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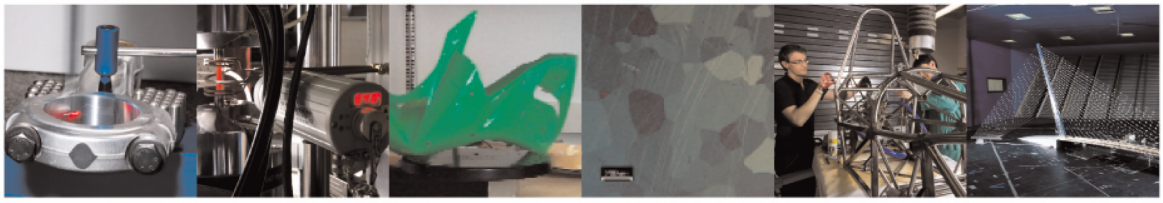

Global reliability sensitivity analysis by sobol-based dynamic adaptive kriging importance sampling

F. Cadini;S. Lombardo;M. Giglio

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Structural Safety

GLOBAL RELIABILITY SENSITIVITY ANALYSIS BY SOBOL-BASED DYNAMIC ADAPTIVE KRIGING IMPORTANCE SAMPLING

--Manuscript Draft--

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Abstract:	The stochastic uncertainties affecting the models used to describe the behavior of structural/mechanical systems may give rise to unfavorable scenarios leading to failures. In this framework, the quantification of the failure probability is a recognized fundamental task for structural safety and reliability analyses. Unfortunately, the estimation of the failure probability of structural/mechanical systems is a computationally demanding task, especially when the failure is a rare event and the computer codes used to model the system response require large computational efforts. One major issue further complicates the estimation process, i.e., the parameters of the probability distributions of the random variables used to describe the uncertainties involved can, in turn, be imprecise, since they are typically estimated by means of statistical inference based on observations and engineering judgment. In this context, reliability sensitivity analysis aims at estimating the influence of this additional source of uncertainty on the system failure probability in order to assess the robustness of the system to the modeling of uncertainties. Intuitively, reliability sensitivity analyses may easily become prohibitive by standard sampling-based methods (e.g., Monte Carlo method), since a nested, second level of uncertainties is involved. To overcome this issue, in this work we embed the efficient AK-IS algorithm for estimating small failure probabilities within an original computational framework that allows to perform a Sobol-based, global sensitivity analysis of the failure probability at an affordable number of computer model evaluations. The algorithm is demonstrated with reference to two case studies of literature of structural/mechanical reliability, often used in the literature as benchmark tests.

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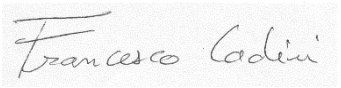
To whom it may concern

Milano, October 2019

The context of the research work presented in this manuscript is that of reliability sensitivity analysis of structural/mechanical systems. It is well known that these kinds of analyses may provide a significant support to any reliability-based design process. However, these tasks are, in general, prohibitive from a computational point of view, due to the complexity of the models used to verify the structures and to the large number of model evaluations required to satisfactorily estimate the (small) failure probabilities involved and to perform global sensitivity analysis. In order to overcome this problem, some researcher have focused on the estimation of the partial derivatives of the failure probability with respect to the model uncertain parameters. Yet, sensitivity analyses based on the estimation of partial derivatives are local in nature, i.e., focused on small variations of the inputs around some nominal values and taking each parameter one at a time. The main original contribution of the present work is then the development of a new algorithm (SDAKIS – Sobol-based Dynamic AKIS) for performing a more complete global reliability sensitivity analysis based on the estimation of the Sobol indexes. The algorithm exploits the efficiency of the AKIS by embedding it within an original computational scheme to provide estimates of the computationally demanding Sobol indexes at an affordable number of computer model evaluations. The main novelties of this work are related to i) the sequential re-use of the surrogate model DOE at each run of the AKIS in correspondence of different realizations of the uncertain distribution parameters and ii) the exploitation of the DOE adaptive refinement scheme of the AKIS for limiting the number of performance function evaluations required for each estimate of the failure probability. The algorithm is demonstrated with reference to two benchmark case studies of literature typically used in structural/mechanical reliability analysis studies. The limited computational power available for this research has suggested to restrict the demonstration of the algorithms (in particular the SDAKIS) to these case studies, which, on the other hand, still offer the possibility to compare and validate the results with respect to crude Monte Carlo simulations, otherwise impossible with more complex applications.

With best regards,



Francesco Cadini

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To: Prof. Bruce R. Ellingwood

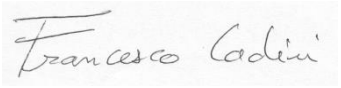
Milano, May 29, 2020

Dear Prof. Ellingwood,

Thank you very much for your email and for sending us the reviewers' reports. We have proceeded to a careful consideration of the comments by the reviewers and have accordingly modified the paper, adding the requested modifications (highlighted in yellow).

Again, we appreciate your collaborative efforts,

With best regards,



Francesco Cadini

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REVIEWER 1

1. "It is not obvious that kriging (a surrogate) is useful when dealing with extreme events, see J. Li and D. Xiu. Evaluation of failure probability via surrogate models. Journal of Computational Physics, pages 89668980, 2010".

Author Reply: We agree with the Reviewer that Kriging, as many other surrogate models, has its own limitations, as also already highlighted in the manuscript, in particular when dealing with very irregular limit states, or discontinuous and ill-behaved performance functions. At the same time, it is our belief that, when the performance functions (and, consequently, the associated limit states) are more regular, which often seems to be the case in real applications, Kriging has been shown to be a robust approximation method, with the additional capability of naturally providing some quantification of the approximation error. This last property, in particular, has been exploited in most works of literature, including the present one, to further refine the surrogate model performances. Finally, additional reasons motivating the use of Kriging-based surrogate models are that (i) they have been shown in several works to significantly reduce the overall number of performance function evaluations, including those required to build the surrogate, which is a fundamental issue especially in the context of global reliability sensitivity analysis, and (ii) they can be quite straightforwardly implemented in computer codes.
To reinforce this idea, the following sentences have been added to the Introduction:

"Indeed kriging, as many other surrogate models, has its own limitations, as epitomized in [19], in particular when dealing with very irregular limit states, or discontinuous and ill-behaved performance functions. At the same time, when the performance functions are more regular, which often seems to be the case in real applications, kriging has been shown to be a robust approximation method, with the additional capability of naturally providing some quantification of the approximation error. This last property, in particular, has been exploited in most works of literature, including the present one, to further refine the surrogate model performances. The popularity of kriging-based surrogate models for structural reliability also owes to the facts that (i) they have been shown to significantly reduce the overall number of performance function evaluations, including those required to build the surrogate, which is a fundamental issue especially in the context of global reliability sensitivity analysis, and (ii) they can be quite straightforwardly implemented in computer codes [7][8][11][12][15][16][17]".

2. "The assumptions that (1) the functional form of the distributions of the input random variables is known and (2) the mapping $X \rightarrow U = T(X)$ is available are rather restrictive. The uncertainty in the model form uncertainty may dominate the parametric uncertainty. The unavailability of analytical form of the mapping $X \rightarrow U = T(X)$ is likely to result in notable errors due requisite approximations. It seems that the examples are initiated in the standard Gaussian space".

Author Reply: We understand the Reviewer's concerns. However, these two assumptions are the common starting point of many works of literature focused on the development of the algorithms for structural reliability, especially those relying on a FORM-based preliminary stage (see, for example, [Echard, B.; Gayton, N.; Lemaire, M.; Relun, N. (2013): A combined Importance Sampling and Kriging reliability method for small failure probabilities with time-demanding numerical models. In Reliability Engineering & System Safety 111, pp. 232–240. DOI: 10.1016/j.ress.2012.10.008.]), which is also the objective of our research. Delving into the foundation of those two assumptions would, in our opinion, require an additional paper by itself, and would be out the scope of the present work. Valuable work addressing these issues can be found, for example, in [[38] V. Dubourg, V. (2011): Adaptive surrogate models for reliability analysis and

reliability-based design optimization, Ph.D. Thesis, Université Blaise Pascal - Clermont-Ferrand II] and [[6] Der Kiureghian, A. (2008): Analysis of structural reliability under parameter uncertainties. In Probabilistic Engineering Mechanics 23 (4), pp. 351–358. DOI: 10.1016/j.probengmech.2007.10.011], to which our manuscript already refers to.
Finally, note that none of the case studies is directly set up in the standard space: the input variable uncertainties are assumed to be described by independent Gaussian probability density functions, but not standard.

3. "Finally, there are many moving parts in the proposed algorithm and there is no proof that the algorithm converges to the target probability under some conditions".

Author Reply: We apologize, but it is not clear what the Reviewer meant by "moving" parts in the algorithm. As for the second statement, the following statements have been added in the Conclusions:

"Note that there is no mathematical proof that the algorithm unbiasedly converges to the target Sobol indices. On the other hand, following a procedure adopted in many works of literature dealing with the same topic, we have provided a successful comparison of the results with those obtained with a brute-force Monte Carlo scheme, which is proven to yield unbiased estimates of the quantities of interest. We acknowledge that such "empirical" comparisons do not have the strength of a mathematical proof and, as such, do not guarantee that some ill-posed conditions may exist under which convergence is not achieved. However, we believe the proposed comparison should serve as a basis to build confidence within the scientific community about the applicability of the approach to the cases most likely encountered in the practice".

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REVIEWER 2

1. "The authors are using the expression "crude MC." This expression is misleading as it implies that crude MC has crude accuracy. A better expression is "brute force MC."".

Author Reply: The expression "crude MC" has been replaced by "brute force MC" all throughout the manuscript.

2. "In the academic example, in Table 2, is there a typographical error or is there a one order of magnitude error in the estimation of the probability of error? If this is indeed an error, why is it so high?"

Author Reply: It was indeed a typo, which has been fixed.

3. "The Ncall value of 262350 for brute force MC is significantly larger that 10+67 for AK-IS, but the computations can still be performed for brute force MC without any real problem (by the way, this applies for both examples). How are these numbers translating to a problem with tens/hundreds of random variables where MC will start having real problems? Where is MC becoming prohibitive/impossible with current computational capabilities, and where is AK-IS?".

Author Reply: Indeed, as stated in the manuscript, the two examples have been chosen so as to be able to compare the results with those obtained with a brute force MC on a normal personal computer. By the way, note that the number of performance functions the Reviewer is pointing at refer to a *single* failure probability estimation, and not to the whole global sensitivity analysis (~$9 \cdot 10^{11}$ evaluations were required in the work [31]).
In order to answer to the two questions, let us first distinguish between the "curse of dimensionality", related to the number of uncertain model inputs, and the computational capabilities, limiting structural reliability analyses.
The estimation "error" of brute force MC schemes does not depend on the dimensionality of the input space, but only on the number of performance function evaluations. More accurate and precise estimates require higher numbers of evaluations and, consequently, higher computational times, which almost linearly increase with the number of model runs. The limitations of brute force MC schemes are thus usually only associated to the computational time required by a single performance function evaluation, and not to the dimensionality of the input space. Note in this regard that modern cloud computation can guarantee very high speeds and parallelization capabilities, which can be effectively exploited by brute force MC schemes, leading to unthinkable possibilities only a decade ago. However, if the performance function to be evaluated involves the solution of a complex structural finite element model taking, say, a few hours to run, then we believe that a few hundred evaluations might be the limit above which no economic advantages can be gained. Unfortunately, for very rare events, as shown in the paper, this number of evaluations is still not enough.
The AK-IS-based methods, on the other hand, might offer significant savings in terms of model evaluation, as shown in this and many other works of literature, but they do not come for free. The major price to be paid is in terms of the dimensionality of the input space. In fact, both the performances of kriging-based meta-modeling (actually of meta-modeling in general) and the choice of a good importance density are significantly affected by the number of input uncertain variables. In the literature, except for some "toy" case

studies, there are very few realistic applications of this kind of algorithms involving some tens of uncertain inputs.
No modifications have been made to the manuscript since we believe these concepts are already summarized in the Conclusions.

4. "Both numerical examples provided involve a very small number of random variables. How is the methodology performing when the number of random variables increases into the tens and hundreds? But maybe the most important question is the following: how can we estimate the accuracy of the proposed methodology when it's not possible anymore to compare with corresponding brute force MC results?".

Author Reply: This issue has been partially addressed in the answer to the previous question by the same Reviewer. Probably a few tens of uncertain inputs can still be manageable.
As far as the accuracy is concerned, the following sentences have been added in the Introduction:

"Finally, note that the proposed SDAKIS method requires that a threshold δ_{th} be set for the maximum coefficient of variation for each estimate of the failure probability required by the global sensitivity analysis (δ_{th}= 20% in the present work). This means that each of the required probabilities is estimated to an accuracy of, at least, 20%. The coefficient of variations actually obtained during the analyses in both case studies were usually much lower, i.e., approximately 5%. Consequently, even if no brute force MC reference is available to assess the reliability of the estimates, the accuracy of the method can be a priori bounded. However, care should be taken in setting this threshold, since, as seen in the results, too low values may require too many AK-IS iterations for the calculation of each failure probability, eventually leading to not affordable computational times".

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REVIEWER 3

1. "In this context, the authors might want to address the "curse of dimensionality", and in particular how their approach could be (adversely) affected by the implementation of dimensionality reduction methods such as the PCA. It also would be helpful if they could provide a simple but size-scalable example which shows the growth of the computational burden with increasing dimensionality".

Author Reply: Several authors working in the field of structural reliability have acknowledged, and are still recognizing, that high dimensionality input space problems are probably the bottleneck limiting the application of most of the recent, advanced methods relying on surrogate modeling (and, to some extent, also on importance sampling) to real industrial applications. As stated in the Conclusions, it was not the aim of this work to address this issue. Besides, the context of structural reliability *sensitivity analysis* is even much less ideal for studying the effects of high dimensionality, due to the very heavy computational load. In this regard, our suggestion and our research objective in the near future, is to tackle the problem in a "simpler" context of single failure probability estimation (no distribution parameter uncertainties), such as, for example, that of the AK-IS method. In this framework, it will be easier to study the effects of the dimensionality on size-scalable simple examples, as already done in some previous works of literature. However , we are personally somewhat skeptical on the effectiveness of those examples (the typical one considers a performance function $G(x_1, \ldots, x_n) = (n+3\sigma n^{0.5})-\Sigma x_i$, where x_i, $i = 1, \ldots, n$, are the uncertain input variables), since the algorithms seem to still perform quite well also for a high number of input variables, but the performance function has a really too simple mathematical form (linear). We believe that many more problems would arise for 50-200 uncertain input variables when using much more complex (like FEM) models.
Finally, further investigation on the problem in these last months has led us conclude that PCA is not readily suitable for the input space dimensionality reduction within the context of structural reliability, unless the inputs are significantly correlated, which might not be the case in practical situations. Instead, the literature has already proposed methods for dimension reduction to exploit some low-dimensional approximation of the function of the quantity of interest (i.e., in the present case, the failure probability), such as sensitivity analysis, sparse grids, analysis of variance (ANOVA), separation of variables or polynomial chaos expansions, etc. Yet, in the context of structural reliability, a trade off should be found between the additional number of performance function evaluations required to perform dimensionality reduction and the advantages offered by a low-dimensional problem. In this respect, the sentence mentioning the PCA-based strategy in the Conclusions has been removed and substituted with the following:

"In this regard, the literature has already proposed methods for dimension reduction to exploit some low-dimensional approximation of the model, such as sensitivity analysis, sparse grids, analysis of variance (ANOVA), separation of variables, polynomial chaos expansions, etc. Yet, in the context of structural reliability, a trade off should be found between the additional number of performance function evaluations required to perform dimensionality reduction and the advantages offered by a low-dimensional problem. This is currently an open topic of debate within the structural reliability scientific community".

GLOBAL RELIABILITY SENSITIVITY ANALYSIS BY SOBOL-BASED DYNAMIC ADAPTIVE KRIGING IMPORTANCE SAMPLING

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Highlights

- We tackle global sensitivity analysis within the structural reliability context
- We propose a new AKIS-based method for global variance-based sensitivity analysis
- AKIS is used dynamically to estimate the Sobol indexes
- The adaptive refinement of the Kriging DoE is key to reduction of model evaluations
- The efficiency of both methods is shown on benchmark case studies of literature

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GLOBAL RELIABILITY SENSITIVITY ANALYSIS BY SOBOL-BASED DYNAMIC ADAPTIVE KRIGING IMPORTANCE SAMPLING

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ABSTRACT

The stochastic uncertainties affecting the models used to describe the behavior of structural/mechanical systems may give rise to unfavorable scenarios leading to failures. In this framework, the quantification of the failure probability is a recognized fundamental task for structural safety and reliability analyses. Unfortunately, the estimation of the failure probability of structural/mechanical systems is a computationally demanding task, especially when the failure is a rare event and the computer codes used to model the system response require large computational efforts. One major issue further complicates the estimation process, i.e., the parameters of the probability distributions of the random variables used to describe the uncertainties involved can, in turn, be imprecise, since they are typically estimated by means of statistical inference based on observations and engineering judgment. In this context, reliability sensitivity analysis aims at estimating the influence of this additional source of uncertainty on the system failure probability in order to assess the robustness of the system to the modeling of uncertainties. Intuitively, reliability sensitivity analyses may easily become prohibitive by standard sampling-based methods (e.g., Monte Carlo method), since a nested, second level of uncertainties is involved. To overcome this issue, in this work we embed the efficient AK-IS algorithm for estimating small failure probabilities within an original computational framework that allows to perform a Sobol-based, global sensitivity analysis of the failure probability at an affordable number of computer model evaluations. The algorithm is demonstrated with reference to two case studies of literature of structural/mechanical reliability, often used in the literature as benchmark tests.

Keywords: reliability sensitivity analysis; adaptive kriging; importance sampling; Sobol indexes.

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1. INTRODUCTION

The economical sustainability and the safety levels of many complex structural engineering applications significantly rely on reliability and risk analyses, which, in turn, are motivated by the presence of several types of uncertainties affecting the material properties, the geometries, the operating/boundary conditions, etc. In general, uncertainties in the design and/or the operation of an engineering system are due to different degrees of lack of knowledge, as, for example, not fully understood physical behaviors, measurement errors or scarce operating experience to characterize it, manufacturing errors/defects, errors and approximations in the mathematical models and the computer codes used to represent its dynamics, unknown or not very well known boundary/initial conditions, unpredictable external events affecting the system performances, etc.

One fundamental issue when dealing with these uncertainties is that, in practical applications, the behavior of a system (a structure, a component, a plant, etc.) may be different from the nominal, or design, one, thus potentially leading to undesired, or even failure, conditions. The evaluation of the effects of uncertainties on the system performances (uncertainty analysis) thus plays an important role for the safe and economically sound management of engineering structures and plants. In order to do so, typically, uncertain quantities (or model inputs) are taken to be stochastic (random) variables, described by properly identified probability distributions (probability density functions – pdfs, or probability mass distributions – pmds) accounting for their possible variability. As a consequence, also the outputs of the models that we use to represent the behavior of the systems under analysis become random variables, described by probability distributions which depend on both the distributions of the model inputs and the model structure. In this framework, it is then possible to quantify the effects of the uncertainties by estimating the probability of occurrence of some undesired events, such as the failure of a structure or of a mechanical component, usually defined as some functional (performance indicator) of the model outcomes overcoming some predefined safety thresholds [1][2][3][4].

Formally, consider a stochastic model whose system response is described by a performance function $G(\boldsymbol{X})$, where the input $\boldsymbol{X} = \{X_1, \dots, X_n\}$ is a n-dimensional random vector with joint probability density function (pdf) $\phi_X(\boldsymbol{x})$. According to [5] and [6], the random variables $\{X_1, \dots, X_n\}$ are also called basic variables, since they are typically assumed to be observable, so that empirical data can be obtained and used for building a statistics of the uncertain input vector $\boldsymbol{X}$. Examples of possible basic variables in structural/mechanical reliability problems include material properties, load features, environmental boundary conditions and geometric properties. On the other hand, the performance function $G(\boldsymbol{X})$ is a derived variable, usually not directly observable, unless specific tests

are carried out; reliability criteria are often set with respect to these derived variables, such as, for example, stresses, strains, displacements, etc. Thus, in general, a failure random event can be defined as:

$$F = \{G(\boldsymbol{X}) \leq 0\} \tag{1}$$

where the set of values $\boldsymbol{X}: G(\boldsymbol{X}) \leq 0$ and $\boldsymbol{X}: G(\boldsymbol{X}) = 0$ are called "failure domain" and "limit-state function", respectively.

By introducing the following failure indicator function:

$$1_F(\boldsymbol{x}) = \begin{cases} 1 & G(\boldsymbol{X}) \leq 0 \\ 0 & G(\boldsymbol{X}) > 0 \end{cases} \tag{2}$$

the failure probability can, then, be written as:

$$p_f = P[G(\boldsymbol{X}) \leq 0] = \int_F \phi_X(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\mathbb{R}^n} 1_F(\boldsymbol{x})\, \phi_X(\boldsymbol{x})\, d\boldsymbol{x} \tag{3}$$

One usual approach adopted for estimating the failure probability p_f in (3) is that of resorting to standard, or brute force, Monte Carlo (MC) simulation by i) sampling N_a values $\boldsymbol{x}^{(i)}$, $i = 1, \cdots, N_a$, of the model input random vector $\boldsymbol{X}$ from $\phi_X(\boldsymbol{x})$ and ii) evaluating the performance function $G(\boldsymbol{X})$, (usually a computer numerical code) in correspondence of the N_a samples of $\boldsymbol{X}$. The failure probability can, then, be estimated by dividing the number of realizations for which $G(\boldsymbol{X}) \leq 0$ by N_a, yielding:

$$\widehat{p_f}_{MCS} = \frac{1}{N_a} \sum_{i=1}^{N_a} 1_F\left(\boldsymbol{x}^{(i)}\right) \tag{4}$$

However, modern structures are very reliable, so that a failure rarely occurs: the estimation of such probabilities would thus require a large number of performance function simulations. This problem is even worse when the computer codes implementing $G(\boldsymbol{X})$ are computationally very expensive, such as, for example, those based on complex finite elements models (FEMs) [2][7][8]. In these cases, the standard MC schemes would require prohibitive computational times.

Various methods have been proposed in the literature to address this problem: the interested

reader may refer to [9][10] for thorough reviews and comparisons of many existing methods. Recently, a general strategy has gained popularity in the scientific community, which amounts to seeking for proper combinations of some "optimization" algorithms (e.g., FORM, genetic algorithm, etc.) with MC variance reduction techniques (e.g., importance sampling, subset sampling, line sampling, etc.) and meta-modeling, i.e., the use of fast surrogate models (e.g., neural networks, kriging, support vector machines, etc.) for approximating the original, computationally expensive, model responses [2][7][11][12][13][14]. In this work, for example, we shall refer to the Adaptive Kriging-Importance Sampling (AK-IS) algorithm, originally proposed by [12], which, by combining kriging meta-modeling to importance sampling, has shown to be a very efficient and flexible method in many different applications, including engineering fields also different from the structural one [15][16][17][18]. Indeed kriging, as many other surrogate models, has its own limitations, as epitomized in [19], in particular when dealing with very irregular limit states, or discontinuous and ill-behaved performance functions. At the same time, when the performance functions are more regular, which often seems to be the case in real applications, kriging has been shown to be a robust approximation method, with the additional capability of naturally providing some quantification of the approximation error. This last property, in particular, has been exploited in most works of literature, including the present one, to further refine the surrogate model performances. The popularity of kriging-based surrogate models for structural reliability also owes to the facts that (i) they have been shown to significantly reduce the overall number of performance function evaluations, including those required to build the surrogate, which is a fundamental issue especially in the context of global reliability sensitivity analysis, and (ii) they can be quite straightforwardly implemented in computer codes [7][8][11][12][15][16][17].

The basic idea of the AK-IS is to assess the failure probability by classifying the elements of a set of samples drawn from a suitable "importance" pdf into some safe ($G(\boldsymbol{X}) > 0$) and failure ($G(\boldsymbol{X}) \leq 0$) groups. More specifically, a kriging meta-model (Gaussian process), trained by a properly chosen set of N_s input/output examples $\left\{\left(\boldsymbol{x}^{(i)}, G\left(\boldsymbol{x}^{(i)}\right)\right); i = 1, \dots, N_s\right\}$ (broadly referred to as Design of Experiment (DoE)), is used to predict the outcomes of $G(\boldsymbol{X} = \boldsymbol{x}^*)$ in correspondence of a new input $\boldsymbol{x}^*$.

Unfortunately, regardless the method used to estimate the failure probability p_f, the robustness of the results significantly depends on an additional type of uncertainty, sometimes referred to in the literature as "epistemic", as opposed to the "aleatory" uncertainty described by the basic random variables $\boldsymbol{X}$. In fact, as effectively described in [6] with a specific focus on structural reliability, or, more generally, in [20][21], in addition to the uncertainties in the basic variables, in

practical applications the joint pdf $\phi_X(\boldsymbol{x})$ is also not precisely known and, as such, affected by uncertainties. In fact, since statistical inference based on actual observations of the system behavior (laboratory or field data) and engineering judgment/analysis are typically used to develop probabilistic models for the basic variables, i.e., $\phi_X(\boldsymbol{x})$, inevitably they introduce additional uncertainties into the stochastic models thus affecting, in turn, the failure probabilities [6]. The quantification of the effects of these uncertainties on the system structural reliability, often referred to in the literature as reliability sensitivity analysis, is an important design task and has stimulated the development of several computational strategies with applications to many engineering fields [16][22][23][24][25].

Let us assume that the functional form of $\phi_X(\boldsymbol{x})$ has been already chosen, and restrict our attention only to the analysis of the effects of the uncertain parameters of $\phi_X(\boldsymbol{x})$ on the failure probability p_f. By introducing $\boldsymbol{\Theta}_\phi$ as the vector of the N_Θ uncertain parameters characterizing the joint distribution $\phi_X(\boldsymbol{x})$ (which will now be formally indicated as $\phi_X(\boldsymbol{x}|\boldsymbol{\Theta}_\phi)$), the failure probability estimation problem can be rewritten as:

$$p_f(\boldsymbol{\Theta}_\phi) = P[G(\boldsymbol{X}) \leq 0|\boldsymbol{\Theta}_\phi] = \int_F \phi_X(\boldsymbol{x}|\boldsymbol{\Theta}_\phi) d\boldsymbol{x} \tag{5}$$

As stated above, one possible approach to quantify the influence of $\boldsymbol{\Theta}_\phi$ on the failure probability p_f is that of resorting to sensitivity analysis techniques. Unfortunately, sensitivity analyses for computationally demanding models easily become prohibitive tasks, due to the very large number of model evaluations required. This problem is even worse in our framework, where a single model evaluation, i.e., a single estimate of the failure probability p_f in correspondence of a sample of the distribution parameters $\boldsymbol{\Theta}_\phi$, already requires many evaluations of the computationally expensive performance function $G(\boldsymbol{X})$ [26].

The problem of quantifying the effects of the parameters $\boldsymbol{\Theta}_\phi$ on the variability of p_f has been tackled in the literature by several authors. A brief review of the existing approaches can be found, for example, in [27], where a method based on the computation of the local gradient of the failure probability p_f is also proposed. In that work, the authors innovatively combined their meta-IS algorithm for efficient small failure probability estimation [7] with the score function approach introduced in [28], in order to perform a local sensitivity analysis of the failure probability p_f with respect to the vector of uncertain parameters $\boldsymbol{\Theta}_\phi$ as a simple post-processing of the sampling-based reliability analysis outcomes, with no additional performance function evaluations required. Many

other methods have been introduced in the literature which exploits the concept of partial derivative of p_f for performing reliability sensitivity analysis, see, for example, [29][30][31]. However, these methods only allow to perform local sensitivity analysis, i.e., they focus on small variations of the input parameters around some nominal values and they evaluate the effects of these variations on the failure probability one at a time, keeping the other parameters fixed at their nominal values. Although some strategy to overcome this issue based on averaging the partial derivatives over regions of space close to the limit state have been proposed, global sensitivity analysis (GSA) methods offer a more complete and useful picture of the dependence of the variability of p_f on the uncertainty of the input parameters, by accounting for the entire range of variability of the inputs and for their possible synergies (or "interactions") [32]. Unfortunately, in general the better quality of a GSA comes at the price of much higher computational expenses to reach a reasonable convergence of the sensitivity indexes. This is already a significant issue for "standard" models (where a single code evaluation is required to obtain a model output), which has stimulated the development of a quite large spectrum of methods aimed at reducing the computational burden of GSA [32][33][34][35]. A brute force application of any of these methods to the problem under study is out of question, due to the large number of performance function (model) evaluations required in order to obtain a single estimate of the failure probability. In this context, an attempt to perform GSA, based on a decomposition of the output variance [32], has been propounded in [36], where the authors propose to combine kriging meta-modeling with the efficient Sobol method for estimating the Sobol sensitivity indexes, i.e., the indicators numerically quantifying the importance of the different inputs in $\boldsymbol{\Theta}_{\phi}$. However, hundreds of thousands of evaluations of the performance function $G(\boldsymbol{X})$ were still required to obtain a suitable training set for the kriging metamodel, thus still leading to prohibitive computational expenses for any realistic application. Moreover, no strategy is proposed in [36] to control/reduce the approximation error of the kriging surrogate model, thus giving rise to rather imprecise estimates of the Sobol sensitivity indexes. In a more recent work [37], the same authors (among others) acknowledged the limitations of the method and proposed a modified version relying on an improved importance sampling scheme, obtaining a significant reduction in the number of performance function evaluations.

In this work, we tackle the computational problem of global reliability sensitivity analysis in the structural reliability context. In particular, we propose to combine the GSA Sobol method to a modified AK-IS algorithm for efficiently estimating the Sobol sensitivity indexes. To this aim, we propose a significant modification of the AK-IS algorithm stemming from the idea that, after the estimate of the first $p_f(\boldsymbol{\theta}_{\phi}^{(1)})$ by a standard AK-IS run (including the FORM stage) is obtained in

correspondence of a first sample $\boldsymbol{\theta}_{\phi}^{(1)}$ of the uncertain parameter vector, all the other required evaluations of p_f in correspondence of different values of the input parameters $\boldsymbol{\Theta}_{\phi}$ for estimating the Sobol sensitivity indexes, are obtained by running several AK-ISs in sequence, whose DoEs $\left\{\left(\boldsymbol{x}^{(i)}, G\left(\boldsymbol{x}^{(i)}\right)\right); i = 1, \dots, N_s\right\}$ are "inherited" from the previous AK-IS run, with just few adaptive additions needed to account for the different values of the input parameter samples $\boldsymbol{\theta}_{\phi}^{(j)}, j = 1, \cdots, N_e$, and which do not require new FORM evaluations. The proposed scheme is based on the assumption that the uncertainty related to the parameter vector $\boldsymbol{\Theta}_{\phi}$ does not significantly influence the outcomes of the iso-probabilistic transformation of the pdf $\phi_X\left(\boldsymbol{x}|\boldsymbol{\Theta}_{\phi}\right)$, so that the kriging surrogate models built by the AK-IS in correspondence of different input values of the parameter vector $\boldsymbol{\Theta}_{\phi}$, are still capable of approximating the limit-state function in largely overlapping regions of the input space.

The main novelties of this method are related to i) the re-use of the DoE for training new surrogate models for estimating different failure probabilities and ii) the adoption of the active learning scheme of the AK-IS to adaptively update the DoE as new samples $\boldsymbol{\theta}_{\phi}^{(j)}$ of the distribution parameter uncertainty are drawn. While the first feature may conceptually resemble the strategy proposed in [37], the second one represents a significant step forward, which allows to further reduce the number of performance function evaluations and, at the same time, to increase the accuracy of the estimates of the failure probabilities $p_f\left(\boldsymbol{\Theta}_{\phi}\right)$, and, hence, of the Sobol sensitivity indexes.

This paper is organized as follows. Section 2 first briefly reviews the basics of the AK-IS method. Then, after recalling the framework for performing the variance-based global sensitivity analysis, the Section presents the novel modification of the AK-IS for computationally affordable GSA of failure probabilities. The novel approach is then demonstrated in Section 3 with reference to an academic example and a popular benchmark case study, both widely used in the literature ([12] and [31], respectively). Finally, some conclusions are drawn in Section 4.

2. AK-IS FOR GLOBAL SENSITIVITY ANALYSES

2.1. Adaptive Kriging – Importance Sampling (AK-IS)

Here, we first briefly recall the basic functioning of the AK-IS. The two main preliminary operative steps of the AK-IS are: i) an iso-probabilistic transformation $\boldsymbol{T}$ must be performed to define the standard random variables as $\boldsymbol{U} = \boldsymbol{T}(\boldsymbol{X})$, so that the performance function G in the standard space

becomes $H(\boldsymbol{U}) = G\left(\boldsymbol{T^{-1}}(\boldsymbol{U})\right)$, and the joint pdf $\phi_X(\boldsymbol{x})$ transforms into a joint Gaussian pdf $\phi_U(\boldsymbol{u})$ in the standard space [38]; ii) the most probable failure point (MPFP), assumed to be unique, must be identified by means of a FORM analysis [12][38]. Then, according to the importance sampling theory, the failure probability p_f can be recast in the standard space as

$$p_f = \int_{\mathbb{R}^n} 1_F(\boldsymbol{u}) \frac{\phi_U(\boldsymbol{u})}{\varphi_U(\boldsymbol{u})} \varphi_U(\boldsymbol{u}) d\boldsymbol{u} \tag{6}$$

where $\varphi_U(\boldsymbol{u})$ is the importance pdf, usually chosen as a joint Gaussian distribution centered in the MPFP and with a standard deviation $\sigma_{IS} = 1$, and $1_F(\boldsymbol{u})$ is the failure indicator function. The integral (4) can be estimated as [12]:

$$\widehat{p_f}_{IS} = \frac{1}{N_{IS}} \sum_{i=1}^{N_{IS}} 1_F\left(\boldsymbol{u}^{(i)}\right) \frac{\phi_U\left(\boldsymbol{u}^{(i)}\right)}{\varphi_U(\boldsymbol{u}^{(i)})} \tag{7}$$

where $\{\boldsymbol{u}^{(i)};\ i = 1, \dots, N_{IS}\}$ is the *i*-th sample drawn from the importance density $\varphi_U(\boldsymbol{u})$. Accordingly, the variance of the failure probability estimator can then be estimated as [12]:

$$Var(\widehat{p_f}_{IS}) = \frac{1}{N_{IS}} \left(\frac{1}{N_{IS}} \sum_{i=1}^{N_{IS}} \left(1_F\left(\boldsymbol{u}^{(i)}\right) \frac{\phi_U\left(\boldsymbol{u}^{(i)}\right)}{\varphi_U(\boldsymbol{u}^{(i)})} \right)^2 - \widehat{p_f}_{IS}^{\,2} \right) \tag{8}$$

and the coefficient of variation δ_{IS} of the failure probability estimator becomes:

$$\delta_{IS} = \frac{\sqrt{Var\left[\widehat{p_f}_{IS}\right]}}{\widehat{p_f}_{IS}} \tag{9}$$

Then, a kriging-based surrogate model is used to approximate the (generally) time consuming performance function $H(\boldsymbol{U})$ in correspondence of a sample $\{\boldsymbol{u}^{(i)}, i = 1, \dots, N_{IS}\}$. The kriging meta-model approximates the function H as a realization of a Gaussian process, whose parameters have to be determined. In correspondence of a sample $\boldsymbol{u}^{(i)}$, the unknown value of $H(\boldsymbol{u}^{(i)})$ is considered to be a realization of a Gaussian distribution characterized by a mean prediction μ_H (kriging mean) and a

kriging variance σ_H^2, which can be taken as a measure of the prediction uncertainty. The AK-IS algorithm exploits the kriging mean μ_H and its uncertainty σ_H to identify the best new sample $\boldsymbol{u}^*$ to add to the DoE within the active/adaptive learning scheme. The adaptive enrichment of the DoE, which is performed during the estimation process, significantly improves the prediction capability of the kriging surrogate model, and consequently, the estimation performances of the AK-IS algorithm. The main operative steps of AK-IS algorithm can be briefly summarized as follows [12]:

1. Generation of N_{IS} input samples $\{\boldsymbol{u}^{(i)}; i = 1, \dots, N_{IS}\}$ in the standard space by sampling $\varphi_U(\boldsymbol{u})$: These samples represent candidate input values at which the deterministic performance function $H(\boldsymbol{U})$ is computed when required by the active learning.
2. Definition of the initial DoE: the initial DoE is made up of the samples at which the performance function $H(\boldsymbol{U})$ is computed during the FORM stage.
3. Training of the kriging model according to the DoE: this step is performed by means of the kriging toolbox DACE [39].
4. Prediction of the kriging meta-model and estimation of the failure probability: the mean prediction μ_H and its variance ${\sigma_H}^2$ are computed in correspondence of the N_{IS} input samples. The failure indicator $1_{\tilde{F}}$, based on the kriging predictions, is used to identify the samples $\boldsymbol{u}^{(i)}$ belonging to failure region. The failure probability p_f can be predicted by replacing 1_F with $1_{\tilde{F}}$ in (6).
5. Active learning and stopping criterion: the active learning process is based on the kriging assumption that $\Phi\left(\frac{|\mu_H(\boldsymbol{u}^{(i)})|}{\sigma_H(\boldsymbol{u}^{(i)})}\right)$ is the probability that $\boldsymbol{u}^{(i)}$ is classified in the correct group (failure or safe) according to the sign of its kriging mean $\mu_H(\boldsymbol{u}^{(i)})$. The best next sample to learn among the N_{IS} available ones, is the one at which the function $\Phi\left(\frac{|\mu_H(\boldsymbol{u}^{(i)})|}{\sigma_H(\boldsymbol{u}^{(i)})}\right)$ is minimized. The stopping condition is $\frac{|\mu_H(\boldsymbol{u}^{(i)})|}{\sigma_H(\boldsymbol{u}^{(i)})} \geq 2$, which means that the sample whose group (or sign) is the most uncertain shows a probability of at least $\Phi(2) = 97{,}7\%$ of being correctly classified according to the sign of its kriging mean.
6. Enrich the DoE with the identified sample: if the stopping condition at the previous step is not satisfied, the learning process continues and the performance function $H(\boldsymbol{U})$ is computed at the best new sample identified. This new observation is added to the DoE and the algorithm goes back to step 3 to build the new kriging model with the enriched DoE.
7. Computation of the coefficient of variation: if the stopping condition in step 5 is satisfied, the learning is stopped and the population N_{IS} is then checked to be large enough to provide an

accurate failure probability prediction. To do so, the coefficient of variation δ_{AK-IS} of the failure probability estimate $\widehat{p_f}$ is calculated by replacing 1_F with $1_{\tilde{F}}$ in (7).

8. Increase of the population: if the coefficient of variation is too high, N_{IS} is augmented with new samples and the AK-IS goes back to step 4 to predict the new population. The active learning method continues until the stopping condition is met again.
9. End of AK-IS: if the coefficient of variation δ_{AK-IS} of the failure probability is small enough, the AK-IS stops and the last $\widehat{p_f}$ is accepted.

The interested reader is referred to [12] for further details on the functioning of the AK-IS.

2.2. Sobol-based dynamic AK-IS (SDAKIS) for reliability sensitivity analysis

Within the probabilistic framework described in the Introduction for the different uncertainties involved in structural/mechanical engineering problems, the failure probability p_f depends on the vector of "second level", epistemically uncertain parameters $\boldsymbol{\Theta}_\phi$. As anticipated in the Introduction, the analysis of the role played by the variability of each distribution parameter Θ_{ϕ_j}, $j = 1, \dots, N_\Theta$, on the variability of the failure probability $p_f(\boldsymbol{\Theta}_\phi)$ is an important engineering task, in particular during the design phase of any complex system (structural, mechanical, energy, etc.). The quantification of these "roles" in the probabilistic framework here adopted often stems from the idea that each input parameter variability contributes to the unconditional variance of the output $\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)$ to different extents [32][33][34][35]. In this regard, it can be shown that the unconditional variance of p_f can be decomposed as:

$$\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f) = \sigma^2_{\Theta_{\phi_j}}\left(\mathbb{E}\left[p_f \middle| \Theta_{\phi_j}\right]\right) + \mathbb{E}_{\Theta_{\phi_j}}\left[\sigma^2\left(p_f \middle| \Theta_{\phi_j}\right)\right] \quad j = 1, \dots, N_\Theta \qquad (10)$$

where the conditional variance $\sigma^2_{\Theta_{\phi_j}}\left(\mathbb{E}\left[p_f \middle| \Theta_{\phi_j}\right]\right)$ is the contribution to the total variance of p_f only due to the variability of the parameter Θ_{ϕ_j} alone, whereas the term $\mathbb{E}_{\Theta_{\phi_j}}\left[\sigma^2\left(p_f \middle| \Theta_{\phi_j}\right)\right]$ includes the contributions related to the variability of all the other parameters. The first term is often called 'main effect' and is used as an indicator of the importance of the variability of Θ_{ϕ_j} in contributing to the total variance of p_f, i.e., the sensitivity of p_f with respect to Θ_{ϕ_j}. Normalizing the main effect by the unconditional variance of the output, we obtain the first order sensitivity indexes [32][33][34][35]:

$$Si_j = \frac{\sigma^2_{\Theta_{\phi_j}}\left(\mathbb{E}\left[p_f \middle| \Theta_{\phi_j}\right]\right)}{\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)} \; ; \qquad j = 1, \dots, N_\Theta \tag{11}$$

The usual approach adopted for estimating the unconditional variance $\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)$ is that of resorting to standard MC simulation by i) sampling N_e samples of the random vector $\boldsymbol{\Theta}_\phi$ from $\phi_{\Theta_\phi}(\boldsymbol{\theta}_\phi)$ and ii) evaluating the corresponding failure probabilities in correspondence of the N_e samples of $\boldsymbol{\Theta}_\phi$. The unconditional variance of the failure probability, $\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)$, can, then, be estimated by resorting to the following MC estimator, i.e.:

$$\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f) = \left(\frac{1}{N_e} \sum_{m=1}^{N_e} p_f\left(\boldsymbol{\theta}_\phi^{(m)}\right)^2 \; - \; \mathbb{E}_{\boldsymbol{\Theta}_\phi}[p_f]^2 \right) \tag{12}$$

where the expected value of the failure probability is:

$$\mathbb{E}_{\boldsymbol{\Theta}_\phi}[p_f] \; = \frac{1}{N_e} \sum_{m=1}^{N_e} p_f(\boldsymbol{\theta}_\phi^{(m)}) \tag{13}$$

The main effect of the input factor Θ_{ϕ_j} can then be efficiently estimated according to [35]:

$$\sigma^2_{\Theta_{\phi_j}} = \left(\frac{1}{N_e} \sum_{m=1}^{N_e} p_f(SM) p_f\left(AM_j\right) - \mathbb{E}_{\boldsymbol{\Theta}_\phi}[p_f]^2 \right) \qquad j = 1, \dots, N_\Theta \tag{14}$$

where SM is the sample matrix and AM_j is the auxiliary matrix associated to the *j*-th distribution parameter Θ_{ϕ_j}, both of size N_e x N_Θ . Note that sample matrix SM is obtained by drawing N_e samples from $\phi_{\Theta_\phi}(\boldsymbol{\theta}_\phi)$, whereas the auxiliary matrices AM_j, $j = 1, \dots, N_\Theta$, are, in turn, built on the basis of an additional sample set made up of N_e samples from $\phi_{\Theta_\phi}(\boldsymbol{\theta}_\phi)$, the so-called resample matrix RM, whose size is then N_e x N_Θ. The auxiliary matrix AM_j is, in fact, obtained by taking all the columns of SM, except for the *j*-th one, which is substituted by the *j*-th column of RM. Normalizing the estimate of the main effects in (14) by the estimate of the unconditional variance in (12), we obtain

the estimate of the first order sensitivity indexes Si_j in (11). In [32], an extension of the Sobol method for the estimation of the total effect of Θ_{ϕ_j} over the output p_f, i.e., the sum of the main effect of Θ_{ϕ_j} and of all its interactions with the other parameters $\Theta_{\phi_{l\neq j}}$ is presented, which allows the computation of the total sensitivity index St_j by normalizing the estimate of the total effect by the unconditional variance:

$$St_j = 1 - \frac{\left(\frac{1}{N_e}\sum_{m=1}^{N_e} p_f(SM)p_f(BM_j) - \mathbb{E}[p_f]^2\right)}{\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)} \qquad j = 1, \dots, N_\Theta \quad (15)$$

The auxiliary matrix BM_j associated to the j-th distribution parameter Θ_{ϕ_j} is obtained by taking all the columns of RM, except for the j-th one, which is substituted by the j-th column of SM. The interested reader is referred to [32] and [35] for details on the derivation of the first order and total Sobol indexes. In our context, by resorting to these approaches, the estimation of the first and the total order sensitivity indexes for N_Θ inputs and N_e samples would require approximately $N_e(2N_\Theta + 1)$ failure probability estimations to reach convergence, with the assumption that either SM or RM is used to compute the unconditional variance $\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)$ in (12).

As anticipated in the Introduction, the major problem of reliability GSA by the Sobol method lies in the fact that the estimation of the failure probability $p_f\left(\boldsymbol{\theta}_\phi^{(m)}\right)$ associated to a generic sample $\boldsymbol{\theta}_\phi^{(m)}, m = 1, \dots, N_e\,(2N_\Theta + 1)$, by brute force MC requires N_a evaluations of the usually time consuming performance function $G(\boldsymbol{X})$, such that the estimation of the first and total order sensitivity indexes requires at least $N_a \cdot N_e\,(2N_\Theta + 1)$ calls to the performance function $G(\boldsymbol{X})$, which easily turns out to be prohibitive. One possible approach to tackle this computational burden may amount to estimating the failure probability $p_f(\boldsymbol{\theta}_\phi^{(m)})$ associated to each sample $\boldsymbol{\theta}_\phi^{(m)}$ obtained according to the Sobol method by resorting to more efficient techniques for the assessment of small failure probabilities. For instance, within the computational framework of the AK-IS mentioned above, the estimation of the Sobol indexes could be performed by repeatedly running the AK-IS algorithm in correspondence of the samples $\{\boldsymbol{\theta}_\phi^{(m)}, m = 1, \dots, N_e(2N_\Theta + 1)\}$ obtained according to the Sobol method and, then, by performing the post-procedure envisaged by the Sobol method (see (14) and (15)). This procedure would entail creating different, "ad-hoc" DoEs, one for each run of the AK-IS, thus still requiring too many performance function evaluations, at least for realistic applications.

On the other hand, it is important to note that the active learning module of the AK-IS tends to enrich the DoE with the samples $\boldsymbol{x}^{(i)}$, $m = 1, \dots, N_a$, of the basic variables, among the N_{IS} available ones, which are closer to the limit-state function. Thus, the several DoEs that would be obtained by repeatedly running the AK-IS algorithm, would be made up of samples not only close to the limit-state function, but, also, rather close to each other. This suggests that the samples of the DoE obtained with a single run of the AK-IS in correspondence of, say, the first sample of the distribution parameter vector $\boldsymbol{\theta}_{\phi}^{(1)}$, could be used to estimate the failure probability corresponding to a new sample $\boldsymbol{\theta}_{\phi}^{(1<m\leq N_e(2N_{\Theta}+1))}$ to an acceptable degree of accuracy. Then, all the other evaluations of p_f in correspondence of different values of the input parameters $\boldsymbol{\Theta}_{\phi}$, required for the estimation of the first order and total Sobol indexes, could be obtained by running several successive AK-ISs, whose DoEs are "inherited" from the previous AK-IS run, with just few adaptive additions needed to account for the different values of the input parameters $\boldsymbol{\Theta}_{\phi}$, but with no additional FORM evaluations.

More formally, let us suppose that a sequence of samples of $\boldsymbol{\Theta}_{\phi}$ is generated according to the Sobol method, i.e. $\{\boldsymbol{\theta}_{\phi}^{(m)}; m = 1, \dots, N_e(2N_{\Theta} + 1)\}$ (see Section 2.1), and that the estimate of the failure probability $p_f^{(1)}$ corresponding to some initial values $\boldsymbol{\theta}_{\phi}^{(1)}$ is obtained by running the AK-IS, including the initial FORM search of the MPFP. The samples of the first $DoE^{(1)}$ created during the first AK-IS run lie close to the limit-state in the standard space $\mathbb{U}$ associated to $\phi_X(\boldsymbol{x}|\boldsymbol{\theta}_{\phi}^{(m)})$, for any sample $\{\boldsymbol{\theta}_{\phi}^{(m)}, m = 1, \dots, N_e(2N_{\Theta} + 1)\}$. Thus, the samples $\boldsymbol{x}^{(i)}$, $m = 1, \dots, N_a$ in the available DoE are assumed to be suitable for building a sufficiently accurate surrogate model of the true (and unknown) limit-state function for any realization of $\boldsymbol{\Theta}_{\phi}$, as it will be heuristically verified in the case studies of the next Section.

The algorithm then proceeds with the estimation of $p_f^{(2)}$ at a new sample $\boldsymbol{\theta}_{\phi}^{(2)}$ by running again the AK-IS, but exploiting the $DoE^{(1)}$ obtained at the end of the previous run, thus avoiding the FORM stage. The new approximation of the MPFP is found by identifying the input sample of the $DoE^{(1)}$ closest to the origin of the standard space. At the end of this step, an estimate $\widehat{p_f}^{(2)}$ of the new failure probability, and a new $DoE^{(2)}$ updated by the active learning scheme of the AK-IS (Section 2.1), are available. The algorithm then estimates all the remaining failure probabilities $\{p_f^{(m)}; m = 3, \dots, N_e(2N_{\Theta} + 1)\}$ in the same fashion, with the m-th $DoE^{(m)}$ slightly increasing in size, as m increases.

The complete algorithm for the GSA of small failure probabilities based on the first order Si_j

and total Sobol St_j indexes can finally be obtained by a simple post-processing of all the p_fs obtained according to the Sobol method. The operative steps of the SDAKIS procedure are described in Table 1.

Table 1: Main steps of the SDAKIS algorithm

1. Draw $2N_e$ samples from the joint pdf $\phi_{\Theta_\phi}(\boldsymbol{\theta}_\phi)$ of vector $\boldsymbol{\Theta}_\phi$, and collect N_e samples in the sample matrix (SM), and N_e samples in the resample matrix (RM).
2. Generate the auxiliary matrices AM_1, AM_2,..., AM_{N_Θ} and BM_1, BM_2,..., BM_{N_Θ} for each input factor Θ_{ϕ_j}, from Θ_{ϕ_1} to $\Theta_{\phi_{N_\Theta}}$:
   i. select the resample matrix (RM),
   ii. replace the *j*−th column, corresponding to the input factor Θ_{ϕ_j}, with the *j*−th column of the sample matrix (SM),
   iii. store the matrix AM_j,
   iv. select the sample matrix (SM),
   v. replace the *j*−th column, corresponding to the input factor Θ_{ϕ_j}, with the *j*−th column of the resample matrix (RM),
   vi. store the matrix BM_j,
3. Concatenate the sample matrix (SM) and the auxiliary matrices (AM_1, ..., AM_{N_Θ}, BM_1,..., BM_{N_Θ}) in order to obtain the sample sequence $\left\{\boldsymbol{\theta}_\phi^{(m)}, m = 1, \dots, (2N_\Theta + 1)N_e\right\}$.
4. Estimate the $p_f^{(1)}$ associated to $\boldsymbol{\theta}_\phi^{(1)}$ by the AK-IS. Perform an inverse iso-probabilistic transformation $\boldsymbol{T}^{-1}$ of the samples of the $DoE^{(1)}$.
5. For any other sample stored in the sequence $\left\{\boldsymbol{\theta}_\phi^{(m)}, m = 2, \dots, (2N_\Theta + 1)N_e\right\}$:
   (a) Perform an iso-probabilistic transformation $\boldsymbol{T}$ on the samples of the $DoE^{(m-1)}$ such that their joint pdf $\phi_X(\boldsymbol{x}|\boldsymbol{\theta}_\phi^{(m)})$ becomes a joint Gaussian standard $\phi_U(\boldsymbol{u})$, and the $DoE^{(m-1)}$ is represented in the standard space $\mathbb{U}$.
   (b) Identify the best approximation of the MPFP by finding the closest failure sample to the origin of the standard space among the ones available in the $DoE^{(m-1)}$.
   (c) Define the importance sampling pdf $\varphi_U(\boldsymbol{u})$ in the standard space $\mathbb{U}$ by "centering"

$\phi_U(\boldsymbol{u})$ on the best approximation of the MPFP.

(d) Estimate $p_f(\boldsymbol{\theta}_\phi^{(m)})$ by means of the main steps of the main steps of the AK-IS method and store it.

(e) Perform an inverse iso-probabilistic transformation $\boldsymbol{T}^{-1}$of the samples of the final $DoE^{(m)}$ (go back to the physical space $\mathbb{X}$) and update the $DoE^{(m-1)}$.

6. Estimate $\mathbb{E}_{\boldsymbol{\Theta}_\phi}[p_f]$ and $\sigma^2_{\boldsymbol{\Theta}_\phi}(p_f)$ according to (12) and (13) respectively. Obtain the estimations of Si_j by combining the estimations of the p_fs according to (14), and of the St_j according to (15).

3. RESULTS

In this Section, the SDAKIS is applied to an academic example of literature and to a benchmark case study of structural reliability analysis.

3.1. Academic Example [12].

A 2D analytical case study is used to illustrate the performances of the SDAKIS algorithm. The performance function $G(\boldsymbol{X})$ is [12]:

$$G(X_1, X_2) = 0.5(X_1 - 2)^2 - 1.5(X_2 - 5)^3 - 3 \quad (16)$$

where the two model inputs (basic variables) X_1 and X_2 are stochastic variables assumed to be described by independent Gaussian distributions with means $\mu_{X_1}$and μ_{X_2}, and standard deviations σ_{X_1} and σ_{X_2}, respectively, so that the distribution parameter vector is $\boldsymbol{\Theta}_\phi = \{\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}\}$. We initially consider the reference value of $\boldsymbol{\Theta}_\phi$ to be $\boldsymbol{\theta}_\phi^{ref} = \{\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}\}^{ref} = \{0, 1.75, 1, 1\}$, where the value of μ_{X_2} is purposedly chosen different from that used in [12] in order to obtain higher failure probabilities, thus speeding up the GSA calculations that otherwise would have been uselessly long. The resulting failure probability $p_f = P[G(\boldsymbol{X}) \leq 0]$ is estimated by a brute force MC simulation (Table 2, column 2), by a FORM analysis (Table 2, column 3) and by a single AK-IS run (Table 2, column 4). The comparison of the results, performed at approximately the same coefficient of variation (δ) (except for the FORM, which does not allow a quantification of the estimation error) shows, as expected, that the AK-IS outperforms the other two methods in terms of the number of calls

to the performance function $G(\boldsymbol{X})$, N_{call}. Table 2 also compares the estimates of all the partial derivatives of the failure probability p_f with respect to the four distribution parameters $\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}$ and σ_{X_2}, obtained with i) the classical score function approach applied to the brute force MC simulation [28] (column 2), ii) the FORM (column 3) (see [40] for a description of the method for obtaining the partial derivatives with the FORM) and iii) again the score function approach applied to the modified AK-IS described in Section 2.2, with $N_{IS} = 10^4$ (column 4). In all the three cases, the estimates are obtained by a simple post-processing of the performance function evaluations performed for estimating the failure probability p_f, so they come at no additional computational expenses. The brute force MC estimates are taken as the reference solutions. Table 2 shows that, even though an increase of the values of any distribution parameters (i.e., $\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}$) would lead to larger failure probabilities (all the partial derivatives of p_f are positive), the distribution parameters of X_2, (i.e., μ_{X_2} and σ_{X_2}), have larger influence on p_f. In fact, the partial derivatives $\partial p_f/\partial \mu_{X_2}$ and $\partial p_f/\partial \sigma_{X_2}$ are approximately one order of magnitude larger than $\partial p_f/\partial \mu_{X_1}$ and $\partial p_f/\partial \sigma_{X_1}$. Note also that the FORM estimates are rather inaccurate, due to the high non-linearity of the limit-state function [40]. On the other hand, the AK-IS yields results in satisfactory agreement with the reference ones. We now assume that the distribution parameters $\boldsymbol{\theta}_\phi = \{\mu_{X_1}, \mu_{X_2}, \sigma_{X_1}, \sigma_{X_2}\}$ are, in turn, affected by uncertainties, their values not being precisely known due to, for example, lack of data or inference approximations. In the absence of any further information, we propose to model the uncertainties affecting the distribution parameter vector $\boldsymbol{\Theta}_\phi$ resorting to uniform distributions $\{U^{(\mu_1)}, U^{(\mu_2)}, U^{(\sigma_1)}, U^{(\sigma_2)}\}$. The parameters of the uniform distributions, shown in Table 3, are chosen on the basis of a trial and error procedure aimed at highlighting the efficiency of the proposed method at reasonable computational times.

Table 4 compares the estimates of some parameters of the empirical pdf of the failure probability p_f obtained with a brute force double-loop MC (column 2) and the SDAKIS algorithm (column 3). Table 5 shows the comparison of the results of the Sobol-based GSA of the failure probability p_f obtained by brute force MC, taken as the reference values (column 2), and by the proposed SDAKIS algorithm (column 3).

According to the Sobol estimation scheme illustrated in Section 2.2, $2N_e$ samples must be drawn from the uniform distributions $\{U^{(\mu_1)}, U^{(\mu_2)}, U^{(\sigma_1)}, U^{(\sigma_2)}\}$ and properly combined to populate the sample and resample matrices SM and RM, yielding eight auxiliary matrices, i.e., $AM_{\mu_{X_1}}, AM_{\mu_{X_2}}, AM_{\sigma_{X_1}}, AM_{\sigma_{X_2}}$, $BM_{\mu_{X_1}}, BM_{\mu_{X_2}}, BM_{\sigma_{X_1}}, BM_{\sigma_{X_2}}$, each made up of N_e samples. The failure probabilities p_f corresponding to each sample $\boldsymbol{\theta}_\phi^{(m)} = \left\{\mu_{X_1}^{(m)}, \mu_{X_2}^{(m)}, \sigma_{X_1}^{(m)}, \sigma_{X_2}^{(m)}\right\}, m =$

$1, \dots, N_e(2N_\Theta + 1)$, in the matrices SM, AM_j and BM_j, are then evaluated by the SDAKIS procedure with a threshold $\delta_{th} = 20\%$ for the coefficient of variation, chosen large enough to guarantee convergence in reasonable computational times. Finally, the first order Si_j and total St_j Sobol indexes are computed by the procedures illustrated in Section 2.2. The estimates of the Sobol indexes obtained by the SDAKIS satisfactorily approach the ones obtained by brute force MC (here taken as reference values), the number of performance function evaluations being orders of magnitude lower. The computational efficiency of the SDAKIS mainly stems from the adaptive increase of the DoE, which allows to estimate new failure probabilities $p_f^{(m)}, m = 1, \dots, N_e(2N_\Theta + 1)$, requiring only a few additional performance function evaluations. This is demonstrated in Figure 1, which shows how the $DoE^{(m)}$ increases in correspondence of four different steps as the SDAKIS algorithm proceeds (black triangles indicate elements of the old DoE, whereas the red circles indicate the new adaptive additions). In order to further clarify, Figure 2 shows the evolution of the size of the $DoE^{(m)}$ as the number m of samples $\boldsymbol{\theta}_\phi^{(m)}$ processed by the SDAKIS algorithm increases. It can be seen that, after a quick increase up to 10^4 samples, the DoE size starts slowing down its growth. Note that the stars in Figure 2 represent the sizes of the DoEs of Figure 1. The global reliability sensitivity analysis provides, even in this simple (two dimensional) case, a complete picture of the dependencies of the failure probability p_f on the distribution parameters $\boldsymbol{\Theta}_\phi$. The analysis, in fact, confirms the importance of μ_{X_2} and σ_{X_2} (found locally also by the partial derivatives in Table 2), even when the entire ranges of variability of the four distribution parameters are explored. However, the most important parameter turns out to be μ_{X_2}, which is somehow in contrast with the results of the local reliability sensitivity analysis (σ_{X_2} was, in fact, the most important parameter). Furthermore, it can be seen that the interactions among the parameters μ_{X_1}, μ_{X_2}, σ_{X_1} and σ_{X_2} (due to the non-linearities in the I/O mapping) also significantly contribute to the variability of p_f.

All the failure probabilities $p_f^{(m)}, m = 1, \dots, N_e(2N_\Theta + 1)$, are estimated with a coefficient of variation $\delta_{p_f}^{(m)}$ much lower than the pre-specified threshold $\delta_{th} = 20\%$, for all the $N_e(2N_\Theta + 1)$ samples, as verified by authors, but not shown here for brevity's sake.

3.2. A roof truss [31].

The problem under analysis is that of assessing the serviceability limit-state of a roof truss with respect to its maximal vertical displacement. The performance function is defined as follows [31]:

$$G(q, l, A_c, A_s, E_c, E_s) = 0.03 - \frac{ql^2}{2}\left(\frac{3.81}{A_c E_c} + \frac{1.13}{A_s E_s}\right) \quad (17)$$

where q is the magnitude of a uniformly distributed load, l is the roof span and (E_s, A_s) and (E_c, A_c) are the Young's moduli and the cross section areas of the steel and the concrete beams, respectively. The uncertainties affecting these quantities are modeled by independent normal random variables whose distribution parameters are given in Table 6. The admissible displacement is arbitrarily fixed at 0.03 m according to the reference work in [31].

The failure probability $p_f = P[G(\boldsymbol{X}) \leq 0]$ in correspondence of the mean values of the basic input parameters (Table 7) is estimated by a brute force MC simulation (Table 7, column 2), by a FORM analysis (Table 7, column 3) and by a single AK-IS-based estimation (Table 7, column 4). The comparison of the results, performed at approximately the same coefficient of variation δ (except for the FORM, as explained before) shows, again, that the AK-IS outperforms the other two methods in terms of the number of calls to the performance N_{call}.

Table 7 also compares the estimates of all the elasticities of the failure probability p_f with respect to the distribution parameters $\boldsymbol{\theta}_\phi = \{\mu_q . \mu_l, \dots, \mu_{E_c}, \sigma_q, \sigma_l, \dots, \sigma_{E_c}\}$ (i.e., a normalized form of the partial derivatives, see [20] for their formal definition) obtained with the classical score function approach applied to the brute force MC simulation (column 2), with the FORM (column 3) and with the score function approach applied to the AK-IS with $N_{IS} = 10^4$ (column 4). The failure probability appears most sensitive to the mean of the roof span μ_l and the load μ_q. Larger values of these two parameters would lead to an increase of the failure probability, since their elasticities are positive. Note that the variability of the standard deviations seem to be not effective in influencing the variability of the failure probability, at least for small variations around the nominal "mean" values.

Now, let us assume that the distribution parameter vector $\boldsymbol{\theta}_\phi = \{\mu_q, \mu_l, \mu_{A_s}, \mu_{A_c}, \mu_{E_s}, \mu_{E_c}\}$, is, in turn, affected by uncertainties. Again, in the absence of reliable data, we propose to model these uncertainties by resorting to uniform distributions $\{U^{(\mu_q)}, U^{(\mu_l)}, U^{(\mu_{A_s})}, U^{(\mu_{A_c})}, U^{(\mu_{E_s})}, U^{(\mu_{E_c})}\}$. The parameters of the six uniform distributions are shown in Table 8. The parameters of the uniform distributions are chosen on the basis of physical/manufacturing considerations. The Sobol estimation scheme of Section 2.2 requires the computation of 12 auxiliary matrices, i.e. $\{AM_{\mu_q}, AM_{\mu_l}, \dots, AM_{\mu_{E_c}}, BM_{\mu_q}, BM_{\mu_l}, \dots, BM_{\mu_{E_c}}\}$, each one made up of N_e samples of $\boldsymbol{\Theta}_\phi$. The failure probability p_f, corresponding to each sample in the matrices $\{SM, AM_{\mu_q}, AM_{\mu_l}, \dots, AM_{\mu_{E_c}}, BM_{\mu_q}, BM_{\mu_l}, \dots, BM_{\mu_{E_c}}\}$, is then evaluated by the SDAKIS procedure with a threshold $\delta_{th} = 20\%$ for the coefficient of

variation, chosen high enough, as before, to guarantee a reasonably fast "convergence" of the algorithm. Finally, the first order and total Sobol indexes are estimated by the procedures described in Section 2.2.

Table 9 and 10 compare the estimates of the parameters of the empirical pdf of the failure probability p_f and the results of the associated GSA, obtained with a brute force MC (column 1) and the SDAKIS (column 2). The estimates of the moments of the empirical pdf of p_f show that the uncertainties in $\boldsymbol{\Theta}_\phi$ significantly influence the magnitude of the failure probability, albeit apparently not very large with respect to the mean, nominal values (see Table 8). This suggests that the roof truss performances are not so robust with respect to the modeling uncertainties, as one could be led to believe by the very small value of the failure probability in correspondence of the mean (nominal) values of the distribution parameters. The global reliability sensitivity analysis confirms the importance of μ_q, even when the entire ranges of variability of the six parameters are taken into account, and shows that the variability of μ_l is almost ineffective in contributing to the p_f variability, which seems in contrast with the results of the local reliability sensitivity analysis. In addition to that, it can be seen that the interactions among the parameters $\mu_q . \mu_l, \dots, \mu_{E_c}$ (due to the mapping non-linearities) also provide a significant contribution to the variability of p_f.

Again, the estimates of the Sobol indexes by the SDAKIS (column 3) are very similar to the reference ones obtained by brute force MC (column 2), but the number of performance function evaluations is orders of magnitude lower. All the failure probabilities $p_f^{(m)}, m = 1, \dots, N_e(2N_\Theta + 1)$, are estimated with a coefficient of variation $\delta_{p_f}^{(m)}$ as verified by authors, but not shown here for brevity's sake, much lower than the pre-specified threshold $\delta_{th} = 20\%$, for all the $N_e(2N_\Theta + 1)$ samples. Note that the number of performance function evaluations required by the Sobol-based GSA is approximately one order of magnitude lower than that required by the method proposed in [37] for the same case study. As anticipated before, this is probably due to the capability of the kriging surrogate model DoE to adapt as a new sample of the distribution parameters is drawn.

Table 2: Estimates of the failure probability w.r.t. the distribution parameters $\boldsymbol{\Theta}_\phi$ at the reference values for the academic example [12].

	Brute force MC	**FORM**	**AK-IS**
p_f	$9.4 \cdot 10^{-3}$	$1.31 \cdot 10^{-2}$	$9.3 \cdot 10^{-3}$
N_{call}	$5 \cdot 10^5$	5	$5 + 10$
δ	1.5%	-	1.8%

$\frac{\partial p_f}{\partial \mu_1}$	$7.8 \cdot 10^{-3}$	$8.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$
$\frac{\partial p_f}{\partial \mu_2}$	$2.4 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	$2.3 \cdot 10^{-2}$
$\frac{\partial p_f}{\partial \sigma_1}$	$1.4 \cdot 10^{-3}$	$4.9 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$
$\frac{\partial p_f}{\partial \sigma_2}$	$5.2 \cdot 10^{-2}$	$7.1 \cdot 10^{-2}$	$5.1 \cdot 10^{-2}$

Table 3: Uniform distributions modeling the second level uncertainty affecting the distribution parameters $\boldsymbol{\Theta_\phi}$ in the academic example [12].

$U \sim U(l, u)$	l	u
$U^{(\mu_1)}$	-2	2
$U^{(\mu_2)}$	-0.25	3.75
$U^{(\sigma_1)}$	0.5	2
$U^{(\sigma_2)}$	0.5	2

Table 4: Estimated parameters of failure probability distribution for the academic example [12] by brute force MC and SDAKIS.

	Brute force MC	**SDAKIS**
$\widehat{\mathbb{E}}[p_f]$	$5 \cdot 10^{-2}$	$4.95 \cdot 10^{-2}$
$\hat{\sigma}(p_f)$	$9.2 \cdot 10^{-2}$	$8.9 \cdot 10^{-2}$
$\widehat{Med}(p_f)$	$5.8 \cdot 10^{-3}$	$5.8 \cdot 10^{-3}$
$\widehat{Min}(p_f)$	$2.4 \cdot 10^{-10}$	$2.2 \cdot 10^{-10}$
$\widehat{Max}(p_f)$	$4.9 \cdot 10^{-1}$	$4.8 \cdot 10^{-1}$

Table 5: Sobol indexes for the academic example [12] estimated by brute force MC and SDAKIS.

	Brute force MC	**SDAKIS**
N_{call}	$9 \cdot 10^4 \cdot 10^8$	1515
$Si_{\mu_{X_1}}$	$9.5 \cdot 10^{-2}$	$9.7 \cdot 10^{-2}$

$Si_{\mu_{X_2}}$	$5.7 \cdot 10^{-1}$	$5.8 \cdot 10^{-1}$
$Si_{\sigma_{X_1}}$	$8.0 \cdot 10^{-3}$	0
$Si_{\sigma_{X_2}}$	$3.7 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$
$St_{\mu_{X_1}}$	$3.4 \cdot 10^{-1}$	$3.5 \cdot 10^{-1}$
$St_{\mu_{X_2}}$	$8.7 \cdot 10^{-1}$	$8.5 \cdot 10^{-1}$
$St_{\sigma_{X_1}}$	$1.0 \cdot 10^{-4}$	0
$St_{\sigma_{X_2}}$	$1.0 \cdot 10^{-1}$	$9.9 \cdot 10^{-2}$
Interactions	$3.0 \cdot 10^{-1}$	$2.9 \cdot 10^{-1}$

Table 6: Normal distribution modeling the first level uncertainty affecting the basic variables for the roof truss case study [31].

Variable	**Mean $\boldsymbol{\mu}$**	**St. dev. $\boldsymbol{\sigma}$ (% $\boldsymbol{\mu}$).**
$q\ (N/m)$	20,000	7%
$l\ (m)$	12	1%
$A_s\ (m^2)$	$9.82 \cdot 10^{-4}$	6%
$A_c\ (m^2)$	$400 \cdot 10^{-4}$	12%
$E_s\ (N/m^2)$	$1 \cdot 10^{11}$	6%
$E_c\ \ (N/m^2)$	$2 \cdot 10^{10}$	6%

Table 7: Estimates of the failure probability w.r.t. the distribution parameters $\boldsymbol{\Theta_\phi}$ at the reference values for the roof truss case study [31].

	Brute force MC	**FORM**	**AK-IS**
p_f	$9{,}44 \cdot 10^{-3}$	$7{,}57 \cdot 10^{-3}$	$9{,}40 \cdot 10^{-3}$
N_{call}	262350	10	$10 + 67$
δ	2 %	-	2 %
e_{μ_q}	23.94	24.43	24.16
e_{μ_l}	49.46	53.77	51.37
$e_{\mu_{A_s}}$	-19.09	-20.34	-19.29

$e_{\mu_{A_c}}$	-9.12	-9.02	-8.48
$e_{\mu_{E_s}}$	-19.63	-20.34	-19.28
$e_{\mu_{E_c}}$	-7.82	-8.20	-7.54
e_{σ_q}	2.46	2.58	2.42
e_{σ_l}	0.20	0.25	0.24
$e_{\sigma_{A_s}}$	1.22	1.31	1.28
$e_{\sigma_{A_c}}$	1.27	1.03	1.28
$e_{\sigma_{E_s}}$	1.30	1.31	1.31
$e_{\sigma_{E_c}}$	0.21	0.21	0.26

Table 8: Uniform distributions modeling the second level uncertainty affecting the distribution parameters $\boldsymbol{\Theta_\phi}$ in the roof truss case study [31].

U ~ U(l, u)	***l***	***u***
$U^{(\mu_q)}$	96.5% $\cdot\ \mu_q$	103.5% $\cdot\ \mu_q$
$U^{(\mu_l)}$	99.5% $\cdot\ \mu_l$	100.5% $\cdot\ \mu_l$
$U^{(\mu_{A_s})}$	97% $\cdot\ \mu_{A_s}$	103% $\cdot\ \mu_{A_s}$
$U^{(\mu_{A_c})}$	94% $\cdot\ \mu_{A_c}$	106% $\cdot\ \mu_{A_c}$
$U^{(\mu_{E_s})}$	97% $\cdot\ \mu_{E_s}$	103% $\cdot\ \mu_{E_s}$
$U^{(\mu_{E_c})}$	97% $\cdot\ \mu_{E_c}$	103% $\cdot\ \mu_{E_c}$

Table 9: Estimated parameters of failure probability distribution in the roof truss case study [31] by brute force MC and SDAKIS.

	Brute force MC	**SDAKIS**
$\widehat{\mathbb{E}}[p_f]$	$1.3 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$
$\hat{\sigma}(p_f)$	$8.9 \cdot 10^{-3}$	$9.3\ \cdot 10^{-2}$
$\widehat{Med}(p_f)$	$9.3 \cdot 10^{-2}$	$9.5\ \cdot 10^{-2}$
$\widehat{Min}(p_f)$	$5.5 \cdot 10^{-4}$	$5.0 \cdot 10^{-4}$
$\widehat{Max}(p_f)$	$6.4 \cdot 10^{-1}$	$6.8 \cdot 10^{-1}$

Table 10: Sobol indexes for the roof truss case study [31] estimated by brute force MC and SDAKIS

	Brute force MC	**SDAKIS**
N_{call}	$9 \cdot 10^3 \cdot 10^8$	1616
Si_{μ_q}	$3{,}9 \cdot 10^{-1}$	$4{,}2 \cdot 10^{-1}$
Si_{μ_l}	$3{,}9 \cdot 10^{-2}$	$4{,}1 \cdot 10^{-2}$
$Si_{\mu_{A_s}}$	$1{,}2 \cdot 10^{-1}$	$1{,}4 \cdot 10^{-1}$
$Si_{\mu_{A_c}}$	$1{,}1 \cdot 10^{-1}$	$1{,}5 \cdot 10^{-1}$
$Si_{\mu_{E_s}}$	$1{,}3 \cdot 10^{-1}$	$1{,}7 \cdot 10^{-1}$
$Si_{\mu_{E_c}}$	$5{,}9 \cdot 10^{-2}$	$2{,}6 \cdot 10^{-2}$
St_{μ_q}	$4{,}5 \cdot 10^{-1}$	$4{,}7 \cdot 10^{-1}$
St_{μ_l}	$3{,}3 \cdot 10^{-2}$	$4{,}6 \cdot 10^{-2}$
$St_{\mu_{A_s}}$	$2{,}4 \cdot 10^{-1}$	$1{,}9 \cdot 10^{-1}$
$St_{\mu_{A_c}}$	$2{,}0 \cdot 10^{-1}$	$1{,}7 \cdot 10^{-1}$
$St_{\mu_{E_s}}$	$2{,}2 \cdot 10^{-1}$	$1{,}8 \cdot 10^{-1}$
$St_{\mu_{E_c}}$	$4{,}2 \cdot 10^{-2}$	$2{,}6 \cdot 10^{-2}$
Interactions	$1{,}5 \cdot 10^{-1}$	$1{,}0 \cdot 10^{-1}$

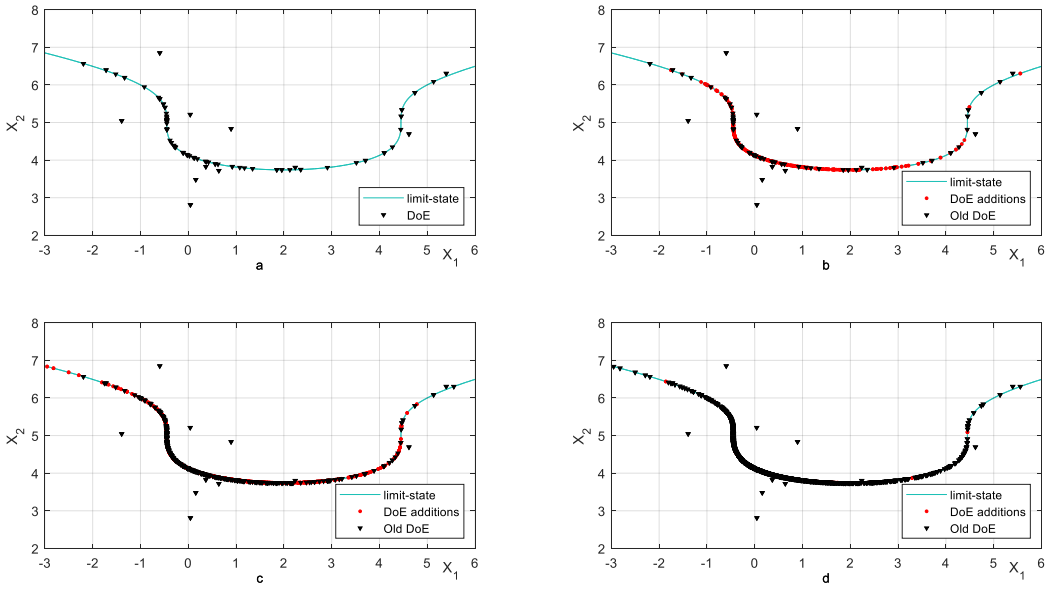

Figure 1: Adaptive increase of the DoE as (a) $m = 500$, (b) $m = 10000$, (c) $m = 60000$, (d) $m = 80000$ samples of Θ_ϕ are processed by the SDAKIS in the academic example [12].

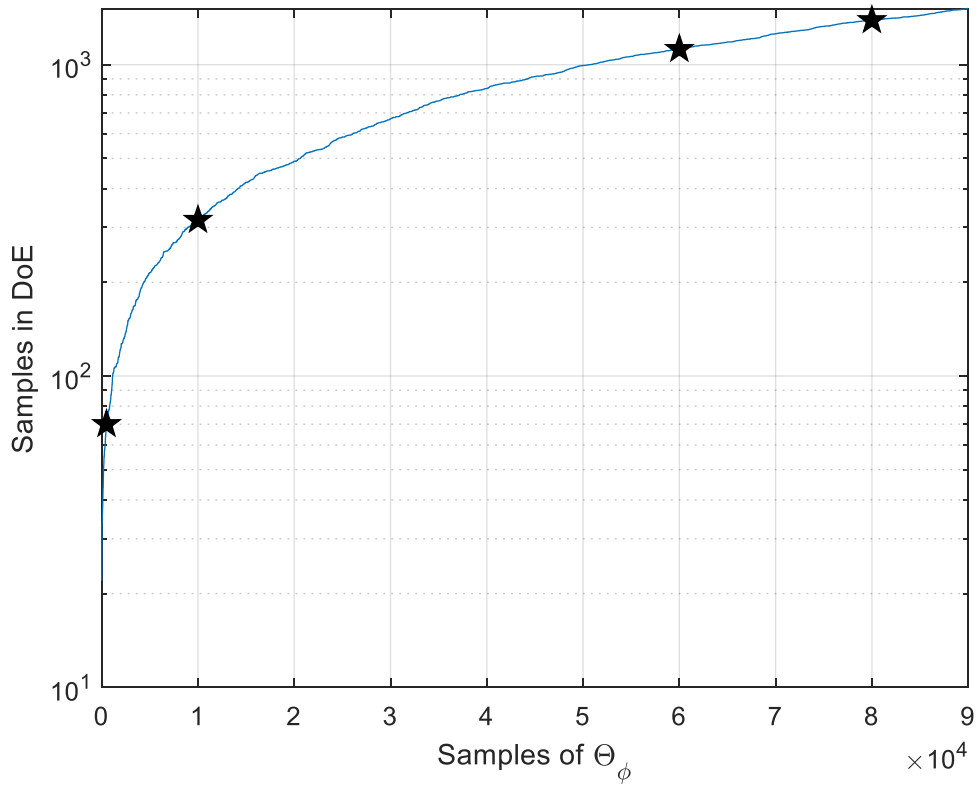

Figure 2: Adaptive increase of the DoE in correspondence of all the samples of Θ_ϕ [12].

4. CONCLUSIONS

In this paper we have proposed a modification of the AK-IS algorithm, originally developed in literature for the estimation of small failure probabilities within the context of structural/mechanical reliability problems, in order to be able to perform efficient reliability sensitivity analysis.

More specifically, we have turned our attention to the challenging variance-based global reliability sensitivity analysis problem by proposing to combine the Sobol method for variance-based GSA with the AK-IS algorithm in an adaptive scheme. This solution stems from the idea that, after the first estimate of the failure probability in correspondence of a sample of the uncertain distribution parameters by a standard AK-IS run (including the initial FORM stage), all the other required evaluations of p_f in correspondence of different samples of the distribution parameters are obtained by running several AK-ISs in sequence, whose DoEs are "inherited" from the previous AK-IS run, with just few adaptive additions needed to account for the different values of the input parameters, and which do not require new FORM evaluations.

The method has been applied to two case studies of literature, offering a significant reduction in the total number of performance function evaluations required, not only with respect to the classical double loop brute force Monte Carlo method for estimating the Sobol sensitivity indexes (implemented in this work for the benchmark examples so as to have a reference solution to compare with), but also with respect to other methods proposed in the literature to tackle the same problem. The key successful idea, besides those of resorting to surrogate model and to a smart use of importance sampling (as proposed also in other works of the literature), has turned out to be that of exploiting, and properly tailoring, the adaptation stage of the AK-IS algorithm. All calculations have been performed on a PC with processor Intel ® Core™ i7 − 3930K − CPU 3.20GHz x 12 and memory RAM 7, 7 GiB in Ubuntu 14.04 LTS environment. Note that there is no mathematical proof that the algorithm unbiasedly converges to the target Sobol indices. On the other hand, following a procedure adopted in many works of literature dealing with the same topic, we have provided a successful comparison of the results with those obtained with a brute-force Monte Carlo scheme, which is proven to yield unbiased estimates of the quantities of interest. We acknowledge that such "empirical" comparisons do not have the strength of a mathematical proof and, as such, do not guarantee that some ill-posed conditions may exist under which convergence is not achieved. However, we believe the proposed comparison should serve as a basis to build confidence within the scientific community about the applicability of the approach to the cases most likely encountered in the practice.

Three main issues remain unresolved.

First, the matrix inversion required for the construction of the kriging-based meta-model usually introduces significant numerical problems when the number of samples in the DoE becomes,

say, larger than $2 \cdot 10^3$, depending on the machines used for the computations. Since the algorithm here introduced relies on an adaptively growing size of the DoE, this issue may render the computations (even for simple, analytic case studies) quite computationally heavy. However, it should be kept in mind that, when dealing with real applications, possibly involving FEM-based performance functions, the only driver of the computational expenses should be the number of performance function evaluations. In any case, in the literature works can be found that address this issue by devising alternative schemes for updating the surrogate model as new data are collected, so there are probably margins of improvements in this regard, but we defer the problem to future work.

Second, the DoE refinement stage of an individual AK-IS may take a lot of time before the stopping criterion is met, slowing down the entire SDAKIS algorithm. This is particularly true when the failure probabilities to be estimated are very low, which might happen quite often, depending on how uncertain the distribution parameters are: the more is the spread, the most likely is to encounter very small failure probabilities. To mitigate this problem additional stopping criteria can be added that limit the number of additions to the DoE at each AK-IS run, but a detailed analysis of this possible solution lies beyond the scope of this work.

The third issue is related to the well-known "curse of dimensionality", which is such that any algorithm involving the construction of a surrogate model suffers when the dimensionality of the input space increases, because the DoE becomes sparser with a power law relationship; this issue is unavoidable and limits the application of surrogate model to engineering problems with a small number of input parameters (say tens), unless some dimensionality reduction strategy is adopted. In this regard, the literature has already proposed methods for dimension reduction to exploit some low-dimensional approximation of the model, such as sensitivity analysis, sparse grids, analysis of variance (ANOVA), separation of variables, polynomial chaos expansions, etc. Yet, in the context of structural reliability, a trade off should be found between the additional number of performance function evaluations required to perform dimensionality reduction and the advantages offered by a low-dimensional problem. This is currently an open topic of debate within the structural reliability scientific community.

Finally, note that the proposed SDAKIS method requires that a threshold δ_{th} be set for the maximum coefficient of variation for each estimate of the failure probability required by the global sensitivity analysis ($\delta_{th} = 20\%$ in the present work). This means that each of the required probabilities is estimated to an accuracy of, at least, 20%. The coefficient of variations actually obtained during the analyses in both case studies were usually much lower, i.e., approximately 5%. Consequently, even if no brute force MC reference is available to assess the reliability of the estimates, the accuracy of the method can be a priori bounded. However, care should be taken in

setting this threshold, since, as seen in the results, too low values may require too many AK-IS iterations for the calculation of each failure probability, eventually leading to not affordable computational times.

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