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Minimum Residual Vibrations for Flexible Satellites with Frequency Uncertainty

Zhili Hou, Yunhai Geng, and Simeng Huang

Abstract—The resonant frequencies will be excited if satellites perform a rapidly maneuver, which will increase the vibration settling time. In order to reduce the maneuver time and the residual vibration after maneuver, a set of shaped angular acceleration profiles are presented, and their analytical solutions are derived by minimizing the time integral of the squared magnitude of the difference between angular acceleration and its mean value subject to that the magnitude of the residual vibrations at several frequencies surrounding the natural frequency are zero. Then, suitable frequency points, where the residual vibrations are constrained to be zero, are chosen to minimize the acceleration time subject to both the residual vibration magnitude limit and the angular acceleration magnitude limit. Finally, three sets of simulations are presented to demonstrate that the shaped angular acceleration profiles can reduce the residual vibration under the frequency uncertainty.

Index Terms—Flexible spacecraft, vibration reduction, frequency uncertainty, trajectory planning.

I. INTRODUCTION

If a satellite performs a rapid maneuver, vibrations in the flexible appendages (solar panels or flexible antennae) will be excited. These vibrations are particularly detrimental at the end of the maneuver, where precision is usually demanded. Some vibration reduction methods have been developed to reduce residual vibrations at the end of maneuvers.

Input shaping[1], a class of widely-used methods, is a command generation technique to reduce residual vibrations that surround natural frequencies and works such as a notch filter that cancels out the decaying sinusoidal response. Singer and Seering[2] presented the ZV (zero vibration) shaper to eliminate residual vibrations at natural frequencies. However, the ZV shaper is limited to applications where the natural frequencies do not change significantly. To overcome this weakness, Singer and Seering[2] presented the ZVD (zero vibration derivative) shaper to improve the robustness for frequency changes. The ZVD shaper was derived by forcing the derivative of the residual vibration with respect to frequency to equal zero at the natural frequencies. Furthermore, Singhose[3–5] presented an approach named the EI (extra insensitive) shaper, which provides extra robustness without increasing the shaper duration when compared with the ZVD shaper. Singhose, Seering and Singer[6] then presented a more general method called the SI (specified insensitive) shaper to suppress a specified range of frequencies. In addition, a summary and comparison of input shaping methods were presented[7, 8].

Input smoothing is another class of effective method was methods that was developed by shaping the input command as a type of a smooth profile to reduce residual vibrations. Meckl[9] developed optimal s-curve motion profiles by minimizing the ramp-up time to achieve fast motions with minimum vibrations. Junkins[10] used the s-curve as input profiles to develop a near minimum time control law for a flexible satellite. The parameters of the s-curve profiles were chosen by minimizing the combination of vibration energy and maneuver time. Swigert[11] developed a set of smooth input torques constructed from a series of specified trigonometric functions. The coefficients of the trigonometric functions were calculated by minimizing the weighted combination of the squared magnitude of the input torque and the sensitivity of the residual vibration to model errors. Meckl[12, 13] presented two types of force profiles constructed from a versine(1-cosine) function and ramped sinusoid function. The excitation in a range of frequencies surrounding the system natural frequency was then minimized to obtain the coefficients. Xie X[14] designed an optimal smooth filter with high robustness at high frequencies compared with a ZVDDD input shaper.

The input smoothing method has an inherent advantage over the input shaping method because it can provide smooth input profiles that are conducive to trajectory tracking control. However, the smooth input profiles were often chosen as certain types of functions such as an s-curve, the summation of versine functions, and the summation of ramped sinusoid functions, which incapable of representing an arbitrary function. In this paper, the input(angular acceleration) profiles are allowed to be any reasonable function, and the analysis expression of this input profile then is derived by minimizing the time integral of the squared magnitude of the difference between the angular acceleration and its mean value, subject to the constraints that the magnitudes of the residual vibrations at several frequency points surrounding the natural frequency are zero. Then, suitable positions of the zero vibration frequencies are chosen to keep both the angular acceleration and residual vibration within acceptable bounds and at the same time, to minimize the acceleration time. In addition, the smooth input profile developed in this paper can also be used in the system where the damping is not zero.
The remainder of this paper contains three main sections. Section II develops the mathematical expression of the angular acceleration and residual vibrations under the assumption that the control error is small. Section III develops the analytical expression for the optimal angular acceleration profiles to minimize the residual vibration. Section IV discusses a set of simulations to verify that the developed angular acceleration profiles are valid to suppress residual vibrations under frequency uncertainty.

II. RELATIONSHIP BETWEEN ANGULAR ACCELERATION PROFILES AND RESIDUAL VIBRATIONS

This section first introduces the dynamic equations for a flexible satellite and attitude tracking controller, presents a derivation of the response of the residual vibration to an arbitrary input, and finally presents an evaluation index to measure the effectiveness of the vibration suppression.

A. Satellite Dynamic Equations and Kinematics

The attitude dynamics of a flexible satellite are given by

\[
\begin{align*}
I \ddot{\omega} + \omega \times I \omega + \omega \times \dot{h}_c + B \ddot{\eta} &= T_c, \\
\dot{\eta} + 2 \Omega \dot{\eta} + \Omega^2 \eta + B^T \omega &= 0
\end{align*}
\]

(1)

where \(I\) is the inertia matrix, \(\omega = [\omega_x, \omega_y, \omega_z]^T\) is the body angular velocity relative to the inertial reference frame, \(h_c\) is the total angular momentum of the flywheel, \(T_c\) is the control torque input vector, \(\eta = [\eta_1, \eta_2, \cdots, \eta_m]^T\) is a generalized coordinate vector, \(m\) is the number of modes, \(\xi\) is the modal damping ratio, \(\Omega\) is a diagonal matrix with entries \(\Omega_j\) which is the \(j\)th natural frequency. \(\Omega^2\) is a diagonal frequency matrix with entries \(\Omega_j^2\), and \(B\) is the coupling matrix between the rigid body and appendages.

The kinematics as described by quaternions are

\[
\begin{align*}
\dot{q}_v &= -\frac{1}{2} \omega \times q_v + \frac{1}{2} q_0 \omega, \text{ and} \\
\dot{q}_0 &= -\frac{1}{2} \omega^T q_v
\end{align*}
\]

(2)

(3)

where \(q_v = [q_1, q_2, q_3]^T\) is the quaternion vector of \(q = [q_0, q_1, q_2, q_3]^T\), which is the quaternion relative to the inertial reference frame.

B. SIMPLIFICATION OF THE VIBRATION EQUATION

Assume that the reference trajectory is scheduled as an eigenaxis rotation. A PD controller is then given by

\[
T_c = I (-k_p q_{ev} - k_d \omega_e) + T_r (t) + \omega \times (I \omega + h_c)
\]

(4)

where \(q_{ev} = [q_{e1}, q_{e2}, q_{e3}]^T\) is the vector part of \(q_e = [q_{e0}, q_{e1}, q_{e2}, q_{e3}]^T\) which represents the error quaternion, \(\omega_e\) is the error angular velocity, and \(T_r (t)\) is the reference torque, which is expressed as

\[
T_r (t) = I e a_r (t)
\]

(5)

where \(a_r (t)\) is the reference angular acceleration to be scheduled, and \(e\) is the fixed eigenaxis. \(q_{ev}\) and \(\omega_e\) can be obtained according to the equations

\[
\omega_e = \omega - \omega_r (t), \text{ and}
\]

\[
q_{ev} = -q_0 q_{ev} (t) + q_{r0} (t) q_v - q_{rv} (t) \times q_v
\]

(6)

(7)

where \(\omega_r (t) = [\omega_{rx} (t), \omega_{ry} (t), \omega_{rz} (t)]^T\) is the reference angular velocity, and the expressions for \(q_{r0} (t)\) and \(q_{rv} (t)\) are

\[
q_{r0} (t) = \cos \left( \frac{\Phi_r (t)}{2} \right), \qquad q_{rv} (t) = e \sin \left( \frac{\Phi_r (t)}{2} \right)
\]

(8)

where \(\Phi_r (t)\) is the reference principal rotation angle. \(\omega_r\) and \(\Phi_r\) can be obtained by integrating \(a_r\) (t).

After substituting Eq. (4) and Eq. (5) into the first equation of Eq. (1), simplifying yields

\[
\ddot{\omega} = -I^{-1} B \ddot{\eta} - k_p q_e - k_d \omega_e + e a_r (t)
\]

(9)

If the satellite attitude can track the reference trajectory well, the actual angular acceleration \(\dot{\omega}\) is approximately equal to the reference angular acceleration \(e a_r (t)\). The second equation of Eq. (1) then can be simplified to

\[
\ddot{\eta} + 2 \xi \Omega \dot{\eta} + \Omega^2 \eta = -B^T e a_r (t)
\]

(10)

where

\[
D = -B^T e
\]

(11)

Remark: Using the approximation \(\dot{\omega} \approx e a_r (t)\) will lead to a small difference between the actual and ideal frequencies. However, this difference can be considered to be a part of the frequency uncertainty that will be considered in the derivation of the optimal angular acceleration profiles.

C. Expression of the Residual Vibrations

With an arbitrary input function \(a_r (t)\), the residual vibration after a maneuver is expressed as

\[
\eta_j (t) = \int_0^{t_f} a_r (\tau) \eta_{j\tau} (t - \tau) d\tau
\]

(12)

where \(t_f\) is the maneuver time, and \(\eta_{j\tau} (t)\) is the impulse response of the \(j\)th modal vibration, whose expression is given by

\[
\eta_{j\tau} (t) = \frac{D_j e^{-\kappa_j \xi \tau} \sin \left( \kappa_j \sqrt{1 - \xi^2} t \right)}{\kappa_j \sqrt{1 - \xi^2}}
\]

(13)

where \(D_j\) is the \(j\)th element of \(D\).

The response of the \(j\)th modal vibration with the input \(a_r (t)\) can be written as Eq. (16). The expression for the maximum amplitude of residual vibration then can be written as Eq. (17). To measure the vibration suppression effectiveness, define the Residual Vibration Ratio as

\[
v = \frac{V (\kappa)}{\eta_{\text{step}} (\kappa)}
\]

(14)

where \(V (\kappa)\) is the amplitude of the residual vibration at frequency \(\kappa\), \(\eta_{\text{step}}\) is the maximum value of the step response of the vibration system shown in Eq. (10), where
\[ \eta_j(t) = D_j \int_0^{t_f} a(t) \eta_{\delta j}(t - t) \, dt = D_j e^{-\kappa_j \xi t} \sin \left( \kappa_j \sqrt{1 - \xi^2} t \right) \int_0^{t_f} a(t) e^{\kappa_j \xi \tau} \cos \left( \kappa_j \sqrt{1 - \xi^2} \tau \right) d\tau - \int_0^{t_f} a(t) e^{\kappa_j \xi \tau} \sin \left( \kappa_j \sqrt{1 - \xi^2} \tau \right) d\tau \]

\[ V_j = D_j e^{-\kappa_j \xi t_f} \sqrt{\int_0^{t_f} a(t) e^{\kappa_j \xi \tau} \cos \left( \kappa_j \sqrt{1 - \xi^2} \tau \right) d\tau + \int_0^{t_f} a(t) e^{\kappa_j \xi \tau} \sin \left( \kappa_j \sqrt{1 - \xi^2} \tau \right) d\tau} \]

where \( \xi = 0 \), and the step amplitude is \( a_{\text{max}} \). The expression for \( \eta_{\text{step}} \) is

\[ \eta_{\text{step}} = \frac{2D_j a_{\text{max}}}{\kappa^2} \]

where \( a_{\text{max}} \) is the allowed maximum angular acceleration.

**III. DEVELOPMENT OF THE OPTIMAL ANGULAR ACCELERATION PROFILES**

The angular acceleration profiles to be developed are intended for angular velocity-limited systems where the angular velocity profile has a constant angular velocity where the speed is saturated. In addition, the angular acceleration profile must contain three regions: acceleration, dwell, and deceleration. In general, the acceleration and deceleration regions have the same shape but opposite sign. Therefore, only the acceleration region needs to be designed, and the other regions then can be generated naturally.

The angular acceleration function in the acceleration region can be chosen as an arbitrary function, denoted by \( a_{\text{ac}}(t) \). The goal is to pick a suitable \( a_{\text{ac}}(t) \) that minimizes the magnitude of the angular acceleration and the magnitude of the residual vibration surrounding the system resonant frequencies. To minimize the magnitude of the angular acceleration, we will minimize the squared magnitude of the difference between the angular acceleration and its mean value. Therefore, the performance to be minimized is

\[ J = \frac{1}{2} \int_0^{t_{\text{ac}}} \left( a_{\text{ac}}(t) - \frac{1}{t_{\text{ac}}} \int_0^{t_{\text{ac}}} a_{\text{ac}}(t) \, dt \right)^2 \, dt \]

where \( t_{\text{ac}} \) is the duration of the acceleration region.

To minimize the magnitude of the residual vibration considering frequency uncertainties, we set the magnitude of the residual vibration surrounding the system resonant frequencies \( \kappa_j \) equal to zero. The zero-vibration points can be chosen as in the following equation

\[ V \left( \kappa_{\text{min},j} + \frac{i \left( \kappa_{\text{max},j} - \kappa_{\text{min},j} \right)}{n_j - 1} \right) = 0 \]

where \( \kappa_{\text{min},j} \) and \( \kappa_{\text{max},j} \), which are symmetrical about \( \kappa_j \), represent the minimum and maximum zero-vibration points around \( \kappa_j \), respectively, and

\[ i = 0, 1, \ldots, n_j - 1 \]

\[ \beta_j = \frac{\kappa_{U,j} - \kappa_{L,j}}{\kappa_j} \]

where \( \beta_j \) is the uncertainty ratio of the \( j \)-th mode, \( \kappa_{U,j} \) is the upper bound of \( \kappa_j \), and \( \kappa_{L,j} \) is the lower bound of \( \kappa_j \). The relationships among \( \kappa_{U,j}, \kappa_{L,j}, \kappa_{\text{max},j}, \kappa_{\text{min},j} \) and \( \kappa_j \) are shown in Fig. 1.

Fig. 1. Frequency bounds and zero-vibration points

To simplify Eq. (19), we define

\[ \kappa_{ij} = \kappa_{\text{min},j} + \frac{i \left( \kappa_{\text{max},j} - \kappa_{\text{min},j} \right)}{n_j - 1} \]

where \( \kappa_{ij} \) represents the \( i \)-th zero point around the \( j \)-th mode frequency \( \kappa_j \).

With Eq. (17), Eq. (19) can be written as the following \( \sum_{j=1}^m n_j \) equations:

\[ \int_0^{t_{\text{ac}}} a_{\text{ac}}(t) e^{\kappa_{ij} \xi t} \cos \left( \kappa_{ij} \sqrt{1 - \xi^2} t \right) dt = 0, \quad \text{and} \quad \int_0^{t_{\text{ac}}} a_{\text{ac}}(t) e^{\kappa_{ij} \xi t} \sin \left( \kappa_{ij} \sqrt{1 - \xi^2} t \right) dt = 0 \]

where \( j = 1, 2, \ldots, m \). To ensure the angular rate at the end of the acceleration equals the maximum angular rate, the following constraint should be added

\[ \omega_{\text{max}} = \int_0^{t_{\text{ac}}} a_{\text{ac}}(t) \, dt \]

where \( \omega_{\text{max}} \) is the maximum angular rate.
Substituting Eq. (25) into Eq. (18), the performance $J$ can be simplified as

$$J = \frac{1}{2} \int_0^{t_{ac}} \left( a_{ac}(t) - \frac{\omega_{\text{max}}}{t_{ac}} \right)^2 dt$$  \hspace{1cm} (26)

Based on the principle of the calculus of variations, to minimize the performance index, Eq. (26) subject to the constraints of Eq. (23), Eq. (24) and Eq. (25), the following Euler-Lagrange equation should be satisfied:

$$\frac{\partial L}{\partial a_{ac}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{a}_{ac}} \right) = 0$$  \hspace{1cm} (27)

where $L$ is the Lagrangian, which is expressed as

$$L = \frac{1}{2} \left( a_{ac}(t) - \frac{\omega_{\text{max}}}{t_{ac}} \right)^2 + \lambda a_{ac}(t)$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left( \lambda_{ij} a_{ac}(t)e^{\kappa_{ij}\xi_i t} \cos \left( \kappa_{ij}\sqrt{1-\xi_i^2} t \right) \right)$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{n_j} \left( \lambda_{ij} a_{ac}(t)e^{\kappa_{ij}\xi_i t} \sin \left( \kappa_{ij}\sqrt{1-\xi_i^2} t \right) \right)$$  \hspace{1cm} (28)

where $\lambda$, $\lambda_{ij}$ and $\lambda_{ij}$ are constant co-state variables.

Substituting Eq. (28) into Eq. (27), the optimal expression of $a_{ac}(t)$ is obtained as

$$a_{ac}(t) = \frac{\omega_{\text{max}}}{t_{ac}} - \lambda$$

$$- \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{ij} e^{\kappa_{ij}\xi_i t} \cos \left( \kappa_{ij}\sqrt{1-\xi_i^2} t \right)$$

$$+ \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{ij} e^{\kappa_{ij}\xi_i t} \sin \left( \kappa_{ij}\sqrt{1-\xi_i^2} t \right)$$  \hspace{1cm} (29)

The expression for $a_{ac}(t)$ was then obtained as shown in Eq. (29), but there are $1 + \sum_{j=1}^{m} n_j$ unknown constants $\lambda$, $\lambda_{ij}$ and $\lambda_{ij}$. The next step is to determine the unknown constants using the constraint equations Eq. (23), Eq. (24) and Eq. (25). First, substituting Eq. (29) into Eq. (23), Eq. (24) and Eq. (25), a group of linear equations of $1 + \sum_{j=1}^{m} n_j$ dimensions is then obtained as Eq. (33), where $l \in \mathbb{N}^+$ vary from 1 to $m$, $k \in \mathbb{N}^+$ vary from 1 to $n_l$, and $e_{ij}$, $s_{ij}$, $c_{ijl}$, $M_{ijkl}$, $N_{ijkl}$ and $S_{ijkl}$ are constants that can be calculated by Eq. (34).

The analytical expressions for $e_{ij}$, $s_{ij}$, $c_{ijl}$, $M_{ijkl}$, $N_{ijkl}$, $S_{ijkl}$ are derived from a simple knowledge of calculus. The unknown variables $\lambda$, $\lambda_{ij}$ and $\lambda_{ij}$ can be uniquely obtained by solving the linear equations of Eq. (33). The derivation of the optimal angular acceleration profile is now finished. In Eq. (29), the variables $t_{ac}$, $\kappa_{\text{min}j}$, and $\kappa_{\text{max}j}$ should be assigned in advance to ensure that the amplitude of the angular acceleration and the residual vibration ratio satisfy the following constraints:

$$\max(a_{ac}(t)) \leq a_{\text{max}}$$ and

$$\max \left( v(\kappa) \right) \leq v_{\text{max}}, \quad \kappa \in \bigcup_{j=1}^{m} \left[ \kappa_{L_j}, \kappa_{U_j} \right]$$  \hspace{1cm} (30)

where $a_{\text{max}}$ represents the maximum allowed angular acceleration, and $v_{\text{max}}$ represents the maximum allowed residual vibration ratio. To describe the changes in $\kappa_{\text{min}j}$ and $\kappa_{\text{max}j}$, we define a variable $\alpha_j$ expressed as

$$\alpha_j = \kappa_{\text{min}j} - \kappa_{L_j} = \kappa_{U_j} - \kappa_{\text{max}j}$$  \hspace{1cm} (32)

The $\alpha_j$ can be suitably and simply selected by setting them to the same value denoted by $\alpha$ in Fig. 1, which means the following equation holds:

$$\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_m$$  \hspace{1cm} (35)

Then, $\max(a_{ac}(t))$ and $\max(v(\kappa))$ are determined uniquely by $t_{ac}$ and $\alpha$. With certain $n_j$, $\kappa_j$ and $\beta$, the relationship between $t_{ac}$, $\alpha$ and $\max(a_{ac}(t))$ and the relationship between $t_{ac}$, $\alpha$ and $\max(v(\kappa))$ can be obtained. These relationships can be visualized via a contour line, and then optimal $t_{ac}$ and $\alpha$ can be found to keep both the angular acceleration and residual vibration within acceptable bounds, at the same time, to minimize the acceleration time.

The following is an example. The parameters are $\kappa_1 = 1\text{rad/s}$, $\zeta = 0$, and $\beta = 0.36$, $a_{\text{max}} = 0.003\text{rad/s}^2$ and $v_{\text{max}} = 0.05$. The contour line of $\max(a_{ac}(t))$ and $\max(v(\kappa))$ with $n_1 = 3$, $n_1 = 4$ and $n_1 = 5$ are then shown in Figs. 2, 3 and 4, respectively. The horizontal ordinate of these figures is $\alpha$, the vertical ordinate is acceleration time, and the counter line represents the maximum acceleration and maximum residual vibration ratio respectively.

Under the constraints of Eq. (30) and Eq. (31), the values of $t_{ac}$ and $\alpha$ corresponding to the minimum acceleration time are

$$t_{ac} = 14.5s, \alpha = 0.043 \quad \text{for} \quad n_1 = 3$$

$$t_{ac} = 13.9s, \alpha = 0.025 \quad \text{for} \quad n_1 = 4$$

$$t_{ac} = 16.5s, \alpha = 0.02 \quad \text{for} \quad n_1 = 5$$  \hspace{1cm} (36)

In these three groups of values, the acceleration time of $n_1 = 4$ is a minimum, the values of $t_{ac}$, $\alpha$ and $n_1$ were therefore chosen to be $t_{ac} = 13.9s$, $\alpha = 0.025$ and $n_1 = 4$. The corresponding acceleration curve and residual vibration ratio are shown in Fig. 5.

From these figures, we can summarize an approximation law: fix $t_{ac}$ and $\max(a_{ac}(t))$ will increase approximately with the increase in $n_j$, but $\max(v(\kappa))$ will decrease approximately with the increase in $n_j$. Therefore, if the constraints shown in Eq. (30) and Eq. (31) are not satisfied, we can adjust $n_j$ according to this law. In addition, from Fig. 5 we found an interesting law: if $\xi = 0$ the angular acceleration in the acceleration region will be symmetrical about line $t = \frac{t_{ac}}{2}$. This law is generally true with other simulations, but it is difficult to prove.

To obtain the integrated angular acceleration profile, the dwell time, which is denoted $t_{\text{dwell}}$, should be calculated first. $t_{\text{dwell}}$ can be calculated by the following steps. The maneuver Euler angle, $\Phi_f$, can be expressed as

$$\Phi_f = \Phi_{ac} + \Phi_{\text{dwell}} + \Phi_{\text{dec}}$$  \hspace{1cm} (37)

where $\Phi_{ac}$ is the rotation angle in the acceleration region, $\Phi_{\text{dwell}}$ is the rotation angle in the dwell region, and
\begin{equation}
\begin{aligned}
-M_{ac} &= \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{cij} C_{ij} - \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{sij} S_{ij} = 0 \\
\left(\omega_{ac}^{\text{max}} - \lambda\right) c_{kl} &= \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{cijkl} - \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{sijkl} M_{ijkl} = 0 \\
\left(\omega_{ac}^{\text{max}} - \lambda\right) s_{kl} &= \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{cijkl} N_{ijkl} - \sum_{j=1}^{m} \sum_{i=1}^{n_j} \lambda_{sijkl} S_{ijkl} = 0
\end{aligned}
\end{equation}

\begin{align}
p_{ij} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} \cos \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) dt \\
p_{ij} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} \sin \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) dt \\
C_{ij kl} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} e^{\kappa_{kl} t} \cos \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) \cos \left(\kappa_{kl} \sqrt{1 - \xi^2 t}\right) dt \\
M_{ijkl} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} e^{\kappa_{kl} t} \sin \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) \cos \left(\kappa_{kl} \sqrt{1 - \xi^2 t}\right) dt \\
N_{ijkl} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} e^{\kappa_{kl} t} \cos \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) \sin \left(\kappa_{kl} \sqrt{1 - \xi^2 t}\right) dt \\
S_{ijkl} &= \int_{0}^{t_{\text{ac}}} e^{\kappa_{ij} t} e^{\kappa_{kl} t} \sin \left(\kappa_{ij} \sqrt{1 - \xi^2 t}\right) \sin \left(\kappa_{kl} \sqrt{1 - \xi^2 t}\right) dt
\end{align}

Fig. 2. Contour line with $n_1 = 3$: a) Maximum amplitude of the angular acceleration profile b) Maximum residual vibration ratio

Fig. 3. Contour line with $n_1 = 4$: a) Maximum amplitude of the angular acceleration profile b) Maximum residual vibration ratio

Fig. 4. Contour line with $n_1 = 5$: a) Maximum amplitude of the angular acceleration profile b) Maximum residual vibration ratio
**Fig. 5.** Shaped angular acceleration profile and residual vibration ratio with \( n_1 = 4: \) a) Shaped angular acceleration profile b) Residual vibration ratio

\( \Phi_{\text{dec}} \) is the rotation angle in the deceleration region. The expression for \( \Phi_{\text{ac}}, \Phi_{\text{dwell}}, \) and \( \Phi_{\text{dec}} \), which are shown in Fig. 6, are

\[
\Phi_{\text{ac}} = \int_0^{t_{\text{ac}}} \left( \int_0^t a_{\text{ac}}(\tau) d\tau \right) dt, \quad (38)
\]

\[
\Phi_{\text{dwell}} = \omega_{\text{max}} t_{\text{dwell}}, \text{ and} \quad (39)
\]

\[
\Phi_{\text{dec}} = \int_{t_2}^{t_3} \left( \omega_{\text{max}} - \int_{t_2}^t a_{\text{ac}}(\tau - t_2) d\tau \right) dt = \omega_{\text{max}} t_{\text{ac}} - \int_0^{t_{\text{ac}}} \left( \int_0^t a_{\text{ac}}(\tau) d\tau \right) dt \quad (40)
\]

where

\[
t_2 = t_{\text{ac}} + t_{\text{dwell}}, \text{ and} \quad (41)
\]

\[
t_3 = 2t_{\text{ac}} + t_{\text{dwell}} \quad (42)
\]

After substituting Eq. (38), Eq. (39), and Eq. (40) into Eq. (37), simplifying yields

\[
t_{\text{dwell}} = \frac{\Phi_f}{\omega_{\text{max}}} - t_{\text{ac}} \quad (43)
\]

To reduce the maneuver time, \( \omega_{\text{max}} \) should be chosen to be as large as possible under the constraint of the actuator capability. However, \( t_{\text{dwell}} \) must equal or be greater than 0, and the following constraint should therefore be satisfied:

\[
\Phi_f \geq \omega_{\text{max}} t_{\text{ac}} \quad (44)
\]

**IV. Simulations**

This section presents a group of simulations to verify that the designed angular acceleration profile can reduce the residual vibration effectively with frequency uncertainty.

**A. Simulation Description**

The simulation parameters are presented in Table I. The lower and upper bounds of the natural frequencies can be calculated as follows: \( \kappa_{L1} = 0.64 \text{rad/s}, \kappa_{U1} = 1.36 \text{rad/s}, \kappa_{L2} = 3.2 \text{rad/s}, \) and \( \kappa_{U2} = 6.8 \text{rad/s}. \) In the simulation, the actual frequencies are selected as their lower bounds, which are 0.64 rad/s and 3.2 rad/s respectively.

**Table I**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>500 20 13 kg · m²</td>
</tr>
<tr>
<td>( \Omega )</td>
<td>diag([1 5]) rad/s</td>
</tr>
<tr>
<td>( B )</td>
<td>[1.1 −2.2 0] −1.6 −10.7 −2.3 T</td>
</tr>
<tr>
<td>( \xi )</td>
<td>0.01</td>
</tr>
<tr>
<td>( \beta )</td>
<td>36%</td>
</tr>
<tr>
<td>( k_p )</td>
<td>0.0016</td>
</tr>
<tr>
<td>( k_d )</td>
<td>0.072</td>
</tr>
<tr>
<td>( a_{\text{max}} )</td>
<td>0.003 rad/s²</td>
</tr>
<tr>
<td>( v_{\text{max}} )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \omega_{\text{max}} )</td>
<td>0.175 rad/s</td>
</tr>
<tr>
<td>( \Phi_f )</td>
<td>1 rad</td>
</tr>
</tbody>
</table>
Considering the actuator capacity, the amplitude of the angular acceleration should satisfy the following requirement:

$$\max (a_{ac}(t)) \leq a_{\text{max}} \quad t \in [0 \quad t_{ac}] \quad (45)$$

In addition, to reduce residual vibrations surrounding the natural frequencies, the maximum amplitude of the residual vibration ratio should satisfy the following requirement:

$$\max(v(\kappa)) < v_{\text{max}} \quad \kappa \in [\kappa_{L1} \quad \kappa_{U1}] \cup [\kappa_{L2} \quad \kappa_{U2}] \quad (46)$$

The optimal values of \( n_1 \) and \( n_2 \) were chosen as \( n_1 = 4 \) and \( n_2 = 14 \), which correspond to the minimum acceleration time. Actually, the contour lines of \( \max(a_{ac}(t)) \) and \( \max(v(\kappa)) \) for all possible sets of \( n_1 \) and \( n_2 \) should be given to choose the optimal \( n_1 \) and \( n_2 \) to minimize the acceleration time under the conditions of Eq. (30) and Eq. (31). However, for simplicity, only the contour lines with optimal \( n_1 \) and \( n_2 \) are given.

Fig. 7(a) shows the contour lines of \( \max(a_{ac}(t)) \), and Fig. 7(b) shows the contour lines of \( \max(v(\kappa)) \). With the requirements shown in inequalities Eq. (30) and Eq. (31), we found that the minimum \( t_{ac} \) is 15.5s, and the corresponding \( \alpha \) is 0.025.

According to the parameters chosen in the previous section, the shaped angular acceleration profile is shown in Fig. 8(a), where the maximum amplitude is less than \( a_{\text{max}} \), and the corresponding residual vibration ratio is shown in Fig. 8(b), where the residual vibration ratio is less than \( v_{\text{max}} \) in regions \([\kappa_{L1} \quad \kappa_{U1}] \) and \([\kappa_{L2} \quad \kappa_{U2}] \). The requirements of Eq. (40) and Eq. (41) are both satisfied.

For comparison, we designed three sets of simulations with different reference trajectories. The first reference trajectory was generated using a step input of magnitude \( a_{\text{max}} \), the second reference trajectory was generated using an s-curve, and the last reference trajectory was generated using the proposed angular acceleration profile. The selection of the s-curve parameters was presented in [9], where the profiles of the angular acceleration and the angular velocity are shown in Fig. 9, where \( t_a \) is the unique parameter of s-curve. According to [9], the optimal value of \( t_a \) is \( \omega_{\text{max}}/a_{\text{max}} \) for this simulation.

The actual frequencies in that set of simulations were set to the lower bounds, \([0.64 \quad 3.2] \text{ rad/s} \). In addition, the simulation time was 150s, and the maneuver began at 50s.

B. Simulation Results

The integrated shaped angular acceleration profile must contain the acceleration, dwell and deceleration regions. So the shaped angular acceleration profile in the acceleration region must be extended by adding the dwell and deceleration regions. For the shaped angular acceleration profile, the dwell time was determined to be \( t_{\text{dwell}} = 41.8 \)s according to Eq. (43). The integrated acceleration profiles for these three sets of simulations were then obtained and are shown in Fig. 10.

The simulation results are shown in Fig. 11, Fig. 12 and Table. II.

Fig. 11 shows the curve of the residual vibrations. The first-order modal vibrations are shown in Fig. 11(a), where the amplitude of the residual vibration is 0.02 by the excitation of the step input, 0.015 by that of the s-curve input, and \( 3 \times 10^{-4} \) by that of the shaped angular acceleration profile. The second-order modal vibrations are shown in Fig. 11(b), where the amplitude of the residual vibration is \( 2.5 \times 10^{-3} \) by the excitation of the step input, \( 1.5 \times 10^{-4} \) by that of the s-curve input, and \( 1 \times 10^{-5} \) by that of the shaped angular acceleration profile. The simulation results shown in Fig. 11 indicate the angular acceleration profile designed by the proposed method can suppress the residual vibration within a certain range. The simulation results also show that the shaped angular acceleration profile has a better vibration suppression effect than the s-curve input, which is because the s-curve input is not suitable for the vibration suppression with a low natural frequency.

The curves for the Euler angle and the projection of angular velocity onto the eigen-axis denoted by \( \omega \) are shown in Fig. 12. The control errors of the Euler angle shown in Fig. 12(a) are \( 1 \times 10^{-4} \text{rad} \), \( 5 \times 10^{-5} \text{rad} \), and...
Fig. 7. Contour lines: a) Maximum amplitude of the angular acceleration profile b) Residual vibration ratio

Fig. 8. Angular acceleration profile and residual vibration ratio a) Angular acceleration profile in the acceleration region b) Residual vibration ratio

TABLE II
Simulation results

<table>
<thead>
<tr>
<th>Input</th>
<th>Maneuver time (s)</th>
<th>$V_1$</th>
<th>$V_2$</th>
<th>Euler angle error (rad)</th>
<th>Angular rate error (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step</td>
<td>100</td>
<td>0.02</td>
<td>2.5 $\times$ 10$^{-3}$</td>
<td>1 $\times$ 10$^{-4}$</td>
<td>5 $\times$ 10$^{-5}$</td>
</tr>
<tr>
<td>S-curve</td>
<td>86.5</td>
<td>0.015</td>
<td>1.5 $\times$ 10$^{-3}$</td>
<td>5 $\times$ 10$^{-5}$</td>
<td>2 $\times$ 10$^{-5}$</td>
</tr>
<tr>
<td>Shaped profile</td>
<td>72.8</td>
<td>3 $\times$ 10$^{-4}$</td>
<td>1 $\times$ 10$^{-5}$</td>
<td>5 $\times$ 10$^{-6}$</td>
<td>3 $\times$ 10$^{-6}$</td>
</tr>
</tbody>
</table>

Fig. 11. Actual modal vibrations: a) First-order natural frequency b) Second-order natural frequency

5 $\times$ 10$^{-6}$ rad using the step input, the s-curve input, and the shaped angular acceleration profile, respectively. The control errors of the angular rate shown in Fig. 12(b) are 5 $\times$ 10$^{-5}$ rad/s, 2 $\times$ 10$^{-5}$ rad, and 3 $\times$ 10$^{-6}$ rad/s using
the step input, the s-curve input, and the shaped angular acceleration profile, respectively.

For the step input, the absolute value of the eigen-angle and angular velocity were within $1 \times 10^{-4}\text{rad}$ and $1 \times 10^{-5}\text{rad/s}$ after 150s, and the corresponding maneuver time was therefore 100s. For the s-curve input, the absolute value of the eigen-angle and angular velocity were within $1 \times 10^{-4}\text{rad}$ and $2 \times 10^{-5}\text{rad/s}$ after 136.5s, and the corresponding maneuver time was therefore 86.5s. For the shaped angular acceleration profile, the absolute values of the eigen-angle and angular velocity were within $1 \times 10^{-4}\text{rad}$ and $2 \times 10^{-5}\text{rad/s}$ after 122.8s, and the corresponding maneuver time was therefore 72.8s.

Obviously, compared with the step input and the s-curve input, the shaped angular acceleration profile was able to reduce the residual vibrations at the end of the maneuver. Therefore, the maneuver using the shaped input had a higher control precision and shorter maneuver time. In fact, the reference trajectory generated by the step input and the s-curve input had a shorter duration than that generated by the shaped input. However, convergence to a certain control precision requires a long time, which would lead to a longer maneuver time.

V. Conclusions

This paper presents a method for designing reference angular acceleration profiles for flexible satellites to reduce residual vibrations at the ends of maneuvers. An analytical expression for the shaped angular acceleration profiles was obtained to minimize the selected performance. The shaped angular acceleration profiles can be used in a multimode system even if the damping ratio is not 0. The simulations show that suitable angular acceleration profiles can be designed using this method, with a consideration of the residual vibration magnitude limit and angular acceleration magnitude limit.

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REFERENCES


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