Constrained Nonlinear Discrete-Time Sliding Mode Control Based on a Receding Horizon Approach

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***Abstract*—In this paper, a discrete-time sliding mode control law is proposed for nonlinear (possibly multi-input) systems, in the presence of mixed input-state constraints and additive bounded disturbances. The control law is defined by formulating a nonlinear predictive control problem aimed at generating a control input that imitates an unconstrained discrete-time sliding mode law. In addition to satisfying input and state constraints, the resulting control law has all the properties of discrete-time sliding mode, and in particular, finite time convergence of the state onto the sliding manifold in the nominal case, or into an a-priori defined boundary layer of the sliding manifold in case bounded disturbances are present.**

***Index Terms*—Sliding mode control, uncertain systems, nonlinear predictive control, constrained control.**

I. Introduction

In continuous-time sliding mode control (SMC), a discontinuous control law enforces convergence of the state on a manifold $\mathcal{S}$ (*sliding manifold*), containing all state values for which a suitable *sliding variable* is equal to zero. By enforcing the state to remain on $\mathcal{S}$, the SMC law aims at state convergence to the origin, exploiting strong robustness properties [1]. In practice, dealing with finite sampling frequencies often leads to defining SMC laws in discrete time, namely *discrete-time SMC (DSMC)*. Different DSMC approaches were proposed, among others, in [1]–[7]. Due to the discretization of the dynamics, the presence of disturbance terms with DSMC laws only allows convergence of the state into a *boundary layer* of $\mathcal{S}$.

To define DSMC laws that account for both input and state constraints, one can combine DSMC with model predictive control (MPC), also known as receding-horizon control: due to the complexity of stabilizing nonlinear MPC schemes, this approach was mostly studied for linear systems. The first works in this field merged DSMC with generalized linear predictive control [8]. The same approach was extended to nonlinear systems, but relying on a linear approximation of the dynamics [9]. As an alternative, in [10], the DSMC reaching law of [3] generates a reference for an MPC controller, for the case of unconstrained single-input uncertain linear systems. Also, [11] proposes an MPC law for single-input perturbed linear systems, which guarantees asymptotic convergence of the state to a boundary layer of $\mathcal{S}$. Moreover, [12] presents single-input MPC laws for unperturbed linear and nonlinear systems, where $\mathcal{S}$ is used to define the terminal constraint of the MPC problem, while [13] proposes a DSMC law for linear multi-input systems based on the solution of a robust linear MPC problem: the resulting control law guarantees *finite-time convergence* of the state onto $\mathcal{S}$ (in case of vanishing disturbance) or into an a-priori determined boundary layer of it (in case of persistent disturbance). DSMC for setpoint tracking in constrained linear systems was studied in [14] and [15]: [14] relies on a dual-mode receding horizon DSMC law which exploits the flatness property of suitably defined sliding hyperplanes, while [15] proposes a second-order DSMC scheme to add virtual reference variables to the receding horizon law.

Rather than merging MPC and DSMC in a single control law, a different approach composes separate DSMC and MPC laws into a single feedback scheme. In particular, in [16], integral SMC provides robustness to an explicit MPC controller, which aims at stabilizing the nominal system. In [17], a multirate scheme is proposed, consisting of a DSMC law which reduces the magnitude of the disturbance terms to be handled by a robust nonlinear MPC controller. The work in [18] introduces an integral SMC term into an MPC law, to compensate matched disturbances, thus maintaining the state inside a boundary layer of $\mathcal{S}$ throughout the entire response; the same idea is extended to unmatched disturbances in [19]. Finally, [20] compares different approaches for combining SMC and MPC.

In this paper, we present a DSMC law for (possibly multi-input) nonlinear systems with unknown but bounded additive disturbances, and inequality constraints on inputs and states. The general case of mixed input-state constraints is considered: this includes the case of separate constraints on inputs and states, considered in the cited works, as a particular case. First, an underlying DSMC law for nonlinear systems with additive disturbances, which does not take the presence of constraints into account, is defined based on the general multi-input formulation described in [1]. A nonlinear MPC law is thus defined to "imitate" the underlying DSMC law while enforcing input and state constraints: the resulting control law, though based on the solution of a nonlinear MPC problem, is proven to be still a DSMC law. Indeed, it guarantees *finite-time convergence* of the state into an a-priori determined boundary layer of $\mathcal{S}$ (in case of persistent disturbances), or onto $\mathcal{S}$ itself (whenever the disturbance terms vanish).

The approach presented in [13] was based on the linear MPC approach of [21], that can be applied for linear system dynamics and linear inequality constraints. Instead, given the need to satisfy arbitrary nonlinear inequality constraints for nonlinear

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dynamical systems in the presence of disturbances, in this work constraint satisfaction is achieved based on a tightening approach, relying on Lipschitz continuity. This approach is inspired by the idea proposed in the fundamental work for robust nonlinear MPC described in [22], which is recast into the SMC framework, giving rise to an original constrained DSMC approach. Indeed, while the MPC law of [22] consisted of a sequence of open-loop control moves, on the other hand imitating a DSMC law requires using state-feedback control policies. As a consequence, several results on the formulation of tightened constraints have to be redefined, as the results of [22] cannot be applied in the considered setting. Finally, the convergence results are obtained by generalizing the ideas defined for linear MPC in [21] and in its extension to mixed input-state constraints [23].

Apart from the listed technical challenges, the main contribution of this work consists of defining a DSMC law for uncertain and possibly multi-input nonlinear systems, directly based on the nonlinear dynamics, that guarantees the satisfaction of both input and state constraints in a general form. Such a result, to the best of our knowledge, is not available in the literature, as the cited papers either deal with linear systems [8], [10], [11], [13]–[15] or linear systems approximations [9], or generate an overall control law that is not a DSMC law [12], or finally use a DSMC controller to enhance the robustness properties of a separate MPC controller [16]–[20]. Notice that, when the state constraints are defined directly on the components of the sliding variable, one could design a sliding mode controller without the need for receding-horizon approaches, so as to force the state to slide on the boundary of the admissible region (i.e., the region of the state space in which all state constraints are satisfied), whenever this boundary is reached, thus satisfying the imposed constraints (see, e.g., [24] for continuous-time SMC). On the other hand, when the inequality constraints are also defined on components of the state vector that cannot be mapped onto the components of the sliding variable (which is the case of this paper), once the boundaries of the constraint set are reached, it will be impossible, in general, to avoid constraint violation: this is why a prediction of the state trajectory (and thus, a receding-horizon control law) is used in this work to define a DSMC controller.

II. Notation

Let $\mathbb{N}_{\geq 0}$ denote the set of non-negative integers. Given two integer values $a_i \leq a_f$, let $\mathbb{N}_{[a_i,a_f]} \triangleq \{a_i, a_i+1, ..., a_f\}$. The interior of a set $\mathcal{Q} \subseteq \mathbb{R}^n$ is denoted as $\text{int}(\mathcal{Q})$. $\mathcal{Q}$ is said to be *compact* if it is closed and bounded. Given a vector $v \in \mathbb{R}^n$, $\|v\|_2$ is its Euclidean norm, while $\|v\|$ can denote any p-norm. However, once a specific norm (e.g., 1-norm) is chosen, it is used throughout the whole development described in the paper whenever the notation $\|v\|$ is present. Given a matrix $M \in \mathbb{R}^{n\times n}$, $\|M\|$ is the induced matrix norm corresponding to $\|v\|$. The Minkowski sum of two sets $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{R}^n$ is defined as $\mathcal{Q}_1 \oplus \mathcal{Q}_2 \triangleq \{x+y : x \in \mathcal{Q}_1, y \in \mathcal{Q}_2\}$, and their Pontryagin difference as $\mathcal{Q}_1 \ominus \mathcal{Q}_2 \triangleq \{x \in \mathbb{R}^n : x+y \in \mathcal{Q}_1,\ \forall y \in \mathcal{Q}_2\}$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *$\mathcal{K}$-function* if it is continuous and strictly increasing, and $\gamma(0) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *$\mathcal{K}_\infty$-function* if it is a $\mathcal{K}$-function and $\lim_{y\to+\infty} \gamma(y) = +\infty$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a *$\mathcal{KL}$-function* if, for any fixed $t \geq 0$, $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for any fixed $y \geq 0$, $\beta(y, \cdot)$ is decreasing and $\lim_{t\to\infty} \beta(y, t) = 0$. To simplify notation, time dependence is omitted when not relevant.

III. The unconstrained DSMC law

Consider the discrete-time nonlinear system

$$x_{t+1} = f(x_t, u_t) + w_t \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector fully available for feedback, $u \in \mathbb{R}^m$ (with $n \geq m$) is the control vector, $f(\cdot,\cdot) : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a smooth vector field, and $w \in \mathbb{R}^n$ is an unknown but bounded disturbance, potentially including both matched and unmatched terms, and satisfying

$$w_t \in \mathcal{W},\ t \in \mathbb{N}_{\geq 0}, \tag{2}$$

where $\mathcal{W} \subset \mathbb{R}^n$ is a compact set that includes the origin. Given the compactness of $\mathcal{W}$, it is always possible to define

$$\bar{w} \triangleq \max_{w\in\mathcal{W}} \|w\|. \tag{3}$$

Without loss of generality, we assume that the initial condition is defined at $t = 0$ as x_0. We define the *sliding variable* as

$$s = h(x) \tag{4}$$

with $h(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ being a smooth nonlinear function determined by the designer of the control system, associated with the sliding manifold $\mathcal{S} \triangleq \{x \in \mathbb{R}^n\ :\ s = h(x) = 0\}$.

Definition 1: A *discrete-time sliding mode* is said to take place on $\mathcal{S}$ if there exists a finite $t_1 \in \mathbb{N}_{\geq 0}$, such that $s_t = 0$, $\forall t \geq t_1$ (adapted from [1, Def. 9.1]). □

Following the basic idea proposed in [1], [4], we define a nonlinear DSMC law $u^{\text{sm}}(x)$ by solving $h(f(x, u^{\text{sm}}(x))) = 0$, assuming that a solution to this system of nonlinear equations can be found analytically $\forall x \in \mathbb{R}^n$. This control law ensures the attainment of a discrete-time sliding mode in one time step when $\mathcal{W} = \{0\}$.

Remark 1: In case of control-affine dynamics, i.e., $f(x, u) = f'(x) + g(x)u$ with $f' : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ smooth vector functions, one can find an expression for $u^{\text{sm}}(x)$, if the sliding variable is chosen as a linear function of the state, namely $h(x) = Cx,\ C \in \mathbb{R}^{m\times n}$. Then, $u^{\text{sm}}(x) = -\left(Cg(x)\right)^{-1} Cf'(x)$, which has solution $\forall x \in \mathbb{R}^n$, assuming invertibility of $Cg(x) \in \mathbb{R}^{m\times m}$, $\forall x \in \mathbb{R}^n$. □

The closed-loop dynamics, now depending only on the initial conditions and on the time evolution of the disturbance term, can be written as

$$x_{t+1} = f(x_t, u^{\text{sm}}(x_t)) + w_t = \phi(x_t) + w_t \tag{5}$$

where $\phi(x_t) \triangleq f(x_t, u^{\text{sm}}(x_t))$ represents the nominal closed-loop system dynamics.

The presence of a non-zero disturbance term (both matched and unmatched) prevents the ideal attainment of a discrete-time sliding mode as described in Definition 1. Instead, one can obtain a practical sliding motion, defined in the following.

Definition 2: A *practical sliding motion* is said to take place on $\mathcal{S}$ if there exists a finite $t_1 \in \mathbb{N}_{\geq 0}$ and a $\mathcal{K}$-function $\varepsilon(\cdot)$ such that $\|s_t\| \leq \varepsilon(\bar{w})$, $\forall t \geq t_1$. □
In other words, during a practical sliding motion, the state is confined in a *boundary layer* of $\mathcal{S}$ whose thickness decreases with the size of $\mathcal{W}$; if $\mathcal{W} = \{0\}$, the practical sliding motion turns into an ideal sliding motion according to Definition 1.

The convergence of the state to (a boundary layer of) $\mathcal{S}$ does not necessarily guarantee any stability property for system (5). In the practice of sliding mode controllers design (see, e.g., [1]), it is always required that the state of system (5) converges to the origin when $\mathcal{W} = \{0\}$, or to a neighborhood of the origin when disturbance terms are present, which is obtained by a careful definition of the sliding variable during the control design phase. We formally state this property by referring to the concept of input-to-state-stability [25].

Assumption 1: System (5) is globally input-to-state-stable (ISS), i.e., there exist a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathcal{K}$-function $\gamma(\cdot)$ such that, for $x_0 \in \mathbb{R}^n$ and $w_t \in \mathcal{W}$,

$$\|x_t\|_2 \leq \beta\left(\|x_0\|_2, t\right) + \gamma(\hat{w}_t), \tag{6}$$

for $t \in \mathbb{N}_{\geq 0}$, where $\hat{w}_t \triangleq \max_{k\in[0,t-1]} \|w_k\|_2$. □
A way to prove that (5) is ISS is to find an ISS Lyapunov function [25, Lem. 3.5]. A standard formulation for it, based on [25, Def. 3.2], is the following.

Definition 3: A continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is an ISS Lyapunov function for system (5) if there exist $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that

$$\alpha_1(\|x\|_2) \leq V(x) \leq \alpha_2(\|x\|_2),\ \forall x \in \mathbb{R}^n \tag{7}$$

and also there exist a $\mathcal{K}_\infty$-function $\alpha_3(\cdot)$ and a $\mathcal{K}$-function $\sigma(\cdot)$, such that

$$V(\phi(x)) - V(x) \leq -\alpha_3(\|x\|_2) + \sigma(\|w\|_2) \tag{8}$$

$\forall x \in \mathbb{R}^n$ and $\forall w \in \mathcal{W}$, with $t \in \mathbb{N}_{\geq 0}$. □
By the concept of *asymptotic gain* [25, Sec. 3.2], the existence of an ISS Lyapunov function $V(x)$ implies that x_t, given dynamics (5), asymptotically converges to a bounded set

$$\Theta_f \triangleq \left\{x \in \mathbb{R}^n \ :\ \|x\|_2 \leq \gamma_a\left(\limsup_{t\to\infty} \|w_t\|_2\right)\right\}, \tag{9}$$

where $\gamma_a(\cdot)$ is a $\mathcal{K}$-function defined based on the functions introduced in Definition 3 (we refer the reader interested in the details of the calculation of $\gamma_a(\cdot)$ to [25, Sec. 3]).

Remark 2: As the application of $u^{\mathrm{sm}}(x)$ can require a large control action, saturation bounds are typically imposed on the components of the control vector when an approach such as that of [1] is followed. As a result the convergence of x to $\mathcal{S}$ is in general attained only after a finite number of steps [1]. □

IV. Nominal and perturbed dynamics

In this paper, mixed inequality constraints are imposed on the discrete-time evolution of control inputs and states, as

$$z \in \mathcal{Z}, \tag{10}$$

with $\mathcal{Z}$ compact set including the origin in its interior, and

$$z \triangleq \begin{bmatrix} x^\top & u^\top \end{bmatrix}^\top \in \mathbb{R}^{n+m}. \tag{11}$$

Since $u^{\mathrm{sm}}(x)$ is not designed taking condition (10) into account, its direct application might lead to constraints violation, even in the absence of disturbance terms.

To define a control strategy to cope with these constraints in the presence of disturbances, one has to bound the discrepancy between nominal (i.e., for $w_t = 0$) and perturbed (i.e., with $w_t \in \mathcal{W}$) dynamics. This can be obtained by introducing sets $\mathcal{X}_z$ and $\mathcal{U}_z$, defined as the *projections* of $\mathcal{Z}$ onto the state space and the input space, repectively [26].

Assumption 2: Function $f(\cdot,\cdot)$ is a Lipschitz function of its first argument, i.e., there exists $L_{f_x} \in \mathbb{R}_{\geq 0}$ such that

$$\|f(x_1, u) - f(x_2, u)\| \leq L_{f_x}\|x_1 - x_2\|, \tag{12}$$

$\forall (x_1, x_2) \in \mathcal{X}_z$, and $\forall u \in \mathcal{U}_z$. □

Assumption 3: Function $f(\cdot,\cdot)$ is a Lipschitz function of its second argument, i.e., there exists $L_{f_u} \in \mathbb{R}_{\geq 0}$ such that

$$\|f(x, u_1) - f(x, u_2)\| \leq L_{f_u}\|u_1 - u_2\| \tag{13}$$

$\forall x \in \mathcal{X}_z$ and $\forall (u_1, u_2) \in \mathcal{U}_z$. □

Assumption 4: Given $x \in \mathcal{X}_z$, $u^{\mathrm{sm}}(x)$ is a Lipschitz function, i.e., there exists $L_u \in \mathbb{R}_{\geq 0}$ such that

$$\|u^{\mathrm{sm}}(x_1) - u^{\mathrm{sm}}(x_2)\| \leq L_u\|x_1 - x_2\| \tag{14}$$

$\forall (x_1, x_2) \in \mathcal{X}_z$. □

The following result analyzes the discrepancy between nominal evolutions of the system, from different initial conditions.

Lemma 1: Suppose Assumptions 2-4 hold. Consider an initial condition $\hat{x}_t$, and define a sequence of control policies

$$\hat{u}_{t+k} = u^{\mathrm{sm}}(\hat{x}_{t+k}) + \Delta u_{t+k},\ k \in \mathbb{R}_{\geq 0}, \tag{15}$$

where Δu_{t+k} are fixed additive control moves. Starting at $\hat{x}_t$, this control sequence generates the nominal state values given by $\hat{x}_{t+k+1} = f(\hat{x}_{t+k}, \hat{u}_{t+k})$, for $k \in \mathbb{R}_{\geq 0}$. The notation $\hat{x}_{t+k}$ simply refers to a specific (nominal) time evolution of x. Introduce another nominal sequence of states $\bar{x}_{t+k}$, defined by

$$\bar{u}_{t+k} = u^{\mathrm{sm}}(\bar{x}_{t+k}) + \Delta u_{t+k}, \tag{16}$$

given Δu_{t+k} as above, and initial condition $\bar{x}_t = \hat{x}_t + \Delta x$, with $\Delta x \in \mathcal{W}$. This generates the sequence of states $\bar{x}_{t+k}$ determined by $\bar{x}_{t+k+1} = f(\bar{x}_{t+k}, \bar{u}_{t+k})$, for $k \in \mathbb{R}_{\geq 0}$. The notation $\bar{x}_{t+k}$, similarly to $\hat{x}_{t+k}$, also refers to a specific (nominal) time evolution of the state x. Assume

$$\hat{z}_{t+k} = \begin{bmatrix} \hat{x}_{t+k} \\ \hat{u}_{t+k} \end{bmatrix} \in \mathcal{Z},\ \bar{z}_{t+k} = \begin{bmatrix} \bar{x}_{t+k} \\ \bar{u}_{t+k} \end{bmatrix} \in \mathcal{Z}, \tag{17}$$

$\forall k \in \mathbb{N}_{\geq 0}$. Then, given $L_x \triangleq L_{f_u} L_u + L_{fx}$,

$$\|\bar{x}_{t+k} - \hat{x}_{t+k}\| \leq L_x^k \bar{w},\ \forall k \in \mathbb{N}_{\geq 0}. \tag{18}$$

Proof: For $k = 0$, we have that $\|\bar{x}_t - \hat{x}_t\| \leq \bar{w}$. For $k = 1$, $\|\bar{x}_{t+1} - \hat{x}_{t+1}\| = \|f(\bar{x}_t, \bar{u}_t) - f(\hat{x}_t, \hat{u}_t)\|$. By adding and subtracting $f(\bar{x}_t, \hat{u}_t) = f(\bar{x}_t, u^{\mathrm{sm}}(\hat{x}_t) + \Delta u_t)$, we obtain $\|\bar{x}_{t+1} - \hat{x}_{t+1}\| = \|f(\bar{x}_t, \bar{u}_t) - f(\hat{x}_t, \hat{u}_t) + f(\bar{x}_t, \hat{u}_t) - f(\bar{x}_t, \hat{u}_t)\|$. Thus, by applying the triangular inequality together with Assumptions 2 and 3 and then Assumption 4, noticing that the terms Δu_t are common to both control policies and can therefore be simplified, we obtain

$$\begin{aligned}&\|\bar{x}_{t+1}-\hat{x}_{t+1}\| \le \|f(\bar{x}_t,\bar{u}_t)-f(\bar{x}_t,\hat{u}_t)\|+\\&+\|f(\hat{x}_t,\hat{u}_t)-f(\bar{x}_t,\hat{u}_t)\| \le L_{f_u}\|\bar{u}_t-\hat{u}_t\|+L_{f_x}\|\bar{x}_t-\hat{x}_t\|\\&=L_{f_u}\|u^{\rm sm}(\bar{x}_t)-u^{\rm sm}(\hat{x}_t)\|+L_{f_x}\|\bar{x}_t-\hat{x}_t\|\\&\le (L_{f_u}L_u+L_{f_x})\bar{w}.\end{aligned} \tag{19}$$

Analogously, for $k=2$ we obtain $\|\bar{x}_{t+2}-\hat{x}_{t+2}\| \le (L_{f_u}L_u+L_{f_x})^2\bar{w}$, and iterating for any step k, $\|\bar{x}_{t+k}-\hat{x}_{t+k}\| \le (L_{f_u}L_u+L_{f_x})^k\bar{w}=L_x^k\bar{w}$, which concludes the proof. ■

Remark 3: Notice that Assumptions 2-3 had to be defined with respect to $\mathcal{X}_z$ and $\mathcal{U}_z$, rather than directly to $\mathcal{Z}$, to prove Lemma 1. Indeed, despite the validity of (17), it is not guaranteed that $\begin{bmatrix}\bar{x}_{t+k}^\top & \hat{u}_{t+k}^\top\end{bmatrix}^\top \in \mathcal{Z}$. However, (17) guarantees that $\bar{x}_{t+k}\in\mathcal{X}_z$ and $\hat{u}_{t+k}\in\mathcal{U}_z$, which permits the application of Assumptions 2-3 to claim that $\|f(\bar{x}_t,\bar{u}_t)-f(\bar{x}_t,\hat{u}_t)\| \le L_{f_u}\|\bar{u}_t-\hat{u}_t\|$ and $\|f(\hat{x}_t,\hat{u})-f(\bar{x}_t,\hat{u}_t) \le L_{f_x}\|\bar{x}_t-\hat{x}_t\|$. These results are used to obtain (19). □

Given Lemma 1, we can prove the next result, which analyzes the discrepancy between nominal and perturbed evolutions of the system, starting from the same initial conditions. The same rationale of [22, Lemma 1] is still valid; the fundamental difference is that the control law applied in [22] consisted of an "open-loop" sequence of control moves, while in our case it is a sequence of state-feedback control policies.

Lemma 2: Suppose Assumptions 2-4 hold. Consider an initial condition $\hat{x}_t$, together with the sequence of control policies $\hat{u}_{t+k}$ and corresponding state evolution $\hat{x}_{t+k}$, with $k\in\mathbb{R}_{\ge 0}$, as defined in Lemma 1. Also, introduce a perturbed evolution x_{t+k}, obtained applying $u_{t+k}=u^{\rm sm}(x_{t+k})+\Delta u_{t+k}$ to the system dynamics (1)-(2), given the same initial condition $x_t=\hat{x}_t$, and the same fixed additive control moves Δu_{t+k}. Assume also that, $\forall k\in\mathbb{N}_{\ge 0}$,

$$\hat{z}_{t+k}=\begin{bmatrix}\hat{x}_{t+k}\\ \hat{u}_{t+k}\end{bmatrix}\in\mathcal{Z},\;\; z_{t+k}=\begin{bmatrix}x_{t+k}\\ u_{t+k}\end{bmatrix}\in\mathcal{Z}. \tag{20}$$

Then, this result holds for $k\in\mathbb{N}_{\ge 0}$:

$$\|x_{t+k}-\hat{x}_{t+k}\| \le \eta_k\bar{w}. \tag{21}$$

with

$$\eta_k=\begin{cases}\frac{L_x^k-1}{L_x-1}, & L_x\ne 1,\\ k, & L_x=1.\end{cases} \tag{22}$$

Proof: For $k=0$, $x_t=\hat{x}_t$ by definition, which satisfies (21). For $k=1$, $\|x_{t+1}-\hat{x}_{t+1}\|=\|w_t\|\le\bar{w}$, which also satisfies (21). For $k=2$, applying equation (19) in Lemma 1, together with the triangular inequality, we obtain $\|x_{t+2}-\hat{x}_{t+2}\| \le L_x\|x_{t+1}-\hat{x}_{t+1}\|+\|w_{t+1}\| \le (L_x+1)\bar{w}$, for which (21) is again satisfied. Generalizing to $k\in\mathbb{N}_{\ge 1}$, $\|x_{t+k}-\hat{x}_{t+k}\| \le L_x\|x_{t+k-1}-\hat{x}_{t+k-1}\|+\|w_{t+k-1}\| \le \sum_{i=0}^{k-1}L_x^i\bar{w}$. Indeed, by explicitly writing the expression of the given geometric series, we obtain

$$\sum_{i=0}^{k-1}L_x^i\bar{w}=\begin{cases}\frac{L_x^k-1}{L_x-1}\bar{w}, & L_x\ne 1,\\ k\bar{w}, & L_x=1,\end{cases}$$

which, given the definition of η_k in (22), is equivalent to inequality (21). ■

The discrepancy (in p-norm) between nominal and perturbed evolutions of system (1) for $k\in\mathbb{N}_{\ge 0}$ therefore belongs to the set

$$\mathcal{B}_x^k \triangleq \{\xi\in\mathbb{R}^n \;:\; \|\xi\|\le\eta_k\bar{w}\}. \tag{23}$$

Given the mixed-constraints formulation (10), results analogous to those in Lemmas 1 and 2, respectively, have to be carried out to extend our analysis from the evolution of x_t to that of z_t.

Lemma 3: Given system (5) satisfying Assumptions 2-4, consider the state-input evolutions $\hat{z}_{t+k}\in\mathcal{Z}$ and $\bar{z}_{t+k}\in\mathcal{Z}$ defined within Lemma 1 in (17), for $k\in\mathbb{N}_{\ge 0}$. After defining

$$\bar{w}_z \triangleq \left\|\begin{bmatrix}1 & L_u\end{bmatrix}\right\|\cdot\bar{w}, \tag{24}$$

we obtain

$$\|\bar{z}_{t+k}-\hat{z}_{t+k}\| \le L_x^k\bar{w}_z. \tag{25}$$

Proof: This lemma can be proven by extending the results in Lemma 1: the proof is omitted due to space limitation. ■

Lemma 4: Given system (5) satisfying Assumptions 2-4, consider the nominal evolution $\hat{z}_{t+k}\in\mathcal{Z}$ and perturbed evolution $z_{t+k}\in\mathcal{Z}$, for $k\in\mathbb{N}_{\ge 0}$, as defined within Lemma 2 in (20). Recalling the definition of $\bar{w}_z$ in (24), we obtain

$$\|z_{t+k}-\hat{z}_{t+k}\| \le \eta_k\bar{w}_z. \tag{26}$$

Proof: This lemma can be proven by extending the results in Lemma 2: the proof is omitted due to space limitation. ■

Similarly to (23), we define

$$\mathcal{B}_z^k \triangleq \left\{\zeta\in\mathbb{R}^{n+m} \;:\; \|\zeta\|\le\eta_k\bar{w}_z\right\}. \tag{27}$$

Having in mind the idea to design a control algorithm to take into account the uncertain terms, in order to guarantee that the constraint $z\in\mathcal{Z}$ is satisfied for any feasible realization of the disturbance, let $\mathcal{Z}_k$ be a tightened set defined as

$$\mathcal{Z}_k=\mathcal{Z}\ominus\mathcal{B}_z^k. \tag{28}$$

From the definition of $\mathcal{B}_z^k$ in (27) and the result of Lemma 4, one can see that

$$\hat{z}_{t+k}\in\mathcal{Z}_k \Rightarrow z_{t+k}\in\mathcal{Z},\;\; w_{t+k}\in\mathcal{W},\; k\in\mathbb{N}_{\ge 0}. \tag{29}$$

V. Properties of the unconstrained DSMC law

In Section IV, we determined how to relate nominal and perturbed dynamics, and system constraints. These results are now used to introduce further properties of the control law $u^{\rm sm}(x)$, and to define new sets useful for the definition of our constrained DSMC law.

According to Assumption 1, in absence of constraints, system (5) is ISS and therefore admits an ISS Lyapunov function $V(x)$ as in Definition 3. Given $V(x)$ and a constant $\alpha_\Theta\in\mathbb{R}_{>0}$, we introduce the invariant set in nominal conditions

$$\Theta \triangleq \{x\in\mathbb{R}^n \;:\; V(x)\le\alpha_\Theta\}, \tag{30}$$

defined such that, given the set Θ_f introduced in (9), one has

$$\Theta_f\subset\mathrm{int}(\Theta), \quad x\in\Theta\Rightarrow\begin{bmatrix}x\\ u^{\rm sm}(x)\end{bmatrix}\in\mathcal{Z}. \tag{31}$$

Another set is defined as

$$\Omega \triangleq \{x \in \mathbb{R}^n \ : V(x) \le \alpha_\Omega\}, \tag{32}$$

with $\alpha_\Omega \in [0, \alpha_\Theta]$, which is required to satisfy

$$x \in \Theta \Rightarrow \phi(x) \in \Omega. \tag{33}$$

Assumption 5: In addition to being positive definite (see, e.g., [25]), $V(\cdot)$ is a Lipschitz function in $\mathcal{A} \triangleq \Theta \cup (\Omega \oplus \mathcal{W})$. More precisely, there exists $L_V \in \mathbb{R}_{\ge 0}$ such that

$$|V(x_1) - V(x_2)| \le L_V \|x_1 - x_2\| \tag{34}$$

$\forall x_1, x_2 \in \mathcal{A}$. □

Remark 4: Notice that the common choice, when possible, of using a quadratic ISS Lyapunov function would automatically satisfy Assumption 5 on any bounded set. However, the Lipschitz continuity is required to hold in $\mathcal{A}$ only. It is in our interest not to consider a region larger than needed, in order to obtain a value of L_V that is as low as possible.

Assumption 6: The following holds:

$$\bar{w} \le \frac{\alpha_\Theta - \alpha_\Omega}{L_V \bar{L}_x}, \tag{35}$$

where $\bar{L}_x \ge 1$. □

Notice that, as the size of Θ_f and the value of $\bar{w}$ depend (in case of persistent disturbances) on how large set $\mathcal{W}$ is, condition (35) can be satisfied as long as the disturbance terms are small enough.

Lemma 5: Under Assumptions 5-6, Θ is a robust positively invariant (RPI) set [26] for the closed-loop system (5), i.e., $x \in \Theta \Rightarrow \phi(x) + w \in \Theta, \ \forall w \in \mathcal{W}$.

Proof: The definition of Ω in (32) ensures that $x \in \Theta \Rightarrow \phi(x) \in \Omega$. As $x \in \Theta \subseteq \mathcal{A}$ and $\phi(x) + w \in \Omega \oplus \mathcal{W} \subseteq \mathcal{A}$, by virtue of Assumptions 5 and 6, and taking into account the definition of $\bar{L}_x$, $|V(\phi(x) + w) - V(\phi(x))| \le L_V \bar{w} \le \frac{\alpha_\Theta - \alpha_\Omega}{\bar{L}_x} \le \alpha_\Theta - \alpha_\Omega$. Starting from this last inequality, we distinguish two cases: (a) If $V(\phi(x)+w) - V(\phi(x)) \ge 0$, then one can claim that $V(\phi(x)+w) - V(\phi(x)) \le \alpha_\Theta - \alpha_\Omega$, which implies $V(\phi(x) + w) - \alpha_\Theta \le V(\phi(x)) - \alpha_\Omega \le 0$, leading to $V(\phi(x) + w) \le \alpha_\Theta$, and thus, by the definition of Θ in (30), $\phi(x) + w \in \Theta$. (b) On the other hand, if $V(\phi(x) + w) - V(\phi(x)) < 0$, i.e., $V(\phi(x) + w) < V(\phi(x))$, then $\phi(x) + w \in \Omega \subseteq \Theta$ follows from the definition of Ω in (32). ■

We now detail how *practical sliding motion* can be achieved by the perturbed system in the presence of constraints by applying $u^{\mathrm{sm}}(x)$ for initial conditions $x_t \in \Theta$.

Assumption 7: Function $h(\cdot)$ in (4) is defined such that

$$\|h(x_1) - h(x_2)\| \le L_h(\|x_1 - x_2\|) \tag{36}$$

$\forall (x_1, x_2) \in \Theta$, where $L_h \in \mathbb{R}_{\ge 0}$ is the corresponding Lipschitz constant. □

Lemma 6: Consider system (5) (i.e., the closed-loop system obtained by applying the underlying DSMC law), with initial condition $x_t \in \Theta$. If Assumptions 5-7 hold, a practical sliding motion takes place as described in Definition 2, with $\varepsilon(\bar{w}) = L_h \bar{w}$, and $z_{k+t} \in \mathcal{Z}$ (constraints satisfaction) for $t \in \mathbb{N}_{\ge 0}$.

Proof: The definition of $u^{\mathrm{sm}}(x_t)$ implies that $h(\phi(x_t)) = 0$. When introducing the disturbance term, we obtain the following by adding and subtracting $h(\phi(x_t))$: $\|h(\phi(x_t) + w_t)\| = \|h(\phi(x_t) + w_t) - h(\phi(x_t)) + h(\phi(x_t))\|$. Then, being $x_t \in \Theta$, thanks to Assumption 7 we obtain $\|h(\phi(x_t) + w_t)\| \le \|h(\phi(x_t)+w_t) - h(\phi(x_t))\| + \|h(\phi(x_t))\| \le L_h(\|\phi(x_t) + w_t - \phi(x_t)\|) \le L_h \bar{w}$. Given the robust invariance of Θ with respect to system (5), guaranteed by Assumptions 5-6, the same results will hold not only at t, but for $t + k$, $k \in \mathbb{N}_{\ge 0}$. This implies on the one hand that a practical sliding motion is taking place, with $\varepsilon(\bar{w}) = L_h \bar{w}$, and on the other hand (given the second condition in (31)) that $z_{k+t} \in \mathcal{Z}$ for $t \in \mathbb{N}_{\ge 0}$. ■

VI. The receding-horizon control law

The scope of this section is to enlarge (beyond Θ) the set of initial states from which constraint satisfaction is ensured and a practical sliding motion takes place. In order to do it, we define $N \in \mathbb{N}_{\ge 1}$ as the *prediction horizon* of the MPC problem, and extend the requirements of the unconstrained DSMC law as follows.

Assumption 8: Let Assumption 5 hold in $\mathcal{A} \triangleq \Theta \cup (\Omega \oplus \mathcal{W}) \cup (\Omega \oplus \mathcal{B}_L)$, where $\mathcal{B}_L \triangleq \{\xi \in \mathbb{R}^n \ : \ \|\xi\| \le L_x^{N-1} \bar{w}\}$. Moreover, $x \in \Theta$ implies $\begin{bmatrix} x & u^{\mathrm{sm}}(x) \end{bmatrix}^\top \in \mathcal{Z}_{N-1}$, with $\mathcal{Z}_{N-1}$ as in (28) for $k = N - 1$. Finally, Assumption 6 is valid for $\bar{L}_x = \max\{1, L_x^{N-1}\}$. □

The new proposed control law consists of a modification of $u^{\mathrm{sm}}(x)$ given by

$$u^{\mathrm{rh}}(x_t) = u^{\mathrm{sm}}(x_t) + c_t^* \tag{37}$$

where $c_t^* \in \mathbb{R}^m$ is an additive term aimed at ensuring constraints satisfaction. This term is obtained by formulating and solving a finite-horizon optimal control problem (FHOCP) at each time instant t. As the FHOCP is based on predictions, for a given initial state x_t, we refer to predictions at subsequent time instants $t+k$ given time t: for instance, $\hat{x}_{t+k|t}$ is the state predicted at $t + k$ starting at t using the nominal dynamics. A general sequence of terms $c_{t+k|t} \in \mathbb{R}^m$, $k = \mathbb{N}_{[0,N-1]}$, is defined as $\mathbf{c}_t \triangleq \begin{bmatrix} c_{t|t}^\top & c_{t+1|t}^\top & \cdots & c_{t+N-1|t}^\top \end{bmatrix}^\top \in \mathbb{R}^{Nm}$, where N is the *prediction horizon*. The optimal value of the sequence, namely $\mathbf{c}_t^* \triangleq \begin{bmatrix} c_{t|t}^{*\top} & c_{t+1|t}^{*\top} & \cdots & c_{t+N-1|t}^{*\top} \end{bmatrix}^\top$, is obtained at each time instant t by solving the above-mentioned FHOCP. Following the so-called receding-horizon approach, the first element of the sequence is applied to the system as $c_t^* = c_{t|t}^*$, and a new sequence $\mathbf{c}_{t+1}^*$ is determined at the next time instant.

After defining a constant matrix $\Psi \in \mathbb{R}^{m \times m}$, $\Psi = \Psi^\top \succ 0$, the FHOCP is introduced as follows:

$$\mathbf{c}_t^* = \arg\min_{\mathbf{c_t}} \sum_{k=0}^{N-1} c_{t+k|t}^\top \Psi c_{t+k|t} \tag{38a}$$

subj. to

$$\hat{x}_{t|t} = x_t, \tag{38b}$$

$$\hat{x}_{t+k+1|t} = f(\hat{x}_{t+k|t}, u^{\mathrm{sm}}(\hat{x}_{t+k|t}) + c_{t+k|t}), \ k \in \mathbb{N}_{[0,N-1]}, \tag{38c}$$

$$\begin{bmatrix} \hat{x}_{t+k|t} \\ u^{\mathrm{sm}}(\hat{x}_{t+k|t}) + c_{t+k|t} \end{bmatrix} \in \mathcal{Z}_k, \ k \in \mathbb{N}_{[0,N-1]}, \tag{38d}$$

$$\hat{x}_{t+N} \in \Omega. \tag{38e}$$

In (38b)-(38c), the system dynamics is initialized at the measured state x_t, and predicted using the nominal dynamics. Each $c^*_{t|t+k} \in \mathbb{R}^m$ is associated with a corresponding optimal control policy

$$u^*(\hat{x}_{t+k|t}) \triangleq u^{\mathrm{sm}}(\hat{x}_{t+k|t}) + c^*_{t+k|t}, \tag{39}$$

with $k \in \mathbb{N}_{[0,N-1]}$, defined based on the nominal dynamics. It is implicitly assumed that $\mathcal{Z}_k \neq \emptyset$ for $k \in \mathbb{N}_{[0,N-1]}$: indeed, if this condition is not satisfied, the FHOCP (38) is always infeasible. Finally, the constraint (38e) is defined with respect to the compact set $\Omega \subset \mathbb{R}^n$, already introduced in (32). The following result on the receding-horizon control law analyzes constraint satisfaction, based on the possibility of finding a feasible solution of the FHOCP, when a feasible solution was available at the previous time instant (*recursive feasibility*). Notice that this result and all the considered assumptions are instrumental to prove the main result of this work, which will be presented in Theorem 2.

Theorem 1: Define $\mathcal{D}_N$ as the set of initial states x_0 for which problem (38) is feasible. If Assumptions 2-6 and 8 are satisfied, then $\mathcal{D}_N$ is an RPI set for the closed-loop system obtained by applying $u_t = u^{\mathrm{rh}}(x_t)$ introduced in (37) to system (1). Moreover, the satisfaction of the mixed input-state constraints defined in (10) is guaranteed for $t \in \mathbb{N}_{\geq 0}$.

Proof: Given $x_{t-1} \in \mathcal{D}_N$ and the corresponding FHOCP solution, we assume to apply $u^{\mathrm{rh}}(x_{t-1})$ to system (1), thus obtaining $x_t = f(x_{t-1}, u^{\mathrm{rh}}(x_{t-1})) + w_{t-1}$. At the next time instant t, we explicitly define the predicted state sequence

$$\bar{x}_{t+k|t} = \begin{cases} x_t, & k = 0 \\ f(\bar{x}_{t+k-1|t}, u^{\mathrm{sm}}(\bar{x}_{t+k|t}) + c^*_{t+k|t-1}), & \\ & k = \mathbb{N}_{[1,N-1]} \\ f(\bar{x}_{t+N-1|t}, u^{\mathrm{sm}}(\bar{x}_{t+N-1|t-1})), & k = N, \end{cases} \tag{40}$$

obtained with the suboptimal control sequence

$$\bar{u}_{t+k|t} = \begin{cases} u^{\mathrm{sm}}(\bar{x}_{t+k|t}) + c^*_{t+k|t-1}, & k = \mathbb{N}_{[0,N-2]} \\ u^{\mathrm{sm}}(\bar{x}_{t+N-1|t-1}), & k = N-1. \end{cases} \tag{41}$$

Our aim is to show that (40)-(41) constitutes a feasible solution of (38), which implies that an optimal solution at time t can also be found: we do that by verifying that all inequality constraints in (38), i.e., (38d) and (38e), are satisfied, as the equality constraints are satisfied by construction.

To prove that (38d) holds for $k \in \mathbb{N}_{[0,N-2]}$, we start defining

$$\hat{z}^*_{t+k|t-1} \triangleq \begin{bmatrix} \hat{x}^*_{t+k|t-1} \\ u^*(\hat{x}^*_{t+k|t-1}) \end{bmatrix}, \; \bar{z}_{t+k|t} \triangleq \begin{bmatrix} \bar{x}_{t+k|t} \\ \bar{u}_{t+k|t} \end{bmatrix}, \tag{42}$$

in which $\hat{x}^*_{t+k|t-1}$ represents the (optimal) nominal state evolution predicted at $t-1$, and $u^*(\hat{x}^*_{t+k|t-1})$ the corresponding optimal control policies. For $k \in \mathbb{N}_{[0,N-2]}$, $\bar{u}_{t+k|t} = u^{\mathrm{sm}}(\bar{x}_{t+k|t}) + c^*_{t+k|t-1}$, and that by virtue of Lemma 1, $\|\bar{x}_{t+k|t} - \hat{x}^*_{t+k|t-1}\| \leq L_x^k \bar{w}$. Notice that, *mutatis mutandis* (i.e., considering predictions rather than actual time evolutions of the system, substituting Δu_{t+k} with $c^*_{t+k|t-1}$, and Δx with w_{t-1}), and given that $\bar{x}_{t|t} = \hat{x}_{t|t-1} + w_{t-1}$, all assumptions needed for applying Lemma 3 hold. Applying Lemma 3 yields

$$\|\bar{z}_{t+k|t} - \hat{z}^*_{t+k|t-1}\| \leq L_x^k \bar{w}_z. \tag{43}$$

In order the new input sequence to be feasible, we need to prove that $\bar{z}_{t+k|t} \in \mathcal{Z}_k$ for $k \in \mathbb{N}_{0,N-1}$. In order to do so, we introduce disturbance terms $\zeta_k \in \mathcal{B}_z^k$, with the aim to prove, by using (43), that $\bar{z}_{t+k|t} + \zeta_k \in \mathcal{Z}$ for $k \in \mathbb{N}_{[0,N-2]}$ and all $\zeta_k \in \mathcal{B}_z^k$, which would imply $\bar{z}_{t+k|t} \in \mathcal{Z}_k$ for $k \in \mathbb{N}_{[0,N-2]}$. We start by defining a variable $\delta_k \in \mathbb{R}^n$, as

$$\delta_k = \bar{z}_{t+k|t} - \hat{z}^*_{t+k|t-1} + \zeta_k. \tag{44}$$

From (43) and from the definition of $\bar{w}_z$ in Lemma 3, it follows that $\|\delta_k\| \leq \|\bar{z}_{t+k|t} - \hat{z}^*_{t+k|t-1}\| + \|\zeta_k\| \leq L_x^k \bar{w}_z + \eta_k \bar{w}_z = \eta_{k+1} \bar{w}_z$, which implies $\delta_k \in \mathcal{B}_z^{k+1}$ (this result generalizes the idea of [22, Lemma 2] to the case of mixed constraints). By feasibility of (38) at time $t-1$, we know that $\hat{z}^*_{t+k|t-1} \in \mathcal{Z}_{k+1}$, for $k \in \mathbb{N}_{[0,N-2]}$. As a consequence, $\hat{z}^*_{t+k|t-1} + \delta_k \in \mathcal{Z}$, but by means of (44) we also get $\bar{z}_{t+k|t} + \zeta_k \in \mathcal{Z}$ for $k \in \mathbb{N}_{[0,N-2]}$ and all $\zeta_k \in \mathcal{B}_z^k$. Hence, $\bar{z}_{t+k|t} \in \mathcal{Z}_k$ for $k \in \mathbb{N}_{[0,N-2]}$.

In order to prove that (38d) is also satisfied in the prediction at time $t+N-1$, we apply Lemma 1, obtaining

$$\|\bar{x}_{t+N-1|t} - \hat{x}^*_{t+N-1|t-1}\| \leq L_x^{N-1} \bar{w}. \tag{45}$$

Assumption 5 modified as in Assumption 8 can be applied for $\bar{x}_{t+N-1|t}$ and $\hat{x}^*_{t+N-1|t-1}$, as $\hat{x}^*_{t+N-1|t-1} \in \Omega \subseteq \Theta \subseteq \mathcal{A}$, and (45) implies that $\bar{x}_{t+N-1|t} \in \Omega \oplus \mathcal{B}_L \subseteq \mathcal{A}$. Thus, applying (45) and recalling (34), we get $|V(\bar{x}_{t+N-1|t}) - V(\hat{x}^*_{t+N-1|t-1})| \leq L_V \|\bar{x}_{t+N-1|t} - \hat{x}^*_{t+N-1|t-1}\| \leq L_V L_x^{N-1} \bar{w}$. We consider two cases: if $V(\bar{x}_{t+N-1|t}) \geq V(\hat{x}^*_{t+N-1|t-1})$, then, by means of Assumption 6 modified as in Assumption 8, one has $V(\bar{x}_{t+N-1|t}) \leq V(\hat{x}^*_{t+N-1|t-1}) + L_V L_x^{N-1} \bar{w} \leq \alpha_\Omega + L_V L_x^{N-1} \bar{w} \leq \alpha_\Theta$, while, if $V(\bar{x}_{t+N-1|t}) \leq V(\hat{x}^*_{t+N-1|t-1})$, one has $V(\bar{x}_{t+N-1|t}) \leq V(\hat{x}^*_{t+N-1|t-1}) \leq \alpha_\Omega \leq \alpha_\Theta$. In both cases, it is ensured that $\bar{x}_{t+N-1|t} \in \Theta$. By definition of $\bar{u}_{t+N-1|t}$, from the second condition in (31) we obtain that $\bar{z}_{t+N-1|t} \in \mathcal{Z}_{N-1}$. Constraint (38d) is therefore satisfied for $k \in \mathbb{N}_{[0,N-1]}$.

Finally, the definition of Ω (in particular, (33)) guarantees that, being $\bar{x}_{t+N-1|t} \in \Theta$, using $u^{\mathrm{sm}}(\bar{x}_{t+N-1|t})$ one has $\phi(\bar{x}_{t+N-1|t}) = \bar{x}_{t+N|t} \in \Omega$, which satisfies (38e). This proves recursive feasibility.

Regarding the satisfaction of constraint (10) for $t \in \mathbb{N}_{\geq 0}$, given the result in (29), the satisfaction of constraint (38d) in the FHOCP implies that any feasible perturbed state evolution satisfies

$$\begin{bmatrix} x_{t+k|t} \\ u^{\mathrm{sm}}(x_{t+k|t}) + c_{t+k|t} \end{bmatrix} \in \mathcal{Z} \tag{46}$$

for $k \in \mathbb{N}_{0,N-1}$, and $\forall w_{t+k|t} \in \mathcal{W}$. This holds in particular for $k = 1$, which represents the time instant at which the FHOCP will be solved again based on the new state measurement. This result, together with the recursive feasibility property, ensures (by induction) that (10) will be satisfied for $t \in \mathbb{N}_{\geq 0}$. ■

Remark 5: One can see that (35) plays a fundamental role in guaranteeing recursive feasibility. If $\bar{w}$ (which provides a measure of the disturbance term) is too large, then constraint satisfaction cannot be ensured. An increasingly large disturbance measure $\bar{w}$ can be allowed for increasingly large values of $\alpha_\Theta - \alpha_\Omega$ (which reflects the ability of the sliding mode control law to steer the state value close to the origin) and increasingly small values of $L_V \bar{L}_x$. □

The following intermediate result will be useful to show that the proposed MPC law is indeed a constrained DSMC law.

Corollary 1: Consider system (1), satisfying Assumptions 2-6 and 8. Then, $x \in \Theta \Rightarrow u^{\mathrm{rh}}(x) = u^{\mathrm{sm}}(x)$.

Proof: It is immediate to see that the unconstrained global minimizer of (38a) is obtained for $c^*_{t|t+k} = 0$, $k \in \mathbb{N}_{[0,N-1]}$. The reason for it is that $\sum_{k=0}^{N-1} c^\top_{t+k|t} \Psi c_{t+k|t} \geq 0$, and for $c^*_{t|t+k} = 0$, $k \in \mathbb{N}_{[0,N-1]}$, its value is equal to zero. We prove now that this is a feasible solution of the FHOCP (38) if $x \in \Theta$. The equality constraints (38b) and (38c) are satisfied by construction. Also, given the definition of Θ in (30), $\hat{z}^*_{t+k} \in \mathcal{Z}_k$, for $k \in \mathbb{N}_{[0,N-1]}$. This implies the satisfaction of (38d). Finally, (38e) is implied by (32), given that $\hat{z}^*_{N-1} \in \mathcal{Z}_{N-1}$. Given the feasibility of the unconstrained global minimizer, we get $c^*_{t|t} = 0$, and thus $u^{\mathrm{rh}}(x) = u^{\mathrm{sm}}(x)$ for $x \in \Theta$. ∎

The following theorem states the fundamental convergence properties of the proposed constrained DSMC law.

Theorem 2: Consider the closed-loop system (1) with $u_t = u^{\mathrm{rh}}(x_t)$ defined in (37), satisfying Assumptions 1-8. Then, for all initial conditions $x \in \mathcal{D}_N$, $\lim_{t\to\infty} x_t \in \Theta_f$, and a practical sliding motion is achieved in finite time, according to Definition 2, with $\varepsilon(\bar{w}) = L_h \bar{w}$. In case there exists t_1 such that $w_t = 0$ for $t \geq t_1$, then $\lim_{t\to\infty} x_t = 0$, and an ideal sliding motion (see Definition 1) is achieved in finite time.

Proof: Given $x_{t-1} \in \mathcal{D}_N$, consider the optimal value of the cost function of the FHOCP (38) at time $t-1$, namely $J^*_{t-1} \triangleq \sum_{k=0}^{N-1} c^{*\top}_{t+k-1|t-1} \Psi c^*_{t+k-1|t-1}$. By recursive feasibility, Theorem 1 guarantees the existence of $J^*_t \triangleq \sum_{k=0}^{N-1} c^{*\top}_{t+k|t} \Psi c^*_{t+k|t}$, but its explicit value cannot be determined before solving the FHOCP (38) at time t. By following the idea first introduced in [21] for linear robust MPC, the suboptimal sequence of control moves $\bar{c}_{t+k|t}$ used in Theorem 1 is explicitly defined as $\bar{c}_{t+k|t} = c^*_{t+k|t-1}$ for $k = 0, \dots, N-2$, and as $\bar{c}_{t+k|t} = 0$ for $k = N-1$. This control sequence generates the suboptimal value of the cost function $\bar{J}_t \triangleq \sum_{k=0}^{N-1} \bar{c}^\top_{t+k|t} \Psi \bar{c}_{t+k|t} = \sum_{k=0}^{N-2} c^{*\top}_{t+k|t-1} \Psi c^*_{t+k|t-1} = J^*_{t-1} - c^{*\top}_{t-1} \Psi c^*_{t-1}$, in which we recall that $c^*_{t-1} \triangleq c^*_{t-1|t-1}$. The cost function $\bar{J}_t$ is associated with the suboptimal control terms $\bar{u}_{t+k|t}$, $k \in \mathbb{N}_{[0,N-1]}$, defined in (41), which constitutes a feasible solution of (38), according to Theorem 1. As $J^*_t \leq \bar{J}_t$, from the definition of $\bar{J}_t$ it follows that $J^*_{t-1} - J^*_t \geq c^{*\top}_{t-1} \Psi c^*_{t-1} \geq 0$, as $\Psi \succ 0$. As a consequence, the sequence of optimal cost functions $\{J^*_t\}$ is a non-increasing and non-negative sequence, which, as $t \to \infty$, necessarily converges to a finite value, namely J^*_∞. Therefore, considering to solve problem (38) starting at time $t = 0$ with $x_0 \in \mathcal{D}_N$, as $J^*_{t-1} - J^*_t \geq 0$, one has that $\infty > \sum_{t=0}^{\infty} J^*_t - J^*_{t+1} = J^*_0 - J^*_\infty \geq \sum_{t=0}^{\infty} c^{*\top}_t \Psi c^*_t \geq 0$. This implies $\lim_{t\to\infty} c^{*\top}_t \Psi c^*_t = 0$. As $\Psi \succ 0$, one also has that $\lim_{t\to\infty} c^*_t = 0$. This in turn implies that $\lim_{t\to\infty} u^{\mathrm{rh}}_t = u^{\mathrm{sm}}_t$, and thus, by definition of Θ_f, $\lim_{t\to\infty} x_t \in \Theta_f$. Being $\Theta_f \subset \mathrm{int}(\Theta)$ by assumption, then x_t converges to Θ in a finite time $\bar{t} \in \mathbb{N}_{\geq 0}$. Once $x_t \in \Theta$, Lemma 5 and Corollary 1 guarantee that $u^{\mathrm{rh}}(x_t) = u^{\mathrm{sm}}(x_t)$ for $t \geq \bar{t}$. This implies, thanks to Lemma 6, that a practical sliding motion takes place for $t \geq \bar{t} + 1$. Finally, if $w_t = 0$ for $t \geq t_1$, Assumption 1 implies that an ideal sliding motion takes place for $t \geq \max(\bar{t}+1, t_1+1)$, and $\lim_{t\to\infty} x_t = 0$. ∎

VII. Illustrative example

Consider the discrete-time, two-dimensional oscillator, obtained by forward difference approximation of the *Duffing equation* with sampling time $T_s = 0.1\,\mathrm{s}$:

$$x_{t+1} = \begin{bmatrix} 1 & T_s \\ -T_s & (1-0.6T_s) \end{bmatrix} x_t + \begin{bmatrix} 0 & 0 \\ -T_s & 0 \end{bmatrix} x_t^3 + \begin{bmatrix} 0 \\ T_s \end{bmatrix} (u_t + d_t). \tag{47}$$

This example was already used in [17] and [27] to test other receding-horizon control algorithms, with a consistent selection of the sampling time. The latter allows a suitable length of the prediction horizon. The uncertain term $d_t \in \mathcal{D} \subset \mathbb{R}$ is generated as a pseudo-random signal, bounded in the set $\mathcal{D} \triangleq [-1.34 \times 10^{-4},\, 1.34 \times 10^{-4}]$. Note that, in compliance with system (1) we pose $w_t \triangleq \begin{bmatrix} 0 & T_s d_t \end{bmatrix}^\top \in \begin{bmatrix} 0 & T_s \mathcal{D} \end{bmatrix}^\top \triangleq \mathcal{W}$, which implies $\bar{w} = 1.34 \times 10^{-5}$, while the other terms are included in vector field $f(\cdot,\cdot)$. The sliding variable in (4) is instead defined as $s = h(x) = cx_1 + x_2$. Hence, by posing $w_t = 0$, and substituting into (47) the unconstrained DSMC law satisfying equation $h(f(x, u^{\mathrm{sm}}(x))) = 0$, that is

$$u^{\mathrm{sm}}(x) = \left[\left(1 - \frac{c}{T_s}\right) x_1 - \left(\frac{1}{T_s} - 0.6 + c\right) x_2 + x_1^3\right], \tag{48}$$

one achieves the closed-loop system in form (5) defined by $\phi(x) = \begin{bmatrix} x_1 + T_s x_2 & -cx_1 - cT_s x_2 \end{bmatrix}^\top$. This specific choice of the sliding variable has led to an LTI closed-loop system, and so that the sliding manifold is properly designed if and only if $0 < c < \frac{2}{T_s}$. Therefore, the constant has been set as $c = 5$, and the system in form (5) is

$$x_{t+1} = \underbrace{\begin{bmatrix} 1 & T_s \\ -c & -cT_s \end{bmatrix}}_{A} x_t + w_t. \tag{49}$$

One can set

$$V = x^\top P x,\; P = \begin{bmatrix} 35.6667 & 3.4667 \\ 3.4667 & 1.3467 \end{bmatrix},$$

where $P \succ 0$ is the solution of the associated discrete-time Lyapunov equation $A^\top P A - A = -Q$, with Q being fixed as a 2-by-2 identity matrix. From the results described in [25, Ex. 3.4], one can claim that $V(x)$ is an ISS Lyapunov function: in particular, referring to Definition 3, we have that $\alpha_1(\|x\|_2) = \lambda_{\min}(P)\|x\|_2^2 = \|x\|_2^2$, $\alpha_2(\|x\|_2) = \lambda_{\max}(P)\|x\|_2^2 = 36.01\|x\|_2^2$, $\alpha_3(\|x\|_2) = \frac{1}{2}\lambda_{\min}(Q)\|x\|_2^2 = \frac{1}{2}\|x\|_2^2$, and $\sigma(\|w\|_2) = \left(\frac{2\|A^\top P\|_2^2}{\lambda_{\min}(Q)} + \|P\|_2^2\right)\|w\|_2^2 = 1.99 \cdot 10^3 \|w\|_2^2$, which implies that system (49) is ISS. The set Θ_f in case of persistent disturbances can be calculated considering that, for system (49), one has $\gamma_a\left(\max_{w\in\mathcal{W}} \|w\|_2\right) = 2.05 \cdot \max_{w\in\mathcal{W}} \|w\|_2$, and thus, given the definition of $\mathcal{W}$ for our case study for which $\max_{w\in\mathcal{W}} \|w\|_2 = 1.31 \cdot 10^{-5}$, $\Theta_f = \left\{x \in \mathbb{R}^n \;:\; \|x\|_2 \leq 2.68 \cdot 10^{-5}\right\}$.

Introducing the augmented state $z = [x_1,\, x_2,\, u]^\top \in \mathbb{R}^3$, the mixed input/state constraints $\mathcal{Z}$ in (10), for this example, are such that $|x_1| \leq 1$, $|x_2| \leq 4$, $|u| \leq 80$, $|-5(x_1+x_2)+u| \leq 80$, and $|5(x_2 - x_1) + u)| \leq 80$. As for Assumptions 2-4, by using the infinity norm and the INTLAB MATLAB-based toolbox [28], the required parameters result as $L_{f_x} = 1.34$, $L_{f_u} = 0.1$, $L_u = 49$, so that $L_x = 6.24$, and $\bar{w}_z = 6.54 \times 10^{-4}$. In order

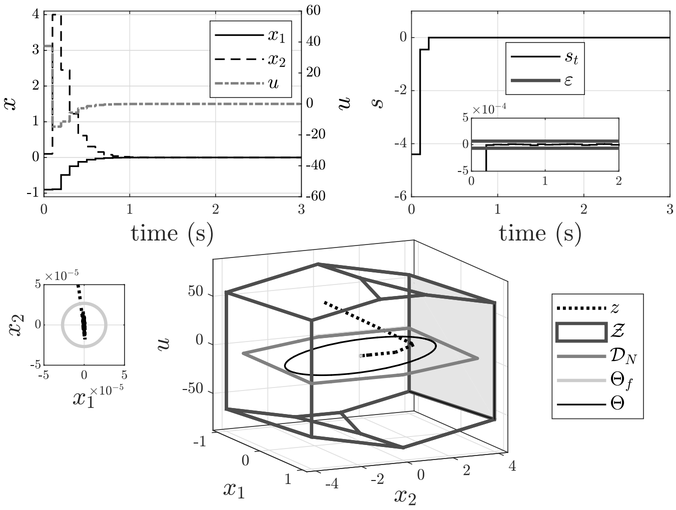

Figure 1. From top left, clockwise. Time evolution of the states x_{1_t}, x_{2_t} (left axis), and of the input $u_t = u_t^{\mathrm{sm}} + c_t^*$ (right axis). Time evolution of the sliding variable s_t. State-space portrait: constraint set $\mathcal{Z}$ (solid black line), the set $\mathcal{D}_N$ (solid gray line), convergence set Θ_f (close-up, solid light gray line), RPI set Θ (solid dark gray line), state trajectory (dotted black line)

to solve the FHOCP, the prediction horizon is set equal to $N = 5$, while the (scalar) input weight is $\Psi = 100$. Set $\mathcal{D}_N$ has been instead computed numerically as a polytope using the MPT3 MATLAB-based toolbox [29]. As for the sets Θ and Ω, they are thus obtained as ellipsoidal sets defined as described in (30) and (32), respectively, with $\alpha_\Theta = 14.34$ and $\alpha_\Omega = 13.35$. Specifically, as for Θ, it is the largest ellipsoid such that the tightened constraints (28) are satisfied. The latter are computed by defining the set $\mathcal{B}_z^k$ in (27) by exploiting the infinity norm. Overall, Assumptions 5 and 6 are satisfied by these values of the parameters, with $L_V = 48.92$. With the chosen sliding variable, the Lipschitz constant in Assumption 7 is instead $L_h = 5$. Finally, the amplitude of the boundary layer around the sliding manifold was determined numerically to be equal to $\varepsilon = 6.6 \times 10^{-5}$. Figure 1 illustrates the behavior of the system controlled via the proposed law (37), as well as the state-space portrait. The state trajectory, differently from the unconstrained classical case, never violates the constraints defined by $\mathcal{Z}$: the initial conditions $x_0 = \begin{bmatrix}-0.9 & 0.1\end{bmatrix}^\top$ are inside the FHOCP feasible set $\mathcal{D}_N$; the state then enters Θ in finite time, and finally converges to the bounded set Θ_f.

VIII. CONCLUSIONS

We developed a receding-horizon control approach to achieve a DSMC law that guarantees the satisfaction of mixed input-state constraints. All theoretical results were rigorously proven, and the algorithm was successfully tested in the simulation of a discretized Duffing equation.

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