (0,+\infty)$$

in the sense of Definition 3.8. Thanks to a Kato-type inequality, since $\bar{u}_r^m(0,t) \leq 0$, we can easily infer that $\bar{u}$ is a supersolution of equation (4.1) in the sense of Definition 3.8. $\qquad\square$

Remark 4.2. *Let*

$$p > m$$

and assumption (2.3) be satisfied. Let $\omega := \frac{C^{m-1}}{a}$. In Theorem 2.1 the precise hypotheses on parameters $C > 0$, $\omega > 0$, $T > 0$ are the following:

$$\frac{p-m}{p-1} \geq \omega\frac{m}{m-1}k_2\frac{1}{\log(r_0)}, \tag{4.16}$$

$$\omega\frac{m}{m-1}\left[k_1(N-2) - \frac{k_2}{(m-1)\log(r_0)}\right] \geq C^{p-1} + \frac{1}{p-1}. \tag{4.17}$$

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Lemma 4.1. *All the conditions in Remark 4.2 can be satisfied simultaneously.*

Proof. Since $p > m$ the left-hand-side of (4.16) is positive. By (2.3), we can select $\omega > 0$ so that (4.16) holds and

$$\omega \frac{m}{m-1} \left[k_1(N-2) - \frac{k_2}{(m-1)\log(r_0)} \right] > \frac{1}{p-1} \, .$$

Then we take $C > 0$ so small that (4.17) holds (and so $a > 0$ is accordingly fixed). $\qquad\square$

Proof of Theorem 2.1. We prove Theorem 2.1 by means of Proposition 4.1. In view of Lemma 4.1, we can assume that alla conditions in Remark 4.2 are fulfilled. Set

$$\zeta = (T+t)^{-\alpha} \, , \quad \eta = (T+t)^{-\beta} \, , \quad \text{for all} \quad t > 0 \, .$$

Consider conditions (4.8), (4.9) of Proposition 4.1 with this choice of $\zeta(t)$ and $\eta(t)$. Therefore we obtain

$$\beta - \frac{C^{m-1}}{a} \frac{m}{m-1} k_2 (T+t)^{-\alpha(m-1)-\beta+1} \geq 0 \tag{4.18}$$

and

$$-\alpha(T+t)^{-\alpha-1} + \frac{C^{m-1}}{a} \frac{m}{m-1} \left[k_1(N-2) - \frac{k_2}{(m-1)\log(r_0)} \right] (T+t)^{-\alpha m - \beta}$$
$$- C^{p-1}(T+t)^{-\alpha p} \geq 0 \, . \tag{4.19}$$

We take

$$\alpha = \frac{1}{p-1} \, , \quad \beta = \frac{p-m}{p-1} \, . \tag{4.20}$$

Due to (4.20), (4.18) and (4.19) become

$$\frac{p-m}{p-1} \geq \frac{C^{m-1}}{a} \frac{m}{m-1} \frac{k_2}{\log(r_0)} \, , \tag{4.21}$$

$$\frac{C^{m-1}}{a} \frac{m}{m-1} \left[k_1(N-2) - \frac{k_2}{(m-1)\log(r_0)} \right] \geq C^{p-1} + \frac{1}{p-1} \, . \tag{4.22}$$

Therefore, (4.8) and (4.9) follow from assumptions (4.16) and (4.17). Thus the conclusion follows by Propositions 4.1 and 3.6. $\qquad\square$

4.2 Order of decaying: $q > 2$

We assume (H), (1.2) and (2.12) for $q > 2$ and (2.13). In order to construct a suitable family of supersolutions of (4.1), we define, for all $(x, t) \in \mathbb{R}^N \times (0, +\infty)$,

$$\bar{u}(x, t) \equiv \bar{u}(r(x), t) := C\zeta(t)(r + r_0)^{-\frac{\bar{b}}{m}}; \tag{4.23}$$

where $\zeta \in C^1([0, +\infty); [0, +\infty))$ and $C > 0$, $r_0 > 0$.

Now, we compute

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p.$$

For any $(x, t) \in [\mathbb{R}^N \setminus \{0\}] \times (0, +\infty)$, we have:

$$\bar{u}_t = C\,\zeta'\,(r + r_0)^{-\frac{\bar{b}}{m}}. \tag{4.24}$$

$$(\bar{u}^m)_r = -\bar{b}\,C^m\,\zeta^m\,(r + r_0)^{-\bar{b}-1}. \tag{4.25}$$

$$(\bar{u}^m)_{rr} = \bar{b}\,(\bar{b}+1)\,C^m\,\zeta^m\,(r + r_0)^{-\bar{b}-2}. \tag{4.26}$$

Proposition 4.3. *Let* $\zeta = \zeta(t) \in C^1[0, +\infty); [0, +\infty)), \zeta' \geq 0$. *Assume* (H), (1.2) *and* (2.12) *for* $q > 2$, (2.13), *and that*

$$\bar{b}k_1(N - 2 - \bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p > 0. \tag{4.27}$$

Then $\bar{u}$ *defined in* (4.23) *is a supersolution of equation* (4.1).

Proof of Proposition 4.3. In view of (4.24), (4.25), (4.26) and the fact that

$$\frac{1}{(r + r_0)^{\bar{b}+1}r} \geq \frac{1}{(r + r_0)^{\bar{b}+2}} \quad \text{for any } x \in \mathbb{R}^N,$$

we get, for any $(x, t) \in (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty)$,

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p$$

$$\geq C\zeta'(r + r_0)^{-\frac{\bar{b}}{m}} + \frac{1}{\rho}\left\{(N - 2 - \bar{b})C^m\zeta^m\bar{b}(r + r_0)^{-\bar{b}-2}\right\} - C^p\zeta^p(r + r_0)^{-\frac{\bar{b}p}{m}}. \tag{4.28}$$

Thanks to hypothesis (1.2), (2.12) and (2.13), we have

$$\frac{(r + r_0)^{-\bar{b}-2}}{\rho} \geq k_1(r + r_0)^{-\bar{b}-2+q} = k_1, \tag{4.29}$$

$$-(r + r_0)^{-\frac{\bar{b}p}{m}} \geq -\bar{c}$$

Since $\zeta' \geq 0$, from (4.28) and (4.29) we get

$$\bar{u}_t - \frac{-}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq k_1\bar{b}(N - 2 - \bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p\,. \tag{4.30}$$

Hence we get the condition

$$k_1\bar{b}(N - 2 - \bar{b})C^m\zeta^m - \bar{c}\,C^p\zeta^p \geq 0\,, \tag{4.31}$$

which is guaranteed by (2.12) and (4.27). Hence we have proved that

$$\bar{u}_t - \frac{1}{\rho}\Delta(\bar{u}^m) - \bar{u}^p \geq 0 \quad \text{in}\ \ (\mathbb{R}^N \setminus \{0\}) \times (0, +\infty)\,.$$

Now observe that
$$\bar{u} \in C(\mathbb{R}^N \times [0, +\infty))\,,$$
$$\bar{u}^m \in C^1([\mathbb{R}^N \setminus \{0\}] \times [0, +\infty))\,,$$
$$\bar{u}_r^m(0, t) \leq 0\,.$$

Hence, thanks to a Kato-type inequality we can infer that $\bar{u}$ is a supersolution to equation (4.1) in the sense of Definition 3.8.

$\square$

Remark 4.4. *Let*
$$q > 2$$
and assumption (2.12) be satisfied. In Theorem 2.3 the precise hypotheses on parameters α, $C > 0$, $T > 0$ are as follows.

(a) Let $p < m$. We require that
$$\alpha > 0, \tag{4.32}$$
$$\bar{b}\,k_1(N - 2 - \bar{b})C^m - \bar{c}\,C^p \geq 0 \tag{4.33}$$

(b) Let $p > m$. We require that
$$\alpha = 0, \tag{4.34}$$
$$\bar{b}\,k_1(N - 2 - \bar{b})C^m - \bar{c}\,C^p \geq 0 \tag{4.35}$$

Lemma 4.2. *All the conditions in Remark 4.4 can hold simultaneously.*

Proof. (a) We observe that, due to (2.12),

$$N - 2 - \bar{b} > 0.$$

Therefore, we can select $C > 0$ sufficiently large to guarantee (4.33).
(b) We choose $C > 0$ sufficiently small to guarantee (4.35). $\square$

Proof of Theorem 2.3. We now prove Theorem 2.3 in view of Proposition 4.3. In view of Lemma 4.2 we can assume that all conditions in Remark 4.4 are fulfilled. Set

$$\zeta(t) = (T+t)^\alpha, \quad \text{for all} \quad t \geq 0.$$

Let $p < m$. Inequality (4.27) reads

$$\bar{b}\, k_1(N-2-\bar{b})C^m(T+t)^{m\alpha} - \bar{c}\, C^p(T+t)^{p\alpha} \geq 0 \quad \text{for all} \quad t > 0.$$

This follows from (4.32) and (4.33), for $T > 1$. Hence, by Propositions 4.3 and 3.5 the thesis follows in this case.

Let $p > m$. Conditions (4.34) and (4.35) are equivalent to (4.27). Hence, by Propositions 4.3 and 3.5 the thesis follows in this case too. The proof is complete.

□

5 Blow-up: proofs

In what follows we set $r \equiv |x|$. We construct a suitable family of subsolutions of equation

$$u_t = \frac{1}{\rho(x)}\Delta(u^m) + u^p \quad \text{in } \mathbb{R}^N \times (0,T). \tag{5.1}$$

5.1 Order of decaying: $q = 2$

Suppose (H), (2.1) and (2.7). To construct a suitable family of subsolution of (5.1), we define, for all $(x,t) \in [\mathbb{R}^N \setminus B_e] \times (0,T)$,

$$\underline{u}(x,t) \equiv \underline{u}(r(x),t) := C\zeta(t)\left[1 - \frac{\log(r)}{a}\eta(t)\right]_+^{\frac{1}{m-1}}, \tag{5.2}$$

and

$$w(x,t) \equiv w(r(x),t) := \begin{cases} \underline{u}(x,t) & \text{in } [\mathbb{R}^N \setminus B_e] \times (0,T), \\ v(x,t) & \text{in } B_e \times (0,T), \end{cases} \tag{5.3}$$

where

$$v(x,t) \equiv v(r(x),t) := C\zeta(t)\left[1 - \frac{r^2+e^2}{2e^2}\frac{\eta}{a}\right]_+^{\frac{1}{m-1}}. \tag{5.4}$$

Let us set

$$F(r,t) := 1 - \frac{\log(r)}{a}\eta(t),$$

and

$$G(r,t) := 1 - \frac{r^2+e^2}{2e^2}\frac{\eta(t)}{a}.$$

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For any $(x,t) \in (\mathbb{R}^N \setminus B_e) \times (0,T)$, we have:

$$\underline{u}_t = C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} F^{\frac{1}{m-1}-1}\left(-\frac{\log(r)}{a}\eta'\right) =$$
$$= C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\left(1 - \frac{\log(r)}{a}\eta\right)\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1} =$$
$$= C\zeta' F^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta} F^{\frac{1}{m-1}-1}.$$
$$(5.5)$$

$$(\underline{u}^m)_r = -\frac{C^m}{a}\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}}\frac{1}{r}\eta. \qquad (5.6)$$

$$(\underline{u}^m)_{rr} = -C^m\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\left(1 - \frac{\log(r)}{a}\eta\right)\eta\frac{1}{r^2\log(r)}$$
$$+ \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\frac{\eta}{r^2\log(r)} + \frac{C^m}{a}\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}}\frac{1}{r^2}\eta =$$
$$= -\frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}}\eta\frac{1}{r^2\log(r)}$$
$$+ \frac{C^m}{a}\zeta^m \frac{m}{(m-1)^2} F^{\frac{1}{m-1}-1}\frac{\eta}{r^2\log(r)} + \frac{C^m}{a}\zeta^m \frac{m}{m-1} F^{\frac{1}{m-1}}\frac{1}{r^2}\eta.$$
$$(5.7)$$

For any $(x,t) \in B_e \times (0,T)$, we have:

$$v_t = C\zeta' G^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1} G^{\frac{1}{m-1}-1}\left(-\frac{r^2+e^2}{2e^2}\frac{\eta'}{a}\right) =$$
$$= C\zeta' G^{\frac{1}{m-1}} + C\frac{\zeta}{m-1}\left(1 - \frac{r^2+e^2}{2e^2}\frac{\eta}{a}\right)\frac{\eta'}{\eta} G^{\frac{1}{m-1}-1} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta} G^{\frac{1}{m-1}-1} =$$
$$= C\zeta' G^{\frac{1}{m-1}} + C\zeta \frac{1}{m-1}\frac{\eta'}{\eta} G^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta} G^{\frac{1}{m-1}-1}.$$
$$(5.8)$$

$$(v^m)_r = -\frac{C^m}{a}\zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}}\frac{r}{e^2}\eta. \qquad (5.9)$$

$$(v^m)_{rr} = -C^m\zeta^m \frac{m}{m-1} G^{\frac{1}{m-1}}\frac{1}{e^2}\frac{\eta}{a} + \frac{C^m}{a^2}\zeta^m \frac{m}{(m-1)^2} G^{\frac{1}{m-1}-1}\eta^2\frac{r^2}{e^4}. \qquad (5.10)$$

We also define

$$\underline{\sigma}(t) := \zeta' + \zeta\frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta k_2\left(N-2+\frac{1}{m-1}\right),$$

$$\underline{\delta}(t) := \zeta\frac{1}{m-1}\frac{\eta'}{\eta},$$

$$\underline{\gamma}(t) := C^{p-1}\zeta^p, \tag{5.11}$$

$$\underline{\sigma}_0(t) := \zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{N}{e^2}\rho_2\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta,$$

$$K := \left(\frac{m-1}{p+m-2}\right)^{\frac{m-1}{p-1}} - \left(\frac{m-1}{p+m-2}\right)^{\frac{p+m-2}{p-1}} > 0.$$

Proposition 5.1. *Let $p > m$. Let $T \in (0, \infty)$, ζ, $\eta \in C^1([0,T); [0,T))$. Let $\underline{\sigma}$, $\underline{\delta}$, $\underline{\gamma}$, $\underline{\sigma}_0$, K be defined in (5.11). Assume that, for all $t \in (0, T)$,*

$$\underline{\sigma}(t) > 0, \quad K[\underline{\sigma}(t)]^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}, \tag{5.12}$$

$$(m-1)\underline{\sigma}(t) \le (p+m-2)\underline{\gamma}(t). \tag{5.13}$$

$$\underline{\sigma}_0(t) > 0, \quad K[\underline{\sigma}_0](t)^{\frac{p+m-2}{p-1}} \le \underline{\delta}(t)\underline{\gamma}(t)^{\frac{m-1}{p-1}}, \tag{5.14}$$

$$(m-1)\underline{\sigma}_0(t) \le (p+m-2)\underline{\gamma}(t). \tag{5.15}$$

Then w defined in (5.3) is a subsolution of equation (5.1).

Proof of Proposition 5.1. Let $\underline{u}$ be as in (5.2) and set

$$D_2 := \left\{(x,t) \in (\mathbb{R}^N \setminus B_e) \times (0,T) \mid 0 < F(r,t) < 1\right\}.$$

In view of (5.5), (5.6), (5.7), we obtain, for all $(x,t) \in D_2$,

$$\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p$$

$$= C\zeta'F^{\frac{1}{m-1}} + C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}} - C\zeta\frac{1}{m-1}\frac{\eta'}{\eta}F^{\frac{1}{m-1}-1}$$

$$+ F^{\frac{1}{m-1}}\frac{C^m}{a}\zeta^m\frac{m}{m-1}\eta\frac{1}{\rho r^2}\left(\frac{1}{(m-1)\log(r)} + N - 1\right) - \frac{C^m}{a}\zeta^m\frac{m}{(m-1)^2}F^{\frac{1}{m-1}-1}\frac{\eta}{\log(r)}\frac{1}{\rho r^2}$$

$$- C^p\zeta^p F^{\frac{p}{m-1}}.$$

In view of hypotheses (2.1) and (2.7), we can infer that

$$\frac{1}{\rho r^2} \le k_2, \quad -\frac{1}{\rho r^2} \le -k_1 \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_e. \tag{5.16}$$

Moreover,

$$-1 \leq -\frac{1}{\log(r)} \leq 0\,, \quad \frac{1}{\log(r)} \leq 1\,, \quad \text{for all} \quad x \in \mathbb{R}^N \setminus B_e\,. \qquad (5.17)$$

From (5.16) and (5.17) we have

$$\begin{aligned}
&\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \\
&\leq CF^{\frac{1}{m-1}-1}\left\{ F\left[\zeta' + \zeta \frac{1}{m-1}\frac{\eta'}{\eta} + \frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\eta\,k_2 \right.\right. \qquad (5.18) \\
&\left.\left. \times \left(N-2+\frac{1}{m-1}\right)\right] - \zeta\frac{1}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p F^{\frac{p+m-2}{m-1}}\right\}.
\end{aligned}$$

Thanks to (5.11) and (5.18)

$$\underline{u}_t - \frac{1}{\rho}\Delta(\underline{u}^m) - \underline{u}^p \leq CF^{\frac{1}{m-1}-1}\varphi(F), \qquad (5.19)$$

where

$$\varphi(F) := \underline{\sigma}(t)F - \underline{\delta}(t) - \underline{\gamma}(t)F^{\frac{p+m-2}{m-1}}. \qquad (5.20)$$

Due to (5.19), our goal is to find suitable $C > 0$, $a > 0$, ζ, η such that

$$\varphi(F) \leq 0\,, \quad \text{for all } F \in (0,1)\,.$$

To this aim, we impose that

$$\sup_{F \in (0,1)} \varphi(F) = \max_{F \in (0,1)} \varphi(F) = \varphi(F_0) \leq 0\,,$$

for some $F_0 \in (0,1)$. We have

$$\begin{aligned}
\frac{d\varphi}{dF} = 0 &\iff \underline{\sigma}(t) - \frac{p+m-2}{m-1}\underline{\gamma}(t)F^{\frac{p-1}{m-1}} = 0 \\
&\iff F_0 = \left[\frac{m-1}{p+m-2}\frac{\underline{\sigma}(t)}{\underline{\gamma}(t)}\right]^{\frac{m-1}{p-1}}.
\end{aligned}$$

Then,

$$\varphi(F_0) = K\frac{\underline{\sigma}(t)^{\frac{p+m-2}{p-1}}}{\underline{\gamma}(t)^{\frac{m-1}{p-1}}} - \underline{\delta}(t)$$

where the coefficient $K = K(m,p)$ has been defined in (5.11). By hypotheses (5.12) and (5.13)

$$\varphi(F_0) \leq 0\,, \quad 0 < F_0 \leq 1\,. \qquad (5.21)$$

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So far, we have proved that

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p \leq 0 \quad \text{in } D_2. \tag{5.22}$$

Furthermore, since $\underline{u}^m \in C^1([\mathbb{R}^N \setminus B_e] \times [0, T))$, due to Lemma 3.1 (applied with $\Omega_1 = D_2, \Omega_2 = \mathbb{R}^N \setminus [B_e \cup D_2], u_1 = \underline{u}, u_2 = 0, u = \underline{u}$), it follows that $\underline{u}$ is a subsolution to equation

$$\underline{u}_t - \frac{1}{\rho(x)}\Delta(\underline{u}^m) - \underline{u}^p = 0 \quad \text{in } [\mathbb{R}^N \setminus B_e] \times (0, T), \tag{5.23}$$

in the sense of Definition 3.8.

Let
$$D_3 := \{(x, t) \in B_e \times (0, T) \mid 0 < G < 1\}.$$

In view of (5.8), (5.9) and (5.10), for all $(x, t) \in D_3$,

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p$$
$$= CG^{\frac{1}{m-1}-1}\left\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \frac{1}{\rho}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{N-1}{e^2}\eta\frac{1}{\rho}\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{1}{e^2}\eta\right]\right.$$
$$\left. + -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - \frac{1}{\rho}\frac{C^{m-1}}{a^2}\zeta^m\frac{m}{(m-1)^2}\frac{r^2}{e^4}\eta^2 - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\right\} \tag{5.24}$$

Using (2.2), (5.24) yield, for all $(x, t) \in D_3$,

$$v_t - \frac{1}{\rho}\Delta(v^m) - v^p$$
$$\leq CG^{\frac{1}{m-1}-1}\left\{G\left[\zeta' + \frac{\zeta}{m-1}\frac{\eta'}{\eta} + \rho_2\frac{C^{m-1}}{a}\zeta^m\frac{m}{m-1}\frac{N}{e^2}\eta\right]\right. \tag{5.25}$$
$$\left. -\frac{\zeta}{m-1}\frac{\eta'}{\eta} - C^{p-1}\zeta^p G^{\frac{p+m-2}{m-1}}\right\}.$$

Thanks to (4.7) and (5.25),

$$v_t - \frac{1}{\rho}\Delta(v^m) - v^p \leq CG^{\frac{1}{m-1}-1}\psi(G), \tag{5.26}$$

where
$$\psi(G) := \underline{\sigma_0}(t)G - \underline{\delta}(t) - \underline{\gamma}(t)G^{\frac{p+m-2}{m-1}}. \tag{5.27}$$

Now, by the same arguments used to obtain (5.23), in view of (5.14) and (5.15) we can infer that

$$\psi(G) \leq 0 \quad 0 < G \leq 1.$$

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So far, due to (5.26), we have proved that

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p \leq 0 \quad \text{for any } (x,t) \in D_3. \qquad (5.28)$$

Moreover, by Lemma 3.1 v is a subsolution of equation

$$v_t - \frac{1}{\rho(x)}\Delta(v^m) - v^p = 0 \quad \text{in } B_e \times (0,T), \qquad (5.29)$$

in the sense of Definition 3.8. Now, observe that $w \in C(\mathbb{R}^N \times [0,T))$, indeed,

$$\underline{u} = v = C\zeta(t)\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e \times (0,T).$$

Moreover, $w^m \in C^1(\mathbb{R}^N \times [0,T))$, indeed,

$$(\underline{u}^m)_r = (v^m)_r = -C^m\zeta(t)^m\frac{m}{m-1}\frac{1}{e}\frac{\eta(t)}{a}\left[1 - \frac{\eta(t)}{a}\right]_+^{\frac{1}{m-1}} \quad \text{in } \partial B_e \times (0,T).$$

Hence, by Lemma 3.1 again, w is a subsolution to equation (5.1) in the sense of Definition 3.8. $\square$

Remark 5.2. *Let*

$$p > m,$$

and assumptions (2.1) and (2.7) be satisfied. Let define $\omega := \frac{C^{m-1}}{a}$. In Theorem 2.2, the precise hypotheses on parameters $C > 0$, $a > 0$, $\omega > 0$ and $T > 0$ is the following.

$$\max\left\{1 + mk_2\frac{C^{m-1}}{a}\left(N - 2 + \frac{1}{m-1}\right); 1 + m\rho_2\frac{C^{m-1}}{a}\frac{N}{e^2}\right\} \leq (p + m - 2)C^{p-1}, \qquad (5.30)$$

$$\frac{K}{(m-1)^{\frac{p+m-2}{p-1}}}\ \max\left\{\left[1 + mk_2\frac{C^{m-1}}{a}\left(N - 2 + \frac{1}{m-1}\right)\right]^{\frac{p+m-2}{p-1}};\right.$$
$$\left.\left(1 + m\rho_2\frac{C^{m-1}}{a}\frac{N}{e^2}\right)^{\frac{p+m-2}{p-1}}\right\} \leq \frac{p-m}{(m-1)(p-1)}C^{m-1}. \qquad (5.31)$$

Lemma 5.1. *All the conditions in Remark 5.2 can hold simultaneously.*

Proof. We can take $\omega > 0$ such that

$$\omega_0 \leq \omega \leq \omega_1$$

for suitable $0 < \omega_0 < \omega_1$ and we can choose $C > 0$ sufficiently large to guarantee (5.30) and (5.31) (so, $a > 0$ is fixed, too). $\square$

Proof of Theorem 2.2. We now prove Theorem 2.2, by means of Proposition 5.1. In view of Lemma 5.1 we can assume that all conditions of Remark 5.2 are fulfilled. Set

$$\zeta = (T - t)^{-\alpha}, \quad \eta = (T - t)^{-\beta}, \quad \text{for all} \quad t > 0,$$

and α and β as defined in (4.20). Then

$$
\begin{aligned}
\underline{\sigma}(t) &:= \left[\frac{1}{m-1} + \frac{C^{m-1}}{a} \frac{m}{m-1} k_2 \left(\frac{1}{m-1} + N - 2 \right) \right] (T - t)^{-\frac{p}{p-1}}, \\
\underline{\delta}(t) &:= \frac{p-m}{(m-1)(p-1)} (T - t)^{-\frac{p}{p-1}}, \\
\underline{\gamma}(t) &:= C^{p-1} (T - t)^{-\frac{p}{p-1}}, \\
\underline{\sigma}_0(t) &:= \frac{1}{m-1} \left[1 + \frac{\rho_2 N m}{e^2} \frac{C^{m-1}}{a} \right] (T - t)^{-\frac{p}{p-1}}.
\end{aligned}
\tag{5.32}
$$

Let $p > m$. Condition (5.30) implies (5.12), (5.13), while condition (5.31) implies (5.14), (5.15). Hence by Propositions 5.1 and 3.7 the thesis follows. $\square$

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