A General Framework for Switched and Variable-Gain Higher-Order Sliding Mode Control

Gian Paolo Incremona, *Member, IEEE*, Matteo Rubagotti, *Senior Member, IEEE*, Mara Tanelli, *Senior Member, IEEE*, and Antonella Ferrara, *Fellow, IEEE*

***Abstract*—Sliding mode control is a widely used approach in different application domains, due to its versatility and ease of implementation. As is well-known, one of its most serious drawbacks is the presence of chattering. To alleviate this problem, higher-order sliding mode approaches have been proposed, which also allow dealing with high relative degree plants. To gain more flexibility in the controller design and to boost performance, switched and variable-gain approaches are being developed for first and second-order sliding mode controllers. This technical note introduces a conceptual framework to merge the two aspects, providing a general methodology for the design and tuning of high-order sliding mode controllers. The main strength of the method is its generality, in that it accommodates a generic order r sliding mode controller, and it encompasses both continuous and discrete variation of the controller parameters, the latter giving rise to switched strategies. The properties of the closed-loop system are formally analyzed, and the effectiveness of the method is demonstrated in simulation on examples of switched and variable-gain higher-order sliding mode controllers.**

***Index Terms*—Sliding mode control, higher-order sliding mode (HOSM), switched control.**

I. Introduction

Variable structure control approaches, and sliding mode (SM) in particular, are recognized as successful methods to control a wide class of nonlinear systems affected by uncertainties that are in general unknown in their structure, but with known bounds. Their ability to provide working solutions in many practical applications, combined with their ease of implementation also with scarce computational power, have made these methods rather pervasive in the control community. Of course, these approaches have also some drawbacks, which originated different lines of research that study how to alleviate them: the main one is the so-called chattering phenomenon, which is a high-frequency oscillatory motion around the sliding manifold due to the discontinuity of the control law. To address this issue, Higher-Order Sliding Mode (HOSM) approaches have been developed, and several HOSM algorithms have been proposed [1]–[7]. Among these, the algorithm obtained as solution of the so-called Fuller's Problem is presented in [8], where different HOSM algorithms are considered, which guarantee a time-optimal reaching of the sliding manifold. A second issue in SM control is the generally high control authority resulting from the application of the method. This is mainly due to the fact that the controller is designed not based on the actual uncertainty level, but rather on a – possibly rather coarse – estimate of its upper bound. A fixed tuning of the controller parameters based on such estimate, in fact, leads in general to an excessive control authority, especially when close to the sliding manifold. To address this issue, several approaches have been proposed. Further, considering that in many applications one has to deal with different degrees of uncertainty and/or different control objectives, which vary according to the region of the state space currently visited by the closed-loop trajectory, switched formulations proved to be an efficient way to achieve performance enhancement. In the SM literature, different approaches leading to time-varying control gains were devised to deal with specific situations and constraints. A possibility is to employ adaptive SM control laws, which are reviewed in [9]: they are typically based either on the so-called σ-adaptation method [10]–[12] or on dynamic adaptation [13], [14]. A different approach was proposed in [15], where, to deal with state-dependent uncertainties and to ensure global convergence properties, a second-order SM (SOSM) control algorithm which modifies the amplitude of the control gain within each time interval between two successive extremal points was proposed, giving rise to a *de facto* switching control law. Within the same context, [16] proposes a hybrid first-order SM control law for second-order systems, which relies on a partitioning of the state space into different regions and on the availability of two scalar control variables. In [17], a time-based adaptation coupled with the additional degrees of freedom given by a switched variation of the controller parameters is proposed for a SOSM control algorithm. Furthermore, [18] considered state-dependent uncertainties associated with each region of the state space, and proposed a switched strategy to define time-varying parameters for a SOSM controller.

The present technical note aims at optimally combine the advantages of HOSM with those of variable gain and switching, providing a comprehensive methodological framework for the design of the resulting controllers. This allows achieving both chattering alleviation and reduced control authority, while leaving freedom to also accommodate different performance specifications for a single system according to the online evolution of the closed-loop trajectory. Specifically, the extension

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G.P. Incremona and M. Tanelli are with the Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milan, Italy (gianpaolo.incremona@polimi.it, mara.tanelli@polimi.it).

M. Rubagotti is with the Department of Robotics and Mechatronics, Nazarbayev University, 53 Kabanbay Batyr Avenue, 010000 Nur-Sultan, Kazakhstan (matteo.rubagotti@nu.edu.kz).

A. Ferrara is with the Dipartimento di Ingegneria Industriale e dell'Informazione, University of Pavia, Via Ferrata 5, 27100 Pavia, Italy (antonella.ferrara@unipv.it).

to HOSM with augmented auxiliary system for chattering reduction and to the case of state constraints as in [19] are discussed. The main result of the proposed approach is the generality of the method, that allows designing a generic order r sliding mode controller (thus allowing one to deal with systems of any relative degree), and the fact that it embodies, as particular cases, both continuous and discrete variation of the controller parameters, the latter giving rise to switched strategies. General conditions for designing the controller parameters are provided independently of the adopted strategy. Moreover, overcoming the explicit definition of a Lyapunov function candidate, which is quite challenging for the generic order r sliding mode, finite time convergence of the sliding variable to the sliding manifold is rigorously proved.

The structure of the technical note is as follows. Section II introduces the needed preliminaries on SM control, states the problem formulation, and provides general theoretical results. Section III presents the proposed variable-gain and switched formulations of the control algorithm, together with the assessment of the closed-loop properties, while in Section IV a numerical example is illustrated. Conclusions are drawn in Section V.

II. General framework

A. Problem formulation

In this technical note, we consider the class of continuous-time SISO uncertain nonlinear systems [8] described by

$$\dot{x}(t) = a(x,t) + b(x,t)u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state (with initial condition $x_0 \triangleq x(t_0)$), $u \in \mathbb{R}$ is the control variable, $a : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $b : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are uncertain and sufficiently smooth vector fields, and the whole state vector x is available for feedback (the explicit dependence of variables on time is omitted when convenient, in order to improve readability). The first step towards designing the proposed HOSM control strategy is to define the system output

$$y(t) = f(x(t)) \tag{2}$$

which coincides with the *sliding variable* to be steered to zero in finite time, with $f : \mathbb{R}^n \to \mathbb{R}$ being a known function. Two assumptions, which are standard in sliding mode control systems, are introduced in the following.

Assumption 1: The relative degree r of system (1)-(2) (i.e., the minimum order of time derivative of $y(t)$ in which $u(t)$ appears explicitly) is uniform, globally well defined, and time-invariant. □

Given suitable functions $h(x,t)$ and $g(x,t)$, one obtains that the r-th order time derivative of $y(t)$, namely $y^{(r)}(t)$, is

$$y^{(r)}(t) = h(x,t) + g(x,t)u(t). \tag{3}$$

Assumption 2: Functions h and g are uncertain, but bounded according to

$$|h(x,t)| \leq C(x,t) \tag{4}$$

$$g(x,t) \in [K_m(x,t), K_M(x,t)] \subseteq \left[\bar{K}_m, \bar{K}_M\right] \tag{5}$$

for all $x \in \mathbb{R}^n$ and all $t \geq t_0$, where $\bar{K}_M \geq \bar{K}_m > 0$ are known scalar constants, while $C(x,t)$, $K_m(x,t)$, and $K_M(x,t)$ are known functions of state and/or time. □

The objective of the HOSM control law is to attain, in finite time, the manifold given by

$$y^{(0)}(t) = y^{(1)}(t) = \ldots = y^{(r-1)}(t) = 0. \tag{6}$$

The system describing the evolution of $y(t)$ and its time derivatives can be directly expressed by defining $\sigma_{i+1}(t) \triangleq y^{(i)}(t)$, obtaining

$$\begin{cases} \dot{\sigma}_1(t) & = \sigma_2(t) \\ & \vdots \\ \dot{\sigma}_{r-1}(t) & = \sigma_r(t) \\ \dot{\sigma}_r(t) & = h(x,t) + g(x,t)u(t) \end{cases} \tag{7}$$

which is referred to as *auxiliary system*. In general, one can define a diffeomorphism $\Phi(x) : \mathbb{R}^n \to \mathbb{R}^n$, which generates both $\sigma \in \mathbb{R}^r$ and the internal state $\zeta \in \mathbb{R}^{n-r}$. The dynamics of $\zeta(t)$ has to satisfy the following assumption.

Assumption 3: The internal dynamics of system (1)-(2), namely

$$\dot{\zeta}(t) = \psi(\sigma(t), \zeta(t)), \tag{8}$$

presents no finite escape time phenomena, i.e., there exists no finite time instant t_e such that $\lim_{t \to t_\mathrm{e}} \|\zeta(t)\| = \infty$, for $t_0 < t_\mathrm{e} < \infty$. Furthermore, the zero dynamics is globally asymptotically stable, which can be proven by the existence of a radially unbounded Lyapunov function $V(\zeta)$ [20]: as a consequence, when $\sigma(t) \equiv 0$, every trajectory $\zeta(t)$ converges to zero as $t \to \infty$. □

Assumption 3 implies that, once $\sigma(t)$ is steered to zero in finite time by the control action, $x(t)$ will converge to zero asymptotically.

Remark 1: Notice that, differently from classical approaches in HOSM, which consider constant bounds on the uncertain terms, the bounds considered in the proposed framework can be dependent on time and on the value of the state. Our formulation only considers the presence of matched disturbances. This is a common assumption in sliding mode control, however one could still employ the same approach detailed in the remainder of the technical note (which only deals with the dynamics of $y(t)$ and its time derivatives) while assuming the presence of unmatched disturbances acting on the internal dynamics. Indeed, as discussed, among others, in [21], due to the definition of $\Phi(x)$, the uncertain function $a(x,t)$ is mapped into matched uncertain terms in the auxiliary system (7), while the residual unmatched terms can affect only the internal dynamics. In this case, the internal dynamics has to be assumed input-to-state stable with respect to the unmatched disturbances, which would be a less restrictive hypothesis than that of Assumption 3. This would imply convergence of $x(t)$ to a bounded set, whose size would depend on the unmatched disturbance realization. □

Following an approach similar to that of [8] (which was however developed for systems with constant bounds on the uncertain terms), the control law $u(t)$ is defined as a discontinuous function of

$$\sigma \triangleq \begin{bmatrix}\sigma_1 & \sigma_2 & \dots & \sigma_r\end{bmatrix}', \tag{9}$$

while the amplitude $U = U(x,t)$ of the discontinuous control variable is assumed to be varying as a function of time, and of the state. No reference to any specific amplitude variation strategy is made here, in order to obtain a general result. However, it is assumed that $U(x,t)$ satisfies the following assumption.

Assumption 4: For all $x \in \mathbb{R}^n$ and all $t \geq t_0$,

$$K_m(x,t)U(x,t) - C(x,t) \geq \epsilon, \tag{10}$$

$$U(x,t) \leq \bar{U}, \tag{11}$$

where $\epsilon \in \mathbb{R}_{>0}$ and $\bar{U} \in \mathbb{R}_{>0}$ are constant terms. □

One can notice that Assumption 4 implies the existence of a global upper bound on $C(x,t)$ which, however, does not need to be explicitly defined. The expression of the control law is

$$u(t) = -U(x,t) \cdot \operatorname{sgn}(s(\sigma)), \tag{12}$$

in which, being $s(\sigma)$ a null-measure set, the value of $u(t)$ for $s(\sigma) = 0$ is determined as the Filippov solution of the discontinuous vector field at σ [22]. As also reported in [8], [19], the use of efficient methods to define an expression of $s(\sigma)$ for a specific higher-order case is still an open problem (see, e.g., [23]), but a general analytical condition from which the expression of $s(\sigma)$ can be obtained is provided in [8].

Remark 2: The expression of $s(\sigma)$ for the general case can be obtained as follows. In [8], it is assumed that C, K_m, and K_M are constant, and the constant *reduced control amplitude* $\alpha_{\mathrm{r}} \in \mathbb{R}_{>0}$ is defined as

$$\alpha_{\mathrm{r}} \triangleq K_m U - C, \tag{13}$$

with U being the constant amplitude of the control variable. The general expression of $s(\sigma)$ is obtained in this work by substituting ϵ to α_{r} in [8]. Thus, ϵ can be interpreted as global lower bound of a "reduced control amplitude". □

We report here the explicit expression of $s(\sigma)$ for $r = 1, 2, 3$, by referring to [8], and substituting ϵ to α_{r}:

$$\sigma \in \mathbb{R}^1 \Rightarrow s(\sigma) = \sigma_1 \tag{14}$$

$$\sigma \in \mathbb{R}^2 \Rightarrow s(\sigma) = \sigma_1 + \tfrac{\sigma_2|\sigma_2|}{2\epsilon} \tag{15}$$

$$\begin{aligned}\sigma \in \mathbb{R}^3 \Rightarrow s(\sigma) = {}& \sigma_1 + \tfrac{\sigma_3^3}{3\epsilon^2} + \operatorname{sgn}\left(\sigma_2 + \tfrac{\sigma_3^2 \operatorname{sgn}(\sigma_3)}{2\epsilon}\right) \times \\ & \times \left[\tfrac{1}{\sqrt{\epsilon}}\left(\operatorname{sgn}\left(\sigma_2 + \tfrac{\sigma_3^2 \operatorname{sgn}(\sigma_3)}{2\epsilon}\right)\sigma_2 + \tfrac{\sigma_3^2}{2\epsilon}\right)^{\frac{3}{2}} + \tfrac{\sigma_2\sigma_3}{\epsilon}\right].\end{aligned} \tag{16}$$

Obtaining the expression of the surface s for the fourth and fifth-order cases is possible in principle: the complexity of the expression of $s(\sigma)$ grows very fast with the order r, but the derivation of efficient numerical or exact-algebraic methods (see the references in [8]), is not investigated in this paper.

Remark 3: The general formulation of the control law in (12) provides the control designer with a wide choice. In particular, the knowledge of the time-varying bounds on the uncertain terms can lead to varying $U(x,t)$, increasing it when large uncertain terms have to be dominated, and reducing it when smaller amplitudes are sufficient to guarantee convergence, while at the same time reducing the chattering effect. The variation can be smooth in time, or the control amplitude can switch between different pre-defined values. In the remainder of this section, a general result will be proven, while more specific results will be described in the following sections. □

B. General result on finite-time convergence

Theorem 1: Given system (1)-(2) satisfying Assumptions 1 and 3, with bounds on the uncertain terms satisfying Assumption 2, a feedback control law of form (12) is defined which satisfies Assumption 4. Then, σ converges to the origin in finite time.

Proof: In order to use a more compact notation, a new term is defined as $v(x,t) \triangleq g(x,t)u(t)$. Being $g(x,t) > 0$ by assumption, given the expression of $u(t)$ in (12), then $v(x,t) = -V(x,t)\operatorname{sgn}(s(\sigma(t)))$, with

$$V(x,t) \in [C(x,t) + \epsilon, K_M(x,t)U(x,t)] > 0, \tag{17}$$

as $C(x,t) + \epsilon \leq K_m(x,t)U(x,t)$ according to (10). Consider system

$$\begin{cases} \dot{z}_1(t) & = z_2(t) \\ & \vdots \\ \dot{z}_{r-1}(t) & = z_r(t) \\ \dot{z}_r(t) & = \phi(x,t) + \gamma(x,t)w(t) \end{cases} \tag{18}$$

with

$$\phi(x,t) \in [-W + \epsilon, W - \epsilon] \tag{19}$$

$$\gamma(x,t) \equiv 1 \tag{20}$$

$$w(t) = -W \operatorname{sgn}(s(z)) \tag{21}$$

being $W \triangleq \bar{K}_M \bar{U}$. Considering that the expression of $s(\cdot)$ is the same for σ or z as argument, one can notice that $s(z)$ is defined by using the *reduced control amplitude*, given by $\alpha_{\mathrm{r}} = 1 \cdot W - (W - \epsilon) = \epsilon$, which can be obtained from the analogous of (13) for system (18). System (18) has constant bounds on the uncertain terms, positive reduced control amplitude, and the expression of $s(z)$, following the expressions (14)-(16) and their generalization to the r-th order, is the same defined in [8]. Therefore, according to Theorem 2 in [8], z converges to the origin in finite time for any initial condition $z(t_0)$.

Now, consider again the auxiliary system (18) with initial condition $\sigma(t_0)$, and take an arbitrary time evolution of $h(x,t)$ and $g(x,t)$ that satisfies (4)-(5). By taking $z(t_0) = \sigma(t_0)$, one will obtain $z(t) = \sigma(t)$ if and only if

$$h(x,t) + g(x,t)u(t) = \phi(x,t) + w(t),$$

which, given the two closed-loop control laws and remembering the definition of $v(x,t)$, is equivalent to imposing

$$\phi(x,t) = h(x,t) + (W - V(x,t))\operatorname{sgn}(s(z(t))). \tag{22}$$

However, this corresponds to a feasible evolution of system (18) only if $\phi(x,t)$ satisfies (19). Two cases have to be analyzed:

- If $s \geq 0$, then

$$\phi(x,t) = h(x,t) + W - V(x,t).$$

Given the bounds provided in (4) and (17) for $h(x,t)$ and $V(x,t)$, respectively, one has

$$\begin{aligned}\phi(x,t) \in & [-C(x,t) + W - W, \\ & C(x,t) + W - C(x,t) - \epsilon] \\ = & [-C(x,t), W - \epsilon] \,. \end{aligned} \tag{23}$$

- If $s < 0$, then

$$\phi(x,t) = h(x,t) - W + V(x,t),$$

which implies

$$\begin{aligned}\phi(x,t) \in & [-C(x,t) - W + C(x,t) + \epsilon, \\ & C(x,t) - W + W] \\ = & [-W + \epsilon, C(x,t)] \,. \end{aligned} \tag{24}$$

Given that, by Assumption 4, $W - C(x,t) = \bar{K}_M \bar{U} - C(x,t) \geq \epsilon$, then $W - \epsilon \geq C(x,t)$, and an inclusion merging (23) and (24) can be written as

$$\phi(x,t) \in [-W + \epsilon, W - \epsilon] \,,$$

which coincides with (19). This implies that every feasible time evolution of system (7) coincides with the time evolution of system (18) for a particular (and feasible) realization of $\phi(x,t)$, which in turn implies that $\sigma(t)$ converges to the origin in finite time. ■

Remark 4: The result proved in Theorem 1 for a general amplitude variation strategy is based on the parallelism with fixed gain approaches as in [8] and [24]. Hence, the presented convergence arguments refer to the construction of a family of non-smooth Lyapunov functions. The choice of the Lyapunov function candidate relies on the analytic expression of the convergence time, whose computation, however, becomes difficult for high values of r. For the simpler case of SOSM control with switched gain, an estimate of the convergence time can be found for instance in [21]. □

Remark 5: In some cases, the definition of the control variable as a discontinuous signal, although of varying amplitude, can induce unacceptable vibrations on the controlled variables, especially when mechanical systems are considered. The proposed framework can be easily extended to the case in which additional states are introduced in the auxiliary system (7), and an m-th order time derivative of $u(t)$ is used as control variable, thus obtaining the value of $u(t)$ by m-fold time integration of a discontinuous control variable. As m increases, the smoothness of the control signal $u(t)$ also increases, so that the frequency of chattering is significantly reduced. On the other hand, the chattering amplitude does not necessarily decrease as m increases. Indeed, as discussed in [25] by exploiting the so-called describing function method, higher-order sliding mode approaches can generate larger chattering amplitude with respect to classical sliding mode control. A higher gain leads to a higher chattering amplitude: therefore, switched/variable gain methods such as those presented in this technical note can successfully tackle this problem. □

Remark 6: If the last equation of the auxiliary system (7) is defined including a known function $\bar{h}(x,t)$, as

$$\dot{\sigma}_r(t) = \bar{h}(x,t) + h(x,t) + g(x,t)u(t), \tag{25}$$

then one can define an additional term in the control law in order to compensate it (see, e.g., [26]), as

$$u(t) = -\bar{h}(x,t) - U(x,t) \cdot \mathrm{sgn}(s(\sigma(t))). \tag{26}$$

It is immediate to see that the resulting closed-loop system is the same considered in Theorem 1, and therefore the same results hold. □

In [19], a general HOSM strategy was proposed in order to cope with the presence of state constraints, and in particular inequality contraints that have to be satisfied point-wise in time on the components of σ. Assume that the satisfaction of the state constraint

$$\sigma(t) \in \mathcal{S}, \; t \geq t_0 \tag{27}$$

is added to the requirements for the closed-loop system, where $\mathcal{S}$ is a compact set including the origin. Then the control law can be expressed as

$$w(t) = \begin{cases} -W(x,t) \cdot \mathrm{sgn}(s(\sigma(t))), \; \sigma \in \mathcal{S} \\ -W(x,t) \cdot \mathrm{sgn}(\sigma_r(t)), \; \sigma \notin \mathcal{S} \,. \end{cases} \tag{28}$$

If the "unconstrained" control law (12) is applied, there exists a set $\mathcal{S}' \subseteq \mathcal{S}$ of initial conditions $\sigma(t_0)$, named *region of attraction*, for which $\sigma(t) \in \mathcal{S}$ for all $t \geq t_0$, and for which $\sigma(t)$ converges to the origin in finite time. The use of the control law defined in (28) is targeted at an enlargement of the region of attraction.

Corollary 1: Given system (1)-(2) satisfying Assumptions 1 and 3 and subject to the state constraints in (27), with bounds on the uncertain terms of the auxiliary system (7) satisfying Assumption 2, a feedback control law of form (28) is defined which satisfies Assumption 4. Then, σ converges to the origin in finite time for all $\sigma(t_0) \in \mathcal{S}'$ with $\mathcal{S}' \subseteq \mathcal{S}$, and x converges asymptotically to the origin.

Proof: The result is immediately proven given the definition of $\mathcal{S}'$ and the results in Theorem 1. ■

The amount of enlargement of the region of attraction $\mathcal{S}$ as compared to $\mathcal{S}'$ strongly depends on the system dynamics and on the type of state constraints. The reader is referred to Section V in [19] for a discussion on this topic in the case of constant control amplitude.

III. Variable-gain and switched HOSM laws

Following the general formulation in Section II, in this section variable-gain and switched HOSM control laws, belonging to the class of algorithms represented by (12), will be introduced. This section will refer to the formulation of Section II, also imposing state constraints.

A. Variable-gain HOSM

Consider the auxiliary system (7) controlled via (12) where the gain $U(x,t)$ varies according to

$$U(x,t) \triangleq \frac{C(x,t)}{K_m(x,t)} + \delta \;, \tag{29}$$

with δ being a positive constant.

Corollary 2: Consider a given auxiliary dynamics (7) with associated constraints (4) and (5) and controlled via (12). Assumptions 3-4 hold. If $U(x,t)$ is defined as in (29) with $\epsilon = K_m(x,t)\delta$, then inequalities (10) and (11) are satisfied, and the results of Theorem 1 apply. □

Proof: Since δ is a positive constant, being $\epsilon = K_m(x,t)\delta$, one has that $K_m(x,t)U(x,t) = C(x,t)+\epsilon$, which directly implies (10). Moreover, since Assumption 2 implies the existence of a global upper bound on $C(x,t)$, and a global positive lower bound on $K_m(x,t)$, there exists a positive constant $\bar{U}$ such that inequality (11) holds. ■

B. Switched HOSM

Consider the auxiliary system (7) with an additional assumption on the partition of the state-space as follows. We assume that the state space $\mathcal{S}$ of system (7) is partitioned into k non-overlapping subsets $\mathcal{S}_i$, $i = 1,\dots,k$, which are such that $\cup_{i=1,\dots,k}\mathcal{S}_i = \mathcal{S}$, and we assume that in each of them we may define different upper and lower bounds for the uncertainties. Note that the subsets $\mathcal{S}_i$, $i = 1,\dots,k$ do not need to be necessarily nested. If this is the case, instead, the state space $\mathcal{S}$ can be partitioned into compact regions $\mathcal{R}_i$, $i = 1,\dots,k$ so that it is possible to introduce the subsets $\mathcal{S}_i = \mathcal{R}_i \setminus \mathcal{R}_{i+1}$, $i = 1,\dots,k-1$ with $\mathcal{S}_k \equiv \mathcal{R}_k$.

More specifically, inside each subset $\mathcal{S}_i$, $i = 1,\dots,k$, constant upper and lower bounds on the uncertain terms are assumed to be known, i.e., $\forall\, i = 1,\dots,k$, and we can write

$$h(x,t) \in [-C_i,\ C_i] \tag{30}$$

$$g(x,t) \in [K_{m,i},\ K_{M,i}] \;. \tag{31}$$

Such upper bounds can be determined by taking into account the shape of the regions, and the fact that within each of them the state of the auxiliary system is bounded.

Consider now the control law (12) where $U(x,t)$ is a switched gain selected as

$$U(x,t) \triangleq \frac{C_i}{K_{m,i}} + \delta_i, \quad i = 1,\dots,k \;, \tag{32}$$

with δ_i being positive constants.

Corollary 3: Consider a given auxiliary dynamics (7) with associated constraints (4) and (5) and controlled via (12). Assumptions 3-4 hold. If $U(x,t)$ is defined as in (32) with $\epsilon = K_{m,i}\delta_i,\ \forall\, i = 1,\dots,k$, then inequalities (10) and (11) are satisfied, and the results of Theorem 1 apply. □

Proof: Since δ_i is a positive constant, and being $\epsilon = K_{m,i}\delta_i,\ \forall\, i = 1,\dots,k$, it holds that $K_{m,i}U(x,t) = C_i + \epsilon$, which directly implies (10). Moreover, since $K_{m,i}$ and C_i are constant inside each subset, it is possible to state that $\bar{U} = C_i/K_{m,i} + \delta_i$ so that also inequality (11) holds. ■

IV. Numerical Example

Consider the nonlinear system given by

$$\dot{x}_1 = -x_1 + e^{2x_2}(u+d) \tag{33a}$$

$$\dot{x}_2 = 2x_1x_2 + \sin x_2 + \frac{1}{2}(u+d) \tag{33b}$$

$$\dot{x}_3 = x_2 \tag{33c}$$

which is adapted from [27, Ch. 6] by adding the unknown matched disturbance term d, detailed in the following using two different formulations. System (33) can be rewritten in form (1) with

$$a(x,t) = \begin{bmatrix} -x_1 + e^{2x_2}d \\ 2x_1x_2 + \sin(x_2) + \frac{1}{2}d \\ 2x_2 \end{bmatrix}, \quad b(x,t) = \begin{bmatrix} e^{2x_2} \\ \frac{1}{2} \\ 0 \end{bmatrix}. \tag{34}$$

By choosing $y = \sigma_1 = x_3$, one has

$$\dot{\sigma}_1 = \sigma_2 = 2x_2 \tag{35}$$

$$\dot{\sigma}_2 = \bar{h} + h + gu = 2(2x_1x_2 + \sin(x_2)) + d + u \tag{36}$$

with relative degree equal to 2. Note that the term $\bar{h} = 2(2x_1x_2 + \sin(x_2))$ represents the known component of the drift term (see Remark 6), while $h = d$ is the unknown counterpart. Furthermore, consider the constraint set as $\mathcal{S} = \{(\sigma_1,\sigma_2) : |\sigma_1| \le 1,\ |\sigma_2| \le 1\}$. Define now the diffeomorphism

$$\Phi(x) = \begin{pmatrix} x_3 \\ 2x_2 \\ 1 + x_1 - e^{2x_2} \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \zeta \end{pmatrix} \tag{37}$$

so that the internal dynamics of the system, that is

$$\dot{\zeta} = \psi(\sigma,\zeta) = (1 - \zeta - e^{\sigma_2})(1 + 2\sigma_2 e^{\sigma_2}) - 2\sin\left(\frac{\sigma_2}{2}\right)e^{\sigma_2} \;, \tag{38}$$

is globally Lipschitz in ζ, thus presenting no finite time escape phenomena [28]. Furthermore, the zero dynamics

$$\dot{\zeta} = \psi(0,\zeta) = -\zeta \tag{39}$$

is asymptotically stable. Thus, Assumption 3 is satisfied.

A. Variable-gain strategy

Now, we would like to design a variable-gain control law, as detailed in the previous section. Assume that $d = \nu(1 + \sin(t) + |x_3 + 4x_2^2|)$, with $\nu \subseteq [-1,\ 1]$ randomly generated. Then, $|h(x,t)| \le 1 + \sin(t) + |x_3 + 4x_2^2| = C(x,t) \le \bar{C} = 4$, while $g(x,t) = 1 = K_m$. Choosing $\delta = 1$ and $U(x,t) = 1 + \sin(t) + |x_3 + 4x_2^2| + 1$, all the assumptions of Corollary 2 are satisfied. Compensating the known dynamics as described in Remark 6, the whole control law is

$$u = -2(2x_1x_2 + \sin(x_2)) - U(x,t)\,\mathrm{sgn}\left(\sigma_1 + \frac{\sigma_2|\sigma_2|}{2\epsilon}\right) \;. \tag{40}$$

Based on the result in Corollary 2, σ converges to the origin in finite time while satisfying the imposed constraints, as can be observed in Figure 1 (lower-left), where the auxiliary state-space and the box constraint $\mathcal{S}$ are illustrated. More specifically, σ_2 slides on its lower bound. Figure 1 (upper-right) reports also the control input $u(t)$ and the uncertain

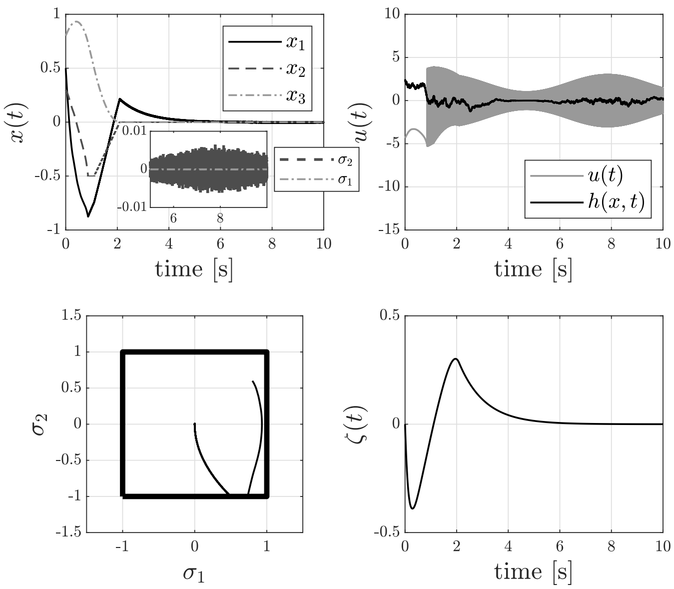

Figure 1. Variable-gain. From the top left: time evolution of the state x; time evolution of the control input u, disturbance h and variable gain $U(x,t)$; auxiliary phase-space with box state constraints $\mathcal{S}$; time evolution of the internal dynamics ζ

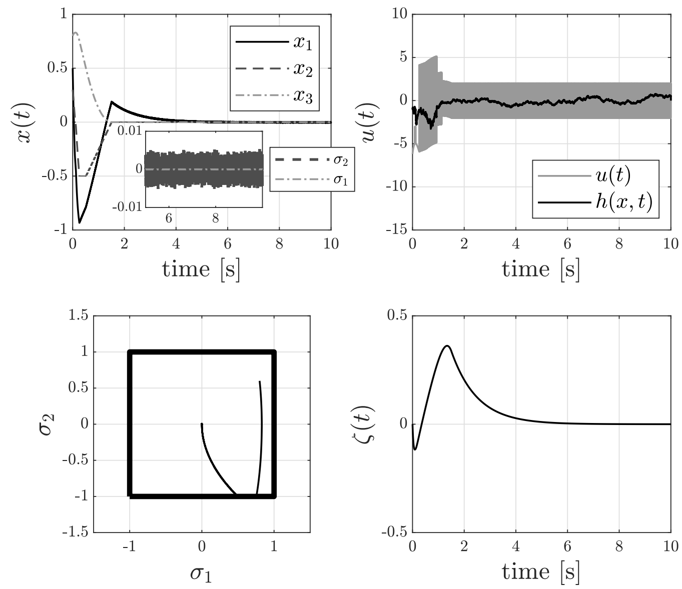

Figure 2. Switched gain. From the top left: time evolution of the state x; time evolution of the control input u, disturbance h and switched-gain $U(x,t)$; auxiliary phase-space with box state constraints $\mathcal{S}$; time evolution of the internal dynamics ζ

term $h(x,t)$. It is apparent how the control gain $U(x,t)$ is continuously varied, while maintaining the reduced control amplitude ϵ constant. Finally, Figure 1 (upper-left) reports the time evolution of the state x, while Figure 1 (lower-right) illustrates the time evolution of the asymptotically stable internal dynamics.

B. Switched-gain strategy

Consider now again the outermost constraint set $\mathcal{S}$, and the auxiliary state space partitioned into ellipsoidal subsets, i.e.,

$$\mathcal{S}_1 \triangleq \{(\sigma_1,\sigma_2)\ : 0.3 \le \sigma' M\sigma \le 0.7\}$$
$$\mathcal{S}_2 \triangleq \{(\sigma_1,\sigma_2)\ : \sigma' M\sigma \le 0.3\}\ ,$$

with $M = \mathrm{diag}\{1,\, 2\}$ such that

$$|h(x,t)| \le \begin{cases} 1 & \text{if } \sigma \in \mathcal{S}_2 \\ 2 & \text{if } \sigma \in \mathcal{S}_1 \\ 4 & \text{otherwise}\ . \end{cases}$$

Choosing $\delta = 1$ and

$$U(x,t) = \begin{cases} 2 & \text{if } \sigma \in \mathcal{S}_2 \\ 3 & \text{if } \sigma \in \mathcal{S}_1 \\ 5 & \text{otherwise}\ , \end{cases}$$

all the assumptions of Corollary 3 are satisfied. Hence, σ converges to the origin in finite time when applying the control law (40) with the switched gain defined above. Analogously to the variable-gain case, Figure 2 reports the time evolution of the state x (upper-left), as well as the auxiliary state-space with the outermost constraint set (lower-left). Notice that σ is steered to zero in finite time, while satisfying the imposed limits, with σ_2 sliding on its lower bound. Moreover Figure 2 (upper-right) reports the control input $u(t)$ and the uncertain term $h(x,t)$. $U(x,t)$ is switched depending on the ellipsoidal subset to which the state belongs, while maintaining the reduced control amplitude ϵ constant.

C. Comparative analysis

As a term of comparison, a standard second-order controller with constant gain $\bar{U} = 5$ and compensation of the known dynamics as in Remark 6, for a fair comparison, has been considered, and the results shown in Figure 3. For the sake of brevity, only the case with the same disturbance as in the variable-gain strategy is illustrated. The amount of chattering (visible in the zoomed portions of Figures 1, 2 and 3) is measured by the L_2 and L_∞-norms of the components of σ in the time interval between 4 s and 10 s, in which ideally $\sigma_1 = \sigma_2 = 0$. As can be seen in Table I, the fixed-gain strategy leads to a visible increase of both the L_2-norm and the L_∞-norm, as compared to the variable-gain strategy and switched one. Hence, our proposed control laws, as expected, present a double advantage: one in terms of control effort reduction, guaranteed by the switched/variable gain nature, and one in terms of the beneficial chattering attenuation property.

V. Conclusions

This technical note proposed a holistic framework for the design of high-order SM controllers with gain depending on time and on the state value. The method allows designing SM controllers with both continuous and switched adaptation under the same conceptual scheme, which enjoy finite time convergence to the sliding manifold. The shown simulation examples confirm the capability of the method to offer the way for combined chattering alleviation, reduction of the control authority and enhanced performance levels.

Table I
CHATTERING EVALUATION

HOSM strategy	Variable	L_2-norm	L_∞-norm
variable-gain	σ_1	7.61×10^{-6}	9.16×10^{-6}
	σ_2	5.5×10^{-3}	6.5×10^{-3}
switched-gain	σ_1	7.94×10^{-6}	1.07×10^{-5}
	σ_2	5.3×10^{-3}	5.3×10^{-3}
fixed-gain	σ_1	9.98×10^{-6}	1.35×10^{-5}
	σ_2	9.5×10^{-3}	9.5×10^{-3}

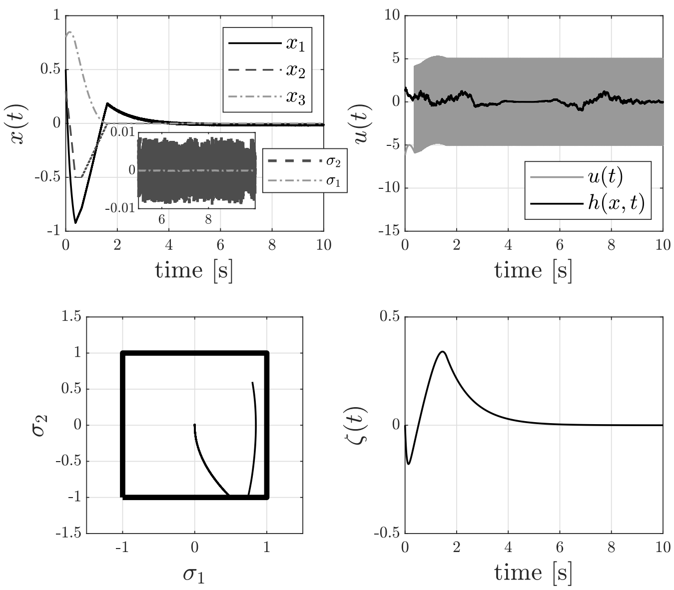

Figure 3. Constant gain. From the top left: time evolution of the state x; time evolution of the control input u, disturbance h and constant gain $\bar{U}$; auxiliary phase-space with box state constraints $\mathcal{S}$; time evolution of the internal dynamics ζ

REFERENCES

[1] G. Bartolini, A. Ferrara, E. Usai, and V. Utkin, "On multi-input chattering-free second-order sliding mode control," *IEEE T Autom Contr*, vol. 45, no. 9, pp. 1711–1717, 2000.

[2] G. Bartolini, A. Ferrara, and E. Usai, "Chattering avoidance by second-order sliding mode control," *IEEE T Autom Contr*, vol. 43, no. 2, pp. 241–246, 1998.

[3] A. Levant, "Higher-order sliding modes, differentiation and output-feedback control," *Int J Control*, vol. 76, no. 9-10, pp. 924–941, 2003.

[4] ——, "Quasi-continuous high-order sliding-mode controllers," *IEEE T Autom Contr*, vol. 50, no. 11, pp. 1812 – 1816, 2005.

[5] T. Floquet, J.-P. Barbot, and W. Perruquetti, "Higher-order sliding mode stabilization for a class of nonholonomic perturbed systems," *Automatica*, vol. 39, no. 6, pp. 1077 – 1083, 2003.

[6] Y. B. Shtessel, L. Fridman, and A. Zinober, "Higher order sliding modes," *Int J Robust Nonlin Contr*, vol. 18, no. 4-5, pp. 381–384, 2008.

[7] Y. Shtessel, C. Edwards, L. Fridman, and A. Levant, "Higher-order sliding mode controllers and differentiators," in *Sliding Mode Control and Observation*. Springer, 2014, pp. 213–249.

[8] F. Dinuzzo and A. Ferrara, "Higher order sliding mode controllers with optimal reaching," *IEEE T Autom Contr*, vol. 54, no. 9, pp. 2126 –2136, 2009.

[9] V. Utkin and A. Poznyak, "Adaptive sliding mode control," in *Advances in Sliding Mode Control*. Springer, 2013, pp. 21–53.

[10] Y. Shtessel, J. A. Moreno, F. Plestan, L. M. Fridman, and A. Poznyak, "Super-twisting adaptive sliding mode control: A Lyapunov design," in *Proc. IEEE Conf Decision Contr*, 2010, pp. 5109–5113.

[11] F. Plestan, Y. Shtessel, V. Bregeault, and A. Poznyak, "New methodologies for adaptive sliding mode control," *Int J Control*, vol. 83, no. 9, pp. 1907–1919, 2010.

[12] Y. Shtessel, M. Taleb, and F. Plestan, "A novel adaptive-gain supertwisting sliding mode controller: Methodology and application," *Automatica*, vol. 48, no. 5, pp. 759–769, 2012.

[13] Y. Huang, T. Kuo, and S. Chang, "Adaptive sliding-mode control for nonlinear systems with uncertain parameters," *IEEE T Sys Man Cybernetics, Part B*, vol. 38, no. 2, pp. 534–539, 2008.

[14] V. Utkin and A. Poznyak, "Adaptive sliding mode control with application to super-twist algorithm: Equivalent control method," *Automatica*, vol. 49, no. 1, pp. 39–47, 2013.

[15] G. Bartolini, A. Pisano, and E. Usai, "Global Stabilization for Nonlinear Uncertain Systems with Unmodeled Actuator Dynamics," *IEEE T Autom Contr*, vol. 46, no. 11, pp. 1826–1832, 2001.

[16] A. Ferrara, L. Magnani, and R. Scattolini, "A Globally Stabilizing Hybrid Variable Structure Control Strategy," *IEEE T Autom Contr*, vol. 47, no. 8, pp. 1334–1337, 2002.

[17] A. Pisano, M. Tanelli, and A. Ferrara, "Switched/time-based adaptation for second-order sliding mode control," *Automatica*, vol. 64, pp. 126–132, 2016.

[18] M. Tanelli and A. Ferrara, "Enhancing Robustness and Performance via Switched Second Order Sliding Mode Control," *IEEE T Autom Contr*, vol. 58, no. 4, pp. 962–974, 2013.

[19] G. P. Incremona, M. Rubagotti, and A. Ferrara, "Sliding mode control of constrained nonlinear systems," *IEEE Trans Autom Contr*, vol. 62, no. 6, pp. 2965–2972, 2017.

[20] A. Isidori, *Nonlinear Control Systems*. London: 3rd Edition, Spriger, 1995.

[21] G. P. Incremona, M. Cucuzzella, and A. Ferrara, "Second order sliding mode control for nonlinear affine systems with quantized uncertainty," *Automatica*, vol. 86, pp. 46–52, 2017.

[22] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, ser. Mathematics and its Applications, F. Arscott, Ed. Springer Netherlands, 1988, vol. 18.

[23] U. Walther, T. T. Georgiou, and A. Tannenbaum, "On the computation of switching surfaces in optimal control: a Gröbner basis approach," *IEEE Trans. Autom. Control*, vol. 46, no. 4, pp. 534–540, 2001.

[24] F. Dinuzzo and A. Ferrara, "Finite-time output stabilization with second order sliding modes," *Automatica*, vol. 45, no. 9, pp. 2169–2171, 2009.

[25] A. Swikir and V. I. Utkin, "Chattering analysis of conventional and super twisting sliding mode control algorithm," in *Proc. Int Workshop on Variable Structure Systems*, 2016, pp. 98–102.

[26] V. I. Utkin, *Sliding modes in control and optimization*. Springer Science & Business Media, 2013.

[27] J. J. E. Slotine and W. Li, *Applied nonlinear control*. Prentice-Hall Englewood Cliffs, NJ, 1991, vol. 60.

[28] H. Khalil, *Nonlinear Systems*. Upper Saddle River, New Jersey: 2nd Edition, Prentice Hall, 1996.