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Identification of the theory of orthogonal polynomials in d-indeterminates with the theory of 3-diagonal symmetric interacting Fock spaces on $\mathbb{C}^d$

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The identification mentioned in the title allows a formulation of the multidimensional Favard lemma different from the ones currently used in the literature and which parallels the original 1-dimensional formulation in the sense that the positive Jacobi sequence is replaced by a sequence of positive Hermitean (square) matrices and the real Jacobi sequence by a sequence of positive definite kernels. The above result opens the way to the program of a purely algebraic classification of probability measures on $\mathbb{R}^d$ with moments of any order and more generally of states on the polynomial algebra on $\mathbb{R}^d$.

The quantum decomposition of classical real-valued random variables with all moments is one of the main ingredients in the proof.

Keywords: Multidimensional orthogonal polynomials; Favard theorem; interacting Fock space; quantum decomposition of a classical random variable.

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1. Introduction

The theory of orthogonal polynomials is one of the classical themes of calculus since almost two centuries and, in the 1-dimensional case, the large literature devoted to this topic has been summarized in several well-known monographs (see for example, Refs. 9, 12, 20 and 21). In this case, even if at analytical level many deep problems remain open, at the algebraic level, the situation is well understood and described by Favard lemma which, to any probability measure μ on the real line with finite moments of any order, associates two sequences, called the Jacobi sequences of μ,

$$\{(\omega_n)_{n\in\mathbb{N}},\, (\alpha_n)_{n\in\mathbb{N}}\}, \quad \omega_n \in \mathbb{R}_+, \quad \alpha_n \in \mathbb{R}, \quad n = 0, 1, 2, \dots \tag{1.1}$$

subjected to the only constraint that, for any $n, k \in \mathbb{N}$,

$$\omega_n = 0 \Rightarrow \omega_{n+k} = 0. \tag{1.2}$$

Conversely, given two such sequences, it gives an inductive way to uniquely reconstruct:

(i) a state on the algebra $\mathcal{P}$ of polynomials in one indeterminate (see Sec. 2.3),
(ii) the orthogonal decomposition of $\mathcal{P}$ canonically associated to this state.

In this sense one can say that the pair of sequences (1.1), subjected to the only constraint (1.2), constitutes a full set of algebraic invariants for the equivalence classes of probability measures on the real line with respect to the equivalence relation $\mu \sim \nu$ if and only if all moments of μ and ν are finite and coincide (moment equivalence of probability measures on $\mathbb{R}$).

Compared to the 1-dimensional case the literature available in the multi-dimensional case is definitively scarce, even if several publications (see e.g., Refs. 10, 13, 17 and 18) show an increasing interest to the problem in the past years, and for several years it has been mainly confined to applied journals, where it emerges in connection with different kinds of approximation problems. The need for an insightful theory was soon perceived by the mathematical community, for example in the 1953 monograph[11] (cited in Ref. 23), the authors claim that "... there does not seem to be an extensive general theory of orthogonal polynomials in several variables ...".

Several progresses followed, both on the analytical front concerning multi-dimensional extensions of Carleman's criteria,[19,22] and on the algebraic front, with the introduction of the matrix approach[16] and the early formulations of the multi-dimensional Favard lemma.[14,15,23]

However, even with these progresses in view, one cannot yet speak of a "general theory of orthogonal polynomials in several variables" and of a multi-dimensional Favard lemma. In fact the importance of Favard lemma consists in the fact that the pair (α_n, ω_n) condensates the *minimal information* gained from the knowledge of the nth moment with respect to the knowledge of all the kth moments with $k \le n - 1$. Here the word *minimal* is essential:

It is exactly this *minimality* that was missing in all the numerous approaches to the multi-dimensional Favard lemma until a couple of years ago.

The more recent multi-dimensional formulations of Favard lemma are based on two sequences of matrices, one of which rectangular, with quadratic constraints among the elements of these sequences (see Refs. 2, 5 and 24, where (see Theorem 2.4) the commutation relations in Refs. 2 and 5 are expressed in terms of a fixed basis of orthogonal polynomials). As mentioned in Ref. 24 such formulations look far from the elegant simplicity of the 1-dimensional Favard lemma.

Since the multi-dimensional analogues of positive (resp. real) numbers are the positive definite (resp. Hermitian) matrices, one would intuitively expect that a multi-dimensional extension of the Favard lemma would replace the sequence (ω_n) by a sequence of positive definite matrices $(\tilde{\Omega}_n)$ and the (α_n)-sequence by a sequence of Hermitian matrices $(a^0_{j,n})$ for each coordinate function X_j on $\mathbb{R}^d$. The precise formulation of this naive conjecture is what we call *the multi-dimensional Favard problem* (see Sec. 3).

The main result of this paper is the proof that *the above-mentioned conjecture is correct*. The *new feature*, *specific of the multi-dimensional case*, is that the two sequences $(\tilde{\Omega}_n)$ and $(a^0_{j,n})_j$ must be constructed recursively, because the choice of $\tilde{\Omega}_{n+1}$ and $a^0_{j,n+1}$ $(j = 1, \ldots, d)$ is constrained by the choices of the previous pairs.

The determination of these constraints, and their recursive formulation, is based on several new results and notions that are of independent interest. In particular:

(1) The identification of the theory of orthogonal polynomials with respect to a state on the algebra of polynomial functions on $\mathbb{R}^d$ with the theory of *symmetric* interacting Fock spaces over $\mathbb{C}^d$ with a 3-diagonal structure (see Sec. 7.1 and Appendix B on interacting Fock spaces).
(2) The explicit form of the above-mentioned minimal set of constraints.

The reconstruction theorem (Theorem 8.2) then shows that the d-dimensional analogue of the principal Jacobi sequence is given by the sequence of *the real parts* $(\tilde{\Omega}_{R,n+1})$ of the positive-definite kernels (block matrices) $(\tilde{\Omega}_{n+1})$, defining the scalar product on the space of orthogonal polynomials of order $n+1$ in terms of the scalar product on the space of order n. In fact, once given this scalar product, $\tilde{\Omega}_{R,n+1}$ is an arbitrary kernel, positive-definite with respect to it. The imaginary part of $\tilde{\Omega}_{n+1}$, on the contrary, is uniquely fixed by the commutation relations and by the nth terms of the secondary Jacobi sequence: $a^0_{j,n+1}$ $(j = 1, \ldots, d)$. The d-dimensional analogue of condition (1.2) consists in the statement that these kernels map zero-norm vectors into zero-norm vectors. In particular, if the nth kernel is identically zero, then the nth space of the gradation consists only of zero vectors, hence the same will be true for all the Nth spaces with $N \geq n$.

The d-dimensional analogue of the secondary Jacobi sequence is given by d sequences of symmetric matrices. These are not arbitrary, but have to satisfy an inductive system of *linear equations*. The fact that this system always admits the zero-solution, corresponding to symmetric states on the polynomial algebra, shows that, in analogy with the one-dimensional case, every class of states on the polynomial algebra in d real variables, for the equivalence relation of having the same sequence of scalar products on the gradation spaces, contains exactly one symmetric measure.

The proof of all the above results heavily relies, on the quantum probabilistic approach to the theory of orthogonal polynomials first proposed, in the 1-dimensional case, in the paper,[1] where the notion of quantum decomposition of a classical random variable was introduced and used to establish a canonical identification between the theory of orthogonal polynomials in one indeterminate and the theory of 1-mode interacting Fock spaces (IFS). One can say that the quantum decomposition of a classical random variable is a reformulation of the Jacobi recurrence relation.

The early extensions of this approach to the multi-dimensional case[5,2,3] constructed the quantum decomposition of the coordinate random varibles in terms of creation, annihilation and preservation operators on an IFS canonically associated to the orthogonal decomposition of the polynomial algebra in d indeterminates $\mathcal{P}_d$ with respect to a given state, however, as mentioned above, for the Favard lemma they used rectangular matrices and quadratic relations. This made explicit construction of the solutions a difficult problem. An important step towards the solution of this problem was done in Ref. 4 where it was proved that the reconstruction of the state on $\mathcal{P}_d$ can be achieved using only the commutators between creation and annihilation operators and the preservation operator. These operators preserve the orthogonal gradation, therefore each of them is determined by a sequence of square matrices. Moreover, the preservation operator, being symmetric, is determined by a sequence of Hermitian matrices while the commutators between creation and annihilation operators are determined by two positive definite matrix valued kernels, respectively $(a_j a_k^+)$ and $(a_k^+ a_j)$ $(j, k \in \{1, \dots, d\})$.

Although this framework was much nearer to the one conjectured in the Favard problem, yet important discrepancies remained, in particular:

(i) While the sequence of Hermitean matrices is only one for each coordinate random variable, as conjectured, the commutators involved are defined by two sequences of positive definite matrix valued kernels, namely the restrictions, to the gradation spaces, of $(a_j a_k^+)$ and $(a_k^+ a_j)$ $(j, k \in \{1, \dots, d\})$.

(ii) Contrarily to the 1-dimensional case, the correspondence between families of orthogonal polynomials in d variables and IFS over $\mathbb{C}^d$ is not one-to-one.

(iii) The multi-dimensional analogue of the compatibility condition (1.2) remained obscure.

(iv) The "minimality condition" mentioned above was not respected (this fact will be clear from this paper).

These problems have been settled in this paper: (i) and (iv) because the sequence defined by the $(a_k^+ a_j)$ is inductively determined, while the sequence defined by the $(a_j a_k^+)$, i.e. $\tilde{\Omega}_{n+1}(j, k)$, has an arbitrary real part; (ii) because the correct one-to-one correspondence is with *symmetric* IFS 3-diagonal interacting Fock spaces over $\mathbb{C}^d$; (iii) for the reasons explained above.

It is precisely the emergence of this universal model for the theory of orthogonal polynomials that justifies our choice to work at a pre-Hilbert rather than Hilbert space level. In fact when quotienting over zero norm vectors, the specific features of the state destroy the universality of the model.

In the present approach the emergence of the symmetric tensor algebra as well as of nontrivial commutation relations are both consequences of the commutativity of the coordinate random variables.

In this sense *a noncommutative structure is canonically deduced from a commutative one.*

From the point of view of physics, this clearly shows the probabilistic origins of the Heisenberg commutation relations, which have been shown to characterize Gaussian measures (see Ref. 4). For classes different from the Gaussian one, we obtain a powerful generalization of the whole mathematical structure of quantum theory that, in its infinite dimensional version, corresponds to an extension of quantum field theory. *Thus the traditional theory of orthogonal polynomials merges with the program of nonlinear first and second quantizations and provides new tools for it.*

2. The Polynomial Algebra

2.1. *Notations*

Throughout this paper, for any $m \in \mathbb{N}$, $\mathbb{C}^m$ (resp. $\mathbb{R}^m$) will denote the m-dimensional complex (resp. real) vector space referred to the canonical basis denoted in both cases (e_j) $(j \in \{1, \ldots, m\})$ and the term *coordinates* will be referred to this basis. Unless otherwise specified, algebras and vector spaces will be complex. Let $D := \{1, \ldots, d\}$ $(d \in \mathbb{N})$ be a finite set and denote

$$\mathcal{P} := \mathcal{P}_D := \mathbb{C}[(X_j)_{j \in D}] \tag{2.1}$$

the complex polynomial algebra in the commuting indeterminates $(X_j)_{j \in D}$ with the $*$-algebra structure uniquely determined by the prescription that the X_j are self-adjoint. The principle of identity of polynomial states that a polynomial is identically zero if and only if all its coefficients are zero. This is equivalent to say that the generators X_j $(j \in D)$ are algebraically independent. These generators will also be called *coordinates*.

By definition $\mathcal{P}$ has an identity, denoted $1_{\mathcal{P}}$, and

$$X_j^0 = 1_{\mathcal{P}}, \quad \forall\, j \in D, \tag{2.2}$$

where $1_{\mathcal{P}}$ denotes the identity of $\mathcal{P}$.

For any vector space V we denote $\mathcal{L}(V)$ the algebra of linear maps of V into itself.

For $F = \{1, \ldots, m\} \subseteq D$ and $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$ we will use the notation:

$$X_v := \sum_{j \in F} v_j X_j.$$

The coordinates $X_j (j \in D)$ define a linear map

$$X : v = \sum_{j\in D} v_j e_j \in \mathbb{R}^d \mapsto X_v := \sum_{j\in D} v_j X_j \in \mathcal{L}(\mathcal{P}).$$

The real linear span $\mathcal{P}_{\mathbb{R}}$ of the generators X_j induces a natural real structure on $\mathcal{P}$ given by

$$\mathcal{P} = \mathcal{P}_{\mathbb{R}} \dot{+} i\mathcal{P}_{\mathbb{R}}, \tag{2.3}$$

where, here and in the following, $\dot{+}$ in (2.3) means direct sum in the real vector space sense. All the properties considered in this section continue to hold if one restricts one's attention to the real algebra $\mathcal{P}_{\mathbb{R}}$.

With the convention (2.2) a *monomial of degree* $n \in \mathbb{N}$ is by definition any product of the form

$$M := \prod_{j\in F} X_j^{n_j}, \tag{2.4}$$

where $F \subseteq D$ is a finite subset, and for any $j \in F,\ n_j \in \mathbb{N}$

$$\sum_{j\in F} n_j = n.$$

The monomial (2.4) is said to be *localized in the subset* $F \subseteq D$.

The algebra generated by such monomials will be denoted

$$\mathcal{P}_F \subseteq \mathcal{P} := \mathcal{P}_D.$$

Notice that, with this definition of localization, if $F \subseteq G \subseteq D$ then any monomial localized in F is also localized in G, i.e.

$$\mathcal{P}_F \subseteq \mathcal{P}_G \subseteq \mathcal{P}.$$

For all $n \in \mathbb{N}$ and for any subset $F \subseteq D$, we use the following notations:

$$\mathcal{M}_{F,n]} := \text{the set of monomials of degree less than or equal to } n \text{ localized in } F \tag{2.5}$$

$$\mathcal{M}_{F,n} := \text{the set of monomials of degree } n \text{ localized in } F \tag{2.6}$$

$$\mathcal{P}_{F,n]} := \text{the vector subspace of } \mathcal{P} \text{ generated by the set } \mathcal{M}_{F,n]} \tag{2.7}$$

$$\mathcal{P}^0_{F,n} := \text{the vector subspace of } \mathcal{P} \text{ generated by the set } \mathcal{M}_{F,n}\}. \tag{2.8}$$

We use the apex 0 in $\mathcal{P}^0_{F,n}$ to distinguish the monomial gradation (see (2.14) below), which is purely algebraic, from the orthogonal gradations, which will be introduced later on and depend on the choice of a state on $\mathcal{P}$. The only monomial of degree $n = 0$ is by definition

$$M_0 := 1_{\mathcal{P}}.$$

Therefore

$$\mathcal{P}^0_{F,0} = \mathcal{P}_{F,0]} = \mathbb{C} \cdot 1_{\mathcal{P}}. \tag{2.9}$$

More generally, if $|F| = m$ then for any $n \in \mathbb{N}$ there are exactly

$$d_n := \binom{n+m-1}{m-1} \tag{2.10}$$

monomials of degree n localized in F and, by the principle of identity of polynomials they are linearly independent. Therefore one has

$$\mathcal{P}^0_{F,n} \equiv \mathbb{C}^{d_n}, \tag{2.11}$$

where the isomorphism is meant in the sense of vector spaces.

For future use it is useful to think of $\mathcal{P}$ as an algebra of operators acting on itself by left multiplication. In the following, when no confusion is possible, we will use the same symbol for an element $Q \in \mathcal{P}$ and for its multiplicative action on $\mathcal{P}$. Sometimes, to emphasize the fact that Q is considered as an element of the vector space $\mathcal{P}$, we will use the notation

$$Q \cdot 1_{\mathcal{P}} =: Q \cdot \Phi_0.$$

The sequence $(\mathcal{P}_{F,n]})_{n\in\mathbb{N}}$ is an increasing filtration of complex finite dimensional $*$-vector subspaces of $\mathcal{P}$, i.e.

$$\mathcal{P}_{F,0]} \subset \mathcal{P}_{F,1]} \subset \mathcal{P}_{F,2]} \subset \cdots \subset \mathcal{P}_{F,n]} \subset \cdots \subset \mathcal{P}_F \subset \mathcal{P}. \tag{2.12}$$

Moreover,

$$\bigcup_{n\in\mathbb{N}} \mathcal{P}_{F,n]} = \mathcal{P}_F \tag{2.13}$$

and, for any m, $n \in \mathbb{N}$ one has

$$\mathcal{P}_{F,m]} \cdot \mathcal{P}_{F,n]} = \mathcal{P}_{F,m+n]}.$$

The sequence $(\mathcal{P}^0_{F,n})_{n\in\mathbb{N}}$ defines a vector space gradation of $\mathcal{P}_F$

$$\mathcal{P}_F = \dot{\sum_{k\in\mathbb{N}}} \mathcal{P}^0_{F,k} \tag{2.14}$$

called the monomial decomposition of $\mathcal{P}$. In (2.14) the symbol $\dot{\sum}$ denotes direct sum in the sense of vector spaces, i.e. elements of $\mathcal{P}$ are finite linear sums of elements in some of the $\mathcal{P}^0_{F,n}$ and

$$m \neq n \Rightarrow \mathcal{P}^0_{F,m} \cap \mathcal{P}^0_{F,n} = \{0\}. \tag{2.15}$$

The gradation (2.14) is compatible with the filtration $(\mathcal{P}_{F,n]})$ in the sense that, for any $n \in \mathbb{N}$,

$$\mathcal{P}_{F,n]} = \dot{\sum_{k\in\{0,1,\ldots,n\}}} \mathcal{P}^0_{F,k}. \tag{2.16}$$

In particular,

$$\mathcal{P}_F = \mathcal{P}_{F,n]} \dotplus \left(\dot{\sum_{k>n}} \mathcal{P}^0_{F,k} \right), \quad \forall n \in \mathbb{N}.$$

Lemma 2.1. (i) *For any vector subspace* $W \subset \mathcal{P}_F$, *the set*

$$XW := \{X_v W : v \in \mathbb{C}^F\} \tag{2.17}$$

is a vector subspace of $\mathcal{P}_F$, *where* $\mathbb{C}^F := \{v \in \mathbb{C}^m : v_j = 0 \text{ if } j \notin F\}$.

(ii) *For each* $n \in \mathbb{N}$, *one has*

$$X\mathcal{P}^0_{F,n} = \mathcal{P}^0_{F,n+1}, \tag{2.18}$$

$$\mathcal{P}_{F,n+1]} = X\mathcal{P}_{F,n]} \dot{+} \mathcal{P}_{F,0]} = \mathcal{P}_{F,n]} \dot{+} \mathcal{P}^0_{F,n+1}. \tag{2.19}$$

(iii) *For* $n \in \mathbb{N}$, *let* $\mathcal{P}_{n+1}$ *be a vector subspace of* $\mathcal{P}_{n+1]}$ *such that*

$$\mathcal{P}_{n]} \dot{+} \mathcal{P}_{n+1} = \mathcal{P}_{n+1]}. \tag{2.20}$$

Then as a vector space $\mathcal{P}_{n+1}$ *is isomorphic to* $\mathcal{P}^0_{n+1}$.

Proof. (i) The set (2.17) coincides with the set

$$\left\{ \sum_{j \in F} X_j \xi^{(j)}_w : \xi^{(j)}_w \in W, \forall\, j \in F \right\}$$

and this is clearly a vector space.

(ii) Since $\mathcal{M}_{F,n}$ is a linear basis of $\mathcal{P}^0_{F,n}$, $\bigcup_{j\in F} X_j \mathcal{M}_{F,n} \subset \mathcal{P}^0_{F,n+1}$ is a system of generators of the subspace $X\mathcal{P}^0_{F,n}$. Hence $X\mathcal{P}^0_{F,n} \subset \mathcal{P}^0_{F,n+1}$. The converse inclusion is clear because $\bigcup_{j\in F} X_j \mathcal{M}_{F,n}$ is also a system of generators of $\mathcal{P}^0_{F,n+1}$. This proves (2.18). (2.19) follows from (2.16) and (2.18).

(iii) Since the sum in (2.20) is direct and the spaces are finite dimensional, one has

$$\dim(\mathcal{P}^0_{n+1}) = \dim(\mathcal{P}_{n+1]}) - \dim(\mathcal{P}_{n]}) = \dim(\mathcal{P}_{n+1}). \qquad \square$$

2.2. $\mathcal{P}$ and the symmetric tensor algebra over $\mathbb{C}^d$

In this paper the number $d \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$ will be fixed and

$$D \equiv \{1, \ldots, d\}$$

in the following the index D will be omitted and we will use the notations:

$$\mathcal{P}_D = \mathcal{P}, \quad \mathcal{P}^0_n := \mathcal{P}^0_{D,n}, \quad \mathcal{P}_{n]} := \mathcal{P}_{D,n]}, \quad n \in \mathbb{N}$$

with the convention

$$\mathcal{P}^0_{-1} = \mathcal{P}_{-1]} = \{0\}.$$

The natural real structure on $\mathbb{C}$ given by $\mathbb{C} = \mathbb{R} \dot{+} i\mathbb{R}$ induces a real structure on $\mathbb{C}^d = \mathbb{R}^d \dot{+} i\mathbb{R}^d$ the associated (componentwise) involution given by complex conjugation:

$$(u + iv)^* := u - iv, \quad u + iv \in \mathbb{C}^d := \mathbb{R}^d \dot{+} i\mathbb{R}^d. \tag{2.21}$$

In the following we fix the choice $V := \mathbb{C}^d$ and we denote $(e_j)_{j\in D}$ the canonical basis of $\mathbb{C}^d$ which is a *real basis*, i.e. a basis of $\mathbb{R}^d \subset \mathbb{C}^d$. $\otimes$ will denote algebraic tensor product and $\widehat{\otimes}$ its symmetrization. The tensor algebra over $\mathbb{C}^d$ is the vector space

$$\mathrm{Tens}(\mathbb{C}^d) := \dot{\sum_{n\in\mathbb{N}}} (\mathbb{C}^d)^{\otimes n}$$

with multiplication given by

$$(u_n \otimes \cdots \otimes u_1) \otimes (v_m \otimes \cdots \otimes v_1) := u_n \otimes \cdots \otimes u_1 \otimes v_m \otimes \cdots \otimes v_1$$

for any m, $n \in \mathbb{N}$ and all u_j, $v_j \in \mathbb{C}^d$. The extension to $\mathbb{C}^d$ of the natural real structure on $\mathbb{C}$ given by $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ and the associated involution, induces a $*$-algebra structure on $\mathcal{T}(\mathbb{C}^d)$ whose involution is characterized by the property that

$$(v_n \otimes \cdots \otimes v_1)^* := v_n^* \otimes \cdots \otimes v_1^*, \quad \forall\, n \in \mathbb{N}, \ \forall\, v \in \mathbb{C}^d. \tag{2.22}$$

For $n \in \mathbb{N}^*$, the $*$-subspace of $(\mathbb{C}^d)^{\otimes n}$ generated by the elements of the form

$$v^{\otimes n} := v \otimes \cdots \otimes v \quad (n \text{ times}), \quad \forall\, n \in \mathbb{N}, \quad \forall\, v \in \mathbb{C}^d \tag{2.23}$$

is called *the symmetric tensor product of n copies of* $\mathbb{C}^d$ and denoted by $(\mathbb{C}^d)^{\widehat{\otimes} n}$. $(\mathbb{C}^d)^{\widehat{\otimes} n}$ coincides with the fixed point subspace of the linear action, on $(\mathbb{C}^d)^{\otimes n}$, of the nth order permutation group $\mathcal{S}_n$ given by

$$\widehat{\sigma}(v_n \otimes v_{n-1} \otimes \cdots \otimes v_1) := v_{\sigma_n} \otimes v_{\sigma_{n-1}} \otimes \cdots \otimes v_{\sigma_1},$$
$$v_n \otimes v_{n-1} \otimes \cdots \otimes v_1 \ \in \ (\mathbb{C}^d)^{\otimes n}, \quad \sigma \in \mathcal{S}_n.$$

By definition:

$$(\mathbb{C}^d)^{\widehat{\otimes} 0} := \mathbb{C},$$

$$\mathrm{Tens}_{\mathrm{sym}}(\mathbb{C}^d) := \dot{\sum_{n\in\mathbb{N}}} (\mathbb{C}^d)^{\widehat{\otimes} n}.$$

$\mathrm{Tens}_{\mathrm{sym}}(\mathbb{C}^d)$ is the graded abelian $*$-sub-algebra of $\mathrm{Tens}(\mathbb{C}^d)$ generated by the elements of the form (2.23) and is called *the symmetric tensor algebra over* $\mathbb{C}^d$.

The following lemma reformulates some known results in a language and with the notations that will be used later.

Lemma 2.2. *Let $(e_j)_{j\in D}$ be the canonical linear basis of $\mathbb{C}^d$. The map*

$$e_j \mapsto X_j, \quad j \in D, \quad 1_{\mathcal{T}_{\mathrm{sym}}(\mathbb{C}^d)} \mapsto 1_{\mathcal{P}} \tag{2.24}$$

extends uniquely to a is a gradation preserving isomorphism of commutative $*$-algebras:

$$S^0 := \dot{\sum_{n\in\mathbb{N}}} S_n^0 \ : \mathrm{Tens}_{\mathrm{sym}}(\mathbb{C}^d) := \dot{\sum_{n\in\mathbb{N}}} (\mathbb{C}^d)^{\widehat{\otimes} n} \ \to \ \dot{\sum_{n\in\mathbb{N}}} \mathcal{P}_n^0 \equiv \mathcal{P}. \tag{2.25}$$

In particular for all $n \in \mathbb{N}^*$ and for all maps $j : \{1, \ldots, n\} \to \{1, \ldots, d\}$:

$$e_{j_n} \widehat{\otimes} \cdots \widehat{\otimes} e_{j_1} \mapsto X_{j_n} \cdots X_{j_1} \tag{2.26}$$

and, in the notations of Sec. B.1 *below*:

$$e_j\widehat{\otimes}(\cdot) = \ell^*_{e_j} = X_j. \tag{2.27}$$

Proof. The thesis follows from the fact that the e_j's (resp. X_j's) $(j \in D)$ are algebraically independent (i.e. the terms appearing in (2.26) and the corresponding identities are linearly independent) self-adjoint generators of the commutative $*$-algebra $\mathcal{T}_{\mathrm{sym}}(\mathbb{C}^d)$ (resp. $\mathcal{P}$) and that the correspondence (2.24) is 1-to-1. □

Remark. In analogy with the identification of X_j with its action as multiplication operator on $\mathcal{P}$, e_j can be identified with the symmetric tensor multiplication by e_j. If confusion may arise, we use the notation

$$\hat{M}_{e_j}(e_{j_n}\widehat{\otimes}\cdots\widehat{\otimes}e_{j_1}) := e_j\widehat{\otimes}e_{j_n}\widehat{\otimes}\cdots\widehat{\otimes}e_{j_1}.$$

With this notation and the corresponding one for the X_j's, one has

$$S^0\hat{M}_{e_j}(S^0)^{-1} = M_{X_j}, \quad j \in D. \tag{2.28}$$

Lemma 2.3. *Let $(\mathcal{P}_n)_{n\in\mathbb{N}}$ be any family of subspaces of $\mathcal{P}$ such that*

$$\mathcal{P}_{k+1]} = \mathcal{P}_{k]}\dot{+}\mathcal{P}_{k+1}, \quad \forall\, k \in \mathbb{N},$$

$$\mathcal{P}_0 = \mathcal{P}_{0]} = \mathcal{P}^0_0 = \mathbb{C}1_{\mathcal{P}}.$$

Then, for all $n \in \mathbb{N}$, there exists a vector space isomorphism

$$S_n : (\mathbb{C}^d)^{\widehat{\otimes}n} \to \mathcal{P}_n \tag{2.29}$$

and the map

$$S := \dot{\sum_{n\in\mathbb{N}}} S_n : \mathrm{Tens}_{\mathrm{sym}}(\mathbb{C}^d) := \dot{\sum_{n\in\mathbb{N}}}(\mathbb{C}^d)^{\widehat{\otimes}n} \ \to \ \dot{\sum_{n\in\mathbb{N}}}\mathcal{P}_n \equiv \mathcal{P} \tag{2.30}$$

is a gradation preserving vector space isomorphism.

Proof. From Lemma 2.1 we know that, for all $n \in \mathbb{N}$, $\mathcal{P}_n$ has the same dimension as $\mathcal{P}^0_n$ (given by (2.10)). Hence there exists a vector space isomorphism

$$T_n : \mathcal{P}^0_n \ \to \ \mathcal{P}_n, \quad \forall\, n \in \mathbb{N}.$$

Defining $S_n := T_n \circ S^0_n$ where S^0_n is given by (2.25), (2.29) follows. This implies that the map defined by (2.30) is a gradation preserving vector space isomorphism.

□

Remark. In general the map defined by (2.30) is not an isomorphism of commutative $*$-algebras in particular the analogue for S of (2.28) does not hold. To obtain this additional property will require a different choice for the vector space isomorphisms T_n (see Sec. 8 below).

2.3. *States on $\mathcal{P}$*

For the terminology on pre-Hilbert spaces we refer to Appendix A.

Denote $\mathcal{S}(\mathcal{P})$ the set of linear functionals on $\mathcal{P}$ that are real on real polynomials, 1 on the identity and positive on polynomials of the form $P = |Q|^2$ with $P, Q \in \mathcal{P}$. Such linear functionals will be called *states*.

Any probability measure on $\mathbb{R}^d$ with all moments induces a state on $\mathcal{P}$. The converse finds an obstruction in the existence, for $d > 1$, of positive polynomials P not expressible in the form $P = |Q|^2$. We refer to the paper[8] for references on this old and deep problem, that is related to the polynomial version of Hilbert's 17th problem.

Even in case of existence and even in the case $d = 1$, there may be many probability measures on $\mathbb{R}^d$ defining the same state on $\mathcal{P}$ (non-uniqueness in the moment problem). On the contrary, the state on $\mathcal{P}$ is uniquely defined. For this reason in the following we restrict our attention to states on $\mathcal{P}$.

As shown in the following of this paper, all the constructions related to orthogonal polynomials are valid in the more general framework of states on $\mathcal{P}_V$. Therefore in the following we will discuss this more general framework.
Any state $\varphi \in \mathcal{S}(\mathcal{P})$ defines a pre-scalar product $\langle \cdot, \cdot \rangle_\varphi$ on $\mathcal{P}$ given by

$$(a, b) \in \mathcal{P} \times \mathcal{P} \mapsto \langle a, b \rangle_\varphi := \varphi(a^* b) \in \mathbb{C} \tag{2.31}$$

satisfying the conditions

$$\begin{aligned} \langle 1_\mathcal{P}, 1_\mathcal{P} \rangle_\varphi &= 1 \\ \langle ab, c \rangle_\varphi &= \langle b, a^* c \rangle_\varphi, \quad \forall\, a,\, b,\, c \in \mathcal{P}, \end{aligned} \tag{2.32}$$

where a^* denotes the adjoint of a in $\mathcal{P}$. In particular the operators X_j are symmetric as pre-Hilbert space operators. Thus the pair

$$(\mathcal{P}, \langle \cdot, \cdot \rangle_\varphi)$$

is a commutative pre-Hilbert algebra.

Lemma 2.4. *For a pre-scalar product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}$ the following statements are* equivalent

(i) *There exists a state φ on $\mathcal{P}$ such that*:

$$\varphi(f^* g) = \langle f, g \rangle, \quad f, g \in \mathcal{P}. \tag{2.33}$$

(ii) *The pre-scalar product $\langle \cdot, \cdot \rangle$ satisfies*

$$\langle 1_\mathcal{P}, 1_\mathcal{P} \rangle = 1 \tag{2.34}$$

and, for each $j \in D$, multiplication by the coordinate X_j is a symmetric linear operator on $\mathcal{P}$ with respect to $\langle \cdot, \cdot \rangle$, i.e.

$$\langle X_j f, g \rangle = \langle f, X_j g \rangle. \tag{2.35}$$

Proof. (ii) ⇒ (i) Every scalar product on $\mathcal{P}$ is induced by the linear functional:

$$\varphi(Q) := \langle 1_{\mathcal{P}}, Q \cdot 1_{\mathcal{P}} \rangle, \quad Q \in \mathcal{P}. \tag{2.36}$$

Condition (2.35) implies that φ is a $*$-functional on $\mathcal{P}$, i.e. for any $Q \in \mathcal{P}$, $\overline{\varphi(Q)} = \varphi(Q^*)$, where $*$ denotes the involution on $\mathcal{P}$. Hence condition (2.36) implies that φ is positive. Then, because of (2.34), φ is a state on $\mathcal{P}$.

(i) ⇒ (ii) This is clear and has already been discussed before the statement of the theorem. □

3. The Multi-Dimensional Favard Problem

3.1. *Fundamental lemmas*

Definition 3.1. For $n \in \mathbb{N}$ we say that a subspace $\mathcal{P}_n \subset \mathcal{P}_{n]}$ is *monic of degree* n if it has a *real* linear basis B_n with the property that for each $b \in B_n$, the highest order term of b is a nonzero multiple of a single monomial of degree n and each monomial of degree n appears exactly once in the basis B_n.

Such a basis is called *a perturbation of the monomial basis of order* n in the coordinates $(X_j)_{j \in D}$ or simply *a monic basis of order* n if no confusion is possible.

Remark. For a monic subspace one has:

$$\mathcal{P}_{n]} = \mathcal{P}_{n-1]} \dot{+} \mathcal{P}_n \tag{3.1}$$

(with the convention $\mathcal{P}_{-1]} = \{0\}$). Notice that monic bases arise naturally in the Gram–Schmidt orthogonalization process of monomials.

Let φ be a state on $\mathcal{P}$ and denote

$$\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_\varphi$$

the corresponding pre-scalar product. When no ambiguity is possible, the elements ξ of $\mathcal{P}$ (resp. $\mathcal{P}_{n]}$, $\mathcal{P}_n^0$) satisfying

$$\langle \xi, \xi \rangle = 0$$

will simply be called *zero norm vectors* without explicitly mentioning the pre-scalar product (or the associated state φ). By the Schwarz inequality the set of zero norm vectors in $\mathcal{P}$ (resp. $\mathcal{P}_{n]}$, $\mathcal{P}_n^0$), denoted $\mathcal{N}_\varphi$ (resp. $\mathcal{N}_{\varphi,n]}$, $\mathcal{N}_{\varphi,n}$) is a $*$-subspace satisfying

$$\mathcal{P}\mathcal{N}_{\varphi,n} \subseteq \mathcal{P}\mathcal{N}_{\varphi,n]} \subseteq \mathcal{P}\mathcal{N}_\varphi \subseteq \mathcal{N}_\varphi. \tag{3.2}$$

In particular, $\mathcal{N}_\varphi$ is a $*$-ideal of $\mathcal{P}$. The monomial decomposition (2.14) is compatible with the filtration $(\mathcal{P}_{F,n]})$ in the sense of (2.16), therefore

$$\mathcal{P} = \mathcal{P}_{n]} \dot{+} \left(\dot{\sum_{k>n}} \mathcal{P}_k^0 \right), \quad \forall\, n \in \mathbb{N}.$$

For reasons that will be clear in the reconstruction theorem of Sec. 7, we want to keep the discussion at a pure vector space, rather than Hilbert space level. In particular, we do not want to quotient out the zero norm vectors. Therefore, rather than the usual Gram–Schmidt orthonormalization procedure, we use its pre-Hilbert space variant, described in Appendix A.

Lemma 3.2. *Let φ be a state on $\mathcal{P}$ and denote $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\varphi$ the associated pre-scalar product. Then there exists a gradation*

$$\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_{n,\varphi} \tag{3.3}$$

called the φ-orthogonal gradation of $\mathcal{P}$, with the following properties:

(i) (3.3) *is orthogonal for the pre-scalar product $\langle\cdot,\cdot\rangle$;*
(ii) (3.3) *is compatible with the filtration $(\mathcal{P}_{n]})$ in the sense that*

$$\mathcal{P}_{k]} = \bigoplus_{h\in\{0,1,\ldots,k\}} \mathcal{P}_{h,\varphi}, \quad \forall\, k \in \mathbb{N}; \tag{3.4}$$

(iii) *for each $n \in \mathbb{N}$ the space $\mathcal{P}_{n,\varphi}$ is monic.*

Conversely, let the following be given:

(j) *a vector space direct sum decomposition of $\mathcal{P}$*

$$\mathcal{P} = \dot{\sum_{n\in\mathbb{N}}} \mathcal{P}_n \tag{3.5}$$

such that $\mathcal{P}_0 = \mathbb{C}\cdot 1_{\mathcal{P}}$, and for each $n \in \mathbb{N}$, $\mathcal{P}_n$ is monic of degree n,
(jj) *for all $n \in \mathbb{N}$ a pre-scalar product $\langle\cdot,\cdot\rangle_n$ on $\mathcal{P}_n$ with the property that $1_{\mathcal{P}}$ has norm 1 and the unique pre-scalar product $\langle\cdot,\cdot\rangle$ on $\mathcal{P}$ defined by the conditions*:

$$\langle\cdot,\cdot\rangle|_{\mathcal{P}_n} = \langle\cdot,\cdot\rangle_n, \quad \forall\, n \in \mathbb{N}\,, \tag{3.6}$$

$$\mathcal{P}_n \perp \mathcal{P}_m, \quad \forall\, m \neq n, \tag{3.7}$$

is such that the operators of multiplication by the coordinates X_j $(j \in D)$ are $\langle\cdot,\cdot\rangle$-symmetric linear operators on $\mathcal{P}$.

Then there exists a state φ on $\mathcal{P}$ such that the decomposition (3.5) *is the orthogonal polynomial decomposition of $\mathcal{P}$ with respect to φ.*

Proof. Let a state φ on $\mathcal{P}$ be given. In the above notations, for each $k \in \mathbb{N}$ define inductively the subspace $\mathcal{P}_{k,\varphi}$ and the two sequences of $\langle\cdot,\cdot\rangle$-orthogonal projectors

$$P_{k],\varphi} : \mathcal{P} \to \mathcal{P}_{k]}, \quad P_{k,\varphi} : \mathcal{P} \to \mathcal{P}_{k,\varphi}, \quad \forall\, k \in \mathbb{N}$$

compatible with the real structures of the corresponding spaces (i.e. $P_{k],\varphi}(\mathcal{P}_\mathbb{R}) \subseteq \mathcal{P}_{\mathbb{R},k]}$, $P_{k,\varphi}(\mathcal{P}_\mathbb{R}) \subseteq \mathcal{P}_{\mathbb{R},k}$, and in this case we speak of *real projectors*) as follows.

For $k = 0$, define $\mathcal{P}_{0,\varphi} := \mathcal{P}_{0]}$ and

$$P_{0,\varphi} := P_{0],\varphi} : Q \in \mathcal{P} \mapsto \varphi(Q)1_{\mathcal{P}} = \langle 1_{\mathcal{P}}, Q \cdot 1_{\mathcal{P}}\rangle 1_{\mathcal{P}} \in \mathcal{P}_{0]} =: \mathcal{P}_{0,\varphi}, \quad \forall Q \in \mathcal{P}.$$

Clearly, $P_{0,\varphi}$ is a real projector. Having defined the real projectors

$$\{P_{0,\varphi}, P_{1,\varphi}, \dots, P_{n,\varphi}\}, \quad \{P_{0],\varphi}, P_{1],\varphi}, \dots, P_{n],\varphi}\}$$

so that for each $k \in \{0, 1, \dots, n\}$ the space $\mathcal{P}_{k,\varphi}$ is monic and (3.4) is satisfied, in the notation (2.6), define

$$\mathcal{P}_{n+1,\varphi} := \text{lin-span}\{M_{n+1} - P_{n],\varphi}(M_{n+1}) : M_{n+1} \in \mathcal{M}_{n+1}\}. \tag{3.8}$$

Then the space $\mathcal{P}_{n+1,\varphi}$ is monic of order $n+1$ since the generating set on the right-hand side of (3.8) is clearly a basis, it is real because such is the projector $P_{n],\varphi}$ and it is a perturbation of the monomial basis of order n because the $P_{n],\varphi}(M_{n+1})$ are polynomials of degree n. In particular, the sum

$$\mathcal{P}_{n+1,\varphi} + \mathcal{P}_{n]} = \mathcal{P}_{n+1]}$$

is direct, hence such is also the decomposition

$$\mathcal{P} = \mathcal{P}_{n+1,\varphi} \dot{+} \mathcal{P}_{n]} \dot{+} \mathcal{P}_{(n+1}$$

($\mathcal{P}_{(n+1}$ denotes the space of polynomials of degree $> n+1$).

Define $\mathcal{K}_{0,1}$ (resp. $\mathcal{K}_{0,0}$) the subspace of $\mathcal{P}_{n+1,\varphi}$ generated by the non-$\langle\cdot,\cdot\rangle$-zero norm (resp. $\langle\cdot,\cdot\rangle$-zero norm) vectors in the set on the right-hand side of (3.8). Since the elements of this set are linearly independent, $\mathcal{K}_{0,1} \cap \mathcal{K}_{0,0} = \{0\}$ and by construction $\mathcal{K}_{0,1} \dot{+} \mathcal{K}_{0,0} = \mathcal{P}_{n+1,\varphi}$. By the induction assumption on the $\langle\cdot,\cdot\rangle_n$, the real structure on $\mathcal{P}$ induces a real structure on $\mathcal{P}_{n+1,\varphi}$.

Applying Corollary A.2 of Appendix A with $\mathcal{K} = \mathcal{P}$, $\mathcal{K}_0 = \mathcal{P}_{n+1,\varphi}$, $\mathcal{K}_1 := \mathcal{P}_{n]} \dot{+} \mathcal{P}_{(n+1}$ and $\mathcal{K}_{0,1}$ any vector space supplement of the $\langle\cdot,\cdot\rangle$-zero norm subspace $\mathcal{K}_{0,0}$ of $\mathcal{P}_{n+1,\varphi}$, we define the orthogonal projection

$$P_{n+1,\varphi} : \mathcal{P} \to \mathcal{P}_{n+1,\varphi}$$

which by construction is onto $\mathcal{P}_{n+1,\varphi}$ hence orthogonal to $P_{n],\varphi}$. Therefore the operator

$$P_{n+1],\varphi} := P_{n],\varphi} + P_{n+1,\varphi}$$

is the orthogonal projection onto $\mathcal{P}_{n+1]}$. Finally, given φ, the conditions of Lemma 2.4 are satisfied by the associated pre-scalar product on $\mathcal{P}$. This completes the induction construction.

To prove the converse, notice that the fact that each $\mathcal{P}_n$ is monic implies that the decomposition (3.5) satisfies condition (3.4). In fact this is true for $\mathcal{P}_0$ by construction and, supposing it is true for $k \in \mathbb{N}$, it follows for $k+1$ from the monicity condition. Thus, by induction, property (3.4) holds for each $n \in \mathbb{N}$. Because of Lemma 2.4, condition (jj) implies that the pre-scalar product $\langle\cdot,\cdot\rangle$ is induced by a state φ in the sense of the identity (2.31). This implies that the decomposition (3.5) is the orthogonal polynomial decomposition of $\mathcal{P}$ with respect to the state φ. □

The following lemma shows that the isomorphism, defined abstractly in Lemma 2.3 can be explicitly constructed if the gradation on $\mathcal{P}$ is the one constructed in Lemma 3.2.

Lemma 3.3. *Let a vector space direct sum decomposition of $\mathcal{P}$ of the form* (3.5) *satisfying conditions* (j) *and* (jj) *of Lemma* 3.2 *be given. Let B_n be a perturbation of the monomial basis in $\mathcal{P}_n$ (see Definition* 3.1*) and for each monomial $M_n \in \mathcal{M}_{D,n}$ denote $p_n(M_n)$ the corresponding element of B_n. Then the map*

$$\pi_n : e_{j_n} \widehat{\otimes} e_{j_{n-1}} \widehat{\otimes} \cdots \widehat{\otimes} e_{j_1} \in (\mathbb{C}^d)^{\widehat{\otimes} n} \mapsto p_n(X_{j_n} X_{j_{n-1}} \cdots X_{j_1}) \cdot 1_{\mathcal{P}} \in \mathcal{P}_n, \tag{3.9}$$

where $n \in \mathbb{N}^*$ ($\pi_0 = id_{\mathbb{C}}$) and $\widehat{\otimes}$ denotes symmetric tensor product, extends to a vector space isomorphism.

Proof. A basis B_n as in the statement of the lemma exists because $\mathcal{P}_n$ is monic. Denoting $j : \{1, \ldots, n\} \to \{1, \ldots, d\}$ a generic function, the map

$$e_{j_n} \otimes e_{j_{n-1}} \otimes \cdots \otimes e_{j_1} \mapsto p_n(X_{j_n} X_{j_{n-1}} \cdots X_{j_1}) \cdot 1_{\mathcal{P}} \in \mathcal{P}_n \tag{3.10}$$

is well-defined on a linear basis of $(\mathbb{C}^d)^{\otimes n}$ because $X_{j_n} X_{j_{n-1}} \cdots X_{j_1}$ is a monomial of degree n. Since both sides in (3.10) are multi-linear, by the universal property of the tensor product it extends to a linear map, denoted $\widehat{\pi}_n$, of $(\mathbb{C}^d)^{\otimes n}$ into $\mathcal{P}_n$. This map is surjective because when j runs over all maps $\{1, \ldots, n\} \to \{1, \ldots, d\}$, $p_n(X_{j_n} X_{j_{n-1}} \cdots X_{j_1}) \cdot 1_{\mathcal{P}}$ runs over a linear basis of $\mathcal{P}_n$. Since the right-hand side of (3.10) is invariant under permutations of the indices $j_n, j_{n-1}, \ldots, j_1$, $\widehat{\pi}_n$ induces a linear map of the vector space of equivalence classes of elements of $(\mathbb{C}^d)^{\otimes n}$ with respect to the equivalence relation induced by the linear action of the permutation group. Since this quotient space is canonically isomorphic to the symmetric tensor product $(\mathbb{C}^d)^{\widehat{\otimes} n}$, this induced map defines a linear extension of the map (3.9).

This extension is an isomorphism because we have already proved that surjectivity and injectivity follow from the fact that the equivalence class under permutations of any n-tuple $(j_n, j_{n-1}, \ldots, j_1)$ defines a unique element of the basis $\{p_n(M_n) \cdot 1_{\mathcal{P}}; M_n \in \mathcal{P}_n\}$ of $\mathcal{P}_n$. □

Remark. The construction of Lemma 3.2 depends on the choice of the vector space supplement of the zero norm subspace of $\mathcal{P}_{n,\varphi}$. However, any vector in another supplement will differ by a zero norm vector from a vector in the previous choice. Therefore, at Hilbert space level, the two choices will coincide.

3.2. *Statement of the multi-dimensional Favard problem*

From Lemma 3.2 we know that the orthogonal polynomial decomposition of $\mathcal{P}$ with respect to a state φ induces a decomposition of $\mathcal{P}$ of the form (3.5). Given such a decomposition, for every $n \in \mathbb{N}$, we can use the vector space isomorphisms π_n defined in Lemma 3.3 to transfer the pre-Hilbert structure of $\mathcal{P}_n$ on the symmetric tensor product space $(\mathbb{C}^d)^{\widehat{\otimes} n}$. Imposing the orthogonality of the $\mathcal{P}_n$'s one obtains a gradation preserving unitary isomorphism between $\mathcal{P}$, with the orthogonal polynomial gradation induced by the state φ, and a symmetric interacting Fock space structure over $\mathbb{C}^d$ (see Appendix B.5). The converse of this statement is at basis of the multi-dimensional Favard problem:

Given a symmetric interacting Fock space structure over $\mathbb{C}^d$ (see Sec. B.5 below)

$$\bigoplus_{n\in\mathbb{N}} \left((\mathbb{C}^d)^{\widehat{\otimes} n}, \langle \cdot, \Omega_n \cdot \rangle_{\widehat{\otimes} n} \right):$$

(i) does there exist a state φ on $\mathcal{P}$ whose associated symmetric IFS is the given one?
(ii) is it possible to parametrize all solutions of problem (i) and to characterize them constructively?

The second part of this paper is devoted to the solution of this problem. Before that, in the following section, we establish some notations and necessary conditions.

4. The Symmetric Jacobi Relations

4.1. *The orthogonal gradation and the three-diagonal recurrence relations*

In this section we fix a state φ on $\mathcal{P}$ and we follow the notations of Lemma 3.2 with the exception that we omit the index φ. Thus we write $\langle \cdot, \cdot \rangle$ for the pre-scalar product $\langle \cdot, \cdot \rangle_\varphi$, $P_{k]} : \mathcal{P} \to \mathcal{P}_{k]}$ $(k \in \mathbb{N})$ for the $\langle \cdot, \cdot \rangle$-orthogonal projector in the pre-Hilbert space sense, constructed in the proof of Lemma 3.2, $\mathcal{P}_{k+1}$ for the space defined by (3.8) and

$$P_n = P_{n]} - P_{n-1]} \tag{4.1}$$

the corresponding projector. We know that

$$P_{n]}(\mathcal{P}_{\mathbb{R}}) \subseteq \mathcal{P}_{\mathbb{R}} \cap \mathcal{P}_{n]} = \mathcal{P}_{\mathbb{R},n]}, \quad \forall\, n \in \mathbb{N}, \tag{4.2}$$

and that the sequence $(\mathcal{P}_{n]})_{n\in\mathbb{N}}$ is an increasing filtration with union $\mathcal{P}$ (see (2.12) and (2.13)). It follows that the sequence of projections (4.1) is a partition of the identity in $(\mathcal{P}, \langle \cdot, \cdot \rangle)$, i.e.

$$P_n P_m = \delta_{mn} P_m, \quad P_n = P_n^*, \quad \forall\, m, n \in \mathbb{N}, \tag{4.3}$$

$$\sum_{n\in\mathbb{N}} P_n = \lim_n P_{n]} = 1_{\mathcal{P}}. \tag{4.4}$$

Lemma 4.1. *Suppose that, for some $m \in \mathbb{N}^*$, the range of P_m is contained in the subspace of zero-norm vectors. Then the same is true for any $n \geq m$, i.e.*

$$P_n(\mathcal{P}) \subseteq \mathcal{N}, \quad \forall\, n \geq m. \tag{4.5}$$

Proof. Under our assumptions for any monomial M_m of degree m, one has $M_m - P_{m-1]}(M_m) \in \mathcal{N}$. This implies that $M_m \in \mathcal{P}_{m-1]} + \mathcal{N}$. Since multiplication by coordinates leaves $\mathcal{N}$ invariant, this implies that for each $j \in D$, $X_j M_m \in \mathcal{P}_{m]} + \mathcal{N}$. Therefore for any monomial M_{m+1} of degree $m+1$, $M_{m+1} \in \mathcal{P}_{m]} + \mathcal{N}$. In particular, $M_{m+1} - P_{m]}(M_{m+1}) \in \mathcal{N}$, i.e. $\mathcal{P}_{m+1} \subseteq \mathcal{N}$ and this is equivalent to the thesis. □

Theorem 4.2. *With the notation*

$$P_{-1]} := 0$$

for any $j \in D$ and any $n \in \mathbb{N}$, one has

$$X_j P_n = P_{n+1} X_j P_n + P_n X_j P_n + P_{n-1} X_j P_n. \tag{4.6}$$

Proof. Because of (4.4), for any $j \in D$,

$$X_j = 1_{\mathcal{P}} \cdot X_j \cdot 1_{\mathcal{P}} = \sum_{m,n \in \mathbb{N}} P_m X_j P_n.$$

Therefore for each $n \in \mathbb{N}$,

$$X_j P_n = \sum_{m \in \mathbb{N}} P_m X_j P_n.$$

Since

$$X_j \mathcal{P}_n \subseteq \mathcal{P}_{n+1]},$$

it follows that

$$X_j P_n = P_{n+1]} X_j P_n.$$

Since $(P_{m]})$ is increasing, if $m > n+1$ then

$$P_{m]} P_{n+1]} = P_{m-1]} P_{n+1]} = P_{n+1]},$$

hence

$$P_m X_j P_n = P_m P_{n+1]} X_j P_n = (P_{m]} - P_{m-1]}) P_{n+1]} X_j P_n = 0.$$

If $m < n-1$, then the first part of the proof implies that

$$P_m X P_n = (P_n X P_m)^* = 0.$$

Summing up: $P_m X_j P_n$ can be nonzero only if $m \in \{n-1, n, n+1\}$ and this proves (4.6). □

Definition 4.3. The identity (4.6) is called the symmetric Jacobi relation.

4.2. *The CAP operators and the quantum decomposition of the coordinates*

For each $n \in \mathbb{N}$ and $j \in D$, define the operators

$$a^{+}_{j|n} := P_{n+1} X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{n+1} \tag{4.7}$$

$$a^{0}_{j|n} := P_{n} X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{n} \tag{4.8}$$

$$a^{-}_{j|n} := P_{n-1} X_j P_n \Big|_{\mathcal{P}_n} : \mathcal{P}_n \to \mathcal{P}_{n-1}. \tag{4.9}$$

Remark. Notice that for each $n \in \mathbb{N}$, $j \in D$ and $\varepsilon \in \{+, 0, -\}$, the operators $a^{\varepsilon}_{j|n}$ map polynomials with real coefficients into polynomials with the same property. In fact both multiplication by coordinates and the projections P_n satisfy this condition (see (4.2)).

Notice that, D being a finite set, the spaces $\mathcal{P}_n$ are finite dimensional. Moreover, in the present algebraic context, the sum

$$\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_n \tag{4.10}$$

is orthogonal and meant *in the following weak sense*, i.e. for each element $Q \in \mathcal{P}$ there is a finite set $I \subset \mathbb{N}$ such that

$$Q = \sum_{n\in I} p_n, \quad p_n \in \mathcal{P}_n. \tag{4.11}$$

Theorem 4.4. *For any $j \in D$, the following operators are well-defined on $\mathcal{P}$:*

$$a^{+}_{j} := \sum_{n\in\mathbb{N}} a^{+}_{j|n},$$

$$a^{0}_{j} := \sum_{n\in\mathbb{N}} a^{0}_{j|n},$$

$$a^{-}_{j} := \sum_{n\in\mathbb{N}} a^{-}_{j|n},$$

and one has

$$X_j = a^{+}_{j} + a^{0}_{j} + a^{-}_{j} \tag{4.12}$$

in the sense that both sides of (4.12) *are well-defined on $\mathcal{P}$ and the equality holds. Moreover, the decomposition on the right-hand side of* (4.12) *is unique in the sense that, if b^{+}_{j}, b^{0}_{j}, b^{-}_{j} are linear operators on $\mathcal{P}$ satisfying* (4.7), (4.8), (4.9), *then they coincide with a^{+}_{j}, a^{0}_{j}, a^{-}_{j} respectively. Finally the operators a^{+}_{j}, a^{0}_{j}, a^{-}_{j} map polynomials with real coefficients into polynomials with the same property.*

Proof. For all $j \in D$, using the symmetric Jacobi relation (4.6), one has

$$\begin{aligned}(a_j^+ + a_j^0 + a_j^-) &= \sum_{n\in\mathbb{N}} (a_{j|n}^+ + a_{j|n}^0 + a_{j|n}^-) \\ &= \sum_{n\in\mathbb{N}} (P_{n+1}X_jP_n + P_nX_jP_n + P_{n-1}X_jP_n) \\ &= \sum_{n\in\mathbb{N}} X_jP_n = X_j.\end{aligned}$$

Finally uniqueness follows from the identity $b_j^+ + b_j^0 + b_j^- = a_j^+ + a_j^0 + a_j^-$ and the fact that, for $\epsilon \neq \epsilon'$ ($\epsilon, \epsilon' \in \{-1, 0, +1\}$) the ranges of the operators $a_j^\epsilon - b_j^\epsilon$ and $a_j^{\epsilon'} - b_j^{\epsilon'}$ are orthogonal. Therefore the operators a_j^ϵ and b_j^ϵ coincide on all n-particle spaces, hence on $\mathcal{P}$. The last statement follows from the Remark after the definition of the operators $a_{j|n}^\varepsilon$. □

Definition 4.5. The identity (4.12) is called the *quantum decomposition of* X_j with respect to the state φ.

Remark. The *quantum decomposition of* X_j with respect to φ allows to extend the map $X : \mathbb{R}^d \to \mathbb{R}^d$ to a map $X : \mathbb{C}^d \to \mathbb{C}^d$ as follows: If $v = (v_1, \ldots, v_d) \in \mathbb{C}^d$, we denote

$$a_v^\varepsilon := \sum_{j\in D} v_j a_j^\varepsilon, \quad \varepsilon \in \{+, 0\}, \quad a_v^- := \sum_{j\in D} \bar{v}_j a_j^-. \tag{4.13}$$

Then one defines, in the notation (2.21)

$$X_v := a_v^+ + a_v^0 + a_{v^*}^-, \quad v \in \mathbb{C}^d. \tag{4.14}$$

With this definition one has

$$(X_v)^* = X_{v^*}.$$

In particular, since the maps $v \in \mathbb{C}^d \mapsto a_v^\varepsilon$ are real linear, the operators

$$F_v := a_v^+ + a_v^-, \quad F_{iv} := a_{iv}^+ + a_{iv}^- = i(a_v^+ - a_v^-), \quad v \in \mathbb{R}^d, \tag{4.15}$$

are symmetric (notice that this would not be true for X_{iv}). These operators are called the *field operators associated to* φ. When φ is the standard Gaussian state and $\mathbb{R}^d$ is replaced by an infinite dimensional real Hilbert space, these are the Fock field operators in quantum field theory.

4.3. *Properties of the quantum decomposition*

Notice that, by construction, for any $j \in D$ and $n \in \mathbb{N}$, the maps

$$a_{j|n}^+ := P_{n+1}X_jP_n$$

satisfy

$$a^+_{j|n}(\mathcal{P}_{\mathbb{R},n}) \subseteq \mathcal{P}_{\mathbb{R},n+1} \tag{4.16}$$

hence in particular

$$a^+_{j|n}(\mathcal{P}_n) \subseteq \mathcal{P}_{n+1} \tag{4.17}$$

and recall that, by construction, the nonzero elements of $\mathcal{P}_{n+1}$ are polynomials of degree $n+1$.

Lemma 4.6. *For any $j \in D$ and $n \in \mathbb{N}$, one has*

$$(a^+_{j|n})^* = a^-_{j|n+1}, \quad (a^+_j)^* = a^-_j;$$
$$(a^0_{j|n})^* = a^0_{j|n}, \quad (a^0_j)^* = a^0_j.$$

Proof. For an arbitrary $j \in D$ and $n \in \mathbb{N}$ we have

$$(a^+_{j|n})^* = (P_{n+1}X_jP_n)^* = P_nX_jP_{n+1} = a^-_{j|n+1}.$$

Recall that, with the notation (4.9),

$$a^-_{j|n} = P_{n-1}X_jP_n : \mathcal{P}_n \to \mathcal{P}_{n-1}.$$

Thus

$$(a^+_j)^* = \left(\sum_{n\in\mathbb{N}} a^+_{j|n}\right)^* = \sum_{n\in\mathbb{N}}(a^+_{j|n})^* = \sum_{n\in\mathbb{N}} a^-_{j|n+1}$$

and, with the change of variables $n+1 =: m \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$, this becomes

$$(a^+_j)^* = \sum_{m\in\mathbb{N}^*} a^-_{j|m} = \sum_{n\in\mathbb{N}} a^-_{j|n} = a^-_j$$

because

$$a^-_{j|0} = 0.$$

Summing up

$$(a^+_j)^* = a^-_j, \quad (a^-_j)^* = ((a^+_j)^*)^* = a^+_j;$$
$$(a^0_{j|n})^* = (P_nX_jP_n)^* = a^0_{j|n};$$
$$(a^0_j)^* = \left(\sum_{n\in\mathbb{N}} a^0_{j|n}\right)^* = \sum_{n\in\mathbb{N}}(a^0_{j|n})^* = \sum_{n\in\mathbb{N}} a^0_{j|n} = a^0_j. \qquad \square$$

Lemma 4.7. *For any $j \in D$, the operators*

$$X_j, \quad a^+_j, \quad a^-_j, \quad a^0_j$$

preserve the space $\mathcal{N}_\varphi$ of zero-norm vectors.

Proof. It is sufficient to show that, for each $n \in \mathbb{N}$ if $\xi \in \mathcal{P}_n$ is a zero-norm vector, then the same is true for the vectors

$$X_j\xi, \quad a^+_{j|n}\xi, \quad a^0_{j|n}\xi, \quad a^-_{j|n}\xi, \quad j \in D.$$

That $X_j\xi$ is a zero-norm vector follows from

$$|\langle X_j\xi, X_j\xi\rangle| = |\langle X_j^2\xi, \xi\rangle| \leq |\langle X_j^2\xi, X_j^2\xi\rangle|^{1/2}|\langle \xi, \xi\rangle|^{1/2} = 0.$$

From this and the quantum decomposition (4.12) it follows that the vector

$$X_jP_n\xi = a^+_{j|n}\xi + a^0_{j|n}\xi + a^-_{j|n}\xi$$

has zero-norm. Since the right-hand side is a sum of three mutually orthogonal vectors, it follows that each of them is a zero norm vector. □

Lemma 4.8. *In the notations of Definition* 7.1, *for* $n \in \mathbb{N}$, *let the following be given*:

(i) *two monic vector subspaces in the coordinates* (X_j) $\mathcal{P}_{n-1} \subset \mathcal{P}_{n-1]}, \mathcal{P}_n \subset \mathcal{P}_{n]}$ *respectively of degree* $n-1$ *and* $n-1$,
(ii) *two arbitrary linear maps*

$$v \in \mathbb{C}^d \mapsto A^0_{v|n} \in \mathcal{L}_a(\mathcal{P}_n, \mathcal{P}_n), \tag{4.18}$$

$$v \in \mathbb{C}^d \mapsto A^-_{v|n} \in \mathcal{L}_a(\mathcal{P}_n, \mathcal{P}_{n-1}). \tag{4.19}$$

Then, defining for any $v \in \mathbb{C}^d$ *the map*

$$A^+_{v|n} := X_v\big|_{\mathcal{P}_n} - A^0_{v|n} - A^-_{v|n}, \tag{4.20}$$

the vector space

$$\tilde{\mathcal{P}}_{n+1} := \{A^+_{v|n}\mathcal{P}_n; v \in \mathbb{C}^d\} \tag{4.21}$$

has the form

$$\tilde{\mathcal{P}}_{n+1} = \mathcal{P}_{n+1} \dotplus (\tilde{\mathcal{P}}_{n+1} \cap \mathcal{P}_{n]}), \tag{4.22}$$

where $\mathcal{P}_{n+1}$ *is a monic vector subspace of degree* $n+1$ *and* $\dotplus$ *denotes direct sum of linear spaces.*

Proof. Since $\mathcal{P}_n$ is monic of degree n in the coordinates (X_j), it has a linear basis $B_n := (\xi_{n,M})_{M \in \mathcal{M}_{e,n}}$ which is a perturbation of the monomial basis of degree n. From the definition (4.20) of $A^+_{v|n}$ we know that, for each $j \in D$ and $M \in \mathcal{M}_{e,n}$, one has

$$A^+_{j|n}\xi_{n,M} = X_j\xi_{n,M} - A^0_{j|n}\xi_{n,M} - A^-_{j|n}\xi_{n,M}. \tag{4.23}$$

The assumptions on $A^0_{j|n}$ and $A^-_{j|n}$ imply that $A^0_{j|n}\xi_{n,M} + A^-_{j|n}\xi_{n,M}$ is a polynomial of degree $\leq n$. Therefore, when $\xi_{n,M}$ varies in B_n and X_j varies among all coordinate functions, $A^+_{j|n}\xi_{n,M}$ defines a set of monic polynomials whose leading terms contain the set of all monomials of degree $n+1$ (with possible repetitions). Therefore from this set one can extract a perturbation of the monomial basis of order $n+1$. Denote by B_{n+1} this basis and $\mathcal{P}_{n+1}$ its linear span. By construction $\mathcal{P}_{n+1}$ is a monic vector subspace of $\mathcal{P}_{n+1]}$. The definition of perturbation of a monomial basis implies that

$$\mathcal{P}_{n+1} \cap (\tilde{\mathcal{P}}_{n+1} \cap \mathcal{P}_{n]}) = \{0\} \tag{4.24}$$

because the nonzero elements of the space $\mathcal{P}_{n+1}$ are polynomials of degree $n+1$. Let us prove that the identity (4.22) holds. To this goal it will be sufficient to prove that the set

$$\{A^+_{j|n}\xi_{n,M} \ ; \ \xi_{n,M} \in B_n\}$$

is contained in the left-hand side of (4.24). By construction $\mathcal{P}_{n+1}$ contains B_{n+1}. Let $\xi_{n,M} \in B_n$ be such that

$$A^+_{j|n}\xi_{n,M} = X_j M + Q_{n]} \notin B_{n+1}.$$

Since B_{n+1} is a perturbation of the monomial basis of order $n+1$ in the (X_j)-coordinates there exist $k \in D$ and $M' \in \mathcal{M}_{e,n}$ such that

$$A^+_{k|n}\xi_{n,M'} = X_k M' + R_{n]} \in B_{n+1}$$

($R_{n]}$ is a polynomial of degree $\leq n$) and

$$X_k M' = X_j M.$$

It follows that

$$A^+_{j|n}M - A^+_{k|n}M' \in \mathcal{P}_{n]} \cap \tilde{\mathcal{P}}_{n+1}.$$

Therefore

$$A^+_{j|n}M = A^+_{k|n}\xi_{n,M'} + (A^+_{j|n}M - A^+_{k|n}M') \in \mathcal{P}_{n+1} \dot{+} (\tilde{\mathcal{P}}_{n+1} \cap \mathcal{P}_{n]}).$$

This proves (4.22). □

Remark. The vector space sum $\tilde{\mathcal{P}}_{n+1} + \mathcal{P}_{n]}$ is not direct. However, the vector space sum $\mathcal{P}_{n+1} + \mathcal{P}_{n]}$ is direct and one has

$$\tilde{\mathcal{P}}_{n+1} + \mathcal{P}_{n]} = \mathcal{P}_{n+1} \dot{+} \mathcal{P}_{n]}.$$

Remark. If the operators $A^{\varepsilon}_{v|n}$ are the CAP operators associated to a given state on $\mathcal{P}$, then the subspace $\mathcal{P}_{n]} \cap \tilde{\mathcal{P}}_{n+1}$ necessarily consists of zero-norm vectors because, in this case, operators in the spaces $A^+_{v|n}\mathcal{P}_n$ are orthogonal to $\mathcal{P}_{n]}$.

4.4. *Commutation relations*

In this section we briefly recall some known facts about commutation relations canonically associated to orthogonal polynomials (see Refs. 2 and 5) which will be used in the following section. We refer the reader to[2] for more detailed analysis.

Theorem 4.9. *Let the following be given*:

— *a pre-Hilbert space* H;
— *an orthogonal gradation of* H :

$$H = \bigoplus_{n \in \mathbb{N}} H_n;$$

— *a family of operators* $a_j^{\pm} : H_n \to H_{n\pm 1}$, $a_j^0 : H_n \to H_n$, $(j \in \{1, \ldots, d\})$

$$a_j^0 = (a_j^0)^*, \quad a_j^- = (a_j^+)^*, \quad j \in \{1, \ldots, d\}.$$

Define the operators Y_j $(j \in \{1, \ldots, d\})$ *on* H *by*

$$Y_j := a_j^+ + a_j^0 + a_j^-, \quad j \in \{1, \ldots, d\}. \tag{4.25}$$

Then the decomposition (4.25) *is unique and the operators* Y_j *commute on the algebraic linear span of the* H_n *if and only if the operators* a_j^+, a_j^0, a_j^- *satisfy the following commutation relations on the same domain*: *for all* $j, k \in \{1, \ldots, d\}$ *such that* $j < k$

$$[a_j^+, a_k^+] = 0, \tag{4.26}$$

$$[a_j^+, a_k^-] + [a_j^0, a_k^0] + [a_j^-, a_k^+] = 0, \tag{4.27}$$

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0. \tag{4.28}$$

Proof. Clearly the operators a_j^+, a_j^0 are well-defined on the algebraic linear span of the H_n and leave this domain invariant. Given (4.25) one has, for each $j, k \in \{1, \ldots, d\}$:

$$\begin{aligned} 0 = [Y_j, Y_k] &= [(a_j^+ + a_j^0 + a_j^-), (a_k^+ + a_k^0 + a_k^-)] \\ &= [a_j^+, a_k^+] + [a_j^+, a_k^0] + [a_j^0, a_k^+] + [a_j^+, a_k^-] \\ &\quad + [a_j^0, a_k^0] + [a_j^-, a_k^+] + [a_j^0, a_k^-] + [a_j^-, a_k^0] + [a_j^-, a_k^-]. \end{aligned} \tag{4.29}$$

The mutual orthogonality of the H_k's and the properties of the a_k^ϵ imply that the commutativity of the Y_js, is equivalent to the fact the expressions on different rows of the right-hand side of (4.29) are separately equal to zero. Since the fifth row is the adjoint of the first one and the fourth row is equivalent to the adjoint of the second one, the vanishing of all the rows is equivalent to (4.26), (4.27), (4.28) for all $j, k \in \{1, \ldots, d\}$. But this is equivalent to the validity of these relations for all $j, k \in \{1, \ldots, d\}$ such that $j < k$ because all the relations are identically satisfied for $j = k$ and, exchanging the roles of j and k, the left-hand sides of (4.26), (4.27) are transformed into its opposite and that of (4.28) remains unaltered.

Finally the uniqueness of the decomposition (4.25) is established as in the proof of Theorem 4.4. □

5. Orthogonal Polynomials and Symmetric Interacting Fock Spaces

The notion of symmetric interacting Fock space is discussed in Appendix B.5 below and in this section we will use freely the definitions and notations of this Appendix.

Theorem 5.1. *Let φ be a state on $\mathcal{P}$ and let $\mathcal{P} = \bigoplus_{n\in\mathbb{N}} \mathcal{P}_n$ its orthogonal polynomial gradation. Denote*:

— *for $n \in \mathbb{N}$, $\langle\cdot,\cdot\rangle_n$ the restriction on $\mathcal{P}_n$ of the pre-scalar product $\langle\cdot,\cdot\rangle$ induced by φ on $\mathcal{P}$*;

— *for $j \in D$*

$$X_j = a_j^+ + a_j^0 + a_j^- \tag{5.1}$$

the quantum decomposition of the coordinate X_j with respect to φ;

— $a^+ : v = \sum_{j\in D} v_j e_j \in \mathbb{C}^d \to a_v^+ := \sum_{j\in D} v_j a_j^+ \in \mathcal{L}_a(\mathcal{P}, \langle\cdot,\cdot\rangle)$ *the creation map.*

Then the pair

$$\big((\mathcal{P}_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\big) \tag{5.2}$$

is a symmetric interacting Fock space with the following properties:

(i) *The restriction on $\mathcal{P}_\mathbb{R}$ of the pre-scalar product $\langle\cdot,\cdot\rangle$ is real-valued and there exists a family of gradation preserving self-adjoint operators $a_j^0 : \mathcal{P}\cdot\Phi_0 \to \mathcal{P}\cdot\Phi_0$ $(j \in D)$ such that*

$$a_j^\varepsilon(\mathcal{P}_\mathbb{R}\cdot\Phi_0) \subseteq \mathcal{P}_\mathbb{R}\cdot\Phi_0, \quad \forall\,\varepsilon \in \{+,0,-\}, \quad j \in D, \tag{5.3}$$

and the coordinate operators X_j mutually commute;

(ii) *the vacuum vector Φ of the IFS* (5.2) (*identified to the vector $\Phi_0 \in \mathcal{P}\cdot\Phi_0$*) *is cyclic for the polynomial algebra generated by the family* (5.1).

Conversely, given a symmetric interacting Fock space on $\mathbb{C}^d$

$$((\hat{\mathcal{P}}_n, \langle\cdot,\cdot\,\rangle_{IFS,n}), \hat{a}^+)$$

and a family of gradation preserving operators $\hat{a}_j^0$ $(j \in D)$ such that the operators

$$\hat{X}_j := \hat{a}_j^+ + \hat{a}_j^0 + (\hat{a}_j^+)^*, \quad j \in D, \tag{5.4}$$

commute and, denoting $\hat{\mathcal{P}}$, (resp. $\hat{\mathcal{P}}_\mathbb{R}$) the $*$-algebra (resp. real $*$-algebra) generated by the $\hat{X}_j$ conditions (i) and (ii) above are satisfied.

Then there exists a unique state φ on $\mathcal{P}$ characterized by the property that for all maps $n : D \to \mathbb{N}$, denoting Φ the vacuum vector of $\hat{\mathcal{P}}$, one has:

$$\varphi(X_1^{n_1}\cdots X_d^{n_d}) = \langle\Phi, \hat{X}_1^{n_1}\cdots\hat{X}_d^{n_d}\Phi\rangle, \quad \forall\, n_1,\ldots n_d \in \mathbb{N}. \tag{5.5}$$

Moreover, the expectation values (5.5) are real-valued.

In particular, there is a symmetric IFS isomorphism (see Definition B.9)

$$U : ((\mathcal{P}_n, \langle \cdot, \cdot \rangle_n), a^+) \to ((\hat{\mathcal{P}}_n, \langle \cdot, \cdot \rangle_{IFS,n}), \hat{a}^+)$$

preserving the real structures of both spaces and such that

$$X_j = U^* \hat{a}_j^+ U + U^* \hat{a}_j^0 U + (U^* \hat{a}_j^+ U)^* \tag{5.6}$$

is the quantum decomposition of the X_j with respect to φ.

Proof. Let φ be a state on $\mathcal{P}$ and let a^ε be the associated CAP operators. Let us first prove the pair (5.2) satisfies the conditions of Definition B.1. We know that $\mathcal{P}_0$ is 1-dimensional with the scalar product uniquely determined by the condition $\|\Phi_0\| = 1$. Lemma 4.6 implies that a^+ is adjointable. Finally

$$\begin{aligned}\mathcal{P}_{n+1} &= P_{n+1}\mathcal{P} \cdot \Phi_0 = (P_{n+1]} - P_{n]})\mathcal{P} \cdot \Phi_0 \\ &= (P_{n+1]} - P_{n]})P_{n+1]}\mathcal{P} \cdot \Phi_0 = P_{n+1}\mathcal{P}_{n+1]} \cdot \Phi_0\end{aligned}$$

and $\mathcal{P}_{n+1]} \cdot \Phi_0$ is the complex linear span of the set $\{X_v \mathcal{P}_{n]} : v \in \mathbb{R}^d\}$. Therefore, to verify condition (B.3) of Definition B.1, it is sufficient to prove that $a^+(V)\mathcal{P}_n$ contains $\{P_{n+1} X_v \mathcal{P}_{n]} \cdot \Phi_0 : v \in \mathbb{R}^d\}$. This follows from the symmetric Jacobi relations because for any $v \in \mathbb{R}^d$:

$$P_{n+1} X_v P_{n]} = P_{n+1} X_v (P_n + P_{n-1]}) = P_{n+1} X_v P_n = a^+_{v|n}.$$

Thus $((\mathcal{P}_n, \langle \,\cdot\, , \,\cdot\, \rangle_n), a^+)$ is an IFS. That it is a symmetric IFS follows from Definition B.9 and the commutativity of the creators, established in Sec. 4.4. Property (i) follows from the quantum decomposition of the coordinates. Property (ii) holds by definition of $\mathcal{P}$.

Conversely, let $((\hat{\mathcal{P}}_n, \langle \cdot, \cdot \rangle_{IFS,n}), \hat{a}^+)$ be an interacting Fock space on $\mathbb{C}^d$ and suppose that conditions (i) and (ii) above are satisfied in the sense specified in the statement of the theorem. Then, since the operators

$$\hat{X}_j := a_j^+ + a_j^0 + (a_j^+)^*, \quad j \in D, \tag{5.7}$$

are self-adjoint, property (i) implies that the complex $*$-algebra $\hat{\mathcal{P}}$ generated by them is commutative.

Since $\mathcal{P}$ is isomorphic to the free abelian $*$-algebra with identity and d self-adjoint generators, there exists a $*$-algebra homomorphism $\pi : \mathcal{P} \to \hat{\mathcal{P}}$ characterized by the property that

$$\pi(X_j) := \widehat{X}_j = \hat{a}_j^+ + \hat{a}_j^0 + (\hat{a}_j^+)^*, \quad j \in D.$$

Denoting φ_F the restriction of the Fock state $\langle \Phi \cdot, \cdot \Phi \rangle$ on $\hat{\mathcal{P}}$, define the state φ on $\mathcal{P}$ by

$$\varphi := \varphi_F \circ \pi. \tag{5.8}$$

Then (5.5) holds by construction. Since the monomials are linearly independent in $\mathcal{P}$, for any map $n : D \to \mathbb{N}$, the map

$$X_1^{n_1} \cdots X_d^{n_d} \Phi_0 \mapsto \widehat{X}_1^{n_1} \cdots \widehat{X}_d^{n_d} \Phi$$

can be extended to a linear map $U : \mathcal{P} \cdot \Phi_0 \to \hat{\mathcal{P}} \cdot \Phi$ which is onto by condition (ii). (5.5) implies that this extension preserves scalar products, therefore U is a unitary isomorphism of pre-Hilbert spaces. It preserves the real structure of the corresponding spaces because of condition (i). Moreover U satisfies, for $j \in D$,

$$\begin{aligned} X_j &= U^*(\hat{a}_j^+ + \hat{a}_j^0 + (\hat{a}_j^+))^* U \\ &= U^* \hat{a}_j^+ U + U^* \hat{a}_j^0 U + U^* (\hat{a}_j^+)^* U \\ &= U^* \hat{a}_j^+ U + U^* \hat{a}_j^0 U + (U^* \hat{a}_j^+ U)^*, \quad j \in D \end{aligned} \tag{5.9}$$

which implies

$$\mathcal{P}_{n]} = U^* \hat{\mathcal{P}}_{n]} U, \quad n \in \mathbb{N}.$$

Therefore, since U is unitary,

$$\mathcal{P}_n = \mathcal{P}_{n-1]}^{\perp} \cap \mathcal{P}_{n]} = U^* \hat{\mathcal{P}}_{n-1]}^{\perp} U \cap U^* \hat{\mathcal{P}}_{n]} U = U^* \hat{\mathcal{P}}_n U, \quad n \in \mathbb{N}.$$

Denote $X_j = a_j^+ + a_j^0 + (a_j^+)^*$ the quantum decomposition of the X_j associated to the state φ defined by (5.8). Then (5.9) implies that

$$X_j = a_j^+ + a_j^0 + (a_j^+)^* = U^* \hat{a}_j^+ U + U^* \hat{a}_j^0 U + (U^* \hat{a}_j^+ U)^*$$

and the operators $a_j^{\pm}$ (resp. a_j^0) and $U^* \hat{a}_j^{\pm} U$ (resp. $U^* \hat{a}_j^0 U$) are of degree ± 1 (resp. 0) with respect to the same orthogonal gradation. From the uniqueness of the quantum decomposition (see Theorem 4.9) we conclude that

$$a_j^{\pm} = U^* \hat{a}_j^{\pm} U, \quad a_j^0 = U^* \hat{a}_j^0 U, \quad j \in D.$$

Thus U is an isomorphism of IFS. Since the X_j commute, we know from Theorem 4.9 that the operators $\hat{a}_j^+$ mutually commute so that the IFS is symmetric (see Definition B.9). □

Theorem 5.1 motivates the following definition.

Definition 5.2. Let $((\hat{\mathcal{P}}_n, \langle \cdot, \cdot \rangle_{IFS,n}), \hat{a}^+)$ be an interacting Fock space on $\mathbb{C}^d$. A family of gradation preserving self-adjoint operators $a_j^0 : \hat{\mathcal{P}}_n \to \hat{\mathcal{P}}_n$ $(j \in D)$ is said to define a 3-*diagonal structure* on $((\hat{\mathcal{P}}_n, \langle \cdot, \cdot \rangle_{IFS,n}), \hat{a}^+)$ if the operators $\hat{X}_j$, defined by (5.7), satisfy conditions (i) and (ii) of the second part of Theorem 5.1.

Remark. From the Remark after Theorem 5.1 it follows that an interacting Fock space with a 3-diagonal structure is necessarily symmetric. Therefore, by Lemma B.10, we can identify it, up to isomorphism, to its symmetric tensor representation (see Lemma B.10).

Remark. The assignment of a gradation preserving self-adjoint operator $a_j^0 : \mathcal{P} \to \mathcal{P}$ $(j \in D)$ is equivalent to the assignment of a sequence of self–adjoint operators $a_{j|n}^0 : \mathcal{P}_n \to \mathcal{P}_n$ $(n \in \mathbb{N})$.

Definition 5.3. Let a finite dimensional vector space V, and a sequence $\tilde{\Omega}^{\widehat{\otimes}} := (\tilde{\Omega}_n^{\widehat{\otimes}})$, inductively defined as in Theorem B.11 be given.

Let $\Gamma(V, \tilde{\Omega}) := ((V^{\widehat{\otimes} n}, \langle\cdot,\cdot\rangle_n), \ell^*)$ be the symmetric IFS on V associated to the pair $(V, (\tilde{\Omega}_n^{\widehat{\otimes}}))$ according to Theorem B.11 and let, for each $n \in \mathbb{N}$ and $j \in D$, $a_{j|n}^0 : (V^{\widehat{\otimes} n}, \langle\cdot,\cdot\rangle_n) \to (V^{\widehat{\otimes} n}, \langle\cdot,\cdot\rangle_n)$ be a sequence of self-adjoint operators.

The pair $(\tilde{\Omega}^{\widehat{\otimes}}\ ,\ (a_{j|n}^0))$ is said to induce a 3-diagonal structure on $\Gamma(V, \tilde{\Omega})$, if the family of gradation preserving self-adjoint operators $a_j^0 : \Gamma(V, \tilde{\Omega}) \to \Gamma(V, \tilde{\Omega})$ $(j \in D)$ is a 3-diagonal structure on $\Gamma(V, \tilde{\Omega})$ in the sense of Definition 5.3.

Theorem 5.4. *In the notations of Theorem* 5.1 *and of Definition* 5.3, *any state* φ *on* $\mathcal{P}$ *uniquely defines a pair* $(\tilde{\Omega}^{\widehat{\otimes}}, (a_{j|n}^0))$ *that induces a* 3*-diagonal structure on* $\Gamma(V, \tilde{\Omega})$.

Conversely, any pair $(\tilde{\Omega}^{\widehat{\otimes}}, (a_{j|n}^0))$ *that induces a* 3*-diagonal structure on* $\Gamma(V, \tilde{\Omega}^{\widehat{\otimes}})$ *uniquely defines a state* φ *on* $\mathcal{P}$.

Proof. Both statements are immediate consequences of the corresponding statements in Theorem 5.1. □

Remark. Theorem 5.4 implies that the (standard) interacting Fock spaces on $\mathbb{C}^d$ of the form

$$\{(V^{\widehat{\otimes} n}, \langle\cdot, \Omega_n \cdot\rangle_{\widehat{\otimes}, n}), \hat{\ell}^*\} \tag{5.10}$$

with a 3-diagonal structure provide a universal model for the theory of orthogonal polynomials in d variables.

Remark. From Sec. 4.4 we know that the operators (5.4) commute if and only if the relations (4.26)–(4.28) hold. On the other hand, from Theorem B.8 we know that IFS on $\mathbb{C}^d$ are characterized by sequences of PD kernels on $\mathbb{C}^d$ and, from the identity (B.26) we know that these PD kernels have the form $a^-(u)a^+(v)$ $(u, v \in \mathbb{C}^d)$. Since products of this form appear in the commutators in (4.26)–(4.28), it follows that these commutation relations create constraints between the kernels defining the scalar products in the IFS and the operators a_j^0. In the following section we will investigate these constraints.

6. Implications of the Commutation Relations

With the notations (4.7)–(4.9), the tri-diagonal relation (4.6) takes the form

$$X_j P_n = a_{j|n}^+ + a_{j|n}^0 + a_{j|n}^-, \quad \forall\, j \in D, \quad \forall\, n \in \mathbb{N},$$

or equivalently, due to Lemma 4.6

$$a^+_{j|n} = X_j P_n - a^0_{j|n} - (a^+_{j|n-1})^*, \quad \forall\, j \in D, \quad \forall\, n \in \mathbb{N}. \tag{6.1}$$

This can be interpreted as an inductive relation that, given $a^+_{j|n-1}$ ($j \in D$), the scalar product on $\mathcal{P}_n$ and $a^0_{j|n}$, uniquely defines $a^+_{j|n}$. Notice that, if $a^0_{j|n}$ is chosen to be a pre-Hilbert space operator, in particular mapping zero norm vectors into zero norm vectors, and if it maps real vectors in $\mathcal{P}_n$ into real vectors, then $a^+_{j|n-1}$ will have the same properties because X_j has these properties and $(a^+_{j|n-1})^*$ has these properties by the induction construction.

In this section we will establish the constraints, imposed by the commutation relations, on the objects that define the induction relation, namely the $a^+_{j|n-1}$ ($j \in D$), the scalar product on $\mathcal{P}_n$ and the $a^0_{j|n}$.

Remark. Recall that, if A is an adjointable operator on a pre-Hilbert space, then its real and imaginary parts are defined by

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*) =: \mathrm{Re}(A) + i\mathrm{Im}(A). \tag{6.2}$$

Similarly, for any PD kernel $\tilde{\Omega}$ one has

$$\tilde{\Omega}(e_j, e_k)^* = \tilde{\Omega}(e_k, e_j)$$

therefore

$$\begin{aligned}\tilde{\Omega}(e_j, e_k) &= \frac{1}{2}((\tilde{\Omega}(e_j, e_k)) + \tilde{\Omega}(e_j, e_k)^*) + \frac{1}{2}((\tilde{\Omega}(e_j, e_k)) - \tilde{\Omega}(e_j, e_k)^*)\\ &= \frac{1}{2}((\tilde{\Omega}(e_j, e_k)) + \tilde{\Omega}(e_k, e_j)) + \frac{1}{2}((\tilde{\Omega}(e_j, e_k)) - \tilde{\Omega}(e_k, e_j))\\ &=: \tilde{\Omega}_R(e_j, e_k) + \tilde{\Omega}_I(e_j, e_k) \end{aligned} \tag{6.3}$$

with

$$\tilde{\Omega}_R(e_j, e_k) = \tilde{\Omega}_R(e_k, e_j) = \tilde{\Omega}_R(e_j, e_k)^*, \quad -\tilde{\Omega}_I(e_j, e_k) = \tilde{\Omega}_I(e_k, e_j) = \tilde{\Omega}_I(e_j, e_k)^*.$$

Thus any PD kernel $\tilde{\Omega}$ is the sum of a symmetric kernel and a symplectic kernel. In this section we will use the notations (4.7)–(4.9) and in the following $(\tilde{\Omega}_n)$ will denote the sequence of positive definite (PD) kernels defined by $\tilde{\Omega}_0 = 1 \in \mathbb{C}$ and

$$\tilde{\Omega}_{n+1}(e_j, e_k) := (a^-_j a^+_k)_{|n} := (a^+_{j|n})^* a^+_{k|n}, \quad \forall\, n \in \mathbb{N}, \quad \forall\, j, k \in D. \tag{6.4}$$

Since the operators $a^+_{k|n}$ map real polynomials into real polynomials, it follows that the operators $\tilde{\Omega}_n$ also have this property. By linearity this is equivalent to say that the $a^+_{k|n}$ map maps real vectors in $\mathcal{P}_n$ into real vectors.

Lemma 6.1. *The commutation relations* (4.27), *i.e.*

$$[a^+_j, a^-_k] + [a^0_j, a^0_k] + [a^-_j, a^+_k] = 0 \tag{6.5}$$

are equivalent to

$$\tilde{\Omega}_1(e_j, e_k) = \tilde{\Omega}_1(e_k, e_j) \in \mathbb{R} \tag{6.6}$$

$$\mathrm{Im}(\tilde{\Omega}_{n+1}(e_j, e_k)) = \mathrm{Im}(a^+_{k|n-1}(a^+_{j|n-1})^*) + \mathrm{Im}(a^0_{k|n}a^0_{j|n}), \quad \forall\, n \geq 1, \tag{6.7}$$

for all $j, k \in D$ such that $j < k$ and all $n \in \mathbb{N}$.

Proof. For j, k and n as in the statement, the commutation relation (4.27) is

$$\begin{aligned}
&[a^+_j, a^-_k] + [a^0_j, a^0_k] + [a^-_j, a^+_k] = 0\\
&\quad \Leftrightarrow [a^+_j a^-_k - a^-_k a^+_j] + [a^0_j a^0_k - a^0_k a^0_j] + [a^-_j a^+_k - a^+_k a^-_j] = 0\\
&\quad \Leftrightarrow (a^+_j)^* a^+_k - (a^+_k)^* a^+_j = a^+_k a^-_j - a^+_j a^-_k + a^0_k a^0_j - a^0_j a^0_k.
\end{aligned} \tag{6.8}$$

These are identically satisfied for $j = k$ and, exchanging j and k, one finds an equivalent relation. Therefore it is sufficient to consider the case $j < k$.

On $\mathcal{P}_0$, (6.8) is equivalent to:

$$\begin{aligned}
(a^+_j)^* a^+_k \Phi_0 - (a^+_k)^* a^+_j \Phi_0 \;&=\; a^+_k a^-_j \Phi_0 - a^+_j a^-_k \Phi_0 + a^0_k a^0_j \Phi_0 - a^0_j a^0_k \Phi_0\\
&\Leftrightarrow (a^+_j)^* a^+_k \Phi_0 - (a^+_k)^* a^+_j \Phi_0 = 0.
\end{aligned}$$

Recalling (6.4) the above identity becomes

$$\tilde{\Omega}_1(e_j, e_k)\Phi_0 - \tilde{\Omega}_1(e_k, e_j)\Phi_0$$

and, since $\tilde{\Omega}_1(e_j, e_k)$ maps $\mathbb{C} \cdot \Phi_0$ into itself, the above identity is equivalent (up to obvious identifications) to

$$\tilde{\Omega}_1(e_j, e_k) = \tilde{\Omega}_1(e_k, e_j) \;\in \mathbb{C}$$

and from condition (5.3) and the identity

$$\tilde{\Omega}_{n+1}(e_j, e_k)^* := (((a^+_j)^* a^+_k)_{|n})^* = ((a^+_k)^* a^+_j)_{|n} = \tilde{\Omega}_{n+1}(e_k, e_j)$$

it follows that $\tilde{\Omega}_1(e_j, e_k) \in \mathbb{R}$. This proves (6.6). Let $n > 0$. From

$$(a^+_k)^* a^+_j = ((a^+_j)^* a^+_k)^*$$

one deduces that for any $\xi_n, \eta_n \in \mathcal{P}_n$

$$\langle (a^+_j)^* a^+_k \xi_n, \eta_n \rangle_n = \langle \xi_n, (a^+_k)^* a^+_j \eta_n \rangle_n \Leftrightarrow ((a^+_k)^* a^+_j)_{|n} = (((a^+_j)^* a^+_k)_{|n})^*.$$

Therefore the identity (6.8), restricted to $\mathcal{P}_n$ is equivalent to the fact that, for each $n \in \mathbb{N}$ and each $j \in D$,

$$(a^+_{j|n})^* a^+_{k|n} - (a^+_{k|n})^* a^+_{j|n} = a^+_{k|n-1} a^-_{j|n} - a^+_{j|n-1} a^-_{k|n} + a^0_{k|n} a^0_{j|n} - a^0_{j|n} a^0_{k|n} \tag{6.9}$$

or equivalently

$$\begin{aligned}
\tilde{\Omega}_{n+1}(e_j, e_k) - \tilde{\Omega}_{n+1}(e_j, e_k)^* &= \tilde{\Omega}_{n+1}(e_j, e_k) - \tilde{\Omega}_{n+1}(e_k, e_j) = 2i\mathrm{Im}(\tilde{\Omega}_{n+1}(e_j, e_k))\\
&= (a^+_k a^-_j)_{|n} - (a^+_j a^-_k)_{|n} + (a^0_k a^0_j)_{|n} - (a^0_j a^0_k)_{|n}.
\end{aligned} \tag{6.10}$$

Now notice that for any $\xi_n, \eta_n \in \mathcal{P}_n$

$$a_k^+ a_j^- \eta_n = a_{k|n-1}^+ a_{j|n}^- \eta_n = a_{k|n-1}^+ (a_{j|n-1}^+)^* \eta_n,$$

i.e.

$$(a_k^+ a_j^-)_{|n} = a_{k|n-1}^+ (a_{j|n-1}^+)^* = (a_{j|n-1}^+ (a_{k|n-1}^+)^*)^*.$$

Since the a_j^0 preserve the gradation and are self-adjoint, $(a_k^0 a_j^0)_{|n} = a_{k|n}^0 a_{j|n}^0$, therefore (6.10) becomes

$$\begin{aligned} 2i\mathrm{Im}(\tilde{\Omega}_{n+1}(e_j, e_k)) &= (a_k^+ a_j^-)_{|n} - (a_j^+ a_k^-)_{|n} + (a_k^0 a_j^0)_{|n} - (a_j^0 a_k^0)_{|n} \\ &= a_{k|n-1}^+ (a_{j|n-1}^+)^* - a_{j|n-1}^+ (a_{k|n-1}^+)^* + a_{k|n}^0 a_{j|n}^0 - a_{j|n}^0 a_{k|n}^0 \\ &= a_{k|n-1}^+ (a_{j|n-1}^+)^* - (a_{k|n-1}^+ (a_{j|n-1}^+)^*)^* + a_{k|n}^0 a_{j|n}^0 - (a_{k|n}^0 a_{j|n}^0)^* \\ &= 2i\mathrm{Im}(a_{k|n-1}^+ (a_{j|n-1}^+)^*) + 2i\mathrm{Im}(a_{k|n}^0 a_{j|n}^0) \end{aligned} \tag{6.11}$$

and this is equivalent to (6.7). □

Remark. Lemma 6.1 implies that the commutation relations (4.27), associated to a state on $\mathcal{P}$, inductively fix the symplectic parts of the kernels $\tilde{\Omega}_{n+1}$. Since, adding a symplectic kernel to any PD kernel, one still obtains a PD kernel, fixing the imaginary part of a PD kernel leaves its symmetric part completely arbitrary up to the conditions of positive-definiteness and of preservation of the real structure.

Lemma 6.2. *The commutation relations* (4.28), *i.e.*

$$[a_j^+, a_k^0] + [a_j^0, a_k^+] = 0 \tag{6.12}$$

are equivalent to

$$a_{j|n+1}^0 a_{k|n}^+ - a_{k|n+1}^0 a_{j|n}^+ = a_{k|n}^+ a_{j|n}^0 - a_{j|n}^+ a_{k|n}^0 \tag{6.13}$$

for all $j, k \in D$ such that $j < k$ and all $n \in \mathbb{N}$.

Proof. The commutation relations (6.12) are identically satisfied for $j = k$ and, exchanging j and k, one finds an equivalent relation. Therefore it is sufficient to consider the case $j < k$. In this case, with arguments similar to those used in the proof of Lemma 6.1, one shows that (6.12) is equivalent to

$$\begin{aligned} & a_j^+ a_k^0 - a_k^0 a_j^+ + a_j^0 a_k^+ - a_k^+ a_j^0 = 0 \\ &\quad \Leftrightarrow (a_j^+ a_k^0)_{|n} - (a_k^0 a_j^+)_{|n} + (a_j^0 a_k^+)_{|n} - (a_k^+ a_j^0)_{|n} = 0, \quad \forall\, n \in \mathbb{N} \\ &\quad \Leftrightarrow a_{j|n}^+ a_{k|n}^0 - a_{k|n+1}^0 a_{j|n}^+ + a_{j|n+1}^0 a_{k|n}^+ - a_{k|n}^+ a_{j|n}^0 = 0 \\ &\quad \Leftrightarrow a_{j|n+1}^0 a_{k|n}^+ - a_{k|n+1}^0 a_{j|n}^+ = a_{k|n}^+ a_{j|n}^0 - a_{j|n}^+ a_{k|n}^0 \end{aligned}$$

that is (6.13). □

Remark. Since the inductive form of the creators is uniquely determined by condition (6.1), the identity (6.13) can be interpreted as a necessary condition to be satisfied by the $a^0_{j|n+1}$ once given the $a^0_{j|n}$ $(j \in D)$. Notice that the inductive system of equations (6.13) always admits the zero solution given by the sequence

$$a^0_{j|n} = 0, \quad \forall\, j \in D, \quad \forall\, n \in \mathbb{N}.$$

Lemma 6.3. *The commutation relations* (4.26) (*commutativity of creators*) *are equivalent to the following identities*

$$\begin{aligned} &a^0_{k|n+1}a^+_{j|n} - a^0_{j|n+1}a^+_{k|n} \\ &\qquad = X_k a^+_{j|n} - X_j a^+_{k|n} + 2i\mathrm{Im}(a^+_{k|n-1}(a^+_{j|n-1})^*) + 2i\mathrm{Im}(a^0_{k|n}a^0_{j|n}) \end{aligned} \tag{6.14}$$

for all $j, k \in D$ such that $j < k$ and all $n \in \mathbb{N}$.

Proof. The commutativity of creators is identically satisfied for $j = k$ and, exchanging j and k, one finds the same relation up to a common sign. Therefore it is sufficient to consider the case $j < k$.

Due to (6.1), the commutativity of creators is equivalent to

$$\begin{aligned} a^+_j a^+_k = a^+_k a^+_j &\Leftrightarrow a^+_j a^+_k P_n = a^+_k a^+_j P_n \\ &\Leftrightarrow a^+_{j|n+1}a^+_{k|n} = a^+_{k|n+1}a^+_{j|n} \quad \forall\, j \in D, \quad \forall\, n \in \mathbb{N}. \end{aligned}$$

Using the quantum decomposition of the X_j this becomes equivalent to

$$\begin{aligned} &(X_j - a^0_j - (a^+_j)^*)a^+_{k|n} = (X_{k|n+1} - a^0_{k|n+1} - (a^+_{k|n})^*)a^+_{j|n} \\ &\quad\Leftrightarrow X_j a^+_{k|n} - a^0_{j|n+1}a^+_{k|n} - (a^+_{j|n})^* a^+_{k|n} = X_k a^+_{j|n} - a^0_{k|n+1}a^+_{j|n} - (a^+_{k|n})^* a^+_{j|n} \\ &\quad\Leftrightarrow a^0_{k|n+1}a^+_{j|n} - a^0_{j|n+1}a^+_{k|n} = X_k a^+_{j|n} - X_j a^+_{k|n} + (a^+_{j|n})^* a^+_{k|n} - (a^+_{k|n})^* a^+_{j|n} \\ &\quad\Leftrightarrow a^0_{k|n+1}a^+_{j|n} - a^0_{j|n+1}a^+_{k|n} = X_k a^+_{j|n} - X_j a^+_{k|n} + 2i\mathrm{Im}(\tilde{\Omega}_{n+1}(e_j, e_k)). \end{aligned}$$

Using (6.7) this becomes

$$\begin{aligned} &a^0_{k|n+1}a^+_{j|n} - a^0_{j|n+1}a^+_{k|n} \\ &\qquad = X_k a^+_{j|n} - X_j a^+_{k|n} + 2i\mathrm{Im}(a^+_{k|n-1}(a^+_{j|n-1})^*) + 2i\mathrm{Im}(a^0_{k|n}a^0_{j|n}) \end{aligned}$$

which is equivalent to (6.14). □

Lemma 6.4. *The linear system in the unknowns* $(a^0_{k|n})$, *given by Eqs.* (6.13) *and* (6.14), *i.e.*

$$a^0_{k|n}a^+_{j|n-1} - a^0_{j|n}a^+_{k|n-1} = a^+_{j|n-1}a^0_{k|n-1} - a^+_{k|n-1}a^0_{j|n-1}, \tag{6.15}$$

$$\begin{aligned} a^0_{k|n}a^+_{j|n-1} - a^0_{j|n}a^+_{k|n-1} &= X_k a^+_{j|n-1} - X_j a^+_{k|n-1} + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) \\ &\quad + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1}) \end{aligned} \tag{6.16}$$

($j, k \in D$, $j < k$) is equivalent to the single linear system given by (6.15).

Proof. Since the left-hand sides of (6.15) and (6.16) are equal, the same must be true for the right-hand sides, therefore one must have

$$a^+_{j|n-1}a^0_{k|n-1} - a^+_{k|n-1}a^0_{j|n-1} = X_k a^+_{j|n-1} - X_j a^+_{k|n-1} + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1}). \tag{6.17}$$

Conversely, if (6.17) holds, then also the right-hand sides of (6.15) and (6.16) are equal, hence the system (6.15), (6.16) is equivalent to the single system (6.15).

Now notice that right-hand side of (6.17) is equal to

$$\begin{aligned}
&X_k a^+_{j|n-1} - X_j a^+_{k|n-1} + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1})\\
&\quad = X_k(X_{j|n-1} - a^0_{j|n-1} - (a^+_{j|n-2})^*) - X_j(X_{k|n-1} - a^0_{k|n-1} - (a^+_{k|n-2})^*)\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1})\\
&\quad = X_k X_{j|n-1} - X_k a^0_{j|n-1} - X_k(a^+_{j|n-2})^*\\
&\qquad - X_j X_{k|n-1} + X_j a^0_{k|n-1} + X_j(a^+_{k|n-2})^*\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1})\\
&\quad = X_j(a^+_{k|n-2})^* - X_k(a^+_{j|n-2})^* + X_j a^0_{k|n-1} - X_k a^0_{j|n-1}\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1}).
\end{aligned}$$

With similar arguments, the left-hand side of (6.17) is equal to

$$\begin{aligned}
&a^+_{j|n-1}a^0_{k|n-1} - a^+_{k|n-1}a^0_{j|n-1}\\
&\quad = (X_{j|n-1} - a^0_{j|n-1} - (a^+_{j|n-2})^*)a^0_{k|n-1}\\
&\qquad - (X_{k|n-1} - a^0_{k|n-1} - (a^+_{k|n-2})^*)a^0_{j|n-1}\\
&\quad = X_{j|n-1}a^0_{k|n-1} - a^0_{j|n-1}a^0_{k|n-1} - (a^+_{j|n-2})^* a^0_{k|n-1} - X_{k|n-1}a^0_{j|n-1}\\
&\qquad + a^0_{k|n-1}a^0_{j|n-1} + (a^+_{k|n-2})^* a^0_{j|n-1}\\
&\quad = X_{j|n-1}a^0_{k|n-1} - X_{k|n-1}a^0_{j|n-1} + a^0_{k|n-1}a^0_{j|n-1} - a^0_{j|n-1}a^0_{k|n-1}\\
&\qquad + (a^+_{k|n-2})^* a^0_{j|n-1} - (a^+_{j|n-2})^* a^0_{k|n-1}.
\end{aligned}$$

Therefore the identity (6.17) holds iff

$$\begin{aligned}
&X_{j|n-1}a^0_{k|n-1} - X_{k|n-1}a^0_{j|n-1} + a^0_{k|n-1}a^0_{j|n-1} - a^0_{j|n-1}a^0_{k|n-1} + (a^+_{k|n-2})^* a^0_{j|n-1}\\
&\quad - (a^+_{j|n-2})^* a^0_{k|n-1} = X_j(a^+_{k|n-2})^* - X_k(a^+_{j|n-2})^* + X_j a^0_{k|n-1} - X_k a^0_{j|n-1}\\
&\quad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1})
\end{aligned}$$

$$\Leftrightarrow +2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1}) + (a^+_{k|n-2})^* a^0_{j|n-1} - (a^+_{j|n-2})^* a^0_{k|n-1}$$
$$= X_j(a^+_{k|n-2})^* - X_k(a^+_{j|n-2})^* + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*) + 2i\mathrm{Im}(a^0_{k|n-1}a^0_{j|n-1}).$$

Thus, using the quantum decomposition, the identity (6.17) can be rewritten in the form

$$\begin{aligned}
&(a^+_{k|n-2})^* a^0_{j|n-1} - (a^+_{j|n-2})^* a^0_{k|n-1}\\
&\quad = X_j(a^+_{k|n-2})^* - X_k(a^+_{j|n-2})^* + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = X_{j|n-2}(a^+_{k|n-2})^* - X_{k|n-2}(a^+_{j|n-2})^* + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = (a^+_{j|n-2} + a^0_{j|n-2} + (a^+_{j|n-3})^*)(a^+_{k|n-2})^*\\
&\qquad - (a^+_{k|n-2} + a^0_{k|n-2} + (a^+_{k|n-3})^*)(a^+_{j|n-2})^* + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = a^+_{j|n-2}(a^+_{k|n-2})^* + a^0_{j|n-2}(a^+_{k|n-2})^* + (a^+_{j|n-3})^*(a^+_{k|n-2})^*\\
&\qquad - a^+_{k|n-2}(a^+_{j|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^* - (a^+_{k|n-3})^*(a^+_{j|n-2})^*\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = a^+_{j|n-2}(a^+_{k|n-2})^* + a^0_{j|n-2}(a^+_{k|n-2})^* - a^+_{k|n-2}(a^+_{j|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^*\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = a^+_{j|n-2}(a^+_{k|n-2})^* - a^+_{k|n-2}(a^+_{j|n-2})^* + a^0_{j|n-2}(a^+_{k|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^*\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = 2i\mathrm{Im}(a^+_{j|n-2}(a^+_{k|n-2})^*) + a^0_{j|n-2}(a^+_{k|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^*\\
&\qquad + 2i\mathrm{Im}(a^+_{k|n-2}(a^+_{j|n-2})^*)\\
&\quad = a^0_{j|n-2}(a^+_{k|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^*
\end{aligned}$$

or equivalently:

$$(a^+_{k|n-2})^* a^0_{j|n-1} - (a^+_{j|n-2})^* a^0_{k|n-1} = a^0_{j|n-2}(a^+_{k|n-2})^* - a^0_{k|n-2}(a^+_{j|n-2})^*. \tag{6.18}$$

Taking the adjoint of the identity

$$(a^+_k)^* a^0_j - (a^+_j)^* a^0_k = a^0_j(a^+_k)^* - a^0_k(a^+_j)^*$$

one finds

$$a^0_j a^+_k - a^0_k a^+_j = a^+_k a^0_j - a^+_j a^0_k.$$

Restricting to $\mathcal{P}_{n-1}$ one obtains

$$a^0_{j|n} a^+_{k|n-1} - a^0_{k|n} a^+_{j|n-1} = a^+_{k|n-1} a^0_{j|n-1} - a^+_{j|n-1} a^0_{k|n-1}$$

which gives the adjoint of (6.18). Since this is equivalent to (6.15), we conclude that the identity (6.17) holds if and only if (6.15) holds. This proves the statement. □

Lemma 6.5. *The inductive system of Eqs.* (6.13) *and* (6.14) *in the unknowns* $a^0_{j|n}$, *always admit the zero solution, given by the sequence*

$$a^0_{j|n} = 0, \quad \forall\, j \in D, \quad \forall\, n \in \mathbb{N}. \tag{6.19}$$

Proof. If $a^0_j = 0$, (6.13), i.e. (6.15) is identically satisfied. Therefore the result follows from Lemma 6.4. □

7. The Reconstruction Theorem

7.1. 3-*diagonal decompositions of* $\mathcal{P}$

The goal of this section is to abstract, from a given orthogonal gradation, a minimal set of characteristics that allow an inductive reconstruction of this gradation.

For two pre-Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we denote $\mathcal{L}_a(\mathcal{H}, \mathcal{K})$ the $*$-algebra of all adjointable linear operators from $\mathcal{H}$ to $\mathcal{K}$ (see Appendix A).

Recall that $(e_j)_{j\in D}$ is the canonical basis of $\mathbb{C}^d$ and that we use the notation

$$a^{\varepsilon}_{j|k} := a^{\varepsilon}_{e_j|k}, \quad j \in D, \quad \varepsilon \in \{+, 0, -\}.$$

Definition 7.1. For $n \in \mathbb{N}^*$, a 3-*diagonal decomposition of* $\mathcal{P}_{n]}$ is defined by:

(i) a vector space direct sum decomposition of $\mathcal{P}_{n]}$

$$\mathcal{P}_{k]} = \dot{\sum_{h\in\{0,\ldots,k\}}} \mathcal{P}_h, \quad \forall\, k \in \{0, 1, \ldots, n\}, \tag{7.1}$$

such that each $\mathcal{P}_h$ is monic of order h,

(ii) for each $k \in \{0, 1, \ldots, n\}$, a pre-scalar product $\langle\cdot,\cdot\rangle_k$ on $\mathcal{P}_k$, such that, denoting $\langle\cdot,\cdot\rangle_{n]}$ the unique scalar product on $\mathcal{P}_{n]}$ characterized by the conditions that the vector space decompositions (7.1) are orthogonal for the restriction of $\langle\cdot,\cdot\rangle_{n]}$ on each $\mathcal{P}_{k]}$:

$$\mathcal{P}_{k]} = \bigoplus_{h\in\{0,\ldots,k\}} \mathcal{P}_h, \quad \forall\, k \in \{0, 1, \ldots, n\}, \tag{7.2}$$

and for all $k \in \{0, 1, \ldots, n\}$

$$\langle\cdot,\cdot\rangle_{n]}\big|_{\mathcal{P}_k} = \langle\cdot,\cdot\rangle_k \tag{7.3}$$

the restrictions of the operators X_{e_j} on $\mathcal{P}_{n-1]}$ are symmetric:

$$\langle X_{e_j}\xi, \eta\rangle_{n]} = \langle \xi, X_{e_j}\eta\rangle_{n]}, \quad \xi, \eta \in \mathcal{P}_{n-1]}, \quad j \in D, \tag{7.4}$$

(iii) two families of pre-Hilbert space linear maps (see Appendix A for the notations)

$$a^+_{e_j|k} \in \mathcal{L}_a((\mathcal{P}_k, \langle\cdot,\cdot\rangle_k), (\mathcal{P}_{k+1}, \langle\cdot,\cdot\,\rangle_{k+1})), \quad k \in \{0, 1, \ldots, n-1\}, \tag{7.5}$$

$$a^0_{e_j|k} \in \mathcal{L}_a((\mathcal{P}_k, \langle\cdot,\cdot\rangle_k), (\mathcal{P}_k, \langle\cdot,\cdot\rangle_k)), \quad k \in \{0, 1, \ldots, n-1\}, \tag{7.6}$$

$j \in D$, such that:

(iii.1) for all $k \in \{1, \ldots, n-1\}$ and $j \in D$, $a^0_{e_j|k}$ is self-adjoint in the pre-Hilbert space sense;

(iii.2) the following identity is satisfied:

$$X_{e_j|k} = a^+_{e_j|k} + a^0_{e_j|k} + a^-_{e_j|k}, \quad k \in \{0, 1, \ldots, n-1\}, \quad j \in D, \tag{7.7}$$

with the convention that $a^+_{e_j|-1} = 0$, and

$$\begin{aligned} a^-_{e_j|k} := (a^+_{e_j|k-1})^* &: (\mathcal{P}_k, \langle\cdot,\cdot\rangle_k) \\ &\to (\mathcal{P}_{k-1}, \langle\cdot,\cdot\rangle_{k-1}), \quad k \in \{0, 1, \ldots, n-1\}, \end{aligned} \tag{7.8}$$

where $(a^+_{e_j|k-1})^*$ denotes, when no confusion is possible, the pre-Hilbert space adjoint of $a^+_{e_j|k-1}$.

(iii.3) The operators $a^\pm_{e_j|k}$, $a^0_{e_j|k}$ satisfy the commutation relations (6.6), (6.7), (6.15).

Remark. (1) In the following, if no confusion can arise, we will simply say that

$$\left\{(\mathcal{P}_k, \langle\cdot,\cdot\rangle_k)^n_{k=0}, (a^+_{\cdot\ |k})^{n-1}_{k=0}, (a^0_{\cdot\ |k})^{n-1}_{k=0}\right\} \tag{7.9}$$

is a 3-diagonal decomposition of $\mathcal{P}_{n]}$.

(2) Note that *a priori* all the objects defining a 3-diagonal decomposition of $\mathcal{P}_{n]}$ may depend on $n \in \mathbb{N}$.

Definition 7.2. (i) A 3-diagonal decomposition of $\mathcal{P}_{n+1]}$

$$\left\{(\mathcal{P}_k(n+1), \langle\cdot,\cdot\rangle_{n+1,k})^{n+1}_{k=0}, (a^+_{\cdot\ |k}(n+1))^n_{k=0}, (a^0_{\cdot|k}(n+1))^n_{k=0}\right\}$$

is called *an extension of a 3-diagonal decomposition* of $\mathcal{P}_{n]}$

$$\left\{(\mathcal{P}_k(n), \langle\cdot,\cdot\,\rangle_{n,k})^n_{k=0}, (a^+_{\cdot|k}(n))^{n-1}_{k=0}, (a^0_{\cdot|k}(n))^{n-1}_{k=0}\right\}$$

if, in obvious notations

$$\begin{aligned} \mathcal{P}_k(n) &= \mathcal{P}_k(n+1), \quad \forall\, k \in \{0, \ldots, n\}, \\ \langle\cdot,\cdot\rangle_{n+1]}\big|_{\mathcal{P}_{n]}} &= \langle\cdot,\cdot\,\rangle_{n]} \\ a^0_{\cdot|k}(n+1) &= a^0_{\cdot|k}(n), \quad \forall\, k \in \{0, \ldots, n\}, \\ a^+_{\cdot|k}(n+1) &= a^+_{\cdot|k}(n), \quad \forall\, k \in \{0, \ldots, n-1\}, \end{aligned}$$

(ii) A 3-*diagonal decomposition of* $\mathcal{P}$ is a sequence of 3-diagonal decompositions

$$D_n := \left\{ (\mathcal{P}_k(n), \langle\cdot,\cdot\rangle_{n,k})_{k=0}^{n}, (a^+_{\cdot|k}(n))_{k=0}^{n-1}, (a^0_{\cdot|k}(n))_{k=0}^{n-1} \right\}, \quad n \in \mathbb{N},$$

such that, for each $n \in \mathbb{N}$, D_{n+1} is an extension of D_n. In this case one simply writes

$$\{(\mathcal{P}_n, \langle\cdot,\cdot\rangle_n), a^+_{\cdot|n}, a^0_{\cdot|n}\}_{n\in\mathbb{N}}. \tag{7.10}$$

Remark. Any 3-diagonal decomposition of $\mathcal{P}_{n]}$ induces, by restriction, a 3-diagonal decomposition of $\mathcal{P}_{k]}$ for any $k \le n$.

The following theorem motivates the introduction of the notion of 3-diagonal decomposition given above.

Theorem 7.3. *Every state* φ *on* $\mathcal{P}$ *uniquely defines a* 3*-diagonal decomposition of* $\mathcal{P}$. *Conversely, given a* 3*-diagonal decomposition of* $\mathcal{P}$, *there exists a unique state* φ *on* $\mathcal{P}$ *such that the* 3*-diagonal decomposition of* $\mathcal{P}$, *associated to* φ *according to the first part of the theorem, is the given one.*

Proof. If the pre-scalar product on $\mathcal{P}$ is induced by a state φ on $\mathcal{P}$, then by Lemma 2.4 the operators of multiplication by the coordinates are symmetric for this pre-scalar product and the quantum decompositions of the random variables X_j $(i \in D)$ constructed in Sec. 4 provide a 3-diagonal decomposition of $\mathcal{P}$. The uniqueness of the quantum decomposition implies the uniqueness of the corresponding 3-diagonal decomposition of $\mathcal{P}$.

Conversely, let a 3-diagonal decomposition of $\mathcal{P}$ be given and denote $\langle\cdot,\cdot\rangle$ the pre-scalar product induced by it on $\mathcal{P}$. Then, by condition (ii) of Definition 7.1 and condition (ii) of Definition 7.2, for each $n \in \mathbb{N}$, the restriction of the operator X_{e_j} $(j \in D)$ on $\mathcal{P}_{n-1]}$ is symmetric with respect to the restriction of $\langle\cdot,\cdot\rangle$ on $\mathcal{P}_{n-1]}$. Since $\bigcup_{k\in\mathbb{N}} \mathcal{P}_{k]} = \mathcal{P}$, the operators X_{e_j} are $\langle\cdot,\cdot\rangle$-symmetric on $\mathcal{P}$.

Lemma 2.4 then implies that the pre-scalar product on $\mathcal{P}$ is induced by some state φ on $\mathcal{P}$ and this concludes the proof. □

7.2. *Structure of 3-diagonal decompositions of* $\mathcal{P}$

Having established the equivalence between 3-diagonal decomposition of $\mathcal{P}$ and orthogonal gradations induced by states on $\mathcal{P}$, our next goal is to produce a characterization of the 3-diagonal decomposition of $\mathcal{P}$. As a first step towards this goal in this section we discuss the following problem:
given a 3*-diagonal decomposition of* $\mathcal{P}_{n]}$, *classify all its possible extensions in the sense of Definition* 7.2.

Lemma 7.4. *Let, for* $n \in \mathbb{N}^*$,

$$\left\{ (\mathcal{P}_k, \langle\cdot,\cdot\rangle_k)_{k=0}^{n}, (a^+_{\cdot\ |k})_{k=0}^{n-1}, (a^0_{\cdot|k})_{k=0}^{n-1} \right\} \tag{7.11}$$

be a 3*-diagonal decomposition of* $\mathcal{P}_{n]}$ (*see* (7.9)). *Any* 3*-diagonal extension of* (7.11) *defines a pair*

$$(\tilde{\Omega}_{n+1}, a^0_{\cdot|n}) \tag{7.12}$$

with the following properties:

(i) $a^0_{\cdot|n}$ *is a linear map*

$$a^0_{\cdot|n} : v \in \mathbb{C}^d \mapsto a^0_{v|n} \in \mathcal{L}_a(\mathcal{P}_n, \langle\cdot,\cdot\rangle_n) \tag{7.13}$$

such that:

— *for all* $v \in \mathbb{R}^d$, $a^0_{v|n}$ *is a self-adjoint operator on the pre-Hilbert space* $(\mathcal{P}_n, \langle\cdot,\cdot\rangle_n)$;

(ii) *For each* $n \in \mathbb{N}$ *a* $\mathcal{L}_a((\mathcal{P}_n, \langle\cdot,\cdot\rangle_n)$*-valued positive definite kernel on* $\mathbb{C}^d$, *denoted* $\tilde{\Omega}_n$, *mapping real vectors onto real vectors and such that* $\tilde{\Omega}_0 \equiv 1$, $\tilde{\Omega}_1$ *is arbitrary and, for* $n > 1$ *the pair*

$$((\tilde{\Omega}_n)_{n\in\mathbb{N}}, (a^0_{e_j|n})_{j\in D})$$

is a solution of the joint system of inductive equations (6.6), (6.7), (6.13) *and* (6.14) *where the* $a^+_{j|n}$ *are defined by* (6.1) *and the* $(a^+_{j|n})^*$ *by the right-hand side of* (B.20).

Conversely any pair of the form (7.12), *satisfying conditions* (i) *and* (ii) *above, defines a* 3*-diagonal decomposition of* $\mathcal{P}$.

Proof. Definition 7.1 implies that any 3-diagonal decompositions of $\mathcal{P}_{n+1]}$ extending the given one determines a pair (7.12) with a self-adjoint operator $a^0_{\cdot|n}$ and with positive definite kernel $(\tilde{\Omega}_n(e_j, e_h))$ defined by

$$\tilde{\Omega}_{n+1}(e_j, e_h) := a^-_{e_j|n+1}a^+_{e_h|n} \in \mathcal{L}_a(\mathcal{P}_n, \langle\cdot, \ ,\cdot\ \rangle_n), \quad j, h \in D, \quad n \in \mathbb{N}.$$

Lemma 6.1 implies that the $\tilde{\Omega}_n$ satisfy conditions (6.6), (6.7); Lemma 6.2 implies that the $a^0_{j|n+1}$ satisfy condition (6.13); Lemma 6.3 implies that the $a^0_{j|n+1}$ satisfy condition (6.14). Therefore properties (i) and (ii) above are satisfied.

Conversely, given $n \in \mathbb{N}^*$, the 3-diagonal decomposition (7.11) of $\mathcal{P}_{n]}$, and a pair of the form (7.12), satisfying conditions (i) and (ii) above, define for each $j \in D$ the linear maps

$$a^+_{e_j|n} : \mathcal{P}_n \to \mathcal{P}_{n+1]} \tag{7.14}$$

by the condition

$$a^+_{e_j|n} := \left. X_j \right|_{\mathcal{P}_n} - a^0_{e_j|n} - (a^+_{e_j|n-1})^* \tag{7.15}$$

and let $\mathcal{P}_{n+1}$ be the vector space constructed in Lemma 4.8 with the choices

$$A^0_{e_j|n+1} := a^0_{e_j|n+1} \quad \text{and} \quad A^-_{e_j|n+1} := a^-_{e_j|n+1} = (a^+_{e_j|n})^*.$$

That $\mathcal{P}_{n+1}$ is a monic subspace of order $n+1$, of $\mathcal{P}_{n+1]}$ follows from Lemma 4.8. This proves that condition (i) of Definition 7.1 is satisfied.

Let $\langle\cdot,\cdot\rangle_{n+1}$ be the pre-scalar product on $\mathcal{P}_{n+1}$, induced by the positive definite kernel $(\tilde{\Omega}_{n+1}(e_j,e_h))$ through the identity:

$$\sum_{j,h\in D}\langle a^+_{e_j|n}\xi_j, a^+_{e_h|n}\eta_h\rangle_{n+1} := \sum_{j,h\in D}\langle \xi_j, \tilde{\Omega}_{n+1}(e_j,e_h)\eta_h\rangle_n, \quad \xi_j,\eta_h\in\mathcal{P}_n,$$

and let $\xi\in\mathcal{P}_n$ be a zero norm vector. Then for each $j\in D$ and $\xi\in\mathcal{P}_n$

$$\|a^+_{e_j|n}\xi\|^2_{n+1} = \langle a^+_{e_j|n}\xi, a^+_{e_jh|n}\xi\rangle_{n+1} = \langle\xi,\tilde{\Omega}_{n+1}(e_j,e_h)\xi\rangle_n = 0.$$

Thus the operators $a^+_{e_j|n}$ are pre-Hilbert space operators in the sense of Definition A.1.

Let us prove that for each $j\in\{1,\dots,d\}$, the restriction on $\mathcal{P}_{n]}$ of the multiplication operator by X_{e_j} is symmetric, i.e. that for each $\xi,\eta\in\mathcal{P}_{n]}$, one has

$$\langle X_{e_j}\xi,\eta\rangle_{n+1]} = \langle\xi, X_{e_j}\eta\rangle_{n+1]}. \tag{7.16}$$

From (7.15) we know that

$$a^+_{e_j|n} + a^0_{e_j|n} + (a^+_{e_j|n-1})^* = X_{e_j|n}, \tag{7.17}$$

where the restriction is meant in the sense of right multiplication by the projection onto $\mathcal{P}_n$, so that both sides are zero outside $\mathcal{P}_n$. This implies in particular that, for each $k\le n$

$$X_{e_j|k}:\mathcal{P}_k\to\mathcal{P}_{k+1}\oplus\mathcal{P}_k\oplus\mathcal{P}_{k-1}.$$

If both $\xi,\eta\in\mathcal{P}_{n-1]}$, then the identity (7.16) is reduced to the identity

$$\langle X_{e_j}\xi,\eta\rangle_{n]} = \langle\xi, X_{e_j}\eta\rangle_{n]}$$

which holds because (7.4) is a 3-diagonal decomposition of $\mathcal{P}_{n]}$.

Therefore it is sufficient to consider the case in which $\xi,\eta\in\mathcal{P}_n\oplus\mathcal{P}_{n-1}$.

By symmetry the problem is reduced to the two cases:

$$\eta\in\mathcal{P}_{n-1} \quad\text{and}\quad \xi\in\mathcal{P}_n$$
$$\eta\in\mathcal{P}_{n} \quad\text{and}\quad \xi\in\mathcal{P}_n.$$

Case 1. $\eta\in\mathcal{P}_{n-1};\ \xi\in\mathcal{P}_n$.

Using the mutual orthogonality of the spaces $\mathcal{P}_k$ for $k\le n+1$, one finds:

$$\begin{aligned}
\langle X_{e_j}\xi,\eta\rangle_{n+1]} &= \langle\xi, X_{e_j}\eta\rangle_{n+1]}\\
&\Leftrightarrow \langle(a^+_{e_j|n} + a^0_{e_j|n} + (a^+_{e_j|n-1})^*)\xi,\eta\rangle_{n+1]}\\
&= \langle\xi,(a^+_{e_j|n-1} + a^0_{e_j|n-1} + (a^+_{e_j|n-2})^*)\eta\rangle_{n+1]}\\
&\Leftrightarrow \langle a^+_{e_j|n}\xi,\eta\rangle_{n+1]} + \langle a^0_{e_j|n}\xi,\eta\rangle_{n+1]} + \langle (a^+_{e_j|n-1})^*\xi,\eta\rangle_{n+1]}\\
&= \langle\xi, a^+_{e_j|n-1}\eta\rangle_{n+1]} + \langle\xi, a^0_{e_j|n-1}\eta\rangle_{n+1]} + \langle\xi,(a^+_{e_j|n-2})^*\eta\rangle_{n+1]}\\
&\Leftrightarrow \langle (a^+_{e_j|n-1})^*\xi,\eta\rangle_{n-1} = \langle\xi, a^+_{e_j|n-1}\eta\rangle_n
\end{aligned}$$

that is identically satisfied because (7.4) is a 3-diagonal decomposition of $\mathcal{P}_{n]}$.

Case 2. $\eta \in \mathcal{P}_n;\ \xi \in \mathcal{P}_n$

$$\begin{aligned}\langle X_{e_j}\xi, \eta\rangle_{n+1]} &= \langle \xi, X_{e_j}\eta\rangle_{n+1]} \\ &\Leftrightarrow \langle (a^+_{e_j|n} + a^0_{e_j|n} + (a^+_{e_j|n-1})^*)\xi, \eta\rangle_{n+1]} \\ &= \langle \xi, (a^+_{e_j|n} + a^0_{e_j|n} + (a^+_{e_j|n-1})^*)\eta\rangle_{n+1]} \\ &\Leftrightarrow \langle a^+_{e_j|n}\xi, \eta\rangle_{n+1]} + \langle a^0_{e_j|n}\xi, \eta\rangle_{n+1]} + \langle (a^+_{e_j|n-1})^*\xi, \eta\rangle_{n+1]} \\ &= \langle \xi, a^+_{e_j|n}\eta\rangle_{n+1]} + \langle \xi, a^0_{e_j|n}\eta\rangle_{n+1]} + \langle \xi, (a^+_{e_j|n-1})^*\eta\rangle_{n+1]} \\ &\Leftrightarrow \langle a^0_{e_j|n}\xi, \eta\rangle_n = \langle \xi, a^0_{e_j|n}\eta\rangle_n\end{aligned}$$

that is identically satisfied because, by assumption, $a^0_{e_j|n}$ is self-adjoint for the $\langle\cdot,\cdot\rangle_n$-scalar product. Therefore the restriction on $\mathcal{P}_{n]}$, of the multiplication operator by X_{e_j} is symmetric, i.e. condition (ii) of Definition 7.1 is satisfied.

The linear maps $(a^0_{\cdot|n+1})$ are self-adjoint for the pre-scalar product $\langle\cdot,\cdot\rangle_{n+1}$ because of assumption (i). This is equivalent to condition (iii.1) of Definition 7.1.

(7.15) implies that condition (iii.2) of Definition 7.1 is satisfied.

Finally condition (iii.3) of the same definition is satisfied because of condition (ii).

In conclusion: for any choice of the pair (7.12), satisfying conditions (i) and (ii) above, the triple

$$\left\{(\mathcal{P}_k, \langle\cdot,\cdot\rangle_k)_{k=0}^{n+1}, (a^+_{\cdot|k})_{k=0}^{n-1}, (a^0_{\cdot|k})_{k=0}^{n-1}\right\}$$

is a 3-diagonal decomposition of $\mathcal{P}_{n+1]}$ extending the given one (7.11). This concludes the proof. □

8. The *d*-Dimensional Favard Lemma

We have seen that the d-dimensional analogue of the principal Jacobi sequence (ω_n) of a state on $\mathcal{P}$ is the sequence of positive definite kernels $(\tilde{\Omega}_n)$ and the d-dimensional analogue of the secondary Jacobi sequence (α_n) is the set of sequences of self-adjoint operators $(a^0_{j|n})$ $(j \in D)$ (in this section we often use the notation $a^\varepsilon_{j|n} = a^\varepsilon_{e_j|n}$ for $\varepsilon \in \{+, 0, -\}$, $j \in D$). In the 1-dimensional case, the (ω_n) have the only constraint $\omega_n = 0 \Rightarrow \omega_{n+k} = 0$, while the (α_n) are arbitrary real numbers. In the d-dimensional case we have seen in Sec. 6 that the commutation relations impose constraints both on the $(\tilde{\Omega}_n)$ and on the $(a^0_{j|n})$ $(j \in D)$. Fortunately, when written in inductive form, these constraints, turn out to be *linear*. In order to obtain the inductive formulation of the d-dimensional extension of Favard lemma we introduce the following definition, that expresses in a precise way the basic idea of these inductive relations, namely that: given the $a^+_{j|n-1}$ $(j \in D)$ and the scalar product on $\mathcal{P}_n$ one chooses the $a^0_{j|n}$, compatibly with the linear constraints and this uniquely defines the $a^+_{j|n}$. The choice of the $a^+_{j|n}$ uniquely defines the vector space $\mathcal{P}_{n+1}$ and, since the imaginary part of the kernel $(\tilde{\Omega}_{n+1})$ is uniquely determined by the constraints, its real part is only subject to the constraints of positive-definitness and of mapping real vectors of $\mathcal{P}_n$ into real vectors.

Definition 8.1. Given a linear basis (e_j) of $\mathbb{R}^d$, a *recursive* 3*-diagonal structure on* $\mathcal{P}$ *with respect to the basis* (e_j) is defined by the following procedure.

(i) Define the vector subspace with real structure

$$\mathcal{P}_0 := \mathbb{C} \cdot \Phi_0 \equiv (\mathbb{R} \oplus i\mathbb{R}) \cdot \Phi_0 =: \mathcal{P}_{R,0} \dot{+} i\mathcal{P}_{R,0}$$

and the scalar product $\langle \cdot, \cdot \rangle_0$ on it uniquely determined by the condition $\|\Phi_0\| := 1$.

(ii) For each $j \in D$, choose arbitarily a self-adjoint operator

$$a^0_{j|0} : (\mathcal{P}_0, \langle \cdot, \cdot \rangle_0) \to (\mathcal{P}_0, \langle \cdot, \cdot \rangle_0),$$

i.e. a real number $\tilde{a}^0_{j|0} \in \mathbb{R}$ characterized by $a^0_{j|0}\Phi_0 =: \tilde{a}^0_{j|0}\Phi_0$.

(iii) For each $j \in D$, define the linear operator $a^+_{j|0} : \mathcal{P}_0 \to \mathcal{P}_1$ by

$$a^+_{j|0} := X_j - a^0_{j|0}$$

and the vector spaces

$$\begin{aligned} \mathcal{P}_{R,1} &:= \mathbb{R}\text{-lin-span of } \{a^+_{j|0}\mathcal{P}_{R,0} : j \in D\} \\ &= \mathbb{R}\text{-lin-span of } \{X_j - a^0_{j|0}\Phi_0 : j \in D\} \end{aligned} \tag{8.1}$$

$$\mathcal{P}_1 := \mathbb{C}\text{-lin-span of } \{a^+_{j|0}\mathcal{P}_{R,0} : j \in D\} = \mathcal{P}_{R,1} + i\mathcal{P}_{R,1}. \tag{8.2}$$

(iv) Choose arbitrarily an $\mathcal{L}_a((\mathcal{P}_0, \langle \cdot, \cdot \rangle_0))$-valued positive definite kernel $\tilde{\Omega}_{R,1}$ on $\mathbb{C}^d \equiv \mathbb{R}^d \oplus i\mathbb{R}^d$ such that, for any $u, v \in \mathbb{R}^d$, $\tilde{\Omega}_{R,1}(u, v)$ maps real vectors of $\mathcal{P}_0$ into real vectors. Equivalently, choose arbitrarily a pre-scalar product on $\mathcal{P}_1$, real-valued on $\mathcal{P}_{R,1}$. Define $\tilde{\Omega}_1 := \tilde{\Omega}_{R,1}$ and the pre-scalar product $\langle \cdot, \cdot \rangle_1$ on $\mathcal{P}_1$, by

$$\langle a^+_{j|0}\Phi_0, a^+_{k|0}\Phi_0 \rangle_1 := \langle \Phi_0, \tilde{\Omega}_1(e_j, e_k)\Phi_0 \rangle_0.$$

(v) Having defined, for $1 \le k \le n$, the pre-Hilbert space $(\mathcal{P}_k, \langle \cdot, \cdot \rangle_k)$ with real structure $\mathcal{P}_k = \mathcal{P}_{R,k} + i\mathcal{P}_{R,k}$, the linear operators $a^+_{j|n-1} : \mathcal{P}_{n-1} \to \mathcal{P}_n$, and the self-adjoint operators $a^0_{j|n-1} : (\mathcal{P}_{R,n-1}, \langle \cdot, \cdot \rangle_{n-1}) \to (\mathcal{P}_{R,n-1}, \langle \cdot, \cdot \rangle_{n-1})$ $(j \in D)$, choose arbitrarily a self-adjoint solution $a^0_{j|n} : (\mathcal{P}_{R,n}, \langle \cdot, \cdot \rangle_n) \to (\mathcal{P}_{R,n}, \langle \cdot, \cdot \rangle_n)$ $(j \in D)$ of the linear system

$$a^0_{k|n}a^+_{j|n-1} - a^0_{j|n}a^+_{k|n-1} = a^+_{j|n-1}a^0_{k|n-1} - a^+_{k|n-1}a^0_{j|n-1} \tag{8.3}$$

for all $j, k \in D$ such that $j < k$ (such solutions exist by Lemma 6.5).

(vi) Define the linear operator

$$a^+_{j|n} := X_{j|n} - a^0_{j|n} - (a^+_{j|n-1})^* : \mathcal{P}_n \to \mathcal{P}$$

and the vector spaces

$$\mathcal{P}_{R,n+1} := \mathbb{R}\text{-lin-span of } \{a^+_{j|n}\mathcal{P}_{R,n} : j \in D\} \tag{8.4}$$

$$\mathcal{P}_{n+1} := \mathbb{C}\text{-lin-span of } \{a^+_{j|n}\mathcal{P}_{R,n} : j \in D\} = \mathcal{P}_{R,n+1} + i\mathcal{P}_{R,n+1}. \tag{8.5}$$

(vii) Choose *arbitrarily* an $\mathcal{L}_a((\mathcal{P}_n, \langle\cdot,\cdot\rangle_n))$-valued positive definite kernel $\tilde{\Omega}_{R,n+1}$ on $\mathbb{C}^d \equiv \mathbb{R}^d \oplus i\mathbb{R}^d$ such that, for any $u, v \in \mathbb{R}^d$, $\tilde{\Omega}_{R,n+1}(u,v)$ maps $\mathcal{P}_{R,n}$ into itself and define $\tilde{\Omega}_{n+1}(e_j, e_k)$ by:

$$\tilde{\Omega}_{n+1}(e_j, e_k) := \tilde{\Omega}_{R,n+1}(e_j, e_k) + \mathrm{Im}(a^+_{k|n-1}(a^+_{j|n-1})^*) + \mathrm{Im}((a^0_{k|n})a^0_{j|n}) \tag{8.6}$$

and the pre-scalar product $\langle\cdot,\cdot\rangle_{n+1}$ on $\mathcal{P}_{n+1}$ by:

$$\langle a^+_{j|0}\xi_n, a^+_{k|0}\eta_n\rangle_{n+1} := \langle \xi_n, \tilde{\Omega}_{n+1}(e_j, e_k)\eta_n\rangle_n, \quad \xi_n, \eta_n \in \mathcal{P}_n.$$

(viii) Having defined the pre-Hilbert space $(\mathcal{P}_{n+1}, \langle\cdot,\cdot\rangle_{n+1})$ with real structure $\mathcal{P}_{n+1} = \mathcal{P}_{R,n+1} + i\mathcal{P}_{R,n+1}$, the $a^+_{j|n}$ and the $a^0_{j|n}$ ($j \in D$), one can iterate the construction of item (v) above.

Theorem 8.2. (d-Dimensional Favard Lemma) *For any linear basis (e_j) of $\mathbb{R}^d$, there is a one-to-one correspondence between states on $\mathcal{P}$ and recursive 3-diagonal structures on $\mathcal{P}$ with respect to the basis (e_j).*

Remark. Since, by adding an arbitrary symplectic kernel to a positive definite kernel, the result is still a positive definite kernel, Eq. (8.6) does not introduce additional constraints on the $\tilde{\Omega}_{R,n+1}$.

Proof. *Necessity.* Let φ be a state on $\mathcal{P}$. From the results in Sec. 6, it follows that the 3-diagonal decomposition of $\mathcal{P}$ associated to the pair $(\mathcal{P}, \varphi)$ according to Theorem 7.3, defines a recursive 3-diagonal structure on $\mathcal{P}$ with respect to the basis (e_j).

Sufficiency. Given a recursive 3-diagonal structure on $\mathcal{P}$ with respect to the basis (e_j), denote

$$\left\{(\mathcal{P}_k, \langle\cdot,\cdot\rangle_k), (a^+_{\cdot\,|k}), (a^0_{\cdot|k})\right\}_{k\in\mathbb{N}} \tag{8.7}$$

the 3-diagonal decomposition of $\mathcal{P}$ associated to it, i.e.

$$a^+_{\cdot} := \sum_{n\in\mathbb{N}} a^+_{\cdot|n}, \quad a^0_{\cdot} := \sum_{n\in\mathbb{N}} a^0_{\cdot\ |n}, \quad a^-_{\cdot} := (a^+_{\cdot})^*.$$

Then the commutation relations (6.13) are satisfied because of (8.3) and Lemma 6.2.

The commutation relations (6.14) are satisfied because of (6.16), Lemma 6.4 and Lemma 6.3.

The commutation relations (6.7) are satisfied because of (8.6), Lemma 6.1 and the remark following it. Since the 3-diagonal decomposition (8.7) is uniquely defined by the recursive 3-diagonal structure, it follows that the same is true for the unique state on $\mathcal{P}$ defined by it according to Theorem 7.3. This proves the statement. □

Appendix A. Orthogonal Projectors and Adjoints on Pre-Hilbert Spaces

Definition A.1. We use the following terminology:

(1) A pre-scalar product on a vector space V is a non-identically zero positive definite Hermitian form on V.
(2) A scalar product on a vector space V is a non-degenerate pre-scalar product on V.
(3) A pre-Hilbert space is a vector space equipped with a pre-scalar product.
(4) A Hilbert space is a vector space equipped with a scalar product and complete with respect to the topology induced by it.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ and $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$, be two pre-Hilbert spaces. In the following, when no confusion is possible, we will omit the label from the two scalar products.
$\mathcal{L}_a(\mathcal{H},\mathcal{K})$ denotes the space of all adjointable linear operators from $\mathcal{H}$ to $\mathcal{K}$.
By definition, $A \in \mathcal{L}_a(\mathcal{H},\mathcal{K})$ if and only if:

— A is a linear operator operator everywhere defined on $\mathcal{H}$;
— A maps zero norm vectors of $\mathcal{H}$ into zero norm vectors of $\mathcal{K}$;
— there exists a linear operator $A^* : \mathcal{K} \to \mathcal{H}$ such that

$$\langle Ah, k\rangle_{\mathcal{K}} = \langle h, A^*k\rangle_{\mathcal{H}}, \quad \forall\, h \in \mathcal{H}, \quad \forall\, k \in \mathcal{K}.$$

In this case A^* is called an adjoint of A and A is called self-adjoint if $A = A^*$ for some choice of A^*.

Remark. If A^* and A^+ are two adjoints of A, then the range of the operator $A^+ - A^*$ is contained in the zero-norm subspace because

$$\langle (A^+ - A^*)k, h\rangle = \langle k, Ah\rangle - \langle k, Ah\rangle = 0, \quad \forall\, h \in \mathcal{H}, \quad \forall\, k \in \mathcal{K}.$$

Lemma A.2. *Let $\mathcal{H}$ be a pre-Hilbert space and let $\mathcal{K}$ be a finite dimensional subspace of $\mathcal{H}$. Denote $\mathcal{K}_0$ the subspace of the zero-norm vectors in $\mathcal{H}$.*

Then, for any choice of:

— *a linear complement $\mathcal{H}_1$ of $\mathcal{K}_0$ in $\mathcal{H}$,*
— *a linear complement $\mathcal{K}_1$ of $\mathcal{K} \cap \mathcal{K}_0$ in $\mathcal{K}$,*
— *a linear complement $\mathcal{K}_{0,1}$ of $\mathcal{K} \cap \mathcal{K}_0$ in $\mathcal{K}_0$,*

there exists a self-adjoint projections $P_{\mathcal{K}}$ from $\mathcal{H}$ onto $\mathcal{K}$.

If $\mathcal{H}_1'$, $\mathcal{K}_1'$, $\mathcal{K}_{0,1}'$ are other choices of the above-mentioned complements then, denoting $P_{\mathcal{K}}'$ the orthogonal projection onto $\mathcal{K}$, defined by the first part of the theorem, the range of $P_{\mathcal{K}} - P_{\mathcal{K}}'$ is contained in the zero norm subspace of $\mathcal{H}$.

If $\mathcal{K}_1$ has an orthogonal basis B and $\mathcal{H}$ a linear basis C such that the scalar products of elements of B with elements of C are real, then the projection $P_{\mathcal{K}}$ can *be chosen so that the real linear span of C is mapped onto the real linear span of B.*

Proof. The assumptions imply the decompositions

$$\mathcal{H} = (\mathcal{K}_0 \cap \mathcal{K}) \oplus \mathcal{K}_{0,1} \oplus \mathcal{H}_1, \quad \mathcal{K} = (\mathcal{K}_0 \cap \mathcal{K}) \oplus \mathcal{K}_1, \tag{A.1}$$

that are orthogonal because $\mathcal{K}_0$ is orthogonal to all vectors. Let $(k_j)_{j\in D_1}$, D_1 a finite set, be a linear basis of $\mathcal{K}_1$. Since by assumption $\mathcal{K}_0 \cap \mathcal{K}_1 = \{0\}$, the orthonormalization procedure can be applied to the set $(k_j)_{j\in D_1}$ leading to an orthonormal basis $(e_j)_{j\in D_1}$ of $\mathcal{K}_1$. Any vector $h \in \mathcal{H}$ can be written in a unique way as

$$h = h_1 + k_0 + k_{0,1} \quad \text{with } h_1 \in \mathcal{H}_1, \quad k_0 \in (\mathcal{K}_0 \cap \mathcal{K}), \quad k_{0,1} \in \mathcal{K}_{0,1}.$$

The linear map defined by

$$P_{\mathcal{K}}(h) := \sum_{j\in D_1} \langle e_j, h\rangle e_j + k_0 = \sum_{j\in D_1} \langle e_j, h_1\rangle e_j + k_0 \tag{A.2}$$

is clearly a pre-Hilbert space projection from $\mathcal{H}$ onto $\mathcal{K}$ and

$$\langle P_{\mathcal{K}}(h), h'\rangle = \sum_{j\in D_1} \overline{\langle e_j, h_1\rangle}\langle e_j, h'\rangle + \langle k_0, h'\rangle = \sum_{j\in D_1} \langle e_j, h_1\rangle\langle e_j, h'\rangle = \langle h, P_{\mathcal{K}}(h')\rangle.$$

Therefore $P_{\mathcal{K}}$ is self-adjoint. By inspection from (A.2) it follows that $P_{\mathcal{K}}$ does not depend on the choice of the orthonormal basis (e_j) of $\mathcal{K}_1$.

Let $\mathcal{H}'_1$, $\mathcal{K}'_1$, $\mathcal{K}'_{0,1}$ be as in the statement of theorem. Then any vector $h \in \mathcal{H}$ has two decompositions

$$h = h_1 + k_0 + k_{0,1} = h'_1 + k'_0 + k'_{0,1}, \quad h_1 \in \mathcal{H}, \quad k'_1 \in \mathcal{K}'_1, \quad k_0, k'_0 \in \mathcal{K}_0,$$
$$k_{0,1}, k'_{0,1} \in \mathcal{K}_{0,1}$$

hence h_1 differs from h'_1 by a zero-norm vector. A similar argument shows that, for each e_j in the basis (e_j) of $\mathcal{K}_1$, there exists $k_{0,j} \in \mathcal{K}_0 \cap \mathcal{K}$ and $e'_j \in \mathcal{K}'_1$ such that

$$e_j := e'_j + k_{0,j}.$$

The e'_j are clearly orthonormal and they are a basis of $\mathcal{K}'_1$ because it has the same (finite) dimension as $\mathcal{K}_1$. Moreover, one has

$$\begin{aligned} P_{\mathcal{K}}(h) - k_0 &= \sum_{j\in D_1} \langle e_j, h\rangle e_j = \sum_{j\in D_1} \langle e'_j + k_{0,j}, h_1\rangle (e'_j + k_{0,j}) \\ &= \sum_{j\in D_1} \langle e'_j, h\rangle\langle e'_j, h'\rangle e'_j + \sum_{j\in D_1} \langle e'_j, h\rangle k_{0,j} + k'_0 = P'_{\mathcal{K}}(h) \\ &\quad + \left(-k'_0 + \sum_{j\in D_1} \langle e'_j, h\rangle k_{0,j}\right) \end{aligned}$$

which shows that the range of $P_{\mathcal{K}} - P'_{\mathcal{K}}$ is contained in $\mathcal{K}_0$.

The last statement of the theorem is clear. □

Definition A.3. The projection $P_{\mathcal{K}_0}$, defined in Lemma A.2, will be called the orthogonal projection onto $\mathcal{K}_0$ associated to the decompositions (A.1).

Lemma A.4. *Let $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$, $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ be pre-Hilbert spaces, suppose that $\mathcal{H}$ is finite dimensional and let*

$$A : (\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}}) \to (\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$$

be a linear operator. Denote $\mathcal{H}_0$ (resp. $\mathcal{K}_0$) the zero-norm subspace of $\mathcal{H}$ (resp. $\mathcal{K}$). Suppose that A has the property that $A\mathcal{H}_0 \subseteq \mathcal{K}_0$. Then for any vector space complement $\mathcal{H}_1$ of $\mathcal{H}_0$ there exists an adjoint of A.

Proof. For any $k \in \mathcal{K}$, the map

$$h \in \mathcal{H} \mapsto \langle Ah, k\rangle_{\mathcal{K}} = \langle k, Ah\rangle_{\mathcal{K}}$$

is a linear functional on $\mathcal{H}$, therefore it defines an element of $\mathcal{H}^*$, the algebraic dual of $\mathcal{H}$, denoted $\hat{A}k$ and characterized by the property

$$\hat{A}k(h) = \langle k, Ah\rangle_{\mathcal{K}}. \tag{A.3}$$

By assumption

$$(\hat{A}k)(\mathcal{H}_0) = \{0\},$$

therefore $\hat{A}k$ induces a linear functional on $\mathcal{H}\backslash\mathcal{H}_0$.

Let $\mathcal{H}_1$ be a vector space complement of $\mathcal{H}_0$ so that $\mathcal{H} = \mathcal{H}_1 \dot{+} \mathcal{H}_0$. Then $\mathcal{H}_1$ is isomorphic to $\mathcal{H}\backslash\mathcal{H}_0$ as a linear space and, through this isomorphism, it becomes a Hilbert space, because $\mathcal{H}$ is finite dimensional. Therefore any linear functional f_1 on $\mathcal{H}_1$ is determined by an element of $h_1 \in \mathcal{H}_1$ through the identity

$$f_1(h_2) = \langle h_1, h_2\rangle_{\mathcal{H}_1}, \quad h_2 \in \mathcal{H}_1.$$

For any $k \in \mathcal{K}$, define A^*k the element of $\mathcal{H}_1$ corresponding to $\hat{A}k$ in $\mathcal{H}_1$. Then

$$\langle A^*k, h\rangle = \hat{A}k(h) = \langle k, Ah\rangle_{\mathcal{K}}. \tag{A.4}$$

Thus the linear operator $k \in \mathcal{K} \mapsto A^*k \in \mathcal{H}_1$ is an adjoint of A. This proves the statement. □

Definition A.5. In the notations and assumptions of Lemma A.4, the pre-Hilbert space linear operator A^* defined in Lemma A.4 is called the adjoint of A with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \dot{+} \mathcal{H}_0$.

Appendix B. Interacting Fock Spaces[6]

All constructions used in the following, like direct sums and tensor products, are algebraic. For any pair of pre-Hilbert spaces $(H, \langle\cdot,\cdot\rangle_H)$, $(K, \langle\cdot,\cdot\rangle_K)$, $\mathcal{L}_a((H, \langle\cdot,\cdot\rangle_H), (K, \langle\cdot,\cdot\rangle_K))$, or simply when no confusion is possible $\mathcal{L}_a(H, K)$, denotes the space of all adjointable pre-Hilbert space maps $A : H \to K$, such that there exists a linear map $A^* : K \to H$ satisfying

$$\langle f, Ag\rangle_K = \langle A^* f, g\rangle_H, \quad \forall g \in H, \quad \forall f \in K.$$

If $H = K$ $\mathcal{L}(K, \langle\cdot,\cdot\rangle_K)$ has a natural structure of $*$-algebra and we simply write $\mathcal{L}_a(K)$.

Definition B.1. Let V be a vector space. An *interacting Fock space on* V is a pair:

$$\left\{(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\right\} \tag{B.1}$$

such that:

— $(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}$ is a sequence of pre-Hilbert spaces with

$$H_0 =: \mathbb{C}\cdot\Phi_0, \quad \|\Phi_0\| = 1,$$

Φ_0 is called the *vacuum or Fock vector*;

— denoting $\langle\cdot,\cdot\rangle$ the unique pre-Hilbert space scalar product on the vector space direct sum of the family $(H_n)_{n\in\mathbb{N}}$ which makes this direct sum

$$H := \bigoplus_{n\in\mathbb{N}}(H_n, \langle\cdot,\cdot\rangle_n) \tag{B.2}$$

an orthogonal sum, the linear operator

$$a^+ : V \to \mathcal{L}_a((H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}})$$

satisfies the following conditions:

$$H_{n+1} = \text{lin-span}\{a^+(V)H_n\}, \quad \forall\, n\in\mathbb{N}. \tag{B.3}$$

For each $v\in V$, one fixes a choice of adjoint of $a^+(v)$ denoted by $a^-(v)$ (or simply a_v) so that

$$a(v)\Phi_0 = 0 \text{ Fock prescription}, \quad \forall\, v\in V. \tag{B.4}$$

The operators $a^+(v)$ $(f\in V)$ are called *creators* and their adjoints $a(v)$–*annihilators*. The spaces $(H_n)_{n\in\mathbb{N}}$ are called the *n-particle spaces*, if $n = 0$ one speaks of the vacuum space. If

$$\{(H_{1,n}, \langle\cdot,\cdot\rangle_{1,n})_{n\in\mathbb{N}}, a_1^*\}$$

is another IFS on a vector space V_1, a *morphism* from $\{(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\}$ to $\{(H_{1,n}, \langle\ \cdot\ ,\ \cdot\ \rangle_{1,n})_{n\in\mathbb{N}}, a_1^*\}$ is a linear map $U_1 : V\to V_1$ and a linear isometry

$$U : \bigoplus_{n\in\mathbb{N}}(H_n, \langle\cdot,\cdot\rangle_n) \to \bigoplus_{n\in\mathbb{N}}(H_{1,n}, \langle\cdot,\cdot\rangle_{1,n})$$

such that U is gradation preserving and

$$Ua_v^+U^* = a^*_{1,U_1v}, \quad \forall\, v\in V.$$

The pair (U_1, U) is an *isomorphism* if U_1 is invertible and U is onto up to vectors of norm zero.

Remark. For any $f\in V$, since the annihilator $a(f)$ is defined as the adjoint of the creator $a^+(f)$, its action on Φ_0 is not defined. However, *since the gradation* (B.2) *is* 1-*sided*, the only possible way to define it compatibly with the condition that $a(f) = (a^+(f))^*$, is to define

$$H_{-1} := \{0\} \tag{B.5}$$

or equivalently to introduce the Fock prescription (B.4).

Remark. Recall that, by definition of pre-Hilbert space linear map, each $a^+(f)$ $(f \in V)$ maps zero-norm vectors into zero-norm vectors. The existence of a pre-Hilbert space adjoint of $a^+(v)$ with respect to the pre-scalar product (B.16), which by definition must be defined on the whole space $H^{\otimes(n+1)}$, is equivalent to the condition that for any $\xi_{n+1} \in H^{\otimes(n+1)}$ the map

$$\eta_n \in (H^{\otimes n}, \langle\cdot,\cdot\rangle_n) \mapsto \langle \xi_{n+1}, a_v^+ \eta_n \rangle_{n+1}$$

can be extended to a continuous linear functional on the domain of $a^+(v)$, which by definition is the whole algebraic tensor product $H^{\otimes n}$. In the case of Hilbert spaces this happens if and only if there are constants $c_{\xi_{n+1},v}$ such that

$$|\langle \xi_{n+1}, v \otimes \eta_n \rangle_{n+1}| \le c_{\xi_{n+1},v} \|\eta_n\|_n \tag{B.6}$$

but in the infinite dimensional case the condition that the whole algebraic tensor product $H^{\otimes n}$ is in the domain of the adjoint, is not automatically guaranteed.

B.1. *Example*: *The full Fock space*

The *full Fock space* $\mathcal{F}(V)$ on a pre-Hilbert space $(V, \langle\cdot,\cdot\rangle_V)$ is obtained by setting $H_n = V^{\otimes n}$ equipped with natural inner product given by the n-fold tensor product:

$$\langle f_n \otimes \cdots \otimes f_1, g_n \otimes \cdots \otimes g_1 \rangle_{\otimes n} := \langle f_n, g_n \rangle_V \langle f_{n-1}, g_{n-1} \rangle_V \cdots \langle f_1, g_1 \rangle_V \tag{B.7}$$

$f_n, \dots, f_1, g_n, \dots, g_1 \in V$. Creators on the full Fock space are denoted by $\ell^*(f)$ $(f \in V)$ and their action on each H_n is defined by setting

$$\ell^*(f) f_n \otimes \cdots \otimes f_1 := f \otimes f_n \otimes \cdots \otimes f_1 \tag{B.8}$$

$$\ell_f^* \Phi_0 := \ell^*(f)\Phi_0 = f.$$

The adjoint of $\ell(f)$, with respect to the pre-scalar product (B.7), is:

$$\ell(f) f_n \otimes \cdots \otimes f_1 = \langle f, f_n \rangle f_{n-1} \otimes \cdots \otimes f_1,$$

$$\ell(f)\Phi_0 = 0.$$

B.2. *The tensor representation of an IFS*

Lemma B.2. *Every IFS*

$$\{(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\} \tag{B.9}$$

on a vector space V *is isomorphic, in the sense of Definition* B.1, *to an IFS of the form*

$$\{(V^{\otimes n}, \langle\cdot,\cdot\rangle_{\otimes,n}), \ell^*\}, \tag{B.10}$$

where the pre-scalar products $\langle\cdot,\cdot\rangle_{\otimes,n}$ *are given by*

$$\begin{aligned}&\langle u_n \otimes \cdots \otimes u_1, v_n \otimes \cdots \otimes v_1\rangle_{\otimes,n}\\&\quad := \langle a^+(u_n)\cdots a^+(u_1)\Phi_0, a^+(v_n)\cdots a^+(v_1)\Phi_0\rangle_n\end{aligned} \tag{B.11}$$

$(u_n, v_n, \ldots, u_1, v_1 \in V)$ *and the operator* ℓ^* *is defined, in the notation* (B.8), *by*

$$T^{-1}a^+(v)T = \ell^*(v), \quad \forall\, v \in V. \tag{B.12}$$

Proof. By the universal property of the tensor product, for each $n \in \mathbb{N}$, the map

$$v \otimes h_n \in V \otimes H_n \to a^+(v)h_n \in H_{n+1} \tag{B.13}$$

has a unique linear extension denoted $T_{n,n+1} : V \otimes H_n \to H_{n+1}$.

One easily verifies that the left-hand side of (B.3) is a vector space.

Iterating the maps (B.13), one sees that the linear extensions of the maps

$$T_n : v_n \otimes \cdots \otimes v_1 \in V^{\otimes n} \to a^+(v_n)\cdots a^+(v_1)\Phi_0 \in H_n \tag{B.14}$$

$(n \in \mathbb{N})$ are well-defined and define a graded vector space homomorphism

$$T := \bigoplus_n T_n : \mathrm{Tens}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} \to \bigoplus_{n\in\mathbb{N}} H_n \tag{B.15}$$

which, by construction, satisfies (B.12).

Defining the pre-scalar products $\langle\cdot,\cdot\rangle_{\otimes,n}$ by (B.11), the maps T_n become pre-Hilbert space unitary isomorphisms, hence T an IFS isomorphism. This defines the IFS (B.10). □

Definition B.3. The isomorphic realization (B.10), of the IFS on V given by (B.1), is called the *tensor representation* of the IFS (B.1).

B.3. *Standard interacting Fock spaces*

Definition B.4. An IFS $\{(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\}$ *on a pre-Hilbert space* $(H, \langle\cdot,\cdot\rangle_H)$ is called *standard* if, in its tensor representation (B.10) (with $V = H$), the pre-scalar products have the form

$$\langle \cdot, \cdot \rangle_{\otimes,n} = \langle \cdot, \Omega_n \cdot \rangle_{H^{\otimes n}}, \tag{B.16}$$

where, for $f_j, g_j \in H$ $(j = 1, \ldots, n)$

$$\langle f_n \otimes \cdots \otimes f_1, g_n \otimes \cdots \otimes g_1 \rangle_{H^{\otimes n}} := \langle f_n, g_n \rangle_H \langle f_{n-1}, g_{n-1} \rangle_H \cdots \langle f_1, g_1 \rangle_H \tag{B.17}$$

is the natural scalar product on $H^{\otimes n}$ and

$$\Omega_n : H^{\otimes n} \to H^{\otimes n}$$

is a positive linear operator.

Remark. If H is finite dimensional, then every IFS on H is standard.

B.4. *Interacting Fock space and positive definite operator-valued kernels*

The existence of the creation and annihilation operators poses some restrictions on the sequence of scalar products defining an IFS. To describe this restrictions we introduce the following definition.

Definition B.5. Let S be a set and B a $*$-algebra. A map $\Omega : S \times S \to B$ is called a *B-valued positive definite kernel on S* if, for any finite subset $F \subseteq S$ and any map $b : F \to B$, one has

$$\sum_{s,t \in F} b_s^* \Omega_{s,t} b_t \geq 0$$

Ω is called *linear* if S is a vector space and the map $(s,t) \in S \times S \mapsto \Omega_{s,t} \in B$ is sesquilinear. If

$$B := \mathcal{L}_a((H, \langle \cdot, \cdot \rangle))$$

is the $*$-algebra of adjointable operators on a pre-Hilbert space $(H, \langle \cdot, \cdot \rangle)$, we simply speak of a *positive definite linear kernel on S based on* $(H, \langle \cdot, \cdot \rangle)$

Remark. Any B-valued positive definite kernel on S defines a linear kernel on the free vector space V_S generated by S. Conversely, if V is a vector space, a B-valued positive definite linear kernel on V is uniquely determined by its values on a Hamel basis of V.

Remark. From now on we restrict our attention to the case of interest for the present paper, namely that in which all IFS are based on finite dimensional vector spaces.

For a discussion of the general case we refer to Ref. 7 where the notion of positive definite kernel with values in a $*$-algebra was introduced.

Lemma B.6. *Let the following be given*:

— *a finite dimensional pre-Hilbert space* $(\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})$;
— *two finite-dimensional vector spaces* W, V;

— *an* $\mathcal{L}_a(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$*-valued PD Kernel* $\tilde{\Omega}$ *on* V*;*
— *a linear map* $a^+ : V \to \mathcal{L}_a((\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}}), W)$ *such that*

$$\text{lin-span}(a_V^+\mathcal{K}) = W.$$

Then there exists a unique pre-scalar product $\langle\cdot,\cdot\rangle_W$ *on* W *such that*

$$\langle a_u^+\xi, a_v^+\eta\rangle_W = \langle\xi, \tilde{\Omega}(u,v)\eta\rangle_{\mathcal{K}}, \quad \forall\, u,v \in V, \xi,\eta \in \mathcal{K}. \tag{B.18}$$

Moreover, the adjoint of a_u^+*, denoted* $(a_u^+)^* : (W, \langle\cdot,\cdot\rangle_W) \to (\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ *satisfies*

$$\tilde{\Omega}(u,v) = (a_u^+)^* a_v^+. \tag{B.19}$$

In particular, the action of $(a_u^+)^*$ *on* W *is given, up to addition of vectors of zero norm, by the identity*

$$(a_u^+)^*\Phi = \sum_{j\in D} \tilde{\Omega}(u, e_j)\xi_j, \quad \Phi = \sum_{j\in D} a_{e_j}^+\xi_j. \tag{B.20}$$

Proof. Let $\mathcal{K}_0 \subseteq \mathcal{K}$ and $\mathcal{H}_1 \subseteq W$ be subspaces as in Lemma A.2 with $\mathcal{H}$ replaced by W. Let $e \equiv (e_j)_{j\in D}$ be a linear basis of V and $(P_{\mathcal{K},k})_{k\in D_{\mathcal{K}}}$ an orthonormal basis of $\mathcal{K}$. The set

$$\{a_{e_j}^+ P_{\mathcal{K},k} : j \in D, k \in D_{\mathcal{K}}\}$$

is a system of generators of W. Therefore there exist sets

$$D_0 \subseteq D, \quad D_{0\mathcal{K}} \subseteq D_{\mathcal{K}},$$

such that the set

$$\{a_{e_j}^+ \Phi_k : j \in D_0, k \in D_{0,\mathcal{K}}\} \tag{B.21}$$

is a linear basis of W. Define $\forall\, j, j' \in D_0$, $\forall\, k, k' \in D_{0,\mathcal{K}}$

$$\langle a_{e_j}^+\Phi_k, a_{j'}^+\Phi_{k'}\rangle_W := \langle\Phi_k, \tilde{\Omega}(e_j, e_{j'})\Phi_{k'}\rangle_{\mathcal{K}}. \tag{B.22}$$

Then there exists a unique pre-scalar product $\langle\cdot,\cdot\rangle_W$ on W such that its restriction on the linear basis (B.21) is given by (B.22). By sesquilinearity $\langle\cdot,\cdot\rangle_W$ satisfies (B.18).

We know from Lemma (A.4) that the map a_u^+ is adjointable and is a pre-Hilbert space operator. Moreover, any adjoint of a_n^+ satisfies

$$(a_n^+)^*\Phi = \sum_{j\in F} (a_n^+)^* a_{e_j}^+(\xi_j), \quad \Phi = \sum_{j\in F} a_{e_j}^+(\xi_j).$$

By definition of $\tilde{\Omega}$ this implies that, for any $\Psi, \Phi \in \mathcal{K}$ and any $u, v \in V$, one has

$$\langle\Psi, \tilde{\Omega}(u,v)\Phi\rangle_W = \langle a_u^+\Psi, a_v^+\Phi\rangle_W = \langle\Psi, (a_u^+)^* a_v^+\Phi\rangle_W.$$

This implies that the identity (B.19) is satisfied up to addition of a zero-norm vector. But we know that any vector $\Phi \in a_V^+\mathcal{K}$ has the form $\Phi = \sum_{j\in D} a_{e_j}^+\xi_j$ for some vectors $\xi_j \in \mathcal{K}$. Therefore up to addition of a zero-norm vector

$$(a_n^+)^*\Phi = \sum_{j\in D}(a_u^+)^* a_{e_j}^+\xi_j = \sum_{j\in D}\tilde{\Omega}(u,e_j)\xi_j$$

and this proves (B.20). □

Remark. Every IFS on a vector space V

$$\{(H_n, \langle\cdot,\cdot\rangle_n)_{n\in\mathbb{N}}, a^+\} \tag{B.23}$$

defines a sequence $(\tilde{\Omega}_n)$ with the following properties:

$$\tilde{\Omega}_0 \equiv 1 \tag{B.24}$$

is the constant kernel equal to 1 on the Hilbert space

$$(H_0, \langle\cdot,\cdot\rangle_0) := (\mathbb{C}, \langle z,w\rangle_0 := \bar{z}w\ (z,w\in\mathbb{C})). \tag{B.25}$$

For $n\in\mathbb{N}$, $\tilde{\Omega}_{n+1}$ is the $\mathcal{L}_a((H_n,\langle\cdot,\cdot\rangle_n))$-valued *linear kernel* on V defined by

$$\tilde{\Omega}_{n+1}(u,v) := a(u)a^+(v)\big|_{H_n} \in \mathcal{L}_a((H_n,\langle\cdot,\cdot\rangle_n)), \quad u,v\in V. \tag{B.26}$$

Because of (B.3), the positive definite kernel $\tilde{\Omega}_{n+1}$ uniquely determines the pre-scalar product $\langle\cdot,\cdot\rangle_{n+1}$ through the identity

$$\begin{aligned}\langle a^+(u)h_n, a^+(v)h_n'\rangle_{n+1} &= \langle h_n, a(u)a^+(v)h_n'\rangle_n\\ &=: \langle h_n, \tilde{\Omega}_{n+1}(u,v)h_n'\rangle_n \quad (u,v\in V,\ h_n,h_n'\in H_n).\end{aligned} \tag{B.27}$$

Remark. The converse of this statement is most conveniently formulated using the tensor representation of the IFS (B.23) and its proof is based on the following result.

Lemma B.7. *Let V be a finite dimensional vector space.*

(i) *Any pair of pre-scalar products $\langle\cdot,\cdot\rangle_{V^{\otimes(n+1)}}$, $\langle\cdot,\cdot\rangle_{V^{\otimes n}}$, on $V^{\otimes(n+1)}$, $V^{\otimes n}$ respectively, defines, through the prescription*

$$\langle u\otimes h_n, v\otimes h_n'\rangle_{V^{\otimes(n+1)}} = \langle h_n, \tilde{\Omega}^{\otimes}_{n+1}(u,v)h_n'\rangle_{V^{\otimes n}} \tag{B.28}$$

$(u,v\in V,\ h_n,h_n'\in V^{\otimes n})$ *an $\mathcal{L}_a((V^{\otimes n},\langle\cdot,\cdot\rangle_{V^{\otimes n}}))$-valued PD kernel $\tilde{\Omega}^{\otimes}_{n+1}$ on V such that*

$$\tilde{\Omega}^{\otimes}_{n+1}(u,v) = \ell_{n+1}(u)\ell_n^*(v), \tag{B.29}$$

where $\ell^*(u)$ is the restriction on H_n of the operator defined by (B.8) *and $\ell_{n+1}(u)$ denotes the adjoint of the pre-Hilbert space linear map*

$$\ell_n^*(u) := \ell^*(u)\big|_{V^{\otimes n}} : (V^{\otimes n},\langle\cdot,\cdot\rangle_{V^{\otimes n}}) \to (V^{\otimes(n+1)},\langle\cdot,\cdot\rangle_{V^{\otimes(n+1)}}). \tag{B.30}$$

(ii) *Conversely, any pair* $(\tilde{\Omega}^{\otimes}_{n+1}, \langle\cdot,\cdot\rangle_{V^{\otimes n}})$, *where* $\langle\cdot,\cdot\rangle_{V^{\otimes n}}$ *is a pre-scalar product on* $V^{\otimes n}$ *and* $\tilde{\Omega}^{\otimes}_{n+1}$ *is a* $\mathcal{L}_a((H_n, \langle\cdot,\cdot\rangle_n))$*-valued PD kernel on* V *defines, by the prescription* (B.28), *a pre-scalar product* $\langle\cdot,\cdot\rangle_{V^{\otimes(n+1)}}$ *on* $V^{\otimes(n+1)}$ *satisfying* (B.29).

Proof. (i) Given $\langle\cdot,\cdot\rangle_{V^{\otimes(n+1)}}$, for each $u, v \in V$ the map

$$(h_n, h'_n) \in V^{\otimes n} \times V^{\otimes n} \mapsto \langle u \otimes h_n, v \otimes h'_n\rangle_{V^{\otimes(n+1)}} \tag{B.31}$$

is sesquilinear. Since $V^{\otimes n}$ is finite dimensional, the map (B.31) defines, for each $u, v \in V$ a linear map

$$\tilde{\Omega}^{\otimes}_{n+1}(u, v) : H_n \to H_n$$

that, by construction, satisfies (B.28) and is adjointable because of finite dimensionality. Again by finite dimensionality the map (B.30) is adjointable and satisfies (B.29).

(ii) Conversely, given $\tilde{\Omega}^{\otimes}_{n+1}$, define $\langle\cdot,\cdot\rangle_{V\otimes H_n}$ by the right-hand side of (B.28). By definition of $\mathcal{L}_a((H_n, \langle\cdot,\cdot\rangle_n))$-valued PD kern $\Omega^{\otimes}_{n+1}$ on V, this gives a pre-scalar product on $V \otimes H_n$. The same argument as in the proof of (i) shows that the map (B.30) is adjointable and satisfies (B.28) and therefore (B.29). This proves (ii).

□

Theorem B.8. *Let* $(H_n, \langle\cdot,\cdot\rangle_n)$ *be an IFS on a finite dimensional vector space* V *and let*

$$\{(V^{\otimes n}, \langle\cdot,\cdot\,\rangle_{\otimes,n}), \ell^*\} \tag{B.32}$$

be its tensor representation defined by Lemma B.2. *Then the sequence of pre-scalar products* $(\langle\cdot,\cdot\rangle_{\otimes,n})$ *is uniquely defined by a sequence* $(\tilde{\Omega}^{\otimes}_n)$ *with the follwing properties*:

$$\tilde{\Omega}^{\otimes}_0 \equiv 1 \tag{B.33}$$

is the constant kernel on V, *identically equal to* 1, *based on the Hilbert space*

$$(H_0, \langle\cdot,\cdot\rangle_0) := (\mathbb{C}, \langle z, w\rangle_0 := \bar{z}w \ (z, w \in \mathbb{C})) \tag{B.34}$$

and $\tilde{\Omega}^{\otimes}_{n+1}$ *is the* $\mathcal{L}_a((V^{\otimes n}, \langle\cdot,\cdot\rangle_{\otimes n}))$*-valued PD kernel on* V *defined by* (B.29).

Conversely, let the sequence $(\tilde{\Omega}^{\otimes}_n)$ be inductively defined as follows: $\tilde{\Omega}^{\otimes}_0$ and $(H_0, \langle\cdot,\cdot\rangle_0)$ are defined respectively by (B.33) and (B.34).

Having defined, for $0 \le m \le n$, the pre-scalar product $\langle\cdot,\cdot\rangle_{\otimes,m}$ on $V^{\otimes m}$, $\tilde{\Omega}^{\otimes}_{n+1}$ is an arbitrary $\mathcal{L}_a(V^{\otimes n}, \langle\cdot,\cdot\rangle_{\otimes,n})$-valued kernel on V.

Then, with ℓ^* defined by (B.8), $((V^{\otimes n}, \langle\cdot,\cdot\rangle_{\otimes,n})_{n\in\mathbb{N}}, \ell^*)$ is an IFS on V.

Proof. Applying the Remark after Definition B.5 to the tensor representation of $(H_n, \langle\cdot,\cdot\rangle_n)$, one obtains the required sequence $(\tilde{\Omega}^{\otimes}_n)$.

Conversely, if the sequence $(\tilde{\Omega}^{\otimes}_n)$ is defined as in the second part of the theorem then, according to Lemma B.7, the pair $(\tilde{\Omega}^{\otimes}_{n+1}, \langle\cdot,\cdot\,\rangle_{\otimes,n})$ defines, by the prescription (B.28), a pre-scalar product $\langle\cdot,\cdot\rangle_{V^{\otimes(n+1)}}$ on $V^{\otimes(n+1)}$ satisfying (B.29). □

B.5. *Symmetric interacting Fock spaces*

Definition B.9. An IFS on a vector space V is called *symmetric*, if the creators commute.

The following lemma shows that, in the tensor representation of a symmetric IFS, the tensor algebra can be replaced by the symmetric tensor algebra.

Lemma B.10. *Every symmetric IFS*

$$\{(H_n, \langle\cdot,\cdot\,\rangle_n)_{n\in\mathbb{N}}, a^+\} \tag{B.35}$$

on a vector space V is isomorphic, in the sense of Definition B.1 *to an IFS of the form*

$$\{(V^{\widehat{\otimes}n}, \langle\cdot,\cdot\rangle_{\widehat{\otimes},n}), \hat{\ell}^*\}, \tag{B.36}$$

where

— *for all $n \in \mathbb{N}$, $V^{\widehat{\otimes}n}$ denotes the nth symmetric algebraic tensor power of V and by definition*

$$V^{\widehat{\otimes}0} := \mathbb{C}\cdot\Phi, \quad \langle\Phi,\Phi\rangle_0 = 1, \tag{B.37}$$

— *the isomorphism is given by the unique linear extension of the map*

$$\hat{T}(u_n\widehat{\otimes}\cdots\widehat{\otimes}u_1) := a^+(u_n)\cdots a^+(u_1)\Phi, \quad n\in\mathbb{N}, \quad u_j\in V, \tag{B.38}$$

— *the pre-scalar products $\langle\cdot,\cdot\rangle_{\widehat{\otimes},n}$ are given, for any $n\in\mathbb{N}$ and $u_1, v_1, \ldots, u_n, v_n \in V$, by*

$$\begin{aligned}&\langle u_n\widehat{\otimes}\cdots\widehat{\otimes}u_1, v_n\widehat{\otimes}\cdots\widehat{\otimes}v_1\rangle_{\otimes,n}\\ &\qquad := \langle a^+(u_n)\cdots a^+(u_1)\Phi, a^+(v_n)\cdots a^+(v_1)\Phi\rangle_n,\end{aligned} \tag{B.39}$$

— *the operator $\hat{\ell}^*$ defined by*

$$\hat{\ell}^*(v)(u_n\widehat{\otimes}\cdots\widehat{\otimes}u_1) := v\widehat{\otimes}u_n\widehat{\otimes}\cdots\widehat{\otimes}u_1, \quad \forall\, v, u_n, \ldots, u_1 \in V, \tag{B.40}$$

up to addition of zero-norm vectors satisfies

$$\hat{T}^{-1}a^+(v)\hat{T} = \hat{\ell}^*(v), \quad \forall\, v\in V. \tag{B.41}$$

Proof. The mutual commutativity of the creators implies that, in the notations of Lemma B.2, the maps T_n $(n\in\mathbb{N})$ satisfy

$$T_n(v_n\otimes\cdots\otimes v_1) = T_n(v_n\widehat{\otimes}\cdots\widehat{\otimes}v_1) =: \hat{T}_n(v_n\widehat{\otimes}\cdots\widehat{\otimes}v_1), \quad n\in\mathbb{N}. \tag{B.42}$$

This shows that the graded vector space homomorphism T, defined by (B.15) can be restricted to the symmetric tensor algebra $\hat{T}ens(V)$ thus defining the graded vector space homomorphism

$$\hat{T} := \bigoplus_n \hat{T}_n : \hat{T}ens(V) = \bigoplus_{n\in\mathbb{N}} V^{\widehat{\otimes}n} \to \bigoplus_{n\in\mathbb{N}} H_n, \tag{B.43}$$

where $\hat{T}_n$ is given by (B.38). In this restriction the pre-scalar products defined by (B.11) become (B.39) and the condition that the spaces $V^{\widehat{\otimes}n}$ are mutually orthogonal uniquely defines the pre-scalar product on $\hat{T}ens(V)$. With this scalar product T becomes a unitary gradation preserving isomorphism. Therefore, with $\ell^*_{v_n}$ given by (B.40), the identity (B.39) can be rewritten in the form

$$\begin{aligned}
&\langle u_n \widehat{\otimes} \cdots \widehat{\otimes} u_1, \widehat{\ell}^*_{v_n} v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1 \rangle_{\widehat{\otimes},n} \\
&\quad = \langle a^+(u_n) \cdots a^+(u_1)\Phi, a^+(v_n) a^+(v_{n-1}) \cdots a^+(v_1)\Phi \rangle_n \\
&\quad = \langle T(u_n \widehat{\otimes} \cdots \widehat{\otimes} u_1), T(T^{-1} a^+(v_n) T) v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1 \rangle_n \\
&\quad = \langle u_n \widehat{\otimes} \cdots \widehat{\otimes} u_1, (T^{-1} a^+(v_n) T) v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1 \rangle_{\widehat{\otimes},n}.
\end{aligned}$$

Therefore

$$\widehat{\ell}^*_{v_n} v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1 - (T a^+(v_n) T^{-1}) T v_{n-1} \widehat{\otimes} \cdots \widehat{\otimes} v_1$$

is a zero-norm vector and this proves (B.41).

The unitarity of T and (B.41) imply the adjointability of the maps $\ell^*(v)$ $(v \in V)$ because the maps $a^+(v)$ admit pre-Hilbert space adjoints by definition. Therefore, with this definition $\hat{T}$ becomes an isomorphism of IFS. □

Theorem B.11. *Every symmetric IFS* $(H_n, \langle \cdot, \cdot \rangle_n)$ *on a finite dimensional vector space* V *uniquely defines a sequence* $(\tilde{\Omega}_n^{\widehat{\otimes}})$ *with the following properties*:

$$\tilde{\Omega}_0^{\widehat{\otimes}} \equiv 1 \tag{B.44}$$

is the constant kernel on V, *identically equal to* 1, *on the Hilbert space*

$$(H_0, \langle \cdot, \cdot \rangle_0) := (\mathbb{C}, \langle z, w \rangle_0 := \bar{z} w \ (z, w \in \mathbb{C})) \tag{B.45}$$

and $\tilde{\Omega}_{n+1}^{\widehat{\otimes}}$ *is the* $\mathcal{L}_a((V^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{\widehat{\otimes} n}))$*-valued PD kernel on* V *defined by* (B.29)

$$\langle u \widehat{\otimes} h_n, v \widehat{\otimes} h'_n \rangle_{V^{\widehat{\otimes}(n+1)}} = \langle h_n, \tilde{\Omega}_{n+1}^{\widehat{\otimes}}(u, v) h'_n \rangle_{V^{\widehat{\otimes} n}} \tag{B.46}$$

$(u, v \in V,\ h_n, h'_n \in V^{\widehat{\otimes} n})$, *where* $\langle \cdot, \cdot \rangle_{\widehat{\otimes} n}$ *is the pre-scalar product induced on* $V^{\widehat{\otimes} n}$ *by the symmetric tensor representation of* $(H_n, \langle \cdot, \cdot \rangle_n)$.

Conversely, let the sequence $(\tilde{\Omega}_n^{\widehat{\otimes}})$ be inductively defined as follows: $\tilde{\Omega}_0^{\widehat{\otimes}}$ and $(H_0, \langle \cdot, \cdot \rangle_0)$ are defined respectively by (B.44) and (B.45). Having defined, for $0 \le m \le n$, the pre-scalar product $\langle \cdot, \cdot \rangle_{\widehat{\otimes},m}$ on $V^{\widehat{\otimes} n}$, $\tilde{\Omega}_{n+1}^{\widehat{\otimes}}$ is an arbitrary $\mathcal{L}_a(V^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{\widehat{\otimes} m})$-valued kernel on V. Then $((V^{\widehat{\otimes} n}, \langle \cdot, \cdot \rangle_{\widehat{\otimes},n}), \ell^*)$, where $\hat{\ell}^*$ is defined by

$$\ell^*(f) f_n \widehat{\otimes} \cdots \widehat{\otimes} f_1 := f \widehat{\otimes} f_n \widehat{\otimes} \cdots \widehat{\otimes} f_1 \tag{B.47}$$

is a symmetric IFS on V.

Proof. The proof is based on the remark that Lemma B.7 and Theorem B.8 continue to hold for symmetric tensor products and their proofs are just verbal adjustments of those in the nonsymmetric case. □

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