Journal of Statistical Physics manuscript No.
(will be inserted by the editor)

C^*–quadratic quantization

Luigi Accardi · Ameur Dhahri · Habib Rebei.

Received: date / Accepted: date

Abstract In the first part of the paper we introduce a new parametrization for the manifold underlying quadratic analogue of the usual Heisenberg group introduced in [11] which makes the composition law much more transparent. In the second part of the paper the new coordinates are used to construct an inductive system of $*$–algebras each of which is isomorphic to a finite tensor product of copies of the one–mode quadratic Weyl algebra. We prove that the inductive limit $*$–algebra is factorizable and has a natural localization given by a family of $*$–sub-algebras each of which is localized on a bounded Borel subset of $\mathbb{R}$. Moreover, we prove that the family of quadratic analogues of the Fock states, defined on the inductive family of $*$–algebras, is projective hence it defines a unique state on the limit $*$–algebra. Finally we complete this $*$–algebra under the (minimal regular) C^*–norm thus obtaining a C^*–algebra.

Keywords Renormalized square of white noise · Quadratic Weyl C^*–algebra · Quadratic Fock states

PACS 60J65

Mathematics Subject Classification (2000) 60J45 · 60H40

L. Accardi
Volterra Center, University of Tor Vergata Roma.II
Via Columbia 2, 00133 Roma, Italy
E-mail: accardi@volterra.uniroma2.it

A. Dhahri
Department of Mathematics, Chungbuk National University
1 Chungdae-ro, Seowon-gu, Cheongju, Korea, 362-763
E-mail: ameur@chungbuk.ac.kr

H. Rebei
Department of Mathematics, College of Science
Qassim University, P.O.Box 5155- code No 51422, Kingdom of Saudi Arabia
E-mail: habib.rebei@ipein.rnu.tn

1 Introduction

The renormalized square of white noise (RSWN) algebra was first introduced in [10]. The construction of the Fock representation of the RSWN algebra motivated a large number of papers extending it in different directions and exhibiting connections with almost all fields of Mathematics, see for example [21] for the case of free white noise; [14] for the connection with infinite divisibility and for the identification of the vacuum distributions of the generalized fields with the three nonstandard Meixner classes; [1] and [18] for finite temperature representations; [6], [7], [15], [16] for the construction of the Fock functor; the survey [3] and the paper [4] for the connections with conformal field theory and with the Virasoro–Zamolodchikov hierarchy; [5] for the connections between renormalization and central extensions; [11] for the construction of the quadratic Weyl operators and their associated quadratic Heisenberg group; [12], [13] for the study of the RSWN quantum time shift and quantum Markov semi–groups.
The RSWN is the first and at the moment best understood example illustrating the program of non–linear quantization. The starting point of quadratic quantization is the known fact that the smallest $*$–Lie algebra containing the squares of the creation and annihilation operators in a 1–mode Fock representation is isomorphic to the 1–dimensional (trivial) central extension of $sl(2,\mathbb{R})$, that we denote RSWN(1). Then one tries to find, by heuristic considerations, a re–normalization rule that gives a meaning to the $*$–Lie algebra generated by symbols of the form $(a_x^+)^2$, a_x^2 $(x \in \mathbb{R}^d)$ with $[a_x, a_x^+] = \delta(x-y)$. Once this is done, the most difficult step is to construct some $*$–representation of this $*$–Lie algebra as operators on a Hilbert space with a common domain and with *good* continuity properties. The third step is to exponentiate the field operators thus obtaining non–linear analogues of the Weyl representation of the CCR.
The obstruction to this program is the possible emergence of *ghosts*, i.e. the non existence of a scalar product compatible with the combination of the $*$–Lie algebra structure with the additional prescriptions used to construct the representation, e.g. the existence of a natural analogue of the vacuum vector. In the homogeneous quadratic case, this problem does not arise, but already in the attempt to combine first and second powers, it appears (see [21], [14]). The reason why this approach leads to results more plausible than previous attempts to give a meaning to squares of quantum fields are discussed in [2] where the decades long history of these attempts is briefly reviewed. In particular this method is robust under change of dimensions, i.e. the replacement of $\mathbb{R}$ by $\mathbb{R}^d$ introduces no additional difficulty.
From the physical point of view, squeezed states are widely in quantum optics and solid state physics, but a *squeezed quantum field theory* was missing before the construction of the Fock representation of the RSWN. From the list of papers mentioned at the beginning of this section one can see how different is the quadratic Fock space from the usual one and at the same time how the greater complexity due to non–linearity can be handled leading to explicit formulas

such as, for example, the vacuum distributions of the quadratic fields (which are not Gaussian). More generally from these papers one can see how fruitful the non–linear quantization program has been, leading to unexpected connections with objects widely studied in theoretical and mathematical physics such as the Virasoro–Zamolodchikov hierarchy and to the solution of open problems related to it.

Another example of non–linear quantization was studied in the paper [8] which deals with the case of polynomial extensions of the Heisenberg algebra. Even if this example does not require re–normalization, the existence for it of an analogue of the Fock representation for it is still an open problem. As an intermediate step, in the paper [9] an analogue of the Weyl C^*–algebra was constructed.
In 1–st order case, i.e. usual quantization, the great advantage of the Weyl form of the Boson commutation relations with respect to Heisenberg ones is that, in finite dimensions, there is a unique, up to isomorphism, strongly continuous representation of the former relations, while uniqueness is known to fail for the latter. More generally the framework of Weyl algebras is usually preferred in mathematical physics because it allows to avoid the insidious ground of algebras of unbounded operators. In infinite dimensions, uniqueness is known to fail even for the Weyl relations. However a weaker form of uniqueness survives even in infinite dimensions, namely **uniqueness at a C^*–algebra level** (see [17], Theorem 2.1). A natural conjecture is that, also in the case of polynomial extensions of the Heisenberg algebra, as well as for the quadratic Weyl relations studied in the present paper, uniqueness holds. Since no proof of this conjecture is available at the moment, even in the one–dimensional case, we constructed a representation of this algebra in the usual 1–mode Boson Fock space, proved the exponentiability of its formally self–adjoint elements and used it to construct an inductive system of $*$–algebras. The construction exploits the fact that, if π is a finite Borel partition of a bounded Borel subset of $\mathbb{R}$, then there is a natural way to give a meaning to the generalized Weyl algebra with test functions constant on the sets of π. This is based on the identification of this algebra with the tensor product of $|\pi|$ (cardinality of π) copies of the one–mode generalized Weyl algebra (see section 8). Using this we construct an inductive system of $*$–algebras each of which is isomorphic to a finite tensor product of copies of the one mode generalized Weyl algebra, but the embeddings defining the inductive system are not the usual tensor product embedding.
On the inductive limit of this system we introduce a C^*–norm and take completion. This provides a natural analogue of the usual, 1–st order, Weyl C^*–algebra in the quadratic case (see formula (1)).

In the paper [11] a quadratic analogue of the usual Heisenberg group, naturally arising in first order quantization, was constructed starting from the realization of the one-mode quadratic Weyl C^*–algebra in usual 1–mode Boson Fock space. This group is called the **one-mode quadratic Heisenberg group**

and denoted by QHeis$(1, F)$.
Its Lie algebra is RSWN(1) and the quadratic Weyl operators are a unitary representation of this group acting on the quadratic Fock space.
In [11] it was proved that QHeis$(1, F)$ is a Lie group and its underlying manifold was identified with the region of $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ enclosed between two sheets of an hyperboloid (of which only one belongs to the manifold) (see formula (16)).
In the first part of this paper we introduce a new parametrization for this manifold (the cardinal coordinates (20)) leading to a drastic simplification of the composition law on QHeis$(1, F)$ much more transparent than the one in [11] (see Theorem 5.1). Given the huge literature on the Heisenberg group one could expect at least some interest in its more sophisticated and richer non–linear generalizations. On the other hand, due to non–linearity the composition laws of these groups are more complex. This motivates our effort to simplify them as much as possible. It should be added that, even in the immense literature on representations of $sl(2, \mathbb{R})$, these formulas are new. For these reasons we included their explicit derivation.
In this paper we have included only those parts of the proof that drastically simplify the corresponding parts in [11].

In the second part of this paper we construct a **quadratic Weyl C^*–algebra** by adapting to the present case the technique, developed in [9].
The main difference between the present paper and [9] is that the construction of the inductive system is realized here at a $*$–algebra level and the C^*–norm is introduced after taking the inductive limit. This simplifies the proof not requiring the proof of the continuity of the inductive embeddings at each step. Since, as already mentioned, at the moment there is no quadratic analogue of the uniqueness theorem for the Weyl C^*–algebra, this construction necessarily involves some arbitrary choices. For example since, both in the Fock case [14] and in the equilibrium case [1], quadratic fields can be exponentiated, one can construct **quadratic Weyl C^*–algebras** associated to these representations. In the case of polynomial extensions of the Heisenberg algebra no such representation are known at the moment (even in the quadratic case) and this motivated the inductive construction in [9]. We use a modification of this construction here because the main motivation of the present paper is to understand the structural roots of the main difference between the quadratic Weyl algebra and the algebras based on polynomial extensions of the Heisenberg algebra, namely that the family of the quadratic Fock states, defined on the inductive family of $*$–algebras, is always projective in the case of quadratic quantization, while for the n–th degree Heisenberg group this is true only for $n = 1$, i.e. for the usual Weyl algebra (see [9]).
It is now clear that the main difference between the two cases is in the structure of the building block for the construction of the inductive embeddings (see Proposition 2 below). What is not yet clear is if there exist other embeddings of the polynomial extensions of the Heisenberg algebra that avoid the non–projectivity problem or if, more probably, the non–projectivity result can

be used for a direct proof of the non-existence of the Fock representation for this class of algebras. This issue is presently under investigation.

We recall from [14] that the re–normalized square of white noise algebra (RSWN) with test functions in a $*$–sub–algebra $\mathcal{K}$ of the Hilbert algebra $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ (for the point–wise operations) is the $*$–Lie algebra with linearly independent generators

$$\{b_\varphi^+ \quad ; \quad b_\varphi^- \quad ; \quad n_\varphi \, , \, \varphi \in \mathcal{K} \quad ; \quad \mathbf{1}\,(\text{central element})\}$$

(in the sense that $\varphi_0\mathbf{1}+b^+_{\varphi_1}+b^-_{\varphi_2}+n_{\varphi_4} = 0$ with $\varphi_0 \in \mathbb{C}$ and $\varphi_j \in \mathcal{K}$ $(j = 1, 2, 3)$ if and only if $\varphi_0 = 0$ and $\varphi_1 = \varphi_2 = \varphi_3 = 0$ Lebesgue–a.e.) and commutation relations and involution given by:

$$[b_\varphi^-, b_\phi^+] = \frac{c}{2}\langle\varphi, \phi\rangle\mathbf{1} + n_{\bar{\varphi}\phi}$$

$$[n_\phi, b_\varphi^+] = 2b^+_{\phi\varphi}$$

$$[n_\phi, b_\varphi^-] = -2b^-_{\bar{\phi}\varphi}$$

$$[n_\phi, n_\varphi] = [b_\phi^+, b_\varphi^+] = [b_\phi^-, b_\varphi^-] = 0$$

$$(n_\phi)^* = n_{\bar{\phi}} \quad ; \quad (b_\phi^-)^* = b_\phi^+$$

The choice of the test function algebra $\mathcal{K}$ is a matter of convenience, as long as it is *sufficiently large*. In this paper we have chosen as $\mathcal{K}$ the algebra of step functions from $\mathbb{R}$ to $\mathbb{C}$ with bounded support and finitely many values. In any concrete representation with good continuity property (such as the Fock one) this algebra can be enlarged with standard approximation techniques.
Starting from section 5, we construct a C^*–algebra whose generators can be *naturally* identified with the *quadratic Weyl operators*:

$$e^{i(b^+_\varphi + b^-_\varphi + n_\phi + \langle\psi\rangle\mathbf{1})} \qquad ; \qquad \langle\psi\rangle := \int_{\mathbb{R}} \psi(x)dx \ , \ \varphi \in \mathcal{K} \ , \ \phi, \psi \in \mathcal{K}_{\mathbb{R}} \tag{1}$$

with test functions in $\mathcal{K}$ independently of any (infinite dimensional) representation.

2 The one-mode quadratic Fock space

Definition 1 The $sl(2, \mathbb{C})$ algebra is the complex $*$–Lie algebra with generators $\{M, B^\pm\}$ and relations

$$[B^-, B^+] = M \quad ; \quad [M, B^\pm] = \pm 2B^\pm \tag{2}$$

$$(B^-)^* = B^+ \quad ; \quad M^* = M \tag{3}$$

The central extension of $sl(2, \mathbb{C})$, denoted by $sl(2, \mathbb{C}, E)$, is the $*$–Lie algebra generated by $sl(2, \mathbb{C})$ and the central self-adjoint element E, i.e., commuting with all other generators of $sl(2, \mathbb{C})$ and satisfying $E^* = E$.

It is known (see [11]) that for all $\mu > 0$, there exists a unique irreducible $*$–representation of $sl(2,\mathbb{C})$, called the Fock representation and realized on a Hilbert space Γ, with an orthonormal basis $\{\Phi_n, n \in \mathbb{N}\}$ on which $B^{\pm}$ and M act as follows (we identify elements of $sl(2,\mathbb{C})$ with their images in the representation):

$$B^{+}\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1} \quad , \quad n \in \mathbb{N}, \tag{4}$$

$$B^{-}\Phi_n = \sqrt{\omega_n}\Phi_{n-1} \quad , \quad n \in \mathbb{N}^{*}, \tag{5}$$

where $\Phi_{-1} = 0$ and $\Phi_0 =: \Phi$ is the vacuum vector,

$$M\Phi_n = (2n+\mu)\Phi_n \quad , \quad n \in \mathbb{N} \tag{6}$$

The sequence (ω_n) is uniquely determined by

$$\omega_n = n(\mu + n - 1) \quad , \quad n = 1, 2, \dots \quad , \quad \omega_0 := 1 \tag{7}$$

Defining the number operator by

$$N\Phi_n = 2n\Phi_n \tag{8}$$

this representation can be extended to $sl(2,\mathbb{C},E)$ with the prescription

$$E\Phi_n = \Phi_n \tag{9}$$

i.e., in this identification, $E \equiv 1$. In this case

$$N + \mu E = M \tag{10}$$

Definition 2 The space Γ is called the one-mode quadratic Fock space.

3 The quadratic Weyl relations and the group QHeis$(1, F)$

In this section, we recall the quadratic analogue of the Weyl relations, then we give the composition law associated with the quadratic Heisenberg group QHeis$(1,F)$. It was proved in [11], that for all $Z := (z,\lambda) \in \mathbb{C} \times \mathbb{R}$, the operators

$$F_0(Z) := zB^{+} + \bar{z}B^{-} + \lambda M \tag{11}$$

called the **(one-mode) quadratic field operators**, are essentially self–adjoint

Definition 3 Denoting the closures of the operators (11) with the same symbol, the **one-mode quadratic Weyl operators** and the **re–scaled quadratic Weyl operators** are defined respectively by

$$W(Z) := e^{iH_0(Z)} = e^{i(zB^{+}+\bar{z}B^{-}+\lambda N)} \tag{12}$$

$$W_r(Z) := e^{i\mu\lambda}W(Z) = e^{i(zB^{+}+\bar{z}B^{-}+\lambda N+\lambda\mu 1)} = e^{iF_0(Z)} \tag{13}$$

where $F_0(Z)$ is defined by (11). They are unitary operators acting on Γ. We denote $\mathcal{B}(\Gamma)$, the bounded operators on the 1–mode quadratic Fock space Γ and

$$\begin{aligned} \mathcal{W}_2^0(\Gamma) := &\text{ the } *\text{–algebra generated by } \{W(Z) \ : \ Z \in \mathfrak{D}_+\} \\ = &\text{ the } *\text{–algebra generated by } \{W_r(Z) \ : \ Z \in \mathfrak{D}_+\} \subset \mathcal{B}(\Gamma) \end{aligned} \tag{14}$$

Remark 1 It is clear, from Definition 3, that for each $Z = (z, \lambda) \in \mathbb{C} \times \mathbb{R}$, the one-mode quadratic Weyl operator $W(Z)$ is well defined. However the map $Z \in \mathbb{C} \times \mathbb{R} \mapsto W(Z)$ is not injective. In the following we introduce a subset of $\mathbb{C} \times \mathbb{R}$ which enjoys this property. This will be the group manifold of the quadratic Heisenberg group.

Notations

1. For $Z = (z, \lambda) \in \mathbb{C} \times \mathbb{R}$, we define
$$w_Z := \sqrt{|z|^2 - \lambda^2} \ \in \ \mathbb{R}_+ \cup i\mathbb{R}_+, \tag{15}$$
where by definition, if $|z|^2 - \lambda^2 \geq 0$, then $\sqrt{|z|^2 - \lambda^2} \in \mathbb{R}_+$ and, if $|z|^2 - \lambda^2 < 0$, then $\sqrt{|z|^2 - \lambda^2} \in i\mathbb{R}_+$.
In Proposition 4.1 of [11] it was proved that the re–scaled quadratic Weyl operators are parameterized, in a one–to–one way, by the set
$$\mathfrak{D}_+ := \{Z = (z, \lambda) \in \mathbb{C} \times \mathbb{R} \equiv \mathbb{R}^3 \ : \ w_Z \in \mathbb{R}_+ \cup i(0, \frac{\pi}{2}] \text{ with } \lambda > 0 \text{ if } w_Z = i\frac{\pi}{2}\} \tag{16}$$
$$\equiv \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 - z^2 \geq -\frac{\pi^2}{4}, \text{ with } z > 0 \text{ if } x^2 + y^2 - z^2 = -\frac{\pi^2}{4}\}$$
in the sense that, for all $Z = (z, \lambda) \in \mathbb{C} \times \mathbb{R}$, there exists a unique $Z_0 = (z_0, \lambda_0) \in \mathfrak{D}_+$, such that
$$W_r(Z) = W_r(Z_0)$$
$\mathfrak{D}_+$ is called the **principal domain** of the one-mode quadratic Weyl operators. It is a manifold, with boundary, embedded in $\mathbb{R}^3$.
2. Recall that the **cardinal tangent** of $w \in \mathbb{R} \cup i\mathbb{R} \setminus i(\frac{\pi}{2} + \pi\mathbb{Z})$ is given by
$$\text{thc}(w) := \frac{\tanh(w)}{w} \tag{17}$$
where $\tanh(w)$ denotes the hyperbolic tangent of w. Moreover $\text{thc}(w)$ is defined by continuity at $w = 0$ to be equal to 1.
3. Notice that
$$\begin{aligned} &\text{thc}(w) \in \mathbb{R} \quad ; \quad \forall w \in \mathbb{R} \cup i\mathbb{R} \quad , \quad w \notin i\frac{\pi}{2}(2\mathbb{Z} + 1) \\ &\text{thc}(w) := \text{thc}(-w) \quad ; \quad \forall w \in \mathbb{R} \cup i\mathbb{R} \quad , \quad w \notin i\frac{\pi}{2}(2\mathbb{Z} + 1) \\ &\text{thc}(w) = 0, \ w \in i\pi\mathbb{Z}^* \end{aligned} \tag{18}$$
$$\text{thc}(iw') = \frac{\tanh(iw')}{iw'} = \frac{\tan(w')}{w'} \quad ; \quad \forall iw' \in i\mathbb{R} \setminus i(\frac{\pi}{2} + \pi\mathbb{Z})$$
4. Because of the symmetry (18), the quantity
$$T_Z := \frac{1}{\text{thc}(w_Z)} = \frac{w_Z}{\tanh(w_Z)} \in \mathbb{R}_+ \cup i\mathbb{R}_+ \tag{19}$$
does not depend on the choice of the sign of the square root in (15). Moreover, we have
$$T_Z = 0 \iff w_Z = i(\frac{\pi}{2} + k\pi), \ k \in \mathbb{Z}$$
5. It is convenient (see Theorem 1 below) to introduce the following coordinates on $\mathfrak{D}_+$
$$M_Z := \frac{\sinh(w_Z)}{w_Z} z = \frac{\cosh(w_Z)}{T_Z} z \ ; \ \mu_Z := \frac{\sinh(w_Z)}{w_Z} \lambda = \frac{\cosh(w_Z)}{T_Z} \lambda \tag{20}$$
From now the coordinates (M_Z, μ_Z) will be called the **cardinal coordinates** of Z.

6. Notice that
$$\frac{\sinh(w_Z)}{w_Z} > 0 \qquad ; \qquad \forall Z = (z, \lambda) \in \mathfrak{D}_+ \tag{21}$$
In the following a pair $Z = (z, \lambda) \in \mathfrak{D}_+$ will be identified with the corresponding pair of cardinal coordinates (M_Z, μ_Z) and we write
$$Z \equiv (M_Z, \mu_Z)$$
The inverse transformation of (20) is obtained by observing that, given (M_Z, μ_Z), then w_Z is uniquely given by the identity
$$\sinh^2(w_Z) = \mid M_Z \mid^2 - \mu_Z^2 \tag{22}$$
hence, using (20), one finds
$$z = \frac{w_Z}{\sinh(w_Z)} M_Z \quad ; \quad \lambda = \frac{w_Z}{\sinh(w_Z)} \mu_Z$$
7. For simplicity, when $Z_1, Z_2 \in \mathfrak{D}_+$, we use the notation $Z_j \equiv (M_j, \mu_j),\ j = 1, 2$ instead of $Z_j \equiv (M_{Z_j}, \mu_{Z_j})$. Moreover, we often write w_j instead of w_{Z_j}.

Theorem 1 For all $Z_1 \equiv (M_1, \mu_1),\ Z_2 \equiv (M_2, \mu_2) \in \mathfrak{D}_+$, there exists a unique $Z =: Z_1 \circ Z_2 \equiv (M_Z, \mu_Z) \in \mathfrak{D}_+$ such that
$$W_r(Z_1)W_r(Z_2) = W_r(Z_1 \circ Z_2) = W_r(Z) \tag{23}$$
where the cardinal coordinates of Z are given by
$$M_Z = \varepsilon\Big(\cosh(w_1)M_2 + \cosh(w_2)M_1 + i(\mu_2 M_1 - \mu_1 M_2)\Big) \tag{24}$$
$$\mu_Z = \varepsilon\Big(\cosh(w_1)\mu_2 + \cosh(w_2)\mu_1 + \mathrm{Im}(\overline{M}_1 M_2)\Big) \tag{25}$$
and ε is defined by
$$\varepsilon = \begin{cases} sgn\big(\mathrm{Re}(Q)\big) & ;\text{ if } \mathrm{Re}(Q) \neq 0, \\ sgn\big(\mathrm{Im}(Q)\big) & ;\text{ if } \mathrm{Re}(Q) = 0 \end{cases} \tag{26}$$
$$Q = (\cosh(w_1) + i\mu_1)(\cosh(w_2) + i\mu_2) + \overline{M}_1 M_2 \tag{27}$$

Proof. In the proof of Theorem 5.1 in [11] it was shown (see Equations (5.34), (5.35) and (5.36)) that, in the notation (19) , the following relations hold
$$\frac{z}{T_Z - i\lambda} = \frac{z_2(T_{Z_1} - i\lambda_1) + z_1(T_{Z_2} + i\lambda_2)}{(T_{Z_1} - i\lambda_1)(T_{Z_2} - i\lambda_2) + \bar{z}_2 z_1} \tag{28}$$
$$\frac{\bar{z}}{T_Z - i\lambda} = \frac{\bar{z}_2(T_{Z_1} + i\lambda_1) + \bar{z}_1(T_{Z_2} - i\lambda_2)}{(T_{Z_1} - i\lambda_1)(T_{Z_2} - i\lambda_2) + \bar{z}_2 z_1} \tag{29}$$
$$\frac{T_Z^2 - w_Z^2}{(T_Z - i\lambda)^2} = \frac{(T_{Z_1}^2 - w_1^2)(T_{Z_2}^2 - w_2^2)}{\big((T_{Z_1} - i\lambda_1)(T_{Z_2} - i\lambda_2) + \bar{z}_2 z_1\big)^2} \tag{30}$$

Multiplying numerator and denominator of the left hand side of Equation (28) by $\frac{\cosh(w_Z)}{T_Z}$ and taking (20) into account, we get

$$\frac{z}{T_Z - i\lambda} = \frac{\frac{\cosh(w_Z)z}{T_Z}}{\cosh(w_Z) - i\frac{\cosh(w_Z)\lambda}{T_Z}} = \frac{M_Z}{\cosh(w_Z) - i\mu_Z}. \tag{31}$$

For the right hand side of (28), a similar operation can be applied with $\frac{\cosh(w_1)\cosh(w_2)}{T_{Z_1}T_{Z_2}}$ replacing $\frac{\cosh(w_Z)}{T_Z}$. This gives

$$\frac{z_2(T_{Z_1} - i\lambda_1) + z_1(T_{Z_2} + i\lambda_2)}{(T_{Z_1} - i\lambda_1)(T_{Z_2} - i\lambda_2) + \bar{z}_2 z_1} = \frac{M_2(\cosh(w_1) - i\mu_1) + M_1(\cosh(w_2) + i\mu_2)}{(\cosh(w_1) - i\mu_1)(\cosh(w_2) - i\mu_2) + \overline{M}_2 M_1}. \tag{32}$$

Injecting the expressions (31) and (32) in (28), we obtain

$$\frac{M_Z}{\cosh(w_Z) - i\mu_Z} = \frac{M_2(\cosh(w_1) - i\mu_1) + M_1(\cosh(w_2) + i\mu_2)}{(\cosh(w_1) - i\mu_1)(\cosh(w_2) - i\mu_2) + \overline{M}_2 M_1}. \tag{33}$$

Repeating the same steps with Equation (29), one finds

$$\begin{aligned} \frac{\overline{M}_Z}{\cosh(w_Z) - i\mu_Z} &= \frac{\overline{M}_2(\cosh(w_1) + i\mu_1) + \overline{M}_1(\cosh(w_2) - i\mu_2)}{(\cosh(w_1) - i\mu_1)(\cosh(w_2) - i\mu_2) + \overline{M}_2 M_1} \\ \iff \frac{M_Z}{\cosh(w_Z) + i\mu_Z} &= \frac{M_2(\cosh(w_1) - i\mu_1) + M_1(\cosh(w_2) + i\mu_2)}{(\cosh(w_1) + i\mu_1)(\cosh(w_2) + i\mu_2) + \overline{M}_1 M_2} \end{aligned} \tag{34}$$

For the left hand side of (30), we have

$$\begin{aligned} \frac{T_Z^2 - w_Z^2}{(T_Z - i\lambda)^2} &= \frac{T_Z^2\big(1 - (\frac{w_Z}{T_Z})^2\big)}{T_Z^2(1 - i\frac{\lambda}{T_Z})^2} = \frac{1 - (w_Z \mathrm{thc}(w_Z))^2}{(1 - i\frac{\lambda}{T_Z})^2} \\ &= \frac{1 - \tanh^2(w_Z)}{(1 - i\frac{\lambda}{T_Z})^2} = \frac{1}{\cosh^2(w_Z)(1 - i\frac{\lambda}{T_Z})^2} \\ &= \frac{1}{(\cosh(w_Z) - i\frac{\lambda\cosh(w_Z)}{T_Z})^2} = \frac{1}{(\cosh(w_Z) - i\mu_Z)^2} \end{aligned} \tag{35}$$

A similar computations applied to the right hand side of (30) gives

$$\frac{(T_{Z_1}^2 - w_1^2)(T_{Z_2}^2 - w_2^2)}{\Big((T_{Z_1} - i\lambda_1)(T_{Z_2} - i\lambda_2) + \bar{z}_2 z_1\Big)^2} = \frac{1}{\Big((\cosh(w_1) - i\mu_1)(\cosh(w_2) - i\mu_2) + \overline{M}_2 M_1\Big)^2}. \tag{36}$$

Combining Equations (35) with (36), we get

$$\begin{aligned} (\cosh(w_Z) - i\mu_Z)^2 &= \Big((\cosh(w_1) - i\mu_1)(\cosh(w_2) - i\mu_2) + \overline{M}_2 M_1\Big)^2 \\ \iff (\cosh(w_Z) + i\mu_Z)^2 &= \Big((\cosh(w_1) + i\mu_1)(\cosh(w_2) + i\mu_2) + \overline{M}_1 M_2\Big)^2 \end{aligned} \tag{37}$$

Let us denote

$$X = M_2(\cosh(w_1) - i\mu_1) + M_1(\cosh(w_2) + i\mu_2)$$

and

$$Q = (\cosh(w_1) + i\mu_1)(\cosh(w_2) + i\mu_2) + \overline{M}_1 M_2$$

Let us prove that $Q \neq 0$. Suppose, by contradiction, that $Q = 0$. Since from (16), (20) and (21) it follows that $\cosh(w_Z)$ and μ_Z are always real. Therefore (37) implies that

$$Q = 0 \iff \cosh(w_Z) = \mu_Z = 0 \iff w_Z = i\frac{\pi}{2}$$

But, due to (22), this would imply

$$-1 = \sinh^2(w_Z) =\mid M_Z \mid^2 - \mu_Z^2 =\mid M_Z \mid^2$$

which is impossible. Therefore $Q \neq 0$. It follows that equations (33), (34) and (37) can be written in the form:

$$\frac{M_Z}{\cosh(w_Z) - i\mu_Z} = \frac{X}{\overline{Q}} \tag{38}$$

$$\frac{M_Z}{\cosh(w_Z) + i\mu_Z} = \frac{X}{Q} \tag{39}$$

$$(\cosh(w_Z) + i\mu_Z)^2 = Q^2 \tag{40}$$

(40) implies that

$$\cosh(w_Z) + i\mu_Z = \varepsilon Q \tag{41}$$

with $\varepsilon = \pm 1$ and this implies that $\mu_Z = \varepsilon \mathrm{Im}(Q)$, which is equivalent to (25). Using (41) in (39), one finds

$$\begin{aligned} M_Z &= \varepsilon X = \varepsilon\big(M_2(\cosh(w_1) - i\mu_1) + M_1(\cosh(w_2) + i\mu_2)\big) \\ &= \varepsilon\big(\cosh(w_1)M_2 + \cosh(w_2)M_1 + i(\mu_2 M_1 - \mu_1 M_2)\big) \end{aligned}$$

which is (24). To determine ε notice that (41) implies that $\cosh(w_Z) = \varepsilon \mathrm{Re}(Q)$. Therefore, since $\cosh(w_Z) \geq 0$, $\varepsilon = sgn(\mathrm{Re}(Q))$ when $\mathrm{Re}(Q)$ is non zero.
When $\mathrm{Re}(Q) = 0$, then $\cosh(w_Z) = 0$ and we have seen that this implies $w_Z = i\frac{\pi}{2}$. In this case, λ must be > 0 because of (16). Since $\mu_Z = \frac{\sinh(w_Z)}{w_Z}\lambda$, then μ_Z is also positive. But we have seen that $\mu_Z = \varepsilon \mathrm{Im}(Q)$, and this implies $\varepsilon = sgn(\mathrm{Im}(Q))$.

Corollary 1 *For all $Z_1, Z_2 \in \mathfrak{D}_+$, we have*

$$W(Z_1)W(Z_2) = W(Z_1 \circ Z_2)e^{i\mu(\lambda - \lambda_1 - \lambda_2)} \tag{42}$$

where $Z_1 \circ Z_2 \equiv (M_Z, \mu_Z) \in \mathfrak{D}_+$ is given in Theorem 1.

Remark. As proved in [11] (Cor. 5.1) $\mathfrak{D}_+$ is a Lie group, denoted QHeis(1), with composition law $(Z_1, Z_2) \equiv ((M_1, \mu_1), (M_2, \mu_2)) \mapsto Z_1 \circ Z_2 \equiv (M_Z, \mu_Z)$ defined by (24), (25), (26), (27). identity given by $(0, 0)$ and inverse given by $(z, \lambda)^{-1} = (\bar{z}, -\lambda)$. The map $Z \mapsto W_r(Z)$ is a unitary representation of this group (the Fock representation with parameter μ_0) and the range of this representation is denoted QHeis$(1, F)$ (or QHeis$(1, F, \mu_0)$ when the role of the parameter μ_0 is relevant).

4 The vacuum Fock state on the one-mode quadratic Weyl $*$–algebra $\mathcal{W}_2^0(\Gamma)$

The following Proposition expresses the normally ordered form of the re–scaled quadratic Weyl operators (splitting formula) in terms of cardinal coordinates.

Proposition 1 *For* $Z = (z, \lambda) \equiv (M_Z, \mu_Z) \in \mathfrak{D}_+$, *denote* $\overline{Z} = (\bar{z}, \lambda) \equiv (\overline{M}_Z, \mu_Z) \in \mathfrak{D}_+$. *Then the following identity holds:*

$$W_r(Z) = e^{F(Z)B^+} e^{G(Z)M} e^{F(\overline{Z})B^-} \tag{43}$$

where the functions F, G *are uniquely defined by:*

$$F(Z) := \frac{iM_Z}{\cosh(w_Z) - i\mu_Z}, \tag{44}$$

$$G(Z) := -\log(\cosh(w_Z) - i\mu_Z), \tag{45}$$

where log *denotes the principal determination of the Logarithm.*

Proof. From [11], Theorem 3.1, we have two cases:
Case.I: $w_Z \neq i\frac{\pi}{2}$. In this case, we have

$$\begin{aligned} F(Z) &= \frac{iz\mathrm{thc}(w_Z)}{1 - i\lambda\mathrm{thc}(w_Z)} = \frac{iz\cosh(w_Z)\mathrm{thc}(w_Z)}{\cosh(w_Z) - i\lambda\cosh(w_Z)\mathrm{thc}(w_Z)} \\ &= \frac{iz\sinh(w_Z)/w_Z}{\cosh(w_Z) - i\lambda\sinh(w_Z)/w_Z} = \frac{iM_Z}{\cosh(w_Z) - i\mu_Z} \end{aligned} \tag{46}$$

$$\begin{aligned} G(Z) &= \frac{1}{2}\log\Big(\frac{1 - \tanh^2(w_Z)}{(1 - i\lambda\mathrm{thc}(w_Z))^2}\Big) = \frac{1}{2}\log\Big(\frac{\frac{1}{\cosh^2(w_Z)}}{(1 - i\lambda\mathrm{thc}(w_Z))^2}\Big) \\ &= \frac{1}{2}\log\Big(\frac{1}{(\cosh(w_Z) - i\lambda\cosh(w_Z)\mathrm{thc}(w_Z))^2}\Big) \\ &= \frac{1}{2}\log\Big(\frac{1}{(\cosh(w_Z) - i\mu_Z)^2}\Big) = -\frac{1}{2}\log\big((\cosh(w_Z) - i\mu_Z)^2\big) \\ &= -\log\big(\cosh(w_Z) - i\mu_Z\big) \end{aligned} \tag{47}$$

where in the last step we have used the fact that

$$\log(\frac{1}{h}) = -\log(h) \quad \forall h \in \mathbb{C} \quad : \quad -\pi < \arg(h) < \pi$$

with the choice $h = (\cosh(w_Z) - i\mu_Z)^2$ and, in Equation (47), we have used

$$\log(h^2) = 2\log(h) \quad \forall h \in \mathbb{C} \quad : \quad -\frac{\pi}{2} < \arg(h) < \frac{\pi}{2}$$

for the choice $h = \cosh(w_Z) - i\mu_Z$.

Case.II: $w_Z = i\frac{\pi}{2}$. Taking $w_Z = i\frac{\pi}{2}$ in equation (46), we get

$$F(Z) = \frac{-M_Z}{\mu_Z} = -\frac{\frac{z\sinh(w_z)}{w_Z}}{\lambda\frac{z\sinh(w_z)}{w_Z}} = -\frac{z}{\lambda} \tag{48}$$

Putting $w_Z = i\frac{\pi}{2}$ in equation (47) and taking into account that $\mu_Z > 0$, we obtain

$$\begin{aligned} G(Z) &= -\log(-i\mu_Z) = -\big(\ln(\mu_Z) - i\frac{\pi}{2}\big) = -\ln(\mu_Z) + i\frac{\pi}{2} \\ &= \frac{1}{2}\big(\ln(\mu_Z^{-2}) + i\pi\big) = \frac{1}{2}\log(-\frac{1}{\mu_Z^2}) = \frac{1}{2}\log\big(\frac{-w_Z^2}{\sinh^2(i\frac{\pi}{2})\lambda^2}\big) \\ &= \frac{1}{2}\log\big(\frac{-w_Z^2}{i^2\sin^2(\frac{\pi}{2})\lambda^2}\big) = \frac{1}{2}\log(\frac{w_Z^2}{\lambda^2}) = \frac{1}{2}\log\big(\frac{\mid z\mid^2 - \lambda^2}{\lambda^2}\big) \end{aligned} \tag{49}$$

where ln denotes the Neperian Logarithm. Comparing the results in equations (48) and (49) with what was proved in [11], Theorem 3.1, we deduce that the expressions (44) and (45) are valid also for $w_Z = i\frac{\pi}{2}$. This ends the proof.

We know from (16) that the one-mode quadratic Weyl operators are parameterized, in a one–to–one way by the manifold $\mathfrak{D}_+$. Clearly the $*$–algebra generated by the operators

$$\{W(Z)\ ;\ Z \in \mathfrak{D}_+\} \tag{50}$$

coincides with $\mathcal{W}_2^0(\Gamma)$ and with the $*$–algebra generated by the re–scaled quadratic Weyl operators (13).

Remark 2 As in the first order case, the transition from quadratic Weyl operators to re–scaled ones corresponds to the transition from projective to bona fide unitary representations.

Corollary 2 *Let φ_μ be the restriction on $\mathcal{W}_2(\Gamma)$ of the Fock state $\langle\Phi,\ \cdot\ \Phi\rangle$ on $\mathcal{B}(\Gamma)$. Then, in the notations (13), (45), one has*

$$\varphi_\mu(W_r(Z)) = e^{\mu G(Z)} \tag{51}$$

Proof. From the identities (5) and (8), one has $B^-\Phi = 0$ and $M\Phi = \mu\Phi$. Therefore, one gets

$$e^{uB^-}\Phi = \Phi \quad ; \quad e^{vM}\Phi = e^{\mu v}\Phi \quad ; \quad \forall\ u, v \in \mathbb{C} \tag{52}$$

Using the definition of φ_μ and the splitting formula (43), one has

$$\begin{aligned} \varphi_\mu(W_r(Z)) &= \langle\Phi, W_r(Z)\Phi\rangle = \langle\Phi, e^{F(Z)B^+}e^{G(Z)M}e^{F(\bar{Z})B^-}\Phi\rangle \\ &= \langle e^{\overline{F(Z)}B^-}\Phi, e^{G(Z)M}e^{F(\bar{Z})B^-}\Phi\rangle = \langle\Phi, e^{G(Z)M}\Phi\rangle \\ &= \langle\Phi, e^{\mu G(Z)}\Phi\rangle = e^{\mu G(Z)} \end{aligned}$$

5 The current algebra of $sl(2, \mathbb{C}, E)$ over $\mathbb{R}$

Denote

$$\mathcal{H}_0(\mathbb{R}) := L^1_{\mathbb{C}}(\mathbb{R}) \cap L^\infty_{\mathbb{C}}(\mathbb{R}) = \bigcap_{1\le p\le\infty} L^p_{\mathbb{C}}(\mathbb{R})$$

$\mathcal{H}_0(\mathbb{R})$ has a natural structure of pre–Hilbert algebra with the point–wise operations and the L^2–scalar product.

Lemma 1 *For any $*$–sub–algebra $\mathcal{T}$ of $\mathcal{H}_0(\mathbb{R})$, there exists a unique, up to isomorphism, complex $*$–Lie algebra with generators*

$$\{B^\pm(f)\ ,\ N(f)\ ,\ E(f)\ :\ f \in \mathcal{T}\}$$

involution given by

$$(B^+(f))^* = B^-(f) \quad , \quad (N(f))^* = N(\bar{f}) \quad , \quad \forall f \in \mathcal{T}$$

relations given by the complex linearity of the maps $f \in \mathcal{T} \mapsto B^+(f)\ ,\ N(f)$, and Lie brackets given by:

$$[B^-(f), B^+(g)] = \frac{c}{2}E(\bar{f}g) + N(\bar{f}g) \quad , \quad \forall f, g \in \mathcal{T}$$

$$[N(f), B^+(g)] = 2B^+(fg) \quad ; \quad [N(f), B^-(g)] = -2B^-(\bar{f}g)$$

$$[E(f), N(g)] = [E(f), B^\pm(g)] = 0$$

$$[N(f), N(g)] = [B^+(f), B^+(g)] = [B^-(f), B^-(g)] = 0$$

Proof. A straightforward computation.

Definition 4 The $*$–Lie algebra defined in Lemma 1 is called the **Renormalized Square of White Noise (RSWN) algebra** with test function algebra $\mathcal{T}$ and renormalization constant $c/2$ and denoted by RSWN($\mathcal{T}$) (or RSWN($\mathcal{T}(\mathbb{R})$), if confusion may arise).

In the notations of the previous section and of Definition 4, for a bounded Borel subset $I \subset \mathbb{R}$, we denote by

$$\mathbb{C}\chi_I := \{\text{the algebra of complex multiples of } \chi_I\} \tag{53}$$

Thus

$$\mathrm{RSWN}(\mathbb{C}\chi_I) \subset \mathrm{RSWN}(\mathcal{H}_0(\mathbb{R})) \tag{54}$$

is the Lie sub–$*$–algebra of RSWN($\mathcal{H}_0(\mathbb{R})$) with linear generators

$$\{B^+_I := B^+(\chi_I), \quad B^-_I := B^-(\chi_I), \quad N_I := N(\chi_I), \quad E_I := E(\chi_I)\} \tag{55}$$

The commutation relations in RSWN($\mathbb{C}\chi_I$) are

$$[B^-_I, B^+_I] = \frac{c}{2}E_I + N_I =: M_I \quad ; \quad [M_I, B^\pm_I] = \pm 2B^\pm_I \tag{56}$$

the other commutators vanish.
In view of (56), the sub-Lie algebra $\mathrm{RSWN}_0(\mathbb{C}\chi_I) \subset \mathrm{RSWN}(\mathbb{C}\chi_I)$, generated by the set $\{B_I^+, B_I^-, M_I\}$ satisfies the same commutation relation of $sl(2,\mathbb{C})$. Thus it can be identified with it and the algebra $\mathrm{RSWN}(\mathbb{C}\chi_I)$ will be identified with the central extension of $sl(2,\mathbb{C})$. Furthermore the representation of $sl(2,\mathbb{C})$ given by the relations from (4) to (6) with $\mu = c/2$ induces a $*$–representation of $\mathrm{RSWN}_0(\mathbb{C}\chi_I)$ which can be extended to a $*$–representation of $\mathrm{RSWN}(\mathbb{C}\chi_I)$ on the same Hilbert space.

6 Isomorphism between $\mathrm{RSWN}(\mathbb{C}\chi_I)$ and $sl(2,\mathbb{C},E)$

In line with the previous remark, the following Proposition, realizes a $*$–Lie algebra isomorphism from $\mathrm{RSWN}(\mathbb{C}\chi_I)$ to $sl(2,\mathbb{C},E)$. Its restriction on $\mathrm{RSWN}_0(\mathbb{C}\chi_I)$ will be a $*$–Lie algebra isomorphism from $\mathrm{RSWN}_0(\mathbb{C}\chi_I)$ to $sl(2,\mathbb{C})$.

Proposition 2 *Let $I \subset \mathbb{R}$ be a bounded Borel set and let*

$$\hat{s}_I : \mathrm{RSWN}(\mathbb{C}\chi_I) \to sl(2,\mathbb{C},E)$$

be the unique linear extension of the map:

$$\hat{s}_I(B_I^-) = \alpha_I^- B^- \ , \ \hat{s}_I(B_I^+) = \alpha_I^+ B^+ \ , \ \hat{s}_I(N_I) = \beta_I N \ , \ \hat{s}_I(E_I) = \gamma_I E \tag{57}$$

where $\alpha_I^\pm, \beta_I, \gamma_I$ are complex numbers. Then $\hat{s}_I$ is a $*$–Lie algebra isomorphism if and only if there exists $\theta_I \in \mathbb{R}$ such that

$$\beta_I = 1 \qquad ; \qquad \gamma_I = \frac{2\mu}{c} \qquad ; \qquad \alpha_I^+ = e^{i\theta_I} \qquad ; \qquad \alpha_I^- = e^{-i\theta_I} \tag{58}$$

In this case, in the notation (56), we have

$$\hat{s}_I(M_I) = M \tag{59}$$

Proof. The isomorphism condition requires that

$$\left[\hat{s}_I(B_I^-), \hat{s}_I(B_I^+)\right] = \hat{s}_I(\left[B_I^-, B_I^+\right]) \tag{60}$$

Since

$$\begin{aligned}\left[\hat{s}_I(B_I^-), \hat{s}_I(B_I^+)\right] &= \alpha_I^- \alpha_I^+ \left[B^-, B^+\right] = \alpha_I^- \alpha_I^+ (\mu E + N) \\ \hat{s}_I(\left[B_I^-, B_I^+\right]) &= \hat{s}_I(\frac{c}{2} E_I + N_I) = \frac{c}{2}\gamma_I E + \beta_I N\end{aligned}$$

the linear independence of E and N implies that (60) is equivalent to:

$$\alpha_I^- \alpha_I^+ \mu = \frac{c}{2}\gamma_I \qquad ; \qquad \alpha_I^- \alpha_I^+ = \beta_I \tag{61}$$

Similarly, identifying

$$\begin{aligned}\left[\hat{s}_I(N_I), \hat{s}_I(B_I^+)\right] &= \beta_I \alpha_I^+ \left[N, B^+\right] = 2\beta_I \alpha_I^+ B^+ \\ \hat{s}_I(\left[N_I, B_I^+\right]) &= 2\hat{s}_I(B_I^+) = 2\alpha_I^+ B^+\end{aligned}$$

one obtains: $\alpha_I^+ = \alpha_I^+ \beta_I$. Since the isomorphism assumption implies that $\alpha_I^+ \neq 0$, this implies that $\beta_I = 1$. Therefore (61) implies

$$\alpha_I^- \alpha_I^+ = 1 \quad ; \quad \gamma_I = \frac{2\mu}{c} \tag{62}$$

Since $\hat{s}_I$ is a $*$–map, one has

$$\begin{aligned} (\hat{s}_I(B_I^-))^* &= (\alpha_I^- B^-)^* = (\alpha_I^-)^* B^+ = \hat{s}_I(B_I^+) = \alpha_I^+ B^+ \\ \iff & (\alpha_I^-)^* = \overline{\alpha_I^-} = \alpha_I^+ \end{aligned} \tag{63}$$

Given (63), (62) becomes

$$|\alpha_I^+|^2 = 1$$

We conclude that, for some $\theta_I \in \mathbb{R}$ (58) holds.
Conversely, if α_I^+ , α_I^- , β_I , γ_I are given by (58), then (57) becomes

$$\hat{s}_I(B_I^-) = e^{-i\theta_I} B^- \ , \ \hat{s}_I(B_I^+) = e^{i\theta_I} B^+ \ , \ \hat{s}_I(N_I) = N \ , \ \hat{s}_I(E_I) = \frac{2\mu}{c} E$$

and one easily verifies that this defines a $*$–isomorphism. Equation (59), follows because

$$\hat{s}_I(M_I) = \hat{s}_I(\frac{c}{2} E_I + N_I) = \frac{c}{2} \hat{s}_I(E_I) + \hat{s}_I(N_I) = \frac{c}{2} \frac{2\mu}{c} E + N = \mu E + N = M$$

7 $*$–algebras isomorphisms

In the notations and assumptions of Section 6, and in analogy with (11) , denote

$$F_I(Z) := z B_I^+ + \bar{z} B_I^- + \lambda M_I \ ; \ Z = (z, \lambda) \in \mathfrak{D}_+ \tag{64}$$

the generalized fields. With the choice $\theta_I = 0$ for all I, the image of such an element under the isomorphism $\hat{s}_I$, defined in Proposition 2, is

$$\hat{s}_I(F_I(Z)) = F_0(Z) \tag{65}$$

Denote $\mathrm{QHeis}(1, \mathbb{C}\chi_I)$ the Lie group of $\mathrm{RSWN}_0(\mathbb{C}\chi_I)$. This is an isomorphic copy of $\mathrm{QHeis}(1, F)$. In analogy with the notation (13), the generic element of $\mathrm{QHeis}(1, \mathbb{C}\chi_I)$ will be denoted

$$W_{r,I}(Z) := e^{iF_I(Z)} \qquad ; \qquad Z \in \mathfrak{D}_+ \tag{66}$$

Lemma 2 *In the notations (11), (64) and (66), the map*

$$s_I \ : \ e^{iF_I(Z)} \in \mathrm{QHeis}(1, \mathbb{C}\chi_I) \ \mapsto \ e^{i\hat{s}_I(F_I(Z))} = e^{iF_0(Z)} \in \mathrm{QHeis}(1) \tag{67}$$

where $\hat{s}_I$ the isomorphism is given by Proposition 2, is well defined and is a Lie group isomorphism.

Proof. According to Proposition 2, for each bounded Borel set I, $\hat{s}_I : \mathrm{RSWN}_0(\mathbb{C}\chi_I) \to sl(2,\mathbb{C})$ is a $*$–Lie algebra isomorphism. Since $\mathfrak{D}_+$ is connected and simply connected, $\hat{s}_I$ implements a Lie group isomorphism from $\mathrm{QHeis}(1,\mathbb{C}\chi_I)$ to $\mathrm{QHeis}(1,F)$, denoted s_I, such that

$$s_I\big(e^{iF_I(Z)}\big) = e^{i\hat{s}_I(F_I(Z))} \equiv e^{i\chi_I F_0(Z)} \tag{68}$$

Remark 3 Since the one-mode quadratic Weyl operators (50) are linearly independent (see [19], Theorem 12), the same is true for the set of the re–scaled quadratic Weyl operators $\{W_r(Z) \ : \ Z \in \mathfrak{D}_+\}$. This linear independence and the composition law (23) imply that the group isomorphism given by (67) extends to a $*$–isomorphism, still denoted s_I, from the group–$*$–algebra $\mathcal{QW}^0_{I,1}$ generated by $\mathrm{QHeis}(1,\mathbb{C}\chi_I)$ to the $*$–algebra $\mathcal{W}^0_2(\Gamma)$ generated by $\mathrm{QHeis}(1,F)$. In the notation (66), the explicit form of this extension is given by the unique linear extension of

$$s_I : W_{r,I}(Z) \in \mathcal{QW}^0_{I,1} \longmapsto W_r(Z) \in \mathcal{W}^0_2(\Gamma) \tag{69}$$

This isomorphism gives a meaning to the intuitive identification:

$$s_I\big(e^{iF_I(Z)}\big) = e^{i\hat{s}_I(F_I(Z))} \equiv e^{i\chi_I F_0(Z)} \tag{70}$$

8 The inductive limit

Now, for I as above and for any finite partition π of I, we want to give a meaning to the intuitive identifications

$$e^{i\chi_I F_0(Z)} = e^{i\left(b^+(z\chi_I)+b^-(z\chi_I)+n(\lambda\chi_I)+\langle\frac{c}{2}\lambda\chi_I\rangle\mathbf{1}\right)} = e^{i\sum_{I'\in\pi}\chi_{I'}F_0(Z)} \tag{71}$$

$$= \prod_{I'\in\pi} e^{i\chi_{I'}F_0(Z)} \equiv \bigotimes_{I'\in\pi} e^{i\left(b^+(z\chi_{I'})+b^-(z\chi_{I'})+n(\lambda\chi_{I'})+\langle\frac{c}{2}\lambda\chi_{I'}\rangle\mathbf{1}\right)} = \bigotimes_{I'\in\pi} e^{i\chi_{I'}F_0(Z)}$$

Notice that, for any $n \in \mathbb{N}$, the tensor power $W_r^{\otimes n}$ of the unitary representation W_r is well defined and is a unitary representation of the same group. Because of the linear independence of the one-mode quadratic Weyl operators (50), the map $W_r(Z) \in \mathcal{W}^0_2(\Gamma) \mapsto W_r(Z)^{\otimes n} \in (\mathcal{W}^0_2(\Gamma))^{\otimes n}$ (algebraic tensor product) has a unique linear extension

$$d_n : \mathcal{W}^0_2(\Gamma) \to (\mathcal{W}^0_2(\Gamma))^{\otimes n}$$

called the **diagonal n–th embedding**. Since the $W_r(Z)$ (hence the $W_r(Z)^{\otimes n}$) are linearly independent, d_n is a $*$–isomorphism onto its image and we denote

$$\mathcal{W}^0_{2;n} := \text{ image of } \mathcal{W}^0_2(\Gamma) \text{ under the diagonal } n\text{–embedding} \subset (\mathcal{W}^0_2(\Gamma))^{\otimes n}$$

Therefore, for any bounded Borel set I and any Borel partition π of I, of cardinality n_π,

$$\Theta_{I;\pi} := \left(\bigotimes_{I'\in\pi} s^{-1}_{I'}\right) \circ d_{n_\pi} \circ s_I : \mathcal{QW}^0_{I,1} \to \mathcal{QW}^0_{I;\pi} \subset \bigotimes_{I'\in\pi} \mathcal{QW}^0_{I',1} \tag{72}$$

is a $*$–isomorphism of $\mathcal{QW}^0_{I,1}$ onto its image $\mathcal{QW}^0_{I;\pi}$, called the **diagonal** π–**embedding** of $\mathcal{QW}^0_{I,1}$ into $\bigotimes_{I'\in\pi}\mathcal{QW}^0_{I',1}$. Its explicit action on the linear generators is:

$$\begin{aligned}
\Theta_{I;\pi}\big(W_{r,I}(Z)\big) &= \left(\Big(\bigotimes_{I'\in\pi} s_{I'}^{-1}\Big)\circ d_{n_\pi}\circ s_I\right)\big(W_{r,I}(Z)\big)\\
&= \left(\Big(\bigotimes_{I'\in\pi} s_{I'}^{-1}\Big)\circ d_{n_\pi}\right) s_I\big(W_{r,I}(Z)\big)\\
&= \left(\Big(\bigotimes_{I'\in\pi} s_{I'}^{-1}\Big)\circ d_{n_\pi}\right)\big(W_r(Z)\big)\\
&= \Big(\bigotimes_{I'\in\pi} s_{I'}^{-1}\Big) d_{n_\pi}\big(W_r(Z)\big) = \Big(\bigotimes_{I'\in\pi} s_{I'}^{-1}\Big)\big(W_r(Z)\big)^{\otimes n_\pi}\\
&= \bigotimes_{I'\in\pi} s_{I'}^{-1}\big(W_r(Z)\big) = \bigotimes_{I'\in\pi} W_{r,I'}(Z) \in \mathcal{QW}^0_{I;\pi}
\end{aligned}$$

In terms of the intuitive interpretation (70) this corresponds to

$$s_I\big(e^{iF_I(Z)}\big) \equiv e^{i\chi_I F_0(Z)}$$

$$= e^{i\sum_{I'\in\pi}\chi_{I'}F_0(Z)} \equiv \bigotimes_{I'\in\pi} e^{i\chi_{I'}F_0(Z)} \equiv \bigotimes_{I'\in\pi} s_{I'}\big(e^{iF_{I'}(Z)}\big) \tag{73}$$

For a bounded Borel subset I of $\mathbb{R}$, we denote $\mathcal{P}_f(I)$ the set of finite partitions of I partially ordered by refinement and we denote $\prec$ this partial order.
Given a finite Borel partition π of I, we want to construct an algebra $\mathcal{QW}_{I;\pi}$ where the intuitive identifications

$$e^{i\sum_{I'\in\pi}\chi_{I'}F_0(Z_{I'})} \equiv \bigotimes_{I'\in\pi} e^{i\chi_{I'}F_0(Z_{I'})} \tag{74}$$

have a meaning for each choice of $\pi\in\mathcal{P}_f(I)$ and of the $Z_{I'}\in\mathfrak{D}_+$. In view of the factorization property, it is natural to take this algebra as

$$\mathcal{QW}_{I;\pi} := \bigotimes_{I'\in\pi}\mathcal{QW}^0_{I',1} \quad \text{(algebraic tensor product)} \tag{75}$$

For $\pi,\pi'\in\mathcal{P}_f(I)$ such that $\pi\prec\pi'$ we know that for each $J\in\pi$ the family

$$\pi'_J := \{I'\in\pi' \ : \ I'\subseteq J\} \tag{76}$$

is a partition of J and

$$\bigcup_{J\in\pi}\pi'_J = \pi' \tag{77}$$

Therefore, for any $\pi\prec\pi'\in\mathcal{P}_f(I)$, the map $\Theta_{I;\pi,\pi'} := \bigotimes_{J\in\pi}\Theta_{J;\pi'_J}$ given by

$$\Theta_{I;\pi,\pi'} : \mathcal{QW}_{I;\pi} = \bigotimes_{J\in\pi}\mathcal{QW}^0_{J,1} \longrightarrow \bigotimes_{J\in\pi}\Theta_{J;\pi'_J}(\mathcal{QW}^0_{J,1}) = \bigotimes_{J\in\pi}\mathcal{QW}^0_{J;\pi'_J}$$

is well defined and is a $*$–isomorphism being an algebraic tensor product of $*$–isomorphisms. From (72) we know that, for each $J \in \pi$,

$$\mathcal{QW}^0_{J;\pi'_J} \subset \bigotimes_{I' \in \pi'_J} \mathcal{QW}^0_{I',1}$$

therefore

$$\begin{aligned}\bigotimes_{J\in\pi} \Theta_{J;\pi'_J} : \mathcal{QW}_{I;\pi} \longrightarrow \bigotimes_{J\in\pi} \mathcal{QW}^0_{J;\pi'_J} \subset \bigotimes_{J\in\pi} \bigotimes_{I'\in\pi'_J} \mathcal{QW}^0_{I',1} &= \bigotimes_{I'\in\pi'} \mathcal{QW}^0_{I',1} \\ &= \mathcal{QW}_{I;\pi'}\end{aligned}$$

so that the map

$$\Theta_{I;\pi,\pi'} = \bigotimes_{J\in\pi} \Theta_{J;\pi'_J} : \mathcal{QW}_{I;\pi} \to \mathcal{QW}_{I;\pi'}$$

is a $*$–embedding which is a $*$–isomorphism onto its image.
In terms of the intuitive interpretation (73) this corresponds to

$$\begin{aligned}\bigotimes_{J\in\pi} s_J\big(e^{iF_J(Z_J)}\big) \equiv \bigotimes_{J\in\pi} e^{i\chi_J F_0(Z_J)} &= \bigotimes_{J\in\pi} e^{i\sum_{J'\in\pi'_J} \chi_{J'} F_0(Z_J)} \\ &\equiv \bigotimes_{J\in\pi} \bigotimes_{J'\in\pi'_J} e^{i\chi_{J'} F_0(Z_J)} \qquad (78)\end{aligned}$$

For $\pi \prec \pi' \prec \pi''$, one has

$$\begin{aligned}
&\Theta_{I;\pi',\pi''} \circ \Theta_{I;\pi,\pi'} = \bigotimes_{I'\in\pi'} \Theta_{I';\pi''_{I'}} \circ \bigotimes_{J\in\pi} \Theta_{J;\pi'_J} \\
&= \bigotimes_{J\in\pi} \bigotimes_{I'\in\pi'_J} \Theta_{I';\pi''_{I'}} \circ \bigotimes_{J\in\pi} \Theta_{J;\pi'_J} = \bigotimes_{J\in\pi} \left(\bigotimes_{I'\in\pi'_J} \Theta_{I';\pi''_{I'}} \circ \Theta_{J;\pi'_J} \right) \\
&= \bigotimes_{J\in\pi} \left(\left(\bigotimes_{I'\in\pi'_J} \left(\left(\bigotimes_{I''\in\pi''_{I'}} s^{-1}_{I''} \right) \circ d_{n_{\pi''_{I'}}} \circ s_{I'} \right) \right) \circ \left(\left(\bigotimes_{I'\in\pi'_J} s^{-1}_{I'} \right) \circ d_{n_{\pi'_J}} \circ s_J \right) \right) \\
&= \bigotimes_{J\in\pi} \bigotimes_{I'\in\pi'_J} \left(\left(\bigotimes_{I''\in\pi''_{I'}} s^{-1}_{I''} \right) \circ d_{n_{\pi''_{I'}}} \circ s_{I'} \right) \circ \left(\left(s^{-1}_{I'} \right) \circ d_{n_{\pi'_J}} \circ s_J \right) \\
&= \bigotimes_{J\in\pi} \bigotimes_{I'\in\pi'_J} \left(\left(\bigotimes_{I''\in\pi''_{I'}} s^{-1}_{I''} \right) \circ d_{n_{\pi''_{I'}}} \circ s_{I'} \circ \left(s^{-1}_{I'} \right) \right) \circ d_{n_{\pi'_J}} \circ s_J \\
&= \bigotimes_{J\in\pi} \bigotimes_{I'\in\pi'_J} \left(\bigotimes_{I''\in\pi''_{I'}} s^{-1}_{I''} \circ d_{n_{\pi''_{I'}}} \circ d_{n_{\pi'_J}} \right) \circ s_J \\
&= \bigotimes_{J\in\pi} \left(\bigotimes_{I'\in\pi'_J} \bigotimes_{I''\in\pi''_{I'}} s^{-1}_{I''} \right) \circ \left(\bigotimes_{I'\in\pi'_J} d_{n_{\pi''_{I'}}} \right) \circ d_{n_{\pi'_J}} \circ s_J
\end{aligned}$$

$$= \bigotimes_{J\in\pi} \left(\bigotimes_{I''\in\pi''_J} s_{I''}^{-1} \right) \circ d_{n_{\pi''_J}} \circ s_J = \Theta_{I;\pi,\pi''}$$

where we have used (77) and the identity

$$\left(\bigotimes_{I'\in\pi'_J} d_{n_{\pi''_{I'}}} \right) \circ d_{n_{\pi'_J}} = d_{\sum_{I'\in\pi'_J} n_{\pi''_{I'}}} = d_{n_{\pi''_J}}$$

This gives an inductive system of $*$–algebras in which the embeddings are $*$–isomorphisms onto their images. We denote

$$\left\{ \mathcal{W}_I, (\tilde{\Theta}_{I;\pi})_{\pi\in\mathcal{P}_f(I)} \right\} := \lim_{\rightarrow} \{ (\mathcal{QW}_{I;\pi}), (\Theta_{I;\pi,\pi'})_{\pi\prec\pi'\in\mathcal{P}_f(I)} \}$$

the inductive limit of this inductive system. Thus for any $\pi \in \mathcal{P}_f(I)$ the map

$$\tilde{\Theta}_{I;\pi} : \mathcal{QW}_{I;\pi} \to \mathcal{W}_I$$

is a $*$–isomorphism satisfying

$$\tilde{\Theta}_{I;\pi'} \Theta_{I;\pi,\pi'} = \tilde{\Theta}_{I;\pi} \qquad ; \qquad \forall \pi \prec \pi' \in \mathcal{P}_f(I)$$

9 Factorizable families of $*$–algebras

Definition 5 A family of $*$–algebras (resp. C^*–algebras) $\{\mathcal{W}_I\}$, indexed by the bounded Borel subsets of $\mathbb{R}$, is called *factorizable* if, for every bounded Borel $I \subset \mathbb{R}$ and every Borel partition π of I, there is an $*$–isomorphism (resp. C^*–isomorphism for some cross–norm)

$$u_{I,\pi} : \bigotimes_{I_j\in\pi} \mathcal{W}_{I_j} \to \mathcal{W}_I$$

In this case, each algebra $\mathcal{W}_I$ is the closed linear span of a family of *factorizable elements* i.e. operators $w_I \in \mathcal{W}_I$ with the property that there exist operators $w_{I_j} \in \mathcal{W}_{I_j}$ $(I_j \in \pi)$ such that

$$u_{I,\pi}^{-1}(w_I) = \bigotimes_{I_j\in\pi} w_{I_j} \tag{79}$$

Note that if I, J are disjoint bounded Borel sets in $\mathbb{R}$, then the map

$$(\pi_I, \pi_J) \in \mathcal{P}_f(I) \times \mathcal{P}_f(J) \;\mapsto\; \pi_{I\cup J} := \{\pi_I \cup \pi_J\} \in \mathcal{P}_f(I \cup J) \tag{80}$$

defines a canonical injection of $\mathcal{P}_f(I) \times \mathcal{P}_f(J)$ into $\mathcal{P}_f(I \cup J)$ which is order preserving, in the sense that, if $\pi_I \prec \pi'_I \in \mathcal{P}_f(I)$ and $\pi_J \prec \pi'_J \in \mathcal{P}_f(J)$, then $\pi_{I\cup J} \prec \pi'_{I\cup J} \in \mathcal{P}_f(I \cup J)$. By definition $\pi_{I\cup J}$ is the partition of $I \cup J$ defined by $\{I_1, \dots, I_{n_I}, J_1, \dots, J_{n_J}\}$ where $\pi_I = \{I_1, \dots, I_{n_I}\}$, $\pi_J = \{J_1, \dots, J_{n_J}\}$.
The injection (80) is not surjective, but the image of $\mathcal{P}_f(I) \times \mathcal{P}_f(J)$ under

(80) is a co–final set, i.e. for each $\pi \in \mathcal{P}_f(I \cup J)$ there exist $\pi_I \in \mathcal{P}_f(I)$ and $\pi_J \in \mathcal{P}_f(J)$ such that $\pi \prec \pi_{I \cup J}$.
It is known (see [22]) that the inductive limit of any inductive system over a net $\mathcal{N}$ is isomorphic to the inductive limit of the restriction of the original system to any co–final sub–net of $\mathcal{N}$.

Lemma 3 *Let I, J be disjoint bounded Borel sets in $\mathbb{R}$. Then the inductive system of $*$–algebras*

$$\Big\{(\mathcal{QW}_{I\cup J;\pi_{I\cup J}})_{\pi_{I\cup J}\in\mathcal{P}_f(I\cup J)}\ ,\ (\Theta_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}})_{\pi_{I\cup J}\prec\pi'_{I\cup J}\in\mathcal{P}_f(I\cup J)}\Big\} \tag{81}$$

is isomorphic to the inductive system of $*$–algebras

$$\Big\{(\mathcal{QW}_{I;\pi_I}\otimes\mathcal{QW}_{J;\pi_J})_{(\pi_I,\pi_J)\in\mathcal{P}_f(I)\times\mathcal{P}_f(J)}\ ,$$

$$(\Theta_{I;\pi_I,\pi'_I}\otimes\Theta_{J;\pi_J,\pi'_J})_{\pi_I\prec\pi'_I\in\mathcal{P}_f(I),\pi_J\prec\pi'_J\in\mathcal{P}_f(J)}\Big\} \tag{82}$$

in the sense that, for each $\pi_I \prec \pi'_I \in \mathcal{P}_f(I)$ and $\pi_J \prec \pi'_J \in \mathcal{P}_f(J)$, there exists a $*$–algebra isomorphism

$$u_{I,J,\pi_I,\pi_J}\ :\ \mathcal{QW}_{I;\pi_I}\otimes\mathcal{QW}_{J;\pi_J}\to\mathcal{QW}_{I\cup J;\pi_{I\cup J}}$$

such that, in the notation (80),

$$u_{I,J,\pi'_I,\pi'_J}\circ(\Theta_{I;\pi_I,\pi'_I}\otimes\Theta_{J;\pi_J,\pi'_J})=\Theta_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}} \tag{83}$$

Proof. In the above notations, let

$$s_{\pi_I} := \bigotimes_{I'\in\pi_I} s_{I'} : \mathcal{QW}_{I;\pi_I}\to(\mathcal{W}_2^0(\Gamma))^{\otimes|\pi_I|}$$

be the $*$–isomorphism (75) ($|\pi_I|$:= cardinality of π_I). Then we have

$$\begin{aligned} s_{\pi_I}\otimes s_{\pi_J} : \mathcal{QW}_{I;\pi_I}\otimes\mathcal{QW}_{J;\pi_J} &\to (\mathcal{W}_2^0(\Gamma))^{\otimes|\pi_I|}\otimes(\mathcal{W}_2^0(\Gamma))^{\otimes|\pi_J|}\\ &\equiv (\mathcal{W}_2^0(\Gamma))^{\otimes(|\pi_I|+|\pi_J|)}\\ &= s_{\pi_I\cup\pi_J}(\mathcal{QW}_{I\cup J;\pi_I\cup\pi_J}) \end{aligned}$$

Let us consider the map

$$u_{I,J,\pi_I,\pi_J} := (s_{\pi_I\cup\pi_J})^{-1}\circ(s_{\pi_I}\otimes s_{\pi_J})$$

Clearly

$$u_{I,J,\pi_I,\pi_J}\ :\ \mathcal{QW}_{I;\pi_I}\otimes\mathcal{QW}_{J;\pi_J}\longrightarrow\mathcal{QW}_{I\cup J;\pi_{I\cup J}} \tag{84}$$

is a $*$–isomorphism.
If $\pi_I \prec \pi'_I \in \mathcal{P}_f(I)$ and $\pi_J \prec \pi'_J \in \mathcal{P}_f(J)$, then we know that $\pi_{I\cup J} \prec \pi'_{I\cup J} \in \mathcal{P}_f(I \cup J)$. The generic element of $\mathcal{QW}_{I;\pi_I}$ has the form

$$\bigotimes_{I_0\in\pi_I} s_{I_0}^{-1}(W_r(Z_{I_0})) = \Big(\bigotimes_{I_0\in\pi_I} s_{I_0}^{-1}\Big)\Big(\bigotimes_{I_0\in\pi_I} W_r(Z_{I_0})\Big)\in\mathcal{QW}_{I;\pi_I}=\bigotimes_{I_0\in\pi_I}\mathcal{QW}^0_{I_0,1}$$

In the notation (76) and with : $I' \in (\pi'_I)_{I_0} \Rightarrow Z_{I'} = Z_{I_0}$, one has

$$\Theta_{I;\pi_I,\pi'_I}\left(\bigotimes_{I_0\in\pi_I} s_{I_0}^{-1}(W_r(Z_{I_0}))\right) = \bigotimes_{I'\in\pi'_I} s_{I'}^{-1}(W_r(Z_{I'})) \in \mathcal{QW}_{I;\pi'_I}$$

Therefore

$$u_{I,J,\pi'_I,\pi'_J}\circ(\Theta_{I;\pi_I,\pi'_I}\otimes\Theta_{J;\pi_J,\pi'_J})\left(\left(\bigotimes_{I_0\in\pi_I} s_{I_0}^{-1}(W_r(Z_{I_0}))\right)\otimes\left(\bigotimes_{J_0\in\pi_J} s_{J_0}^{-1}(W_r(Z_{J_0}))\right)\right)$$
$$= (s_{\pi'_I\cup\pi'_J})^{-1}\circ(s_{\pi'_I}\otimes s_{\pi'_J})\left(\left(\bigotimes_{I'\in\pi'_I} s_{I'}^{-1}(W_r(Z_{I'}))\right)\otimes\left(\bigotimes_{J'\in\pi'_J} s_{J'}^{-1}(W_r(Z_{J'}))\right)\right)$$
$$= (s_{\pi'_I\cup\pi'_J})^{-1}\left(\left(\bigotimes_{I'\in\pi'_I} W_r(Z_{I'})\right)\otimes\left(\bigotimes_{J'\in\pi'_J} W_r(Z_{J'})\right)\right)$$
$$= (s_{\pi'_I\cup\pi'_J})^{-1}\left(\bigotimes_{K'\in\pi'_I\cup\pi'_J} W_r(Z_{K'})\right)$$
$$= \bigotimes_{K'\in\pi'_I\cup\pi'_J} s_{K'}^{-1}(W_r(Z_{K'})) = \Theta_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}}\left(u_{I,J,\pi_I,\pi_J}\left(\bigotimes_{K\in\pi_I\cup\pi_J} s_K^{-1}(W_r(Z_K))\right)\right)$$

Therefore

$$u_{I,J,\pi'_I,\pi'_J}\circ(\Theta_{I;\pi_I,\pi'_I}\otimes\Theta_{J;\pi_J,\pi'_J}) = \Theta_{I\cup J;\pi_{I\cup J},\pi'_{I\cup J}}\circ u_{I,J,\pi_I,\pi_J}$$

which proves (83).

Theorem 2 *The family of $*$–algebras $\{\mathcal{W}_I\ :\ I-$bounded Borel subset of $\mathbb{R}\}$ is factorizable.*

Proof. We apply Definition 5 to the case in which the family $\mathcal{F}$ is the family of bounded Borel sets in $\mathbb{R}$. By induction it will be sufficient to prove that, if I, J are disjoint bounded Borel sets in $\mathbb{R}$, then there exists a $*$–isomorphism

$$u_{I,J}\ :\ \mathcal{W}_I\otimes\mathcal{W}_J\to\mathcal{W}_{I\cup J} \tag{85}$$

Since $\mathcal{W}_I\otimes\mathcal{W}_J$ is the inductive limit of the system (82) and $\mathcal{W}_{I\cup J}$ is the inductive limit of the system (81), the statement follows from Lemma 3 because isomorphic inductive systems have isomorphic inductive limits.

From Theorem 2 it follows that, if $I\subset J$ are bounded Borel sets in $\mathbb{R}$, then the map

$$j_{I,J}\ :\ w_I\in\mathcal{W}_I\to w_I\otimes 1_{J\setminus I}\in\mathcal{W}_J \tag{86}$$

is a $*$–algebra isomorphism onto its image. Since clearly, for $I\subset J\subset K$ bounded Borel sets in $\mathbb{R}$, $1_{J\setminus I}\otimes 1_{K\setminus J}\equiv 1_{K\setminus I}$, it follows that

$$\{(\mathcal{W}_I)\ ,\ (j_{I,J})\ ,\ I\subset J\in\text{bounded Borel sets in }\mathbb{R}\} \tag{87}$$

is an inductive system of $*$–algebras.

Definition 6 The inductive limit of the system (87) will be denoted

$$\left\{\mathcal{W}^0_{\mathbb{R}}\ ,\ (j_I)\ ,\ I \in \text{bounded Borel sets in } \mathbb{R}\right\}$$

Since the $j_I : \mathcal{W}_I \to \mathcal{W}^0_{\mathbb{R}}$ are injective embeddings, the family $(j_I(\mathcal{W}_I))$ is factorizable and one can introduce the more intuitive notation:

$$j_I(\mathcal{W}_I) \equiv \mathcal{W}_I \otimes 1_{I^c} \tag{88}$$

On $\mathcal{W}^0_{\mathbb{R}}$ define the semi–norm

$$\|x\| := \sup\{\|\pi(x)\|\ :\ \pi \in \{*\text{–representations of } \mathcal{W}^0_{\mathbb{R}}\}\}\ ;\ x \in \mathcal{W}^0_{\mathbb{R}} \tag{89}$$

Proposition 3 The semi–norm (89) is a norm.

Proof. The limit $*$–algebra $\mathcal{W}^0_{\mathbb{R}}$ is linearly generated by unitaries. Hence the sup in (89) is finite for any $x \in \mathcal{W}^0_{\mathbb{R}}$. To prove that this sup is $\neq 0$ for any element different from zero, it is sufficient to show that, on $\mathcal{W}^0_{\mathbb{R}}$ there is a faithful state. Since the inductive limit of faithful states is faithful, it is sufficient to show that, for each bounded Borel set $I \subset \mathbb{R}$ there is a faithful state on $\mathcal{W}^0_I$ and the same inductive limit argument shows that $\mathcal{W}^0_I$ can be replaced by $\mathcal{QW}_{I;\pi}$ where π is any finite Borel partition of I. But from (75) and the definition of $\mathcal{QW}^0_{I',1}$, we know that $\mathcal{QW}_{I;\pi}$ is isomorphic to the $|\pi|$–th tensor power of the group–$*$–algebra generated by QHeis$(1,F)$, whose elements are unitaries. But on any group–$*$–algebra of a unitary representation the tracial state (which in this case is defined on the generators by $\tau\left(W_{g_1} \otimes \cdots \otimes W_{g_{|\pi|}}\right) := \prod_{j=1}^{|\pi|} \delta_{e,g_j}$ is faithful. This proves the statement.

The norm (89) is called the minimal regular C^*–norm. The completion of $\mathcal{W}^0_{\mathbb{R}}$ under this norm is a C^*–algebra denoted $\mathcal{W}_{\mathbb{R}}$.
Each $*$–algebra $\mathcal{W}_I$ ($I \subset \mathbb{R}$ bounded Borel set) can be identified to a $*$–sub–algebra of $\mathcal{W}^0_{\mathbb{R}}$ hence its completion, still denoted with the same symbol, can be identified to a C^*–sub–algebra of $\mathcal{W}_{\mathbb{R}}$. The family $(\mathcal{W}_I)$ is factorizable as completion of a factorizable family.

10 Existence of factorizable states on $\mathcal{W}_{\mathbb{R}}$

From the factorizability property of the family $(\mathcal{W}_I)$ and of the corresponding generators, for any $I \subset \mathbb{R}$ bounded Borel and any finite partition π of I, we will use the identifications

$$\begin{aligned} \mathcal{W}_I &\equiv j_I(\mathcal{W}_I) \equiv \mathcal{W}_I \otimes 1_{I^c} \subset \mathcal{W}_{\mathbb{R}} \\ W_{r,I}(Z) &\equiv \bigotimes_{I' \in \pi} W_{r,I'}(Z) \qquad ; \qquad \forall Z \in \mathfrak{D}^+ \end{aligned} \tag{90}$$

omitting from the notations the isomorphisms implementing these identifications.

Definition 7 A state φ on $\mathcal{W}_{\mathbb{R}}$ is called factorizable if for every bounded Borel set $I \subset \mathbb{R}$, for every finite partition π of I and every family $Z_{I'} \in \mathfrak{D}_+$ $(I' \in \pi)$, one has:

$$\varphi\left(\prod_{I'\in\pi} W_{r,I'}(Z_{I'})\right) = \prod_{I'\in\pi} \varphi(W_{r,I'}(Z_{I'})) \tag{91}$$

Theorem 3 There exists a unique factorizable state φ on $\mathcal{W}_{\mathbb{R}}$ such that, for each bounded Borel set $I \subset \mathbb{R}$, one has

$$\varphi(W_{r,I}(Z)) = \varphi_{\mu_I}(W_r(Z)) \qquad ; \qquad \forall Z \in \mathfrak{D}_+ \tag{92}$$

where $\mu_I = \frac{c|I|}{2}$ ($\mid I \mid$ the Lebesgue measure of I) and φ_{μ_I} is given by (51).

Proof. Let I be a bounded Borel set of $\mathbb{R}$ and let $\pi = (I_j)_{j\in F}$, $\pi' = (J_h)_{h\in G}$ be finite partitions of I such that $\pi \prec \pi'$. Since $\pi \prec \pi'$, then for all $j \in F$ there exists $G_j \subset G$ such that $I_j = \bigcup_{h\in G_j} J_h$. To prove the existence of such φ, let us consider the family of states

$$\varphi_{I,\pi} : \mathcal{QW}_{I;\pi} \longrightarrow \mathbb{C}$$

given by

$$\varphi_{I,\pi} := \bigotimes_{j\in F} \varphi_{\mu_{I_j}}, \tag{93}$$

with φ_{μ_I} as in (92). First we prove that the family of states $\varphi_{I,\pi}$ is projective w.r.t $\Theta_{I;\pi,\pi'}$, i.e.:

$$\varphi_{I,\pi'} \circ \Theta_{I;\pi,\pi'} = \varphi_{I,\pi} \qquad ; \quad \pi \prec \pi' \tag{94}$$

Denoting, for $I' \in \pi$, $\pi'_{I'}$ the partition of I' induced by π', this follows from the identity

$$\begin{aligned}
\varphi_{I,\pi'} \circ \Theta_{I;\pi,\pi'}\Big(\bigotimes_{I'\in\pi} W_{r,I_j}(Z_{I'})\Big) &= \varphi_{I,\pi'}\Big(\bigotimes_{I'\in\pi}\bigotimes_{J\in\pi'_{I'}} W_{r,J}(Z_{I'})\Big)\\
&= \Big(\bigotimes \bigotimes_{J\in\pi'_{I'}} \varphi_{\mu_J}\Big)\Big(\bigotimes_{I'\in\pi}\bigotimes_{J\in\pi'_{I'}} W_{r,J}(Z_{I'})\Big)\\
&= \prod_{I'\in\pi}\prod_{J\in\pi'_{I'}} \varphi_{\mu_J}\big(W_{r,J}(Z_{I'})\big)\\
&= \prod_{I'\in\pi}\prod_{J\in\pi'_{I'}} e^{\mu_J G(Z_{I'})}\\
&= \prod_{I'\in\pi} e^{(\sum_{J\in\pi'_{I'}} \mu_J)G(Z_{I'})}\\
&= \prod_{I'\in\pi} e^{\mu_{I_j} G(Z_{I'})}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{I' \in \pi} \varphi_{\mu_{I_j}} \big(W_{r,I_j}(Z_{I'}) \big) \\
&= \Big(\bigotimes_{I' \in \pi} \varphi_{\mu_{I_j}} \Big) \Big(\bigotimes_{I' \in \pi} W_{r,I_j}(Z_{I'}) \Big) \\
&= \varphi_{I,\pi} \big(\bigotimes_{I' \in \pi} W_{r,I_j}(Z_{I'}) \big)
\end{aligned}
$$

Projectivity implies that there exists a unique state φ_I on $\mathcal{W}_I$ whose restriction on each $\mathcal{W}_{I;\pi}$ coincides with $\varphi_{I,\pi}$. From the definition of $\varphi_{I,\pi}$ in (93), the family (φ_I) satisfies

$$
\varphi_I \otimes \varphi_J = \varphi_{I \cup J} \circ u_{I,J} \tag{95}
$$

for any pair of disjoint bounded Borel sets $I, J \subset \mathbb{R}$. Then (95) is equivalent to

$$
\varphi_I \otimes \varphi_{J \setminus I} = \varphi_J \circ u_{I,J \setminus I} \,, \quad \forall I \subset J
$$

But we have

$$
(\varphi_I \otimes \varphi_{J \setminus I})(w_I \otimes 1_{J \setminus I}) = \varphi_I(w_I)\varphi_{J \setminus I}(1_{J \setminus I}) = \varphi_I(w_I)
$$

On the other hand

$$
(\varphi_J \circ u_{I,J \setminus I})(w_I \otimes 1_{J \setminus I}) = (\varphi_J \circ j_{I,J})(w_I)
$$

Then

$$
(\varphi_J \circ j_{I,J})(w_I) = \varphi_I(w_I)
$$

which proves that the family of states (φ_I) is projective w.r.t the embeddings $(j_{I,J})$. This implies the existence of a unique state φ on the inductive limit $\mathcal{W}_{\mathbb{R}}$, i.e,. satisfying

$$
\varphi \circ j_I = \varphi_I
$$

for all bounded Borel set I. (95) implies factorizability because for all bounded disjoint Borel sets I, J and all $Z_I, Z_J \in \mathfrak{D}_+$

$$
\begin{aligned}
\varphi_I(W_{r,I}(Z_I))\varphi_J(W_{r,J}(Z_J)) &= (\varphi_I \otimes \varphi_J)\big(W_{r,I}(Z_I) \otimes W_{r,J}(Z_J)\big) \\
&= \varphi_{I \cup J} \circ u_{I,J}\big(W_{r,I}(Z_I) \otimes W_{r,J}(Z_J)\big) \\
&= \varphi_{I \cup J}\big(W_{r,I}(Z_I) W_{r,J}(Z_J)\big)
\end{aligned}
$$

A similar argument shows that (90), i.e. the factorizability of φ, holds on $\mathcal{W}^0_{\mathbb{R}}$. Let ψ be any state on $\mathcal{W}^0_{\mathbb{R}}$ and let $(\mathcal{W}, \pi_\psi, 1_\psi)$ denote its cyclic representation. Then (89) implies that

$$
\psi(a^*a) = \langle 1_\psi, \pi_\psi(a^*a)1_\psi \rangle \le \|\pi_\psi(a)\|^2 \le \|a\|
$$

By the Schwartz inequality if (a_n) is a Cauchy sequence in $\mathcal{W}^0_{\mathbb{R}}$, $(\psi(a_n))$ is a Cauchy sequence. Therefore any state on $\mathcal{W}^0_{\mathbb{R}}$ has a unique extension to $\mathcal{W}_{\mathbb{R}}$. If $I, J \subset \mathbb{R}$ are disjoint bounded Borel sets and $a_{I \cup J} = a_I a_J \in \mathcal{W}_{I \cup J}$ is a factorizable element, then there exist sequences $(a_{I,n}) \subset \mathcal{W}^0_I$ and $(a_{J,n}) \subset \mathcal{W}^0_J$ converging to a_I and a_J respectively. Therefore, if φ is the factorizable state on $\mathcal{W}_{\mathbb{R}}^0$ defined in the first part of the theorem, denoting with the same symbol its extension on $\mathcal{W}_{\mathbb{R}}$, one has

$$\varphi(a_{I\cup J}) = \varphi(a_I a_J) = \lim_{n\to\infty} \varphi(a_{I,n} a_{J,n}) = \lim_{n\to\infty} \varphi(a_{I,n})\varphi(a_{J,n}) = \varphi(a_I)\varphi(a_J)$$

and by induction this implies that φ is a factorizable state on $\mathcal{W}_{\mathbb{R}}$.

Acknowledgements The authors express their deep gratitude to Gerardo Morsella for precious suggestions that allowed to considerably improve the present paper.

L. Accardi acknowledges support by the Russian Science Foundation N. RSF 14-11-00687, Steklov Mathematical Institute.
A. Dhahri acknowledges support by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (grant 2016R1C1B1010008).
H.Rebei gratefully acknowledges Qassim University, represented by the Deanship of Scientific Research, on the support for this research.

References

1. L. Accardi, G. Amosov, U. Franz: *Second quantized automorphisms of the renormalized square of white noise (RSWN) algebra*, IDA–QP (Infinite Dimensional Analysis, Quantum Probability and Related Topics) 7 (2) (2004) 183–194 Preprint Volterra (2002)
2. L. Accardi, R. Roschin:
Renormalized squares of Boson fields,
Infin. Dimens. Anal. Quantum Probab. Relat. Top. (IDA-QP) 8 (2) (2005) 307-326
3. L. Accardi, A. Boukas: *Quantum probability, renormalization and infinite dimensional ∗–Lie algebras*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications) 5 (2009), electronic journal, Special issue on: Kac-Moody Algebras and Applications http://www.emis.de/journals/SIGMA/Kac-Moody_algebras.html
4. Accardi L., Boukas A.: *Cohomology of the Virasoro-Zamolodchikov and Renormalized higher powers of white noise ∗-Lie algebras*, IDA–QP (Infinite Dimensional Analysis, Quantum Probability and Related Topics) 12 (2) (2009) 193–212.
5. L. Accardi, A. Boukas: *Renormalization and central extensions* p-Adic Numbers, Ultrametric Analysis and Applications 4 (2) (2012) 89–101 Special issue dedicated to Igor V. Volovich.
6. L. Accardi, A. Dhahri: *Quadratic exponential vectors*, J. Math. Phys. 50 (2009) 122103.
7. L. Accardi, A. Dhahri: *The quadratic Fock functor*, J. Math. Phys, Vol 51 022105, 2010.
8. L. Accardi, A. Dhahri: *Polynomial extensions of the Weyl C^*-algebra*, Open systems and Information dynamics, Vol 22, 3 (2015).
9. L. Accardi, A. Dhahri: C^*-non linear second quantization, *Annales Henri Poincaré* 17 (7) (2016) 1883-1907
10. L. Accardi, Y.G Lu, I.V. Volovich: *White noise approach to classical and quantum stochastic calculi*, Lecture Notes given at the Volterra–CIRM International School with the same title, Trento, Italy, 1999, Volterra Preprint N. 375 July (1999) (unpublished).
11. L. Accardi, H. Ouerdiane, H. Rebei: *On the quadratic Heisenberg group*, Infin. Dimens. Anal.Quantum Probab. Relat. Top., Vol.13 (4), 551-587 (2010)
12. L. Accardi, H. Ouerdiane and H. Rebei, *Lévy processes through time schift on oscillator Weyl algebra*, Comminications on stochastic analysis, Vol. 6, No. 1 (2012), 125-155.
13. L. Accardi, H. Ouerdiane, H. Rebei: *Renormalized square of white noise quantum time shift*, Communications on Stochastic Analysis, Vol.6 (2), 177-191 (2012)
14. L. Accardi, U. Franz, M. Skeide: *Renormalized squares of white noise and other non-Gaussian as Lévy processes on real Lie algebras*, Comm. Math. Phys., Vol.228, 123-150 (2002)

15. A. Dhahri: *On The quadratic Fock functor*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, Hi Vol 15, No 2, 1250012, 2012.
16. A. Dhahri: *Self-adjointness and boundedness in quadratic quantization*, J. Math. Phys, Vol 55, 052103, 2014.
17. D. Petz, An Invitation to the Algebra of Canonical Commutation Relations, Leuven University Press (1989).
18. D.B. Prohorenko: *On the positivity of KMS functionals on the algebra of the square of white noise*, IDA–QP (4) (2006) Preprint 25-4-2005.
19. H. Rebei: *On the one-mode quadratic Weyl operators* , Journal of Mathematical Analysis and Applications, Vol. 439, Issue 1, 1 July 2016, 135–153, doi:10.1016/j.jmaa.2016.02.040.
20. H. Rebei: *The generalized Heisenberg group arising from Weyl relations*, Quantum Studies. Mathematics and Foundations Vol. 2, No. 4 (2015), pp. 323-350.
21. P. Śniady: *Quadratic bosonic and free white noises*, Commun. Math. Phys. 211 (3) (2000) 615–628.
22. Z. Takeda, Inductive limit and infinite direct product of operator algebras, Mathematical institute, Tohoku Math. J. 7 (2) (1955) 67–86